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RATIONAL POINTS OF RATIONALLY SIMPLY CONNECTED VARIETIES OVER GLOBAL FUNCTION FIELDS

POINTS RATIONNELS DE VARIÉTÉS RATIONNELLEMENT SIMPLEMENT CONNEXES DÉFINIES SUR DES CORPS GLOBAUX DE CARACTÉRISTIQUE NON NULLE

ABSTRACT. — For a complex projective manifold that is rationally connected, resp. rationally simply connected, every finite subset is connected by a rational curve, resp. the spaces parameterizing these connecting rational curves are themselves rationally connected. We prove that a projective scheme over a global function field has a rational point if it deforms to a rationally simply connected variety in characteristic 0 with vanishing *elementary obstruction*. This gives new, uniform proofs over these fields of the Period-Index Theorem, the quasi-split case of Serre's "Conjecture II", and Lang's C_2 property.

RÉSUMÉ. — Si une variété projective complexe est rationnellement connexe, chaque ensemble fini de points est contenu dans une courbe rationnelle ; si elle est rationnellement simplement connexe, les espaces paramétrant ces courbes rationnelles sont eux-mêmes rationnellement connexes. Nous montrons qu'un schéma projectif sur un corps global de caractéristique non nulle possède un point rationnel s'il se déforme en une variété rationnellement simplement connexe de caractéristique zéro dont l'*obstruction élémentaire* s'évanouit. Pour de tels corps, on obtient ainsi des preuves uniformes du théorème période-indice, du cas quasi-déployé de la « Conjecture II » de Serre, et de la propriété C_2 de Lang.

1. Statement of Results

For a field F and a projective F-scheme X_F, does X_F have an F-rational point? Colliot-Thélène and Sansuc defined an obstruction to existence of an F-rational point, the *elementary obstruction*, [CTS87, Section 2.2]. If $X_F \times_{\mathrm{Spec}\,F} \mathrm{Spec}\,\overline{F}$ is integral and smooth, the elementary obstruction vanishes if and only if X_F admits a *universal torsor*, cf. [CTS87, Proposition 2.2.3], [Sko01, Theorem 2.3.4]. For geometrically integral and smooth X_F, the universal torsor is a torsor over X_F for a multiplicative group F-scheme Q such that for every field extension E/F, for every multiplicative group F-scheme R, every R-torsor over $X_F \times_{\mathrm{Spec}\,F} \mathrm{Spec}\,E$ arises from the universal torsor by a unique morphism of group E-schemes, $Q \times_{\mathrm{Spec}\,F} \mathrm{Spec}\,E \to R$. For our geometric applications, it is important to extend these notions to a flat, projective scheme over a base *scheme* rather than a base *field*. This has been developed in [Pir12, Definition, p. 468] and in [Zhu19]. We review below the constructions from [Zhu19].

When is vanishing of the elementary obstruction a *sufficient* condition for existence of an F-rational point? Let $\mathbb{F}(\eta)$ be a *global function field*: the function field of a smooth projective curve over a finite field $\mathbb{F}$. For a projective $\mathbb{F}(\eta)$-variety $f_\eta : X_\eta \to \mathrm{Spec}\,\mathbb{F}(\eta)$, our main theorem gives an $\mathbb{F}(\eta)$-rational point of X_η provided that there exists a lift to characteristic 0 of a projective model that has vanishing elementary obstruction and that is a *rationally simply connected fibration*. A key role is played by Esnault's theory of rational points of specializations over finite fields of varieties of coniveau $\geqslant 1$, specifically [Esn07, Corollary 1.2].

1.1. Rationally simply connected fibration

Let us first fix some notation: Our main theorem will be for degenerations over $\mathbb{F}(\eta)$ that can be lift to rationally simply connected varieties over a field of characteristic 0, which is the function field of a curve C_K over a field $K = K(R)$ where R is a Henselian discrete valuation ring (DVR) (see [Gre66] for the definition and examples of Henselian DVR) with residue field $\mathbb{F}$.

The following is a precise definition of lifts to characteristic 0.

DEFINITION 1.1. — *A* projective extension over R of X_η *is a surjective morphism,*

$$f_R : X_R \to C_R,$$

between projective, flat, R-schemes satisfying the following.

(1) *The morphism $C_R \to \operatorname{Spec} R$ has connected geometric fibers.*
(2) *The R-smooth locus $C_{R,\mathrm{sm}}$ of C_R contains both the generic fiber C_K and one of the generic points η of the closed fiber $C_\mathbb{F}$ whose residue field equals $\mathbb{F}(\eta)$.*
(3) *The flat locus in C_R of the morphism f_R contains both the generic fiber C_K and η.*
(4) *The fiber of f_R over η equals f_η.*

Now, we explain what we mean by simply rationally connected fibrations. While a rationally connected variety X being *rationally simply connected* roughly means the space of rational curves on X is rationally connected, to give a precise definition is a lot more technical. For our purpose in this note, we will consider two cases that the "rational simple connectedness" has been confirmed.

HYPOTHESIS 1.2 ([dJHS11]). — *Let $\mathcal{L}_K$ be an f_K-ample line bundle on $\frac{X_K}{C_K}$. We assume the following.*

(1) *There exists an open subset $C_K^\circ \subset C_K$ and an open set U of $X^\circ := X \times_{C_K} C_K^\circ$ surjecting to C° such that the geometric fibres of the evaluation morphism from the Kontsevich moduli space with the vertical curve classes intersection $\mathcal{L}$ being 1*

$$\operatorname{ev} \colon \overline{\mathcal{M}}_{0,1}\left(\frac{X_K^\circ}{C_K^\circ}, 1\right) \to X_K^\circ$$

over U are nonempty, irreducible and rationally connected.
(2) *There exists a positive integer m_0 and an open subset V of $X_K^\circ \times_{C_K^\circ} X_K^\circ$ surjecting to C_K° and such that the geometric fibres of the evaluation morphism (See [dJHS11, P. 30])*

$$\operatorname{ev}_{1,m_0+1} \colon \operatorname{FreeChain}_2\left(\frac{X_K^\circ}{C_K^\circ}, m_0\right) \to X_K^\circ \times_{C_K^\circ} X_K^\circ$$

over V are nonempty, irreducible and birationally rationally connected.
(3) *There exists a geometric fibre of $X_K \to C_K$ which has a very twisting scroll (see [dJHS11, Definition 12.3]).*

The above Definition 1.1 can be applied to fibrations of homogeneous spaces with Picard number 1, see [dJHS11, Section 15]. In [Zhu19], the setting is extended to fibrations of homogeneous spaces with general Picard number in the following way.

HYPOTHESIS 1.3 ([Zhu19, 5.1]). — *Let $f_K \colon X_K \to C_K$ be a smooth family of projective homogeneous spaces. Assume that the relative Picard number, i.e., the rank of $\operatorname{Pic}_{X_K/C_K}(C_K)$ is one. Assume that the Picard number of the geometric generic fiber of f_K is r. Let S be the character C_K-group scheme of $\operatorname{Pic}_{X_K/C_K}$. Assume that the relative universal S-torsor T exists for the family.*

1.1.1. Main Theorem

The main theorem is as follows.

THEOREM 1.4. — *For every projective $\mathbb{F}(\eta)$-scheme X_η, for every projective extension over R of X_η, if the generic fiber of f_K has vanishing elementary obstruction, [CTS87, Section 2.2], and if f_K is either a rationally simply connected fibration as in Hypothesis 1.2 or a Homogeneous spaces fibration satisfying Hypothesis 1.3, then X_η has an $\mathbb{F}(\eta)$-point.*

Remark 1.5. — One main application of the theory of rationally simply connected varieties is that one can extend various results on the existence of rational points on rationally connected varieties over certain C_1 fields, to rationally simply connected varieties over C_2 fields obtained by adding one variable.

Theorem 1.4 can be considered to roughly say that for $\mathbb{F}(\eta)$, the degenerations of simply rationally connected varieties also has a rational point. The proof will use a similar result for degenerations of rationally connected varieties over $\mathbb{F}$(see [Esn07, EX09] or Section 2.2). In characteristic 0, there is indeed a purely geometric explanation on why in terms of the existence of rational points, the degeneration of rationally connected varieties admit the same property as rationally connected varieties, see [HX09].

The corollaries of Theorem 1.4 include several well-known existence theorems for rational points, as well as some new consequences.

