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BOUNDED SETS OF SHEAVES ON RELATIVE ANALYTIC SPACES

ENSEMBLES LIMITÉS DE FAISCEAUX COHÉRENTS SUR LES ESPACES COMPLEXES RELATIFS

Abstract. — We extend previous results on boundedness of sets of coherent sheaves on a compact Kähler manifold to the relative and not necessarily smooth case. This enlarged context allows us to prove properness properties of the relative Douady space as well as results related to semistability of sheaves such as the existence of relative Harder–Narasimhan filtrations.

Résumé. — Nous étendons au cas relatif et pas nécessairement lisse les résultats précédemment obtenus sur les ensembles limités de faisceaux cohérents sur une variété kählérienne compacte. Ce contexte élargi nous permet de démontrer des énoncés de propreté pour l'espace de Douady relatif ainsi que des résultats liés à la semi-stabilité des faisceaux cohérents, comme l'existence des filtrations de Harder–Narasimhan relatives.

1. Introduction

In his paper [Gro61] constructing the Hilbert scheme Grothendieck introduced the notion of a bounded set $\mathfrak{E}$ of isomorphism classes of coherent sheaves on the fibers of X/S, where X is a scheme of finite type over a noetherian scheme S. Roughly expressed this means that the elements of $\mathfrak{E}$ are among the fibres of an algebraic

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family of coherent sheaves in the fibers of $X_{S'}/S'$ after base change to a scheme S' of finite type over S. Thus in the absolute case, i.e. when $S = \mathrm{Spec}\ k$, this boils down to the existence of an algebraic family of coherent sheaves parameterized by a scheme of finite type over k whose set of isomorphism classes includes $\mathfrak{E}$. When X is projective over S with relatively very ample sheaf $\mathcal{O}(1)$ Grothendieck further gave a criterion [Gro61, Théorème 2.1] for a set $\mathfrak{E}$ of isomorphism classes of coherent sheaves on the fibers of X/S to be bounded saying that this is the case if and only if all elements $E \in \mathfrak{E}$ appear as quotients of a uniform sheaf of the form $\mathcal{O}(-n)^{\oplus N}$ and their Hilbert polynomials range within a finite set of polynomials. This criterion is essential in proving that the connected components of the Hilbert scheme of a projective scheme are projective but also in the theory of moduli spaces of semi-stable sheaves.

This paper deals with boundedness for sets $\mathfrak{E}$ of isomorphism classes of analytic coherent sheaves over relative analytic spaces X/S. One may rewrite almost literally Grothendieck's definition of boundedness in this new set-up, but this will find little use in applications. What is important in our analytic context is the aspect of relative compactness hidden behind that definition. We will therefore say that a set $\mathfrak{E}$ of isomorphism classes of analytic coherent sheaves on the fibers of a relative analytic space X/S is bounded if it can be viewed as a relatively compact subset of a suitable analytic parameter space; see the precise Definition 5.2. Our main result, Theorem 5.5, then gives an analogue of Grothendieck's boundedness criterion [Gro61, Théorème 2.1] in this set-up. For its formulation we replace the Hilbert polynomials by collections of degree functions, Definitions 2.8 and 2.9. We apply the boundedness criterion to properness questions on the Douady space as well as to problems arising in the study of semistable sheaves, such as the existence of relative Harder–Narasimhan filtrations and openness of semistability. Note that in a previous paper [Tom16] we proved properness of the connected components of the Douady space albeit in an absolute smooth Kähler context. To obtain applications to moduli spaces of semistable sheaves however, boundedness results in the relative, not necessarily smooth case were needed and this is precisely what the present paper gives.

An important device which allowed us to prove the main result of this paper as well as [Tom16, Theorem 4.1] was the passage from bounding quotient sheaves of a fixed coherent sheaf to bounding volumes of their projectivizations and applying Bishop's Theorem to the corresponding cycle spaces. However, unlike in [Tom16], where the volume calculations were done globally by intersecting appropriate cohomology classes on compact Kähler manifolds, we perform here finer local volume computations which allow the needed flexibility. Another new feature of this paper is the use of degree systems which do not necessarily come from a fixed Kähler polarization. The consideration of more general degree systems is natural when working for instance with mobile classes on projective manifolds as in [CCP19, CP19, GT17], or [GRT16], and imposes itself in the treatment of stability of coherent sheaves over compact smooth complex non-Kähler surfaces, cf. [LT95, Tom20].

