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PROPER ACTIONS ON ℓ^p -SPACES FOR RELATIVELY HYPERBOLIC GROUPS

DES ACTIONS PROPRES DE GROUPES
RELATIVEMENT HYPERBOLIQUES SUR DES
ESPACES ℓ^p

ABSTRACT. — We show that for any group G that is hyperbolic relative to subgroups that admit a proper affine isometric action on a uniformly convex Banach space, then G acts properly on a uniformly convex Banach space as well.

RÉSUMÉ. — Nous démontrons que si G est un groupe relativement hyperbolique dont les groupes périphériques peuvent être munis d'actions affines propres sur des espaces de Banach uniformément convexes, alors G lui aussi, peut être muni d'une action propre sur un (autre) espace de Banach uniformément convexe.

Introduction

Kazhdan's property (T) was introduced in 1967 by D. Kazhdan in [Kaz67] and since then has been intensively studied. A reformulation of Kazhdan property (T), following from work of Delorme–Guichardet, is that any action on a Hilbert space has a fixed point. Groups with property (T) include higher rank lattices in simple

Lie groups, such as $SL(n, \mathbf{Z})$ for $n \geq 3$, but also some lattices in rank one Lie groups, such as lattices in $Sp(n, 1)$ for $n \geq 3$. The uniform lattices in $Sp(n, 1)$ being hyperbolic, it shows that some hyperbolic groups can have property (T), as opposed to some others, like free groups or surface groups, that admit a proper action on a Hilbert space. The latter is called Haagerup property or amenability. The first natural generalization of a Hilbert space are L^p -space and it is implicit in P. Pansu's results in [Pan90], that lattices in $Sp(n, 1)$ admit an action without fix points on an L^p -space. G. Yu in [Yu05] shows that in fact any hyperbolic group admits a proper action on an ℓ^p -space for p large enough. M. Bourdon in [Bou16], B. Nica in [Nic13] and more recently A. Alvarez and V. Lafforgue in [AL17] gave an alternative proof of Yu's theorem. On the contrary, higher rank lattices are known to not act on any L^p or uniformly convex Banach space ([BFGM07] and [Laf08]).

Relatively hyperbolic groups are a geometric generalization of hyperbolic groups and have been studied a lot. They allow interesting group constructions while retaining a lot of the geometry of hyperbolic spaces. Relatively hyperbolic groups can also have property (T), but their peripheral subgroups can forbid any action on a uniformly convex Banach space. We show that this is in fact the only obstruction. Working with Alvarez–Lafforgue's construction, we prove that any relatively hyperbolic group has a non-trivial affine isometric action on an ℓ^p -space, for some p large enough. If furthermore its peripheral subgroups act properly by affine isometries on some uniformly convex Banach space, then the whole group also acts properly on a uniformly convex Banach space. It is the case for instance for Haagerup peripheral subgroups, in particular for all amenable ones. Our main result is the following.

THEOREM A. — *Let G be a relatively hyperbolic group, non-elementary. Then, for p sufficiently large, G admits an isometric action on an ℓ^p -space with an unbounded orbit.*

Moreover, if each of the parabolic subgroups acts properly on a uniformly convex Banach space (respectively, on an ℓ^p -space), then so does G (respectively, G acts on an ℓ^q -space for q large enough).

Even if they do not appear explicitly in print, some parts, and particular cases of this statement were known to experts. For instance, putting together results of Puls on L^p -cohomology [Pul07], and of Gerasimov [Ger12], on the Floyd boundary of relatively hyperbolic groups, one can show that non-elementary relatively hyperbolic groups act on some L^p -space without a fixed point. E. Guentner, E. Reckwerdt and R. Tessera obtained a version of Theorem A for groups that are relatively hyperbolic with virtually nilpotent parabolic subgroups [GRT] and recently, E. Guentner and R. Tessera announced in talks the case of amenable parabolic subgroups.

Let us comment on the general strategy, from Alvarez–Lafforgue in [AL17], Yu [Yu05] and Mineyev [Min01]. Given a representation π of a group G in the isometry group of a normed vector space V , one can obtain an affine isometric action by constructing a cocycle $c : G \rightarrow V$. The action is given by $g \cdot v = \pi(g)(v) + c(g)$.

For instance, consider $W = \ell^1(G, \mathbb{R})$, and $V_\infty = \ell^\infty(G, W)$. Then consider the representation $\pi : G \rightarrow \text{Isom}(V_\infty)$ by precomposition of the maps in ℓ^∞ . If G is discrete, for all p , $V_p = \ell^p(G, W)$ is a subspace of V_∞ , preserved by π , and on which $\pi(G)$ consists of isometries. For an affine action on V_p , one thus needs a cocycle

$c : G \rightarrow V_\infty$ so that, for all G , $c(g)$ is ℓ^p -summable (in other words, for all g , the sum $\sum_{h \in G} \|(c(g))(h)\|_W^p$ must be finite).

A traditional example is when G is free. In that case, let $d \in V_\infty$ be such that $d(h)$ is the indicator of the unique neighbor of h closer to 1 in the word metric (and $d(1) = 0$). One may see this as the arrival point at h of the geodesic flow from 1. Then $c(g) = d - \pi(g)d$ is also in V_∞ , and is supported by the interval $[1, g]$ in the Cayley tree of G . Since it has finite support, it is in $V_2 = \ell^2(G, W)$. The map c is from G to V_2 , and it is a cocycle for the representation π , because it is the coboundary of d in V_∞ (which makes the formal cocycle relation hold). One thus obtains an affine action of the free group on $\ell^2(G, W)$. The knowledgeable reader may have noticed that this is a variation on the more usual action on $\ell^2(E, \mathbb{R})$ where E is the set of oriented edges of a Cayley tree of the free group. In this more classical (arguably simpler) action, the function d is the indicator of the set of all oriented edges pointing toward 1 (those whose end point is closer to 1 than their starting point).

Defining an analogous cocycle c for other groups is more difficult, sometimes impossible, and is the reason of our seemingly complicated presentation of the free group case. One may opt for trying to determine how much should $h \in G$ be thought in the interval $[1, g]$. For this, one may use a geodesic flow, and test whether the flow from 1 and the flow from g toward h tend to arrive to h similarly or differently. Writing $\mu_1(h), \mu_g(h)$ the final values of such flows (in a suitable normed vector space W), the measurement of this difference defines $c(g) = (\mu_1 - \mu_g) : G \rightarrow W$, a map from G in W . It provides $c : G \rightarrow \ell^p(G, W)$ if the confluence is fast enough (exponential) for almost all $h \in G$, and if p is chosen sufficiently large to compensate the exponential growth of G .

Our argument is then through the construction of a flow as done by Alvarez–Lafforgue in [AL17]. This flow takes an initial point, and a target point (in the coned-off Cayley graph, or more generally in any fine δ -hyperbolic graph), and flows the initial point toward the target point, using a family of “reasonable” paths: those were “ α -geodesics” in the work of Alvarez–Lafforgue [AL17], but to handle the lack of local finiteness we need *conical α -geodesics*. During the flow, the position of the flowed initial point becomes a probability measure, called the *mask of the target point for the initial point* (that is what the target point needs to mask the initial point from its sight and is reminiscent of the notion of wall). The whole argument is to control stability of the flow (along the way, it won’t escape an a-priori controlled zone) and confluence of the flow (two sources close to each other, with same target, will flow to the same positions with positive probability at each step). In a hyperbolic group, it is done by Alvarez–Lafforgue in [AL17], and relies on the uniform local finiteness. Here, we work in highly non-locally finite setting. Using the tools developed to handle this non-local finiteness in [Dah03], we manage to construct a flow. The key here is that, sometimes, due to the angular geometry of relative hyperbolicity, the flow steps are so much confluent that they become equal in one step, which compensates the lack of local finiteness. This flow, which we think of as some kind of geodesic flow, provides a cocycle c , so that $c(g)(h)$ is the difference between the masks of h for 1 and for g , that is ℓ^p -summable (for h ranging over G), hence a nice action.

The flow also provides, not coset representatives for peripheral subgroups, but *random coset representatives*, which are probability measures for coset representatives (see Definition 4.3). This allows us to construct a cocycle for the induced representation, if ever a peripheral subgroup does act. Random coset representatives are not restricted to relatively hyperbolic groups, and for instance in the case of the Haagerup property, we obtain a criteria (Proposition 5.2), which generalizes the fact that free product with amalgamations over finite groups of groups with the Haagerup property have the Haagerup property themselves (and similarly for actions on an ℓ^p -space, see [Arn15]). An interesting corollary is the context of small cancellation groups $C'(\lambda)$ over free products (see [LS77, Chapter V.9]), for small λ .

COROLLARY B. — *Let $\lambda < 1/6$. A group that is small cancellation $C'(\lambda)$ over a free product of Haagerup groups (respectively, of groups acting properly on an ℓ^p -space for some $p > 1$), is itself Haagerup (respectively, acts on an ℓ^p -space for that same p).*

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1. Working in relatively hyperbolic groups

To fix notations we recall that a *graph* X is a set $X^{(0)}$, the *vertex set*, with a set $X^{(1)} \subseteq X^{(0)} \times X^{(0)}$, the *oriented edges*, endowed with an *origin map* $o : X^{(1)} \rightarrow X^{(0)}$ and a fixed-point free involution reversing the edges $\bar{\cdot} : X^{(1)} \rightarrow X^{(1)}$. A pair $\{e, \bar{e}\}$ is an *unoriented edge*. We denote by $t : X^{(1)} \rightarrow X^{(0)}$ the composition \bar{o} , this is the *terminus map*. We assume the reader familiar with paths, connectedness, length, geodesics, in graphs, as well with hyperbolicity. All our graphs are considered with their graph metric where an edge has length one.

DEFINITION 1.1. — *Let X be a graph. Given a vertex v and two oriented edges e_1, e_2 such that $o(e_2) = t(e_1) = v$, the angle between e_1 and e_2 at v , denoted by $\angle_v(e_1, e_2)$, is the infimum (in $\mathbb{R}_+ \cup \{+\infty\}$) of length of paths from $o(e_1)$ to $t(e_2)$ that avoid v . In other words,*

$$\angle_v(e_1, e_2) = d_{X \setminus \{v\}}(o(e_1), t(e_2)).$$

By convention, we will use the following abuse of notation: if x_1, x_2, c are vertices in X , we say that $\angle_c(x_1, x_2) > \theta$ if there exists two edges e_1, e_2 with $t(e_1) = o(e_2) = c$, with e_i on a geodesic between c and x_i ($i = 1, 2$) and such that $\angle_c(e_1, e_2) > \theta$. We

say that $\angle_c(x_1, x_2) \leq \theta$ otherwise, namely, if for any geodesic between c and x_i and starting by and edge $e_i (i = 1, 2)$, then $\angle_c(e_1, e_2) \leq \theta$.

Large angles at a vertex in a δ -hyperbolic graph will force geodesics through that vertex. More precisely we have the following.

PROPOSITION 1.2. — *Let X be a δ -hyperbolic graph, and let $a, b, c \in X^{(0)}$.*

- (1) *If $\angle_c(a, b) > 12\delta$, then all geodesics from a to b contain c .*
- (2) *If e, e' are two edges originating at c and that are on geodesics from c to a , then $\angle_c(e, \bar{e}') \leq 6\delta$.*
- (3) *For any $\theta \geq 0$, if $\angle_c(x, y) > \theta + 12\delta$ then for all choice of edges ϵ_1, ϵ_2 at c starting geodesics respectively to x and to y , $\angle_c(\epsilon_1, \bar{\epsilon}_2) \geq \theta$.*

Proof. —

(1). — Applying δ -thinness of triangles on the geodesics $[c, a]$ and $[c, b]$ at distance 2δ from c , we obtain a path of length at most 12δ , containing an arc of $[a, b]$, and that goes from the end of the first edge of $[c, a]$ to the end of the first edge of $[c, b]$. If the angle at c is larger than 12δ this path cannot avoid c . But only the arc on $[a, b]$ can meet c again. Hence the first claim.

(2). — Any geodesic bigon between c and a must be δ -thin, so if we take a geodesic bigon starting with the edges e and e' and look at the points at distance 2δ from c , we get a path of length 6δ at most and that avoids c .

(3). — This is a consequence of (1) and (2), combined with the triangular inequality for angles. □

DEFINITION 1.3. — *A graph is called fine if for each edge e , and for all $L \geq 0$, the set of edges*

$$\{e' \mid t(e') = o(e), \angle_{o(e)}(e, e') \leq L\}$$

is finite.

