



ANNALES
HENRI LEBESGUE

DMITRY DOLGOPYAT

OMRI SARIG

NO TEMPORAL DISTRIBUTIONAL LIMIT THEOREM FOR A.E. IRRATIONAL TRANSLATION

UNE TRANSLATION IRRATIONNELLE
TYPIQUE NE SATISFAIT PAS DE
THÉORÈME LIMITE TEMPOREL

ABSTRACT. — Bromberg and Ulcigrai constructed piecewise smooth functions on the circle such that the set of α for which the sum $\sum_{k=0}^{n-1} f(x + k\alpha \bmod 1)$ satisfies a temporal distributional limit theorem along the orbit of a.e. x has Hausdorff dimension one. We show that the Lebesgue measure of this set is equal to zero.

RÉSUMÉ. — Bromberg et Ulcigrai ont construit des fonctions lisses par morceaux sur le cercle pour lesquelles l'ensemble des α tels que la somme $\sum_{k=0}^{n-1} f(x + k\alpha \bmod 1)$ satisfait un théorème limite temporel le long de l'orbite de presque tout x est un ensemble de dimension de Hausdorff 1. Nous montrons que cet ensemble est de mesure nulle.

1. Introduction and statement of main result

1.1. Background

Suppose $T : X \rightarrow X$ is a map, $f : X \rightarrow \mathbb{R}$ is a function, and $x_0 \in X$ is a *fixed* initial condition. We say that the *T-ergodic sums* $S_n = f(x_0) + f(Tx_0) + \cdots + f(T^{n-1}x_0)$

2010 *Mathematics Subject Classification*: 37D25, 37D35.

(*) This work was partially supported by the ISF grants 199/14, 1149/18 and by the BSF grant 2016105. The authors thank Adam Kanigowski for simplifying the statement and the proof of Lemma 3.2.

satisfy a *temporal distributional limit theorem (TDLT) on the orbit of x_0* , if there exists a non-constant real valued random variable Y , centering constants $A_N \in \mathbb{R}$ and scaling constants $B_N \rightarrow \infty$ s.t.

$$(1.1) \quad \frac{S_n - A_N}{B_N} \xrightarrow{N \rightarrow \infty} Y \text{ in distribution,}$$

when n is sampled uniformly from $\{1, \dots, N\}$ and x_0 is fixed. Equivalently, for every Borel set $E \subset \mathbb{R}$ s.t. $\mathbb{P}(Y \in \partial E) = 0$,

$$\frac{1}{N} \text{Card} \left\{ 1 \leq n \leq N : \frac{S_n - A_N}{B_N} \in E \right\} \xrightarrow{N \rightarrow \infty} \mathbb{P}(Y \in E).$$

We allow and expect A_N, B_N, Y to depend on T, f, x_0 .

Such limit theorems have been discovered for several zero entropy uniquely ergodic transformations, including systems where the more traditional spatial limit theorems, with x_0 sampled from a measure on X , fail [Bec10, Bec11, ADDS15, DS17b, PS17, DS18]. Of particular interest are TDLT for

$$R_\alpha : [0, 1] \rightarrow [0, 1], \quad R_\alpha(x) = x + \alpha \pmod{1}, \quad f_\beta(x) := 1_{[0, \beta)}(x) - \beta,$$

because the R_α -ergodic sums of f_β along the orbit of x represent the discrepancy of the sequence $x + n\alpha \pmod{1}$ with respect to $[0, \beta)$ [Sch78, CK76, Bec10]. Another source of interest is the connection to the “deterministic random walk” [AK82, ADDS15].

The validity of the TDLT for R_α and f_β depends on the diophantine properties of α and β . Recall that $\alpha \in (0, 1)$ is *badly approximable* if for some $c > 0$, $|q\alpha - p| \geq c/|q|$ for all irreducible fractions p/q . Equivalently, the digits in the continued fraction expansion of α are bounded [Khi63]. Say that $\beta \in (0, 1)$ is *badly approximable with respect to α* if for some $C > 0$, $|q\alpha - \beta - p| > C/|q|$ for all $p, q \in \mathbb{Z}, q \neq 0$. If α is badly approximable then every $\beta \in \mathbb{Q} \cap (0, 1)$ is badly approximable with respect to α . The recent paper [BU18] shows:

THEOREM 1.1 (Bromberg–Ulcigrai [BU18]). — *Suppose α is badly approximable and β is badly approximable with respect to α , e.g. $\beta \in \mathbb{Q} \cap (0, 1)$. Then the R_α -ergodic sums of f_β satisfy a temporal distributional limit theorem with Gaussian limit on the orbit of every initial condition.*

The set of badly approximable α has Hausdorff dimension one [Jar29], but Lebesgue measure zero [Khi24]. This leads to the following question: *Is there a β s.t. the R_α -ergodic sums of f_β satisfy a temporal distributional limit theorem for a.e. α and a.e. initial condition?*

In this paper we answer this question negatively.

1.2. Main result

To state our result in its most general form, we need the following terminology.

Let $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. We say that $f : \mathbb{T} \rightarrow \mathbb{R}$ is *piecewise smooth* if there exists a finite set $\mathfrak{S} \subset \mathbb{T}$ s.t. f is continuously differentiable on $\mathbb{T} \setminus \mathfrak{S}$ and $\exists \psi : \mathbb{T} \rightarrow \mathbb{R}$ with bounded variation s.t. $f' = \psi$ on $\mathbb{T} \setminus \mathfrak{S}$. For example: $f_\beta(x) = 1_{[0, \beta)}(x) - \beta$ (take $\mathfrak{S} = \{0, \beta\}$, $\psi \equiv 0$). We show:

THEOREM 1.2. — *Let f be a piecewise smooth function of zero mean. Then there is a set of full measure $\mathcal{E} \subset \mathbb{T} \times \mathbb{T}$ s.t. if $(\alpha, x) \in \mathcal{E}$ then the R_α -ergodic sums of f do not satisfy a TDLT on the orbit of x .*

The condition $\int_{\mathbb{T}} f = 0$ is necessary: By Weyl’s equidistribution theorem, for every $\alpha \notin \mathbb{Q}$, f Riemann integrable s.t. $\int_{\mathbb{T}} f = 1$, and $x_0 \in \mathbb{T}$, $S_n/N \xrightarrow[N \rightarrow \infty]{\text{dist}} \text{U}[0, 1]$ as $n \sim \text{U}(1, \dots, N)$. See Section 1.4 for the notation.

This paper has a companion [DS17a] which gives a different proof of Theorem 1.2, in the special case $f(x) = \{x\} - \frac{1}{2}$. Unlike the proof given below, [DS17a] does not identify the set of α where the TDLT fails, but it does give more information on the different scaling limits for the distributions of S_n , $n \sim \text{U}(1, \dots, N_k)$ along different subsequences $N_k \rightarrow \infty$. [DS17a] also shows that if we randomize both n and α by sampling (n, α) uniformly from $\{1, \dots, N\} \times \mathbb{T}$, then $(S_n - \frac{1}{N} \sum_{k=1}^N S_k) / \sqrt{\ln N}$ converges in distribution to the Cauchy distribution.

The methods of [DS17a] are specific for $f(x) = \{x\} - \frac{1}{2}$, and we do not know how to apply them to other functions such as $f_\beta(x) = 1_{[0,\beta)}(x) - \beta$.

1.3. The structure of the proof

Suppose f is piecewise smooth and has mean zero.

We shall see below that if f is continuous, then for a.e. α , f is an R_α -coboundary, therefore S_n are bounded, hence (1.1) cannot hold with $B_N \rightarrow \infty$, Y non-constant. We remark that (1.1) does hold with $B_N \equiv 1$, $A_N = f(x_0)$, $Y =$ distribution of minus the transfer function, but this is not a TDLT since no actual scaling is involved.

The heart of the proof is to show that if f is discontinuous, then for a.e. α , the temporal distributions of the ergodic sums have different asymptotic scaling behavior on different subsequences. The proof of this has three independent parts:

- (1) A reduction to the case $f(x) = \sum_{m=1}^d b_m h(x + \beta_m)$, $h(x) := \{x\} - \frac{1}{2}$.
- (2) A proof that if $\mathcal{N} \subset \mathbb{N}$ has positive lower density, then there exists $M \geq 1$ s.t. the following set has full Lebesgue measure in $(0, 1)$:

$$\mathcal{A}(\mathcal{N}, M) := \left\{ \alpha \in (0, 1) : \begin{array}{l} \exists n_k \uparrow \infty, r_k \leq M \text{ s.t. for all } k: \\ r_k q_{n_k} \in \mathcal{N}, a_{n_k+1} / (a_1 + \dots + a_{n_k}) \rightarrow \infty \end{array} \right\}.$$

Here a_n and q_n are the partial quotients and principal denominators of α , see Section 3.1.

- (3) Construction of $\mathcal{N} = \mathcal{N}(b_1, \dots, b_d; \beta_1, \dots, \beta_d) \subseteq \mathbb{N}$ with positive density, s.t. for every $\alpha \in \mathcal{A}(\mathcal{N}, M)$ and a.e. x , one can analyze the temporal distributions of the Birkhoff sums of $\sum_{m=1}^d b_m h(x + \beta_m)$.

1.4. Notation

$n \sim \text{U}(1, \dots, N)$ means that n is a random variable taking values in $\{1, \dots, N\}$, each with probability $\frac{1}{N}$. $\text{U}[a, b]$ is the uniform distribution on $[a, b]$. Lebesgue’s measure is denoted by mes . $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $x \in \mathbb{R}$, then $\|x\| := \text{dist}(x, \mathbb{Z})$ and $\{x\}$ is the unique number in $[0, 1)$ s.t. $x \in \{x\} + \mathbb{Z}$. $\text{Card}(\cdot)$ is the cardinality. If $\varepsilon > 0$, then $a = b \pm \varepsilon$ means that $|a - b| \leq \varepsilon$.

2. Reduction to the case $f(x) = \sum_{m=1}^d b_m h(x + \beta_m)$

Let $h(x) = \{x\} - \frac{1}{2}$, and let \mathcal{G} denote the collection of all non-identically zero functions of the form $f(x) = \sum_{m=1}^d b_m h(x + \beta_m)$, where $d \in \mathbb{N}$, $b_i, \beta_i \in \mathbb{R}$. We explain how to reduce the proof of Theorem 1.2 from the case of a general piecewise smooth $f(x)$ to the case $f \in \mathcal{G}$.

The following proposition was proved in [DS17a]. Let $C(\mathbb{T})$ denote the space of continuous real-valued functions on \mathbb{T} with the sup norm.

PROPOSITION 2.1. — *If $f(t)$ is differentiable on $\mathbb{T} \setminus \{\beta_1, \dots, \beta_d\}$ and f' extends to a function with bounded variation on \mathbb{T} , then there are $d \in \mathbb{N}_0$, $b_1, \dots, b_d \in \mathbb{R}$ s.t. for a.e. $\alpha \in \mathbb{T}$ there is $\varphi_\alpha \in C(\mathbb{T})$ s.t.*

$$f(x) = \sum_{i=1}^d b_i h(x + \beta_i) + \int_{\mathbb{T}} f(t) dt + \varphi_\alpha(x) - \varphi_\alpha(x + \alpha) \quad (x \neq \beta_1, \dots, \beta_d).$$

The following proposition was proved in [DS17b]. Let $(\Omega, \mathcal{B}, \mu)$ be a probability space, and let $T : \Omega \rightarrow \Omega$ be a probability preserving map.