The paper is organized as follows. We start by introducing degree systems for relative analytic spaces and in particular those degree systems arising from integration of polarizing differential forms along homology Todd classes of coherent sheaves.

In Sections 3 and 4 two ingredients of the proof of our main result are presented in detail: the local volume computations in Section 3, which roughly ensure that in order to control a true volume of a projectivized bundle a bound on a certain pseudovolume suffices, and in Section 4 the description of the passage from quotient sheaves with bounded degrees to cycles in projectivized bundles with bounded volume and back. Follow the presentation of the boundedness notion and of the main boundedness criteria in Section 5. In the final Section 6 applications are given to properness properties of relative Douady spaces and to the existence of relative Harder–Narasimhan filtrations.

2. Preliminaries

2.1. Kähler morphisms

We shall review the definition of a Kähler metric on a possibly singular complex analytic space X in the absolute and in the relative case and shall then introduce some weaker versions of such metrics.

We start by recalling the definition of differential forms on complex spaces which we shall use. For this definition differential forms are induced from local embeddings of X in a domain V of $\mathbb{C}^n$ and it is then checked that they are well defined.

Definition 2.1 (Fujiki [Fuj78, 1.1]). — *Suppose that X is embedded in a domain V of $\mathbb{C}^n$ and let $\mathcal{I}_X$ be its ideal sheaf. If $\mathcal{A}_V^m$ denotes the sheaf of differential m-forms on V, one defines the* sheaves of differential m-forms *by*

$$\mathcal{A}_X = \mathcal{A}_X^0 := \mathcal{A}_V^0 \Big/ \left(\mathcal{I}_X + \bar{\mathcal{I}}_X\right) \mathcal{A}_V^0$$

and for $m > 0$

$$\mathcal{A}_X^m := \mathcal{A}_V^m \Big/ \left(\left(\mathcal{I}_X + \bar{\mathcal{I}}_X\right) \mathcal{A}_V^m + \mathrm{d} \left(\mathcal{I}_X + \bar{\mathcal{I}}_X\right) \mathcal{A}_V^{m-1} \right).$$

For further properties of these sheaves we refer the reader to [Var89, 1.1], [BM14, III.2.4]. We just mention the fact that type decomposition as well as exterior differentiation and the operators ∂ and $\bar{\partial}$ are well defined.

Definition 2.2. — *The subsheaves of $\mathcal{A}_X$ of functions induced locally from smooth strongly plurisubharmonic (respectively from pluriharmonic) functions on V will be denoted by $SPSH_X$ (resp. by PH_X). If $f : X \to S$ is a morphism of complex spaces we define the sheaves $SPSH_f$ and PH_f as the subsheaves of functions of $\mathcal{A}_X$ which are strongly plurisubharmonic (respectively pluriharmonic) on the fibers of f. We will call such functions* plurisubharmonic (resp. pluriharmonic) relatively with respect to f.

Definition 2.3 ([Fuj82, Definition 2.1],[Bin83, 5.1]). — *A morphism $f : X \to S$ of complex spaces is said to be* Kähler *if there exist an open cover $(U_i)_{i\in I}$ of X and sections $\phi_i \in SPSH_f(U_i)$ such that $(\phi_i - \phi_j)|_{U_i\cap U_j} \in PH(U_i \cap U_j)$ for all $i, j \in I$. When the differences $\phi_i - \phi_j$ are only required to be in $PH_f(U_i \cap U_j)$ we say that f is* weakly Kähler. *Such a data $(U_i, \phi_i)_{i\in I}$ is called* a Kähler metric *(resp.* a weakly

Kähler metric*) for* f*. When* S *is a point the two notions coincide and we say then that* X *is a* Kähler space*.*

If $(U_i,\phi_i)_{i\in I}$ is a Kähler metric for $f:X\to S$ then the $(1,1)$-forms $i\partial\bar\partial\phi_j$ glue together to give a closed $(1,1)$-form ω on X which we call *the Kähler form of the metric*. If $(U_i,\phi_i)_{i\in I}$ is only weakly Kähler, we only get a "relative" $(1,1)$-form, cf. [Bin83]. Note that in both cases the restriction of ω to the fibers of f is *strictly positive*, in the sense that for local embeddings of the fibers in open domains of number spaces the form appears as the restriction of a strictly positive form. We shall use the analogous definition of positivity (in Lelong's sense) more generally for (p,p)-forms and in particular for the powers of ω, cf. [BM14].