This definition of fine graphs is equivalent to Bowditch's definition that, for all edge, for all number, there are only finitely many simple loops of this length passing through this edge, [Bow12].

DEFINITION 1.4. — *Let G be a finitely generated group, and H_1, \dots, H_k , subgroups of G . Consider a Cayley graph $\text{Cay}(G)$ of G (over a finite generating set), and construct the coned-off over H_1, \dots, H_k as follows: for all i and all left coset gH_i of H_i , add a vertex $\widehat{gH_i}$ and for each $h \in H_i$, add an edge between $\widehat{gH_i}$ and gh . We denote by $\widehat{\text{Cay}}(G)$ this graph, called a coned-off Cayley graph (of G with respect to the subgroups H_1, \dots, H_k).*

We can now recall the definition of Bowditch of relative hyperbolicity from [Bow12] (his original definition is about G -graphs, but this is an equivalent formulation from [Dah03, Appendix]).

DEFINITION 1.5. — *A pair $(G, \{H_1, \dots, H_k\})$ of a group G with a collection of subgroups, is relatively hyperbolic, if a (equivalently any) coned-off Cayley graph $\widehat{\text{Cay}}(G)$ over H_1, \dots, H_k is hyperbolic and fine. The groups H_i are called peripheral subgroups.*

A difficulty of working in relatively hyperbolic groups is the lack of local finiteness in the coned-off graph. The angles, and below the cones, are useful tools for working around this. The following estimate says that the word metric is equivalent to the coned-off metric to which we add the sum of the angles at infinite valence vertices. We will be needing the first inequality only.

PROPOSITION 1.6. — *Let G be a group hyperbolic relative to $\{H_1, \dots, H_k\}$ with d_w a word metric, for a finite generating set containing generating sets of each H_i . Let $\widehat{\text{Cay}}(G)$ be its coned-off Cayley graph, for its relatively hyperbolic structure, with \hat{d} the graph metric.*

There are $A, B > 1$ for which, for all $g \in G$, for each choice of geodesic $[1, g]$ in the coned-off graph $\widehat{\text{Cay}}(G)$, if $\sum \angle([1, g])$ denotes the sum of angles of edges of $[1, g]$ at vertices of infinite valence, then

$$\frac{1}{A}d_w(1, g) - B \leq \hat{d}(1, g) + \sum \angle([1, g]) \leq A d_w(1, g) + B.$$

Proof. — First, given a geodesic $[1, g]$ in the coned-off graph $\widehat{\text{Cay}}(G)$, we construct a path in the Cayley graph (whose length will be bounded from above by $d_w(1, g)$) by replacing each pair of consecutive edges e, e' at an infinite valence vertex, by a path in the corresponding coset xH_i of H_i . We remark that we can choose this path of length bounded above in terms of the angle, say θ . Indeed, by definition of angle, there exists a path in $\widehat{\text{Cay}}(G)$ of length at most θ avoiding $\widehat{xH_i}$, from $o(e)$ to $t(e')$. Project each vertex in this path on the 1-neighborhood of $\widehat{xH_i}$. One gets a sequence of θ points y_1, \dots, y_θ in xH_i , from $o(e)$ to $t(e')$, two consecutive points being at an angle of at most 2δ at $\widehat{xH_i}$. It follows that there are only boundedly many possible transitions $y_i^{-1}y_{i+1}$ in H_i , and we have our bound on the word length in H_i in terms of the angle θ at $\widehat{xH_i}$. Thus $\frac{1}{A}d_w(1, g) - B \leq \hat{d}(1, g) + \sum \angle([1, g])$ (for some A and B).

For the second inequality, consider a geodesic from 1 to g in the word metric. It produces an injective continuous path q in $\widehat{\text{Cay}}(G)$, that fellow-travels a geodesic $[1, g]$ in the coned-off graph (see for instance [DS05, Prop. 8.25]). The path q does not contain any vertex of infinite valence, therefore each time $[1, g]$ contains a vertex of infinite valence, we can use the proximity of $[1, g]$ with q and the fact that q does not contain this point, to estimate the angles in terms of the length of a subsegment of q . By thinness of the graph, each subsegment of q is only used at most a uniformly bounded amount of times. Therefore, the total sum of the angles at infinite valence points is at most a certain multiple of the length of q , which is the word length of g . \square

1.1. Cones

The notion of angles will now allow us to talk about cones in any graph X . Cones are a useful tool for working in relatively hyperbolic groups as they allow to take neighborhoods of geodesics in a finite way.

DEFINITION 1.7. — *Let X be a graph. Given an oriented edge e and a number $\theta > 0$, the cone of parameter θ around the oriented edge e , denoted $\text{Cone}_\theta(e)$ is the*

subset of vertices v and edges ϵ of X such that there is a path from $o(e)$, starting by e , and containing v or ϵ , that is of length at most θ and for which two consecutive edges make an angle at most θ .

For vertices of finite valence we can define the cone at a vertex of parameter θ by the union of all the cones of parameter θ around the edges adjacent to the vertex.

If angles inside a cone of parameter θ are by definition bounded by θ , angles at vertices close to the cone are also controlled in terms of θ . Precisely we have the following estimate.

LEMMA 1.8. — *In any graph X , for any oriented edge e and $\theta \geq 0$, if $v \in \text{Cone}_\theta(e)$ then, for all $c \in X^{(0)}$ at distance $\leq \theta/10$ from both v and e , then $\angle_c(o(e), v) \leq (\theta^2 + 3\theta)/2$.*

Proof. — Consider c with two edges ϵ_1, ϵ_2 starting at c , on geodesics g_1, g_2 toward $o(e)$, and v respectively, of length $\leq \theta/10$. We also are given a path p starting by e , going to v of length at most θ , and maximal angle at most θ between consecutive edges. Concatenating the paths g_1, p, \bar{g}_2 (orientation reversed), we have a loop at c , starting by ϵ_1 , and ending by ϵ_2 and of length at most $6\theta/5$. If this loop doesn't pass by c (except starting and ending point), then by definition of angle we have that $\angle_c(\epsilon_1, \bar{\epsilon}_2) \leq 6\theta/5$, which would prove the claim. If this path passes at least twice at c , only the segment p can use c , and it can do so at most $(\theta - d(o(e), c) - d(c, v))/2$ times, each time realizing an angle between incident and exiting edge of at most θ . By the triangular inequality on angles, this path gives a bound on the angle $\angle_c(\epsilon_1, \bar{\epsilon}_2)$ of at most $\theta(\theta - d(o(e), c) - d(c, v))/2 + 6\theta/5 \leq (\theta^2 + 3\theta)/2$. This establishes the claim. \square

Cones also can be composed in an obvious way.

PROPOSITION 1.9. — *Let $\alpha, \beta \geq 0$, if e, e', e'' are three edges of a graph and if $e'' \in \text{Cone}_\alpha(e')$ and $e' \in \text{Cone}_\beta(e)$ then $e'' \in \text{Cone}_{\alpha+\beta}(e)$.*

Proof. — We have a path of length and maximal angle bounded by β that starts by e and ends by e' or \bar{e}' , and another of length and maximal angle at most β that starts at e' and contains e'' or \bar{e}'' . Concatenate the two paths, possibly overlapping at e' , or possibly cancelling a backtrack $e'\bar{e}'$, produces a path of length at most $\alpha + \beta$ and maximal angle at most $\max\{\alpha, \beta\}$. \square

Finally, the following proposition shows that cones are well-suited to work in hyperbolic and fine graphs.

PROPOSITION 1.10 ([Dah03], [DY08, §2.3]). — *In any fine graph, cones are finite.*

In any δ -hyperbolic graph, geodesic triangles are conically thin: for any geodesic triangle $([a, b], [b, c], [a, c])$, any edge e on $[a, b]$ is contained in a cone of parameter 50δ around an edge e' that is either in $[a, c]$ at same distance from a than e , or in $[b, c]$ at same distance from b than e .

The statement above differs slightly from that in [DY08, §2.3], however the proof is the same. Let us present it, as it is very simple. Consider a subsegment σ_1 of $[a, b]$ that contains e , and extends to a distance 3δ from both its end points (unless it reaches a or c before, a case we leave to the reader). Then append, at both ends, at

most 2δ long segments σ_0 and σ_2 in order to reach $[a, c] \cup [b, c]$ (by usual thin triangles property). If both σ_0 and σ_2 reach $[a, c]$, or if both reach $[b, c]$, then take the arc σ_3 of that segment, that closes a loop $\sigma_0\sigma_1\sigma_2\sigma_3$. Notice that σ_3 is of length at most 11δ (and set σ_4, σ_5 to be trivial in order to continue syntactically the argument). If the reached segments are different, close the loop as an hexagon around the center of the triangle, by an arc σ_3 of length 10δ on $[b, c]$ of the triangle, an arc σ_4 of at most 2δ between $[a, c]$ and $[b, c]$, and an arc σ_5 back to the endpoint of σ_2 on $[c, a]$, hence of length at most 21δ . We thus get a loop $(\sigma_0\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5)$ containing e . Triangle inequality forbids e to be in the transitions arcs $\sigma_0, \sigma_2, \sigma_4$. If it is either in σ_1 or σ_3 there is actually nothing to prove. Thus, we can extract the simple loop containing e . Again triangular inequality ensures some distance between $\sigma_0, \sigma_2, \sigma_4$, and this simple loop must contain an edge e' of σ_2 or σ_4 at same distance from respectively a or b than e . The length of the loop being at most 42δ , its maximal angle is bounded by this quantity. This makes e belong to the cone of parameter 42δ around e' , and the proposition is proved.

The above proposition allows us to deduce that intervals are finite in coned-off graphs, even if those graphs are not locally finite.

PROPOSITION 1.11. — *In any hyperbolic fine graph, between any pair of points a, x the set of points in a geodesic between a and x is finite.*

A more important use of cones was explored in [Dah06], and follows from the fact that in a hyperbolic graph, quasi-geodesics “without detours” are conically close to geodesic ([Dah06, Prop. 1.11]). We will use a variant of it, exposed in the next section.

1.2. Conical α -geodesics

In order to define a flow in the next section, we need to be able to travel coarsely from a point toward another point in a uniformly finite way. In [AL17], the authors use the concept of α -geodesic, for some $\alpha \geq 0$. Precisely, in a metric space, given two points x and a , we say that t belongs to an α -geodesic between x and a if

$$d(x, t) + d(t, a) \leq d(x, a) + \alpha.$$

Note that being on a 0-geodesic is being on a geodesic. However, when the space is not uniformly locally finite, the set of points in a α -geodesic between x and a can be infinite in general, if $\alpha \geq 1$, even if we assume the graph to be fine. For instance, on the coned-off graph of a relatively hyperbolic group, when a geodesic passes through a cone point, any point on the coset of that cone point will be on a 2-geodesic between the endpoints of that geodesic. But the coset points that are close (in the coset metric) to the entrance or exit points of the geodesic in the coset, will make small angles with some edges of the geodesic, whereas the other ones will make a large angle with any edge of the geodesic. We will use a variant of the notion of α -geodesics, that takes angles into account. Constants are not really relevant, but they need to be fixed by convention.

DEFINITION 1.12. — Let X be a graph, and $x, a \in X^{(0)}$. For $\rho \geq 0$ we denote by $\mathcal{E}_{a,x}(\rho)$ the set of edges e with $d(a, o(e)) = \rho$ that are contained in geodesics from a to x , or from x to a . We say that $t \in X^{(0)}$ belongs to an α -conical-geodesic between x and a if t is on an α -geodesic, $d(a, t) \leq d(a, x)$, and

$$t \in \bigcap_{e \in \mathcal{E}_{a,x}(d(a,t))} \text{Cone}_{40\alpha}(e).$$

In other words, points in an α -conical-geodesic are off a geodesic only by a small angle. We will denote $U_\alpha[a, x]$ the set of points belonging to an α -conical-geodesic between x and a .

For each $\rho \geq 0$, we denote by $\mathcal{S}_{a,x}(\rho)$ the slice of $U_\alpha[a, x]$ at distance ρ from a , that is the set

$$\mathcal{S}_{a,x}(\rho) = \{t \in U_\alpha[a, x] \mid d(t, a) = \rho\}$$

of points in $U_\alpha[a, x]$ at distance ρ from a .

We will now see some useful properties, namely that these sets are finite, stable (analogue of [AL17, 3.3] and [AL17, 3.4]), non-empty and slices at large angles are reduced to a point.

