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CAUCHY PROBLEM FOR THE NONLINEAR SCHRÖDINGER EQUATION COUPLED WITH THE MAXWELL EQUATION

PROBLÈME DE CAUCHY POUR UNE ÉQUATION DE SCHRÖDINGER NONLINÉAIRE COUPLÉE AVEC LES ÉQUATIONS DE MAXWELL

Abstract. — In this paper, we study the nonlinear Schrödinger equation coupled with the Maxwell equation. Using energy methods, we obtain a local existence result for the Cauchy problem.

Résumé. — Dans cet article, nous nous intéressons au couplage entre une équation de Schrödinger nonlinéaire et les équations de Maxwell. En utilisant des méthodes d'énergie, nous montrons que le problème de Cauchy est localement bien posé.

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1. Introduction

In this paper, we consider the following nonlinear Schrödinger equation coupled with Maxwell equation stated in $\mathbb{R}_+ \times \mathbb{R}^3$:

$$i\psi_t + \Delta\psi = e\phi\psi + e^2|\mathbf{A}|^2\psi + 2ie\nabla\psi \cdot \mathbf{A} + ie\psi \operatorname{div} \mathbf{A} - g(|\psi|^2)\psi, \tag{1.1}$$

$$\mathbf{A}_{tt} - \Delta\mathbf{A} = e \operatorname{Im}(\bar{\psi}\nabla\psi) - e^2|\psi|^2\mathbf{A} - \nabla\phi_t - \nabla \operatorname{div} \mathbf{A}, \tag{1.2}$$

$$-\Delta\phi = \frac{e}{2}|\psi|^2 + \operatorname{div} \mathbf{A}_t, \tag{1.3}$$

where $\psi : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{C}$, $\mathbf{A} : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^3$, $\phi : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$, $e \in \mathbb{R}$ and i denotes the unit complex number, that is, $i^2 = -1$. In this setting, ψ is an electrically charged field and $(\phi, \mathbf{A})$ represents a gauge potential of an electromagnetic field. System (1.1)–(1.3) describes the interaction of this Schrödinger wave function ψ with the Maxwell gauge potential. The constant e represents the strength of the interaction. For more details and physical backgrounds, we refer to [Fel98].

Since we are interested in the Cauchy Problem, let us consider the following set of initial data:

$$\psi(0, x) = \psi_{(0)}(x), \quad \mathbf{A}(0, x) = \mathbf{A}_{(0)}(x), \quad \mathbf{A}_t(0, x) = \mathbf{A}_{(1)}(x), \tag{1.4}$$

where the regularity of each functions is given in Theorem 1.1. It is known that System (1.1)–(1.3) has a so-called *gauge ambiguity*. Namely if $(\psi, \mathbf{A}, \phi)$ is a solution of (1.1)–(1.3), then $(\exp(ie\chi)\psi, \mathbf{A} + \nabla\chi, \phi - \chi_t)$ is also a solution of (1.1)–(1.3) for any smooth function $\chi : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$. To push out this ambiguity, we adopt in the sequel the Coulomb gauge:

$$\operatorname{div} \mathbf{A} = 0, \tag{1.5}$$

which is propagated by the set of Equations (1.1)–(1.3). Indeed, if initially

$$\operatorname{div} \mathbf{A}(0, \cdot\,) = \operatorname{div} \mathbf{A}_t(0, \cdot\,) = 0,$$

then (1.5) holds for all $t > 0$. (See e.g. [CW16] for the proof.) In this setting, the last Equation (1.3) can be solved explicitly and the solution is given by

$$\phi = \frac{e}{2}(-\Delta)^{-1}|\psi|^2,$$

which imposes that

$$\phi(0, x) = \frac{e}{2}(-\Delta)^{-1}|\psi_{(0)}(x)|^2.$$

From (1.5), we also observe that (1.1) can be written as

$$i\psi_t + L_A\psi - V(x)\psi + g(|\psi|^2)\psi = 0, \tag{1.6}$$

where V is the non-local potential: $V(x) = \frac{e^2}{2}(-\Delta)^{-1}|\psi|^2$ and L_A is the *magnetic Schrödinger operator* which is defined by $\mathbf{A} = (A_1, A_2, A_3)$ and

$$L_A\psi := \sum_{j=1}^{3} \left(\frac{\partial}{\partial x_j} - ieA_j(x) \right)^2 \psi = \Delta\psi - 2ie\nabla\psi \cdot \mathbf{A} - e^2|\mathbf{A}|^2\psi. \tag{1.7}$$

In this context, the two conserved quantities of the Schrödinger–Maxwell system are the charge Q and the energy E:

$$Q(\psi) = \int_{\mathbb{R}^3} |\psi|^2 \,\mathrm{d}x, \tag{1.8}$$

$$\begin{aligned} E(\psi, \mathbf{A}, \phi) = \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla\psi - ie\mathbf{A}\psi|^2 + |\nabla \mathbf{A}|^2 + |\partial_t \mathbf{A}|^2 \right) \mathrm{d}x \\ + \frac{e^2}{4} \int_{\mathbb{R}^3} \phi|\psi|^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} G(|\psi|^2) \,\mathrm{d}x, \end{aligned} \tag{1.9}$$

where $G(t) = \int_0^t g(s)\mathrm{d}s$. To prove that (1.8) is formally conserved, one has to multiply Equation (1.1) by $\overline{\psi}$, integrate over $\mathbb{R}^3$ and take the imaginary part of the resulting equation. In a similar way, the conservation of (1.9) can be proved by multiplying (1.1)–(1.3) by $\partial_t\overline{\psi}$, $\partial_t\mathbf{A}$ and $\partial_t\phi$ respectively. This conserved quantities play a fundamental role if one wants to investigate the stability properties of such system, which is one of our main motivations. Indeed, in a previous paper [CW17], we have showed that for small $e > 0$, System (1.1)–(1.3) admits a unique orbitally stable ground state of the form:

$$(\psi_{e,\omega}, \mathbf{A}_{e,\omega}, \phi_{e,\omega}) := \left(\exp(i\omega t) u_{e,\omega},\, \mathbf{0},\, \frac{e}{2}(-\Delta)^{-1}|u_{e,\omega}|^2 \right). \tag{1.10}$$

In order to investigate the stability of such standing waves $(\psi_{e,\omega}, \mathbf{A}_{e,\omega}, \phi_{e,\omega})$ as performed in [CW17], it is necessary to prove that the Cauchy Problem (1.1)–(1.3) is almost locally well-posed around $(\psi_{e,\omega}, \mathbf{A}_{e,\omega}, \phi_{e,\omega})$.

In a previous paper [CW16], we have proved the local existence of solutions for the nonlinear Klein–Gordon–Maxwell system in Sobolev spaces of high regularity. The method was to convert the Klein–Gordon–Maxwell system into a symmetric hyperbolic system and apply the standard energy estimate. Although our Schrödinger–Maxwell system (1.1)–(1.3) looks similar, especially Equation (1.2) is completely the same, the usual reduction tools does not lead us to a symmetric hyperbolic system, which causes the necessity of a new strategy.

Let us also introduce results concerning the solvability of the Cauchy problem related to (1.1)–(1.3). In [BT09], [NW07], the linear Schrödinger equation ($g \equiv 0$) coupled with the Maxwell equations has been studied. Using the Strichartz estimate, the authors obtained the global well-posedness in the energy space. Recently in [ADM17], it was shown, by using the Strichartz estimate obtained in [NW07], that the system (1.1)–(1.3) is locally well-posed in $H^2 \times H^{\frac{3}{2}} \times H^{\frac{1}{2}}$ and the global existence holds for finite energy weak solutions, when the nonlinear term g is *defocusing* (namely the case with $+|\psi|^{p-1}\psi$ in (1.1)). We also mention the paper [NT86], where the Cauchy problem of the Schrödinger–Maxwell system in the Lorentz gauge has been studied by using the energy method. On the other hand, a huge attention has been paid in the magnetic Schrödinger equation (1.6). Especially in [Mic08], the local well-posedness for (1.6) in the energy space has been established in the case $V \equiv 0$. However, in this situation, the magnetic potential $\mathbf{A}$ is *given* and was assumed to be C^∞, which cannot be expected a priori in our case. We also refer to [DFVV10] for the Strichartz estimate for the magnetic Schrödinger operator (1.7) in the case $\mathbf{A} \in L^2_{loc}(\mathbb{R}^3)$.

We mention that if we look for the standing wave (1.10), we are led to the following non-local elliptic problem:

$$-\Delta u+\omega u+\left(\frac{e^2}{8\pi|x|}*|u|^2\right)u=g(|u|^2)u\quad\text{in }\mathbb{R}^3, \tag{1.11}$$

which is referred as the *Schrödinger–Poisson(–Slater)* equation. The existence of ground states of (1.11) as well as their orbital stability have been widely studied (see [AP08], [BF14], [BS11], [CDSS13], [Kik07] and references therein). Finally, the orbital stability of standing waves for the magnetic Schrödinger equation (1.6) has been considered in [CE88], [GR91]. Our study on the solvability of the Cauchy problem for (1.1)–(1.3) and the result established in [CW17] enable us to generalize these previous results to the full Schrödinger–Maxwell system.