2.2. Homology classes and degrees

The Grothendieck–Riemann–Roch theorem for singular varieties was proved by Baum, Fulton and MacPherson [BFM75, BFM79] in the projective case and by Levy [Lev87] in the complex analytic case. One way to formulate it is that there exists a natural transformation of functors $\tau:K_0\to H_{\mathbb{Q}}:=H_{2*}(\ ;\mathbb{Q})$ such that for any compact complex space X the diagram

$$\begin{tikzcd} K^0X\otimes K_0X \arrow[r, "\otimes"] \arrow[d, "\mathrm{ch}\otimes\tau"] & K_0X \arrow[d, "\tau"] \\ H^{2*}(X;\mathbb{Q})\otimes H_{2*}(X;\mathbb{Q}) \arrow[r, "\frown"] & H_{2*}(X;\mathbb{Q}) \end{tikzcd}$$

commutes and if X is nonsingular then $\tau(\mathcal{O}_X)=\mathrm{Td}(X)\frown[X]$, where K^0X, K_0X are the Grothendieck groups generated by holomorphic vector bundles and coherent sheaves respectively and $\mathrm{Td}(X)$ is the (cohomology) Todd class of the tangent bundle to X. Naturality means that for each proper morphism $f:X\to Y$ of complex spaces the diagram

$$\begin{tikzcd} K_0X \arrow[r, "\tau"] \arrow[d, "f_!"] & H_{\mathbb{Q}}(X) \arrow[d, "f_*"] \\ K_0Y \arrow[r, "\tau"] & H_{\mathbb{Q}}(Y) \end{tikzcd}$$

commutes, where $f_!$ is defined by $f_!([\mathcal{F}])=\sum_i(-1)^i[R^if_*(\mathcal{F})]$ for any coherent sheaf $\mathcal{F}$ on X. (In the non-compact case τ takes values in the Borel–Moore homology, [Ive86], [Ful95, 19.1].) It is furthermore shown in [Ful95] in the algebraic case and in [Lev08] in the complex analytic case that τ may be extended as a Grothendieck transformation between bivariant theories in the sense of [FM81]. We refer to the original papers and to the books [DV76, FL85, Ful95], for a thorough treatment of these facts.

For a coherent sheaf $\mathcal{F}$ on a compact complex space X we shall call $\tau(\mathcal{F}):=\tau([\mathcal{F}])$ the *homology Todd class* of $\mathcal{F}$. The following Proposition gathers some of its properties, cf. [Tom16, Section 2].

Proposition 2.4. —
(1) *If $\mathcal{F}$ is locally free and X is smooth and connected, $\tau(\mathcal{F})$ is the Poincaré dual of $ch(\mathcal{F}) \cdot \mathrm{Td}(X) \in H^{2*}(X;\mathbb{Q})$.*
(2) *If $f : X \to Y$ is an embedding then $\tau(f_*\mathcal{F}) = f_*(\tau(\mathcal{F}))$.*
(3) *$\tau(\mathcal{F})_r = 0$ for $r > \dim \operatorname{Supp} \mathcal{F}$.*
(4) *τ is additive on exact sequences.*
(5) *If X is irreducible then $\tau(\mathcal{F})_{\dim X} = \mathrm{rank}(\mathcal{F})[X] \in H_{2\dim X}(X;\mathbb{Q})$.*
(6) *The component of $\tau(\mathcal{F})$ in degree $\dim \operatorname{Supp} \mathcal{F}$ is the homology class of an effective analytic cycle.*

Definition 2.5. — *Let X be an analytic space of dimension n endowed with a system of differential forms $(\omega_1, \ldots, \omega_{\dim(X)})$ such that each ω_p is* d*-closed of degree $2p$ with $(p.p)$-component $\omega_p^{p,p}$ which is strictly positive in Lelong's sense; cf.* [BM14, III.2.4, IV.10.6]. *We call such an $(X, \omega_1, \ldots, \omega_{\dim(X)})$ a* multi-polarized analytic space. *We will moreover always put $\omega_p = 0$ if $p > \dim(X)$ and $\omega_0 = 1$. For any coherent sheaf $\mathcal{F}$ with compact support on a multi-polarized analytic space $(X, \omega_1, \ldots, \omega_{\dim(X)})$ and any non-negative p we define the p-*degree *of $\mathcal{F}$ by $\deg_p(\mathcal{F}) := \int_{\tau_p(\mathcal{F})} \omega_p$, where the integral is computed on a semianalytic representative of $\tau_p(\mathcal{F})$, cf.* [BH69, DP75], [AG06, Thm. 7.22], [Gor81, 8.4],[Her66].

