Abstract. — Bromberg and Ulcigrai constructed piecewise smooth functions on the circle such that the set of $\alpha$ for which the sum $\sum_{k=0}^{n-1} f(x + k\alpha \mod 1)$ satisfies a temporal distributional limit theorem along the orbit of a.e. $x$ has Hausdorff dimension one. We show that the Lebesgue measure of this set is equal to zero.

1. Introduction and statement of main result

1.1. Background

Suppose $T : X \to X$ is a map, $f : X \to \mathbb{R}$ is a function, and $x_0 \in X$ is a fixed initial condition. We say that the $T$-ergodic sums $S_n = f(x_0) + f(Tx_0) + \cdots + f(T^{n-1}x_0)$

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satisfy a temporal distributional limit theorem (TDLT) on the orbit of \( x_0 \), if there exists a non-constant real valued random variable \( Y \), centering constants \( A_N \in \mathbb{R} \) and scaling constants \( B_N \to \infty \) s.t.

\[
S_n - A_N B_N^{-1} \to^d Y \quad \text{as} \quad n \to \infty
\]

when \( n \) is sampled uniformly from \( \{1, \ldots, N\} \) and \( x_0 \) is fixed. Equivalently, for every Borel set \( E \subset \mathbb{R} \) s.t. \( \mathbb{P}(Y \in \partial E) = 0 \),

\[
\frac{1}{N} \text{Card} \left\{ 1 \leq n \leq N : \frac{S_n - A_N}{B_N} \in E \right\} \to \mathbb{P}(Y \in E).
\]

We allow and expect \( A_N, B_N, Y \) to depend on \( T, f, x_0 \).

Such limit theorems have been discovered for several zero entropy uniquely ergodic transformations, including systems where the more traditional spatial limit theorems, with \( x_0 \) is sampled from a measure on \( X \), fail [Bec10, Bec11, ADDS15, DS17b, PS17, DS18]. Of particular interest are TDLT for \( R^\alpha \colon [0,1] \to [0,1] \), \( R^\alpha(x) = x + \alpha \mod 1 \), \( f_\beta(x) := 1_{[0,\beta)}(x) - \beta \), because the \( R^\alpha \)-ergodic sums of \( f_\beta \) along the orbit of \( x \) represent the discrepancy of the sequence \( x + n\alpha \mod 1 \) with respect to \( [0,\beta) \) [Sch78, CK76, Bec10]. Another source of interest is the connection to the “deterministic random walk” [AK82, ADDS15].

The validity of the TDLT for \( R^\alpha \) and \( f_\beta \) depends on the diophantine properties of \( \alpha \) and \( \beta \). Recall that \( \alpha \in (0,1) \) is badly approximable if for some \( c > 0 \), \( |qa - p| > c/q \) for all irreducible fractions \( p/q \). Equivalently, the digits in the continued fraction expansion of \( \alpha \) are bounded [Khi63]. Say that \( \beta \in (0,1) \) is badly approximable with respect to \( \alpha \) if for some \( C > 0 \), \( |qa - \beta - p| > C/q \) for all \( p, q \in \mathbb{Z}, q \neq 0 \). If \( \alpha \) is badly approximable then every \( \beta \in \mathbb{Q} \cap (0,1) \) is badly approximable with respect to \( \alpha \). The recent paper [BU18] shows:

**Theorem 1.1** (Bromberg–Ulcigrai [BU18]). — Suppose \( \alpha \) is badly approximable and \( \beta \) is badly approximable with respect to \( \alpha \), e.g. \( \beta \in \mathbb{Q} \cap (0,1) \). Then the \( R^\alpha \)-ergodic sums of \( f_\beta \) satisfy a temporal distributional limit theorem with Gaussian limit on the orbit of every initial condition.

The set of badly approximable \( \alpha \) has Hausdorff dimension one [Jar29], but Lebesgue measure zero [Khi24]. This leads to the following question: Is there a \( \beta \) s.t. the \( R^\alpha \)-ergodic sums of \( f_\beta \) satisfy a temporal distributional limit theorem for a.e. \( \alpha \) and a.e. initial condition?

In this paper we answer this question negatively.

### 1.2. Main result

To state our result in its most general form, we need the following terminology.

Let \( T := \mathbb{R}/\mathbb{Z} \). We say that \( f : T \to \mathbb{R} \) is piecewise smooth if there exists a finite set \( \mathcal{G} \subset T \) s.t. \( f \) is continuously differentiable on \( T \setminus \mathcal{G} \) and \( \exists \psi : T \to \mathbb{R} \) with bounded variation s.t. \( f' = \psi \) on \( T \setminus \mathcal{G} \). For example: \( f_\beta(x) = 1_{[0,\beta)}(x) - \beta \) (take \( \mathcal{G} = \{0,\beta\}, \psi \equiv 0 \)). We show:
**Theorem 1.2.** — Let \( f \) be a piecewise smooth function of zero mean. Then there is a set of full measure \( \mathcal{E} \subset \mathbb{T} \times \mathbb{T} \) s.t. if \((\alpha, x) \in \mathcal{E} \) then the \( R_\alpha \)-ergodic sums of \( f \) do not satisfy a TDLT on the orbit of \( x \).

The condition \( f_\tau f = 0 \) is necessary: By Weyl’s equidistribution theorem, for every \( \alpha \not\in \mathbb{Q} \), \( f \) Riemann integrable s.t. \( f_\tau f = 1 \), and \( x_0 \in \mathbb{T} \), \( S_n/N \xrightarrow{\text{dist}} U[0, 1] \) as \( n \sim U(1, \ldots, N) \). See Section 1.4 for the notation.

This paper has a companion [DS17a] which gives a different proof of Theorem 1.2, \( \sim \) means that \( f_\tau f = 0 \) can hold with \( f \) Riemann integrable s.t. \( f_\tau f = 1 \), and \( x_0 \in \mathbb{T} \), \( S_n/N \xrightarrow{\text{dist}} U[0, 1] \) as \( n \sim U(1, \ldots, N) \). See Section 1.4 for the notation.

The methods of [DS17a] are specific for the transfer function, but this is not a TDLT since no actual scaling is involved. The heart of the proof is to show that if \( f \) is discontinuous, then for a.e. \( \alpha \), the temporal distributions of the ergodic sums have different asymptotic scaling behavior on different subsequences. The proof of this has three independent parts:

1. A reduction to the case \( f(x) = \sum_{m=1}^d b_m h(x + \beta_m) \), \( h(x) := \{x\} - \frac{1}{2} \).
2. A proof that if \( N \subset \mathbb{N} \) has positive lower density, then there exists \( M \geq 1 \) s.t. the following set has full Lebesgue measure in \((0, 1)\):

\[
\mathcal{A}(N, M) := \left\{ \alpha \in (0, 1) : \exists n_k \uparrow \infty, \ r_k \leq M \ \text{s.t. for all } k : \ r_k q_{n_k} \in N, \ a_{n_{k+1}}/(a_1 + \cdots + a_{n_k}) \to \infty \right\}.
\]

Here \( a_n \) and \( q_n \) are the partial quotients and principal denominators of \( \alpha \), see Section 3.1.

3. Construction of \( N = N(b_1, \ldots, b_d; \beta_1, \ldots, \beta_d) \subset \mathbb{N} \) with positive density, s.t. for every \( \alpha \in \mathcal{A}(N, M) \) and a.e. \( x \), one can analyze the temporal distributions of the Birkhoff sums of \( \sum_{m=1}^d b_m h(x + \beta_m) \).

**1.3. The structure of the proof**

Suppose \( f \) is piecewise smooth and has mean zero.

We shall see below that if \( f \) is continuous, then for a.e. \( \alpha \), \( f \) is an \( R_\alpha \)-coboundary, therefore \( S_n \) are bounded, hence (1.1) cannot hold with \( B_N \to \infty \), \( Y \) non-constant. We remark that (1.1) does hold with \( B_N \equiv 1 \), \( A_N = f(x_0) \), \( Y = \text{distribution of minus the transfer function} \), but this is not a TDLT since no actual scaling is involved.

The heart of the proof is to show that if \( f \) is discontinuous, then for a.e. \( \alpha \), the temporal distributions of the ergodic sums have different asymptotic scaling behavior on different subsequences. The proof of this has three independent parts:

1. A reduction to the case \( f(x) = \sum_{m=1}^d b_m h(x + \beta_m) \), \( h(x) := \{x\} - \frac{1}{2} \).
2. A proof that if \( N \subset \mathbb{N} \) has positive lower density, then there exists \( M \geq 1 \) s.t. the following set has full Lebesgue measure in \((0, 1)\):

\[
\mathcal{A}(N, M) := \left\{ \alpha \in (0, 1) : \exists n_k \uparrow \infty, \ r_k \leq M \ \text{s.t. for all } k : \ r_k q_{n_k} \in N, \ a_{n_{k+1}}/(a_1 + \cdots + a_{n_k}) \to \infty \right\}.
\]

Here \( a_n \) and \( q_n \) are the partial quotients and principal denominators of \( \alpha \), see Section 3.1.

3. Construction of \( N = N(b_1, \ldots, b_d; \beta_1, \ldots, \beta_d) \subset \mathbb{N} \) with positive density, s.t. for every \( \alpha \in \mathcal{A}(N, M) \) and a.e. \( x \), one can analyze the temporal distributions of the Birkhoff sums of \( \sum_{m=1}^d b_m h(x + \beta_m) \).

**1.4. Notation**

\( n \sim U(1, \ldots, N) \) means that \( n \) is a random variable taking values in \( \{1, \ldots, N\} \), each with probability \( \frac{1}{N} \). \( U[a, b] \) is the uniform distribution on \( [a, b] \). Lebesgue’s measure is denoted by \( \text{mes} \). \( \mathbb{N} = \{1, 2, 3, \ldots\} \) and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). If \( x \in \mathbb{R} \), then \( \|x\| := \text{dist}(x, \mathbb{Z}) \) and \( \{x\} \) is the unique number in \( [0, 1) \) s.t. \( x \in \{x\} + \mathbb{Z} \). Card(\( \cdot \)) is the cardinality. If \( \varepsilon > 0 \), then \( a = b \pm \varepsilon \) means that \( |a - b| \leq \varepsilon \).
2. Reduction to the case $f(x) = \sum_{m=1}^{d} b_m h(x + \beta_m)$

Let $h(x) = \{x\} - \frac{1}{7}$, and let $\mathcal{G}$ denote the collection of all non-identically zero functions of the form $f(x) = \sum_{m=1}^{d} b_m h(x + \beta_m)$, where $d \in \mathbb{N}, b_m, \beta_m \in \mathbb{R}$. We explain how to reduce the proof of Theorem 1.2 from the case of a general piecewise smooth function $f(x)$ to the case $f \in \mathcal{G}$.

The following proposition was proved in [DS17a]. Let $C(\mathbb{T})$ denote the space of continuous real-valued functions on $\mathbb{T}$ with the sup norm.