Before stating the main result of this paper, we introduce the following notations. As usual, $L^p(\mathbb{R}^3)$ denotes the usual Lebesgue space:

$$L^p(\mathbb{R}^3)=\left\{u\in\mathcal{S}'(\mathbb{R}^3)\ ;\ \|u\|_{L^p}<+\infty\right\},$$

where

$$\|u\|_{L^p}=\left(\int_{\mathbb{R}^3}|u(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\quad\text{if }1\leqslant p<+\infty$$

and

$$\|u\|_{L^\infty}=\operatorname{ess\,sup}\left\{|u(x)|\ ;\ x\in\mathbb{R}^3\right\}.$$

We define the Sobolev space $H^s(\mathbb{R}^3)$ as follows:

$$H^s(\mathbb{R}^3)=\left\{u\in\mathcal{S}'(\mathbb{R}^3)\ ;\ \|u\|^2_{H^s(\mathbb{R}^3)}=\int_{\mathbb{R}^3}(1+|\xi|^2)^s|\mathcal{F}(u)(\xi)|^2\,\mathrm{d}\xi<+\infty\right\},$$

where $\mathcal{F}(u)(\xi)$ is the Fourier transform of u. We also introduce the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^3)$ as being the completion of $C_0^\infty(\mathbb{R}^3,\mathbb{C})$ for the norm $u\to\||\xi|\mathcal{F}(u)(\xi)\|_{L^2(\mathbb{R}^3)}$. Recall that the space $\dot{H}^1(\mathbb{R}^3)$ is continuously embedded into $L^6(\mathbb{R}^3)$. Finally let $C(I,E)$ be the space of continuous functions from an interval I of $\mathbb{R}$ to a Banach space E. For $1\leqslant j\leqslant 3$, we set $\partial_{x_j}=\frac{\partial}{\partial x_j}$ and $\partial_t=\frac{\partial}{\partial t}$. For $k\in\mathbb{N}^3$, $k=(k_1,k_2,k_3)$, we denote $D^ku=\partial_{x_1}^{k_1}\partial_{x_2}^{k_2}\partial_{x_3}^{k_3}u$ and for a non-negative integer s, D^s denotes the set of all partial space derivatives of order s. Different positive constants might be denoted by the same letter C. We also denote by $\mathrm{Re}(u)$ and $\mathrm{Im}(u)$ the real part and the imaginary part of u respectively.

We assume that g satisfies

$$g\in C^{m+1}(\mathbb{R},\mathbb{R})\ \text{ and } g(0)=0, \tag{1.12}$$

for some $m\in\mathbb{N}$ with $m\geqslant 2$, so that the function $W:\mathbb{C}\to\mathbb{C}$ defined by $W(u):=g(|u|^2)u$ satisfies $W\in C^{m+1}(\mathbb{C},\mathbb{C})$, $W(0)=W'(0)=0$. Some typical examples of the nonlinear term g are the power nonlinearity $g(s)=\pm s^{\frac{p-1}{2}}$ with $[p]\geqslant 2m+3$ ($[p]$ denotes the integer part of p), or the *cubic-quintic* nonlinearity $g(s)=s-\lambda s^2$ for $\lambda>0$, which frequently appears in the study of solitons in physical literatures. (See [RV08] for example.) In this setting, we prove the following result.

Theorem 1.1. — *Let s be any integer larger than $\frac{3}{2}$ and assume that* $\psi_0 \in H^{s+2}(\mathbb{R}^3, \mathbb{C})$, $\mathbf{A}_{(0)} \in H^{s+2}(\mathbb{R}^3, \mathbb{R}^3)$, $\mathbf{A}_{(1)} \in H^{s+1}(\mathbb{R}^3, \mathbb{R}^3)$ *with* $\operatorname{div} \mathbf{A}_{(0)} = 0$, $\operatorname{div} \mathbf{A}_{(1)} = 0$ *and g satisfies* (1.12). *Then there exist $T^* > 0$ and a unique solution $(\psi, \mathbf{A}, \phi)$ to System* (1.1)–(1.3) *satisfying the initial condition* (1.4) *such that*

$$\psi \in C([0,T^*]; H^{s+2}(\mathbb{R}^3)) \cap C^1([0,T^*]; H^s(\mathbb{R}^3)),$$
$$\mathbf{A} \in C([0,T^*]; H^{s+2}(\mathbb{R}^3)) \cap C^1([0,T^*]; H^{s+1}(\mathbb{R}^3)),$$
$$\phi \in C([0,T^*]; \dot{H}^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)),\ \ \nabla\phi \in C([0,T^*]; H^{s+1}(\mathbb{R}^3)),$$
$$\phi_t \in C([0,T^*]; \dot{H}^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)),\ \ \nabla\phi_t \in C([0,T^*]; H^{s+1}(\mathbb{R}^3)).$$

The proof of Theorem 1.1 is based on energy estimates and particularly, on the strategies developed in [Col02] and [CC04]. Note also that to overcome the loss of derivatives embedded in Equation (1.2), we use the original idea of Ozawa and Tsutsumi presented in [OT92].

The paper is organized as follows. In Section 2, we transform System (1.1)–(1.3) into a system to which we can apply the usual energy method. Section 3 is devoted to the proof of Theorem 1.1.

2. Transformation of the equations

In this section, we transform the original System (1.1)–(1.3) into a new symmetric system to which we can apply an energy method. In order to overcome the loss of derivatives contained in Equations (1.1)–(1.2), we introduce the following new unknowns (see [OT92]):

$$\Psi = \partial_t \psi \quad \text{and} \quad \Phi = \partial_t \phi.$$

Let us first derive equations for Ψ and Φ. Differentiating Equation (1.1) with respect to t, one obtains

$$\begin{aligned} i\Psi_t + \Delta\Psi = e\Phi\psi + e\phi\Psi + e^2|A|^2\Psi + 2e^2\psi\mathbf{A}\cdot\mathbf{A}_t + 2ie\nabla\Psi\cdot\mathbf{A} + 2ie\nabla\psi\cdot\mathbf{A}_t \\ - g'(|\psi|^2)(|\psi|^2\Psi + \psi^2\overline{\Psi}) - g(|\psi|^2)\Psi. \end{aligned}$$

Taking advantage of the new unknown Ψ, we also transform Equation (1.1) into an elliptic version

$$i\Psi + \Delta\psi = e\phi\psi + e^2|\mathbf{A}|^2\psi + 2ie\nabla\psi\cdot\mathbf{A} - g(|\psi|^2)\psi.$$

Moreover, we derive an equation for Φ by applying ∂_t on Equation (1.3):

$$-\Delta\Phi = \frac{e}{2}\Big(\psi\overline{\Psi} + \Psi\overline{\psi}\Big).$$

Next, in order to ensure the Coulomb condition on $\mathbf{A}$ for all $t > 0$, we introduce the projection operator $\mathbb{P}$ on divergence free vector fields :

$$\begin{aligned} \mathbb{P} \ :\ \Big(L^2(\mathbb{R}^3)\Big)^3 &\longrightarrow \Big(L^2(\mathbb{R}^3)\Big)^3 \\ \mathbf{A} \quad &\longmapsto \quad \mathbb{P}\mathbf{A} = \Big(-\Delta\Big)^{-1}\operatorname{rot}\operatorname{rot}\mathbf{A}, \end{aligned}$$

so that if $\operatorname{div} \mathbf{A} = 0$, then $\mathbb{P}\mathbf{A} = \mathbf{A}$. Thus applying $\mathbb{P}$ on Equation (1.2), we derive

$$\mathbf{A}_{tt} - \Delta \mathbf{A} = \mathbb{P}\Big(e \operatorname{Im}(\bar{\psi}\nabla\psi) - e^2|\psi|^2\mathbf{A} - \nabla\Phi \Big). \tag{2.1}$$

Note that any solution to (2.1) satisfying

$$\operatorname{div} \mathbf{A}(0,\cdot\,) = 0 \quad \text{and} \quad \operatorname{div} \mathbf{A}_t(0,\cdot\,) = 0,$$

obviously satisfies

$$\operatorname{div} \mathbf{A}(t,\cdot\,) = 0 \quad \text{for all } t > 0.$$

At this step, we have transformed System (1.1)–(1.3) into

$$i\Psi + \Delta\psi = e\phi\psi + e^2|\mathbf{A}|^2\psi + 2ie\nabla\psi\cdot\mathbf{A} - g(|\psi|^2)\psi, \tag{2.2}$$

$$\begin{aligned} i\Psi_t + \Delta\Psi = {} & e\Phi\psi + e\phi\Psi + e^2|\mathbf{A}|^2\Psi + 2e^2\psi\mathbf{A}\cdot\mathbf{A}_t + 2ie\nabla\Psi\cdot\mathbf{A} \\ & + 2ie\nabla\psi\cdot\mathbf{A}_t - g'(|\psi|^2)(|\psi|^2\Psi + \psi^2\overline{\Psi}) - g(|\psi|^2)\Psi, \end{aligned} \tag{2.3}$$

$$\mathbf{A}_{tt} - \Delta \mathbf{A} = \mathbb{P}\Big(e \operatorname{Im}(\bar{\psi}\nabla\psi) - e^2|\psi|^2\mathbf{A} - \nabla\Phi \Big), \tag{2.4}$$

$$-\Delta\phi = \frac{e}{2}|\psi|^2, \tag{2.5}$$

$$-\Delta\Phi = \frac{e}{2}\Big(\psi\overline{\Psi} + \Psi\overline{\psi}\Big). \tag{2.6}$$