Occasionally we will use positive (p,p)-forms instead of strictly positive forms. In such cases we will speak of *pseudo-degrees* and *pseudo-volumes* instead of degrees and volumes.

When (X, ω) is a Kähler space, cf. [Var89, II.1.2], we will consider its standard multi-polarization given by $(\omega, \omega^2, \ldots, \omega^{\dim(X)})$. When $(X, \mathcal{O}_X(1))$ is a projective variety endowed with an ample line bundle, by taking ω a strictly positive curvature form of $\mathcal{O}_X(1)$, we recover the coefficient of the Hilbert polynomial of $\mathcal{F}$ in degree p as

$$\frac{\deg_p(\mathcal{F})}{p!}.$$

One further property we shall need is the invariance of degrees in a flat family of coherent sheaves. For this we need some preparations. We start with a statement which is an analogue of [Ful95, Example 18.3.8] in our context. We shall work on the category $\mathcal{C}$ of complex analytic spaces which have the topology of finite CW-complexes. For the bivariant theories we will consider on $\mathcal{C}$ the *confined morphisms* will be the proper morphisms in $\mathcal{C}$ and the *independent squares* will be the fiber squares in $\mathcal{C}$; see [FM81] for the terminology. We will use the two operational bivariant theories K and H on $\mathcal{C}$ induced respectively by the homological theories K_0 and $H_{\mathbb{Q}}$ as in [FM81, I.7]. By [Lev08] τ extends to a Grothendieck transformation between these bivariant theories, which we denote again by τ. Restriction to fibers will be understood with respect to either of these bivariant theories. Note that a topological bivariant theory on $\mathcal{C}$ may be also constructed from the standard cohomology theory and induces the usual homology theory, see [FM81, Remark 3.1.9]. In particular specialization in this theory as described in [FM81, I.3.4.4] induces specialization in the theory H; cf. [FM81, I.8.2].

Proposition 2.6. — *Let $X \to T$ be a proper morphism in $\mathcal{C}$ with T smooth and let $\mathcal{E}$ be a coherent sheaf on X flat over T. Then $\tau(\mathcal{E}_t) = \tau(\mathcal{E})_t$ in $H_{\mathbb{Q}}(X)$ for all $t \in T$.*

Proof. — The sheaf $\mathcal{E}$ defines a class $\alpha = [\mathcal{E}]$ in $K(X \to T)$ in the following way. For any fiber square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ T' \arrow[r, "g"] & T \end{tikzcd}$$

we define $\alpha_{Y'} = [\mathcal{E}]_{Y'} : K_0(Y') \to K_0(X')$ by $\mathcal{F} \mapsto \mathcal{F} \otimes_Y \mathcal{E} := \mathcal{F}_{X'} \otimes \mathcal{E}_{X'}$. In order for α to be a class in $K(X \to T)$ the following compatibility conditions for the morphisms $\alpha_{Y'}$ need to be satisfied: if

$$\begin{tikzcd} X'' \arrow[r, "h'"] \arrow[d, "f''"] & X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ T'' \arrow[r, "h"] & T' \arrow[r, "g"] & T \end{tikzcd}$$

is a fiber diagram with h proper, then

$$\begin{tikzcd} K_0(T'') \arrow[r, "\alpha_{T''}"] \arrow[d, "h_*"] & K_0(X'') \arrow[d, "h'_*"] \\ K_0(T') \arrow[r, "\alpha_{T'}"] & K_0(X') \end{tikzcd}$$

must commute. This holds because by the flatness of $\mathcal{E}$ over T one has canonical isomorphisms on X'

$$\mathcal{E}_{X'} \otimes f'^* \left(R^i h_* \mathcal{F}\right) \cong R^i h'_* \left(\mathcal{E}_{X''} \otimes \mathcal{F}_{X''}\right)$$

for all coherent sheaves $\mathcal{F}$ on X'' and all $i \in \mathbb{N}$. The proof of this fact goes exactly as that of [Har77, Proposition III.9.3] using the flatness of $\mathcal{E}_{X'}$ over T' instead of the flatness of $\mathcal{O}_{X'}$.