PROPOSITION 1.13. — Let X be a δ -hyperbolic fine graph, and let $a, x \in X$

- (1) (Finiteness) For any $\alpha > 0$ the set $U_\alpha[a, x]$ is finite. Each slice is contained in a cone of parameter 40α .
- (2) (Stability) If $\alpha = 2\delta$, and if $u \in U_{4\delta}[a, x]$ and $v \in U_{2\delta}[a, u]$ and if $d(u, v) \geq 5\delta$ then $v \in U_{4\delta}[a, x]$.
- (3) For all v on a geodesic between a and x , the slice of $U_{2\delta}[a, x]$ at distance $d(a, v)$ from a contains v , hence is non empty.
- (4) If $\angle_c(a, x) > (1000\delta)^2$, and $\alpha = 2\delta$, then $\mathcal{S}_{a,x}(d(a, c)) = \{c\}$.

Proof. —

(1). — By definition, $U_\alpha[a, x]$ is contained in an intersection of cones, which are finite according to Proposition 1.10. Each slice, by definition, is contained in $U_\alpha[a, x]$, which is an intersection of a family of cones of parameter 40α .

(2). — That v satisfies $d(a, v) + d(v, x) \leq d(a, x) + 4\delta$ is [AL17, 3.4]. Consider y in a geodesic $[a, u]$ such that $d(a, y) = d(a, v)$, and an edge e of $[u, a]$ with origin y . Then $v \in \text{Cone}_{80\delta}(e)$.

We now apply conical thinness of triangles (Proposition 1.10), to the triangle $([a, x], [x, u], [u, a])$, for the edge $e \in [a, u]$. Since e is at least 5δ -far from u , the proposition ensures that there exists an edge e' of $[a, x]$ at same distance from a than e , for which $e \in \text{Cone}_{50\delta}(e')$. It follows, by Proposition 1.9, that $v \in \text{Cone}_{130\delta}(e')$, which establishes the claim.

(3). — Call $[a, x]_1$ the given geodesic containing v . The claimed statement amounts to check that, if $[a, x]_2$ is any geodesic between a and x , and e is an edge of $[a, x]_2$ starting at distance $d(a, v)$ from a , then v is in $\text{Cone}_{80\delta}(e)$. This is an application of the conical thinness of geodesic triangles, Proposition 1.10 to the degenerate triangle $([a, x]_1, [x, x], [x, a]_2)$ (where $[x, a]_2$ is $[a, x]_2$ with reverse orientation).

(4). — Consider two edges e, e' at c (i.e. $o(e) = o(e') = c$) on a geodesic $[a, x]$, where $t(e)$ is closer to a and $t(e')$ is closer to x . By definition of slices, one has $\mathcal{S}_{a,x}(d(a, c)) \subseteq \text{Cone}_{80\delta}(e) \cap \text{Cone}_{80\delta}(e') \cap \{t, d(c, t) \leq 20\delta\}$. Note that it implies $\mathcal{S}_{a,x}(d(a, c)) \subseteq \text{Cone}_{80\delta+1}(\bar{e}) \cap \text{Cone}_{80\delta+1}(\bar{e}') \cap \{t, d(c, t) \leq 20\delta\}$. We will prove that this intersection is reduced to $\{c\}$.

By Proposition 1.2, $\angle_c(e, e') > (999\delta)^2$. We apply Lemma 1.8 for $\theta = 200\delta$. For all $v \in \text{Cone}_{200\delta}(\bar{e})$ different of c , but at distance at most 20δ from it, one has $\angle_c(o(\bar{e}), v) \leq ((200\delta)^2 + 600\delta)/2$. If v is in $\text{Cone}_{200\delta}(\bar{e}')$ as well, then $\angle_c(o(\bar{e}), v) \leq ((200\delta)^2 + 600\delta)/2$. By triangular inequality of angles at c , $\angle_c(\bar{e}, \bar{e}') \leq (200\delta)^2 + 600\delta$, and this contradicts the previous lower bound of $(999\delta)^2$. Thus, the intersection of $\text{Cone}_{200\delta}(\bar{e})$, $\text{Cone}_{200\delta}(\bar{e}')$ and of the ball of radius 20δ around c is reduced to $\{c\}$.

Thus, only c can be in the slice $\mathcal{S}_{a,x}(d(a, c))$. Since the slice is non-empty, we have the result. \square

2. The flow on the coned-off graph

Let us recall that, given a set Y and an additive semigroup V , a *flow* is a map $\mathcal{F} : Y \times V \rightarrow Y$ such that for all y , $\mathcal{F}(y, 0) = y$ and $\mathcal{F}(y, t_1 + t_2) = \mathcal{F}(\mathcal{F}(y, t_1), t_2)$. In this definition, the semi-group V is usually \mathbb{R}_+ , but we will use \mathbb{N} (by convention $0 \in \mathbb{N}$), and in this case the flow is entirely determined by the *flow step*, which is the map $\mathcal{T} : Y \rightarrow Y$ defined by $\mathcal{T}(y) = \mathcal{F}(y, 1)$.

In the following, X will be a δ -hyperbolic fine graph (for some integer $\delta > 0$), and $\text{Proba}(X)$ is the space of probability measures supported on $X^{(0)}$. We will define a flow for the semi-group \mathbb{N} , on the set $Y = \text{Proba}(X) \times X^{(0)}$, which is equivariant by the group of isometries of X . We think of it as a version of a geodesic flow.

When X is countable (which is the case in our context) elements of $\text{Proba}(X)$ are just functions from $X^{(0)}$ to \mathbb{R}_+ of sum 1, but we find it convenient and enlightening to see them as probability measures.

2.1. The flow step

In this subsection we will define the flow step

$$T : \begin{cases} \text{Proba}(X) \times X^{(0)} \rightarrow \text{Proba}(X) \times X^{(0)} \\ (\eta, a) \mapsto (T_a(\eta), a) \end{cases}$$

First let us introduce some vocabulary. Given a and x , we say that the step T_a at x is

- *initial* if $d(a, x) > 5\delta$, and $d(a, x)$ is not a multiple of 5δ ,
- *regular* if $d(a, x) > 5\delta$ and $d(a, x)$ is a multiple of 5δ ,
- *ending* if a is of infinite valence and $1 < d(a, x) \leq 5\delta$, or if a has finite valence, $d(a, x) \leq 5\delta$ and there exists c such that $\angle_c(a, x) > 1000\delta$,
- *stationary* in all other cases, i.e. in any of the following three cases
 - $x = a$,

- a is of infinite valence and x is a neighbor a ,
- a is of finite valence, $d(a, x) \leq 5\delta$ and for all c , $\angle_c(a, x) \leq 1000\delta$.

We denote by δ_x the Dirac mass at x and define the flow step T by defining $T_a(\delta_x)$, for a, x two vertices of X and then extend by linearity on probability measures:

$$T_a \left(\sum_x \lambda_x \delta_x \right) = \sum_x \lambda_x T_a(\delta_x).$$

For the following definition, we set the parameter α of the definition of slices, to be $\alpha = 2\delta$.

DEFINITION 2.1. — *Let X be a δ -hyperbolic fine graph, and let $a, x \in X^{(0)}$. We set $r_{a,x}$ to be the largest integer r such that $5\delta r < d(a, x)$.*

- *If the step T_a at x is either initial or regular, then $T_a(\delta_x)$ is the uniform probability measure supported by the slice $\mathcal{S}_{a,x} = \mathcal{S}_{a,x}(5\delta r_{a,x})$ (as in Definition 1.12).*
- *If the step T_a at x is ending, and a has infinite valence then $T_a(\delta_x)$ is the uniform probability measure supported by the slice $\mathcal{S}_{a,x} = \mathcal{S}_{a,x}(1)$ at distance 1 from a . If the step is ending and a has finite valence, then $T_a(\delta_x)$ is the Dirac mass supported on the unique c minimizing $d(a, c)$ such that $\angle_c(a, x) > 900\delta$.*
- *If the step T_a at x is stationary, then $T_a(\delta_x) = \delta_x$ (and $\mathcal{S}_{a,x} = \{x\}$).*

Let us check that this definition makes sense. The set $\mathcal{S}_{a,x}$ is well defined: in the case of an ending step, any vertex c such that $\angle_c(a, x) > 900\delta$ is on every geodesic between a and x , thus there is a unique one that is closest to a . Then the set $\mathcal{S}_{a,x}$ is never empty by Proposition 1.13 (3) and it is always finite, by part (1) of that same proposition. Observe that among iterates of the flow step applied at (δ_x, a) , only the first iteration can be initial, only one step among the iterates can be ending, and after the ending step, all steps are stationary.

This defines uniquely the value of $T_a(\delta_x)$ for all x and all a and so completely defines the flow step T , hence also the flow

$$\begin{aligned} \mathcal{F} : \text{Proba}(X) \times X^{(0)} \times \mathbb{N} &\rightarrow \text{Proba}(X) \times X^{(0)} \\ (\eta, a, k) &\mapsto (T_a^k(\eta), a). \end{aligned}$$

2.2. Stationarity and mask

For two points x and a in $X^{(0)}$, the flow step $T_a(\delta_x)$ is a probability measure whose support is closer to a than δ_x , so the flow step has pushed the point x towards a . Iterating this flow step will eventually stall and give a measure that is close to a .

PROPOSITION 2.2. — *For X a δ -hyperbolic fine graph and all $a, x \in X^{(0)}$, the flow step from Definition 2.1 is stationary in k : if $k > r_{a,x} + 1$, one has $T_a^{r_{a,x}+1}(\delta_x) = T_a^k(\delta_x)$.*

Proof. — If y, y' are in the support of $T_a(\delta_x)$, one has $d(a, y) = d(a, y') \leq d(a, x)$. In case of equality, then $T_a(x) = \delta_x$. Indeed, if the flow step is initial or regular, then $d(a, y) = d(a, y') < d(a, x)$, since by definition, $5\delta r_{a,x} < d(a, x)$. The ending step case, also induces a strict inequality, and the stationary step case induces the equality $T_a(x) = \delta_x$. Hence the flow eventually become stationary for k large enough. \square

DEFINITION 2.3. — Given any two vertices a and x of X a δ -hyperbolic fine graph, the mask of a for x is the probability measure given by

$$\mu_x(a) = T_a^{R(a,x)}(\delta_x) = \lim_{k \rightarrow \infty} T_a^k(\delta_x),$$

where $R(a, x)$ the minimal number of iterations of T_a on δ_x so that the stationary step is reached.

Let us observe the following.

PROPOSITION 2.4. — Let X be a δ -hyperbolic and fine graph. For $a, x \in X^{(0)}$, then

- (1) If a is of infinite valence, for all x , the measure $\mu_x(a)$ is supported by the 1-neighborhood of a . Moreover, if e is the first edge of a geodesic segment $[a, x]$, the measure $\mu_x(a)$ is supported in $\text{Cone}_{160\delta}(e)$.
- (2) If a is of finite valence, for all x , the measure $\mu_x(a)$ is supported by $\text{Cone}_{\theta_0}(a)$ for $\theta_0 = 1160\delta$.

The last estimate is very crude, and with some closer look, one is likely to get a much more precise control.

Proof. —

(1). — If a has infinite valence, by definition of the ending step in this case the measure is in the slice at distance 1 from a . By Proposition 1.13(2), the last slice used is in a 4δ -conical geodesic from a to x . So, given e which is first edge on a geodesic from a to x , by the definition of slices, this last slice used is contained in $\text{Cone}_{160\delta}(e)$.

(2). — In the case of a finite valence vertex a , the condition for the step being stationary is that there are no angle larger than 1000δ between a and a point in the support. Again, by Proposition 1.13(2), the support is contained in a slice is in a 4δ -conical geodesic from a to x , hence in a cone $\text{Cone}_{160\delta}(e_0)$ centered at an edge in some geodesic $[a, x]$, at distance at most 5δ from a . We moreover know, by definition of the ending step, that there is no angle larger than 1000δ between a and $o(e_0)$. Thus, by composition of cones, Proposition 1.9, the support is contained in $\text{Cone}_{1160\delta}(e)$ for an initial edge of a geodesic $[a, x]$. \square

The definition of the flow step being in a purely metric way, as a consequence the masks are equivariant with respect to the isometry group.

PROPOSITION 2.5. — Let X be a δ -hyperbolic fine graph. The map

$$X^{(0)} \times X^{(0)} \rightarrow \text{Proba}(X), (x, a) \mapsto \mu_x(a)$$

that associates to a pair of points (x, a) the mask of a for x is equivariant with respect to the isometry group of X . More precisely, if ϕ is an isometry of X , one has $\phi_*\mu_x(a) = \mu_{\phi(x)}(\phi(a))$.