In order to take advantage of elliptic regularity properties, we transform Equations (2.2) by adding $-\alpha\psi$ ($\alpha > 0$ will be chosen in Lemma 3.3 below) to both sides of the equation to obtain:

$$(-\Delta + \alpha)\psi = i\Psi - e\phi\psi - e^2|\mathbf{A}|^2\psi - 2ie\nabla\psi\cdot\mathbf{A} + g(|\psi|^2)\psi + \alpha\psi. \tag{2.7}$$

For simplicity, introduce $\mathcal{U} = (\phi, \Phi)$ and rewrite Equations (2.5) and (2.6) as

$$-\Delta\mathcal{U} = F_1(\psi, \Psi), \tag{2.8}$$

where

$$F_1(\psi,\Psi) = \frac{e}{2}\begin{pmatrix} |\psi|^2 \\ \psi\overline{\Psi} + \Psi\overline{\psi} \end{pmatrix}.$$

Equation (2.3) is then transformed into

$$i\partial_t\Psi + \Delta\Psi = 2ie\nabla\Psi\cdot\mathbf{A} + 2ie\nabla\psi\cdot\mathbf{A}_t + F_2(\mathcal{U},\psi,\Psi,\mathbf{A},\mathbf{A}_t), \tag{2.9}$$

where

$$\begin{aligned} F_2(\mathcal{U},\psi,\Psi,\mathbf{A},\mathbf{A}_t) = {} & e\Phi\psi + e\phi\Psi + e^2|\mathbf{A}|^2\Psi + 2e^2\psi\mathbf{A}\cdot\mathbf{A}_t \\ & - g'(|\psi|^2)(|\psi|^2\Psi + \psi^2\overline{\Psi}) - g(|\psi|^2)\Psi. \end{aligned}$$

It is then necessary to work with $\mathbf{A}_t$ as new unknown. We recall first that $\mathbf{A} = (a_1, a_2, a_3)$. To properly write the equations on $\mathbf{A}$ and $\mathbf{A}_t$, for $j = 1,2,3$, $k = 1,2,3$

and $\ell = 1, 2, 3$, we introduce

$$\begin{aligned} p_{j,k} &= \partial_{x_k} a_j, \\ q_j &= \Delta a_j, \\ r_j &= \partial_t a_j, \\ \lambda_{j,k} &= \partial_{x_k} \Delta^{-1} \partial_t a_j, \\ \mu_{j,k,\ell} &= \partial_{x_\ell} \lambda_{j,k} = \partial_{x_\ell} \partial_{x_k} \Delta^{-1} \partial_t a_j, \\ \nu_{j,k} &= \Delta \lambda_{j,k} = \partial_{x_k} \partial_t a_j, \\ \tau_{j,k} &= \partial_t \lambda_{j,k} = \partial_{x_k} \Delta^{-1} \partial_t^2 a_j, \end{aligned}$$

and set $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ with $\mathcal{A}_j : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^{24}$ and

$$\mathcal{A}_j = {}^t(a_j, p_{j,k}, q_j, r_j, \lambda_{j,k}, \mu_{j,k,\ell}, \nu_{j,k}, \tau_{j,k}).$$

We also need to give some details on the projection operator $\mathbb{P}$. For that purpose, we introduce the Riesz transform $\mathbb{R}_j$ from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$ which is given by

$$\mathbb{R}_j = \partial_{x_j} (-\Delta)^{-\frac{1}{2}} \quad \text{for } j = 1, 2, 3.$$

Then, $\mathbb{P}$ can be rewritten as $\mathbb{P} = (\mathbb{P}_{j,m})_{1 \leqslant j,m \leqslant 3}$ where

$$\mathbb{P}_{j,m} = \delta_{j,m} + \mathbb{R}_j \mathbb{R}_m.$$

Now we compute the equations for each components of $\mathcal{A}_j$. First by the definitions of $\mathcal{A}_j$, one finds that

$$\begin{aligned} \partial_t a_j &= \Delta \Delta^{-1} \partial_t a_j = \sum_{k=1}^3 \partial_{x_k} (\partial_{x_k} \Delta^{-1} \partial_t a_j) = \sum_{k=1}^3 \partial_{x_k} \lambda_{j,k}, \\ \partial_t p_{j,k} &= \partial_t \partial_{x_k} a_j = \Delta(\partial_{x_k} \Delta^{-1} \partial_t a_j) = \Delta \lambda_{j,k} = \sum_{\ell=1}^3 \partial_{x_\ell} (\partial_{x_\ell} \lambda_{j,k}) = \sum_{\ell=1}^3 \partial_{x_\ell} \mu_{j,k,\ell}, \\ \partial_t q_j &= \partial_t \Delta a_j = \sum_{k=1}^3 \partial_{x_k} \Delta(\partial_{x_k} \Delta^{-1} \partial_t a_j) = \sum_{k=1}^3 \partial_{x_k} \Delta \lambda_{j,k} = \sum_{k=1}^3 \partial_{x_k} \nu_{j,k}, \\ \partial_t r_j &= \partial_t^2 a_j = \Delta \Delta^{-1} \partial_t^2 a_j = \sum_{k=1}^3 \partial_{x_k} \partial_t (\partial_{x_k} \Delta^{-1} \partial_t a_j) \\ &= \sum_{k=1}^3 \partial_{x_k} \partial_t \lambda_{j,k} = \sum_{k=1}^3 \partial_{x_k} \tau_{j,k}. \end{aligned}$$

Next from Equation (2.4), we have

$$\partial_t^2 a_j = \Delta a_j + \sum_{m=1}^3 \mathbb{P}_{j,m} \Big(e \operatorname{Im}(\bar{\psi} \partial_{x_m} \psi) - e^2 |\psi|^2 a_m - \partial_{x_m} \Phi \Big),$$

which provides

$$\begin{aligned}\partial_t\lambda_{j,k} &= \partial_t\partial_{x_k}\Delta^{-1}\partial_t a_j = \partial_{x_k}\Delta^{-1}\partial_t^2 a_j\\ &= \partial_{x_k}\Delta^{-1}\Big(\Delta a_j + \sum_{m=1}^{3}\mathbb{P}_{j,m}\Big(e\,\mathrm{Im}(\bar{\psi}\partial_{x_m}\psi) - e^2|\psi|^2 a_m - \partial_{x_m}\Phi\Big)\Big)\\ &= \partial_{x_k}a_j + h^1_{j,k}(\psi,\mathbf{A}),\end{aligned}$$

$$\begin{aligned}\partial_t\mu_{j,k,\ell} &= \partial_t\partial_{x_\ell}\partial_{x_k}\Delta^{-1}\partial_t a_j = \partial_{x_\ell}\partial_{x_k}\Delta^{-1}(\partial_t^2 a_j)\\ &= \partial_{x_\ell}\partial_{x_k}\Delta^{-1}\Big(\Delta a_j + \sum_{m=1}^{3}\mathbb{P}_{j,m}\Big(e\,\mathrm{Im}(\bar{\psi}\partial_{x_m}\psi) - e^2|\psi|^2 a_m - \partial_{x_m}\Phi\Big)\Big)\\ &= \partial_{x_\ell}p_{j,k} + h^2_{j,k,\ell}(\psi,\mathbf{A}),\end{aligned}$$

$$\begin{aligned}\partial_t\nu_{j,k} &= \partial_{x_k}\partial_t^2 a_j\\ &= \partial_{x_k}\Big(\Delta a_j + \sum_{m=1}^{3}\mathbb{P}_{j,m}\Big(e\,\mathrm{Im}(\bar{\psi}\partial_{x_m}\psi) - e^2|\psi|^2 a_m - \partial_{x_m}\Phi\Big)\Big)\\ &= \partial_{x_k}q_j + h^3_{j,k}(\psi,\mathbf{A}),\end{aligned}$$

where $h^1_{j,k}$, $h^2_{j,k,\ell}$, $h^3_{j,k}$ are non-local functions defined as follows:

$$\begin{aligned}h^1_{j,k}(\psi,\mathbf{A}) &= \partial_{x_k}\Delta^{-1}\sum_{m=1}^{3}\mathbb{P}_{j,m}\Big(e\mathrm{Im}(\overline{\psi}\partial_{x_m}\psi) - e^2|\psi|^2 a_m - \partial_{x_m}\Phi\Big),\\ h^2_{j,k,\ell}(\psi,\mathbf{A}) &= \partial_{x_\ell}\partial_{x_k}\Delta^{-1}\sum_{m=1}^{3}\mathbb{P}_{j,m}\Big(e\mathrm{Im}(\overline{\psi}\partial_{x_m}\psi) - e^2|\psi|^2 a_m - \partial_{x_m}\Phi\Big),\\ h^3_{j,k}(\psi,\mathbf{A}) &= \partial_{x_k}\sum_{m=1}^{3}\mathbb{P}_{j,m}\Big(e\mathrm{Im}(\overline{\psi}\partial_{x_m}\psi) - e^2|\psi|^2 a_m - \partial_{x_m}\Phi\Big).\end{aligned}$$

Finally one has

$$\begin{aligned}\partial_t\tau_{j,k} &= \partial_{x_k}\Delta^{-1}\partial_t(\partial_t^2 a_j)\\ &= \partial_{x_k}\Delta^{-1}\partial_t\Big(\Delta a_j + \sum_{m=1}^{3}\mathbb{P}_{j,m}\Big(e\,\mathrm{Im}(\bar{\psi}\partial_{x_m}\psi) - e^2|\psi|^2 a_m - \partial_{x_m}\Phi\Big)\Big).\end{aligned}$$