PROPOSITION 2.2. — *Suppose $f = g + \varphi - \varphi \circ T$ μ -a.e. with $f, g, \varphi : \Omega \rightarrow \mathbb{R}$ measurable. If the ergodic sums of g satisfy a TDLT along the orbit of a.e. x , then so do the ergodic sums of f .*

These results show that if Theorem 1.2 holds for every $f \in \mathcal{G}$, then Theorem 1.2 holds for any *discontinuous* piecewise smooth function with zero mean. As for *continuous* piecewise smooth functions with zero mean, these are R_α -cohomologous to $g \equiv 0$ for a.e. α because the b_i in Proposition 2.1 must all vanish. Since the zero function does not satisfy the TDLT, continuous piecewise smooth functions do not satisfy a TDLT.

3. The set \mathcal{A} has full measure

3.1. Statement and plan of proof

Let α be an irrational number, with continued fraction expansion denoted by $[a_0; a_1, a_2, a_3, \dots] := a_0 + \frac{1}{a_1 + \dots}$, $a_0 \in \mathbb{Z}$, $a_i \in \mathbb{N}$ ($i \geq 1$). We call a_n the *quotients* of α . Let p_n/q_n denote the *principal convergents* of α , determined recursively by

$$q_{n+1} = a_{n+1}q_n + q_{n-1}, \quad p_{n+1} = a_{n+1}p_n + p_{n-1}$$

and $p_0 = a_0, q_0 = 1; p_1 = 1 + a_1 a_0, q_1 = a_1$. We call q_n the *principal denominators* and a_i the *partial quotients* of α . Sometimes – but not always! – we will write $q_k = q_k(\alpha)$, $p_k = p_k(\alpha)$, $a_k = a_k(\alpha)$.

Given $\mathcal{N} \subset \mathbb{N}$ and $M \geq 1$, let $\mathcal{A} = \mathcal{A}(\mathcal{N}, M) \subset (0, 1)$ denote the set of irrational $\alpha \in (0, 1)$ s.t. for some subsequence $n_k \uparrow \infty$,

$$(3.1) \quad \exists r_k \leq M \text{ s.t. } r_k q_{n_k} \in \mathcal{N}, \quad \frac{a_{n_k+1}}{(a_0 + \dots + a_{n_k})} \xrightarrow[k \rightarrow \infty]{} \infty.$$

The *lower density* of \mathcal{N} is $d(\mathcal{N}) := \liminf \frac{1}{N} \text{Card}(\mathcal{N} \cap [1, N])$. The purpose of this section is to prove:

THEOREM 3.1. — *If a set \mathcal{N} has positive lower density, then there exists M such that $\mathcal{A}(\mathcal{N}, M)$ has full Lebesgue measure in $(0, 1)$.*

The proof consists of the following three lemmas:

LEMMA 3.2. — *For almost all α there is $n_0 = n_0(\alpha)$ s.t. if $k \geq n_0$ and $a_{k+1} > \frac{1}{4}k(\ln k)(\ln \ln k)$, then $a_{k+1}/(a_1 + \dots + a_k) \geq \frac{1}{8} \ln \ln k$.*

LEMMA 3.3. — *Suppose $\alpha \in (0, 1) \setminus \mathbb{Q}$ and $(p, q) \in \mathbb{N}_0 \times \mathbb{N}$ satisfy $\gcd(p, q) = 1$ and $|q\alpha - p| \leq \frac{1}{qL}$ where $L \geq 4$. Then there exists k s.t. $q = q_k(\alpha)$ and $a_{k+1}(\alpha) \geq \frac{1}{2}L$.*

LEMMA 3.4. — *Suppose $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a non-decreasing function s.t.*

$$(3.2) \quad \sum_n \frac{1}{n\psi(n)} = \infty.$$

Suppose $\mathcal{N} \subset \mathbb{N}$ has positive lower density. For all M sufficiently large, for a.e. $\alpha \in (0, 1)$ there are infinitely many pairs $(m, n) \in \mathbb{N}_0 \times \mathbb{N}$ s.t. $n \in \mathcal{N}$, $\gcd(m, n) \leq M$, and $|n\alpha - m| \leq \frac{1}{n\psi(n)}$.

Remark 3.5. — By the monotonicity of ψ , if $e^{k-1} < n < e^k$ then $\psi(e^{k-1}) \leq \psi(n) \leq \psi(e^k)$. Hence (3.2) holds iff $\sum_k \frac{1}{\psi(e^k)} = \infty$.

Remark 3.6. — If $\mathcal{N} = \mathbb{N}$, then Lemma 3.4 holds with $M = 1$ by the classical Khinchine Theorem. We do not know if Lemma 3.4 holds with $M = 1$ for every set \mathcal{N} with positive lower density.

Proof of Theorem 3.1 given Lemmas 3.2–3.4. — We apply these lemmas with $\psi(t) = c(\ln t)(\ln \ln t)(\ln \ln \ln t)$ and $c > 1/\ln(\frac{1+\sqrt{5}}{2})$.

Fix $M > 1$ as in Lemma 3.4. Then $\exists \Omega \subset (0, 1)$ of full measure s.t. for every $\alpha \in \Omega$ there are infinitely many $(m, n) \in \mathbb{N}_0 \times \mathbb{N}$ as follows. Let $m^* := m/\gcd(m, n)$, $n^* := n/\gcd(m, n)$, $p := \gcd(m, n)$, then

- (1) $pn^* \in \mathcal{N}$, $p \leq M$, $|n^*\alpha - m^*| = \frac{|n\alpha - m|}{p} \leq \frac{1}{n^*\psi(n^*)}$ ($\because n^* \leq n$);
- (2) $\exists k$ s.t. $n^* = q_k(\alpha)$ and $a_{k+1}(\alpha) \geq \frac{1}{2}\psi(q_k)$ (\because Lemma 3.3). By its recursive definition, $q_k \geq k$ -th Fibonacci number $\geq \frac{1}{3}(\frac{1+\sqrt{5}}{3})^k$. So for all k large enough, $a_{k+1}(\alpha) \geq \frac{1}{2}\psi(q_k) > \frac{1}{4}k(\ln k)(\ln \ln k)$;
- (3) $a_{k+1}/(a_1 + \dots + a_k) \geq \frac{1}{8} \ln \ln k \rightarrow \infty$ (\because Lemma 3.2).

So every $\alpha \in \Omega$ belongs to $\mathcal{A} = \mathcal{A}(\mathcal{N}, M)$, and \mathcal{A} has full measure. □

Next we prove Lemmas 3.2–3.4.

3.2. Proof of Lemma 3.2

By [DV86], for almost every α

$$\frac{(a_1 + \dots + a_{k+1}) - \max_{j \leq k+1} a_j}{k \ln k} \rightarrow \frac{1}{\ln 2} < 2.$$

So if k is large enough, and $a_{k+1} > \frac{1}{4}k(\ln k)(\ln \ln k)$ then

$$\max_{j \leq k+1} a_j = a_{k+1}, \quad \frac{a_1 + \dots + a_k}{k \ln k} \leq 2, \quad \text{and} \quad \frac{a_{k+1}}{a_1 + \dots + a_k} > \frac{1}{8} \ln \ln k. \quad \square$$

3.3. Proof of Lemma 3.3

For every (p, q) as in the lemma, $|q\alpha - p| < \frac{1}{2q}$. A classical result in the theory of continued fractions [Khi63, Theorem 19] says that in this case $\exists k$ s.t. $q = q_k(\alpha), p = p_k(\alpha)$.

To estimate $a_{k+1} = a_{k+1}(\alpha)$ we recall the following facts, valid for the principal denominators of any irrational $\alpha \in (0, 1)$ [Khi63]:

- (1) $|q_k\alpha - p_k| > \frac{1}{q_k + q_{k+1}}$;
- (2) $q_{k+1} + q_k < (a_{k+1} + 2)q_k$, whence by (a) $a_{k+1} > \frac{1}{q_k|q_k\alpha - p_k|} - 2$.

In our case, $|q_k\alpha - p_k| = |q\alpha - p| \leq \frac{1}{q_k L}$, so $a_{k+1} > L - 2 \geq \frac{L}{2}$. \square

3.4. Preparations for the proof of Lemma 3.4

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $A_k \in \mathcal{F}$ be measurable events. Given $D > 1$, we say that A_k are D -quasi-independent, if

$$(3.3) \quad \mathbb{P}(A_{k_1} \cap A_{k_2}) \leq D\mathbb{P}(A_{k_1})\mathbb{P}(A_{k_2}) \text{ for all } k_1 \neq k_2.$$

The following proposition is a slight variation on Sullivan's Borel–Cantelli Lemma from ([Sul82]):

PROPOSITION 3.7. — *For every $D \geq 1$ there exists a constant $\delta(D) > 0$ such that the following holds in any probability space:*

- (a) *If A_k are D -quasi-independent measurable events s.t. $\lim_{k \rightarrow \infty} \mathbb{P}(A_k) = 0$ but $\sum_k \mathbb{P}(A_k) = \infty$, then $\mathbb{P}(A_k \text{ occurs infinitely often}) \geq \delta(D)$.*
- (b) *The quasi-independence assumption in (a) can be weakened to the assumption that for some $r \in \mathbb{N}$, $\mathbb{P}(A_{k_1} \cap A_{k_2}) \leq D\mathbb{P}(A_{k_1})\mathbb{P}(A_{k_2})$ for all $|k_2 - k_1| \geq r$.*
- (c) *One can take $\delta(D) = \frac{1}{2D}$.*

Proof. — Since $\mathbb{P}(A_k) \rightarrow 0$ but $\sum \mathbb{P}(A_k) = \infty$, there is an increasing sequence N_j such that $\lim_{j \rightarrow \infty} \sum_{k=N_j+1}^{N_{j+1}} \mathbb{P}(A_k) = \frac{1}{D}$.

Let B_j be the event that at least one of events $\{A_k\}_{k=N_j+1}^{N_{j+1}}$ occurs. Since $B_j = \bigcup_{k=N_j+1}^{N_{j+1}} (A_k \setminus \bigcup_{j=N_j+1}^{k-1} A_j)$,

$$\begin{aligned} \mathbb{P}(B_j) &\geq \sum_{k=N_j+1}^{N_{j+1}} \mathbb{P}(A_k) - \sum_{N_j+1 \leq k_1 < k_2 \leq N_{j+1}} \mathbb{P}(A_{k_1} \cap A_{k_2}) \\ &\geq \sum_{k=N_j+1}^{N_{j+1}} \mathbb{P}(A_k) - D \sum_{N_j+1 \leq k_1 < k_2 \leq N_{j+1}} \mathbb{P}(A_{k_1})\mathbb{P}(A_{k_2}) \\ &\geq \sum_{k=N_j+1}^{N_{j+1}} \mathbb{P}(A_k) - \frac{D}{2} \left(\sum_{k=N_j+1}^{N_{j+1}} \mathbb{P}(A_k) \right)^2. \end{aligned}$$

Since $\lim_{j \rightarrow \infty} \sum_{k=N_j+1}^{N_{j+1}} \mathbb{P}(A_k) = \frac{1}{D}$ and $D \geq 1$, $\liminf \mathbb{P}(B_j) \geq \frac{1}{2D}$.

Let E denote the event that A_j happens infinitely often. E is also the event that B_j happens infinitely often, therefore $E = \bigcap_{n=1}^{\infty} \bigcup_{j=n+1}^{\infty} B_j$. In a probability space,

the measure of a decreasing intersection of sets is the limit of the measure of these sets. So $\mathbb{P}(E) \geq \liminf \mathbb{P}(B_j) \geq \frac{1}{2D}$, proving (a) and (c).

Part (b) follows from part (a) by applying it to the sets $\{A_{kr+\ell}\}$ where $0 \leq \ell \leq r-1$ is chosen to get $\sum_k \mathbb{P}(A_{kr+\ell}) = \infty$. □

The *multiplicity* of a collection of measurable sets $\{E_k\}$ is defined to be the largest K s.t. there are K different k_i with $\mathbb{P}(\bigcap_{i=1}^K E_{k_i}) > 0$.