2.3. The flow stays focused

In this subsection we will be recording the behavior of the flow step at different stages

LEMMA 2.6. — *Let X be a δ -hyperbolic fine graph, and for $a \in X^{(0)}$ let T_a denote the flow step as in Definition 2.1. Let $x, x' \in X^{(0)}$.*

- (1) *If x is such that the k first iterates of T_a are initial or regular steps, then, for any two points y, y' in the support of $T_a^k(\delta_x)$, one has $d(y, y') \leq 5\delta$. Moreover the support of $T_a^k(\delta_x)$ is contained in a cone of parameter 160δ .*
- (2) *If x, x' are neighbors, and $5k\delta < d(a, x) \leq d(a, x') < 5(k+1)\delta$, then the union of the supports of $T_a(\delta_x)$ and $T_a(\delta_{x'})$ has diameter at most $8\delta + 1$.*
- (3) *If $d(x, x') \leq 8\delta$, with $d(x, a) = d(x', a)$ and assume that for both of them, the flow step is regular. Then the union of the supports of both measures $T_a(\delta_x)$ and $T_a(\delta_{x'})$ has diameter at most 8δ . Moreover, the union of these supports is contained in a cone of parameter 170δ , and their intersection is not empty.*
- (4) *If x, x' are neighbors and both at distance larger than 5δ from a , then there are two exponents ϵ, ϵ' that are either 1 or 0, such that $T_a^\epsilon(\delta_x)$ and $T_a^{\epsilon'}(\delta_{x'})$ have support on a sphere centered at a of radius $5k\delta$ for some k , and the diameter of their union is at most $8\delta + 2$.*

Proof. —

(1). — By Proposition 1.13(2) y and y' are on $U_{4\delta}[a, x]$. As in [AL17] we can estimate:

$$d(a, x) + d(y, y') \leq \max\{d(a, y) + d(y', x), d(a, y') + d(y, x)\} + \delta \leq d(a, x) + 4\delta + \delta.$$

Hence $d(y, y') \leq 5\delta$. Then, observe that by Proposition 1.13(2) the support of $T_a^k(\delta_x)$ is in a slice of $U_{4\delta}[a, x]$. Proposition 1.13(1) concludes.

(2). — By assumption the steps of the flow are initial. Let y_0 and y'_0 be respectively on geodesics $[x, a]$ and $[x', a]$ at distance $5k\delta$ from a . By δ -hyperbolicity, they are at distance less than $2\delta + 1$ from each other. Let y, y' respectively be in the supports of $T_a(\delta_x)$ and $T_a(\delta_{x'})$. Apply the argument of part (1) using that y and y' are on 2δ -geodesics (there is only one step of flow). One obtains that $d(y, y_0) \leq 3\delta$ and $d(y', y'_0) \leq 3\delta$, hence $d(y, y') \leq 8\delta + 1$.

(3). — Here x and x' are not assumed neighbors, but at distance $5(k+1)\delta$ from a . Let y_0 and y'_0 respectively be on geodesics $[x, a]$ and $[x', a]$ at distance $5k\delta$ from a . By δ -hyperbolicity, they are at distance less than 2δ from each other and the same argument than part (2) gives a bound of 8δ on the union of the supports of $T_a(\delta_x)$ and $T_a(\delta_{x'})$.

Given two geodesics $[a, x]$ and $[a, x']$, we denote by e, e' the edges of these geodesics starting at distance $(d(a, x) - 5\delta)$ from a . By δ -hyperbolicity, the two segments stay 2δ -close for a length of at least δ after e and e' . One deduces (as in the proof of Proposition 1.2) that $e \in \text{Cone}_{10\delta}(e')$. Therefore, by composition of cones (Proposition 1.9) $\text{Cone}_{160\delta}(e) \subseteq \text{Cone}_{170\delta}(e')$. Moreover, $o(e)$ is at distance at most δ from $[a, x']$, so is on a 2δ -geodesic between a and x' . Also, as we discussed, it is in $\text{Cone}_{10\delta}(e')$, for any edge e' on any geodesic $[a, x']$ satisfying that e' starts at $(d(a, x) - 5\delta)$ from a . Thus, $o(e) \in U_{2\delta}[a, x']$ and hence by Proposition 1.13(3) the intersection of the supports is non-empty.

(4). — There are several cases, depending whether the flow from x , respectively from x' , needs an initial step or not, in other words, whether x , respectively x' , are at distance strictly greater than $5k\delta$ or equal to $5k\delta$ from a .

If the first step of the flow is initial for both (δ_x, a) and $(\delta_{x'}, a)$, then part (2) says that the union of the supports of $T_a(\delta_x)$ and $T_a(\delta_{x'})$ has diameter at most 8δ . The same is true according to part (3) if both first steps of the flow are regular.

Assume that $d(a, x) = 5k\delta$ and $d(a, x') > 5k\delta$. Then we compare δ_x (i.e. $\epsilon = 0$) with $T_a(\delta_{x'})$ (i.e. $\epsilon' = 1$). According to part (1) the support of $T_a(\delta_{x'})$ has diameter at most 5δ , and by Proposition 1.13(3) it contains a point at distance 1 from x' , hence at distance less than 2 from x , the union of the supports of $T_a^\epsilon(\delta_x)$ and $T_a^{\epsilon'}(\delta_{x'})$ has diameter at most $5\delta + 2$, hence the result. \square

2.4. Check points at large angles

The following proposition says that when the flow passes a large angle, this large angle acts as a check point: The flow could have started at this large angle, and still give the same result.

PROPOSITION 2.7. — *Let (X, d) a δ -hyperbolic fine graph. Let a, x be vertices in X . If there exists $c \in [a, x]$ such that $\angle_c(a, x) > (2000\delta)^2$, then for any x' at distance 1 from x , we have that*

$$\mu_x(a) = \mu_{x'}(a) = \mu_c(a).$$

Proof. — Define $r(c)$ to be the largest r so that $(r_{a,x} - r)5\delta \geq d(a, c)$. First notice that for all $r \leq r(c)$, and all b in the support of $T_a^r(\delta_x)$, one has

$$\angle_c(a, b) > (1910\delta)^2.$$

Indeed, let b in the support of $T_a^r(\delta_x)$. Then $d(a, b) = (r_{a,x} - r)5\delta > d(a, c)$, and by Proposition 1.13(3) there is b' on the geodesic $[a, x]$ realizing the maximal angle at c , that is in the support of $T_a^r(x)$. Hence $\angle_c(a, b') > (2000\delta)^2$. Since b' and b are in a same cone of parameter 160δ (Proposition 1.13(1)), the maximal angle between them is at most $2(160\delta)^2$ (by Lemma 1.8). Thus, $\angle_c(a, b) > (1987\delta)^2 > (1910\delta)^2$ as claimed.

We now show that $T_a^{r(c)+1}(\delta_x) = T_a(\delta_c)$. If $d(a, c)$ is a multiple of 5δ , for any point b in the support of $T_a^{r(c)-1}(\delta_x)$ then $\angle_c(a, b) > (1910\delta)^2$. Thus, all geodesics from a to b pass through c and make an angle at least $(1900\delta)^2$. It follows that the intersection of all cones of parameter 80δ centered at an edge of $\mathcal{E}_{a,b}(d(a, c))$ is reduced to $\{c\}$ (by Proposition 1.13(4)), hence $T_a(\delta_b) = \delta_c$, which means that $T_a^{r(c)}(\delta_x) = \delta_c$, and $T_a^{r(c)+1}(\delta_x) = T_a(\delta_c)$ as claimed. When $d(a, c)$ is not a multiple of 5δ , apply T_a to all points b in the support of $T_a^{r(c)}(\delta_x)$. But since we saw that $\angle_c(a, b) > (1910\delta)^2$, all geodesics from a to b pass through c and make an angle at least $(1900\delta)^2$. The points on 2δ -geodesics from a to c are exactly the points on 2δ -geodesics from a to b that are at distance at most $d(a, c)$ from a , and we have equality of the sets $\mathcal{E}_{a,b}((r_{a,x} - r(c))5\delta) = \mathcal{E}_{a,c}((r_{a,x} - r(c))5\delta)$. Hence $T_a(\delta_b) = T_a(\delta_c)$, and $T_a^{r(c)+1}(\delta_x) = T_a(\delta_c)$ again as claimed.

Finally, if $x' \neq c$ and x' at distance 1 from x , then, for $r'(c)$ defined similarly to $r(c)$ for x' we have

$$T_a^{r(c)+1}(\delta_x) = T_a^{r'(c)+1}(\delta_{x'}) = T_a(\delta_c).$$

The angle at c between a and x' can be reduced by one unit, which is still well above the threshold to apply the previous argument. Iterating the flow step gives the conclusion of the proposition. \square

2.5. Confluence

We will need to control that, given a , and x, x' close to each other, but “far” from a (in terms of distance or of angles), each of the regular flow steps from (δ_x, a) and from $(\delta_{x'}, a)$, are uniformly close. To do that, we will be talking of confluence. Recall that the norm $\|\eta\|$ of a measure is defined as its ℓ^1 -norm $\|\eta\| = \sum_{x \in X} |\eta(x)|$. For η, η' two probability measures on X , denote by $\mathcal{M}(\eta, \eta')$ the measure defined by

$$\mathcal{M}(\eta, \eta')(\{y\}) = \min\{\eta(\{y\}), \eta'(\{y\})\}.$$

We also define the *symmetric difference* of η and η' as

$$(\eta \Delta \eta') = \eta + \eta' - 2\mathcal{M}(\eta, \eta'),$$

which is always a positive measure since η, η' are probability measures. Moreover, $\|\eta \Delta \eta'\| = \|\eta - \eta'\|$.

DEFINITION 2.8. — *Let $\beta \geq 0$. We say that the flow step T_a is β -confluent on (η, η') if*

$$\|\mathcal{M}(T_a(\eta), T_a(\eta'))\| \geq \|\mathcal{M}(\eta, \eta')\| + \beta\|\eta \Delta \eta'\|.$$

Remark 2.9. — Notice that the flow step T_a is β -confluent on (η, η') if and only if

$$\|T_a(\eta) \Delta T_a(\eta')\| \leq (1 - 2\beta)\|\eta \Delta \eta'\|.$$

Indeed, observe that $\|T_a(\eta) \Delta T_a(\eta')\| = 2 - 2\|\mathcal{M}(T_a(\eta), T_a(\eta'))\|$, and similarly $\|\eta \Delta \eta'\| = 2 - 2\|\mathcal{M}(\eta, \eta')\|$. After substitution, and dividing by 2, the above condition reads:

$$1 - \|\mathcal{M}(T_a(\eta), T_a(\eta'))\| \leq (1 - \|\mathcal{M}(\eta, \eta')\|) - \beta\|\eta \Delta \eta'\|$$

which is equivalent to the inequality of the definition.

PROPOSITION 2.10. — *Let X be a fine δ -hyperbolic graph with a co-finite group action. Let $k \geq 1$, and η, η' be two measures whose supports are on $S(a, 5(k + 1)\delta)$, and have union of diameter less than 8δ .*

Then, there is a constant $C > 1$ such that the k consecutive flow steps on (η, a) and (η', a) are $\frac{1}{C}$ -confluent. In particular, for all $m \geq k$,

$$\|T_a^m(\eta) \Delta T_a^m(\eta')\| \leq \left(1 - \frac{2}{C}\right)^k \|\eta \Delta \eta'\|.$$

Proof. — First notice that the assumption that X is a fine δ -hyperbolic graph with a co-finite group action implies that here are only finitely many orbits of cones of given parameter, and in each of these orbits, cones are finite (Proposition 1.10), with same cardinality. Hence, there exists $C > 1$ that is an upper bound on the cardinality of each cone of parameter 160δ in X .

We first proceed by induction to prove that for each $i \leq k$, the union of supports of the measures $T_a^i(\eta)$ and $T_a^i(\eta')$ have diameter at most 8δ . There is nothing to prove for $i = 0$. For $i > 0$, assuming $T_a^{i-1}(\eta)$ and $T_a^{i-1}(\eta')$ have support whose union has diameter at most 8δ , we can say that, since all steps of the flow so far were regular, these supports are on the sphere $S(a, 5(k+2-i)\delta)$. We apply Lemma 2.6 (3) to obtain that the union of supports of $T_a^i(\eta)$ and $T_a^i(\eta')$ have diameter at most 8δ .