Computing separately each term of the right-hand side of the previous equation, we obtain

$$\begin{aligned}\partial_t(\overline{\psi}\partial_{x_m}\psi) &= \overline{\Psi}\partial_{x_m}\psi + \overline{\psi}\partial_{x_m}\Psi,\\ \partial_t(|\psi|^2 a_m) &= (\Psi\overline{\psi} + \psi\overline{\Psi})a_m + |\psi|^2 r_m.\end{aligned}$$

Moreover from (2.3) and (2.6), one finds that

$$\begin{aligned}\partial_t(\partial_{x_m}\Phi) &= \partial_t\Big(\frac{e}{2}\partial_{x_m}(-\Delta)^{-1}(\psi\overline{\Psi}+\Psi\overline{\psi})\Big)\\ &= \frac{e}{2}\partial_{x_m}(-\Delta)^{-1}(2|\Psi|^2+\psi\partial_t\overline{\Psi}+\overline{\psi}\partial_t\Psi)\\ &= e\partial_{x_m}(-\Delta)^{-1}|\Psi|^2+e\partial_{x_m}(-\Delta)^{-1}\operatorname{Im}(i\partial_t\Psi\overline{\psi})\\ &= e\partial_{x_m}(-\Delta)^{-1}\Big\{|\Psi|^2+\operatorname{Im}\Big(-\overline{\psi}\Delta\Psi+e\Phi|\psi|^2+e\phi\overline{\psi}\Psi+e^2|\mathbf{A}|^2\overline{\psi}\Psi\\ &\qquad +2e^2|\psi|^2\mathbf{A}\cdot\mathbf{A}_t+2ie\overline{\psi}\nabla\Psi\cdot\mathbf{A}+2ie\overline{\psi}\nabla\psi\cdot\mathbf{A}_t\\ &\qquad -g'(|\psi|^2)|\psi|^2(\overline{\psi}\Psi+\psi\overline{\Psi})-g(|\psi|^2)\overline{\psi}\Psi\Big)\Big\},\end{aligned}$$

from which we conclude that

$$\partial_t\tau_{j,k}=\partial_{x_k}r_j+h^4_{j,k}(\psi,\Psi,\mathbf{A},\mathbf{R}),$$

where $\mathbf{R}=(r_1,r_2,r_3)$ and

$$\begin{aligned}&h^4_{j,k}(\psi,\Psi,\mathbf{A},\mathbf{R})\\ &=\partial_{x_k}\Delta^{-1}\sum_{m=1}^{3}\mathbb{P}_{j,m}\Big(e\operatorname{Im}(\overline{\Psi}\partial_{x_m}\psi+\overline{\psi}\partial_{x_m}\Psi)-e^2(|\psi|^2r_m+(\Psi\overline{\psi}+\psi\overline{\Psi})a_m)\Big)\\ &\quad-\partial_{x_k}\Delta^{-1}\sum_{m=1}^{3}\mathbb{P}_{j,m}\Big[e\partial_{x_m}(-\Delta)^{-1}\Big\{|\Psi|^2+\operatorname{Im}\Big(-\overline{\psi}\Delta\Psi+e\Phi|\psi|^2+e\phi\overline{\psi}\Psi\\ &\qquad+e^2|\mathbf{A}|^2\overline{\psi}\Psi+2e^2|\psi|^2\mathbf{A}\cdot\mathbf{R}+2ie\overline{\psi}\nabla\Psi\cdot\mathbf{A}+2ie\overline{\psi}\nabla\psi\cdot\mathbf{R}\\ &\qquad-g'(|\psi|^2)|\psi|^2(\overline{\psi}\Psi+\psi\overline{\Psi})-g(|\psi|^2)\overline{\psi}\Psi\Big)\Big\}\Big].\end{aligned}$$

The equation on $\mathcal{A}_j$ can be written as a symmetric system of the form

$$\partial_t\mathcal{A}_j+\mathcal{M}_j(\nabla)\mathcal{A}_j+H_j(\psi,\Psi,\mathbf{A},\mathbf{R})=0\quad(j=1,2,3),$$

where $H_j={}^t(0,0,0,0,h^1_{j,k},h^2_{j,k,\ell},h^3_{j,k},h^4_{j,k})$, $\mathcal{M}_j(\nabla)=\sum_{k=1}^{3}\widetilde{M_j}\partial_{x_k}$ are 24×24 symmetric matrices. Recalling that $\mathcal{A}_j=(a_j,p_j,q_j,r_j,\lambda_j,\mu_j,\nu_j,\tau_j)$, where a_j, q_j, r_j are scalar functions, p_j, λ_j, ν_j and τ_j are functions with values in $\mathbb{R}^3$ and μ_j is a function with values in $\mathbb{R}^9$, $\mathcal{M}_j$ can be simply written by blocks:

$$\mathcal{M}_j(\nabla)=\begin{pmatrix}0&0&0&0&\nabla\cdot&0&0&0\\0&0&0&0&0&\nabla\cdot&0&0\\0&0&0&0&0&0&\nabla\cdot&0\\0&0&0&0&0&0&0&\nabla\cdot\\\nabla&0&0&0&0&0&0&0\\0&N(\nabla)&0&0&0&0&0&0\\0&0&\nabla&0&0&0&0&0\\0&0&0&\nabla&0&0&0&0\end{pmatrix},$$

with

$$N(\nabla)=\begin{pmatrix}\nabla&0&0\\0&\nabla&0\\0&0&\nabla\end{pmatrix}.$$

Note that $\widetilde{M_j}$ are 24×24 symmetric matrices whose components are all constants.

Thus from (2.7), (2.8) and (2.9), we have transformed Equations (1.1)–(1.3) into the following system:

$$-\Delta \mathcal{U} = F_1(\psi, \Psi), \tag{2.10}$$

$$(-\Delta + \alpha)\psi + 2ie\nabla\psi \cdot \mathbf{A} = i\Psi - e\phi\psi - e^2|\mathbf{A}|^2\psi + g(|\psi|^2)\psi + \alpha\psi, \tag{2.11}$$

$$i\partial_t \Psi + \Delta\Psi - 2ie\nabla\Psi \cdot \mathbf{A} = 2ie\nabla\psi \cdot \mathbf{R} + F_2(\mathcal{U}, \psi, \Psi, \mathbf{A}, \mathbf{R}), \tag{2.12}$$

$$0 = \partial_t \mathcal{A}_j + \mathcal{M}_j(\nabla)\mathcal{A}_j + H_j(\psi, \Psi, \mathbf{A}, \mathbf{R}). \tag{2.13}$$

3. Solvability of the Cauchy Problem

The aim of this section is to prove Theorem 1.1. To this end, we use a fix-point argument on a suitable version of System (2.10)–(2.13). In this procedure, the necessary estimates follow from the application of the usual energy methods.

For $s \in \mathbb{N}$ with $s > \frac{3}{2}$, take an initial data

$$\psi_{(0)} \in H^{s+2}(\mathbb{R}^3, \mathbb{C}),$$

$$\mathbf{A}_{(0)} = (a_{1(0)}, a_{2(0)}, a_{3(0)}) \in H^{s+2}(\mathbb{R}^3, \mathbb{R}^3),$$

and

$$\mathbf{A}_{(1)} = (r_{1(0)}, r_{2(0)}, r_{3(0)}) \in H^{s+1}(\mathbb{R}^3, \mathbb{R}^3),$$

satisfying

$$\operatorname{div} \mathbf{A}_{(0)} = 0, \ \operatorname{div} \mathbf{A}_{(1)} = 0.$$

Let us define $\Psi_{(0)} \in H^s(\mathbb{R}^3, \mathbb{C})$ by

$$\begin{aligned} \Psi_{(0)} &= i\Big(\Delta\psi_{(0)} - e\phi_{(0)}\psi_{(0)} - e^2|\mathbf{A}_{(0)}|^2\psi_{(0)} - 2ie\nabla\psi_{(0)} \cdot \mathbf{A}_{(0)} - g(|\psi_{(0)}|^2)\psi_{(0)}\Big), \\ \Phi_{(0)} &= \frac{e}{2}(-\Delta)^{-1}(\psi_{(0)}\overline{\Psi}_{(0)} + \Psi_{(0)}\overline{\psi}_{(0)}), \end{aligned} \tag{3.1}$$

where $\phi_{(0)} = \frac{e}{2}(-\Delta)^{-1}|\psi_{(0)}|^2$. We also put

$$\mathcal{A}_{(0)} = (\mathcal{A}_{1(0)}, \mathcal{A}_{2(0)}, \mathcal{A}_{3(0)}) \in H^s(\mathbb{R}^3),$$

for $i, j, k, l = 1, 2, 3$,

$$\begin{gathered} \mathcal{A}_{j(0)} = {}^t(a_{j(0)}, p_{j,k(0)}, q_{j(0)}, r_{j(0)}, \lambda_{j,k(0)}, \mu_{j,k,\ell(0)}, \nu_{j,k(0)}, \tau_{j,k(0)}), \\ p_{j,k(0)} = \partial_{x_k} a_{j(0)}, \ q_{j(0)} = \Delta a_{j(0)}, \ \lambda_{j,k(0)} = \partial_{x_k}\Delta^{-1} r_{j(0)}, \\ \mu_{j,k,\ell(0)} = \partial_{x_k}\partial_{x_\ell}\Delta^{-1} r_{j(0)}, \ \nu_{j,k(0)} = \partial_{x_k} r_{j(0)} \end{gathered}$$

and

$$\tau_{j,k(0)} = \partial_{x_k} a_{j(0)} + \partial_{x_k}\Delta^{-1}\sum_{m=1}^{3} \mathbb{P}_{j,m}\Big(e\operatorname{Im}(\overline{\psi}_{(0)}\partial_{x_m}\psi_{(0)}) - e^2|\psi_{(0)}|^2 a_{m(0)} - \partial_{x_m}\Phi_{(0)}\Big).$$

We introduce $R = 2\Big(\|\psi_{(0)}\|_{H^s} + \|\Psi_{(0)}\|_{H^s} + \|\mathcal{A}_{(0)}\|_{H^s}\Big)$ and let $B(R)$ be the ball of radius R in $C\Big([0, T]; (H^s(\mathbb{R}^3)^2\Big)$ for $T > 0$.