**Proposition 2.1.** — If $f(t)$ is differentiable on $\mathbb{T} \setminus \{\beta_1, \ldots, \beta_d\}$ and $f'$ extends to a function with bounded variation on $\mathbb{T}$, then there are $d \in \mathbb{N}_0, b_1, \ldots, b_d \in \mathbb{R}$ s.t. for a.e. $\alpha \in \mathbb{T}$ there is $\varphi_\alpha \in C(\mathbb{T})$ s.t.

$$f(x) = \sum_{i=1}^{d} b_i h(x + \beta_i) + \int_{\mathbb{T}} f(t) \, dt + \varphi_\alpha(x) - \varphi_\alpha(x + \alpha) \quad (x \neq \beta_1, \ldots, \beta_d).$$

The following proposition was proved in [DS17b]. Let $(\Omega, \mathcal{B}, \mu)$ be a probability space, and let $T : \Omega \to \Omega$ be a probability preserving map.

**Proposition 2.2.** — Suppose $f = g + \varphi - \varphi \circ T$ $\mu$-a.e. with $g, \varphi : \Omega \to \mathbb{R}$ measurable. If the ergodic sums of $g$ satisfy a TDLT along the orbit of a.e. $x$, then so do the ergodic sums of $f$.

These results show that if Theorem 1.2 holds for every $f \in \mathcal{G}$, then Theorem 1.2 holds for any discontinuous piecewise smooth function with zero mean. As for continuous piecewise smooth functions with zero mean, these are $R_\alpha$-cohomologous to $g \equiv 0$ for a.e. $\alpha$ because the $b_i$ in Proposition 2.1 must all vanish. Since the zero function does not satisfy the TDLT, continuous piecewise smooth functions do not satisfy a TDLT.

3. The set $\mathcal{A}$ has full measure

3.1. Statement and plan of proof

Let $\alpha$ be an irrational number, with continued fraction expansion denoted by $[a_0; a_1, a_2, a_3, \ldots] := a_0 + \frac{1}{a_1 + \cdots},$ $a_0 \in \mathbb{Z}, a_i \in \mathbb{N} \ (i \geq 1)$. We call $a_n$ the quotients of $\alpha$. Let $p_n/q_n$ denote the principal convergents of $\alpha$, determined recursively by

$$q_{n+1} = a_{n+1} q_n + q_{n-1}, \quad p_{n+1} = a_{n+1} p_n + p_{n-1}$$

and $p_0 = a_0, q_0 = 1; p_1 = 1 + a_1 a_0, q_1 = a_1$. We call $q_n$ the principal denominators and $a_i$ the partial quotients of $\alpha$. Sometimes – but not always! – we will write $q_k = q_k(\alpha)$, $p_k = p_k(\alpha)$, $a_k = a_k(\alpha)$.

Given $\mathcal{N} \subset \mathbb{N}$ and $M \geq 1$, let $\mathcal{A} = \mathcal{A}(\mathcal{N}, M) \subset (0, 1)$ denote the set of irrational $\alpha \in (0, 1)$ s.t. for some subsequence $n_k \uparrow \infty$,

$$\exists \ r_k \leq M \ s.t. \ r_k q_{n_k} \in \mathcal{N}, \quad \frac{a_{n_k+1}}{(a_0 + \cdots + a_{n_k})} \xrightarrow{k \to \infty} \infty. \tag{3.1}$$

The lower density of $\mathcal{N}$ is $d(\mathcal{N}) := \liminf \frac{1}{N} \text{Card}(\mathcal{N} \cap [1, N])$. The purpose of this section is to prove:
THEOREM 3.1. — If a set \( \mathcal{N} \) has positive lower density, then there exists \( M \) such that \( \mathcal{A}(\mathcal{N}, M) \) has full Lebesgue measure in \((0,1)\).

The proof consists of the following three lemmas:

**LEMMA 3.2.** — For almost all \( \alpha \) there is \( n_0 = n_0(\alpha) \) s.t. if \( k \geq n_0 \) and \( a_{k+1} > \frac{1}{4}(\ln k)(\ln \ln k) \), then \( a_{k+1} / (a_1 + \cdots + a_k) \geq \frac{1}{8} \ln \ln k \).

**LEMMA 3.3.** — Suppose \( \alpha \in (0,1) \setminus \mathbb{Q} \) and \((p,q) \in \mathbb{N}_0 \times \mathbb{N} \) satisfy \( \gcd(p,q) = 1 \) and \(|q\alpha - p| \leq \frac{1}{qL} \) where \( L \geq 4 \). Then there exists \( k \) s.t. \( q = q_k(\alpha) \) and \( a_{k+1}(\alpha) \geq \frac{1}{2}L \).

**LEMMA 3.4.** — Suppose \( \psi : \mathbb{R}_+ \to \mathbb{R} \) is a non-decreasing function s.t.

\[
\sum_{n} \frac{1}{n\psi(n)} = \infty.
\]

Suppose \( \mathcal{N} \subset \mathbb{N} \) has positive lower density. For all \( M \) sufficiently large, for a.e. \( \alpha \in (0,1) \) there are infinitely many pairs \((m,n) \in \mathbb{N}_0 \times \mathbb{N} \) s.t. \( n \in \mathcal{N} \), \( \gcd(m,n) \leq M \), and \( |n\alpha - m| \leq \frac{1}{n\psi(n)} \).

**Remark 3.5.** — By the monotonicity of \( \psi \), if \( e^{k-1} < n < e^k \) then \( \psi(e^{k-1}) \leq \psi(n) \leq \psi(e^k) \). Hence (3.2) holds iff \( \sum_k \frac{1}{\psi(e^k)} = \infty \).

**Remark 3.6.** — If \( \mathcal{N} = \mathbb{N} \), then Lemma 3.4 holds with \( M = 1 \) by the classical Khinchine Theorem. We do not know if Lemma 3.4 holds with \( M = 1 \) for every set \( \mathcal{N} \) with positive lower density.

**Proof of Theorem 3.1 given Lemmas 3.2–3.4.** — We apply these lemmas with

\( \psi(t) = c(\ln t)(\ln \ln t)(\ln \ln \ln t) \) and \( c > 1 / \ln(1 + \sqrt{5}) \).

Fix \( M > 1 \) as in Lemma 3.4. Then \( \exists \Omega \subset (0,1) \) of full measure s.t. for every \( \alpha \in \Omega \) there are infinitely many \((m,n) \in \mathbb{N}_0 \times \mathbb{N} \) as follows. Let \( m^* := m / \gcd(m,n) \), \( n^* := n / \gcd(m,n) \), \( p := \gcd(m,n) \), then

1. \( pm^* \in \mathcal{N} \), \( p \leq M \), \( |n^* \alpha - m^*| = \frac{|n\alpha - m|}{p} \leq \frac{1}{n\psi(n^*)} \), \( n^* \leq n \);
2. \( \exists k \) s.t. \( n^* = q_k(\alpha) \) and \( a_{k+1}(\alpha) \geq \frac{1}{2}\psi(q_k) \) (\( \therefore \) Lemma 3.3). By its recursive definition, \( q_k \geq k \)-th Fibonacci number \( \geq \frac{1}{3}(1 + \sqrt{5})^k \). So for all \( k \) large enough,
   \[
a_{k+1}(\alpha) \geq \frac{1}{2}\psi(q_k) > \frac{1}{2}(\ln k)(\ln \ln k); \]
3. \( a_{k+1}(\alpha) \geq \frac{1}{2}\psi(q_k) > \frac{1}{2}(\ln k)(\ln \ln k) \to \infty \) (\( \therefore \) Lemma 3.2).

So every \( \alpha \in \Omega \) belongs to \( \mathcal{A} = \mathcal{A}(\mathcal{N}, M) \), and \( \mathcal{A} \) has full measure.

Next we prove Lemmas 3.2–3.4.

**3.2. Proof of Lemma 3.2**

By [DV86], for almost every \( \alpha \)

\[
\frac{(a_1 + \cdots + a_{k+1}) - \max_{j \leq k+1} a_j}{k \ln k} \to \frac{1}{\ln 2} < 2.
\]

So if \( k \) is large enough, and \( a_{k+1} > \frac{1}{4}(\ln k)(\ln \ln k) \) then

\[
\max_{j \leq k+1} a_j = a_{k+1}, \quad \frac{a_1 + \cdots + a_k}{k \ln k} \leq 2, \quad \text{and} \quad \frac{a_{k+1}}{a_1 + \cdots + a_k} > \frac{1}{8} \ln \ln k.
\]
3.3. Proof of Lemma 3.3

For every \( (p, q) \) as in the lemma, \(|q\alpha - p| < \frac{1}{2q}\). A classical result in the theory of continued fractions [Khi63, Theorem 19] says that in this case \( \exists k \text{ s.t. } q = q_k(\alpha), p = p_k(\alpha) \).

To estimate \( a_{k+1} = a_{k+1}(\alpha) \) we recall the following facts, valid for the principal denominators of any irrational \( \alpha \in (0, 1) \) [Khi63]:

1. \(|q_k\alpha - p_k| > \frac{1}{q_k+q_{k+1}};\)
2. \( q_{k+1} + q_k < (a_{k+1} + 2)q_k \), whence by (a) \( a_{k+1} > \frac{1}{q_k|q_k\alpha - p_k|} - 2 \).

In our case, \(|q_k\alpha - p_k| = |q\alpha - p| \leq \frac{1}{qL}, \) so \( a_{k+1} > L - 2 \geq \frac{L}{2} \). \( \Box \)

3.4. Preparations for the proof of Lemma 3.4

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space, and \( A_k \in \mathcal{F} \) be measurable events. Given \( D > 1 \), we say that \( A_k \) are \( D \)-quasi-independent, if

\[
\mathbb{P}(A_k \cap A_{k_2}) \leq D \mathbb{P}(A_k) \mathbb{P}(A_{k_2}) \quad \text{for all } k \neq k_2.
\]

The following proposition is a slight variation on Sullivan’s Borel–Cantelli Lemma from ([Sul82]):

**Proposition 3.7.** — For every \( D \geq 1 \) there exists a constant \( \delta(D) > 0 \) such that the following holds in any probability space:

(a) If \( A_k \) are \( D \)-quasi-independent measurable events s.t. \( \lim_{k \to \infty} \mathbb{P}(A_k) = 0 \) but \( \sum_k \mathbb{P}(A_k) = \infty, \) then \( \mathbb{P}(A_k \text{ occurs infinitely often}) \geq \delta(D). \)

(b) The quasi-independence assumption in (a) can be weakened to the assumption that for some \( r \in \mathbb{N}, \) \( \mathbb{P}(A_k \cap A_{k_2}) \leq D \mathbb{P}(A_k) \mathbb{P}(A_{k_2}) \) for all \(|k_2 - k_1| \geq r. \)

(c) One can take \( \delta(D) = \frac{1}{2D}. \)

**Proof.** — Since \( \mathbb{P}(A_k) \to 0 \) but \( \sum \mathbb{P}(A_k) = \infty, \) there is an increasing sequence \( N_j \) such that \( \lim_{j \to \infty} \sum_{k=N_{j+1}}^{N_{j+1}} \mathbb{P}(A_k) = \frac{1}{D}. \)