Applying now the Grothendieck transformation $\tau : K \to H$ to α, the class of $\mathcal{O}_T$ and the map $t \to T$ gives the desired equality of classes. $\square$

Corollary 2.7. — *Let $f : X \to T$ be a proper morphism of reduced irreducible complex spaces and let $\mathcal{E}$ be a coherent sheaf on X which is flat over T. Suppose that X is endowed with a system $(\omega_j)_{j \geqslant 1}$ of closed differential forms inducing multi-polarizations on each fibre X_t of f. Then the degree functions $t \mapsto \deg_p(\mathcal{E}_t)$ computed with respect to these polarizations are constant on T for each positive p.*

Proof. — The statement is local on T. In fact for our purposes T may be assumed to be the complex disc Δ. Moreover using Thom's First Isotopy Lemma we may also assume that the induced map $X \setminus f^{-1}(0) \to \Delta \setminus \{0\}$ is a topologically locally trivial fibration, cf. [Dim92, 1.3.6], [Dim04, p. 179]. By the Proposition $\tau(\mathcal{E}_t) = \tau(\mathcal{E})_t$ in $H_{\mathbb{Q}}(X)$. We have to check that $\deg_p(\mathcal{E}_t) = \deg_p(\mathcal{E}_0)$ for any $t \in \Delta \setminus \{0\}$. This follows now using the description of specialization of homology given in [FM81, I.8.2], the fact that the forms ω_p are closed and Stokes' theorem in this context, see e.g. [Gor81, 8.4]. □

Note that a system of closed differential forms as in the above Corollary is directly obtained as soon as the morphism f is Kähler; see Definition 2.3 and comments thereafter.

2.3. Degree systems

For technical reasons mainly related to our local volume computations the degrees of coherent sheaves used in this paper will be given by fixed positive differential forms on the base complex spaces. In this subsection we will nevertheless attempt at a more general definition which we believe to be naturally adapted to our considerations.

Let X be a compact analytic space of dimension n and let d, d' be integers satisfying $n \geqslant d \geqslant d' \geqslant 0$. We denote by $[F]$ the class in $K_0(X)$ of a coherent sheaf F. If F has dimension at most p, we write $\mathrm{cycle}_p(F)$ for the p-cycle associated to F.

Definition 2.8 (Degrees). — *A group morphism* $\deg_p : K_0(X) \to \mathbb{R}$ *will be called a* degree function in dimension p *on* X *if it enjoys the following properties:*

(1) $\deg_p$ *induces a positive map on non-zero p-cycles when putting* $\deg_p(Z) := \deg_p([\mathcal{O}_Z])$ *for irreducible p-cycles Z*
(2) $\deg_p([F]) = \deg_p(\mathrm{cycle}_p(F))$ *for any coherent sheaf F of dimension at most p on X,*
(3) *if a set of positive p-cycles is such that* $\deg_p$ *is bounded on it, then* $\deg_p$ *takes only finitely many values on this set,*
(4) $\deg_p$ *is continuous on flat families of sheaves.*

If $\deg_p$ *is even locally constant on flat families of sheaves, we will say that it is a* strong degree function. *We will write for simplicity* $\deg_p(F) = \deg_p([F])$ *for any* $F \in \mathrm{Coh}(X)$.

A collection $(\deg_d, \ldots, \deg_{d'})$ *of (strong) degree functions in dimensions d to d' on X will be called a* (strong) (d, d')-degree system. *An $(n, 0)$-degree system will be called a* complete degree system.

Note that for any degree function $\deg_p$ in dimension p on X one has $\deg_p(F) > 0$ if F is p-dimensional, and $\deg_p(F) = 0$ if F is at most $(p-1)$-dimensional.

Definition 2.9 (Relative degrees). — *If X is a complex space proper over a complex space S a family* $(\deg_{d,s}, \ldots, \deg_{d',s})_{s \in S}$ *of degree systems parameterized by S will be called a* relative degree system *on X/S if each degree function* $\deg_{\delta,s}$, $d \geqslant \delta \geqslant d'$, *is continuous on flat families of sheaves over S, i.e. on flat families over any S' after any base change $S' \to S$. Such a relative degree system will be said to be* strong *if its degree functions are locally constant on flat families of sheaves and it will be said to be* complete *if the dimensions of the degree functions range from the relative dimension of X/S to* 0.