We prove the existence of a solution $(\mathcal{U}, \psi, \Psi, \mathcal{A}_j)$ of (2.10)–(2.13) by the following procedure. Take $(\Psi, \mathcal{A}) \in B(R)$ with $\operatorname{div} \mathbf{A} = 0$ arbitrarily and construct new functions $\mathcal{Q}$ and $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ as follows.

First we define $\psi \in C([0,T]; H^s(\mathbb{R}^3, \mathbb{C}))$ by

$$\psi(t,x) := \psi_{(0)}(x) + \int_0^t \Psi(s,x)\,\mathrm{d}s. \tag{3.2}$$

Then by the construction of ψ, one finds that, for T small enough,

$$\|\psi\|_{L^\infty[0,T];H^s)} \leqslant R.$$

Next let $\mathcal{U} \in C([0,T]; \dot{H}^1(\mathbb{R}^3))$ be a solution to

$$-\Delta\mathcal{U} = F_1(\psi, \Psi). \tag{3.3}$$

We note that $\mathcal{U} \in C([0,T]; L^\infty(\mathbb{R}^3))$ and $\nabla\mathcal{U} \in C([0,T]; H^{s+1}(\mathbb{R}^3))$. (See Lemma 3.2 below.) Next we introduce the solution $\chi \in C([0,T]; H^{s+2}(\mathbb{R}^3, \mathbb{C}))$ of the following elliptic equation:

$$(-\Delta + \alpha)\chi + 2ie\nabla\chi \cdot \mathbf{A} = i\Psi - e\phi\psi - e^2|\mathbf{A}|^2\psi + g(|\psi|^2)\psi + \alpha\psi. \tag{3.4}$$

We now consider a linearized version of (2.12)–(2.13). We take $(\mathcal{Q}, \mathcal{B}) \in H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3)$ solutions to

$$\begin{cases} i\partial_t\mathcal{Q} + \Delta\mathcal{Q} - 2ie\nabla\mathcal{Q} \cdot \mathbf{A} = 2ie\nabla\chi \cdot \mathbf{R} + F_2(\mathcal{U}, \chi, \Psi, \mathbf{A}, \mathbf{R}), \\ \mathcal{Q}(0,x) = \Psi_{(0)}, \end{cases} \tag{3.5}$$

$$\begin{cases} \partial_t\mathcal{B}_j + \mathcal{M}_j(\nabla)\mathcal{B}_j + H_j(\chi, \Psi, \mathbf{A}, \mathbf{R}) = 0, \\ \mathcal{B}_j(0,x) = \mathcal{A}_{j(0)}. \end{cases} \tag{3.6}$$

Let

$$\mathcal{S} \;:\; (\Psi, \mathcal{A}) \longmapsto (\mathcal{Q}, \mathcal{B}).$$

Our strategy consists in showing that $\mathcal{S}$ is a contraction mapping on $B(R)$, provided that $T > 0$ is sufficiently small and to prove that $\chi = \psi$, from which we obtain the existence of a solution $(\mathcal{U}, \psi, \Psi, \mathcal{A}_j)$ of (2.10)–(2.13) and complete the proof of Theorem 1.1.

The proof is divided into 6 steps. We first recall the following classical lemma. (See e.g. [AG91, Proposition 2.1.1, p. 98] for the proof.)

Lemma 3.1. — *Let u, $v \in L^\infty(\mathbb{R}^3) \cap H^s(\mathbb{R}^3)$ for $s \in \mathbb{N}$. Then for all $(m_1, m_2) \in \mathbb{N}^3 \times \mathbb{N}^3$ with $|m_1| + |m_2| = s$, one has*

$$\|D^{m_1}uD^{m_2}v\|_{L^2} \leqslant C\Big(\|u\|_{L^\infty}\|v\|_{H^s} + \|v\|_{L^\infty}\|u\|_{H^s}\Big).$$

Step 1: Solving the elliptic equation (3.3)

Lemma 3.2. — *There exists a unique solution $\mathcal{U} \in C([0,T]; \dot{H}^1(\mathbb{R}^3))$ of (3.3). Moreover, $\mathcal{U} = (\phi, \Phi)$ satisfies the following estimates.*

$$\|\nabla\phi\|_{L^\infty([0,T];H^{s+1})} \leqslant C_1(R), \quad \|\phi\|_{L^\infty([0,T];L^\infty)} \leqslant C_2(R), \tag{3.7}$$

$$\|\nabla\Phi\|_{L^\infty([0,T];H^{s+1})} \leqslant C_3(R), \quad \|\Phi\|_{L^\infty([0,T];L^\infty)} \leqslant C_4(R), \tag{3.8}$$

where C_1, C_2, C_3 and C_4 are positive constants depending only on R.

Proof. — First we note that the bilinear form

$$a(u,v) := \int_{\mathbb{R}^3} \nabla u \cdot \nabla v \,\mathrm{d}x$$

is continuous and elliptic on $\dot{H}^1(\mathbb{R}^3,\mathbb{R}) \times \dot{H}^1(\mathbb{R}^3,\mathbb{R})$. Moreover since $\psi \in H^s(\mathbb{R}^3)$ and $\Psi \in H^s(\mathbb{R}^3)$, a direct computation gives

$$\left\| |\psi|^2 \right\|_{L^{\frac{6}{5}}} = \|\psi\|^2_{L^{\frac{12}{5}}} \leqslant C\|\psi\|^2_{H^1},$$

$$\|\psi\overline{\Psi} + \Psi\overline{\psi}\|_{L^{\frac{6}{5}}} \leqslant C\|\psi\|_{L^3}\|\Psi\|_{L^2} \leqslant C\|\psi\|_{H^1}\|\Psi\|_{L^2}.$$

Then by the Sobolev embedding $L^{\frac{6}{5}}(\mathbb{R}^3) \hookrightarrow \left(\dot{H}^1(\mathbb{R}^3)\right)^*$ and the Lax–Milgram theorem, we deduce that there exists a unique solution $\mathcal{U} \in C([0,T];\dot{H}^1(\mathbb{R}^3))$ of (3.3).

Next for $0 \leqslant k \leqslant s$, we apply D^{k+1} to the first line of (3.3), multiply the resulting equation by $D^{k+1}\phi$ and make an integration by parts to obtain

$$\begin{aligned}\|\nabla(D^{k+1}\phi)\|^2_{L^2} &= \frac{e}{2}\left|\int_{\mathbb{R}^3} D^{k+1}|\psi|^2 D^{k+1}\phi \,\mathrm{d}x\right| \\ &\leqslant C\int_{\mathbb{R}^3} \left|D^k|\psi|^2\right||D^{k+2}\phi| \,\mathrm{d}x.\end{aligned}$$

Using the Leibniz rule, Lemma 3.1 and the Schwarz inequality, one has

$$\|\nabla\phi(t,\cdot)\|_{H^{k+1}} \leqslant C\|\psi(t,\cdot)\|^2_{H^k} \quad \text{for all } t \in [0,T]. \tag{3.9}$$

Summing up the inequalities (3.9) from $k = 0$ to s and recalling the fact that $\|\psi\|_{L^\infty([0,T];H^s)} \leqslant R$, we obtain

$$\|\nabla\phi\|_{L^\infty([0,T];H^{s+1})} \leqslant C_1(R),$$

where $C_1(R)$ is a constant depending only on R.

Finally, the Sobolev embedding $W^{1,6}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ provides that

$$\begin{aligned}\|\phi(t,\cdot)\|_{L^\infty} &\leqslant C\left(\sum_{k=1}^{3}\|\partial_{x_k}\phi(t,\cdot)\|_{L^6} + \|\phi(t,\cdot)\|_{L^6}\right) \\ &\leqslant C\left(\sum_{k=1}^{3}\|\nabla(\partial_{x_k}\phi)(t,\cdot)\|_{L^2} + \|\nabla\phi(t,\cdot)\|_{L^2}\right),\end{aligned}$$

from which we deduce that there exists a constant $C_2(R)$ depending only on R such that

$$\|\phi\|_{L^\infty([0,T];L^\infty)} \leqslant C_2(R),$$

which ends the proof of (3.7). The proof of estimates (3.8) is similar and we omit the details. $\square$

Step 2: Solving the elliptic equation (3.4)

Lemma 3.3. — *Suppose that* $\mathbf{A} \in H^s(\mathbb{R}^3, \mathbb{R}^3)$, $s > \frac{3}{2}$ *and* $\operatorname{div} \mathbf{A} = 0$. *Then for sufficiently large* $\alpha > 0$, *the bilinear form*

$$b(u,v) := \int_{\mathbb{R}^3} (\nabla u \cdot \nabla \overline{v} + \alpha u \overline{v} + 2ie \nabla u \cdot \mathbf{A} \overline{v}) \,\mathrm{d}x$$

is hermitian, continuous and elliptic on $H^1(\mathbb{R}^3, \mathbb{C}) \times H^1(\mathbb{R}^3, \mathbb{C})$.