Let \( B_j \) be the event that at least one of events \( \{A_k\}_{k=N_{j+1}}^{N_{j+1}+1} \) occurs. Since \( B_j = \bigcup_{k=N_{j+1}}^{N_{j+1}+1} (A_k \setminus \bigcup_{k=N_{j+1}}^{k-1} A_j) \),

\[
\mathbb{P}(B_j) \geq \sum_{k=N_{j+1}}^{N_{j+1}+1} \mathbb{P}(A_k) - \sum_{k=N_{j+1}}^{N_{j+1}+1} \mathbb{P}(A_k \cap A_{k_2})
= \sum_{k=N_{j+1}}^{N_{j+1}+1} \mathbb{P}(A_k) - D \sum_{k=N_{j+1}}^{N_{j+1}+1} \mathbb{P}(A_k) \mathbb{P}(A_{k_2})
\geq \sum_{k=N_{j+1}}^{N_{j+1}+1} \mathbb{P}(A_k) - \frac{D}{2} \left( \sum_{k=N_{j+1}}^{N_{j+1}+1} \mathbb{P}(A_k) \right)^2.
\]

Since \( \lim_{j \to \infty} \sum_{k=N_{j+1}}^{N_{j+1}+1} \mathbb{P}(A_k) = \frac{1}{D} \) and \( D \geq 1, \) \( \lim \inf \mathbb{P}(B_j) \geq \frac{1}{2D}. \)

Let \( E \) denote the event that \( A_j \) happens infinitely often. \( E \) is also the event that \( B_j \) happens infinitely often, therefore \( E = \bigcap_{n=1}^{\infty} \bigcup_{j=n+1}^{\infty} B_j. \) In a probability space,
the measure of a decreasing intersection of sets is the limit of the measure of these sets. So \( \mathbb{P}(E) = \liminf \mathbb{P}(B_j) \geq \frac{1}{2D} \), proving (a) and (c).

Part (b) follows from part (a) by applying it to the sets \( \{A_{kr+\ell}\} \) where \( 0 \leq \ell \leq r-1 \) is chosen to get
\[
\sum_k \mathbb{P}(A_{kr+\ell}) = \infty.
\]

The multiplicity of a collection of measurable sets \( \{E_k\} \) is defined to be the largest \( K \) s.t. there are \( K \) different \( k_i \) with \( \mathbb{P}(\bigcap_{i=1}^K E_{k_i}) > 0 \).

**PROPOSITION 3.8.** — Let \( E_k \) be measurable sets in a finite measure space. If the multiplicity of \( \{E_k\} \) is less than \( K \), then
\[
\text{mes} \left( \bigcup_k E_k \right) \geq \frac{1}{K} \sum_k \text{mes}(E_k).
\]

*Proof.* — \( 1_{\bigcup_k E_k} \geq \frac{1}{K} \sum_i 1_{E_i} \), almost everywhere. \( \Box \)

**PROPOSITION 3.9.** — For every non-empty open interval \( I \subset [0,1] \),
\[
\text{Card} \left\{ (m,n) \in \{0,\ldots,N\}^2 : \frac{m}{n} \in I, \gcd(m,n) = 1 \right\} \sim 3 \text{mes}(I)N^2/\pi^2, \text{ as } N \to \infty.
\]

*Proof.* — This classical fact due to Dirichlet follows from the inclusion-exclusion principle and the identity \( \zeta(2) = \pi^2/6 \), see [HW08, Theorem 459]. \( \Box \)

**PROPOSITION 3.10.** — Suppose \( \alpha = [0;a_1,a_2,\ldots] \) and \( \bar{x} = [0;a_{\ell+1},a_{\ell+2},\ldots] \). Then the principal convergents \( p_{\ell}/q_{\ell} \) of \( \alpha \) and the principal convergents \( p_{\ell+1}/q_{\ell+1} \) of \( \alpha \) are related by
\[
\begin{pmatrix}
p_{\ell+1} & p_{\ell+1} \\
q_{\ell+1} & q_{\ell+1}
\end{pmatrix}
= \begin{pmatrix}
p_{\ell-1} & p_{\ell} \\
q_{\ell-1} & q_{\ell}
\end{pmatrix}
\begin{pmatrix}
\bar{p}_{\ell} & \bar{p}_{\ell+1} \\
\bar{q}_{\ell} & \bar{q}_{\ell+1}
\end{pmatrix}.
\]

*Proof.* — Since \( a_0 = 0 \), the recurrence relations for \( p_n/q_n \) imply
\[
\begin{pmatrix}
p_n & p_{n+1} \\
q_n & q_{n+1}
\end{pmatrix}
= \begin{pmatrix}
p_{n-1} & p_n \\
q_{n-1} & q_n
\end{pmatrix}
\begin{pmatrix}0 & 1 \\1 & a_{n+1}\end{pmatrix},
\begin{pmatrix}0 & 1 \\1 & a_1\end{pmatrix}.
\]
So
\[
\begin{pmatrix}0 & 1 \\1 & a_1\end{pmatrix} \cdots \begin{pmatrix}0 & 1 \\1 & a_{n+1}\end{pmatrix}.
\]
It follows that
\[
\begin{pmatrix}
p_{\ell+1} & p_{\ell+1} \\
q_{\ell+1} & q_{\ell+1}
\end{pmatrix}
= \begin{pmatrix}
p_{\ell-1} & p_{\ell} \\
q_{\ell-1} & q_{\ell}
\end{pmatrix}
\begin{pmatrix}
\bar{p}_{\ell} & \bar{p}_{\ell+1} \\
\bar{q}_{\ell} & \bar{q}_{\ell+1}
\end{pmatrix},
\]
where \( \bar{p}_i/\bar{q}_i \) are the principal convergents of \( \bar{x} := [0;a_{\ell+1},a_{\ell+2},\ldots] \). \( \Box \)

### 3.5. Proof of Lemma 3.4

Without loss of generality, \( \lim_{t \to \infty} \psi(t) = \infty \), otherwise replace \( \psi(t) \) by the bigger monotone function \( \psi(t) + \ln t \).
Fix $M > 1$, to be determined later. Let

$$
\Omega_k := \{(m, n) \in \mathbb{N}^2 : n \in \mathcal{N}, n \in [e^{k-1}, e^k], 0 < m < n, \gcd(m, n) \leq M\},
$$

$$
A_{m,n,k} := \left\{ \alpha \in \mathbb{T} : |n\alpha - m| \leq \frac{1}{e^{k}\psi(e^k)} \right\},
$$

$$
\mathcal{A}_k := \bigcup_{(m,n) \in \Omega_k} A_{m,n,k},
$$

$$
\mathcal{A} := \{\alpha \in \mathbb{T} : \alpha \text{ belongs to infinitely many } \mathcal{A}_k\}.
$$

The lemma is equivalent to saying that $\mathcal{A}$ has full Lebesgue measure for a suitable choice of $M$.

We will prove a slightly different statement. Fix $\varepsilon > 0$ small. Given an non-empty interval $I \subset [\varepsilon, 1-\varepsilon]$, let

$$
\Omega_k(I) := \left\{ (m, n) \in \Omega_k : \frac{m}{n} \in I \right\},
$$

$$
\mathcal{A}_k(I) := \bigcup_{(m,n) \in \Omega_k(I)} A_{m,n,k},
$$

$$
\mathcal{A}(I) := \{\alpha \in \mathbb{T} : \alpha \text{ belongs to infinitely many } \mathcal{A}_k(I)\}.
$$

We will prove that there exists a positive constant $\delta = \delta(\varepsilon, M)$ s.t. for all intervals $I \subset [\varepsilon, 1-\varepsilon]$, $\mes(\mathcal{A}(I) \cap I) \geq \delta \mes(I)$. It then follows by a standard density point argument (see below) that $\mathcal{A} \cap [\varepsilon, 1-\varepsilon]$ has full measure. Since $\varepsilon$ is arbitrary, the lemma is proved.

**Claim 1.** There exists $K = K(\varepsilon)$ such that for every $k > K$, the multiplicity of $\{A_{m,n,k} \}_{(m,n) \in \Omega_k(I)}$ is uniformly bounded by $M$.

**Proof.** Suppose $(m_i, n_i) \in \Omega_k(I)$ ($i = 1, 2$) and $A_{m_1,n_1,k} \cap A_{m_2,n_2,k} \neq \emptyset$. Then there is $\alpha$ s.t. $|n_i\alpha - m_i| \leq \delta_k := \frac{1}{e^{k}\psi(e^k)}$ ($i = 1, 2$). Choose $K = K(\varepsilon)$ so large that $k > K \Rightarrow \delta_k < \frac{\varepsilon}{4e\varepsilon}$.

If $k > K$, then $\alpha \geq \frac{m_i}{n_i} - \delta_k > \min I - \frac{\varepsilon}{2} > \frac{\varepsilon}{2}$. Let $r_i := \gcd(m_i, n_i)$ and $(n_i^*, m_i^*) := \frac{1}{r_i}(n_i, m_i)$. Then $|n_i^*\alpha - m_i^*| \leq \delta_k$ and $m_i^* \leq n_i^* \leq n_i \leq e^k$, so $|n_i^* m_i^* - n_i^* m_i^*| = \frac{1}{\alpha}|m_i^*(n_i^*\alpha - m_i^*) - m_i^*(n_i^*\alpha - m_i^*)| \leq \frac{2\delta_k}{\varepsilon^2} < 1$. So $n_i^* m_i^* = n_i^* m_i^*$. Since $\gcd(n_i^*, m_i^*) = 1$, $(n_i^*, m_i^*) = (n_i^*, m_i^*)$. It follows that $(n_2, m_2) \in \{(rn_i^*, rm_i^*) : r = 1, \ldots, M\}$. So the multiplicity of $\{A_{m,n,k} \}_{(m,n) \in \Omega_k(I)}$ is uniformly bounded by $M$. \hfill \Box

**Claim 2.** Let $d(\mathcal{N}) := \liminf \frac{1}{N} \Card(\mathcal{N} \cap [1, N]) > 0$, then there exists $M = M(\mathcal{N})$ and $\tilde{K} = \tilde{K}(\varepsilon, \mathcal{N}, |I|)$ s.t. for all $k > \tilde{K}$,

$$
d(\mathcal{N}) \frac{\mes(\mathcal{A}(I))}{4M\psi(e^k)} \leq \mes(\mathcal{A}_k(I)) \leq \frac{6 \mes(I)}{\psi(e^k)}.
$$

In particular, $\mes(\mathcal{A}_k(I)) \rightarrow 0$ as $k \rightarrow \infty$ and $\sum \mes(\mathcal{A}_k(I)) = \infty$.

**Proof.** $\mes(A_{m,n,k}) = \mes \left( \left[ \frac{m}{n} - r_{m,n}, \frac{m}{n} + r_{m,n} \right] \right) = 2r_{m,n}$ where $r_{m,n} = \frac{1}{ne^k\psi(e^k)}$.