PROPOSITION 3.8. — *Let E_k be measurable sets in a finite measure space. If the multiplicity of $\{E_k\}$ is less than K , then*

$$\text{mes} \left(\bigcup_k E_k \right) \geq \frac{1}{K} \sum_k \text{mes}(E_k).$$

Proof. — $1_{\bigcup_i E_i} \geq \frac{1}{K} \sum_i 1_{E_i}$ almost everywhere. □

PROPOSITION 3.9. — *For every non-empty open interval $I \subset [0, 1]$,*

$$\text{Card} \left\{ (m, n) \in \{0, \dots, N\}^2 : \frac{m}{n} \in I, \text{gcd}(m, n) = 1 \right\} \sim 3 \text{mes}(I) N^2 / \pi^2, \text{ as } N \rightarrow \infty.$$

Proof. — This classical fact due to Dirichlet follows from the inclusion-exclusion principle and the identity $\zeta(2) = \pi^2/6$, see [HW08, Theorem 459]. □

PROPOSITION 3.10. — *Suppose $\alpha = [0; a_1, a_2, \dots]$ and $\bar{\alpha} = [0; a_{\ell+1}, a_{\ell+2}, \dots]$. Then the principal convergents $\bar{p}_\ell/\bar{q}_\ell$ of $\bar{\alpha}$ and the principal convergents p_ℓ/q_ℓ of α are related by*

$$\begin{pmatrix} p_{l+\bar{l}} & p_{l+\bar{l}+1} \\ q_{l+\bar{l}} & q_{l+\bar{l}+1} \end{pmatrix} = \begin{pmatrix} p_{l-1} & p_l \\ q_{l-1} & q_l \end{pmatrix} \begin{pmatrix} \bar{p}_\ell & \bar{p}_{\ell+1} \\ \bar{q}_\ell & \bar{q}_{\ell+1} \end{pmatrix}.$$

Proof. — Since $a_0 = 0$, the recurrence relations for p_n/q_n imply

$$\begin{pmatrix} p_n & p_{n+1} \\ q_n & q_{n+1} \end{pmatrix} = \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_{n+1} \end{pmatrix}, \quad \begin{pmatrix} p_0 & p_1 \\ q_0 & q_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix}.$$

So

$$\begin{pmatrix} p_n & p_{n+1} \\ q_n & q_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_{n+1} \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} p_{l+\bar{l}} & p_{l+\bar{l}+1} \\ q_{l+\bar{l}} & q_{l+\bar{l}+1} \end{pmatrix} = \begin{pmatrix} p_{l-1} & p_l \\ q_{l-1} & q_l \end{pmatrix} \begin{pmatrix} \bar{p}_\ell & \bar{p}_{\ell+1} \\ \bar{q}_\ell & \bar{q}_{\ell+1} \end{pmatrix},$$

where \bar{p}_i/\bar{q}_i are the principal convergents of $\bar{\alpha} := [0; a_{\ell+1}, a_{\ell+2}, \dots]$. □

3.5. Proof of Lemma 3.4

Without loss of generality, $\lim_{t \rightarrow \infty} \psi(t) = \infty$, otherwise replace $\psi(t)$ by the bigger monotone function $\psi(t) + \ln t$.

Fix $M > 1$, to be determined later. Let

$$\begin{aligned}\Omega_k &:= \{(m, n) \in \mathbb{N}^2 : n \in \mathcal{N}, n \in [e^{k-1}, e^k], 0 < m < n, \gcd(m, n) \leq M\}, \\ A_{m,n,k} &:= \left\{ \alpha \in \mathbb{T} : |n\alpha - m| \leq \frac{1}{e^k \psi(e^k)} \right\}, \\ \mathcal{A}_k &:= \bigcup_{(m,n) \in \Omega_k} A_{m,n,k}, \\ \mathcal{A} &:= \{ \alpha \in \mathbb{T} : \alpha \text{ belongs to infinitely many } \mathcal{A}_k \}.\end{aligned}$$

The lemma is equivalent to saying that \mathcal{A} has full Lebesgue measure for a suitable choice of M .

We will prove a slightly different statement. Fix $\varepsilon > 0$ small. Given an non-empty interval $I \subset [\varepsilon, 1 - \varepsilon]$, let

$$\begin{aligned}\Omega_k(I) &:= \left\{ (m, n) \in \Omega_k : \frac{m}{n} \in I \right\}, \\ \mathcal{A}_k(I) &:= \bigcup_{(m,n) \in \Omega_k(I)} A_{m,n,k}, \\ \mathcal{A}(I) &:= \{ \alpha \in \mathbb{T} : \alpha \text{ belongs to infinitely many } \mathcal{A}_k(I) \}.\end{aligned}$$

We will prove that there exists a positive constant $\delta = \delta(\varepsilon, M)$ s.t. for all intervals $I \subset [\varepsilon, 1 - \varepsilon]$, $\text{mes}(\mathcal{A}(I) \cap I) \geq \delta \text{mes}(I)$. It then follows by a standard density point argument (see below) that $\mathcal{A} \cap [\varepsilon, 1 - \varepsilon]$ has full measure. Since ε is arbitrary, the lemma is proved.

CLAIM 1. — *There exists $K = K(\varepsilon)$ such that for every $k > K$, the multiplicity of $\{A_{m,n,k}\}_{(m,n) \in \Omega_k(I)}$ is uniformly bounded by M .*

Proof. — Suppose $(m_i, n_i) \in \Omega_k(I)$ ($i = 1, 2$) and $A_{m_1, n_1, k} \cap A_{m_2, n_2, k} \neq \emptyset$. Then there is α s.t. $|n_i \alpha - m_i| \leq \delta_k := \frac{1}{e^k \psi(e^k)}$ ($i = 1, 2$). Choose $K = K(\varepsilon)$ so large that $k > K \Rightarrow \delta_k < \frac{\varepsilon}{4e^k}$.

If $k > K$, then $\alpha \geq \frac{m_i}{n_i} - \delta_k > \min I - \frac{\varepsilon}{2} > \frac{\varepsilon}{2}$. Let $r_i := \gcd(m_i, n_i)$ and $(n_i^*, m_i^*) := \frac{1}{r_i}(n_i, m_i)$. Then $|n_i^* \alpha - m_i^*| \leq \delta_k$ and $m_i^* \leq n_i^* \leq n_i \leq e^k$, so $|n_2^* m_1^* - n_1^* m_2^*| = \frac{1}{\alpha} |m_1^* (n_2^* \alpha - m_2^*) - m_2^* (n_1^* \alpha - m_1^*)| \leq \frac{2e^k \delta_k}{\varepsilon/2} < 1$. So $n_2^* m_1^* = n_1^* m_2^*$. Since $\gcd(n_i^*, m_i^*) = 1$, $(n_1^*, m_1^*) = (n_2^*, m_2^*)$. It follows that $(n_2, m_2) \in \{(rn_1^*, rm_1^*) : r = 1, \dots, M\}$. So the multiplicity of $\{A_{m,n,k}\}_{(m,n) \in \Omega_k(I)}$ is uniformly bounded by M . \square

CLAIM 2. — *Let $d(\mathcal{N}) := \liminf \frac{1}{N} \text{Card}(\mathcal{N} \cap [1, N]) > 0$, then there exists $M = M(\mathcal{N})$ and $\widetilde{K} = \widetilde{K}(\varepsilon, \mathcal{N}, |I|)$ s.t. for all $k > \widetilde{K}$,*

$$(3.4) \quad \frac{d(\mathcal{N}) \text{mes}(I)}{4M\psi(e^k)} \leq \text{mes}(\mathcal{A}_k(I)) \leq \frac{6 \text{mes}(I)}{\psi(e^k)}.$$

In particular, $\text{mes}(\mathcal{A}_k(I)) \xrightarrow[k \rightarrow \infty]{} 0$ and $\sum \text{mes}(\mathcal{A}_k(I)) = \infty$.

Proof. — $\text{mes}(A_{m,n,k}) = \text{mes}\left(\left[\frac{m}{n} - r_{m,n}, \frac{m}{n} + r_{m,n}\right]\right) = 2r_{m,n}$ where $r_{m,n} = \frac{1}{ne^k \psi(e^k)}$. Since $n \in [e^{k-1}, e^k]$,

$$(3.5) \quad \frac{\text{Card}(\Omega_k(I))}{Me^{2k}\psi(e^k)} \leq \text{mes}(\mathcal{A}_k(I)) \leq \frac{e \text{Card}(\Omega_k(I))}{e^{2k}\psi(e^k)},$$

where the lower bound uses Claim 1 and Proposition 3.8.

$\text{Card}(\Omega_k(I))$ satisfies the bounds $A - B \leq \text{Card}(\Omega_k(I)) \leq A$ where

$$A := \text{Card} \left\{ (m, n) : n \in \mathcal{N}, n \in [e^{k-1}, e^k], \frac{m}{n} \in I \right\}$$

$$B := \text{Card} \left\{ (m, n) : n \in \mathcal{N}, n \in [e^{k-1}, e^k], \frac{m}{n} \in I, \text{gcd}(m, n) \geq M \right\}.$$

Choose $\tilde{K} = \tilde{K}(\varepsilon, \mathcal{N}, |I|) > K(\varepsilon)$ s.t. for all $k > \tilde{K}$

- (1) $\text{Card}\{n \in \mathcal{N} : 0 \leq n \leq e^k\} \geq \frac{1}{\sqrt{2}}d(\mathcal{N})$
- (2) $\text{Card}\{n \in [e^{k-1}, e^k] \cap \mathbb{N} : p|n\} \leq 2(e^k - e^{k-1})/p$ for all $p \geq 1$;
- (3) For all $n > e^{\tilde{K}-1}, p \geq 1$,

$$\frac{n}{p\sqrt{2}} \text{mes}(I) \leq \text{Card} \left\{ m \in \mathbb{N} : \frac{m}{n} \in I, p|m \right\} \leq \frac{2n}{p} \text{mes}(I).$$

If $k > \tilde{K}$, then $\frac{1}{2}d(\mathcal{N})e^{2k} \text{mes}(I) \leq A \leq 2e^{2k} \text{mes}(I)$ and

$$B \leq \sum_{p=M}^{\infty} \text{Card} \left\{ (m, n) : n \in [e^{k-1}, e^k], \frac{m}{n} \in I, p|m, p|n \right\}$$

$$\leq \sum_{p=M}^{\infty} \frac{2(e^k - e^{k-1})}{p} \cdot \frac{2e^k \text{mes}(I)}{p} < 4e^{2k} \text{mes}(I) \sum_{p=M}^{\infty} \frac{1}{p^2}$$

$$\leq \frac{1}{4}d(\mathcal{N})e^{2k} \text{mes}(I), \quad \text{provided we choose } M \text{ s.t. } \sum_{p=M}^{\infty} p^{-2} < \frac{1}{16}d(\mathcal{N}).$$

Together we get $\frac{1}{4}d(\mathcal{N})e^{2k} \text{mes}(I) \leq \text{Card}(\Omega_k(I)) \leq 2e^{2k} \text{mes}(I)$. The claim now follows from (3.5). □

CLAIM 3. — *There exists $D = D(\mathcal{N}, M)$, $r = r(M)$, and $\hat{K} = \hat{K}(\varepsilon, \mathcal{N}, I)$ s.t. for all $k_1, k_2 > \hat{K}$ s.t. $|k_1 - k_2| > r(M)$,*

$$(3.6) \quad \text{mes}(\mathcal{A}_{k_1}(I) \cap \mathcal{A}_{k_2}(I)|I) \leq D \text{mes}(\mathcal{A}_{k_1}(I)|I) \text{mes}(\mathcal{A}_{k_2}(I)|I)$$

Proof. — By Claim 2, if k_1, k_2 are large enough, then

$$(3.7) \quad \text{mes}(\mathcal{A}_{k_1}(I)|I) \text{mes}(\mathcal{A}_{k_2}(I)|I) \geq \left(\frac{d(\mathcal{N})}{5M} \right)^2 \frac{1}{\psi(e^{k_1})\psi(e^{k_2})},$$

where we put 5 instead of 4 in the denominator to deal with edge effects arising from $\text{mes}(\mathcal{A}_k(I) \setminus I) = O\left(\frac{1}{e^k\psi(e^k)}\right)$.