As a consequence, there exists a unique solution $\chi(t, \cdot\,) \in H^1(\mathbb{R}^3, \mathbb{C})$ *to* (3.4) *and there exists a constant* $C_5(R)$ *such that*

$$\|\chi\|_{L^\infty([0,T];H^{s+2})} \leqslant C_5(R).$$

Proof. — First we note that b is hermitian by the condition $\operatorname{div} \mathbf{A} = 0$. Indeed, one has

$$\begin{aligned} 2ie \int_{\mathbb{R}^3} (\nabla u \cdot \mathbf{A}) \overline{v} \,\mathrm{d}x &= -2ie \int_{\mathbb{R}^3} \operatorname{div} \mathbf{A} u \overline{v} \,\mathrm{d}x - 2ie \int_{\mathbb{R}^3} (\nabla \overline{v} \cdot \mathbf{A}) u \,\mathrm{d}x \\ &= \overline{2ie \int_{\mathbb{R}^3} (\nabla v \cdot \mathbf{A}) \overline{u} \,\mathrm{d}x}, \end{aligned}$$

from which it follows directly that $b(u,v) = \overline{b(v,u)}$. The continuity is a direct consequence of the Cauchy–Schwarz inequality and the fact that $\mathbf{A} \in H^s(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$. Finally for all $u \in H^1(\mathbb{R}^3, \mathbb{C})$, we have

$$\begin{aligned} \left| 2ie \int_{\mathbb{R}^3} (\nabla u \cdot \mathbf{A}) \overline{u} \,\mathrm{d}x \right| &\leqslant 2e \|\mathbf{A}\|_{L^\infty} \|\nabla u\|_{L^2} \|u\|_{L^2} \\ &\leqslant \frac{1}{2} \|\nabla u\|_{L^2}^2 + 2e^2 \|\mathbf{A}\|_{L^\infty}^2 \|u\|_{L^2}^2 \\ &\leqslant \frac{1}{2} \|\nabla u\|_{L^2}^2 + Ce^2 \|\mathbf{A}\|_{H^s}^2 \|u\|_{L^2}^2. \end{aligned}$$

Taking $\alpha \geqslant 2Ce^2 \|\mathbf{A}\|_{H^s}^2$, one gets

$$b(u,u) \geqslant \frac{1}{2} \|\nabla u\|_{L^2}^2 + \frac{\alpha}{2} \|u\|_{L^2}^2.$$

This shows that b is elliptic on $H^1(\mathbb{R}^3, \mathbb{C}) \times H^1(\mathbb{R}^3, \mathbb{C})$.

Now since ψ, $\Psi \in H^s(\mathbb{R}^3)$ and $\phi \in \dot{H}^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, it is obvious that $i\Psi - e\phi\psi - e^2|\mathbf{A}|^2\psi - g(|\psi|^2)\psi + \alpha\psi$ belongs to $L^2(\mathbb{R}^3) \hookrightarrow (H^1(\mathbb{R}^3))^*$. Then the Lax–Milgram theorem ensures the existence of a unique solution χ to (3.4) in $H^1(\mathbb{R}^3)$. Using the elliptic regularity theory and recalling that

$$(\psi, \Psi) \in C([0,T], H^s(\mathbb{R}^3))^2,\ \phi \in C(([0,T]; \dot{H}^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)),$$

$$\nabla\phi \in C([0,T]; H^s(\mathbb{R}^3)),\ \mathbf{A} \in C([0,T], H^s(\mathbb{R}^3)),$$

one gets

$$\|\chi\|_{L^\infty([0,T];H^{s+2})} \leqslant C_5(R),$$

where $C_5(R)$ is a constant depending only on R. This ends the proof of Lemma 3.3. ☐

Step 3: Solving the Schrödinger equation (3.5)

For convenience, we introduce the real form of Equation (3.5). Denote $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2) = (\operatorname{Re} \mathcal{Q}, \operatorname{Im} \mathcal{Q})$ and write

$$\partial_t \mathcal{R} + \mathcal{J} \Delta \mathcal{R} - 2e \sum_{j=1}^{3} \mathcal{K}_j(\mathbf{A}) \partial_{x_j} \mathcal{R} = \mathcal{L}_1(\nabla \chi, \mathbf{R}) + \mathcal{L}_2(\mathcal{U}, \chi, \Psi, \mathbf{A}, \mathbf{R}), \tag{3.10}$$

$$\mathcal{R}(0,x) = \Big(\operatorname{Re} \Psi_0(x), \operatorname{Im} \Psi_0(x)\Big),$$

where

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{K}_j(\mathbf{A}) = \begin{pmatrix} a_j & 0 \\ 0 & a_j \end{pmatrix},$$

$$\mathcal{L}_1(\nabla\chi, \mathbf{R}) = \begin{pmatrix} \operatorname{Im}(2ie\nabla\chi \cdot \mathbf{R}) \\ -\operatorname{Re}(2ie\nabla\chi \cdot \mathbf{R}) \end{pmatrix},$$

$$\mathcal{L}_2(\mathcal{U}, \chi, \Psi, \mathbf{A}, \mathbf{R}) = \begin{pmatrix} \operatorname{Im} F_2(\mathcal{U}, \chi, \Psi, \mathbf{A}, \mathbf{R}) \\ -\operatorname{Re} F_2(\mathcal{U}, \chi, \Psi, \mathbf{A}, \mathbf{R}) \end{pmatrix}.$$

Now for $\varepsilon > 0$, we consider a long-wave type regularization of (3.10) (see [CG01]):

$$\partial_t (1 - \varepsilon\Delta) \mathcal{R}_\varepsilon + \mathcal{J} \Delta \mathcal{R}_\varepsilon - 2e \sum_{j=1}^{3} \mathcal{K}_j(\mathbf{A}) \partial_{x_j} \mathcal{R}_\varepsilon = \mathcal{L}_1 + \mathcal{L}_2, \tag{3.11}$$

with $\mathcal{R}_\varepsilon(0) = (1-\varepsilon\Delta)^{-1}(\operatorname{Re}\Psi_0, \operatorname{Im}\Psi_0)$. Since Equation (3.11) is linear and contains differential operator in space of at most zero order, one can show that there exists a unique solution $\mathcal{R}_\varepsilon \in C([0,T]; H^s(\mathbb{R}^3))$ to Equation (3.11). Furthermore, we have the following estimate.

Lemma 3.4. — *Let $\mathcal{R}_\varepsilon$ be the unique solution of Equation* (3.11). *Then there exist constants $C_6(R)$, $C_7(R)$ independent of ε such that*

$$\|\mathcal{R}_\varepsilon\|_{L^\infty([0,T];H^s)} \leqslant e^{C_6(R)T} \|\Psi_0\|_{H^s} + \left(e^{C_7(R)T} - 1\right)^{\frac{1}{2}}.$$

Proof. — We first begin with the L^2-estimate. We multiply (3.11) by $\mathcal{R}_\varepsilon$ and integrate over $\mathbb{R}^3$. Since $\mathcal{J}$ is skew-symmetric, one obtains

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\frac{1}{2}\int_{\mathbb{R}^3} \Big(|\mathcal{R}_\varepsilon|^2 + \varepsilon|\nabla\mathcal{R}_\varepsilon|^2\Big)\,\mathrm{d}x\right) \\ &\qquad = 2e \int_{\mathbb{R}^3} \sum_{j=1}^{3} \mathcal{K}_j(\mathbf{A}) \partial_{x_j} \mathcal{R}_\varepsilon \cdot \mathcal{R}_\varepsilon \,\mathrm{d}x + \int_{\mathbb{R}^3} \mathcal{L}_1 \cdot \mathcal{R}_\varepsilon \,\mathrm{d}x + \int_{\mathbb{R}^3} \mathcal{L}_2 \cdot \mathcal{R}_\varepsilon \,\mathrm{d}x. \end{aligned} \tag{3.12}$$

For $j = 1,2,3$, we have from $\|\partial_{x_j} a_j\|_{H^s} \leqslant R$ that

$$\begin{aligned} \left|\int_{\mathbb{R}^3} \mathcal{K}_j(\mathbf{A}) \partial_{x_j} \mathcal{R}_\varepsilon \cdot \mathcal{R}_\varepsilon \,\mathrm{d}x\right| &= \left|\frac{1}{2}\int_{\mathbb{R}^3} a_j \partial_{x_j} |\mathcal{R}_\varepsilon|^2 \,\mathrm{d}x\right| = \left|\frac{1}{2}\int_{\mathbb{R}^3} \partial_{x_j} a_j |\mathcal{R}_\varepsilon|^2 \,\mathrm{d}x\right| \\ &\leqslant \frac{1}{2}\|\partial_{x_j} a_j\|_{L^\infty} \|\mathcal{R}_\varepsilon\|_{L^2}^2 \leqslant C(R) \|\mathcal{R}_\varepsilon\|_{L^2}^2. \end{aligned} \tag{3.13}$$