COROLLARY 1.6. — [Lan52, Corollary, p. 378] *Every intersection $X_{\eta,1} \cap \ldots \cap X_{\eta,b}$ in $\mathbb{P}^n_{\mathbb{F}(\eta)}$ of hypersurfaces of degrees $(d_1, \ldots, d_b)$ satisfying $d_1^2 + \ldots + d_b^2 \leqslant n$ has an $\mathbb{F}(\eta)$-point.*

COROLLARY 1.7 (Period-Index Theorem). — *Every central simple algebra over $\mathbb{F}(\eta)$ has period equal to the index. This also follows from the Brauer–Hasse–Noether theorem and Hasse's Global Structure Theorem over $\mathbb{F}(\eta)$, cf. [Roq05, pp. 40–41].*

COROLLARY 1.8. — [Har75] *For every simply connected, semisimple group $\mathbb{F}(\eta)$-scheme G that is quasi-split, every G-torsor over $\mathbb{F}(\eta)$ has an $\mathbb{F}(\eta)$-point, i.e., the quasi-split case of Serre's "Conjecture II" holds for $\mathbb{F}(\eta)$.*

One new consequence follows from Theorem 1.4 and Robert Findley's work on rational simple connectedness of hypersurfaces in Grassmannians, [Fin10, Theorem 1.8]. Let $\ell > 0$ and $m \geqslant 2\ell$ be integers. Denote by $G_{\ell,m}$ the Grassmannian scheme over $\mathbb{F}(\eta)$ parameterizing ℓ-dimensional quotient vector spaces of a fixed vector space of dimension m (up to replacing ℓ by $m - \ell$, we can always assume that $m \geqslant 2\ell$).

COROLLARY 1.9. — *Let $\ell > 0$ and $m \geqslant 2\ell$ be integers. Let $\mathcal{G}$ be an $\mathbb{F}(\eta)$-scheme whose elementary obstruction vanishes and with $\mathcal{G} \times_{\mathrm{Spec}\,\mathbb{F}(\eta)} \mathrm{Spec}\,\overline{\mathbb{F}(\eta)}$ isomorphic to $G_{\ell,m} \times_{\mathrm{Spec}\,\mathbb{F}(\eta)} \mathrm{Spec}\,\overline{\mathbb{F}(\eta)}$. Every hypersurface in $\mathcal{G}$ of Plücker degree d satisfying $(3\ell - 1)d^2 - d < m - 4\ell - 1$ has an $\mathbb{F}(\eta)$-point.*

Unfortunately, the above bound is likely not the optimal one. For instance, it does not specialise to the bound in Corollary 1.6 when $G_{\ell,m}$ is a projective space.

2. The Proof of the Main Theorem

In this section we review Esnault's theory, [Esn07], we review the constructions of Abel sequences from [dJHS11] and [Zhu19], and we prove Theorem 1.4. In the next

section, we review the technique of *discriminant avoidance* from [SdJ10] and [dJHS11, Section 16], and we prove the corollaries. The proof of Theorem 1.4 has three components. The main new component is the use of Esnault's Theorem, [Esn07, Corollary 1.2].

2.1. Review of Esnault's Theorem.

As above, let R be a Henselian DVR with finite residue field $\mathbb{F}$ and with fraction field K. For the following discussion, we *do not* assume that K has characteristic 0, although in applications K does have characteristic 0. Let $\widetilde{Z}_R \to \operatorname{Spec} R$ be a projective, flat morphism such that the generic fiber $\widetilde{Z}_K$ is smooth and geometrically connected over K. The following Theorem 2.1 was first proved in mixed characteristic in [Esn07]. The equicharacteristic result was proved in [EX09].

THEOREM 2.1. — [Esn07, Theorem 1.1] [EX09, Theorem 1.1] *Assume that the ℓ-adic cohomology of the geometric generic fiber $H^i(X_K \times_{\operatorname{Spec} K} \operatorname{Spec} \overline{K})$ is supported in codimension $\geqslant 1$ for all $i \geqslant 1$. There exists a surjective, projective morphism $\mathcal{Z}_R \to \widetilde{Z}_R$ such that the set of $\mathbb{F}$-rational points of the closed fiber $\mathcal{Z}_{\mathbb{F}}$ is congruent to 1 modulo $q = |\mathbb{F}|$.*

Recall that the ℓ-adic cohomology $H^i(\widetilde{Z}_K \otimes_{\operatorname{Spec} K} \operatorname{Spec} \overline{K})$ is supported in codimension $\geqslant 1$ if for every α in the cohomology group, there exists a closed subset of codimension $\geqslant 1$, $D \subset \widetilde{Z}_K$, such that the restriction of α vanishes in the cohomology group $H^i((\widetilde{Z} \setminus D) \times_{\operatorname{Spec} K} \operatorname{Spec} \overline{K})$. Via the technique of Bloch–Srinivas, Esnault proved, [Esn03], that whenever $\widetilde{Z}_K \times_{\operatorname{Spec} K} \operatorname{Spec} \overline{K}$ is *rationally chain connected*, then for every $i \geqslant 1$, the cohomology is supported in codimension $\geqslant 1$, and thus Theorem 2.1 applies. A smooth, projective variety $\widetilde{Z}_K$ over K is *rationally chain connected* if for every algebraically closed field extension k/K, for every pair of k-points of $\widetilde{Z}_K$ there exists a connected, finite union of images of k-morphisms from $\mathbb{P}^1_k$ to the variety such that both k-points are contained in the union. If K has characteristic 0, then $\widetilde{Z}_K$ is rationally chain connected if and only if $\widetilde{Z}_K$ is *rationally connected*, i.e., for every pair of k-points, there exists a single k-morphism from $\mathbb{P}^1_k$ to the variety whose image contains both k-points.

The proof of Theorem 2.1 is a combination of several techniques. The authors encourage readers to refer to the original articles [Esn07, EX09], but we summarize the method. The proof is more direct when K has characteristic 0, as suffices for this article. First, by de Jong's alteration of singularities, [dJ97], there exists a birational, projective morphism $\mathcal{Z}_R \to \widetilde{Z}_R$ of flat R-schemes such that $\mathcal{Z}_R$ is the geometric quotient of a finite group Γ acting by R-automorphisms on a flat, projective R-scheme $\mathcal{W}_R$ that is regular. In particular, if K has characteristic 0 (the case of interest in what follows), the cohomology $H^i(\mathcal{Z}_K \times_{\operatorname{Spec} K} \operatorname{Spec} \overline{K})$ equals the Γ-invariants $H^i(\mathcal{W}_K \times_{\operatorname{Spec} K} \operatorname{Spec} \overline{K})^{\Gamma}$. Using Esnault's earlier work, [Esn06], and Gabber's purity theorem, cf. [Fuj02], it follows that for every $i \geqslant 1$, the eigenvalues of the geometric Frobenius acting on $H^i(\mathcal{Z}_K \times_{\operatorname{Spec} K} \operatorname{Spec} K^u)$ and on the relative cohomology group $H^i_{\mathcal{Z}_{\overline{\mathbb{F}}}}(\mathcal{Z}_K \times_{\operatorname{Spec} R} R^u)$ are algebraic integers divisible by q. Here R^u denotes the maximal

unramified extension of $\overline{k}$ (usually the ring of Witt vectors of $\overline{\mathbb{F}}$) and K^u is the fraction field of R^u. Finally, the relative cohomology exact sequence implies that for every $i \geqslant 1$, the eigenvalues of the geometric Frobenius acting on $H^i(\mathcal{Z}_{\overline{\mathbb{F}}})$ are algebraic integers divisible by q. By the Lefschetz trace formula, [Jou77, Corollary III.4.8, p. 109], this implies the congruence of the number of $\mathbb{F}$-points of $\mathcal{Z}_{\mathbb{F}}$. In particular, $\mathcal{Z}_{\mathbb{F}}$ has at least one $\mathbb{F}$-point. Since there is an $\mathbb{F}$-morphism from $\mathcal{Z}_{\mathbb{F}}$ to $\widetilde{Z}_{\mathbb{F}}$, it follows that $\widetilde{Z}_{\mathbb{F}}$ has at least one $\mathbb{F}$-point (without stronger hypotheses, however, we lose the more refined congruence for the number of $\mathbb{F}$-points modulo q).