We then prove the confluence. For each Dirac masses $\lambda_v\delta_v$ and $\lambda_{v'}\delta_{v'}$, respectively in the support of $T_a^{i-1}(\eta) - \mathcal{M}(T_a^{i-1}(\eta), T_a^{i-1}(\eta'))$ and of $T_a^{i-1}(\eta') - \mathcal{M}(T_a^{i-1}(\eta), T_a^{i-1}(\eta'))$, the regular step of the flow sends each of these Dirac masses on a uniform measure on a subset of a slice, in a cone of parameter 160δ , by Proposition 1.13 (1). Moreover the respective supports of these uniform measures share at least one point, by Lemma 2.6 (3). Therefore, this step is $\frac{1}{C}$ -confluent on the Dirac masses $\lambda_v\delta_v$ and $\lambda_{v'}\delta_{v'}$ (see Definition 2.8 above), where C is the upper bound on the cardinality of cones of parameter 160δ in X . Precisely, according to Remark 2.9 we have that

$$\|T_a(\delta_v)\Delta T_a(\delta_{v'})\| \leq \left(1 - \frac{2}{C}\right) \|\delta_v\Delta\delta_{v'}\|.$$

Since the flow step on measures is defined by linearity on Dirac masses, this confluence on all masses in these supports gives that

$$\|T_a^i(\eta)\Delta T_a^i(\eta')\| = \|T(T_a^{i-1}(\eta))\Delta T(T_a^{i-1}(\eta'))\| \leq \left(1 - \frac{2}{C}\right) \|T_a^{i-1}(\eta)\Delta T_a^{i-1}(\eta')\| \quad \square$$

PROPOSITION 2.11. — *Let X be a hyperbolic fine graph, with a cofinite group action. There exists a constant $\kappa < 1$ such that, for all a , for all x_1, x_2 at distance 1 from each other, if x is among x_1, x_2 , closest to a , and if Θ denotes the minimal sum of angles at infinite valence vertices on a geodesic $[a, x]$, then for $d(a, x) + \Theta$ sufficiently large, one has*

$$\|\mu_{x_1}(a)\Delta\mu_{x_2}(a)\| \leq \kappa^{d(a,x)+\Theta}$$

Observe that the statement would hold also for Θ defined as the sum of all consecutive angles, and also if it was taken to be the maximum of these sums.

Proof. — If one of the angles on $[a, x]$ is larger than $(2000\delta)^2$ then by Proposition 2.7, $\mu_{x_1}(a)\Delta\mu_{x_2}(a) = 0$. If all the angles on $[a, x]$ are smaller than $(2000\delta)^2$, then

$$d(a, x) \leq d(a, x) + \Theta \leq (2000\delta)^2 d(a, x).$$

It follows that $(\kappa)^{d(a,x)+\Theta} \geq (\kappa^{(2000\delta)^2 6\delta})^{d(a,x)/6\delta}$. We then choose $\kappa < 1$ sufficiently close to 1 so that $\kappa^{(2000\delta)^2 6\delta} > (1 - \frac{2}{C})$ from Proposition 2.10. If $d(a, x)$ is sufficiently large (uniformly), $d(a, x)/6\delta$ is larger than the number of regular step of the flow applied to (δ_x, a) . Thus, Proposition 2.10, and Lemma 2.6 (4) give the conclusion. \square

2.6. Non-confluence

In this subsection we will be showing that for two given sources of the flow, any point on a geodesic between those two sources has disjoint masks for each of those sources.

PROPOSITION 2.12. — *Let X be a δ -hyperbolic and fine graph, and let μ be the mask of Definition 2.3.*

- (1) *If $d(x, x') \geq 10\delta$ and if a is of finite valence, at distance at least 5δ from both x and x' , and $a \in [x, x']$ for some geodesic, then $\|\mu_x(a)\Delta\mu_{x'}(a)\| = 2$.*
- (2) *If a has infinite valence, and $\angle_a(x, x') > 10(160\delta)^2$ then $\|\mu_x(a)\Delta\mu_{x'}(a)\| = 2$.*

Proof. —

(1). — We proceed by contradiction. Assume that v is in the support of both $\mu_x(a)$ and $\mu_{x'}(a)$.

In the first case, assume that no step in the flow was *ending*. Then v is at distance 5δ from a and is in 4δ -geodesics between both a, x and a, x' (by Proposition 1.13 (2)). This is not possible.

Assume that one step of the flow for δ_x was ending. If there was no step of the flow of type ending for $\delta_{x'}$, then the supports of $\mu_x(a)$ and $\mu_{x'}(a)$ are on different spheres around a , so they are disjoint. If there was an ending step for $\delta_{x'}$, then, since a is of finite valence, $\mu_x(a)$ and $\mu_{x'}(a)$ are Dirac masses carried by vertices y, y' on which, respectively, $\angle_y(a, x)$ and $\angle_y(a, x')$ are larger than 20δ . Since a is on a geodesic from x to x' , by Proposition 1.2 one has $y \neq y'$.

(2). — The assumption indicates that the flow from δ_x to a and from $\delta_{x'}$ to a has an ending step. It follows from Proposition 2.4 that the measures $\mu_x(a)$ and $\mu_{x'}(a)$ have supports contained in two cones of parameter 160δ centered respectively on an edge starting a geodesic $[a, x]$ and starting a geodesic $[a, x']$. Let e, e' be these edges. One has $\angle_a(e, e') > 9(160\delta)^2$. It follows from Lemma 1.8 that the intersection of the cones is reduced to $\{a\}$. Thus the intersection of the supports is empty. \square

COROLLARY 2.13. — *Let X be the coned-off graph of a finitely generated relatively hyperbolic group G , over its family of peripheral subgroups. There exists $p > 1$, and $\varepsilon > 0$ such that for all x*

$$\sum_{a \in X} \|\mu_1(a)\Delta\mu_x(a)\|^p$$

is convergent, and if x is at distance at least 10δ from 1, the sum is larger than $\varepsilon d(1, x)$.

Proof. — We identify G with the image of its Cayley graph in X . For the first statement, let $\Theta(x, x')$ denote the minimal sum of angles at infinite valence vertices on a geodesic $[x, x']$. Define $d' : X \times X \rightarrow \mathbb{N}$ as $d'(g_1, g_2) = d_X(g_1, g_2) + \Theta(x, x')$. By Proposition 1.6, d' is coarsely larger than a word metric d_w on G . In particular, there is $A \geq 1$ and $\gamma_G > 1$ such that $S_{d'}(R, G) = \{g \in G, d'(1, g) = R\}$ has cardinality less than $A\gamma_G^R$. Denote by $S_{d'}(R)$ the union of $S_{d'}(R, G)$ and of the vertices x of X of infinite valence such that $d'(1, x) = R$. One can check that its cardinality is at most

$A'\gamma_G^R$ for a certain constant A' (related to the maximal order of intersection of two peripheral subgroups).

Recall that we defined a constant κ in Proposition 2.11. We choose p such that $\kappa^p < 1/\gamma_G$. Then, fixing x , we have the bound

$$\sum_{a \in S_{d'}(R)} \|\mu_1(a)\Delta\mu_x(a)\| \leq \kappa^{pR} A'\gamma_G^R$$

This defines a summable family of numbers therefore $\sum_{a \in X} \|\mu_1(a)\Delta\mu_x(a)\|^p$ is convergent.

The lower bound is given by Proposition 2.12 and the fact that there are at least $d(x, x')/2 - 1$ vertices of finite valence between x and x' . \square

3. An action that is proper in the coned-off distance

In this section X is the coned-off Cayley graph of a relatively hyperbolic group G , and $\mathcal{V} = \mathbb{R}X^{(0)}$ is the vector space of all functions from $X^{(0)}$ to \mathbb{R} with finite support, that we endow with the ℓ^1 -norm. For all $p > 1$ or $p = \infty$, we consider $\mathcal{W}_p = \ell^p(X^{(0)}, \mathcal{V})$. In other words, an element ω of \mathcal{W}_p is the data, for all $a \in X^{(0)}$, of a vector $\omega_a \in \mathcal{V}$, such that its norm

$$\|\omega\|_{\mathcal{W}_p} = \left(\sum_a \|\omega_a\|_{\mathcal{V}}^p \right)^{\frac{1}{p}} = \left(\sum_a \left(\sum_v |\omega_a(v)| \right)^p \right)^{\frac{1}{p}}$$

is finite. Let \mathcal{W} be one of the \mathcal{W}_p . We denote by $\overrightarrow{\text{Isom}}\mathcal{W}$ the unitary group of \mathcal{W} , i.e. the subgroup of linear automorphisms of \mathcal{W} that are isometries for the norm. The group G admits an isometric linear action on \mathcal{W} by precomposition by isometries of X . Let us denote $\pi : G \rightarrow \overrightarrow{\text{Isom}}\mathcal{W}$ this action. We want to promote it into an action by affine isometries that has no fixed point. For that we need a cocycle, as we recall in the next subsection.

3.1. Generalities on cocycles and actions

Recall that we may see \mathcal{W} as an affine space, and also as an abelian group (for the addition of vectors). The group of affine isometries of (the affine space) \mathcal{W} is the semidirect product

$$\text{Aff Isom}\mathcal{W} = \mathcal{W} \rtimes \overrightarrow{\text{Isom}}\mathcal{W}$$

for the natural action of $\overrightarrow{\text{Isom}}\mathcal{W}$ on \mathcal{W} . Given a group G , any homomorphism $\phi : G \rightarrow \text{Aff Isom}\mathcal{W}$ has an image in $\vec{\phi} : G \rightarrow \overrightarrow{\text{Isom}}\mathcal{W}$ through the quotient map, and, given this homomorphism $\vec{\phi}$, the homomorphism ϕ is characterised by a map $c : G \rightarrow \mathcal{W}$ recording $c(g) = \phi(h)(\vec{0}_{\mathcal{W}})$. Applying the law of the semidirect product reveals that c satisfy the cocycle relation

$$c(g_1g_2) = c(g_1) + \vec{\phi}(g_1)(c(g_2)),$$

for all $g_1, g_2 \in G$. The cocycle is actually the difference between ϕ and a given section of $\vec{\phi}$ in the semi-direct product. Conversely, whenever one has $\vec{\phi} : G \rightarrow \overrightarrow{\text{Isom}}\mathscr{W}$ and $c : G \rightarrow \mathscr{W}$ satisfying the cocycle relation for $\vec{\phi}$, one can define the map

$$\phi : G \rightarrow \text{Aff Isom}\mathscr{W}, g \mapsto \phi(g) = (c(g), \vec{\phi}(g)),$$

which is a homomorphism.

DEFINITION 3.1. — *For a finitely generated group G , endowed with a (locally finite) word metric d , the homomorphism ϕ is called proper if c is proper, or equivalently if $\|c(g_i)\|$ goes to infinity for any sequence (g_i) of elements in G for which $d(1, g_i)$ goes to infinity in \mathbb{R} .*

If d is a left invariant metric obtained from a coned-off Cayley graph over the cosets of subgroups, (or, in other words, a relative word metric), then ϕ is relatively proper if $\|c(g_i)\|$ goes to infinity for any sequence (g_i) of elements in G for which $d(1, g_i)$ goes to infinity.

3.2. A relatively proper affine isometric action

We return to our initial context. We have G a relatively hyperbolic group, and $\pi : G \rightarrow \overrightarrow{\text{Isom}}\mathscr{W}$ for our relatively hyperbolic group G , which we want to promote to a homomorphism in $\text{AffIsom}\mathscr{W}$. We thus need a cocycle for π . We identify G with the vertices of its Cayley graph $\text{Cay}(G)$ in the coned-off Cayley graph X . We define the map c as follows

$$c : \begin{cases} G \rightarrow \mathscr{W}_\infty \\ g \mapsto \{a \mapsto \mu_1(a) - \mu_g(a)\}. \end{cases}$$

where $\mu_x(a)$ is the probability measure from Definition 2.3. Recall that $\mu_1(a) - \mu_g(a)$ has same ℓ^1 -norm as the symmetric difference $\mu_1(a)\Delta\mu_g(a)$ because the latter is the absolute value of the former. The following proves the first part of our main result, Theorem A

THEOREM 3.2. — *Let G be a relatively hyperbolic group and let X be the coned-off Cayley graph of G with respect to its peripheral subgroups. For $p > 1$ large enough, c has its values in $\mathscr{W}_p = \ell^p(X^{(0)}, \mathscr{V})$, and is a cocycle for the representation π on \mathscr{W}_p .*

Moreover, the homomorphism $(c, \pi) : G \rightarrow \text{AffIsom}\mathscr{W}_p$ is relatively proper for the metric on G induced by the coned-off Cayley graph. Precisely, there is $\epsilon > 0$ and a constant k_0 , for which for all $g \in G$, $\|g(\vec{0})\|^p \geq \epsilon d_X(1, g) - k_0$.