Since Ψ, $\mathbf{R} \in H^s$ and $\mathbf{A} \in H^{s+1}$, using Lemmas 3.2-3.3, one can also compute as follows :

$$\left|\int_{\mathbb{R}^3} \mathcal{L}_1(\nabla\chi, \mathbf{R}) \cdot \mathcal{R}_\varepsilon \,\mathrm{d}x\right| \leqslant C(R)\|\mathcal{R}_\varepsilon\|_{L^2},$$

$$\left|\int_{\mathbb{R}^3} \mathcal{L}_2(\mathcal{U}, \chi, \Psi, \mathbf{A}, \mathbf{R}) \cdot \mathcal{R}_\varepsilon \,\mathrm{d}x\right| \leqslant C(R)\|\mathcal{R}_\varepsilon\|_{L^2}. \tag{3.14}$$

Collecting (3.12)–(3.14), we derive

$$\frac{\partial}{\partial t}\|\mathcal{R}_\varepsilon\|_{L^2}^2 \leqslant C(R)\|\mathcal{R}_\varepsilon\|_{L^2}^2 + C(R).$$

By the Gronwall inequality and from

$$\|\mathcal{R}_\varepsilon(0, \cdot)\|_{L^2} = \|(1 - \varepsilon\Delta)^{-1}(\operatorname{Re}\Psi_0, \operatorname{Im}\Psi_0)\|_{L^2} \leqslant \|\Psi_0\|_{L^2},$$

it follows that

$$\begin{aligned}
\|\mathcal{R}_\varepsilon(t, \cdot)\|_{L^2}^2 &\leqslant e^{C(R)t}\Big(\|\mathcal{R}_\varepsilon(0, \cdot)\|_{L^2}^2 + 1 - e^{-C(R)t}\Big) \\
&\leqslant e^{C(R)T}\|\Psi_0\|_{L^2}^2 + e^{C(R)T} - 1 \\
&\leqslant \left(e^{\frac{C(R)T}{2}}\|\Psi_0\|_{L^2} + (e^{C(R)T} - 1)^{\frac{1}{2}}\right)^2 \quad \text{for all } t \in [0, T].
\end{aligned}$$

Next we perform the H^s-estimate. We apply D^s on (3.11), multiply the resulting equation by $D^s\mathcal{R}_\varepsilon$, integrate over $\mathbb{R}^3$ and use the Gronwall inequality. We limit our attention to non-trivial terms. Recalling that $\chi \in C([0, T]; H^{s+2})$ and using Lemma 3.3, we obtain

$$\left|\int_{\mathbb{R}^3} D^s\mathcal{L}_1(\nabla\chi, \mathbf{R}) \cdot D^s\mathcal{R}_\varepsilon \,\mathrm{d}x\right| \leqslant C(R)\|\mathcal{R}_\varepsilon\|_{H^s}.$$

Moreover, one gets

$$\begin{aligned}
\left|\int_{\mathbb{R}^3} \mathcal{K}_j(\mathbf{A})\partial_{x_j} D^s\mathcal{R}_\varepsilon \cdot D^s\mathcal{R}_\varepsilon \,\mathrm{d}x\right| &= \left|\frac{1}{2}\int_{\mathbb{R}^3} a_j\partial_{x_j}|D^s\mathcal{R}_\varepsilon|^2 \,\mathrm{d}x\right| \\
&= \left|\frac{1}{2}\int_{\mathbb{R}^3} \partial_{x_j}a_j|D^s\mathcal{R}_\varepsilon|^2 \,\mathrm{d}x\right| \\
&\leqslant \frac{1}{2}\|\partial_{x_j}a_j\|_{L^\infty}\|\mathcal{R}_\varepsilon\|_{H^s}^2.
\end{aligned}$$

Arguing similarly as above, one finds that

$$\|\mathcal{R}_\varepsilon\|_{L^\infty([0,T];H^s)} \leqslant e^{C_6(R)T}\|\Psi_0\|_{H^s} + (e^{C_7(R)T} - 1)^{\frac{1}{2}},$$

which ends the proof of Lemma 3.4. □

Now we argue as in [BdBS95], [CG01] and we perform the limit $\varepsilon \to 0$. By Lemma 3.4, we know that $\mathcal{R}_\varepsilon$ is uniformly bounded in $L^\infty([0, T], H^s)$. From (3.11), one also has

$$\begin{aligned}
\partial_t\mathcal{R}_\varepsilon = -(1 - \varepsilon\Delta)^{-1}\mathcal{J}\Delta\mathcal{R}_\varepsilon + 2e(1 - \varepsilon\Delta)^{-1}\sum_{j=1}^{3}\mathcal{K}_j(\mathbf{A})\partial_{x_j}\mathcal{R}_\varepsilon& \\
+ (1 - \varepsilon\Delta)^{-1}\mathcal{L}_1 + (1 - \varepsilon\Delta)^{-1}\mathcal{L}_2&.
\end{aligned}$$

This implies that

$$\|\partial_t \mathcal{R}_\varepsilon\|_{H^{s-2}} \leqslant C(R)\|\mathcal{R}_\varepsilon\|_{H^s} + \|\mathcal{L}_1\|_{H^{s-2}} + \|\mathcal{L}_2\|_{H^{s-2}} \leqslant C \quad \text{for all } t \geqslant 0 \text{ and } \varepsilon \in (0,1].$$

Thus passing to a subsequence, we may assume that

$$\mathcal{R}_\varepsilon \to \mathcal{R} \in L^\infty([0,T], H^s), \partial_t \mathcal{R}_\varepsilon \to \partial_t \mathcal{R} \in L^\infty([0,T], H^{s-2}) \text{ in the weak * topology.}$$

From (3.11), one can see that $\mathcal{R}$ is a solution of Equation (3.10) and satisfies

$$\|\mathcal{R}\|_{L^\infty([0,T];H^s)} \leqslant e^{C_6(R)T}\|\Psi_0\|_{H^s} + (e^{C_7(R)T} - 1)^{\frac{1}{2}}.$$

Moreover since $\mathcal{R}_\varepsilon(0) \to (\operatorname{Re}\Psi_0, \operatorname{Im}\Psi_0)$, we get $\mathcal{R}(0) = (\operatorname{Re}\Psi_0, \operatorname{Im}\Psi_0)$. We then deduce the existence of a solution $\mathcal{Q}$ to Equation (3.5) satisfying

$$\|\mathcal{Q}\|_{L^\infty([0,T];H^s)} \leqslant e^{C_6(R)T}\|\Psi_0\|_{H^s} + (e^{C_7(R)T} - 1)^{\frac{1}{2}}. \tag{3.15}$$

Step 4: Solving the symmetric system (3.6)

First we note that it is straightforward to prove the existence of a unique solution $\mathcal{B}_j$ to Equation (3.6). (We refer to [AG91, Proposition 1.2, p. 115] for the proof.) Furthermore, by using the Fourier transform $\mathcal{F}$, one has directly

$$\mathbb{R}_j u = \mathcal{F}^{-1}\left(i\frac{\xi_j}{|\xi|}\mathcal{F}(u)\right) \quad \text{for } j = 1,2,3,$$

from which we deduce that $\mathbb{R}_j$ and hence $\mathbb{P}_{j,m}$ are bounded from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. As a consequence, using the fact $\Psi \in H^s$, $\mathbf{A} \in H^{s+2}$, $\mathbf{R} \in H^{s+1}$ and $\chi \in H^{s+2}$, one can prove that

$$\|H_j(\chi, \Psi, \mathbf{A}, \mathbf{R})\|_{L^\infty([0,T];H^s)} \leqslant C(R).$$

Thus applying the energy estimate to (3.6), recalling that $M_j(\nabla) = \sum_{k=1}^3 \widetilde{M}_j \partial_{x_k}$ is symmetric and using the fact $\widetilde{M}_j$ consists of constant elements, we get

$$\frac{\partial}{\partial t}\|\mathcal{B}_j\|_{H^s}^2 \leqslant \|\mathcal{B}_j\|_{H^s}^2 + C(R).$$

Then by the Gronwall inequality, we obtain the following estimate.

Lemma 3.5. — *Let $\mathcal{B}_j$ be the unique solution of* (3.6). *Then there exists a constant $C_8(R) > 0$ such that*

$$\|\mathcal{B}_j\|_{L^\infty([0,T];H^s)} \leqslant e^{\frac{T}{2}}\|\mathcal{A}_{j(0)}\|_{H^s} + C_8(R)(e^T - 1)^{\frac{1}{2}}. \tag{3.16}$$

Collecting (3.15) and (3.16), we can state the following result.

Proposition 3.6. — *There exists $\hat{T} > 0$ such that for $0 < T \leqslant \hat{T}$, $\mathcal{S}$ maps $B(R)$ into itself.*

Step 5: Contraction mapping

Now we establish the following result.