2.2. Application of Esnault's Theorem to the Hilbert scheme.

We apply Theorem 2.1 to a scheme obtained as a blowing up of a closed subscheme of the relative Hilbert scheme of X_R over $\operatorname{Spec} R$. Since f_R is projective and since C_R is projective over $\operatorname{Spec} R$, the relative Hilbert scheme $\operatorname{Hilb}_{X_R/R}$ has connected components that are projective over $\operatorname{Spec} R$. There is an open subscheme $\operatorname{Sec}(X_R/C_R/R)$ of $\operatorname{Hilb}_{X_R/R}$ parameterizing closed subschemes of X_R arising as closed images of sections of f_R, cf. [Gro62, Part IV.4.c, p. 221–219]. Denote by $\Sigma(X_R/C_R/R)$ the closure in $\operatorname{Hilb}_{X_R/R}$ of this open subscheme. A *smooth pseudosection* of $\operatorname{Sec}(X_K/C_K/K)$ is a projective K-morphism $\nu_K : \widetilde{Z}_{K,I} \to \Sigma(X_K/C_K/K)$ pulling back $\operatorname{Sec}(X_K/C_K/K)$ to a dense open subscheme of $\widetilde{Z}_{K,I}$ and such that $\widetilde{Z}_{K,I}$ is a smooth, projective K-scheme whose base change to $\operatorname{Spec} \overline{K}$ is rationally connected, cf. [GHMS05, Definition 1.2]. Esnault's Theorem reduces Theorem 1.4 to existence of a smooth pseudosection.

PROPOSITION 2.2. — *For every projective $\mathbb{F}(\eta)$-scheme X_η, for every projective extension over R of X_η, for every smooth pseudosection of $\operatorname{Sec}(X_K/C_K/K)$, there exists an $\mathbb{F}(\eta)$-point of X_η, a finite flat extension of DVRs $R \to R'$ with fraction field extension $K \to K'$, and a K'-point of the pseudosection parameterizing a section $\sigma_{K'}$ whose specialization over $\eta \times_{\operatorname{Spec} R} \operatorname{Spec} R'$ equals the base change of the $\mathbb{F}(\eta)$-point of X_η.*

Proof. — By taking closures in an appropriate projective space, there exists a flat, projective R-scheme $\widetilde{Z}_{R,I}$ and an R-morphism $\nu_R : \widetilde{Z}_{R,I} \to \operatorname{Hilb}_{X_R/R}$ whose K-fiber is ν_K. By Theorem 2.1 for the characteristic 0 field K, there exists an $\mathbb{F}$-rational point of $\widetilde{Z}_{\mathbb{F},I}$ mapping to an $\mathbb{F}$-rational point of $\Sigma(X_R/C_R/R)$ corresponding to a closed subscheme $\mathcal{C}_{\mathbb{F}}$ of $X_{\mathbb{F}}$.

Since $\Sigma(X_R/C_R/R)$ is the closure of $\operatorname{Sec}(X_R/C_R/R)$, there exists a finite flat extension of DVRs, $R \to R'$, and a lifting of the $\mathbb{F}$-point to an R'-point of $\Sigma(X_R/C_R/R)$ whose generic point, $\operatorname{Spec} K'$, maps to $\operatorname{Sec}(X_R/C_R/R)$. This K'-point parameterizes the closed image of a section $\sigma_{K'}$ of $f_{K'} : X_{K'} \to C_{K'}$.

Since $X_{R'} \to C_{R'}$ is proper, by the valuative criterion of properness, the rational section $\sigma_{K'}$ extends to every codimension one point of the normal locus of $C_{R'}$. By Item (2) of Definition 1.1, every point of $C_{R'}$ lying over η is a codimension one point at which $C_{R'} \to \operatorname{Spec} R'$ is smooth, and hence at which $C_{R'}$ is normal. Thus, the closure of the image of $\sigma_{K'}$ in $X_{K'}$ restricts over η to the closed image of a rational section. Since this closed subset is the base change of $\mathcal{C}_{\mathbb{F}}$, also the restriction

of $\mathcal{C}_{\mathbb{F}}$ over η is the closed image of a rational section. This rational section is an $\mathbb{F}(\eta)$-rational point of X_η. $\qquad\square$

2.3. The associated Abelian variety, Abel torsors, and Abel maps.

In order to apply Proposition 2.2, we need closed subschemes that are potentially equal to pseudosections. Every morphism from a pseudosection to a torsor for an Abelian K-variety is constant. Thus, every pseudosection must be contained in a fiber for every morphism from the Hilbert scheme to a torsor for an Abelian K-variety. The open subscheme $\mathrm{Sec}(X_R/C_R/R)$ is contained in the maximal open subscheme $\mathrm{Hilb}_{X_R/R,\mathrm{CM}}$ of the Hilbert scheme parameterizing Cohen–Macaulay closed subschemes of pure relative dimension 1. Thus, every pseudosection also has dense intersection with this open subscheme. So every pseudosection is contained in a fiber of every K-morphism from $\mathrm{Hilb}^0_{X_R/R}$ to a torsor for an Abelian K-variety. What is the "universal" such morphism? There is a natural answer to this question under the following hypothesis.

DEFINITION 2.3. — [Zhu19, Definition 2.10] *For a Noetherian scheme T, a T-scheme $f : X_T \to T$ that is projective locally over T is* Picard-free *if*

(i) *f is flat with integral geometric fibers X_t,*
(ii) *every X_t is locally complete intersection and is smooth in codimension $\leqslant 3$,*
(iii) *$h^1(X_t, \mathcal{O}_{X_t})$ and $h^2(X_t, \mathcal{O}_{X_t})$ equal 0, and*
(iv) *every finite, Abelian, étale cover of X_t is trivial.*

The Condition (ii) above implies each finite type closed subscheme of $\mathrm{Pic}_{X_T/T}$ is proper over T (see [Gro05, XI, 3.13]).

Remark 2.4. — Assume that f_K is Picard-free, e.g., a rationally simply connected fibration also satisfying Condition (ii).

(i) The relative Picard scheme $P = \mathrm{Pic}_{f_K}$ over C_K is étale locally a constant group scheme $\mathbb{Z}^\rho$ in the sense of [DG70, Definition X.5.1], cf. [Zhu19, Proposition 2.9].
(ii) The Cartier dual group scheme $Q = D(P) = \mathrm{Hom}_{C_K-\mathrm{gp}}(P, \mathbb{G}_{m,C_K})$, is étale locally isomorphic to $\mathbb{G}^\rho_{m,C_K}$, cf. [DG70, Proposition X.5.3].
(iii) The K-stack $BQ_{C_K/K}$ of Q-torsors over C_K is a locally finitely presented, algebraic K-stack, [Lie06, Proposition 2.3.4], equal to a gerbe over the coarse moduli space $|BQ_{C_K/K}|$, which is a smooth group K-scheme whose identity component $|BQ^0_{C_K/K}|$ is an Abelian K-variety.
(iv) The component group $\Lambda(Q/C_K/K)$ of $|BQ_{C_K/K}|$ is an étale group K-scheme whose associated Galois module $\Lambda(Q/C_K/K)(\mathrm{Spec}\,\overline{K})$ is the kernel of the map of curve classes, $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}(X_{\overline{K}}), \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}(C_{\overline{K}}), \mathbb{Z})$.
(v) For every f_K-ample divisor class H on the generic fiber Y of f_K, for the associated complete intersection curve class $\theta = (H \cdots H)_Y$, the extension of smooth group K-schemes,

$$0 \to |BQ^0_{\frac{C_K}{K}}| \to |BQ_{\frac{C_K}{K}}| \to \Lambda(Q/C_K/K) \to 0,$$

has split restriction on the infinite subgroup $\mathbb{Z} \cdot \theta \subset \Lambda(Q/C_K/K)(K)$.

(vi) For every universal f_K-torsor $\mathcal{T}$ on X_K, there is an associated K-morphism

$$\alpha_{\mathcal{T}} : \mathrm{Hilb}_{\frac{X_K}{K},\,\mathrm{CM}} \to |BQ_{C_K/K}|,$$

whose restriction to $\mathrm{Sec}(X_K/C_K/K)$ sends each section σ to $\sigma^*\mathcal{T}$ (see [Zhu19, Proposition 3.3]).