To prove the claim, it remains to bound $\text{mes}(\mathcal{A}_{k_1}(I) \cap \mathcal{A}_{k_2}(I) | I)$ from above by $\frac{\text{const}}{R_1 R_2}$, where $R_i := \psi(e^{k_i})$.

A *cylinder* is a set of the form

$$\llbracket a_1, \dots, a_n \rrbracket = \{ \alpha \in (0, 1) \setminus \mathbb{Q} : a_i(\alpha) = a_i \quad (1 \leq i \leq n) \}.$$

Equivalently, $\alpha \in \llbracket a_1, \dots, a_n \rrbracket$ iff α has an infinite continued fraction expansion of the form $\alpha = [0; a_1, \dots, a_n, *, *, \dots]$.

Our plan is to cover $\mathcal{A}_{k_i}(I)$ by unions of cylinders of total measure $O(1/R_i)$, and then use the following well-known fact: There is a constant $G > 1$ s.t. for any $(a_1, \dots, a_n, b_1, \dots, b_m) \in \mathbb{N}^{n+m}$,

$$(3.8) \quad G^{-1} \leq \frac{\text{mes}[[a_1, \dots, a_n; b_1, \dots, b_m]]}{\text{mes}[[a_1, \dots, a_n]] \text{mes}[[b_1, \dots, b_m]]} \leq G.$$

This is because the invariant measure $\frac{1}{\ln 2} \frac{dx}{1+x}$ of $T : (0, 1) \rightarrow (0, 1)$, $T(x) = \{\frac{1}{x}\}$ (the Gauss map) is a Gibbs–Markov measure, thanks to the bounded distortion of T , see [ADU93, Section 2].

To cover $\mathcal{A}_{k_i}(I)$ by cylinders, it is enough to cover A_{m,n,k_i} by cylinders for every $(m, n) \in \Omega_{k_i}(I)$. Suppose $\alpha \in A_{m,n,k_i}$. Then $r := \gcd(m, n) \leq M$ and $(m^*, n^*) := \frac{1}{r}(m, n)$ satisfies

$$\gcd(m^*, n^*) = 1, \quad n^* \in \bigcup_{|k_i^* - k_i| \leq \ln M} [e^{k_i^* - 1}, e^{k_i^*}], \quad |n^* \alpha - m^*| < \frac{1}{n^* R_i}.$$

Assume k_i is so large that $R_i = \psi(e^{k_i}) \geq 4$. Then Lemma 3.3 gives $a_{\ell+1} > \frac{R_i}{2}$. Thus $\mathcal{A}_{k_i}(I) \subset \mathcal{C}_{k_i}(I, R_i)$ where

$$\mathcal{C}_k(I, R) := \bigcup_{k^* \in [k - \ln M, k]} \left\{ \alpha \in (0, 1) \setminus \mathbb{Q} : \exists \ell \text{ s.t. } \begin{array}{l} q_\ell(\alpha) \in [e^{k^* - 1}, e^{k^*}], \\ a_{\ell+1}(\alpha) \geq R/2 \\ p_\ell(\alpha)/q_\ell(\alpha) \in I \end{array} \right\}.$$

This is a union of cylinders, because $q_\ell(\alpha), p_\ell(\alpha), a_{\ell+1}(\alpha)$ are constant on cylinders of length $\ell + 1$.

We claim that for some $c^*(M)$ which only depends on M , for all k_i large enough,

$$(3.9) \quad \text{mes}(\mathcal{C}_{k_i}(I, R_i)) \leq \frac{c^*(M) \text{mes}(I)}{R_i}.$$

Every rational $\frac{m}{n} \in (0, 1)$ has two finite continued fraction expansions: $[0; a_1, \dots, a_\ell]$ and $[0; a_1, \dots, a_\ell - 1, 1]$ with $a_\ell > 1$. We write $\ell = \ell(\frac{m}{n})$ and $a_i = a_i(\frac{m}{n})$. With this notation

$$\begin{aligned} \mathcal{C}_{k_i}(I, R_i) = & \bigcup_{\substack{k_i^* \in [k_i - \ln M, k_i] \\ n \in [e^{k_i^* - 1}, e^{k_i^*}]}} \bigcup_{\substack{\gcd(m,n)=1 \\ m/n \in I}} \bigcup_{b > R_i/2} \left[\left[a_1 \left(\frac{m}{n} \right), \dots, a_\ell \left(\frac{m}{n} \right), b \right] \right. \\ & \left. \cup \left[\left[a_1 \left(\frac{m}{n} \right), \dots, a_\ell \left(\frac{m}{n} \right) - 1, 1, b \right] \right]. \end{aligned}$$

We have $[[a_1, \dots, a_\ell]] = (\frac{p_\ell + p_{\ell-1}}{q_\ell + q_{\ell-1}}, \frac{p_\ell}{q_\ell})$ or $(\frac{p_\ell}{q_\ell}, \frac{p_\ell + p_{\ell-1}}{q_\ell + q_{\ell-1}})$, depending on the parity of ℓ , see for instance [Khi63]. Since $|p_\ell q_{\ell-1} - p_{\ell-1} q_\ell| = 1$ and $q_{\ell+1} = a_{\ell+1} q_\ell + q_{\ell-1}$, we have $\text{mes}([[a_1, \dots, a_\ell, b]]) = \frac{1}{q_{\ell+1}(q_{\ell+1} + q_\ell)} = \frac{1}{(b q_\ell + q_{\ell-1})(b + 1) q_\ell + q_{\ell-1}} \leq \frac{1}{b(b+1)q_\ell^2}$, leading to

$$\begin{aligned} \text{mes}(\mathcal{C}_{k_i}(I, R_i)) & \leq \sum_{k_i^* \in [k_i - \ln M, k_i]} \sum_{n \in [e^{k_i^* - 1}, e^{k_i^*}]} \sum_{\substack{\gcd(m,n)=1 \\ m/n \in I}} \sum_{b > R_i/2} \frac{2}{n^2 b(b+1)} \\ & \leq \frac{8 \ln M}{e^{2(k_i - 1 - \ln M)} R_i} \sum_{n=1}^{e^{k_i} M} \# \left\{ m \in \mathbb{N} : \frac{m}{n} \in I, \gcd(m, n) = 1 \right\} \leq \frac{c^*(M)}{R_i} \text{mes}(I) \end{aligned}$$

where $c^*(M)$ only depends on M . The last step uses Proposition 3.9.

Next we cover $\mathcal{A}_{k_1}(I) \cap \mathcal{A}_{k_2}(I)$ by cylinders. Suppose without loss of generality that $k_2 > k_1$. Arguing as before one sees that if

$$(3.10) \quad k_2 > k_1 + \ln M + 1,$$

then $\mathcal{A}_{k_1}(I) \cap \mathcal{A}_{k_2}(I)$ can be covered by sets $\llbracket a_1, \dots, a_\ell, b, \bar{a}_1, \dots, \bar{a}_{\bar{l}}, \bar{b} \rrbracket$ as follows: The convergents p_i/q_i of (every) α in $\llbracket a_1, \dots, a_\ell, b, \bar{a}_1, \dots, \bar{a}_{\bar{l}}, \bar{b} \rrbracket$, ($1 \leq i \leq l + \bar{l} + 2$), satisfy

- (1) $q_l \in [e^{k_1^*-1}, e^{k_1^*}]$, $k_1^* \in [k_1 - \ln M, k_1]$, $p_l/q_l \in I$, $b \geq R_1/2$;
- (2) $q_{l+\bar{l}+1} \in [e^{k_2^*-1}, e^{k_2^*}]$, $k_2^* \in [k_2 - \ln M, k_2]$, $p_{\bar{l}}/q_{\bar{l}} \in I$, $\bar{b} \geq R_2/2$
- (3) $k_2^* > k_1^*$ (this is where (3.10) is used).

We claim that

$$(3.11) \quad \llbracket a_1, \dots, a_\ell, b \rrbracket \subset \mathcal{C}_{k_1}(I),$$

$$(3.12) \quad b \leq e^{k_2^* - k_1^* + 1},$$

$$(3.13) \quad \llbracket \bar{a}_1, \dots, \bar{a}_{\bar{l}}, \bar{b} \rrbracket \subset \bigcup_{|r| \leq 3} \mathcal{C}_{k_2 - k_1 + r - \ln b}([0, 1], R_2).$$

(3.11) follows from (1). Next, $e^{k_2^*} \geq q_{l+1} \geq bq_l \geq be^{k_1^*-1}$ proving (3.12). To prove (3.13), let \bar{p}_i/\bar{q}_i , $1 \leq i \leq \bar{l} + 2$, be the principal convergents of (every) $\bar{\alpha} \in \llbracket b, \bar{a}_1, \dots, \bar{a}_{\bar{l}}, \bar{b} \rrbracket$. By Proposition 3.10, $q_{l+1+\bar{l}} = q_{l-1}\bar{p}_{\bar{l}+1} + q_l\bar{q}_{\bar{l}+1}$, whence $q_l\bar{q}_{\bar{l}+1} \leq q_{l+1+\bar{l}} \leq 2q_l\bar{q}_{\bar{l}+1}$. Since $q_l \in [e^{k_1^*-1}, e^{k_1^*}]$ and $q_{l+\bar{l}+1} \in [e^{k_2^*-1}, e^{k_2^*}]$,

$$(3.14) \quad e^{k_2^* - k_1^* - 2} \leq \frac{q_{l+1+\bar{l}}}{2q_l} \leq \bar{q}_{\bar{l}+1} \leq \frac{q_{l+1+\bar{l}}}{q_l} \leq e^{k_2^* - k_1^* + 1}.$$

Next, let \tilde{p}_i/\tilde{q}_i ($1 \leq i \leq \bar{l}$) denote the principal convergents of (every) $\tilde{\alpha} \in \llbracket \bar{a}_1, \dots, \bar{a}_{\bar{l}}, \bar{b} \rrbracket$. Then $\frac{\tilde{p}_{\bar{l}+1}}{\tilde{q}_{\bar{l}+1}} = 1/(b + \frac{\tilde{p}_{\bar{l}}}{\tilde{q}_{\bar{l}}})$, so $\bar{q}_{\bar{l}+1} = b\tilde{q}_{\bar{l}} + \tilde{p}_{\bar{l}}$, whence $b\tilde{q}_{\bar{l}} \leq \bar{q}_{\bar{l}+1} \leq (b + 1)\tilde{q}_{\bar{l}}$. Thus $\tilde{q}_{\bar{l}} \in [(b + 1)^{-1}\bar{q}_{\bar{l}+1}, b^{-1}\bar{q}_{\bar{l}+1}]$. It follows that the \bar{l} -th principal convergent of every $\tilde{\alpha} \in \llbracket \bar{a}_1, \dots, \bar{a}_{\bar{l}}, \bar{b} \rrbracket$ satisfies

$$(3.15) \quad \tilde{q}_{\bar{l}} \in [e^{k_2^* - k_1^* - 3 - \ln b}, e^{k_2^* - k_1^* + 1 - \ln b}].$$

It is now easy to see (3.13).