Since $n \in [e^{k-1}, e^k]$,

$$
\frac{\Card(\Omega_k(I))}{M e^{2k}\psi(e^k)} \leq \mes(\mathcal{A}(I)) \leq \frac{e \Card(\Omega_k(I))}{e^{2k}\psi(e^k)},
$$
where the lower bound uses Claim 1 and Proposition 3.8.

Card(Ω_k(I)) satisfies the bounds A - B ≤ Card(Ω_k(I)) ≤ A where

\[
A := \text{Card}\left\{ (m,n) : n \in \mathcal{N}, \ n \in [e^{k-1}, e^k], \ \frac{m}{n} \in I \right\}
\]

\[
B := \text{Card}\left\{ (m,n) : n \in \mathcal{N}, \ n \in [e^{k-1}, e^k], \ \frac{m}{n} \in I, \gcd(m,n) \geq M \right\}.
\]

Choose \( \tilde{K} = \bar{K}(\varepsilon, \mathcal{N}, |I|) > K(\varepsilon) \) s.t. for all \( k > \tilde{K} \)

1. Card\{\( n \in \mathcal{N} : 0 < n < e^k \}\} ≥ \( \frac{1}{\sqrt{2}} d(\mathcal{N}) \)
2. Card\{\( n \in [e^{k-1}, e^k] \cap \mathbb{N} : p|n \}\} ≤ 2(e^k - e^{k-1})/p for all \( p \geq 1 \);
3. For all \( n > e^{\tilde{K}-1}, p \geq 1, \)
   \[
   \frac{n}{p^{1/2}} \text{mes}(I) \leq \text{Card}\left\{ m \in \mathbb{N} : \frac{m}{n} \in I, p|m \right\} \leq \frac{2n}{p} \text{mes}(I).
   \]

If \( k > \tilde{K} \), then \( \frac{1}{2} d(\mathcal{N}) e^{2k} \text{mes}(I) \leq A \leq 2e^{2k} \text{mes}(I) \) and

\[
B \leq \sum_{p=M}^{\infty} \text{Card}\left\{ (m,n) : n \in [e^{k-1}, e^k], \ \frac{m}{n} \in I, p|m,p|n \right\}
\]

\[
\leq \sum_{p=M}^{\infty} \frac{2(e^k - e^{k-1})}{p} \cdot \frac{2e^k \text{mes}(I)}{p} < 4e^{2k} \text{mes}(I) \sum_{p=M}^{\infty} \frac{1}{p^2}
\]

\[
\leq \frac{1}{4} d(\mathcal{N}) e^{2k} \text{mes}(I), \quad \text{provided we choose } M \text{ s.t. } \sum_{p=M}^{\infty} \frac{1}{p^2} < \frac{1}{16} d(\mathcal{N}).
\]

Together we get \( \frac{1}{4} d(\mathcal{N}) e^{2k} \text{mes}(I) \leq \text{Card}(\Omega_k(I)) \leq 2e^{2k} \text{mes}(I) \). The claim now follows from (3.5).

**Claim 3.** — There exists \( D = D(\mathcal{N}, M), r = r(M), \) and \( \tilde{K} = \bar{K}(\varepsilon, \mathcal{N}, I) \) s.t. for all \( k_1, k_2 > \tilde{K} \) s.t. \( |k_1 - k_2| > r(M) \),

\[
\text{mes}(\mathcal{A}_{k_1}(I) \cap \mathcal{A}_{k_2}(I)|I) \leq D \text{mes}(\mathcal{A}_{k_1}(I)|I) \text{mes}(\mathcal{A}_{k_2}(I)|I)
\]

**Proof.** — By Claim 2, if \( k_1, k_2 \) are large enough, then

\[
\text{mes}(\mathcal{A}_{k_1}(I)|I) \text{mes}(\mathcal{A}_{k_2}(I)|I) \geq \left( \frac{d(\mathcal{N})}{5M} \right)^2 \frac{1}{\psi(e^{k_1})\psi(e^{k_2})},
\]

where we put 5 instead of 4 in the denominator to deal with edge effects arising from \( \text{mes}(\mathcal{A}_k(I) \setminus I) = O\left( \frac{1}{e^\varepsilon \psi(e^\varepsilon)} \right) \).

To prove the claim, it remains to bound \( \text{mes}(\mathcal{A}_{k_1}(I) \cap \mathcal{A}_{k_2}(I)|I) \) from above by \( \frac{\text{const}}{R_1 R_2^3} \), where \( R_i := \psi(e^{k_i}) \).

A cylinder is a set of the form

\[
[a_1, \ldots, a_n] = \{ \alpha \in (0,1) \setminus \mathbb{Q} : a_i(\alpha) = a_i \ (1 \leq i \leq n) \}.
\]

Equivalently, \( \alpha \in [a_1, \ldots, a_n] \) iff \( \alpha \) has an infinite continued fraction expansion of the form \( \alpha = [0; a_1, \ldots, a_n, *, *, \ldots] \).
Our plan is to cover $A_{k_i}(I)$ by unions of cylinders of total measure $O(1/R_i)$, and then use the following well-known fact: There is a constant $G > 1$ s.t. for any $(a_1, \ldots, a_n, b_1, \ldots, b_m) \in \mathbb{N}^{n+m}$,

$$G^{-1} \leq \frac{\text{mes}(a_1, \ldots, a_n; b_1, \ldots, b_m)}{\text{mes}[a_1, \ldots, a_n] \text{mes}[b_1, \ldots, b_m]} \leq G.$$  

(3.8)

This is because the invariant measure $\frac{1}{\ln 2} \frac{dx}{1+2}$ of $T : (0, 1) \to (0, 1)$, $T(x) = \{\frac{1}{2} x\}$ (the Gauss map) is a Gibbs–Markov measure, thanks to the bounded distortion of $T$, see [ADU93, Section 2].

To cover $A_{k_i}(I)$ by cylinders, it is enough to cover $A_{m,n,k_i}$ by cylinders for every $(m, n) \in \Omega_{k_i}(I)$. Suppose $\alpha \in A_{m,n,k_i}$. Then $r := \gcd(m, n) \leq M$ and $(m^*, n^*) := \frac{1}{r}(m, n)$ satisfies

$$\gcd(m^*, n^*) = 1,$$

$n^* \in \bigcup_{|k_i^*-k_i| \leq n M} [e^{k_i^*-1}, e^{k_i^*}], |n^* \alpha - m^*| < \frac{1}{n^* R_i}$.

Assume $k_i$ is so large that $R_i = \psi(e^{k_i}) > 4$. Then Lemma 3.3 gives $a_{l+1} > \frac{R_i}{2}$. Thus $A_{k_i}(I) \subset C_{k_i}(I, R_i)$ where

$$C_{k_i}(I, R) := \bigcup_{k^* \in [k-i, k+i]} \left\{ \alpha \in (0, 1) \setminus \mathbb{Q} : \exists \ell \text{ s.t. } a_{\ell+1}^{\cdot} \alpha, p_{\ell}^{\cdot} \alpha, a_{\ell+1}^{\cdot} \alpha \in I \right\}.$$  

This is a union of cylinders, because $q_{\ell}(\alpha), p_{\ell}(\alpha), a_{\ell+1}^{\cdot} \alpha$ are constant on cylinders of length $\ell + 1$.

We claim that for some $c^*(M)$ which only depends on $M$, for all $k_i$ large enough,

$$\text{mes}(C_{k_i}(I, R_i)) \leq \frac{c^*(M) \text{mes}(I)}{R_i}.$$  

(3.9)

Every rational $\frac{m}{n} \in (0, 1)$ has two finite continued fraction expansions: $[0; a_1, \ldots, a_\ell]$ and $[0; a_1, \ldots, a_\ell - 1, 1]$ with $a_\ell > 1$. We write $\ell = \ell(\frac{m}{n})$ and $a_i = a_i(\frac{m}{n})$. With this notation

$$C_{k_i}(I, R_i) = \bigcup_{k^* \in [k_i, k_i]} \bigcup_{n \in [e^{k_i^*-1}, e^{k_i^*}], m/n \in I} \left[ a_1(\frac{m}{n}), \ldots, a_\ell(\frac{m}{n}) \right].$$  

We have $[a_1, \ldots, a_\ell] = [p_{\ell} + p_{\ell-1}/q_{\ell-1}, p_{\ell}/q_{\ell}]$ or $[p_{\ell} + p_{\ell-1}/q_{\ell-1}, p_{\ell}/q_{\ell}]$, depending on the parity of $\ell$, see for instance [Khi63]. Since $|pq_{\ell+1} - p_{\ell-1}q_{\ell}| = 1$ and $q_{\ell+1} = a_{\ell+1}q_{\ell} + q_{\ell-1}$, we have

$$\text{mes}([a_1, \ldots, a_\ell, b]) = \frac{1}{q_{\ell+1}(q_{\ell+1} + q_{\ell-1})} = \frac{1}{(bq_{\ell} + q_{\ell-1})(bq_{\ell} + q_{\ell-1})} \leq \frac{1}{n(b+1)q_{\ell}^2},$$

leading to

$$\text{mes}(C_{k_i}(I, R_i)) \leq \sum_{k^* \in [k_i, k_i]} \sum_{n \in [e^{k_i^*-1}, e^{k_i^*}]} \sum_{m/n \in I} \sum_{b > R_i/2} \frac{2}{n^2 b(b+1)} \leq \frac{8 \ln M}{e^{2(k_i - 1 - \ln M)} R_{k_i}} \sum_{n=1}^{\# \left\{ m \in \mathbb{N} : \frac{m}{n} \in I, \gcd(m, n) = 1 \right\}} \leq \frac{c^*(M)}{R_i} \text{mes}(I)$$

where $c^*(M)$ only depends on $M$. The last step uses Proposition 3.9.
Next we cover \( \mathcal{A}_{k_1}(I) \cap \mathcal{A}_{k_2}(I) \) by cylinders. Suppose without loss of generality that \( k_2 > k_1 \). Arguing as before one sees that if

\[
(3.10) \quad k_2 > k_1 + \ln M + 1,
\]

then \( \mathcal{A}_{k_1}(I) \cap \mathcal{A}_{k_2}(I) \) can be covered by sets \([a_1, \ldots, a_t, b, \pi_1, \ldots, \pi_{l-1}, \bar{b}]\) as follows: The convergents \( p_i/q_i \) of (every) \( \alpha \) in \([a_1, \ldots, a_t, b, \pi_1, \ldots, \pi_{l-1}, \bar{b}]\), \( 1 \leq i \leq l + I + 2 \), satisfy

(1) \( q_i \in [e^{k_1^{l-1}}, e^{k_1^i}], k_1^i \in [k_1 - \ln M, k_1], p_i/q_i \in I, b \geq R_1/2; \)

(2) \( q_{l+1} \in [e^{k_2^{l-1}}, e^{k_2^i}], k_2^i \in [k_2 - \ln M, k_2], p_{l+1}/q_i \in I, \bar{b} \geq R_2/2 \)

(3) \( k_2^i > k_1^i \) (this is where (3.10) is used).