Proof. — That c is a cocycle for π is a standard computation using Proposition 2.5. It takes its values in $\ell^p(X^{(0)}, \mathscr{V})$, by Corollary 2.13. The estimate on $\|g(\vec{0})\|$ is a consequence of the last part of Corollary 2.13. \square

4. Random coset representatives and subgroup properness

We will now recall some basic notions on induced representations from a subgroup H to a larger group G , and discuss random coset representatives (see Definition 4.3). Our main result is that, in case those random coset representatives that are ℓ^p -almost G -invariant exist (Definition 4.6), we can define a cocycle on G for the induced representation from H , that will have the same H -properness than the cocycle we started with (Proposition 4.14). We start by recalling some basic notions on Banach spaces.

DEFINITION 4.1. — *A Banach space B is called uniformly convex if for any $0 < \epsilon \leq 2$, there is $\delta(\epsilon) > 0$ such that for any $x, y \in B$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$, then*

$$\frac{\|x + y\|}{2} \leq 1 - \delta(\epsilon).$$

Suppose that a finite number of Banach spaces, B_1, B_2, \dots, B_k are given, and that B is their product. We shall call B a uniformly convex product of the B_i if the norm of an element $x = (x_1, x_2, \dots, x_k)$ of B is defined by

$$\|x\| = N(\|x_1\|, \|x_2\|, \dots, \|x_k\|),$$

where N is a continuous non-negative function, homogeneous, strictly convex and strictly increasing.

A classical example of uniformly convex product is an ℓ^p -combination. According to Clarkson [Cla36], ℓ^p -spaces are uniformly convex, and any uniformly convex product of uniformly convex Banach spaces is again uniformly convex. According to Day [Day41], if B is a uniformly convex Banach space, then the B -valued functions that are ℓ^p are uniformly convex as well.

4.1. Induced representations

Assume that H is a subgroup of a group G . Assume that we have a representation $\pi_H : H \rightarrow \overrightarrow{\text{Isom}} \mathcal{V}$ of H in the linear isometry group of a normed vector space \mathcal{V} in other words, a unitary representation on \mathcal{V}). Then, there is a normed vector space \mathcal{V}' , with an isometric embedding of \mathcal{V} in it, and a unitary representation $\pi_G : G \rightarrow \overrightarrow{\text{Isom}} \mathcal{V}'$, called the induced representation of H to G , such that the restriction to H preserves \mathcal{V} and induces π_H . We refer the reader to [BdlHV08, §E.1] for the material recalled below. First, let us recall the notion of induced representation.

DEFINITION 4.2. — *Given a representation (π_H, V) of a group H that happens to be a subgroup of a bigger group G , we define*

$$\mathcal{A}_\infty = \{\varphi : G \rightarrow V \mid \varphi(gh) = \pi_H(h^{-1})\varphi(g) \text{ for all } g \in G, h \in H\}.$$

We can endow the vector space \mathcal{A}_∞ with the induced action from π_H

$$\begin{aligned} \pi_G(g) : \mathcal{A}_\infty &\rightarrow \mathcal{A}_\infty \\ \varphi &\mapsto \{x \mapsto \varphi(g^{-1}x)\} \end{aligned}$$

If $p : G \rightarrow G/H$ denotes the quotient map, we further define

$$\mathcal{A}_0 = \{\varphi \in \mathcal{A}_\infty \text{ and } |p(\text{supp } \varphi)| < \infty\}$$

and check that the induced representation π_G stabilizes \mathcal{A}_0 . Given any norm on V , any element $\varphi \in \mathcal{A}_\infty$ induces a well-defined map

$$N_\varphi : \begin{cases} G/H \rightarrow \mathbb{R} \\ gH \mapsto \|\varphi(g)\|. \end{cases}$$

This map is indeed well-defined by definition of \mathcal{A}_∞ , and because π_H is a representation in the isometry group. If the norm on V is an ℓ^p -norm, then this map induces a norm on \mathcal{A}_0 , by the formula

$$N(\varphi) = \left(\sum_{G/H} N_\varphi(gH)^p \right)^{1/p}$$

that is an ℓ^p -norm. One then defines \mathcal{A}_p to be the ℓ^p -completion of \mathcal{A}_0 with respect to N .

Notice that for all $\varphi \in \mathcal{A}_\infty$, for all $g \in G$, the value of φ at g determines the restriction of φ on gH . Thus any system of representatives for G/H gives an identification of \mathcal{A}_0 with finitely V -valued supported functions on G/H , and \mathcal{A}_p is the inverse image of $\ell^p(G/H, V)$ under this identification. If the space V we started with is uniformly convex, then so will \mathcal{A}_p be according to the above cited result of Day [Day41]. When G is a relatively hyperbolic group, and H is a peripheral subgroup that acts properly on an ℓ^p -space, we will use this induced representation to produce a action of G on an ℓ^p -space that is proper in a sense of angles. To do that, we will need to produce a cocycle of G for this induced representation.

4.2. The example of a free product

In this subsection we are given a group H , a unitary representation (π_H, V) of H on a space V , and a cocycle $c : H \rightarrow V$. We consider $G = K * H$ for an arbitrary group K . We want to produce a cocycle C for G in the space \mathcal{A}_0 for the induced representation (hence an affine isometric action of G on \mathcal{A}_0). This is a well-known construction but we present it here in the way needed for its generalization to relatively hyperbolic groups.

We consider $\widetilde{G/H} \subseteq G$, the coset representatives of H in G given by normal forms in the free product. More precisely, any element $g \in G$ is uniquely of the form $g = k_1 h_1 \dots k_n h_n$ with $k_i \in K$, $h_i \in H$ and $k_2, \dots, k_n, h_1, \dots, h_{n-1} \neq e$. For a coset gH , we write $g = k_1 h_1 \dots k_n h_n$, and we choose the representative $\tilde{g} = k_1 h_1 \dots k_n$. Note that if $g' \in gH$, then $\tilde{g}' = \tilde{g}$, as it should be. One can picture that \tilde{g} corresponds to the projection of 1 on the coset gH for any word metric adapted to the free product. We define, for each $\gamma \in G$ the vector $C_\gamma \in \mathcal{A}_0$ as follows

$$C_\gamma : \begin{cases} G \rightarrow V \\ g \mapsto c(g^{-1}\tilde{g}) - c(g^{-1}\gamma(\widetilde{\gamma^{-1}g})). \end{cases}$$

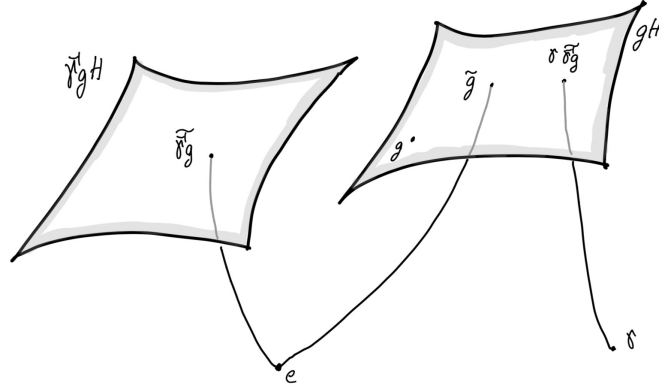


Figure 4.1. Cosets seen from 1 and from γ

Let us first check that for all γ , C_γ is in \mathcal{A}_∞ . Let us compute $C_\gamma(gh)$ for arbitrary $g \in G, h \in H$. In the following computation, the first line holds because \widetilde{gh} and \tilde{g} are equal since they are both the representative of the coset gH , and the second line holds by the cocycle relation for c .

$$\begin{aligned}
 C_\gamma(gh) &= c(h^{-1}g^{-1}(\widetilde{gh})) - c(h^{-1}g^{-1}\gamma(\widetilde{\gamma^{-1}gh})) \\
 &= c(h^{-1}g^{-1}(\tilde{g})) - c(h^{-1}g^{-1}\gamma(\widetilde{\gamma^{-1}g})) \\
 &= c(h^{-1}) + \pi_H(h^{-1})c(g^{-1}(\tilde{g})) - c(h^{-1}) - \pi_H(h^{-1})c(g^{-1}\gamma(\widetilde{\gamma^{-1}g})) \\
 &= \pi_H(h^{-1})C_\gamma(g).
 \end{aligned}$$

Therefore, for all γ , C_γ is in \mathcal{A}_∞ .

Geometrically, for any fixed g , this records the difference (seen through the cocycle c) of the projection of 1 and the projection of γ , on the coset gH (see Figure 4.2 for an illustration).

Since we are in a free product, only finitely many such differences are non-zero, and hence the map $\{\gamma \mapsto C_\gamma\}$ takes its values in \mathcal{A}_0 .

Now we check that C is a cocycle of G for the induced representation. Let us introduce $d : G \rightarrow V$ by $d(g) = c(g^{-1}\tilde{g})$. Then, for $\gamma \in G$, the map $\pi_G(\gamma)d : G \rightarrow V$ is, by definition of the representation, so that $(\pi_G(\gamma)d)(g) = d(\gamma^{-1}g)$ for all $g \in G$. This means that $(\pi_G(\gamma)d)(g) = c(g^{-1}\gamma(\widetilde{\gamma^{-1}g}))$, and we see that $C_\gamma = d - \pi_G(\gamma)d$ for all γ . The representation π_G is a homomorphism, therefore, this immediately implies the cocycle relation $C_{\gamma_1\gamma_2} = C_{\gamma_1} + \pi_G(\gamma_1)C_{\gamma_2}$. Actually, one can see C as the coboundary of d in the space \mathcal{A}_∞ , however, it is no longer a coboundary in \mathcal{A}_0 , but the cocycle relation still holds.

Notice that the subspace \mathcal{A}_H of \mathcal{A}_0 consisting of functions φ whose support is contained in H , is isomorphic with V , and on this subspace π_G and C coincide with π_H and c . We thus see that one can extend any affine isometric action of H on an

ℓ^p -space to an action of $K * H$ on an ℓ^p -space, such that the restrictions to each conjugate of H are all different and all isomorphic to the initial action of H .

4.3. Random representatives

When G is a relatively hyperbolic group, and H is a peripheral subgroup, we don't have in general sufficiently stable coset representatives for H to argue like in the free product case. However, as we will see, the flow of the previous section allows to define, for each coset, a probability measure whose support is in the coset, which we think of as a random representative of the coset, that enjoy an ℓ^p version of the stability of the canonical normal forms of free products. We develop here the needed vocabulary on random representatives.

DEFINITION 4.3. — Given a discrete group G and a subgroup $H < G$, a random set of representatives for G/H is a section

$$\begin{aligned} \nu : G/H &\rightarrow \text{Proba}(G) \\ gH &\mapsto \nu^{gH} \end{aligned}$$

for the map $p : \text{Proba}(G) \rightarrow \mathcal{P}(G/H)$ that assigns to a probability measure on G , the projection of its support in G/H . We will say that random set of representatives ν is finite if all the measures ν^{gH} have finite support, and uniformly finite if all those supports have cardinality uniformly bounded.

Notice that a set of representatives in the usual sense is a section whose image consists of Dirac masses and that for any coset $gH \in G/H$, the probability measure ν^{gH} is supported on the coset $gH \subseteq G$.

DEFINITION 4.4. — For a group G and a subgroup H , a random set of representatives ν will be almost G -invariant if for any $\gamma \in G$, then

$$\nu^{gH} = \gamma \nu^{\gamma^{-1}gH}$$

except for finitely many cosets $gH \in G/H$ at most (see Figure 4.4 for an illustration).

Remarks 4.5. —

- (1) For a group G and a subgroup H , a random set of representatives ν will in general not be G -invariant. Indeed, G -invariance means that for any $\gamma \in G$ and any coset $gH \in G/H$, any $x \in gH$ one has

$$\nu^{gH}(x) = \nu^{\gamma^{-1}gH}(\gamma^{-1}x)$$

so in particular for $\gamma \in H$ and the coset $gH = H$, for any $x \in H$ we have $\nu^H(x) = \nu^H(\gamma^{-1}x)$, meaning that the measure ν^H is an H -invariant probability measure on H , forcing H to be finite and ν^H to be the uniform measure on H .