Even without the universal torsor $\mathcal{T}$, there is still a natural morphism α defined on $\mathrm{Hilb}_{X_K/K,\mathrm{CM}}$ mapping every connected component to a torsor (over some finite extension of $\mathrm{Spec}\,K$) for $|BQ_{C_K/K}^0|$. This is the *Abel torsor*. Since every pseudosection in the connected component maps to a K-point of the Abel torsor, there exists a pseudosection only if the Abel torsor is a trivial torsor for $|BQ_{C_K/K}^0|$ over $\mathrm{Spec}\,K$. The *construction* of $\alpha_{\mathcal{T}}$ using a universal torsor shows that, for every e in the infinite subgroup $\mathbb{Z} \cdot \theta$, the Abel torsor $|BQ_{C_K/K}^e|$ is actually a trivial torsor, i.e., it has K-points I. Because K is the fraction field of a complete DVR, if there exists at least one K-point I, then every dense, Zariski open subset of $|BQ_{C_K/K}^e|$ contains a Zariski dense set of K-points.

2.4. Smoothness of the Abel maps.

Denote by g the genus of the smooth, projective, geometrically connected K-curve C_K. For every algebraically closed extension k/K, a k-morphism $\sigma : C_k \to X_k$ is (g)-*free* if the image is contained in the smooth locus of f_k and for every invertible sheaf $\mathcal{L}$ on C_k of degree $d \leqslant \max(2g, 0)$, the locally free sheaf $\mathrm{Hom}_{\mathcal{O}_{C_k}}(\sigma^*\Omega_{f_k}, \mathcal{L})$ has vanishing h^1. There is an open subscheme $\mathrm{Sec}(X_K/C_K/K)_{\mathrm{free}}$ of $\mathrm{Sec}(X_K/C_K/K)$ such that for every k/K, the k-points of $\mathrm{Sec}(X_K/C_K/K)_{\mathrm{free}}(k)$ are precisely the (g)-free sections. For every (g)-free section σ, for every closed k-curve $\widetilde{C}$ in X_k obtained by gluing to $\mathrm{Image}(\sigma_k)$ at δ distinct k-points a free, genus 0 curve in the smooth locus of the corresponding fiber of f_k, the point $[\widetilde{C}]$ of the Hilbert scheme is a smooth point of $\Sigma(X_R/C_R/R)(\mathrm{Spec}\,k)$, and all formal deformations of $[\widetilde{C}]$ to sections parameterized by $\mathrm{Sec}(X_R/C_R/R)(\mathrm{Spec}\,k(\!(r)\!))$ are also (g)-free sections. Finally, the Abel map is smooth at $[\widetilde{C}]$ under a weak hypothesis. Let $C_K' \to C_K$ be a quasi-finite, étale morphism such that $P \times_{C_K} C_K'$ is trivial, $\mathbb{Z}^\rho$. For each glued free rational curve, choose a lift of the attaching point to C_K' so that the curve class is well defined in the constant group scheme $\mathrm{Hom}_{\mathbb{Z}}(P \times_{C_K} C_K', \mathbb{Z})$ with fiber $N_1 \cong \mathbb{Z}^\rho$. The Abel map is smooth at $[\widetilde{C}]$ if there are at least $(2g - 2)\rho$ free curves that can be partitioned into $\geqslant 2g - 2$ subsets of ρ free rational curves whose classes in $N_1 \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^\rho$ span as a $\mathbb{Q}$-vector space.

2.5. Abel sequences.

There is a "positive structure" on $\Lambda(Q/C_K/K)$. This is the unique étale semigroup subscheme $\Lambda^+(Q/C_K/K)$ of $\Lambda(Q/C_K/K)$ whose associated subsemigroup $\Lambda^+(Q/C_K/K)(\mathrm{Spec}\,\overline{K})$ equals the semigroup of "effective" curve classes, i.e., those curve classes having non-negative pairing with all $f_{\overline{K}}$-ample invertible sheaves on

$X_{\overline{K}}$. If the Mori cone of the geometric generic fiber of f_K is finitely generated, then also $\Lambda^+(Q/C_K/K)(\operatorname{Spec} \overline{K})$ is finitely generated: it equals the invariant subsemigroup of the Mori cone for the (finite) action of the Galois group of $\overline{K}(C_{\overline{K}})$. For every $e_0 \in \Lambda^+(Q/C_K/K)(\operatorname{Spec} K)$, denote by $\Lambda^{\geqslant e_0}(Q/C_K/K)$ the translate of $\Lambda^+(Q/C_K/K)$ by e_0. Denote by $\Sigma^{\geqslant e_0}(Q/C_K/K)$ the inverse image in $\Sigma(Q/C_K/K)$ of $\Lambda^{\geqslant e_0}(Q/C_K/K)$.

DEFINITION 2.5. — *A sequence of* (g)-*free sections is an* K-*rational point* e_0 *of* $\Lambda(Q/C_K/K)$ *and a closed subscheme* $Z_{\geqslant e_0}$ *of* $\Sigma^{\geqslant e_0}(X_K/C_K/K)$

  (i) *whose open subscheme* $Z_{\geqslant e_0} \cap \operatorname{Sec}(X_K/C_K/K)_{\mathrm{free}}$ *is dense,*
  (ii) *whose fiber* Z_e *over every geometric point* e *of* $\Lambda^{\geqslant e_0}(Q/C_K/K)$ *is an irreducible component of* $\Sigma^e(X_K/C_K/K)$, *and*
  (iii) *for every geometric point* $[\sigma]$ *of* $\operatorname{Sec}(X_K/C_K/K)_{\mathrm{free}}$, *for all* $\delta \geqslant \delta_0$, *the scheme* $Z_{\geqslant e_0}$ *parameterizes every curve* $\widetilde{C}$ *obtained by gluing to* $\operatorname{Image}(\sigma)$ *at* δ *distinct points a free, genus 0 curve in the smooth locus of the fiber of* f_K.

The sequence is a pseudo Abel sequence, *resp. an* Abel sequence, *if for the morphism,*

$$\alpha_{\mathcal{T}_K, Z_e} : Z_{\geqslant e_0} \cap \operatorname{Sec}(X_K/C_K/K) \to |BQ_{C_K/K}^{\geqslant e_0}|,$$

every geometric generic fiber is nonempty and integral, resp. "birationally rationally connected" (every projective model is rationally connected). Define $Z_{R, \geqslant e_0}$ *to be the closure of* $Z_{\geqslant e_0}$ *in* $\Sigma(X_R/C_R/R)$.

PROPOSITION 2.6. — *For every projective* $\mathbb{F}(\eta)$-*scheme* X_η, *for every projective extension over* R *of* X_η *whose* K-*fiber* $f_K : X_K \to C_K$ *is a Picard-free morphism admitting a universal torsor* $\mathcal{T}$ *(i.e., the elementary obstruction vanishes), if there exists an Abel sequence* $Z_{\geqslant e_0}$, *then for every* K-*point* e *of* $\Lambda(Q/C_K/K)$ *and for every general* K-*point* I *of* $|BQ_{C_K/K}^e|$, *there exists an* $\mathbb{F}(\eta)$-*point of* X_η, *a finite flat extension of DVRs* $R \to R'$ *with fraction field extension* $K \to K'$, *and a* K'-*point* $[\sigma_{K'}] \in Z_e$ *whose image under* $\alpha_{\mathcal{T}}$ *equals the base change of* I *and whose specialization over* $\eta \times_{\operatorname{Spec} R} \operatorname{Spec} R'$ *equals the base change of the* $\mathbb{F}(\eta)$-*point of* X_η.

Proof. — Since K has characteristic 0, by [Hir64], there exists a projective, birational morphism $\nu : \widetilde{Z}_{\geqslant e_0} \to Z_{\geqslant e_0}$ such that $\widetilde{Z}_{\geqslant e_0}$ is K-smooth. Moreover, there exists such ν for which the Abel map $\alpha_{\mathcal{T}}$ extends to a projective morphism,

$$\widetilde{\alpha}_{\mathcal{T}} : \widetilde{Z}_{\geqslant e_0} \to |BQ_{C_K/K}^{\geqslant e_0}|.$$

By generic smoothness, there exists a dense open subscheme of $|BQ_{C_K/K}^{\geqslant e_0}|$ over which $\widetilde{\alpha}_{\mathcal{T}}$ is smooth. Over this open subscheme, since every geometric generic fiber is rationally connected, every fiber is rationally connected. For I in this open subscheme, denote by $\widetilde{Z}_{K,I}$ the fiber of $\widetilde{\alpha}_{\mathcal{T}}$ over I, and denote by $\nu_K : \widetilde{Z}_{K,I} \to \Sigma(X_K/C_K/K)$ the restriction of ν. This is a smooth pseudosection of $\operatorname{Sec}(X_K/C_K/K)$. Now apply Proposition 2.2. □

Proof of Theorem 1.4. — By hypothesis, f_K is either a rationally simply connected fibration of Picard rank 1 as in [dJHS11, Theorem 13.1], or a rationally simply connected fibration as in [Zhu19, 5.1]. Then by [dJHS11, Theorem 13.1], resp. by [Zhu19, Theorem 5.12], there exists an Abel sequence. By Remark 2.4 (v), for every f_K-ample

divisor class H on the generic fiber of f_K, for $\theta = (H \cdots H)$, for every integer $d \gg 0$, the class $e = d \cdot \theta$ gives a K-point of $\Lambda^{\geqslant e_0}(Q/C_K/K)$ such that $|BQ^e_{C_K/K}|$ has an K-point. For every general K-point I of $|BQ^e_{C_K/K}|$, by Proposition 2.6, there exists an $\mathbb{F}(\eta)$-point of X_η whose base change over $\eta \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, R'$ equals the specialization of a section $\sigma_{K'}$ whose Abel image equals the base change of I. $\qquad \square$

3. Projective Extensions

For each of the four corollaries, we explain how to find projective extensions whose generic fiber is a rationally simply connected fibration over a curve. Combined with Theorem 1.4, this produces rational points.