By (3.13), $\mathcal{A}_{k_1}(I) \cap \mathcal{A}_{k_2}(I) \cap I \subset \bigcup_{|r| \leq 3} \mathfrak{U}_{[a,b] \subset \mathcal{C}_{k_1}(I)} \mathfrak{U}_{[a',b'] \subset \mathcal{C}_{k_2 - k_1 + r - \ln b}([0,1])} \llbracket a, b, a', b \rrbracket$. Now arguing as in the proof of (3.9) and using (3.8) we obtain

$$(3.16) \quad \begin{aligned} & \text{mes}(\mathcal{A}_{k_1}(I) \cap \mathcal{A}_{k_2}(I) \cap I) \\ & \leq \sum_{\substack{k_i^* \in [k_i - \ln M, k_i] \\ i=1,2; |r| \leq 3}} \sum_{n \in [e^{k_1^*-1}, e^{k_1^*}]} \sum_{\substack{\gcd(m,n)=1 \\ m/n \in I}} \sum_{b=[R_1/2]}^{[\exp(k_2^* - k_1^* + 1)]} \frac{2G \text{mes}(\mathcal{C}_{k_2 - k_1 + r - \ln b}([0,1], R_2))}{n^2 b(b+1)} \\ & \leq \frac{\text{const mes}(I)}{R_1 R_2}. \end{aligned}$$

(3.16) uses the estimate $\text{mes}(\mathcal{C}_{k_2 - k_1 + r - \ln b}([0, 1], R_2)) = O(1/R_2)$ which is also valid when $k_2 - k_1 + r - \ln b$ is small, provided we choose M large enough so that the asymptotic in Proposition 3.9 holds for all $N > M$ with $I = [0, 1]$. See the proof of (3.9).

Combining (3.16) with (3.7), we find that under (3.10) $\mathcal{A}_{k_i}(I)$ are D -quasi-independent for sufficiently large D , proving Claim 3. \square

Claims 2 and 3 allow us to apply Sullivan’s Borel–Cantelli Lemma (Proposition 3.7). We obtain $\delta = \delta(M)$ s.t. for every interval $I \subset [\varepsilon, 1 - \varepsilon]$, $\text{mes}(\mathcal{A} \cap I) \geq \delta \text{mes}(I)$. This means that $[\varepsilon, 1 - \varepsilon] \setminus \mathcal{A}$ has no Lebesgue density points, and therefore must have measure zero. So \mathcal{A} has full measure in $[\varepsilon, 1 - \varepsilon]$. Since ε is arbitrary, \mathcal{A} has full measure. \square

4. Proof of Theorem 1.2

As explained in Section 2, it is enough to prove Theorem 1.2 for $f(x)$ of the form $\sum_{m=1}^d b_m h(x + \beta_m) \not\equiv 0$ with $h(x) = \{x\} - \frac{1}{2}$. Without loss of generality, β_i are different and $b_i \neq 0$. Notice that

$$f(x) = - \sum_{m=1}^d b_m \sum_{j=1}^{\infty} \frac{\sin(2\pi j(x + \beta_m))}{\pi j}.$$

Therefore $\|f\|_2^2 = \frac{1}{2\pi^2} \sum_n \frac{1}{n^2} D(\beta_1 n, \dots, \beta_d n)$, where $D : \mathbb{T}^d \rightarrow \mathbb{R}$ is

$$(4.1) \quad D(\gamma_1, \dots, \gamma_d) := \int_0^1 \left[\sum_{m=1}^d b_m \sin(2\pi(y + \gamma_m)) \right]^2 dy.$$

Since $f \not\equiv 0$, $D(\beta_1 n, \dots, \beta_d n) > 0$ for some n . Let \mathfrak{T} denote the closure in \mathbb{T}^d of $\mathbb{O} := \{(\beta_1 n, \dots, \beta_d n) \bmod \mathbb{Z} : n \in \mathbb{Z}\}$. This is a minimal set for the translation by $(\beta_1, \dots, \beta_d)$ on \mathbb{T}^d , so a standard compactness argument shows that for every $\varepsilon_0 > 0$, the set

$$(4.2) \quad \mathcal{N} := \{n \in \mathbb{N} : D(\beta_1 n, \dots, \beta_d n) > \varepsilon_0\}$$

is syndetic: its gaps are bounded. Thus \mathcal{N} has positive lower density.

By Theorem 3.1, if M is sufficiently large then the set $\mathcal{A} := \mathcal{A}(\mathcal{N}, M)$ has full measure in \mathbb{T} . Let

$$S_n(\alpha, x) := \sum_{k=0}^{n-1} f(x + k\alpha).$$

The proof of Theorem 1.2 for $f(x)$ above consists of two parts:

THEOREM 4.1. — *Suppose $\alpha \in \mathcal{A}$, then for a.e. $x \in [0, 1)$, there exist $A_k(x) \in \mathbb{R}$ and $B_k(x), N_k(x) \rightarrow \infty$ such that*

$$\frac{S_n(\alpha, x) - A_k(x)}{B_k(x)} \xrightarrow[k \rightarrow \infty]{\text{dist}} \text{U}[0, 1], \text{ as } n \sim \text{U}(0, \dots, N_k(x)).$$

THEOREM 4.2. — *Suppose $\alpha \in \mathcal{A}$, then for a.e. $x \in [0, 1)$, there are no $A_N(x) \in \mathbb{R}$ and $B_N(x) \rightarrow \infty$ such that*

$$\frac{S_n(\alpha, x) - A_N(x)}{B_N(x)} \xrightarrow[N \rightarrow \infty]{\text{dist}} \text{U}[0, 1], \text{ as } n \sim \text{U}(0, \dots, N).$$

4.1. Preliminaries

LEMMA 4.3. — $S_q(\alpha, \cdot) : \mathbb{T} \rightarrow \mathbb{R}$ has dq discontinuities.

Proof. — The discontinuities of S_q are preimages of discontinuities of f by R_α^{-k} with $k = 0, 1, \dots, q - 1$. □

LEMMA 4.4. — Let $C := \sup |f'| \leq |\sum b_m|$. If x', x'' belong to same continuity component of R_α^r then

$$|S_r(\alpha, x') - S_r(\alpha, x'')| \leq Cr|x' - x''|.$$

Proof. — Since $|S'_r| = \left| \sum_{k=0}^{r-1} f'(x + k\alpha) \right| \leq Cr$, the restriction of S_r to on each continuity component is Lipschitz with Lipschitz constant Cr . □

LEMMA 4.5. — There are constants C_1, C_2 such that the following holds. Suppose that q_n is a principal denominator of α , and $q_{n+1} > cq_n$ with $c > 1$. Let $\mu_n(x) := S_{q_n}(\alpha, x)$, then

$$(4.3) \quad \text{mes} \left\{ x : S_{\ell q_n}(\alpha, x) = \ell \mu_n \pm C_1 \frac{\ell^2}{c} \text{ for } \ell = 0, \dots, k \right\} > 1 - C_2 \frac{k}{c}.$$

Proof. — If x and $x + \ell q_n \alpha$ belong to the same continuity interval of $R_\alpha^{q_n}$ for all $\ell = 0, \dots, k$ then we have by Lemma 4.4 that for $\ell \leq k$

$$\begin{aligned} |S_{\ell q_n}(\alpha, x) - \ell \mu_n| &\leq \sum_{j=0}^{\ell-1} |S_{q_n}(\alpha, x + j q_n \alpha) - S_{q_n}(\alpha, x)| \leq C q_n \sum_{j=0}^{\ell-1} \|j q_n \alpha\| \\ &\leq \frac{C q_n}{q_{n+1}} \sum_{j=0}^{\ell-1} j \leq \frac{C_1 \ell^2}{c}, \quad \text{where } C_1 := C/2. \end{aligned}$$

Therefore if $S_{\ell q_n}(\alpha, x) \neq \ell \mu_n \pm C_1 \frac{\ell^2}{c}$ for some $\ell = 0, \dots, k$, then there must exist $0 \leq \ell \leq k$ s.t. $x, R_\alpha^{\ell q_n}(x)$ are separated by a discontinuity of $S_{q_n}(\alpha, \cdot)$. Since $\text{dist}(x, R_\alpha^{\ell q_n}(x)) \leq \ell/q_{n+1}$, x must belong to a ball with radius k/q_{n+1} centered at a discontinuity of $S_{q_n}(\alpha, \cdot)$. By Lemma 4.3, there are dq_n discontinuities, so the measure of such points is less than $dq_n \left(\frac{2k}{q_{n+1}} \right) \leq \frac{2dk}{c}$. The lemma follows with $C_2 := 2d$. □

LEMMA 4.6. — There is a constant $C_3 = C_3(b_1, \dots, b_d)$ s.t. for every $n \geq 1$ and $\alpha = [0; a_1, a_2, \dots]$, $\max\{|S_r(\alpha, x)| : 0 \leq r \leq q_n - 1\} \leq C_3(a_0 + \dots + a_{n-1})$.

Proof. — Let $r = \sum_{j=0}^{n-1} \mathbf{b}_j q_j$ denote the Ostrowski expansion of r . Recall that this means that $0 \leq \mathbf{b}_j \leq a_j$ and $\mathbf{b}_j = a_j \Rightarrow \mathbf{b}_{j-1} = 0$. So

$$S_r = \sum_{k=0}^{\mathbf{b}_{n-1}-1} S_{q_{n-1}} \circ R_\alpha^{q_{n-1}k} + \sum_{k=0}^{\mathbf{b}_{n-2}-1} S_{q_{n-2}} \circ R_\alpha^{q_{n-2}k} + \dots + \sum_{k=0}^{\mathbf{b}_0-1} S_{q_0} \circ R_\alpha^{q_0k}.$$

By the Denjoy–Koksma inequality $|S_r| \leq \sum \mathbf{b}_j \mathbf{V}(f) \leq \mathbf{V}(f) \sum a_j$ where $\mathbf{V}(f) \leq 2 \sum b_i$ is the total variation of f on \mathbb{T} . □

LEMMA 4.7. — There exist positive constants $\varepsilon_1, \varepsilon_2$ such that for every α irrational, if q_n is a principal denominator of α and $q_n r_n \in \mathcal{N}$ with $r_n \leq M$ then $\text{mes}\{x : |S_{q_n}(\alpha, x)| \geq \varepsilon_1\} \geq \varepsilon_2$.

Proof. — We follow an argument from [Bec94]. Suppose q_n is a principal denominator of α and $q_n r_n \in \mathcal{N}$ for some $r_n \leq M$. Let $N = q_n r_n$. Since $f(x) = -\sum_{m=1}^d b_m \sum_{j=1}^{\infty} \frac{\sin(2\pi j(x+\beta_m))}{\pi j}$, for each $j \in \mathbb{N}$

$$\|S_N(\alpha, \cdot)\|_{L^2}^2 \geq \frac{1}{\pi^2 j^2} \int_0^1 \left(\sum_{m=1}^d b_m \sum_{k=0}^{N-1} \sin(2\pi j(x+k\alpha+\beta_m)) \right)^2 dx.$$

Using the identities $\sum_{k=1}^N \sin(y+kx) = \frac{\cos(y+x/2) - \cos(y+(2N+1)x/2)}{2\sin(x/2)}$ and $\cos A - \cos B = 2\sin(\frac{A+B}{2})\sin(\frac{B-A}{2})$ we find that

$$\begin{aligned} \|S_N(\alpha, \cdot)\|_{L^2}^2 &\geq \left(\frac{\sin(\pi N j \alpha)}{\pi j \sin(\pi j \alpha)} \right)^2 \int_0^1 \left(\sum_{m=1}^d b_m \sin \left(2\pi \left(jx + j \frac{(N-1)\alpha}{2} \right) + 2\pi j \beta_m \right) \right)^2 dx \\ &= \left(\frac{\sin(\pi N j \alpha)}{\pi j \sin(\pi j \alpha)} \right)^2 \int_0^1 \left(\sum_{m=1}^d b_m \sin(2\pi(y+j\beta_m)) \right)^2 dy \\ &= \left(\frac{\sin(\pi N j \alpha)}{\pi j \sin(\pi j \alpha)} \right)^2 D(j\beta_1, \dots, j\beta_m) \quad \text{with } D \text{ as in (4.1)}. \end{aligned}$$

We now take $j = N = r_n q_n$. The first term is bounded below because $\|N\alpha\| \leq M\|q_n\alpha\| \leq \frac{M}{q_{n+1}} \leq \frac{M}{a_{n+1}q_n} \leq \frac{M^2}{a_{n+1}N} = o(\frac{1}{N})$, so $\frac{\sin(\pi N^2 \alpha)}{\pi N \sin(\pi N \alpha)} \xrightarrow{n \rightarrow \infty} \pi^{-1}$. The second term is bounded below by ε_0 , because $N = q_n r_n \in \mathcal{N}$. It follows that for all n large enough, $\|S_{r_n q_n}(\alpha, \cdot)\|_2 > \sqrt{\varepsilon_0}/2\pi$.