- (2) If $G = K * H$ is a free product, then the canonical set of representatives of H -cosets, given by the normal form (and used in 4.2) gives a random set of representatives ν that assigns to each coset, the Dirac mass of the canonical representative. This is, as it should be, an almost G -invariant random set of

representatives. Indeed, for any $\gamma \in G$ and any coset gH , if we assume that g is the canonical representative, then $\gamma^{-1}g$ is the canonical representative of $\gamma^{-1}gH$ unless there are cancellations on the normal forms leaving an element of H in the end position. Those cosets gH are vertices on the geodesic in the Bass–Serre tree between e (the base edge of the tree) and γe , so there are only finitely many of those.

- (3) If G is a group obtained by a small cancellation $C'(\lambda)$ quotient over a free product $K * H$ (for $\lambda \ll 1$), consider a generating set consisting of generators of H and of K , and the family (R_i) of relators satisfying the small cancellation condition. For each left coset gH of H we consider a geodesic from 1 to gH , and denote by $g_0 \in gH$ its end point. We distinguish two cases. If the geodesic has a final subsegment labelled by a word σ appearing in a relation R_i or R_i^{-1} , of length at least $2\lambda|R_i|$, then write, up to cyclic permutation $R_i = \sigma h R'_i$ (or $R_i^{-1} = \sigma h R'_i$) where $h \in H$ and the first letter of R'_i is in K , and we declare that the random representative of gH consists of two Dirac masses of weight $\frac{1}{2}$ on the elements g_0 which is the end point of the geodesic, and $g_0 h$. If there is no such segment σ on our chosen geodesic from 1 to gH , then the random representative of gH is g_0 with probability 1. One can prove, using standard small cancellation and hyperbolicity argument (like in [RS95, Appendix]), that in the later case, all geodesics from 1 to gH (and more generally all geodesics from any point to gH , provided they fellow travel $[1, gH]$ for a sufficient length) must enter gH at g_0 . In the former case, by similar arguments, these geodesics must enter gH either on g_0 or on $g_0 h$. For larger λ , up to $\lambda < 1/6$, this is a consequence of [GS18, Prop. 3.6, and Rem. 3.8].

DEFINITION 4.6. — *For a group G and a subgroup H , a random set of representatives ν will be ℓ^p -almost G -invariant if for any $\gamma \in G$, the map*

$$gH \mapsto \|\nu^{gH} - \gamma\nu^{\gamma^{-1}gH}\|_1$$

belongs to $\ell^p(G/H)$.

4.4. Cocycle from random representatives

In this subsection we will see how, given a random set of representatives for a subgroup H in a larger group G , we can construct a cocycle for the induced representation, starting from a cocycle on H . In case where the random set of representatives is ℓ^p -almost G -invariant, we will see that the induced cocycle is actually on the ℓ^p induced representation.

LEMMA 4.7. — *Let G be a group and let H be a subgroup of G , with a unitary representation (π_H, V) of H and $c : H \rightarrow V$ a cocycle for (π_H, V) . Then for any finite random set of representatives ν , the map defined on G by $C : \gamma \mapsto C_\gamma$ through the formula*

$$C_\gamma(g) = \sum_{x \in gH} \nu^{gH}(x)c(g^{-1}x) - \sum_{y \in gH} \nu^{\gamma^{-1}gH}(\gamma^{-1}y)c(g^{-1}y)$$

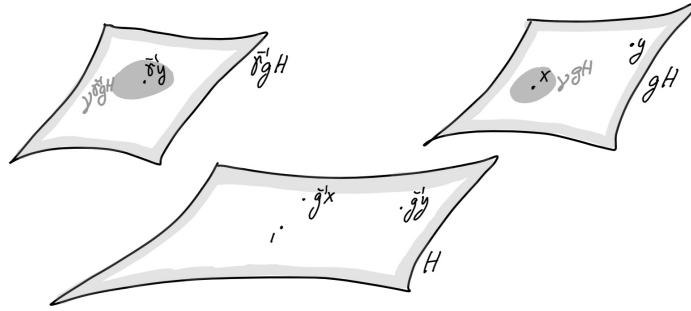


Figure 4.2. Probability measures for coset representatives

is a cocycle on \mathcal{A}_∞ for the induced representation.

Proof. — Since we assumed the random set of representatives to be finite, $C_\gamma(g)$ is well-defined by a finite sum. We first check that $C_\gamma(gh) = \pi_H(h^{-1})C_\gamma(g)$. This is a straightforward computation using first that $gH = ghH$ (hence $\gamma^{-1}gH = \gamma^{-1}ghH$), then the cocycle condition on c . Indeed, one checks

$$\begin{aligned} C_\gamma(gh) &= \sum_{x \in ghH} \nu^{gH}(x)c(h^{-1}g^{-1}x) - \sum_{y \in ghH} \nu^{\gamma^{-1}gH}(\gamma^{-1}y)c(h^{-1}g^{-1}y) \\ &= \sum_{x \in gH} \nu^{gH}(x)(c(h^{-1}) + \pi_H(h^{-1})c(g^{-1}x)) \\ &\quad - \sum_{y \in gH} \nu^{\gamma^{-1}gH}(\gamma^{-1}y)(c(h^{-1}) + \pi_H(h^{-1})c(g^{-1}y)) \\ &= \sum_{x \in gH} \nu^{gH}(x)\pi_H(h^{-1})c(g^{-1}x) - \sum_{y \in gH} \nu^{\gamma^{-1}gH}(\gamma^{-1}y)\pi_H(h^{-1})c(g^{-1}y) \\ &= \pi_H(h^{-1})C_\gamma(g) \end{aligned}$$

Now let us check the cocycle condition, namely that $C_{\gamma_1\gamma_2} = C_{\gamma_1} + \pi_G(\gamma_1)C_{\gamma_2}$. We define

$$d : G \rightarrow V, g \mapsto d(g) = \sum_{x \in gH} \nu^{gH}(x)c(g^{-1}x),$$

and we prove that $C_\gamma = d - \pi_G(\gamma)d$ (which immediately implies the cocycle relation). Let us compute $\pi_G(\gamma)d$. By definition of π_G , one has $\pi_G(\gamma)d(g) = d(\gamma^{-1}g)$, which means $\pi_G(\gamma)d(g) = \sum_{x \in \gamma^{-1}gH} \nu^{\gamma^{-1}gH}(x)c(g^{-1}\gamma x)$. Using the change of variable $y = \gamma x$, one gets $\pi_G(\gamma)d(g) = \sum_{y \in gH} \nu^{\gamma^{-1}gH}(\gamma^{-1}y)c(g^{-1}y)$. Thus, $C_\gamma = d - \pi_G(\gamma)d$ and C is a cocycle. \square

Remark 4.8. — If the random set of representatives ν is uniformly finite and almost G -invariant, then the cocycle defined in Lemma 4.7 is a cocycle on \mathcal{A}_0 and on \mathcal{A}_p for any $p \geq 1$. Indeed, let us check that for any $\gamma \in G$ then $C_\gamma \in \mathcal{A}_0$. Namely, we have to check that the map $\{gH \mapsto \|C_\gamma(g)\|\}$ has finite support for any fixed γ .

Except for the finitely many cosets where $\nu^{gH} \neq \gamma\nu^{\gamma^{-1}gH}$, we have that for $y \in gH$

$$\nu^{\gamma^{-1}gH}(\gamma^{-1}y)c(g^{-1}y) = \nu^{gH}(y)c(g^{-1}y)$$

so we get that $C_\gamma(g) = 0$. It is in general too optimistic to ask for almost G -invariance, this happens for instance in the case where $G = H * K$ or G is hyperbolic relative to H and has a small cancellation property.

LEMMA 4.9. — *Let G be a group and let H be a subgroup of G , with a unitary representation (π, V) of H and $c : H \rightarrow V$ a cocycle for (π, V) . Then for any uniformly finite ℓ^p -almost G -invariant random set of representatives ν , the map defined on G by $C : \gamma \mapsto C_\gamma$ through the formula*

$$C_\gamma(g) = \sum_{x \in gH} \nu^{gH}(x)c(g^{-1}x) - \sum_{y \in gH} \nu^{\gamma^{-1}gH}(\gamma^{-1}y)c(g^{-1}y)$$

is a cocycle on \mathcal{A}_p for the induced representation.

Proof. — We already know from Lemma 4.7 that it is a cocycle on \mathcal{A}_∞ , so it remains only to check that for any $\gamma \in G$ then $C_\gamma \in \mathcal{A}_p$. Namely, we have to check that the map $\{gH \mapsto \|C_\gamma(g)\|\}$ is p -summable for any fixed γ . Because ν is ℓ^p -almost G -invariant, there are only finitely many cosets where $\text{supp}(\nu^{gH}) \cap \text{supp}(\gamma\nu^{\gamma^{-1}gH}) = \emptyset$. For the other cosets, we decompose $\nu^{gH} = \eta + \nu_1^0$ and $\nu^{\gamma^{-1}gH} = \eta + \nu_\gamma^0$ for $\eta = \mathcal{M}\{\nu^{gH}, \gamma\nu^{\gamma^{-1}gH}\}$, so that

$$\begin{aligned} C_\gamma(g) &= \sum_{x \in gH} (\eta(x) + \nu_1^0(x))c(g^{-1}x) - \sum_{y \in gH} (\eta(y) + \nu_\gamma^0(y))c(g^{-1}y) \\ &= \sum_{x \in gH} \nu_1^0(x)c(g^{-1}x) - \sum_{y \in gH} \nu_\gamma^0(y)c(g^{-1}y) \end{aligned}$$

Since $C_\gamma \in \mathcal{A}_\infty$, without loss of generality we can assume that $g \in \text{supp}(\nu^{gH}) \cap \text{supp}(\gamma\nu^{\gamma^{-1}gH}) \neq \emptyset$, so that

$$\max\{\|c(g^{-1}x)\| \mid x \in \text{supp}(\nu^{gH} \cup \gamma\nu^{\gamma^{-1}gH})\} \leq \max\{\|c(z)\| \mid z \in B(e, 2R) \subseteq H\} = M,$$

where R is the uniform bound on the supports of ν and hence M does not depend on g or γ . Since both ν_1^0 and ν_γ^0 are bounded by $\|\nu^{gH} - \gamma\nu^{\gamma^{-1}gH}\|_1$, we deduce that $\|C_\gamma(g)\| \leq 2M\|\nu^{gH} - \gamma\nu^{\gamma^{-1}gH}\|_1$, which allows us to conclude. \square

4.4.1. Contribution and H -properness

We define now the contribution of a coset of H in the value at γ of the cocycle C . Given two coset representatives, x for gH and x' for $\gamma^{-1}gH$, the contribution (for these coset representatives) of gH in C_γ will be the word distance in H between $g^{-1}x$ and $g^{-1}\gamma x'$. It may help to think of these x and $\gamma x'$ as projections on gH of 1 and of γ . In the context of random coset representatives, we will sum over the probability measures.

DEFINITION 4.10. — *Let G be a group, H be a subgroup of G and let ν be a random set of representatives for H in G . Consider a word metric d_H on H . For each*

$\gamma \in G$, let us define the (d_H) -contribution for γ with respect to the random set of representatives ν , to be

$$\max_{gH \in G/H} \left(\sum_{(x,y) \in (gH)^2} (\nu^{gH}(x)\nu^{\gamma^{-1}gH}(\gamma^{-1}y)) d_H(g^{-1}x, g^{-1}y) \right) \quad (\text{Contrib})$$

Observe that in each term of the maximum, the only x and y that can be of interest, are those respectively in the support of ν^{gH} and in the γ -translate of the support of $\nu^{\gamma^{-1}gH}$. We say that a coset gH is involved in the contribution if it realises the maximum in the formula.

Remark 4.11. — Since we will be interested in whether or not the contribution tends to infinity (for a sequence of elements), the choice of d_H among possible word metrics in H is often irrelevant, and we simply talk about the contribution without precisising d_H .

Remark 4.12. — If our random set of representatives consists of Dirac masses at \tilde{g} (for the coset gH), as it is the case for the free products, then the (d_H) -contribution is

$$\max_{G/H} d_H(\tilde{g}, \gamma(\widetilde{\gamma^{-1}g})).$$

In the case of a free product, this is the maximum of the word lengths of the elements h_i of the normal form of γ .

DEFINITION 4.13. — We say that the cocycle C is H -proper if $\|C_\gamma\|$ goes to infinity with the (d_H) -contribution of γ .