Proposition 3.7. — *There exists $T^* \in (0, \hat{T}]$ such that $\mathcal{S}$ is a contraction mapping in the $L^\infty([0,T^*]; L^2(\mathbb{R}^3))$-norm.*

Proof. — The proof is based on the fact that $s > \frac{3}{2}$ and on the fact that all the functions of Equations (3.3)–(3.6) are Lipschitz with respect to their arguments. The proof is classical and we omit the details. □

From Propositions 3.6, 3.7 and by the contraction mapping principle, it follows that there exists a unique $(\Psi, \mathcal{A}) \in B(R)$ such that

$$\mathcal{S}(\Psi, \mathcal{A}) = (\Psi, \mathcal{A}),$$

that is, Ψ is the unique solution of the Schrödinger equation:

$$\begin{cases} i\partial_t \Psi + \Delta\Psi - 2ie\nabla\Psi \cdot \mathbf{A} = 2ie\nabla\chi \cdot \mathbf{R} + F_2(\mathcal{U}, \chi, \Psi, \mathbf{A}, \mathbf{R}), \\ \Psi(0,x) = \Psi_{(0)}, \end{cases}$$

and $\mathcal{A}_j$ $(j = 1,2,3)$ is the unique solution to the symmetric system:

$$\begin{cases} \partial_t \mathcal{A}_j + \mathcal{M}_j(\nabla)\mathcal{A}_j + H_j(\chi, \Psi, \mathbf{A}, \mathbf{R}) = 0, \\ \mathcal{A}_j(0,x) = \mathcal{A}_{j(0)}. \end{cases}$$

Since $\mathcal{A}(0,x) = \mathcal{A}_{(0)}$, we also have $\mathbf{A}(0,x) = \mathbf{A}_{(0)}$ and $\mathbf{A}_t(0,x) = \mathbf{A}_{(1)}$.

Step 6: Proof of Theorem 1.1 completed

Let us now go back to the original problem (1.1)–(1.3). To this end, we first remark that $\partial_t\psi = \Psi$ by (3.2). Applying ∂_t on Equation (3.4), comparing the resulting equation with Equation (2.12) and recalling that $\mathbf{R} = \partial_t \mathbf{A}$, one obtains

$$-\Delta(\partial_t\chi - \partial_t\psi) + 2ie\nabla(\partial_t\chi - \partial_t\psi) \cdot \mathbf{A} = 0. \tag{3.17}$$

By Lemma 3.3, we know that the bilinear form

$$b(u,v) = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla \overline{v} + \alpha u \overline{v} + 2ie\nabla u \cdot \mathbf{A}\overline{v})\,\mathrm{d}x$$

is hermitian, continuous and elliptic on $H^1(\mathbb{R}^3, \mathbb{C}) \times H^1(\mathbb{R}^3, \mathbb{C})$, from which we deduce that Equation (3.17) has a unique solution. Since obviously 0 is a solution to Equation (3.17), one has $\partial_t\chi = \partial_t\psi$. Moreover from (3.2), it follows that $\psi(0,x) = \psi_{(0)}$ and hence $\phi(0,x) = \phi_{(0)}$ by the uniqueness of the solution of (3.3). Thus substituting $t = 0$ into (3.4), we get

$$-\Delta\chi(0) + 2ie\nabla\chi(0) \cdot \mathbf{A}_{(0)} = i\Psi_{(0)} - e\phi_{(0)}\psi_{(0)} - e^2|\mathbf{A}_{(0)}|^2\psi_{(0)} + g(|\psi_{(0)}|^2)\psi_{(0)}.$$

By the definition of $\Psi_{(0)}$ given in (3.1), one finds that

$$-\Delta(\chi(0) - \psi_{(0)}) + 2ie\nabla(\chi(0) - \psi_{(0)}) \cdot \mathbf{A}_{(0)} = 0,$$

yielding that $\chi(0) = \psi_{(0)}$ by Lemma 3.3.

Since $\partial_t \chi = \partial_t \psi$ and $\chi(0) = \psi_{(0)}$, it follows that $\chi = \psi$. As a consequence, ψ is the unique solution to Equation (1.1). Then from (3.4) and (3.5), we conclude that $\mathbf{A}$ and ϕ are the unique solutions to (1.2) and (1.3) respectively. Moreover by the uniqueness of (3.3), one finds that $\partial_t \phi = \Psi$. Finally by Lemmas 3.2, 3.3 and from (3.15), (3.16), $(\psi, \mathbf{A}, \phi)$ has the desired regularity as stated in Theorem 1.1.

BIBLIOGRAPHY

[ADM17] Paolo Antonelli, Michele D'Amico, and Pierangelo Marcati, *Nonlinear Maxwell-Schrödinger system and quantum magneto-hydrodynamics in 3-D*, Commun. Math. Sci. **15** (2017), 451–479. ↑67, 69

[AG91] Serge Alinhac and Patrick Gérard, *Opérateurs pseudo-différentiels et théorème de Nash–Moser*, Savoirs Actuels, Éditions du CNRS, 1991. ↑77, 82

[AP08] Antonio Azzollini and Alessio Pomponio, *Ground state solutions for the nonlinear Schrödinger-Maxwell equations*, J. Math. Anal. Appl. **345** (2008), 90–108. ↑70

[BdBS95] L. Bergé, Anne de Bouard, and Jean-Claude Saut, *Blowing up time-dependent solutions of the planar Chern-Simons gauged nonlinear Schrödinger equation*, Nonlinearity **8** (1995), no. 2, 235–253. ↑81

[BF14] Vieri Benci and Donato Fortunato, *Solitons in Schrödinger-Maxwell equations*, J. Fixed Point Theory Appl. **15** (2014), no. 1, 101–132. ↑70

[BS11] Jacopo Bellazzini and Gaetano Siciliano, *Stable standing waves for a class of nonlinear Schrödinger-Poisson equations*, Z. Angew. Math. Phys. **62** (2011), 267–280. ↑70

[BT09] Ioan Bejenaru and Daniel Tataru, *Global wellposedness in the energy space for the Maxwell-Schrödinger system*, Commun. Math. Phys. **288** (2009), no. 1, 145–198. ↑69

[CC04] Mathieu Colin and Thierry Colin, *On a quasilinear Zakharov system describing laser-plasma interactions*, Differ. Integral Equ. **17** (2004), no. 3-4, 297–330. ↑71

[CDSS13] Isabelle Catto, Jean Dolbeault, Oscar D. Sánchez, and Juan S. Soler, *Existence of steady states for the Maxwell-Schrödinger-Poisson system: exploring the applicability of the concentration-compactness principle*, Math. Models Methods Appl. Sci. **23** (2013), no. 10, 1915–1938. ↑70

[CE88] Thierry Cazenave and Maria J. Esteban, *On the stability of stationary states for nonlinear Schrödinger equations with an external magnetic field*, Mat. Apl. Comput. **7** (1988), no. 3, 155–168. ↑70

[CG01] Thierry Colin and Jean-Michel Ghidaglia, *An initial-boundary value problem for the Korteweg-de-Vries equation posed on a finite interval*, Adv. Differ. Equ. **6** (2001), no. 12, 1463–1492. ↑80, 81

[Col02] Mathieu Colin, *On the local well-posedness of quasilinear Schrödinger equation in arbitrary space dimension*, Commun. Partial Differ. Equations **27** (2002), no. 1-2, 325–354. ↑71

[CW16] Mathieu Colin and Tatsuya Watanabe, *Cauchy problem for the nonlinear Klein-Gordon equation coupled with the Maxwell equation*, J. Math. Anal. Appl. **443** (2016), no. 2, 778–796. ↑68, 69

[CW17] ———, *Standing waves for the nonlinear Schrödinger equation coupled with the Maxwell equation*, Nonlinearity **30** (2017), no. 5, 1920–1947. ↑69, 70

[DFVV10] Piero D'Ancona, Luca Fanelli, Luis Vega, and Nicola Visciglia, *Endpoint Strichartz estimates for the magnetic Schrödinger equation*, J. Funct. Anal. **258** (2010), no. 10, 3227–3240. ↑69

[Fel98] Bjørn Felsager, *Geometry, particles and fields*, Graduate Texts in Contemporary Physics, Springer, 1998. ↑68

[GR91] J. M. Gonçalves Ribeiro, *Instability of symmetric stationary states for some nonlinear Schrödinger equations with an external magnetic field*, Ann. Inst. Henri Poincaré, Phys. Théor. **54** (1991), no. 4, 403–433. ↑70

[Kik07] Hiroaki Kikuchi, *Existence and stability of standing waves for Schrödinger–Poisson–Salter equation*, Adv. Nonlinear Stud. **7** (2007), no. 3, 403–437. ↑70

[Mic08] Laurent Michel, *Remarks on non-linear Schrödinger equation with magnetic fields*, Commun. Partial Differ. Equations **33** (2008), no. 7, 1198–1215. ↑69

[NT86] Kuniaki Nakamitsu and Masayoshi Tsutsumi, *The Cauchy problem for the coupled Maxwell–Schrödinger equations*, J. Math. Phys. **27** (1986), 211–216. ↑69

[NW07] Makoto Nakamura and Takeshi Wada, *Global existence and uniqueness of solutions to the Maxwell–Schrödinger equations*, Commun. Math. Phys. **276** (2007), no. 2, 315–339. ↑69

[OT92] Tohru Ozawa and Yoshio Tsutsumi, *Existence and smoothing effect of solution for the Zakharov equations*, Publ. Res. Inst. Math. Sci. **28** (1992), no. 3, 329–361. ↑71

[RV08] Eugen Radu and Mikhail S. Volkov, *Stationary ring solitons in field theory - Knots and vortons*, Phys. Rep. **468** (2008), no. 4, 101–151. ↑70

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