For any L^2 -function φ and any $\hat{\varepsilon} > 0$,

$$\|\varphi\|_{L^2}^2 \leq \|\varphi\|_{L^\infty}^2 \text{mes}\{x : |\varphi(x)| \geq \hat{\varepsilon}\} + \hat{\varepsilon}^2.$$

Hence $\text{mes}\{x : |\varphi(x)| \geq \hat{\varepsilon}\} \geq \frac{\|\varphi\|_{L^2}^2 - \hat{\varepsilon}^2}{\|\varphi\|_{L^\infty}^2}$. We just saw that for all n large enough, $\|S_{r_n q_n}(\alpha, \cdot)\|_2 > \sqrt{\varepsilon_0}/2\pi$, and by the Denjoy–Koksma inequality $\|S_{r_n q_n}(\alpha, \cdot)\|_{L^\infty} \leq MV(f)$. So for some $\hat{\varepsilon} > 0$ and for all n large enough, $\text{mes}\{x : |S_{r_n q_n}(\alpha, x)| > \hat{\varepsilon}\} \geq \hat{\varepsilon}$.

Looking at the inequality $|S_{r_n q_n}(\alpha, x)| \leq \sum_{k=0}^{r_n-1} |S_{q_n}(\alpha, x+kq_n\alpha)|$, we see that if $|S_{r_n q_n}(\alpha, x)| \geq \hat{\varepsilon}$, then $|S_{q_n}(\alpha, x+kq_n\alpha)| \geq \hat{\varepsilon}/M$ for some $0 \leq k \leq M-1$. So for all n large enough, $\text{mes}\{x : |S_{q_n}(\alpha, x)| > \hat{\varepsilon}/M\} \geq \hat{\varepsilon}/M$. \square

4.2. Proof of Theorem 4.1

Let $\Omega^*(\alpha)$ be the set of x where the conclusion of Theorem 4.1 holds. $\Omega^*(\alpha)$ is R_α -invariant and it is measurable by Lemma A.1 in the appendix. Therefore to show that $\Omega^*(\alpha)$ has full measure, it suffices to show that it has positive measure.

Suppose $\alpha \in \mathcal{A}$ and let $n_k \uparrow \infty$ be a sequence satisfying (3.1) with \mathcal{N} given by (4.2). There is no loss of generality in assuming that

$$\frac{a_{n_{k+1}}}{a_0 + \dots + a_{n_k}} > k^3.$$

So $q_{n_{k+1}} > k^3 L_k q_{n_k}$, where $L_k := a_0 + \dots + a_{n_k}$.

Recall that $\mu_{n_k}(x) = S_{q_{n_k}}(\alpha, x)$. For all k sufficiently large, there is a set A_k of measure at least $\varepsilon_2/2$ such that for all $x \in A_k$,

$$(4.4) \quad S_{\ell q_{n_k}}(\alpha, x) = \ell \left(\mu_{n_k}(x) \pm \frac{C_1 \ell}{k^3 L_k} \right) \text{ for all } \ell = 0, 1, \dots, kL_k,$$

$$(4.5) \quad |\mu_{n_k}(x)| \geq \varepsilon_1.$$

This is because Lemma 4.5 says that the total measure of x for which (4.4) fails is $O(1/k^2)$ while (4.5) holds on the set of measure ε_2 by Lemma 4.7.

It follows that $\text{mes}(\bigcap_{n>1} \bigcup_{k>n} A_k) \geq \varepsilon_2/2$. Therefore there exists x which belongs to infinitely many A_k . After re-indexing n_k , we may assume that (4.4), (4.5) are satisfied for all $k \in \mathbb{N}$. Henceforth, we fix such an x and work with this x . Let

$$N_k(x) := kL_k q_{n_k}, \quad B_k(x) := kL_k |\mu_{n_k}(x)|, \quad A_k(x) := \frac{1}{2}(\text{sgn}(\mu_{n_k}(x)) - 1)B_k.$$

Any $n \leq N_k$ can be written uniquely in the form

$$n = l(n)q_{n_k} + r(n) \quad \text{with} \quad 0 \leq l(n) \leq kL_k \text{ and } 0 \leq r(n) < q_{n_k}.$$

It is easy to see that $\frac{l(n)}{kL_k} \xrightarrow[k \rightarrow \infty]{\text{dist}} \text{U}[0, 1]$ as $n \sim \text{U}(1, \dots, N_k)$.

Writing $S_n(\alpha, x) = S_{l(n)q_{n_k}}(\alpha, x) + S_{r(n)}(\alpha, x + \alpha l(n)q_{n_k})$ we obtain from (4.4) and Lemma 4.6 that

$$S_n(\alpha, x) = l(n)\mu_{n_k}(x) + O(L_k).$$

So $\frac{S_n(x)}{B_k}$ is asymptotically uniform on $[0, 1]$ when $\mu_{n_k} > 0$, and asymptotically uniform on $[-1, 0]$ when $\mu_{n_k} < 0$. So $\frac{S_n(x) - A_k}{B_k} \xrightarrow[k \rightarrow \infty]{} \text{U}[0, 1]$, as $n \sim \text{U}(1, \dots, N_k(x))$. \square

4.3. Proof of Theorem 4.2

Let $\Omega(\alpha)$ denote the set of $x \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$ for which there are $B_N(x) \rightarrow \infty$ and $A_N(x) \in \mathbb{R}$ s.t.

$$(4.6) \quad \frac{S_n(\alpha, x) - A_N(x)}{B_N(x)} \xrightarrow[N \rightarrow \infty]{\text{dist}} \text{U}[0, 1], \text{ as } n \sim \text{U}(1, \dots, N).$$

$\Omega(\alpha)$ is measurable, and $A_n(\cdot), B_n(\cdot)$ can be chosen to be measurable on $\Omega(\alpha)$, see the appendix. Assume by way of contradiction that $\text{mes}[\Omega(\alpha)] \neq 0$ for some $\alpha \in \mathcal{A}$.

$\Omega(\alpha)$ is invariant under $R_\alpha(x) = x + \alpha \pmod 1$ on $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. Since R_α is ergodic, and $\Omega(\alpha)$ is measurable, $\text{mes}[\Omega(\alpha)] = 1$.

Since $\alpha \in \mathcal{A}$, there is an increasing sequence n_k satisfying (3.1) where \mathcal{N} is given by (4.2). We can choose n_k so that $q_{n_k} r_{n_k} \in \mathcal{N}$ for $r_{n_k} \leq M$, and $a_{n_k+1} > k^3 L_k$ where $L_k := a_0 + \dots + a_{n_k}$. In particular, $q_{n_k+1} > k^3 L_k q_{n_k}$.

Recall that $\mu_{n_k}(x) := S_{q_{n_k}}(\alpha, x)$. By Lemma 4.7 we can choose x such that for infinitely many k , $|\mu_{n_k}(x)| \geq \varepsilon_1$. We will suppose that $\mu_{n_k}(x) > 0$ for infinitely many k ; the case where $\mu_{n_k}(x) < 0$ for infinitely many k is similar.

CLAIM 4. — *It is possible to assume without loss of generality that $\|B_{q_{n_k}}\|_\infty := \sup_{x \in \Omega(\alpha)} |B_{q_{n_k}}(x)| \leq 3C_3 L_k$ for all k where C_3 is the constant from Lemma 4.6.*

Proof. — We claim that for every x with (4.6), $B_{q_{n_k}}(x) \leq 3C_3L_k$ for all k large enough. Otherwise, by Lemma 4.6, there are infinitely many k s.t. $B_{q_{n_k}}(x) > 3 \max\{|S_r(\alpha, x)| : r = 0, \dots, q_{n_k-1}\}$, whence $|S_n(\alpha, x)/B_{q_{n_k}}| \leq \frac{1}{3}$ for all $0 \leq n \leq q_{n_k} - 1$. In such circumstances, (4.6) does not hold (the spread is not big enough).

Since $B_{q_{n_k}}(x) \leq 3C_3L_k$ for all k large enough, there is no harm in replacing $B_{q_{n_k}}(x)$ in (4.6) by $\min\{B_{q_{n_k}}(x), 3C_3L_k\}$. \square

CLAIM 5. — Fix $D > C = |\sum b_m|$, and let E_k denote the set of $x \in \Omega(\alpha)$ s.t. $S_r(\alpha, x) = S_r(\alpha, R_\alpha^{\ell q_{n_k}}(x)) \pm \frac{D\ell}{q_{n_k+1}}$ for all $0 \leq \ell \leq B_{q_{n_k}}(x), 0 \leq r < q_{n_k} - 1$. Then $\text{mes}(E_k^c) \leq C_4k^{-3}$.

Proof. — If $x \notin E_k$, then there are $0 \leq \ell \leq B_{q_{n_k}}(x), 0 \leq r < q_{n_k} - 1$ s.t.

$$|S_r(\alpha, x) - S_r(\alpha, x + \ell q_{n_k}\alpha)| \geq \frac{D\ell}{q_{n_k+1}}.$$

By Lemma 4.4, $\{x\}, \{x + \ell q_{n_k}\alpha\}$ are separated by a singularity of $S_r(\alpha, \cdot)$. So x belongs to a ball of radius $2\|\ell q_{n_k}\alpha\|$ centered at one of the dq_{n_k} discontinuities of $S_{q_{n_k}}(\alpha, \dots)$. Thus $\text{mes}(E_k^c) \leq dq_{n_k} \cdot 2\|\ell q_{n_k}\alpha\|$. Now $\|\ell q_{n_k}\alpha\| \leq \ell\|q_{n_k}\alpha\| \leq \frac{\|B_{q_{n_k}}\|_\infty}{q_{n_k+1}} \leq \frac{3C_3L_k}{q_{n_k+1}} \leq \frac{3C_3}{k^3q_{n_k}}$ by our choice of n_k . So $\text{mes}(E_k^c) \leq C_4/k^3$ with $C_4 := 6dC_3$. \square

CLAIM 6. — Let F_k denote the set of $x \in \Omega(\alpha)$ s.t.

$$S_{\ell q_{n_k}}(\alpha, x) = \ell \left(\mu_{n_k}(x) \pm \frac{C_1\ell}{k^3L_k} \right) \text{ for all } 0 \leq \ell \leq B_{q_{n_k}}(x).$$

Then $\text{mes}(F_k^c) \leq C_5k^{-2}$.

Proof. — This follows from Lemma 4.5. \square

By Claims 5 and 6, and a Borel–Cantelli argument, for a.e. x there is $k_0(x)$ s.t. $x \in E_k \cap F_k$ for all $k \geq k_0(x)$.