We claim that

\[
(3.11) \quad [a_1, \ldots, a_t, b] \subset C_{k_1}(I),
\]

\[
(3.12) \quad b \leq e^{k_2^i - k_1^i + 1},
\]

\[
(3.13) \quad [\pi_1, \ldots, \pi_{l-1}, \bar{b}] \subset \bigcup_{|r| \leq 3} C_{k_2 - k_1 + r - \ln b}([0, 1], R_2).
\]

(3.11) follows from (1). Next, \( e^{k_2^i} \geq q_i \geq |\pi_1, \ldots, \pi_{l-1}, \bar{b}| \) proving (3.12). To prove (3.13), let \( \pi_i/q_i, 1 \leq i \leq l + 2 \), be the principal convergents of (every) \( \alpha \in [\pi_1, \ldots, \pi_{l-1}, \bar{b}] \).

By Proposition 3.10, \( q_{l+1+l} = q_{l+1} \pi_{l+1} + q_{l+1} \pi_{l+1} \), whence \( q_{l+1+l} \leq q_{l+1+l} \leq 2 q_{l+1} \).

Since \( q_i \in [e^{k_1^{l-1}}, e^{k_1^i}] \) and \( q_{l+1+l} \in [e^{k_2^{l-1}}, e^{k_2^i}] \),

\[
(3.14) \quad e^{k_2^i - k_1^i - 2} \leq q_{l+1+l} \leq q_{l+1+1} \leq e^{k_2^i - k_1^i + 1}.
\]

Next, let \( \pi_i/q_i, 1 \leq i \leq l + 2 \), denote the principal convergents of (every) \( \alpha \in [\pi_1, \ldots, \pi_{l-1}, \bar{b}] \).

Then \( \pi_{l+1+l} = 1/(b + \bar{b}) \), so \( \pi_{l+1+l} = b_{l+1} + \pi_{l+1} \), whence \( b_{l+1} \leq q_{l+1+l} \leq (b + 1)q_{l+1} \). Thus \( \bar{q}_{l+1} \in [b + 1]^{-1}q_{l+1}, \bar{b} - 1 q_{l+1} \).

It follows that the \( l \)-th principal convergent of every \( \alpha \in [\pi_1, \ldots, \pi_{l-1}, \bar{b}] \) satisfies

\[
(3.15) \quad \bar{q}_{l+1} \in [e^{k_2^i - k_1^i + 1 - \ln b}, e^{k_2^i - k_1^i + 1 - \ln b}].
\]

It is now easy to see (3.13).

By (3.13), \( \mathcal{A}_{k_1}(I) \cap \mathcal{A}_{k_2}(I) \cap I \subset \bigcup_{|r| \leq 3} \bigcup_{[a, b, \alpha] \subset C_{k_1}(I)} \bigcup_{[a', b', \alpha'] \subset C_{k_2 - k_1 + r - \ln b}([0, 1])} [a, b, a', b] \).

Now arguing as in the proof of (3.9) and using (3.8) we obtain

\[
(3.16) \quad \text{mes}(\mathcal{A}_{k_1}(I) \cap \mathcal{A}_{k_2}(I) \cap I) \leq \sum_{k_1^i \in [k_1 - \ln M, k_1]} \sum_{n \in [e^{k_1^i - 1}, e^{k_1^i}]} \sum_{\gcd(m, n) = 1} \sum_{b = \lfloor \sqrt{R_1} \rfloor} 2G \text{mes}(C_{k_2 - k_1 + r - \ln b}([0, 1], R_2))/n^2|b + 1|.
\]

(3.16) uses the estimate \( \text{mes}(C_{k_2 - k_1 + r - \ln b}([0, 1], R_2)) = O(1/R_2) \) which is also valid when \( k_2 - k_1 + r - \ln b \) is small, provided we choose \( M \) large enough so that the asymptotic in Proposition 3.9 holds for all \( N > M \) with \( I = [0, 1] \). See the proof of (3.9).
Combining (3.16) with (3.7), we find that under (3.10) \( A_k(I) \) are \( D \)-quasi-independent for sufficiently large \( D \), proving Claim 3. □

Claims 2 and 3 allow us to apply Sullivan’s Borel–Cantelli Lemma (Proposition 3.7). We obtain \( \delta = \delta(M) \) s.t. for every interval \( I \subset [\varepsilon, 1 - \varepsilon] \), \( \text{mes}(A \cap I) \geq \delta \text{mes}(I) \). This means that \( [\varepsilon, 1 - \varepsilon] \setminus A \) has no Lebesgue density points, and therefore must have measure zero. So \( A \) has full measure in \( [\varepsilon, 1 - \varepsilon] \). Since \( \varepsilon \) is arbitrary, \( A \) has full measure. □

4. Proof of Theorem 1.2

As explained in Section 2, it is enough to prove Theorem 1.2 for \( f(x) \) of the form \( \sum_{m=1}^{d} b_m h(x + \beta_m) \neq 0 \) with \( h(x) = \{x\} - \frac{1}{2} \). Without loss of generality, \( \beta_i \) are different and \( b_i \neq 0 \). Notice that

\[
f(x) = -\sum_{m=1}^{d} b_m \sum_{j=1}^{\infty} \frac{\sin(2\pi j(x + \beta_m))}{\pi j}.
\]

Therefore \( \|f\|_2^2 = \frac{1}{2\pi^2} \sum_{n} \frac{1}{\pi^2} D(\beta_1 n, \ldots, \beta_d n) \), where \( D : \mathbb{T}^d \to \mathbb{R} \)

\[
D(\gamma_1, \ldots, \gamma_d) := \int_0^1 \left[ \sum_{m=1}^{d} b_m \sin(2\pi(y + \gamma_m)) \right]^2 dy.
\]

Since \( f \neq 0 \), \( D(\beta_1 n, \ldots, \beta_d n) > 0 \) for some \( n \). Let \( \mathcal{D} \) denote the closure in \( \mathbb{T}^d \) of \( \mathcal{O} := \{ (\beta_1 n, \ldots, \beta_d n) \mod \mathbb{Z} : n \in \mathbb{Z} \} \). This is a minimal set for the translation by \( (\beta_1, \ldots, \beta_d) \) on \( \mathbb{T}^d \), so a standard compactness argument shows that for every \( \varepsilon_0 > 0 \), the set

\[
\mathcal{N} := \{ n \in \mathbb{N} : D(\beta_1 n, \ldots, \beta_d n) > \varepsilon_0 \}
\]

is syndetic: its gaps are bounded. Thus \( \mathcal{N} \) has positive lower density.

By Theorem 3.1, if \( M \) is sufficiently large then the set \( A := A(\mathcal{N}, M) \) has full measure in \( \mathbb{T} \). Let

\[
S_n(\alpha, x) := \sum_{k=0}^{n-1} f(x + k\alpha).
\]

The proof of Theorem 1.2 for \( f(x) \) above consists of two parts:

**Theorem 4.1.** — Suppose \( \alpha \in A \), then for a.e. \( x \in [0, 1) \), there exist \( A_k(x) \in \mathbb{R} \) and \( B_k(x), N_k(x) \to \infty \) such that

\[
\frac{S_n(\alpha, x) - A_k(x)}{B_k(x)} \xrightarrow{\text{dist} \ k \to \infty} U[0, 1], \text{ as } n \sim U(0, \ldots, N_k(x)).
\]

**Theorem 4.2.** — Suppose \( \alpha \in A \), then for a.e. \( x \in [0, 1) \), there are no \( A_N(x) \in \mathbb{R} \) and \( B_N(x) \to \infty \) such that

\[
\frac{S_n(\alpha, x) - A_N(x)}{B_N(x)} \xrightarrow{\text{dist} \ N \to \infty} U[0, 1], \text{ as } n \sim U(0, \ldots, N).
\]
4.1. Preliminaries

**Lemma 4.3.** — $S_q(\alpha, \cdot) : T \to \mathbb{R}$ has $dq$ discontinuities.

**Proof.** — The discontinuities of $S_q$ are preimages of discontinuities of $f$ by $R_{\alpha}^{-k}$ with $k = 0, 1, \ldots, q - 1$.

**Lemma 4.4.** — Let $C := \sup |f'| \leq |\sum b_m|$. If $x', x''$ belong to same continuity component of $R_{\alpha}^n$ then

$$|S_r(\alpha, x') - S_r(\alpha, x'')| \leq C|x' - x''|.$$

**Proof.** — Since $|S_r'| = \left| \sum_{k=0}^{r-1} f'(x + k\alpha) \right| \leq Cr$, the restriction of $S_r$ to on each continuity component is Lipschitz with Lipschitz constant $Cr$.

**Lemma 4.5.** — There are constants $C_1, C_2$ such that the following holds. Suppose that $q_n$ is a principal denominator of $\alpha$, and $q_{n+1} > cq_n$ with $c > 1$. Let $\mu_n(x) := S_q(x, \alpha)$, then

$$\text{mes} \left\{ x : S_{\ell q_n}(\alpha, x) = \ell \mu_n + C_1 \frac{\ell^2}{c} \text{ for } \ell = 0, \ldots, k \right\} > 1 - C_2 \frac{k}{c}.$$

**Proof.** — If $x$ and $x + \ell q_n \alpha$ belong to the same continuity interval of $R_{\alpha}^n$ for all $\ell = 0, \ldots, k$ then we have by Lemma 4.4 that for $\ell \leq k$

$$|S_{\ell q_n}(\alpha, x) - \ell \mu_n| \leq \sum_{j=0}^{\ell-1} |S_{q_n}(\alpha, x + jq_n \alpha) - S_{q_n}(\alpha, x)| \leq C q_n \sum_{j=0}^{\ell-1} |jq_n \alpha|$$

$$\leq \frac{C q_n}{q_{n+1}} \sum_{j=0}^{\ell-1} j \leq C_1 \frac{\ell^2}{c}, \text{ where } C_1 := C/2.$$

Therefore if $S_{\ell q_n}(\alpha, x) \neq \ell \mu_n + C_1 \frac{\ell^2}{c}$ for some $\ell = 0, \ldots, k$, then there must exist $0 \leq \ell \leq k$ s.t. $x, R_{\alpha}^{q_n}(x)$ are separated by a discontinuity of $S_{q_n}(\alpha, \cdot)$. Since $\text{dist}(x, R_{\alpha}^{q_n}(x)) \leq \ell/q_{n+1}$, $x$ must belong to a ball with radius $k/q_{n+1}$ centered at a discontinuity of $S_{q_n}(\alpha, \cdot)$. By Lemma 4.3, there are $dq_n$ discontinuities, so the measure of such points is less than $dq_n (2k/q_{n+1}) \leq \frac{2ak}{c}$. The lemma follows with $C_2 := 2d$.

**Lemma 4.6.** — There is a constant $C_3 = C_3(b_1, \ldots, b_d)$ s.t. for every $n \geq 1$ and $\alpha = [0; a_1, a_2, \ldots]$, $\max \{|S_r(\alpha, x)| : 0 \leq r \leq q_n - 1 \} \leq C_3(a_0 + \cdots + a_{n-1})$.