Using the geometric idea of projection, it means that if there is a coset of H on which the projection of γ is far from the projection of 1, then C_γ is a large vector. We can now conclude with the main result for this section

PROPOSITION 4.14. — Let G be a group and let H be a subgroup of G , and a unitary representation (π, V) of H and $c : H \rightarrow V$ a proper cocycle for (π, V) . Then for any finite ℓ^p -almost G -invariant random set of representatives ν whose supports are uniformly bounded, the \mathcal{A}_p cocycle for the induced representation defined in Lemma 4.9 is H -proper.

Proof. — Let D be a bound on the diameter of the supports of ν^{gH} . Let $D_c \geq 1$ be a bound on $\|c(h)\|$ for $h \in H$ such that $d_H(1, h) \leq 2D$.

By properness of c , we can choose $R > 2D$ such that $\|c(h)\| > 5D_c$ for each h outside the ball of radius R in (H, d_H) .

Because ν^{gH} is a probability, we can rewrite the sum defining $C_\gamma(g)$ (in Lemma 4.9) as

$$C_\gamma(g) = \sum_{(x,y) \in (gH)^2} (\nu^{gH}(x)\nu^{\gamma^{-1}gH}(\gamma^{-1}y)) (c(g^{-1}x) - c(g^{-1}y))$$

which, by cocycle identity for c gives

$$C_\gamma(g) = \sum_{(x,y) \in (gH)^2} (\nu^{gH}(x)\nu^{\gamma^{-1}gH}(\gamma^{-1}y)) (\pi(g^{-1}y)c(y^{-1}x)).$$

If (x, y) and (x', y') are in $(gH)^2$, then by the cocycle relation we have

$$\pi(g^{-1}y)c(y^{-1}x) = \pi(g^{-1}y)c(y^{-1}y') + \pi(g^{-1}y')c(y'^{-1}x') + \pi(g^{-1}x')c(x'^{-1}x).$$

Let g such that gH is involved in the contribution for γ . We want to bound from below the norm of C_γ in terms of the contribution for γ . Thus, we may assume that the diameter of the union of the supports of ν^{gH} and $\gamma\nu^{\gamma^{-1}gH}$ is larger than R (otherwise this contribution is less than R). Take x_0, y_0 in those supports at maximal distance from each other (hence larger than R). The contribution of gH for γ is at least $d_H(g^{-1}x_0, g^{-1}y_0) - 2D = d_H(1, y_0^{-1}x_0) - 2D$, and at most $d_H(1, y_0^{-1}x_0) + 2D$. Computing $C_\gamma(g)$ we get

$$\begin{aligned} C_\gamma(g) &= \sum_{(x,y) \in (gH)^2} \left(\nu^{gH}(x)\nu^{\gamma^{-1}gH}(\gamma^{-1}y) \right) \\ &\quad \times \left(\pi(g^{-1}y)c(y^{-1}y_0) + \pi(g^{-1}y_0)c(y_0^{-1}x_0) + \pi(g^{-1}x_0)c(x_0^{-1}x) \right). \end{aligned}$$

Hence we can rewrite

$$\begin{aligned} C_\gamma(g) &= \pi(g^{-1}y_0)c(y_0^{-1}x_0) + \sum_{(x,y) \in (gH)^2} \left(\nu^{gH}(x)\nu^{\gamma^{-1}gH}(\gamma^{-1}y) \right) \left(\pi(g^{-1}y)c(y^{-1}y_0) \right) \\ &\quad + \sum_{(x,y) \in (gH)^2} \left(\nu^{gH}(x)\nu^{\gamma^{-1}gH}(\gamma^{-1}y) \right) \left(\pi(g^{-1}x_0)c(x_0^{-1}x) \right). \end{aligned}$$

The first term $\pi(g^{-1}y_0)c(y_0^{-1}x_0)$ has norm at least $5D_c$ by assumption on R , and the result of each sum has norm at most D_c by triangular inequality (recall that the sum of the coefficients is 1) and assumption on D_c . Therefore, in this case, $\|C_\gamma(g)\| \geq \|c(y_0^{-1}x_0)\| - 2D_c \geq 3D_c$.

Recall that the contribution for γ is between $d_H(1, y_0^{-1}x_0) - 2D$ and $d_H(1, y_0^{-1}x_0) + 2D$. By properness of c , the quantity $\|c(y_0^{-1}x_0)\|$ goes to infinity with the contribution for γ , and therefore so does $\|C_\gamma(g)\|$. \square

4.5. Peripherally proper actions

The flow of the previous section provides, in relatively hyperbolic groups, random coset representatives for the peripheral subgroups.

PROPOSITION 4.15. — *Let G be a relatively hyperbolic group and let H_1, \dots, H_k be the peripheral subgroups, and let X be the coned-off Cayley graph of G with respect to H_1, \dots, H_k . Then, for each $i = 1, \dots, n$ the map*

$$\nu : \begin{cases} G/H_i \rightarrow \text{Proba}(G) \\ gH_i \mapsto \mu_1(\widehat{gH_i}) \end{cases}$$

is an ℓ^p -almost G -invariant random set of representatives for H_i , where $\mu_1(\widehat{gH_i})$ is the mask of the infinite valence vertex $\widehat{gH_i}$ from the identity 1, as in Definition 2.3.

If moreover H_i acts properly by affine isometries on an ℓ^p -space, then G acts by affine isometries on an ℓ^p -space, and this action is H_i -proper.

Proof. — That $\nu^{gH_i} = \mu_1(\widehat{gH_i})$ is a probability measure supported on the coset gH_i is the first assertion of Proposition 2.4, hence ν is a random set of representatives for H_i . That this set is ℓ^p -almost G -invariant is the content of Corollary 2.13. Indeed, $\mu_x(\widehat{gH_i}) = x\mu_1(\widehat{x^{-1}gH_i}) = x\nu^{x^{-1}gH_i}$ by Proposition 2.5 (since the flow step is equivariant), hence $\|\mu_1(\widehat{gH_i})\Delta\mu_x(\widehat{gH_i})\| = \|\nu^{gH_i} - x\nu^{x^{-1}gH_i}\|_1$. By Corollary 2.13, this is ℓ^p -summable over the set of left cosets of H .

Now, we can apply Proposition 4.14, and obtain an action of G by affine isometries on an ℓ^p -space that is H_i -proper. Since angles at a vertex $\widehat{gH_i}$ are bounded above by the word-length in H_i , we obtain the stated condition from the definition of H_i -properness. \square

5. Proper actions of relatively hyperbolic groups

To finish the proof of our main result, Theorem A, we start by a well-known remark.

Remark 5.1. — If a group acts properly by affine isometries on an ℓ^p -space, say $\ell^p(X)$ for X a discrete set, then for all $q \geq p$, it also acts properly on $\ell^q(X)$. Indeed, Banach–Lamperti’s theorem says that for $p \neq 2$ the linear part comes from an action of G on X , so yields a linear representation on $\ell^q(X)$ for any $q \geq 1$. Then we have that $\ell^p(X) \subseteq \ell^q(X)$ and hence the cocycle on ℓ^p can be used as is on ℓ^q . Indeed, for any finite sequence (a_i) then

$$\left(\sum_i a_i^q\right)^{\frac{1}{q}} = \left(\left(\sum_i (a_i^p)^{\frac{q}{p}}\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} \leq \left(\sum_i a_i^p\right)^{\frac{1}{p}}$$

(the inequality being Minkowski inequality for $s = q/p \geq 1$) and hence on a discrete set X we have that $\|\cdot\|_q \leq \|\cdot\|_p$, so that $\ell^p(X) \subseteq \ell^q(X)$.

Proof of Theorem A. — The first part of the theorem is the content of Theorem 3.2. Let H_1, \dots, H_k be the peripheral subgroups of G and assume that there are uniformly convex Banach spaces B_i on which H_i acts properly by affine isometries. We use, for each $i = 1, \dots, n$, the action obtained in Proposition 4.15 and denote by \mathcal{V}_i the uniformly convex Banach space obtained. We also use Theorem 3.2, and denote by \mathcal{V}_0 the space obtained. Let $\mathcal{W} = \bigoplus_{i=0}^k \mathcal{V}_i$ be a uniformly convex product of the uniformly convex Banach spaces \mathcal{V}_i , for $i = 0, \dots, k$. The action of G , coordinates by coordinates, is by affine isometries and it remains to check that it is proper. This amounts to checking that the action is proper in one of the coordinates at least. Let g_n be a sequence in G going to infinity. According to Proposition 1.6, then either g_n goes to infinity in the coned-off graph as well, or it remains bounded, and $[1, g_n]$ gets arbitrary large angles at vertices of infinite valence. In the first case $g_n(\vec{0})$ goes to infinity in the \mathcal{V}_0 coordinate. In the second case, $[1, g_n]$ has an arbitrarily large angle at some vertex fixed by conjugates of $H_{i(n)}$. Extracting so that $i(n)$ is constant (which can be done in a way that partitions the sequence) by Proposition 4.15 $g_n(\vec{0})$ goes to infinity in the corresponding coordinate \mathcal{V}_i . In all cases, $g_n(\vec{0})$ goes to infinity in \mathcal{W} .

If the peripheral subgroups act properly by affine isometries on some ℓ^p , taking the maximum over all p 's on which the peripheral subgroups act, and the p obtained in Theorem 3.2, we obtain p large enough and (according to Remark 5.1) a proper action on the ℓ^p -combination of those ℓ^p -spaces. \square

The following statement has an unfortunate technical assumption of compatibility between the CAT(0) cube complex and the system of random coset representatives, through the notion of contribution (see Section 4.4.1). The proof is completely similar and in some sense a particular case of the proof of Theorem A.

PROPOSITION 5.2. — *Let G be a group acting on a CAT(0) cubical complex X , with finitely many orbits, and finite edge stabilizers. Assume that the stabilizer of any point has the Haagerup property (respectively, acts properly on an ℓ^p -space) and admits a random system of coset representatives that is ℓ^2 -almost G -invariant. Assume also that for any sequence (g_n) of elements of G going to infinity in G , either for $x \in X$, $g_n x$, or the contribution for g_n of cosets of stabilizers of vertices of X goes to infinity. Then G has the Haagerup property as well (respectively, G acts properly on an ℓ^p -space).*

Proof. — If we assume the existence of ℓ^2 -almost G -invariant random coset representatives we obtain, according to Proposition 4.15 for each $i = 1, \dots, k$, an action on a Hilbert space (respectively, an ℓ^p -space) denoted \mathcal{V}_i . We also use Niblo–Reeves' construction [NR97] to obtain a Hilbert space, denoted by \mathcal{V}_0 , from the action on a CAT(0) cube complex (and hence also on an ℓ^p -space according to Remark 5.1). Let $\mathcal{W} = \bigoplus_{i=0}^k \mathcal{V}_i$ be the ℓ^2 -sum of the Hilbert spaces \mathcal{V}_i , for $i = 0, \dots, k$, this is again a Hilbert space (respectively, the ℓ^p -sum of the ℓ^p -spaces \mathcal{V}_i , which is again an ℓ^p -space). The action of G , coordinates by coordinates, is by affine isometries and it remains to check that it is proper. This amounts to checking that the action is proper in one of the coordinates at least. Let g_n be a sequence in G going to infinity. Assume first that there is a point x in the CAT(0) cube complex (denoted by X) so that $d(x, g_n(x))$ goes to infinity. Then $g_n(\vec{0})$ goes to infinity in the \mathcal{V}_0 coordinate. Assume now the other case, that $d(x, g_n(x))$ remains bounded for all points x . Then by assumption, the contribution of some coset of vertex stabilizer goes to infinity. \square

Proof of Corollary B. — According to Remarks 4.5(3), in finitely presented small cancellation groups over a free product, the images of the factor groups admit an almost G -invariant random system of coset representatives. Recall also (see [MS17]) that any small cancellation group over a free product acts on a cubical CAT(0) complex X with finitely many orbit, each of which has trivial stabilizer, and whose stabilizer of vertices are images of the free factors (and their conjugates). It is obtained, as in [Wis04], by means of a wallspace structure (or hypergraph system) of an associated 2-dimensional polygonal complex. Then G is hyperbolic relative to these stabilizers of vertices (see for instance [Pan99]), and by [Bow12, Prop 4.13], the 1-skeleton of X is equivariantly quasi-isometric to a coned-off Cayley graph for G over these subgroups. If, for a sequence g_n , and a point $x \in X$, the sequence $g_n x$ is bounded, then the sequence g_n is bounded in the coned-off Cayley graph. If it goes to infinity in the group, then the contribution for it of cosets of stabilizers of vertices goes to infinity. Proposition 5.2 hence applies. \square

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