Suppose $k \geq k_0(x)$, and let $N_k := q_{n_k}B_{q_{n_k}}(x)$. Every $0 \leq n \leq N_k - 1$ can be uniquely represented as $n = \ell q_{n_k} + r$ with $0 \leq \ell \leq B_{q_{n_k}}(x) - 1$ and $0 \leq r \leq q_{n_k} - 1$. Using the bound $\|B_{q_{n_k}}\|_\infty = O(L_k)$, we find:

$$\begin{aligned} & \frac{S_n(\alpha, x) - A_{q_{n_k}}(x)}{B_{q_{n_k}}(x)} \\ &= \frac{S_{\ell q_{n_k}}(\alpha, x)}{B_{q_{n_k}}(x)} + \frac{S_r(\alpha, R_\alpha^{\ell q_{n_k}}x) - A_{q_{n_k}}(x)}{B_{q_{n_k}}(x)} \\ &= \frac{S_{\ell q_{n_k}}(\alpha, x)}{B_{q_{n_k}}(x)} + \frac{S_r(\alpha, x) - A_{q_{n_k}}(x) + o(1)}{B_{q_{n_k}}(x)}, & \text{because } x \in E_k \\ &= \frac{\ell(\mu_{n_k}(x) + o(1))}{B_{q_{n_k}}(x)} + \frac{S_r(\alpha, x) - A_{q_{n_k}}(x) + o(1)}{B_{q_{n_k}}(x)}, & \text{because } x \in F_k. \end{aligned}$$

If $n \sim U(0, \dots, N_k - 1)$, then ℓ, r are independent random variables, $\ell \sim U(0, \dots, B_{q_{n_k}}(x) - 1)$ and $r \sim U(0, \dots, q_{n_k} - 1)$. Thus the distribution of $\frac{\ell(\mu_{n_k}(x) + o(1))}{B_{q_{n_k}}(x)}$ is

close to $U[0, \mu_{n_k}(x)]$, and the distribution of $\frac{S_r(\alpha, x) - A_{q_{n_k}}(x) + o(1)}{B_{q_{n_k}}(x)}$ converges to $U[0, 1]$ (because $x \in \Omega$).

Taking a subsequence such that $\mu_{n_k}(x) \rightarrow \bar{\varepsilon} > \varepsilon_1$ we see that the random variables $\frac{S_n(\alpha, x) - A_{q_{n_k}}(x)}{B_{q_{n_k}}(x)}$, where $n \sim U(0, \dots, n_k - 1)$, converge in distribution to the sum of two independent uniformly distributed random variables. This contradicts to (4.6), because the sum of two independent uniform random variables is not uniform. \square

Appendix A. Measurability concerns

Let $\Omega(\alpha)$ denote the set of $x \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$ such that for some $B_N(x) \rightarrow \infty$ and $A_N(x) \in \mathbb{R}$, $\frac{S_n(\alpha, x) - A_N(x)}{B_N(x)} \xrightarrow[N \rightarrow \infty]{\text{dist}} U[0, 1]$, as $n \sim U(1, \dots, N)$ and let $\Omega^*(\alpha)$ denote the set of $x \in \mathbb{T}$ such that along a subsequence $N_k(x)$ there exist some $B_{N_k}(x) \rightarrow \infty$ and $A_{N_k}(x) \in \mathbb{R}$, $\frac{S_n(\alpha, x) - A_{N_k}(x)}{B_{N_k}(x)} \xrightarrow[k \rightarrow \infty]{\text{dist}} U[0, 1]$, as $n \sim U(1, \dots, N_k)$. We make no assumptions on the measurability of A_N, B_N, N_k as functions of x . The purpose of this section is to prove:

LEMMA A.1. — $\Omega(\alpha)$ and $\Omega^*(\alpha)$ are measurable.

The crux of the argument is to show that $A_N(x), B_N(x)$ can be replaced by measurable functions, defined in terms of the percentiles of the random quantities $S_n(x, \alpha)$, $n \sim U(1, \dots, N)$.

Recall that given $0 < t < 1$, the upper and lower t -percentiles of a random variable X are defined by

$$\begin{aligned} \chi^+(X, t) &:= \inf\{\xi : \Pr(X \leq \xi) > t\} \\ \chi^-(X, t) &:= \sup\{\xi : \Pr(X \leq \xi) < t\} \end{aligned} \quad (0 < t < 1).$$

Notice that $\Pr(X \leq \chi^+(X, t)) \geq t$, $\Pr(X < \chi^-(X, t)) \leq t$, and $\Pr(\chi^-(X, t) < X < \chi^+(X, t)) = 0$. In case X is non-atomic (i.e. $\Pr(X = a) = 0$ for all a), we can say more:

LEMMA A.2. — Suppose X is a non-atomic real valued random variable, fix $0 < t < 1$ and let $\chi_t^\pm := \chi^\pm(X, t)$, then

- (a) $\Pr(X < \chi_t^+) = t$ and $\Pr(X < \chi_t^-) = t$;
- (b) $\forall \varepsilon > 0$, $\Pr(\chi_t^- - \varepsilon < X < \chi_t^-)$, $\Pr(\chi_t^+ < X < \chi_t^+ + \varepsilon)$ are positive;
- (c) $\exists t_1 < t_2$ s.t. $\chi_{t_1}^- < \chi_{t_2}^+$ and $\chi_{t_1}^-, \chi_{t_2}^+$ have the same sign.

Proof. — Since X is non-atomic, $\Pr(X < \chi_t^+) = \Pr(X \leq \chi_t^+) \geq t$ and $\Pr(X < \chi_t^-) \leq t$. If $\chi_t^+ = \chi_t^-$, part (a) holds. If $\chi_t^+ > \chi_t^-$ then for all $h > 0$ small enough $\chi_t^- + h < \chi_t^+ - h$ whence

$$\begin{aligned} 0 \leq \Pr(\chi_t^- < X < \chi_t^+) &= \lim_{h \rightarrow 0^+} \Pr(\chi_t^- + h < X < \chi_t^+ - h) \\ &= \lim_{h \rightarrow 0^+} \Pr(X < \chi_t^+ - h) - \lim_{h \rightarrow 0^+} \Pr(X \leq \chi_t^- + h) \leq t - t = 0. \end{aligned}$$

Necessarily $\lim_{h \rightarrow 0^+} \Pr(X < \chi_t^+ - h) = t$ and $\lim_{h \rightarrow 0^+} \Pr(X \leq \chi_t^- + h) = t$, which gives us $\Pr(X < \chi_t^+) = t$ and $\Pr(X < \chi_t^-) = \Pr(X \leq \chi_t^-) = t$.

For (b) assume by contradiction that $\Pr(\chi_t^- - \varepsilon < X < \chi_t^-) = 0$, then for all $\chi_t^- - \varepsilon < \xi < \chi_t^-$, $\Pr(X \leq \xi) = \Pr(X < \chi_t^-) = t$, whence $\chi_t^- \leq \chi_t^- - \varepsilon$, a contradiction. Similarly, $\Pr(\chi_t^+ < X < \chi_t^+ + \varepsilon) = 0$ is impossible.

To prove (c) note that since X is non-atomic, either $\Pr(X > 0)$ or $\Pr(X < 0)$ is positive. Assume w.l.o.g. that $\Pr(X > 0) \neq 0$. By non-atomicity, there are positive $a < b$ s.t. $\Pr(X \in (0, a)) \neq 0$ and $\Pr(X \in (a, b)) \neq 0$. Take $t_1 := \Pr(X < a)$ and $t_2 := \Pr(X < b)$. \square

From now on fix a non-atomic random variable Y , and choose $0 < t_1 < t_2 < 1$ as in Lemma A.2(c) s.t. $\chi^-(Y, t_1) < \chi^+(Y, t_2)$ and $\text{sgn}(\chi^-(Y, t_1)) = \text{sgn}(\chi^+(Y, t_2))$.

LEMMA A.3. — *Let S_N be (possibly atomic) random variables s.t. for some $A_N \in \mathbb{R}$ and $B_N \rightarrow \infty$, $\frac{S_N - A_N}{B_N} \xrightarrow[N \rightarrow \infty]{\text{dist}} Y$. Then $\frac{S_N - A_N^*}{B_N^*} \xrightarrow[N \rightarrow \infty]{\text{dist}} Y$, where A_N^*, B_N^* are the unique solution to*

$$(A.1) \quad \begin{cases} A_N^* + B_N^* \chi^-(Y, t_1) = \chi^-(S_N, t_1) \\ A_N^* + B_N^* \chi^+(Y, t_2) = \chi^+(S_N, t_2). \end{cases}$$

Proof. — Without loss of generality, $\chi^-(Y, t_1), \chi^+(Y, t_2)$ are both positive.

We need the following fact (which is not automatic since S_N are allowed to be atomic):

$$(A.2) \quad \lim_{N \rightarrow \infty} \Pr(S_N < \chi^-(S_N, t)) = t \text{ for all } 0 < t < 1.$$

Indeed, given $\varepsilon > 0$, let $\xi_N := B_N \chi^-(Y, t - \varepsilon) + A_N$, then

$$\Pr(S_N < \xi_N) = \Pr\left(\frac{S_N - A_N}{B_N} < \chi^-(Y, t - \varepsilon)\right) \xrightarrow[N \rightarrow \infty]{} \Pr(Y < \chi^-(Y, t - \varepsilon)) = t - \varepsilon,$$

by Lemma A.2(a). So for all N large enough, $\xi_N \leq \chi^-(S_N, t)$, whence $\liminf \Pr(S_N < \chi^-(S_N, t)) \geq \lim \Pr(S_N < \xi_N) = t - \varepsilon$. Since ε is arbitrary, $\liminf \Pr(S_N < \chi^-(S_N, t)) \geq t$. The other inequality $\limsup \Pr(S_N < \chi^-(S_N, t)) \leq t$ is clear since $\Pr(S_N < \chi^-(S_N, t)) \leq t$ for all N .

With (A.2) proved, we proceed to prove that

$$(A.3) \quad \frac{A_N^* - A_N}{B_N} \xrightarrow[N \rightarrow \infty]{} 0 \quad \text{and} \quad \frac{B_N^*}{B_N} \xrightarrow[N \rightarrow \infty]{} 1.$$

It will then be obvious that $\frac{S_N - A_N}{B_N} \xrightarrow[N \rightarrow \infty]{\text{dist}} Y$ implies $\frac{S_N - A_N^*}{B_N^*} \xrightarrow[N \rightarrow \infty]{\text{dist}} Y$.

Define two affine transformations, $\varphi_N(t) = \frac{t - A_N}{B_N}$ and $\varphi_N^*(t) = \frac{t - A_N^*}{B_N^*}$. Notice that $(\varphi_N^*)^{-1}(t) = A_N^* + B_N^* t$, so $(\varphi_N^*)^{-1}(\chi^-(Y, t_1)) = \chi^-(S_N, t_1)$, by (A.1). Since $B_N^* = \frac{\chi^+(S_N, t_2) - \chi^-(S_N, t_1)}{\chi^+(Y, t_2) - \chi^-(Y, t_1)} > 0$, φ_N^* is increasing. By (A.2), $\Pr(\varphi_N^*(S_N) < \chi^-(Y, t_1)) = \Pr(S_N < \chi^-(S_N, t_1)) \xrightarrow[N \rightarrow \infty]{} t_1$. So

$$\begin{aligned} t_1 &= \lim_{N \rightarrow \infty} \Pr(\varphi_N^*(S_N) < \chi^-(Y, t_1)) \\ &= \lim_{N \rightarrow \infty} \Pr\left(\varphi_N(S_N) < \varphi_N[(\varphi_N^*)^{-1}(\chi^-(Y, t_1))]\right) \\ &= \lim_{N \rightarrow \infty} \Pr\left(\varphi_N(S_N) < \frac{B_N^*}{B_N} \left(\chi^-(Y, t_1) + \frac{A_N^* - A_N}{B_N}\right)\right). \end{aligned}$$

We claim that this implies that

$$(A.4) \quad \liminf_{N \rightarrow \infty} \frac{B_N^*}{B_N} \left(\chi^-(Y, t_1) + \frac{A_N^* - A_N}{B_N} \right) \geq \chi^-(Y, t_1).$$