**Proof.** — Let $r = \sum_{j=0}^{n-1} b_j q_j$ denote the Ostrowski expansion of $r$. Recall that this means that $0 \leq b_j \leq a_j$ and $b_j = a_j \Rightarrow b_{j-1} = 0$. So

$$S_r = \sum_{k=0}^{b_{n-1}-1} S_{q_n} \circ R_{\alpha}^{q_n} + \sum_{k=0}^{b_{n-2}-1} S_{q_n} \circ R_{\alpha}^{q_n} + \cdots + \sum_{k=0}^{b_0-1} S_{q_0} \circ R_{\alpha}^{q_0}.$$

By the Denjoy–Koksma inequality $|S_r| \leq \sum b_j V(f) \leq V(f) \sum a_j$ where $V(f) \leq 2 \sum b_i$ is the total variation of $f$ on $T$.

**Lemma 4.7.** — There exist positive constants $\varepsilon_1, \varepsilon_2$ such that for every $\alpha$ irrational, if $q_n$ is a principal denominator of $\alpha$ and $q_n r_n \in N$ with $r_n \leq M$ then

$$\text{mes} \{ x : |S_{q_n}(\alpha, x)| \geq \varepsilon_1 \} \geq \varepsilon_2.$$
Proof. — We follow an argument from [Bec94]. Suppose $q_n$ is a principal denominator of $\alpha$ and $q_n r_n \in \mathcal{N}$ for some $r_n \leq M$. Let $N = q_n r_n$. Since $f(x) = -\sum_{m=1}^{d} b_m \sum_{j=1}^{\infty} \frac{\sin(2\pi j (x + \beta_m))}{\pi j}$, for each $j \in \mathbb{N}$

$$\|S_N(\alpha, \cdot)\|_2^2 \geqslant \frac{1}{\pi^2 j^2} \int_0^1 \left( \sum_{m=1}^{d} b_m \sum_{k=0}^{N-1} \sin(2\pi j (x + k\alpha + \beta_m)) \right)^2 dx.$$

Using the identities $\sum_{k=0}^{n} \sin(y + kx) = \frac{\cos(y + x/2) - \cos(y + (2n+1)x/2)}{2\sin(x/2)}$ and $\cos A - \cos B = 2 \sin(\frac{A+B}{2}) \sin(\frac{B-A}{2})$, we find that

$$\|S_N(\alpha, \cdot)\|_2^2 \geqslant \left( \frac{\sin(\pi N j \alpha)}{\pi j \sin(\pi j \alpha)} \right)^2 \int_0^1 \left( \sum_{m=1}^{d} b_m \sin(2\pi (jx + j(j-1)\alpha/2 + 2j\beta_m)) \right)^2 dx$$

$$= \left( \frac{\sin(\pi N j \alpha)}{\pi j \sin(\pi j \alpha)} \right)^2 \int_0^1 \left( \sum_{m=1}^{d} b_m \sin(2\pi (y + j\beta_m)) \right)^2 dy$$

$$= \left( \frac{\sin(\pi N j \alpha)}{\pi j \sin(\pi j \alpha)} \right)^2 D(j\beta_1, \ldots, j\beta_m) \text{ with } D \text{ as in (4.1)}.$$

We now take $j = N = r_n q_n$. The first term is bounded below because $\|N\alpha\| \leq M \|q_n\| \leq M \|q_{n+1}\| \leq M^2 \frac{1}{\alpha_{n+1} N} = o(1)$, so $\frac{\sin(\pi N j \alpha)}{\pi j \sin(\pi j \alpha)} \xrightarrow{\alpha \to \infty} \pi^{-1}$. The second term is bounded below by $\varepsilon_0$, because $N = q_n r_n \in \mathcal{N}$. It follows that for all $n$ large enough, $\|S_{r_n q_n}(\alpha, \cdot)\|_2 > \sqrt{\varepsilon_0}/2\pi$.

For any $L^2$-function $\varphi$ and any $\varepsilon > 0$,

$$\|\varphi\|_2^2 \leq \|\varphi\|_\infty \\text{mes}\{x \mid |\varphi(x)| \geq \varepsilon\} \geq \varepsilon^2.$$

Hence $\text{mes}\{x \mid |\varphi(x)| \geq \varepsilon\} \geq \frac{\|\varphi\|_2^2 - \varepsilon^2}{\|\varphi\|_\infty^2}$. We just saw that for all $n$ large enough, $\|S_{r_n q_n}(\alpha, \cdot)\|_2 > \sqrt{\varepsilon_0}/2\pi$, and by the Denjoy–Koksma inequality $\|S_{r_n q_n}(\alpha, \cdot)\|_{L^\infty} \leq M V(f)$. So for some $\varepsilon > \varepsilon_0$ and for all $n$ large enough, $\text{mes}\{x \mid |S_{r_n q_n}(\alpha, x)| > \varepsilon\} \geq \varepsilon$.

Looking at the inequality $|S_{r_n q_n}(\alpha, x)| \leq \sum_{k=0}^{n-1} |S_q(\alpha, x + k q_n \alpha)|$, we see that if $|S_{r_n q_n}(\alpha, x)| > \varepsilon$, then $|S_q(\alpha, x + k q_n \alpha)| \geq \varepsilon/M$ for some $0 \leq k \leq M - 1$. So for all $n$ large enough, $\text{mes}\{x \mid |S_q(\alpha, x)| > \varepsilon/M\} \geq \varepsilon/M$. \hfill \Box

### 4.2. Proof of Theorem 4.1

Let $\Omega^*(\alpha)$ be the set of $x$ where the conclusion of Theorem 4.1 holds. $\Omega^*(\alpha)$ is $R_\alpha$-invariant and it is measurable by Lemma A.1 in the appendix. Therefore to show that $\Omega^*(\alpha)$ has full measure, it suffices to show that it has positive measure.

Suppose $\alpha \in \mathcal{A}$ and let $n_k \uparrow \infty$ be a sequence satisfying (3.1) with $\mathcal{N}$ given by (4.2). There is no loss of generality in assuming that

$$\frac{a_{n_k+1}}{a_0 + \cdots + a_{n_k}} > k^3.$$

So $q_{n_k+1} > k^3 L_k q_{n_k}$, where $L_k := a_0 + \cdots + a_{n_k}$.
Recall that $\mu_{n_k}(x) = S_{n_k}(\alpha, x)$. For all $k$ sufficiently large, there is a set $A_k$ of measure at least $\varepsilon_2/2$ such that for all $x \in A_k$,

\begin{equation}
S_{lq_n}(\alpha, x) = \ell \left( \mu_{n_k}(x) \pm \frac{C_1 \ell}{k^3 L_k} \right) \quad \text{for all } \ell = 0, 1, \ldots, kL_k, \tag{4.4}
\end{equation}

\begin{equation}
|\mu_{n_k}(x)| \geq \varepsilon_1. \tag{4.5}
\end{equation}

This is because Lemma 4.5 says that the total measure of $x$ for which (4.4) fails is $O(1/k^2)$ while (4.5) holds on the set of measure $\varepsilon_2$ by Lemma 4.7.

It follows that $\text{mes}(\bigcap_{n>1} \bigcup_{k>n} A_k) \geq \varepsilon_2/2$. Therefore there exists $x$ which belongs to infinitely many $A_k$. After re-indexing $n_k$, we may assume that (4.4), (4.5) are satisfied for all $k \in \mathbb{N}$. Henceforth, we fix such an $x$ and work with this $x$. Let

$$N_k(x) := kL_k q_n, \quad B_k(x) := kL_k |\mu_{n_k}(x)|, \quad A_k(x) := \frac{1}{2}(\text{sgn}(\mu_{n_k}(x)) - 1) B_k.$$ 

Any $n \leq N_k$ can be written uniquely in the form

$$n = l(n) q_n + r(n) \quad \text{with} \quad 0 \leq l(n) \leq kL_k \text{ and } 0 \leq r(n) < q_n.$$ 

It is easy to see that $\frac{l(n)}{kL_k} \overset{\text{dist}}{\underset{k \to \infty}{\to}} U[0, 1]$ as $n \sim U(1, \ldots, N_k)$.

Writing $S_n(\alpha, x) = S_{l(n) q_n}(\alpha, x) + S_{r(n)}(\alpha, x + a l(n) q_n)$ we obtain from (4.4) and Lemma 4.6 that

$$S_n(\alpha, x) = l(n) \mu_{n_k}(x) + O(L_k).$$

So $S_{n_k}(\alpha, x) / B_k$ is asymptotically uniform on $[0, 1]$ when $\mu_{n_k} > 0$, and asymptotically uniform on $[-1, 0]$ when $\mu_{n_k} < 0$. So $\frac{S_n(\alpha, x) - A_k}{B_k} \overset{\text{dist}}{\underset{n \to \infty}{\to}} U[0, 1]$, as $n \sim U(1, \ldots, N_k(x))$. \hfill \square

4.3. Proof of Theorem 4.2

Let $\Omega(\alpha)$ denote the set of $x \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$ for which there are $B_N(x) \to \infty$ and $A_N(x) \in \mathbb{R}$ s.t.

\begin{equation}
\frac{S_n(\alpha, x) - A_N(x)}{B_N(x)} \overset{\text{dist}}{\underset{n \to \infty}{\to}} U[0, 1], \quad \text{as } n \sim U(1, \ldots, N), \tag{4.6}
\end{equation}

$\Omega(\alpha)$ is measurable, and $A_n(\cdot), B_n(\cdot)$ can be chosen to be measurable on $\Omega(\alpha)$, see the appendix. Assume by way of contradiction that $\text{mes}[\Omega(\alpha)] \neq 0$ for some $\alpha \in A$.

$\Omega(\alpha)$ is invariant under $R_\alpha(x) = x + \alpha \mod 1$ on $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. Since $R_\alpha$ is ergodic, and $\Omega(\alpha)$ is measurable, $\text{mes}[\Omega(\alpha)] = 1$.

Since $\alpha \in A$, there is an increasing sequence $n_k$ satisfying (3.1) where $N$ is given by (4.2). We can choose $n_k$ so that $q_n r_{n_k} \in N$ for $r_{n_k} \leq M$, and $a_{n_k+1} > k^3 L_k$ where $L_k := a_0 + \cdots + a_{n_k}$. In particular, $q_{n_k} > k^3 L_k q_{n_k}$.

Recall that $\mu_{n_k}(x) := S_{q_{n_k}}(\alpha, x)$. By Lemma 4.7 we can choose $x$ such that for infinitely many $k$, $|\mu_{n_k}(x)| \geq \varepsilon_1$. We will suppose that $\mu_{n_k}(x) > 0$ for infinitely many $k$; the case where $\mu_{n_k}(x) < 0$ for infinitely many $k$ is similar.