3.1. Projective Homogeneous Spaces. Corollaries 1.7 and 1.8.

DEFINITION 3.1. — *For a stack $\mathcal{M}$ over a scheme S, a generic splitting variety of $\mathcal{M}$ (ala Amitsur) is a pair (M, ζ_M) of a locally finitely presented S-scheme M and a 1-morphism over S,*

$$\zeta_M : M \to \mathcal{M},$$

such that for every field E and for every 1-morphism, $\zeta : \mathrm{Spec}\, E \to \mathcal{M}$, there exists a morphism $z : \mathrm{Spec}\, E \to M$ with $\zeta_M \circ z$ equivalent to ζ.

For every fppf group S-scheme G_S, the *classifying stack* BG_S is the stack over the fppf site of S-schemes whose fiber category over an S-scheme T has as objects the G_S-torsors over T and has as morphisms the G_S-equivariant T-morphisms of torsors [LMB00, Proposition 10.13.1]. A smooth, affine group scheme $G_{S,0}$ over S is a *reductive group scheme* if the geometric fibers are connected and (geometrically) reductive, i.e., the unipotent radical is trivial. A flat, affine group scheme G_S over S is *finite-by-reductive* if there exists a surjective, smooth morphism of group S-schemes

$$G_S \twoheadrightarrow \pi_0 G_S$$

where $\pi_0 G_S$ is a finite, flat group S-scheme and the kernel, $G_{S,0}$, is a reductive group S-scheme.

THEOREM 3.2. — [SdJ10] [dJHS11, Section 16] *Let R be a DVR, and let S be $\mathrm{Spec}\, R$ in the above setup. For every "finite-by-reductive" group R-scheme, G_R, for every integer $c \geqslant 1$, there exists a projective, fppf R-scheme $\overline{M}$, an open subscheme M whose closed complement $\partial\overline{M}$ has codimension $> c$ in every fiber $\overline{M}_s$, and a 1-morphism $\zeta_M : M \to BG_R$ that is a generic splitting variety.*

Proof. — There exists a linear representation of G_R on a finite free R-module V whose induced action on $\mathbb{P}_R V$ is free on a dense open subset $(\mathbb{P}_R V)^o$ whose complement in the semistable locus $(\mathbb{P}_R V)^{\mathrm{ss}}$ has codimension $> c$ in every fiber over $\mathrm{Spec}\, R$. By [Ses77], the uniform categorical quotient of the action of G_R on $(\mathbb{P}_R V)^{\mathrm{ss}}$ is a projective R-scheme $\overline{M}$, and there exists a unique open subscheme M such that $(\mathbb{P}_R V)^o$ is a G_R-torsor over M. This defines $\zeta_M : M \to BG_R$.

By construction, $\overline{M}$ is normal and integral. Since it is also projective over the DVR R, and since it dominates the generic point $\operatorname{Spec} K$, $\overline{M}$ is R-flat. Since the complement of $(\mathbb{P}_R V)^o$ has codimension $> c$, also $\partial \overline{M}$ has codimension $> c$. Since $(\mathbb{P}_R V)^\circ$ is R-smooth, and since $(\mathbb{P}_R V)^o \to M$ is flat, also M is R-smooth, [Gro65, Proposition 17.7.7].

For every $\zeta : \operatorname{Spec} E \to BG_R$, the 2-fibered product $M \times_{\zeta_M, \zeta} \operatorname{Spec} E$ with its projection to $\operatorname{Spec} E$ is $(\mathbb{P}_E V_\zeta)^o$ where V_ζ is the E-vector space $\operatorname{Hom}_E(E^{\oplus c+1}, \operatorname{Hom}_E(W_0, W))$ for finite free E-vector spaces W_0, W of equal (positive) rank. The free locus contains the locus parameterizing $(c+1)$-tuples $(\lambda_0, \ldots, \lambda_c)$ of E-linear maps $\lambda_i : W_0 \to W$ that are isomorphisms. This locus has E-rational points. Thus, there exists an R-morphism $\lambda : \operatorname{Spec} E \to M$ and a 2-equivalence of the composition $\zeta_M \circ \lambda$ with ζ. $\square$

Let H_S be a split, connected and simply connected, semisimple group scheme over S. Let $P_S \hookrightarrow H_S$ be a standard parabolic subgroup scheme (containing a specified Borel). This has a Levi decomposition. Denote by $\chi : P_S \to Q_S$ the multiplicative quotient, i.e., the quotient of the Levi factor $L(P_S) = P_S / \operatorname{rad}^u(P_S)$ by its commutator subgroup S-scheme $L^{(1)}(P_S)$. Consider the left regular action of P_S on H_S, $p \cdot h = ph$. Consider the diagonal left action of P_S on $Q_S \times_S H_S$, $p \cdot (q, h) = (q\chi(p)^{-1}, ph)$. The projection morphism $\operatorname{pr} 2 : Q_S \times_S H_S \to H_S$ is P_S-equivariant. Finally, the left regular Q_S-action,

$$Q_S \times_S (Q_S \times_S H_S) \to Q_S \times_S H_S, \ q' \cdot (q, h) = (q'q, h),$$

is P_S-equivariant. By fppf descent, there are affine morphisms

$$H_S \to P_S \backslash H_S, \ \ Q_S \times_S H_S \to P_S \backslash (Q_S \times_S H_S)$$

that are P_S-torsors. The P_S-equivariant morphisms above induce morphisms

$$\operatorname{pr} 2 : P_S \backslash (Q_S \times_S H_S) \to P_S \backslash H_S,$$

$$Q_S \times_S P_S \backslash (Q_S \times_S H_S) \to P_S \backslash (Q_S \times_S H_S).$$

Moreover, the scheme $P_S \backslash H_S$ is projective over S: the dual of the relative dualizing sheaf is relatively very ample, cf. [Dem77, p. 186]. The other schemes are affine over $P_S \backslash H_S$, hence can be constructed by fpqc descent for affine schemes. Altogether these quotients and morphisms make $\mathcal{T}_S := P_S \backslash (Q_S \times_S H_S)$ into a (left) Q_S-torsor over $P_S \backslash H_S$.

DEFINITION 3.3. — *For the pair (H_S, P_S) as above, the associated* projective homogeneous scheme *is $X_{H,P} = P_S \backslash H_S$.*

LEMMA 3.4. — [Zhu19, Construction 5.3], [BCTS08, Lemma 2.2 (vi)], [Gil10, Lemma 6.4] *The Picard torus of $X_{H,P}$ over S is Q_S and the Picard lattice is the character lattice Q_S^D. The Q_S-torsor $\mathcal{T}_S$ is a universal torsor on $P_S \backslash H_S$.*

Since $H_S \to X_{H,P}$ is a P_S-torus, since P_S is S-flat, and since H_S is S-smooth, $X_{H,P}$ is S-smooth, [Gro65, Proposition 17.7.7].

Notation 3.5. — Denote by ρ_X the unique morphism of group S-schemes,

$$\rho_X : Q_S \to \mathbb{G}_{m,S},$$

such that the associated $\mathbb{G}_{m,S}$-torsor $\rho_{X,*} \mathcal{T}_S$ is the $\mathbb{G}_{m,S}$-torsor of the very ample invertible sheaf that is the dual of the relative dualizing sheaf.