Otherwise, $\exists \varepsilon$ s.t. $\liminf_{N \rightarrow \infty} \frac{B_N^*}{B_N} \left(\chi^-(Y, t_1) + \frac{A_N^* - A_N}{B_N} \right) < \chi^-(Y, t_1) - \varepsilon$, so

$$\begin{aligned} t_1 &= \liminf_{N \rightarrow \infty} \Pr \left(\varphi_N(S_N) < \frac{B_N^*}{B_N} \left(\chi^-(Y, t_1) + \frac{A_N^* - A_N}{B_N} \right) \right) \\ &\leq \liminf_{N \rightarrow \infty} \Pr \left(\varphi_N(S_N) < \chi^-(Y, t_1) - \varepsilon \right) = \Pr(Y < \chi^-(Y, t_1) - \varepsilon) \\ &= \Pr(Y < \chi^-(Y, t_1)) - \Pr(\chi^-(Y, t_1) - \varepsilon \leq Y < t_1) \\ &< t_1, \text{ by Lemma A.2(a) and (b).} \end{aligned}$$

Similarly, one shows that

$$(A.5) \quad \limsup_{N \rightarrow \infty} \frac{B_N^*}{B_N} \left(\chi^+(Y, t_2) + \frac{A_N^* - A_N}{B_N} \right) \leq \chi^+(Y, t_2).$$

It remains to see that (A.4) and (A.5) imply (A.3). First we divide (A.4) by (A.5) to obtain

$$\limsup_{N \rightarrow \infty} \frac{\chi^-(Y, t_1) + \frac{A_N^* - A_N}{B_N}}{\chi^+(Y, t_2) + \frac{A_N^* - A_N}{B_N}} \geq \frac{\chi^-(Y, t_1)}{\chi^+(Y, t_2)}.$$

Since $x \mapsto \frac{a+x}{b+x}$ is strictly decreasing on $[0, \infty)$ when $a > b > 0$, this implies that

$$(A.6) \quad \limsup_{N \rightarrow \infty} \frac{A_N^* - A_N}{B_N} \leq 0.$$

Looking at (A.4), and recalling that $\chi^-(Y, t_1) > 0$, we deduce that

$$(A.7) \quad \liminf_{N \rightarrow \infty} \frac{B_N^*}{B_N} \geq 1.$$

Next we look at the difference of (A.4) and (A.5) and obtain

$$\limsup_{N \rightarrow \infty} \frac{B_N^*}{B_N} \left(\chi^+(Y, t_2) - \chi^-(Y, t_1) \right) \leq \chi^+(Y, t_2) - \chi^-(Y, t_1),$$

whence $\limsup_{N \rightarrow \infty} (B_N^*/B_N) \leq 1$. Together with (A.7), this proves that $B_N^*/B_N \xrightarrow{N \rightarrow \infty} 1$.

Substituting this in (A.4), gives $\liminf_{N \rightarrow \infty} \frac{A_N^* - A_N}{B_N} \geq 0$, which, in view of (A.6), implies that $\frac{A_N^* - A_N}{B_N} \xrightarrow{N \rightarrow \infty} 0$. This completes the proof of (A.3), and with it, the lemma. \square

Proof of Lemma A.1. — We begin with the measurability of $\Omega(\alpha)$.

Let $S_N(x)$ denote the random variable equal to $S_n(\alpha, x)$ with probability $\frac{1}{N}$ for each $1 \leq n \leq N$.

We will apply Lemma A.3 with $Y := \text{U}[0, 1]$, $S_N = S_N(x)$ and (say) $t_1 := \frac{1}{3}$, $t_2 := \frac{2}{3}$. It says that

$$\Omega(\alpha) = \left\{ x \in \mathbb{T} : \frac{S_N(x) - A_N^*(x)}{B_N^*(x)} \xrightarrow[N \rightarrow \infty]{\text{dist}} \text{U}[0, 1] \right\},$$

where $A_N^*(x)$ and $B_N^*(x)$ are the unique solutions to (A.1). Since the percentiles of $S_N(x)$ are measurable as functions of x , $A_N^*(x)$, $B_N^*(x)$ are measurable as functions of x .

We claim that $\Omega(\alpha) = \Omega_1(\alpha) \cap \Omega_2(\alpha)$ where

$$\Omega_1(\alpha) := \bigcap_{\ell=1}^{\infty} \bigcup_{M=1}^{\infty} \bigcap_{N=M+1}^{\infty} \left\{ x \in \mathbb{T} : \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{(2,\infty)} \left(\left| \frac{S_n(\alpha, x) - A_N^*(x)}{B_N^*(x)} \right| \right) < \frac{1}{\ell} \right\}$$

$$\Omega_2(\alpha) = \bigcap_{t \in \mathbb{Q} \setminus \{0\}} \left\{ x \in \mathbb{T} : \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{it \left(\frac{S_n(\alpha, x) - A_N^*(x)}{B_N^*(x)} \right)} = \mathbb{E} \left(e^{itY} \right) \right\}.$$

This will prove the lemma, since the measurability of $A_N^*(\cdot)$, $B_N^*(\cdot)$ implies the measurability of $\Omega_i(\alpha)$.

If $x \in \Omega(\alpha)$ then

$$x \in \Omega_1(\alpha), \quad \text{because } \Pr \left[\left| \frac{S_N(x) - A_N^*}{B_N^*} \right| > 2 \right] \xrightarrow{N \rightarrow \infty} 0,$$

$$\text{and } x \in \Omega_2(\alpha), \quad \text{because } \mathbb{E} \left(e^{it \left(\frac{S_N(x) - A_N^*}{B_N^*} \right)} \right) \xrightarrow{N \rightarrow \infty} \mathbb{E} \left(e^{itY} \right) \text{ pointwise.}$$

Conversely, if $x \in \Omega_1(\alpha) \cap \Omega_2(\alpha)$ then it is not difficult to see that

$$\mathbb{E} \left(e^{it \left(\frac{S_N(x) - A_N^*}{B_N^*} \right)} \right) \xrightarrow{N \rightarrow \infty} \mathbb{E} \left(e^{itY} \right)$$

for all $t \in \mathbb{R}$. So $x \in \Omega(\alpha)$ by Lévy's continuity theorem. Thus $\Omega(\alpha) = \Omega_1(\alpha) \cap \Omega_2(\alpha)$, whence $\Omega(\alpha)$ is measurable.

The proof that $\Omega^*(\alpha)$ is measurable is similar. Enumerate $\mathbb{Q} \setminus \{0\} = \{t_n : n \in \mathbb{N}\}$, then $\alpha \in \Omega^*(\alpha)$ iff for every $\ell \in \mathbb{N}$ there exist $M \in \mathbb{N}$ s.t. for some $N > M$

$$\frac{1}{N} \sum_{n=1}^N \mathbf{1}_{(2,\infty)} \left(\left| \frac{S_n(\alpha, x) - A_N^*(x)}{B_N^*(x)} \right| \right) < \frac{1}{\ell},$$

$$\left| \mathbb{E} \left(e^{it_n \left(\frac{S_N(x) - A_N^*(x)}{B_N^*(x)} \right)} \right) - \mathbb{E} \left(e^{it_n Y} \right) \right| < \frac{1}{\ell} \text{ for all } n = 1, \dots, \ell.$$

These are measurable conditions, because $A_N^*(\cdot)$, $B_N^*(\cdot)$ are measurable. So $\Omega^*(\alpha)$ is measurable. \square

BIBLIOGRAPHY

- [ADDS15] Artur Avila, Dmitry Dolgopyat, Eduard Duryev, and Omri Sarig, *The visits to zero of a random walk driven by an irrational rotation*, *Isr. J. Math.* **207** (2015), no. 2, 653–717. MR 3359714 \uparrow 128
- [ADU93] Jon Aaronson, Manfred Denker, and Mariusz Urbański, *Ergodic theory for Markov fibred systems and parabolic rational maps*, *Trans. Am. Math. Soc.* **337** (1993), no. 2, 495–548. MR 1107025 \uparrow 136
- [AK82] Jon Aaronson and Michael Keane, *The visits to zero of some deterministic random walks*, *Proc. Lond. Math. Soc.* **44** (1982), no. 3, 535–553. MR 656248 \uparrow 128

- [Bec94] József Beck, *Probabilistic Diophantine approximation. I. Kronecker sequences*, Ann. Math. **140** (1994), no. 1, 109–160. MR 1289493 ↑140
- [Bec10] ———, *Randomness of the square root of 2 and the giant leap. I*, Period. Math. Hung. **60** (2010), no. 2, 137–242. MR 2647469 ↑128
- [Bec11] ———, *Randomness of the square root of 2 and the giant leap. II*, Period. Math. Hung. **62** (2011), no. 2, 127–246. MR 2805217 ↑128
- [BU18] Michael Bromberg and Corinna Ulcigrai, *A temporal Central Limit Theorem for real-valued cocycles over rotations*, Ann. Inst. Henri Poincaré, Probab. Stat. **54** (2018), no. 4, 2304–2334. ↑128
- [CK76] Jean-Pierre Conze and Michael Keane, *Ergodicité d'un flot cylindrique*, Séminaire de Probabilités, I (Univ. Rennes, Rennes, 1976), Université Rennes, 1976, Exp. no. 5, 7 pages. MR 0584020 ↑128
- [DS17a] Dmitry Dolgopyat and Omri Sarig, *Quenched and annealed temporal limit theorems for circle rotations*, 28 pages, preprint, 2017. ↑129, 130
- [DS17b] ———, *Temporal distributional limit theorems for dynamical systems*, J. Stat. Phys. **166** (2017), no. 3-4, 680–713. MR 3607586 ↑128, 130
- [DS18] ———, *Asymptotic windings of horocycles*, Isr. J. Math. **228** (2018), no. 1, 119–176. ↑128
- [DV86] Harold G. Diamond and Jeffrey D. Vaaler, *Estimates for partial sums of continued fraction partial quotients*, Pac. J. Math. **122** (1986), no. 1, 73–82. MR 825224 ↑131
- [HW08] Godfrey Harold Hardy and Edward Maitland Wright, *An introduction to the theory of numbers*, sixth ed., Oxford University Press, 2008, Revised by D. R. Heath-Brown and J. H. Silverman, With a foreword by Andrew Wiles. MR 2445243 ↑133
- [Jar29] Vojtech Jarník, *Zur metrischen Theorie der diophantischen Approximationen*, Prace Mat.-Fiz. **36** (1929), no. 1, 91–106. ↑128
- [Khi24] Alexander Ya. Khintchine, *Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen*, Math. Ann. **92** (1924), no. 1-2, 115–125. MR 1512207 ↑128
- [Khi63] ———, *Continued fractions*, P. Noordhoff, Ltd., 1963, translated by Peter Wynn. MR 0161834 ↑128, 132, 136
- [PS17] Elliot Paquette and Younghwan Son, *Birkhoff sum fluctuations in substitution dynamical systems*, Ergodic Theory Dyn. Syst. (2017). ↑128
- [Sch78] Klaus Schmidt, *A cylinder flow arising from irregularity of distribution*, Compos. Math. **36** (1978), no. 3, 225–232. MR 515047 ↑128
- [Sul82] Dennis Sullivan, *Disjoint spheres, approximation by imaginary quadratic numbers, and the logarithm law for geodesics*, Acta Math. **149** (1982), no. 3-4, 215–237. MR 688349 ↑132

Manuscript received on 14th March 2018,
revised on 24th October 2018,
accepted on 21st November 2018.

Recommended by Editor S. Gouëzel
Published under license CC BY 4.0



Dmitry DOLGOPYAT
Department of Mathematics
University of Maryland at College Park

College Park, MD 20742 (USA)

dmitry@math.umd.edu

Omri SARIG

Faculty of Mathematics and Computer Science

The Weizmann Institute of Science

234 Herzl Street

7610001 Rehovot (Israel)

omsarig@gmail.com