**Claim 4.** It is possible to assume without loss of generality that $\|B_{q_{n_k}}\|_{\infty} := \sup_{x \in \Omega(\alpha)} |B_{q_{n_k}}(x)| \leq 3C_3 L_k$ for all $k$ where $C_3$ is the constant from Lemma 4.6.
Proof. — We claim that for every $x$ with $(4.6)$, $B_{q_{nk}}(x) \leq 3C_3L_k$ for all $k$ large enough. Otherwise, by Lemma 4.6, there are infinitely many $k$ s.t. $B_{q_{nk}}(x) > 3\max\{|S_r(\alpha,x)| : r = 0, \ldots, q_{nk-1}\}$, whence $|S_n(\alpha,x)/B_{q_{nk}}| \leq \frac{1}{3}$ for all $0 \leq n \leq q_{nk} - 1$. In such circumstances, $(4.6)$ does not hold (the spread is not big enough).

Since $B_{q_{nk}}(x) \leq 3C_3L_k$ for all $k$ large enough, there is no harm in replacing $B_{q_{nk}}(x)$ in $(4.6)$ by $\min\{B_{q_{nk}}(x), 3C_3L_k\}$.

Claim 5. — Fix $D > C = |\sum b_m|$, and let $E_k$ denote the set of $x \in \Omega(\alpha)$ s.t. $S_r(\alpha,x) = S_r(\alpha,R_{q_{nk}}^\alpha(x)) + \frac{D\ell}{q_{nk+1}}$ for all $0 \leq \ell \leq B_{q_{nk}}(x), 0 \leq r < q_{nk} - 1$. Then $\text{mes}(E_k) \leq C_4k^{-3}$.

Proof. — If $x \notin E_k$, then there are $0 \leq \ell \leq B_{q_{nk}}(x), 0 \leq r < q_{nk} - 1$ s.t.

$$|S_r(\alpha,x) - S_r(\alpha,x + \ell q_{nk}\alpha)| \geq \frac{D\ell}{q_{nk+1}}.$$ 

By Lemma 4.4, $\{x\}$, $\{x + \ell q_{nk}\alpha\}$ are separated by a singularity of $S_r(\alpha, \cdot)$. So $x$ belongs to a ball of radius $2\|\ell q_{nk}\alpha\|$ centered at one of the $dq_{nk}$ discontinuities of $S_{q_{nk}}(\alpha, \ldots)$. Thus $\text{mes}(E_k) \leq dq_{nk} \cdot 2\|q_{nk}\alpha\|$. Now $\|q_{nk}\alpha\| \leq \frac{\|B_{q_{nk}}\|_\infty}{q_{nk+1}} \leq \frac{3C_3L_k}{q_{nk+1}} \leq \frac{3C_3}{k^3q_{nk}}$ by our choice of $nk$. So $\text{mes}(E_k) \leq C_4/k^3$ with $C_4 := 6dC_3$.□

Claim 6. — Let $F_k$ denote the set of $x \in \Omega(\alpha)$ s.t.

$$S_{\ell q_{nk}}(\alpha,x) = \ell \left(\frac{\ell q_{nk}(x) + C_1\ell}{k^3L_k}\right) \text{ for all } 0 \leq \ell \leq B_{q_{nk}}(x).$$

Then $\text{mes}(F_k) \leq C_5k^{-2}$.

Proof. — This follows from Lemma 4.5. □

By Claims 5 and 6, and a Borel–Cantelli argument, for a.e. $x$ there is $k_0(x)$ s.t. $x \in E_k \cap F_k$ for all $k \geq k_0(x)$.

Suppose $k \geq k_0(x)$, and let $N_k := q_{nk}B_{q_{nk}}(x)$. Every $0 \leq n \leq N_k - 1$ can be uniquely represented as $n = \ell q_{nk} + r$ with $0 \leq \ell \leq B_{q_{nk}}(x) - 1$ and $0 \leq r \leq q_{nk} - 1$. Using the bound $\|B_{q_{nk}}\|_\infty = O(L_k)$, we find:

$$\frac{S_n(\alpha,x) - A_{q_{nk}}(x)}{B_{q_{nk}}(x)} = \frac{S_{\ell q_{nk}}(\alpha,x)}{B_{q_{nk}}(x)} + \frac{S_r(\alpha,R_{q_{nk}}^\alpha(x) - A_{q_{nk}}(x)}{B_{q_{nk}}(x)}$$

$$= \frac{S_{\ell q_{nk}}(\alpha,x)}{B_{q_{nk}}(x)} + \frac{S_r(\alpha,x) - A_{q_{nk}}(x) + o(1)}{B_{q_{nk}}(x)} \quad \text{because } x \in E_k$$

$$= \frac{\ell(\mu_{nk}(x) + o(1))}{B_{q_{nk}}(x)} + \frac{S_r(\alpha,x) - A_{q_{nk}}(x) + o(1)}{B_{q_{nk}}(x)} \quad \text{because } x \in F_k.$$ 

If $n \sim U(0, \ldots, N_k - 1)$, then $\ell, r$ are independent random variables, $\ell \sim U(0, \ldots, B_{q_{nk}}(x) - 1)$ and $r \sim U(0, \ldots, q_{nk} - 1)$. Thus the distribution of $\frac{\ell(\mu_{nk}(x) + o(1))}{B_{q_{nk}}(x)}$ is...
close to $U[0, \mu_{n_k}(x)]$, and the distribution of $\frac{S_n(x, \alpha) - A_{n_k}(x)}{B_{n_k}(x)}$ converges to $U[0, 1]$ (because $x \in \Omega$).

Taking a subsequence such that $\mu_{n_k}(x) \to \varepsilon > \varepsilon_1$ we see that the random variables $\frac{S_n(x, \alpha) - A_{n_k}(x)}{B_{n_k}(x)}$, where $n \sim U(0, \ldots, n_k - 1)$, converge in distribution to the sum of two independent uniformly distributed random variables. This contradicts to (4.6), because the sum of two independent uniform random variables is not uniform. □

Appendix A. Measurability concerns

Let $\Omega(\alpha)$ denote the set of $x \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$ such that for some $B_N(x) \to \infty$ and $A_N(x) \in \mathbb{R}, \frac{S_n(x, \alpha) - A_N(x)}{B_N(x)} \xrightarrow{\text{dist}} U[0, 1]$, as $n \sim U(1, \ldots, N)$ and let $\Omega^*(\alpha)$ denote the set of $x \in \mathbb{T}$ such that along a subsequence $N_k(x)$ there exist some $B_{N_k}(x) \to \infty$ and $A_{N_k}(x) \in \mathbb{R}, \frac{S_n(x, \alpha) - A_{N_k}(x)}{B_{N_k}(x)} \xrightarrow{\text{dist}} U[0, 1]$, as $n \sim U(1, \ldots, N_k)$.

We make no assumptions on the measurability of $A_N, B_N, N_k$ as functions of $x$. The purpose of this section is to prove:

**Lemma A.1.** — $\Omega(\alpha)$ and $\Omega^*(\alpha)$ are measurable.

The crux of the argument is to show that $A_N(x), B_N(x)$ can be replaced by measurable functions, defined in terms of the percentiles of the random quantities $S_n(x, \alpha), n \sim U(1, \ldots, N)$.

Recall that given $0 < t < 1$, the *upper and lower t-percentiles* of a random variable $X$ are defined by

$$
\begin{align*}
\chi^+(X, t) &:= \inf\{\xi : \Pr(X \leq \xi) > t\} \\
\chi^-(X, t) &:= \sup\{\xi : \Pr(X \leq \xi) < t\}
\end{align*}
$$

(0 < t < 1).

Notice that $\Pr(X \leq \chi^+(X, t)) \geq t, \Pr(X \leq \chi^-(X, t)) \leq t$, and $\Pr(\chi^{-}(X, t) < X < \chi^{+}(X, t)) = 0$. In case $X$ is non-atomic (i.e. $\Pr(X = a) = 0$ for all $a$), we can say more:

**Lemma A.2.** — Suppose $X$ is a non-atomic real valued random variable, fix $0 < t < 1$ and let $\chi^+_t := \chi^+(X, t)$, then

(a) $\Pr(X < \chi^+_t) = t$ and $\Pr(X < \chi^-_t) = t$;
(b) $\forall \varepsilon > 0, \Pr(\chi^-_t - \varepsilon < X < \chi^-_t), \Pr(\chi^+_t < X < \chi^+_t + \varepsilon)$ are positive;
(c) $\exists t_1 < t_2$ s.t. $\chi^-_t < \chi^-_{t_1}$ and $\chi^+_t, \chi^+_t$ have the same sign.

**Proof.** — Since $X$ is non-atomic, $\Pr(X < \chi^+_t) = \Pr(X \leq \chi^+_t) \geq t$ and $\Pr(X < \chi^-_t) \leq t$. If $\chi^-_t = \chi^-_{t_1}$, part (a) holds. If $\chi^+_t > \chi^-_t$ then for all $h > 0$ small enough $\chi^-_t + h < \chi^+_t - h$ whence

$$
0 \leq \Pr(\chi^-_t < X < \chi^+_t) = \lim_{h \to 0^+} \Pr(\chi^-_t + h < X < \chi^+_t) = \lim_{h \to 0^+} \Pr(X < \chi^+_t - h) - \lim_{h \to 0^+} \Pr(X \leq \chi^-_t + h) \leq t - t = 0.
$$

Necessarily $\lim_{h \to 0^+} \Pr(X < \chi^+_t - h) = t$ and $\lim_{h \to 0^+} \Pr(X \leq \chi^-_t + h) = t$, which gives us $\Pr(X < \chi^-_t) = t$ and $\Pr(X < \chi^+_t) = \Pr(X \leq \chi^+_t) = t$. 

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For (b) assume by contradiction that \( \Pr(\chi_i - \varepsilon < X < \chi_i^-) = 0 \), then for all \( \chi_i - \varepsilon < \xi < \chi_i^- \), \( \Pr(X < \xi) = \Pr(X < \chi_i^-) = t \), whence \( \chi_i^- < \chi_i^- - \varepsilon \), a contradiction. Similarly, \( \Pr(\chi_i^+ < X < \chi_i^- + \varepsilon) = 0 \) is impossible.

To prove (c) note that since \( X \) is non-atomic, either \( \Pr(X > 0) \) or \( \Pr(X < 0) \) is positive. Assume w.l.o.g. that \( \Pr(X > 0) \neq 0 \). By non-atomicity, there are positive \( a < b \) s.t. \( \Pr(X \in (0, a)) \neq 0 \) and \( \Pr(X \in (a, b)) \neq 0 \). Take \( t_1 := \Pr(X < a) \) and \( t_2 := \Pr(X < b) \).

From now on fix a non-atomic random variable \( Y \), and choose \( 0 < t_1 < t_2 < 1 \) as in Lemma A.2 (c) s.t. \( \chi^-(Y, t_1) < \chi^+(Y, t_2) \) and \( \sgn(\chi^-(Y, t_1)) = \sgn(\chi^+(Y, t_2)) \).