The automorphism group scheme of the pair $(X_{H,P}, \omega_{X/S}^\vee)$ is an affine group S-scheme, cf. [SdJ10, Section 2.1]. This is essentially just [Gro62, No. 221, Section 4.c], [Gro63, Corollaire 7.7.8], [LMB00, Théorème 4.6.2.1]. The same method proves that the automorphism group scheme G'_S of the pair $(X_{H,P}, \mathcal{T}_S)$ is an affine group S-scheme. Since $\mathcal{T}_S$ is intrinsic, the forgetful morphism

$$\mathrm{Aut}_S(X_{H,P}, \mathcal{T}_S) \to \mathrm{Aut}_S(X_{H,P})$$

is surjective. The domain group scheme, G'_S, is smooth, and the identity component of the target is a semisimple group S-scheme, [Dem77, Proposition 4]. The kernel of the forgetful morphism is Q_S with its induced action on $\mathcal{T}_S$. Thus, G'_S is a smooth, affine group S-scheme whose identity component $G_{S,0}$ is a reductive group scheme.

The quotient $G'_S/G_{S,0}$ is a quasi-finite, étale group scheme $\pi_0 G'_S$ over S. When S is $\mathrm{Spec}\, R$ for a Henselian local ring, there is a unique closed subgroup scheme $\pi_0 G'_{S,\mathrm{fin}}$ of $\pi_0 G'_S$ that is finite, étale over S and whose closed fiber $\pi_0 G'_{F,\mathrm{fin}}$ equals the closed fiber $\pi_0 G'_F$.

By construction G_S is finite-by-reductive. Denote the induced action of G'_S, as a *right* action

$$\gamma_X : X_{H,P} \times_S G'_S \to X_{H,P}, \quad (P_S \cdot h, g) \mapsto (P_S \cdot h) \bullet g.$$
$$\gamma_{\mathcal{T}} : \mathcal{T}_S \times_S G'_S \to \mathcal{T}_S.$$

Let $G_S \subset G'_S$ be a closed subgroup scheme that contains $G_{S,0}$ and that is finite-by-reductive, i.e., $G_{S,0}$ is the kernel of a smooth, surjective group homomorphism $G_S \to \pi_0 G_S$ with $\pi_0 G_S$ finite and flat. The action of G_S on $X_{H,P}$ induces an action on the Picard lattice Q_S^D. This action factors through an action of $\pi_0 G_S$ on Q_S^D.

DEFINITION 3.6. — *The triple (H_S, P_S, G_S) of a split, simply connected, semisimple group S-scheme H_S, a standard parabolic P_S, and a finite-by-reductive subgroup S-scheme of $\mathrm{Aut}_S(X_{H,P}, \mathcal{T}_S)$ containing the identity component is* primitive *if the induced action of $\pi_0 G_S$ on the Picard lattice Q_S^D of $P_S \backslash H_S$ has a rank 1 invariant sublattice.*

Since it is intrinsic, one nonzero element in the invariant lattice is the character ρ_X giving rise to the ample invertible sheaf $\omega_{X/S}^\vee$. Thus, the triple is primitive if and only if every invariant character is commensurate with ρ_X.

For every S-scheme T and for every left G_S-torsor E_T over T,

$$(\mu, \mathrm{pr}\,2) : G_S \times_S E_T \xrightarrow{\cong} E_T \times_T E_T, \quad (g, x) \mapsto (\mu(g, x), x),$$

there are induced left actions of G_S as follows,

$$\widetilde{\mu}_X : G_S \times_S (X_{H,P} \times_S E_T) \to X_{H,P} \times_S E_T, \quad (g, (x, y)) \mapsto (x \gamma_X(g^{-1}), \mu(g)y),$$
$$\widetilde{\mu}_{\mathcal{T}} : G_S \times_S \mathcal{T}_S \times_S E_T \to \mathcal{T}_S \times_S E_T, \quad (g, (q, y)) \mapsto (q \gamma_{\mathcal{T}}(g^{-1}), \mu(g)y).$$

These are free action. Associated to the G_S-invariant cocharacter ρ_X, the action $\widetilde{\mu}_{\mathcal{T}}$ induces a G_S-linearization over $X_{H,P} \times_S E_T$ of the relatively ample invertible sheaf $\mathrm{pr}\,1^* \omega_{X/S}^\vee$. Thus, by fppf descent in the projective case, there is an affine T-morphism that is a G_S-torsor,

$$q_{X,E} : X_{H,P} \times_S E_T \to X_E,$$

together with a G_S-torsor,

$$q_{\mathcal{T},E} : \mathcal{T}_S \times_S E_T \to \mathcal{T}_E,$$

and a quotient morphism

$$\mathcal{T}_E \to X_E$$

that is a Q_S-torsor.

DEFINITION 3.7. — *The pair $(X_E, \mathcal{T}_E)$ is the E-twist of $(X_{H,P}, \mathcal{T}_S)$.*

Notation 3.8. — For a triple (H_S, P_S, G_S), let $(\overline{M}, \mathcal{O}_{\overline{M}}(1), i : M \to \overline{M})$ be a datum with codimension > 1 boundary as in Theorem 3.2. For the 1-morphism $\zeta_M : M \to BG_S$, denote by $f_M : X_M \to M$ and $\mathcal{T}_M$ the associated twist of $(X_{H,P}, \mathcal{T}_S)$.

HYPOTHESIS 3.9. — *With notation as in Section 1, let X_η be an $\mathbb{F}(\eta)$-scheme that is projective homogeneous. Assume that X_η has a universal torsor $\mathcal{T}_\eta$.*

There exists a pair (H_S, P_S) as above such that $X_{H,P} \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, \mathbb{F}(\eta)$ is geometrically isomorphic to X_η. Let Q_η be the Picard torus of X_η.

Notation 3.10. — Denote by $G_S \subset \mathrm{Aut}_S(X_{H,P}, \mathcal{T}_{H,P})$ the unique closed subgroup scheme that is finite-by-reductive, that contains the identity component, and such that $\pi_0 G_\mathbb{F}$ is the group of connected components of $\mathrm{Aut}_\mathbb{F}(X_{H,P}, \mathcal{T}_{H,P})$.

The Isom scheme from $(X_\eta, \mathcal{T}_\eta)$ to $(X_{H,P}, \mathcal{T}_{H,P})$ is a left G_S-torsor over $\mathbb{F}(\eta)$. The S-smooth scheme M from Theorem 3.2, for $c = 1$, is a generic splitting variety. Thus, there is an induced morphism $\zeta : C_\mathbb{F} \to \overline{M}$. If necessary, replace $\overline{M}$ by $\overline{M} \times_S \mathbb{P}^3_S$ so that ζ is a closed immersion. Denote the relative dimension of $\overline{M}$ by m. Denote by $(X_M, \mathcal{T}_M)$ the twist of $(X_{H,P}, \mathcal{T}_{H,P})$ by the torsor $\zeta_M : M \to BG_S$.

THEOREM 3.11. — [Zhu19] *Assume Hypothesis 3.9. For every general $(m-1)$-tuple $(D_1, \ldots, D_{m-1})$ of very ample divisors of $\overline{M}$ that contain $\zeta(C_\mathbb{F})$, the common intersection $C_R = D_1 \cap \ldots \cap D_{m-1}$ gives a projective extension of $X_\eta \to \mathrm{Spec}\, \mathbb{F}(\eta)$. If (H_S, P_S, G_S) is primitive, then $X_K \to C_K$ and $\mathcal{T}_K$ satisfies the hypotheses of [Zhu19, 5.1], and the family has an Abel sequence.*

Proof. — Since $C_\mathbb{F}$ is smooth, and since the image of ζ intersects the smooth open subscheme M, there exist very ample divisors $(D_{1,\mathbb{F}}, \ldots, D_{m-1,\mathbb{F}})$ that contain $\zeta(C_\mathbb{F})$ and whose common intersection $C'_\mathbb{F}$ is a curve containing a dense open subset of $\zeta(C_\mathbb{F})$ as an open subset. Since the boundary $\partial\overline{M}$ has codimension > 1 in $\overline{M}$, for a general lift $(D_1, \ldots, D_{m-1})$ (inside the complete linear system $\mathbb{P}^N_R$ over the Henselian ring R), the curve C_K is a smooth curve that is contained in M. Thus, the restriction of $(X_M, \mathcal{T}_M)$ gives a projective extension of $(X_\eta, \mathcal{T}_\eta)$.

By the Bertini irreducibility theorem, for a general complete intersection curve C_K, the sections of the relative Picard of X_M/M over C_K equal the sections over all of M_K. If (H_S, P_S, G_S) is primitive, the group of sections is a free Abelian group of rank 1. Thus, the restricted family $X_K \to C_K$ with the restricted universal torsor $\mathcal{T}_\eta$ satisfies [Zhu19, 5.1]. By the proof of [Zhu19, Theorem 1.4], this family has an Abel sequence. □

Proofs of Corollaries 1.7 and 1.8. — For Corollary 1.7, let A be a division algebra with center $\mathbb{F}(\eta)$ that has finite dimension n^2 over the center and whose order in the Brauer group of $\mathbb{F}(\eta)$ equals ℓ. Then ℓ divides n; the goal is to prove that ℓ equals n. Let X_η be the generalized Severi–Brauer variety associated to A that represents the functor on $\mathbb{F}(\eta)$-scheme associating to each $\mathbb{F}(\eta)$-scheme T the set of local direct summands of $A \otimes_{\mathbb{F}(\eta)} \mathcal{O}_T$ of rank ℓn over $\mathcal{O}_T$ and that are left ideals. Since the order of the Brauer element equals ℓ, there exists a universal torsor $\mathcal{T}_\eta$ on X_η. Since $X_\eta \times_{\operatorname{Spec}\mathbb{F}(\eta)} \operatorname{Spec}\overline{\mathbb{F}(\eta)}$ is isomorphic to a Grassmannian, which has Picard group isomorphic to $\mathbb{Z}$, the scheme X_η is primitive. By Theorem 3.11, there exists a projective extension that admits an Abel sequence. Thus, by Theorem 1.4, X_η has an $\mathbb{F}(\eta)$-point, i.e., A has an ideal of rank ℓn. The only left ideals in a division algebra are the zero ideal and all of A. Thus, ℓn equals the dimension n^2 of A, i.e., ℓ equals n.

For Corollary 1.8, let E be a left G-torsor over $\operatorname{Spec}\mathbb{F}(\eta)$. Since G is quasi-split, there exists a Borel $B \subseteq G$. Let X_η and $\mathcal{T}_\eta$ be the E-twist of $(X_{G,B}, \mathcal{T}_{\mathbb{F}(\eta)})$. This might *not* be primitive. However, the inductive strategy of [Zhu19, Lemma 12.1] reduces existence of an $\mathbb{F}(\eta)$-point of X_η to existence of an $\mathbb{F}(\eta)$-point of a projective homogeneous space Y over $\mathbb{F}(\eta)$ whose elementary obstruction vanishes and that is primitive. This will imply that the generic fiber of the projective extension of Y is also primitive. In the primitive case, by Theorem 3.11, there exists a projective extension that admits an Abel sequence. Thus, by Theorem 1.4, there is an $\mathbb{F}(\eta)$-point of Y. $\square$

3.2. Complete Intersections. Corollary 1.6.

Notation 3.12. — Let S be a scheme. Let n and $1 \leqslant b \leqslant n$ be positive integers. Let $\underline{d} = (d_1, \ldots, d_b)$ be an ordered b-tuple of integers $d_i \geqslant 2$. For each $j = 1, \ldots, b$, denote by $V_j(d_j)$ the free $\mathcal{O}_S$-module $H^0(\mathbb{P}_S^n, \mathcal{O}_{\mathbb{P}_S^{r_j}}(d_j))$. Denote by $V(\underline{d})$ the direct sum $V_1(d_1) \oplus \ldots \oplus V_b(d_b)$ as a free $\mathcal{O}_S$-module. Denote by $\mathbb{P}_S V(\underline{d})$ the projective space over S on which there is a universal ordered b-tuple $(\phi_1, \ldots, \phi_b)$ of sections of the invertible sheaves $\mathcal{O}_{\mathbb{P}^n}(d_j)$. Precisely, for the product

$$P = \mathbb{P}_S V(\underline{d}) \times_S \mathbb{P}_S^n$$

with its projections

$$\operatorname{pr}_1 : P \to \mathbb{P}_T V(\underline{d}) \text{ and } \operatorname{pr}_2 : P \to \mathbb{P}_T^n,$$

the sequence $(\phi_1, \ldots, \phi_b)$ is a universal homomorphism of coherent sheaves

$$\operatorname{pr}_2^* \mathcal{O}_{\mathbb{P}_S^n}(-d_1) \oplus \ldots \oplus \operatorname{pr}_2^* \mathcal{O}_{\mathbb{P}_S^n}(-d_b) \to \operatorname{pr}_1^* \mathcal{O}_{\mathbb{P}_S V(\underline{d})}(1),$$

or equivalently, a universal homomorphism of coherent sheaves,

$$(\phi_1, \ldots, \phi_b) : \operatorname{pr}_1^* \mathcal{O}_{\mathbb{P}_S V(\underline{d})}(-1) \otimes \left(\operatorname{pr}_2^* \mathcal{O}_{\mathbb{P}_T^n}(-d_1) \oplus \ldots \oplus \operatorname{pr}_2^* \mathcal{O}_{\mathbb{P}_T^n}(-d_b) \right) \to \mathcal{O}_P.$$

For each $j = 1, \ldots, b$, denote by $X_j \subset P$ the effective Cartier divisor defined by ϕ_j. Denote by X_j' the intersection $X_1 \cap \ldots \cap X_j$ as a closed subscheme of P.

Definition 3.13. — *For every $j = 1, \ldots, b$, the* smooth locus $\mathbb{P}_S V(\underline{d})_j^{\mathrm{sm}}$, *resp. the* Lefschetz locus $\mathbb{P}_S V(\underline{d})_j^{\mathrm{Lef}}$, *is the maximal open subscheme of $\mathbb{P}_S V(\underline{d})$ over which X_j' is flat and every geometric fiber is smooth, resp. has at worst a single ordinary double point. The* degenerate locus, *resp. the* badly degenerate locus, *is the closed complement of $\mathbb{P}_S V(\underline{d})_j^{\mathrm{sm}}$, resp. $\mathbb{P}_S V(\underline{d})_j^{\mathrm{Lef}}$.*

Proposition 3.14. — [Del73, Exposé XVII, Théorème 2.5] *Assume that $b < n$. Then for every $j = 1, \ldots, b$, the degenerate locus is a proper closed subset, and the badly degenerate locus of X_j' is a proper closed subset of codimension > 1.*

Proof. — This is proved by induction on j. For $j = 1$, this follows from loc. cit. By way of induction, assume that $j > 1$ and assume that the result is proved for $j - 1$. By loc. cit., the intersection with $\mathbb{P}_S V(\underline{d})_{j-1}^{\mathrm{sm}}$ of the degenerate locus, resp. badly degenerate locus, of X_j' is a proper closed subset, resp. has codimension > 1. It suffices to prove that $\mathbb{P}_S V(\underline{d})_j^{\mathrm{sm}}$ contains every generic point ξ of the degenerate locus of X_{j-1}' that is not in the badly degenerate locus. By hypothesis, $X_{j-1,\xi}'$ has a unique ordinary double point. In $\mathbb{P}_S V(\underline{d})$, it is a codimension 1 condition for ϕ_j to vanish at this point. On the complement of this proper closed subset, the degenerate locus of X_j' is a proper closed subset by Bertini's smoothness theorem [Jou83, Theorème 6.3(2)]. $\square$

Hypothesis 3.15. — *Let S be $\operatorname{Spec} R$ for a Henselian DVR with finite residue field $\mathbb{F}$ and with characteristic 0 fraction field K. Let $\mathbb{F}(\eta)$ be $\mathbb{F}(C_{\mathbb{F}}')$ for a smooth, projective, geometrically connected $\mathbb{F}$-curve $C_{\mathbb{F}}'$. Notations are as in Notation 3.12. Assume that $b < n$. Let $X_\eta' \subset \mathbb{P}_{\mathbb{F}(\eta)}^n$ be an intersection $X_{\eta,1} \cap \ldots \cap X_{\eta,b}$ of hypersurfaces $X_{\eta,i} = \operatorname{Zero}(\phi_{\eta,i})$ of degree d_i. Denote by $\zeta : C_{\mathbb{F}}' \to \mathbb{P}_R V(\underline{d})$ the R-morphism of $(\phi_{\eta,1}, \ldots, \phi_{\eta,b})$. Denote by m the relative dimension of $\mathbb{P}_R V(\underline{d})$ over $\operatorname{Spec} R$.*