**Lemma A.3.** — Let \( S_N \) be (possibly atomic) random variables s.t. for some \( A_N \in \mathbb{R} \) and \( B_N \to \infty \), \( \frac{S_N - A_N}{B_N} \xrightarrow{\text{dist}} N \to \infty Y \). Then \( \frac{S_N - A_N}{B_N} \xrightarrow{\text{dist}} N \to \infty Y \), where \( A_N^*, B_N^* \) are the unique solution to

\[
\begin{cases}
A_N^* + B_N^* \chi^-(Y, t_1) = \chi^-(S_N, t_1) \\
A_N^* + B_N^* \chi^+(Y, t_2) = \chi^+(S_N, t_2).
\end{cases}
\]

**Proof.** — Without loss of generality, \( \chi^-(Y, t_1), \chi^+(Y, t_2) \) are both positive.

We need the following fact (which is not automatic since \( S_N \) are allowed to be atomic):

\[
\lim_{N \to \infty} \Pr(S_N < \chi^-(S_N, t)) = t \text{ for all } 0 < t < 1.
\]

Indeed, given \( \varepsilon > 0 \), let \( \xi_N := B_N \chi^-(Y, t - \varepsilon) + A_N \), then

\[
\Pr(S_N < \xi_N) = \Pr \left( \frac{S_N - A_N}{B_N} < \chi^-(Y, t - \varepsilon) \right) \xrightarrow{N \to \infty} \Pr(Y < \chi^-(Y, t - \varepsilon)) = t - \varepsilon,
\]

by Lemma A.2 (a). So for all \( N \) large enough, \( \xi_N \leq \chi^-(S_N, t) \), whence \( \lim \inf \Pr(S_N < \chi^-(S_N, t)) \geq \lim \Pr(S_N < \xi_N) = t - \varepsilon \). Since \( \varepsilon \) is arbitrary, \( \lim \inf \Pr(S_N < \chi^-(S_N, t)) \geq t \). The other inequality \( \lim \sup \Pr(S_N < \chi^-(S_N, t)) \leq t \) is clear since \( \Pr(S_N < \chi^-(S_N, t)) \leq t \) for all \( N \).

With (A.2) proved, we proceed to prove that

\[
\frac{A_N^* - A_N}{B_N^*} \xrightarrow{N \to \infty} 0 \quad \text{and} \quad \frac{B_N^*}{B_N} \xrightarrow{N \to \infty} 1.
\]

It will then be obvious that \( \frac{S_N - A_N}{B_N} \xrightarrow{\text{dist}} N \to \infty Y \) implies \( \frac{S_N - A_N^*}{B_N^*} \xrightarrow{\text{dist}} N \to \infty Y \).

Define two affine transformations, \( \varphi_N(t) = \frac{t - A_N}{B_N^*} \) and \( \varphi_N^*(t) = \frac{t - A_N^*}{B_N^*} \). Notice that \( (\varphi_N^*)^{-1}(t) = A_N^* + B_N^* t \), so \( (\varphi_N^*)^{-1}(\chi^-(Y, t_1)) = \chi^-(S_N, t_1) \), by (A.1). Since \( B_N^* = \frac{\chi^+(S_N, t_2) - \chi^-(S_N, t_1)}{\chi^+(Y, t_2) - \chi^-(Y, t_1)} > 0 \), \( \varphi_N^* \) is increasing. By (A.2), \( \Pr(\varphi_N(S_N) < \chi^-(Y, t_1)) = \Pr(S_N < \chi^-(S_N, t_1)) \xrightarrow{N \to \infty} t_1 \). So

\[
t_1 = \lim_{N \to \infty} \Pr(\varphi_N(S_N) < \chi^-(Y, t_1))
= \lim_{N \to \infty} \Pr(\varphi_N(S_N) < \varphi_N[(\varphi_N^*)^{-1}(\chi^-(Y, t_1))])
= \lim_{N \to \infty} \Pr(\varphi_N(S_N) < \frac{B_N^*}{B_N^*} \left( \chi^-(Y, t_1) + \frac{A_N^* - A_N}{B_N^*} \right)).
\]
We claim that this implies that

\[(A.4) \liminf_{N \to \infty} \frac{B_N^*}{B_N} \left( \chi^-(Y, t_1) + \frac{A_N^* - A_N}{B_N} \right) \geq \chi^-(Y, t_1). \]

Otherwise, \( \exists \varepsilon \) s.t. \( \liminf_{N \to \infty} \frac{B_N^*}{B_N} \left( \chi^-(Y, t_1) + \frac{A_N^* - A_N}{B_N} \right) < \chi^-(Y, t_1) - \varepsilon \), so

\[t_1 = \liminf_{N \to \infty} \Pr \left( \varphi_N(S_N) < \frac{B_N^*}{B_N} \left( \chi^-(Y, t_1) + \frac{A_N^* - A_N}{B_N} \right) \right) \]

\[\leq \liminf_{N \to \infty} \Pr \left( \varphi_N(S_N) < \chi^-(Y, t_1) - \varepsilon \right) = \Pr(Y < \chi^-(Y, t_1) - \varepsilon) \]

\[= \Pr(Y < \chi^-(Y, t_1)) - \Pr(\chi^-(Y, t_1) - \varepsilon \leq Y < t_1) < t_1, \text{ by Lemma A.2(a) and (b)}. \]

Similarly, one shows that

\[(A.5) \limsup_{N \to \infty} \frac{B_N^*}{B_N} \left( \chi^+(Y, t_2) + \frac{A_N^* - A_N}{B_N} \right) \leq \chi^+(Y, t_2). \]

It remains to see that (A.4) and (A.5) imply (A.3). First we divide (A.4) by (A.5) to obtain

\[\limsup_{N \to \infty} \frac{\chi^-(Y, t_1) + \frac{A_N^* - A_N}{B_N}}{\chi^+(Y, t_2) + \frac{A_N^* - A_N}{B_N}} \geq \frac{\chi^-(Y, t_1)}{\chi^+(Y, t_2)}. \]

Since \( x \mapsto \frac{a+x}{b+x} \) is strictly decreasing on \([0, \infty)\) when \( a > b > 0 \), this implies that

\[(A.6) \limsup_{N \to \infty} \frac{A_N^* - A_N}{B_N} \leq 0. \]

Looking at (A.4), and recalling that \( \chi^-(Y, t_1) > 0 \), we deduce that

\[(A.7) \liminf_{N \to \infty} \frac{B_N^*}{B_N} \geq 1. \]

Next we look at the difference of (A.4) and (A.5) and obtain

\[\limsup_{N \to \infty} \frac{B_N^*}{B_N} \left( \chi^+(Y, t_2) - \chi^-(Y, t_1) \right) \leq \chi^+(Y, t_2) - \chi^-(Y, t_1), \]

whence \( \limsup(B_N^*/B_N) \leq 1 \). Together with (A.7), this proves that \( B_N^*/B_N \to 1 \) as \( N \to \infty \).

Substituting this in (A.4), gives \( \liminf \frac{A_N^* - A_N}{B_N} \geq 0 \), which, in view of (A.6), implies that \( \frac{A_N^* - A_N}{B_N} \to 0 \). This completes the proof of (A.3), and with it, the lemma. \( \square \)

**Proof of Lemma A.1.** — We begin with the measurability of \( \Omega(\alpha) \).

Let \( S_N(x) \) denote the random variable equal to \( S_n(\alpha, x) \) with probability \( \frac{1}{N} \) for each \( 1 \leq n \leq N \).

We will apply Lemma A.3 with \( Y := U[0, 1], S_N = S_n(x) \) and (say) \( t_1 := \frac{1}{3}, t_2 := \frac{2}{3} \).

It says that

\[ \Omega(\alpha) = \left\{ x \in \mathbb{T} : S_N(x) - A_N^*(x) \to \frac{\operatorname{dist}}{N \to \infty} U[0, 1] \right\}, \]

\[ \text{TOME 1 (2018)} \]
where $A_N^*(x)$ and $B_N^*(x)$ are the unique solutions to (A.1). Since the percentiles of $S_N(x)$ are measurable as functions of $x$, $A_N^*(x)$, $B_N^*(x)$ are measurable as functions of $x$.

We claim that $\Omega(\alpha) = \Omega_1(\alpha) \cap \Omega_2(\alpha)$ where

$$\Omega_1(\alpha) := \bigcap_{\ell=1}^{\infty} \bigcap_{M=1}^{\infty} \bigcap_{N=M+1}^{\infty} \left\{ x \in \mathbb{T} : \frac{1}{N} \sum_{n=1}^{N} 1_{(2,\infty)} \left( \left| \frac{S_n(\alpha,x) - A_N^*(x)}{B_N^*(x)} \right| \right) < \frac{1}{\ell} \right\}$$

$$\Omega_2(\alpha) = \bigcap_{t \in \mathbb{Q} \setminus \{0\}} \left\{ x \in \mathbb{T} : \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{it} \left( \frac{S_n(\alpha,x) - A_N^*(x)}{B_N^*(x)} \right) = \mathbb{E} \left( e^{itY} \right) \right\}.$$ 

This will prove the lemma, since the measurability of $A_N^*(\cdot)$, $B_N^*(\cdot)$ implies the measurability of $\Omega_i(\alpha)$.

If $x \in \Omega(\alpha)$ then

$$x \in \Omega_1(\alpha), \quad \text{because } \Pr \left( \left| \frac{S_N(x) - A_N^*}{B_N^*} \right| > 2 \right) \xrightarrow{N \to \infty} 0,$$

and $x \in \Omega_2(\alpha)$, because $\mathbb{E} \left( e^{it} \left( \frac{S_N(x) - A_N^*}{B_N^*} \right) \right) \xrightarrow{N \to \infty} \mathbb{E} \left( e^{itY} \right)$ pointwise.

Conversely, if $x \in \Omega_1(\alpha) \cap \Omega_2(\alpha)$ then it is not difficult to see that

$$\mathbb{E} \left( e^{it} \left( \frac{S_N(x) - A_N^*}{B_N^*} \right) \right) \xrightarrow{N \to \infty} \mathbb{E} \left( e^{itY} \right)$$

for all $t \in \mathbb{R}$. So $x \in \Omega(\alpha)$ by Lévy’s continuity theorem. Thus $\Omega(\alpha) = \Omega_1(\alpha) \cap \Omega_2(\alpha)$, whence $\Omega(\alpha)$ is measurable.

The proof that $\Omega^*(\alpha)$ is measurable is similar. Enumerate $\mathbb{Q} \setminus \{0\} = \{t_n : n \in \mathbb{N}\}$, then $\alpha \in \Omega^*(\alpha)$ iff for every $\ell \in \mathbb{N}$ there exist $M \in \mathbb{N}$ s.t. for some $N > M$

$$\frac{1}{N} \sum_{n=1}^{N} 1_{(2,\infty)} \left( \left| \frac{S_n(\alpha,x) - A_N^*(x)}{B_N^*(x)} \right| \right) < \frac{1}{\ell},$$

$$\left| \mathbb{E} \left( e^{it_n} \left( \frac{S_N(x) - A_N^*(x)}{B_N^*(x)} \right) \right) - \mathbb{E} \left( e^{it_nY} \right) \right| < \frac{1}{\ell} \quad \text{for all } n = 1, \ldots, \ell.$$

These are measurable conditions, because $A_N^*(\cdot)$, $B_N^*(\cdot)$ are measurable. So $\Omega^*(\alpha)$ is measurable. 

□

BIBLIOGRAPHY


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