Theorem 3.16. — [DeL15, dJHS11] *Assume Hypothesis 3.15. For every general $(m-1)$-tuple $(D_1, \ldots, D_{m-1})$ of very ample divisors of $\mathbb{P}_R V(\underline{d})$ that contain $\zeta(C_{\mathbb{F}}')$, for the common intersection $C_R = D_1 \cap \ldots \cap D_{m-1}$, for the restriction X_R' of X_b' over C_R, for the closure X_R in X_R' of the generic fiber $X_{K,b} = X_R' \times_{\operatorname{Spec} R} \operatorname{Spec} K$, the morphism $X_R \to C_R$ is a projective extension over R of the fiber $X_\eta \to \operatorname{Spec} \mathbb{F}(\eta)$. Also, X_η is a closed subscheme of $X_{\eta,b}'$ that has pure dimension $n - b$. The restriction of the $\mathbb{G}_m$-torsor of $\mathcal{O}(1)$ on $\mathbb{P}_R^n$ is a universal torsor. If $d_1^2 + \ldots + d_b^2 \leqslant n$, then $X_{K,b} \to C_K$ is a rationally simply connected fibration (of Picard rank 1) in the sense of [dJHS11, Theorem 13.1], and the family has an Abel sequence.*

Proof. — Existence of the projective extension is basically the same as in the proof of Theorem 3.11. By Proposition 3.14, for a general choice of $(D_1, \ldots, D_{m-1})$, the curve C_K is a smooth, projective, geometrically connected curve contained in the Lefschetz locus of X_b and having dense intersection with the smooth locus.

Now assume that $d_1^2 + \ldots + d_b^2 \leqslant n$. There are three global hypotheses in [dJHS11, Theorem 13.1], and the remaining hypotheses are on the geometric generic fiber of $f_K : X_{K,b} \to C_K$. The first hypothesis is that $X_{K,b}$ is smooth. The projection $X_b' \to \mathbb{P}_R^n$ is a projective space bundle, hence X_b' is R-smooth. By Bertini's theorem, [Jou83, Théorème 6.3(2)], for $D_1, \ldots, D_{m-1}$ general, the inverse image of $C_K = D_{K,1} \cap \ldots \cap D_{K,m-1}$ under the projection $X_b' \to \mathbb{P}_R V(\underline{d})$ is smooth.

The second hypothesis is that every geometric fiber of f_K is irreducible. By hypothesis, every $d_i \geqslant 2$, so that $d_1^2 + \ldots + d_b^2 \geqslant 4b$. Thus, $n \geqslant 4b$, so that also $n - b \geqslant 3b \geqslant 3$.

Every fiber of f_K is a complete intersection of ample divisors of dimension $\geqslant 3$. Thus, the complete intersection is connected, cf. [Gro03, Corollary XII.3.5]. Moreover, the complete intersection has at most a single ordinary double point. Thus, the fiber is normal by Serre's Criterion, cf. [Gro03, XI.3.11], hence it is irreducible.

The third global hypothesis is that $\mathcal{O}(1)$ is f_K-ample. In fact it is f_K-very ample since X_K is a closed subscheme of $C_K \times_{\operatorname{Spec} K} \mathbb{P}^n_K$.

The remaining hypotheses are all hypotheses of the geometric generic fiber of f_K, which is a smooth complete intersection with $d_1^2 + \ldots + d_b^2 \leqslant n$ that is general. These hypotheses are all established by Matt DeLand, [DeL15]. Therefore, by [dJHS11, Theorem 13.1], there exists an Abel sequence.   □

Proof of Corollary 1.6. By Theorem 3.16, there exists a projective extension with vanishing elementary obstruction and with an Abel sequence. By Theorem 1.4, there exists an $\mathbb{F}(\eta)$-point.   □

3.3. Hypersurfaces in Grassmannians. Corollary 1.9.

Notation 3.17. — Let S be an affine scheme. Let $m > 0$ be an integer, and let $\ell > 0$ be an integer such that $m \geqslant 2\ell$ (or else replace ℓ by $m - \ell$). Let $d > 0$ be an integer. Let H_S be $\mathbf{SL}_{m,S}$. Let $P_S \subset H_S$ be the parabolic that preserves the projection $\mathcal{O}_S^{\oplus m} \to \mathcal{O}_S^{\oplus \ell}$ onto the first ℓ factors. Thus, $X_{H,P} = P_S \backslash H_S$ is the Grassmannian $\operatorname{Grass}_S(\ell, \mathcal{O}_S^{\oplus m})$. Also the universal torsor $\mathcal{T}_{H,P}$ is the $\mathbb{G}_m$-torsor associated to the Plücker invertible sheaf $\mathcal{O}(1)$. Let G_S be $\operatorname{Aut}_S(X_{H,P}, \mathcal{T}_{H,P})$. Let $W(d)$ be the free $\mathcal{O}_S$-module $H^0(X_{H,P}, \mathcal{O}(d))$. Denote by $\mathbb{P}_S W(d)^{\operatorname{Lef}}$ the Lefschetz locus parameterizing degree d hypersurfaces that have at worst a single ordinary double point, cf. Definition 3.13.

Let V_S be the linear representation of G_S from the proof of Theorem 3.2 with $c = 1$. Define $\overline{M}'$ to be the uniform categorical quotient of the action of G_S on the semistable locus of $\mathbb{P}_S(V_S) \times_S \mathbb{P}_S W(d)$. By the proofs of Theorem 3.2 and Proposition 3.14, there is an open subscheme $M' \subset \overline{M}'$ whose inverse image equals $\mathbb{P}_S(V_S)^o \times_S \mathbb{P}_S W(d)^{\operatorname{Lef}}$ and that has closed complement $\partial \overline{M}'$ of codimension > 1. Since $\mathbb{P}_S(V_s)^o \times_S \mathbb{P}_S W(d)^{\operatorname{Lef}} \to M'$ is flat, also M' is R-smooth, [Gro65, Proposition 17.7.7]. Denote by $G_{M'} \to M'$ and $\mathcal{T}_{M'}$ the twist of $X_{H,P}$ and $\mathcal{T}_{H,P}$. Denote by $X_{M'} \subset G_{M'}$ the universal degree d hypersurface.

HYPOTHESIS 3.18. — *Let S be $\operatorname{Spec} R$ for a Henselian DVR with finite residue field $\mathbb{F}$ and with characteristic 0 fraction field K. Let $\mathbb{F}(\eta)$ be $\mathbb{F}(C'_{\mathbb{F}})$ for a smooth, projective, geometrically connected $\mathbb{F}$-curve $C'_{\mathbb{F}}$. Let G_η be a smooth, projective $\mathbb{F}(\eta)$-scheme that is geometrically isomorphic to $X_{H,P}$. Assume that there exists a universal torsor $\mathcal{T}_\eta$. Let $X_\eta \subset G_\eta$ be a closed subscheme that is geometrically isomorphic to a degree d hypersurface in $X_{H,P}$. Denote by $\zeta : C'_{\mathbb{F}} \to \overline{M}'_{\mathbb{F}}$ the associated R-morphism. Denote by s the relative dimension of $\overline{M}'$ over $\operatorname{Spec} R$.*

THEOREM 3.19 ([dJHS11, Fin10]). — *Assume Hypothesis 3.18. For every general $(s - 1)$-tuple $(D_1, \ldots, D_{s-1})$ of very ample divisors of $\overline{M}'_R$ that contain $\zeta(C'_{\mathbb{F}})$, the*

common intersection $C_R = D_1 \cap \ldots \cap D_{m-1}$ and the restriction $X_R \to C_R$ of $X_{M'}$ gives a projective extension of $X_\eta \to \operatorname{Spec} \mathbb{F}(\eta)$. The restriction of $\mathcal{T}_{M'}$ is a universal torsor. If $(3\ell - 1)d^2 - d < m - 4\ell - 1$, then $X_{K,b} \to C_K$ is a rationally simply connected fibration (of Picard rank 1) in the sense of [dJHS11, Theorem 13.1]*, and the family has an Abel sequence.*

Proof. — The first part of the proof is precisely the same as in the proof of Theorem 3.16. The hypotheses on the geometric generic fiber of f_K follow from [Fin10, Theorem 1.8]. $\square$

Proof of Corollary 1.9. By Theorem 3.19, there exists a projective extension with vanishing elementary obstruction that has an Abel sequence. By Theorem 1.4, there exists an $\mathbb{F}(\eta)$-point. $\square$

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