One can also think of introducing degree systems depending on real parameters, as in deformation theory for instance, where one defines a notion of a family of complex manifolds over a real manifold. In this paper we will only deal with product situations of this type. In such a case we define continuous families of relative degree systems as follows. If X is a complex space proper over a complex space S and if T is a locally compact topological space a family $(\deg_{d,(s,t)}, \dots, \deg_{d',(s,t)})_{(s,t)\in S\times T}$ of degree systems parameterized by $S \times T$ will be said to be a *continuous family of relative degree systems* if for each $t \in T$ the restricted family is a relative degree system on X/S and for any flat families of sheaves over S the corresponding evaluations of the degree functions of the system are continuous on $S \times T$.

Our basic example of a strong complete degree system is the one defined as in Definition 2.5 on a multi-polarized analytic space $(X, \omega_1, \dots, \omega_{\dim(X)})$. In particular a compact Kähler space (X, ω) has a standard strong complete degree system. The degree functions of such systems satisfy condition (3) of Definition 2.8 since they are constant on connected components of the corresponding cycle spaces and since no analytic space can have an infinite number of irreducible components accumulating at a point. The fact that they are constant on flat families of sheaves follows from Corollary 2.7. Strong complete relative degree systems appear in a situation as in the hypothesis of Corollary 2.7, in particular in the case of proper Kähler morphisms. Weakly Kähler morphisms give rise only to complete relative degree systems.

The condition on the positive differential forms defining degree functions of being d-closed may be relaxed to asking only $\partial\bar{\partial}$-closure when one works on an ambient compact complex manifold. In such a situation using Property (1) of Proposition 2.4 one can replace in the definition of degrees homology Todd classes of coherent sheaves by Chern classes in Bott–Chern cohomology. One can see that Condition (3) of Definition 2.8 is satisfied by the same argument as in the Kähler case. Indeed such a function is pluriharmonic on the cycle space by [Bar78, Proposition 1] and attains its minimum on any closed subset of the cycle space by Bishop's theorem. It is therefore constant on any irreducible component of this space. In particular any smooth compact complex surface may be endowed with a complete degree system by fixing a Gauduchon metric on it.

Whenever degree functions or systems are induced by fixing strictly positive differential forms as in the above cases, we will say that they *come from differential forms*. The results of this paper will be established only for such degree systems, since essential use will be made of local volume computations as well as of (co)-homological projection formulas and duality as in Section 4. It would be interesting to formulate abstract conditions for degree functions allowing to prove similar results.

3. Metric computations

In this section we will consider positivity issues for some forms appearing naturally on total spaces of Kähler mappings, particularly on projectivized bundles. Positivity and strict positivity for (p,p)-forms on a manifold will be considered in the sense of Lelong, i.e. by asking non-negativity, respectively positivity, on each complex tangent p-plane, cf. [BM14, Définition 10.6.1].

The main objective is to show that it is enough to bound a certain pseudo-volume of projectivized bundles in order to get a true volume bound. More precisely, we fix a compact irreducible analytic space X of dimension n and a coherent sheaf G on X. (Actually the irreducibility assumption is made here only for simplicity.) In Proposition 3.2 we endow $\mathbb{P}(G)$ with a certain positive, not necessarily strictly positive, differential form Ω which depends on two fixed strictly positive forms ω_n, ω_{n-1} on X and on a metric h on G. For each rank one quotient F of G we consider the pseudo-volume of the subspace $\mathbb{P}(F)$ of $\mathbb{P}(G)$ computed by integrating Ω on $\mathbb{P}(F)$. We want to show that a uniform bound of this pseudo-volume for all rank one quotients of G leads to a uniform bound on a true volume of the subspaces $\mathbb{P}(F) \subset \mathbb{P}(G)$. One problem appearing when trying to compute metrics on such objects is the fact that the sheaves involved and in particular G may be singular. Moreover in general G is not even a global quotient of a locally free sheaf. But it is so locally. Hence the idea to go to a local situation over some open subset U of X where a locally free sheaf V exists together with a local epimorphism $V \to G_U$. Proposition 3.3 says that we can always lift a hermitian metric from G_U to V. We will work with such a lifted metric on V when we compute the pseudo-volume of the restricted set $\mathbb{P}(F_U) \subset \mathbb{P}(G_U) \subset \mathbb{P}(V)$. If V is of rank $r+1$ we associate to the quotient $V \to F_U$ the sheaf $F' = \operatorname{Coker}(\bigwedge^r \operatorname{Ker}(V \to F_U) \to \bigwedge^r V)$, show that the volume of $\mathbb{P}(F')$ is controlled by the pseudo-volume of $\mathbb{P}(F_U)$ and finally show that the volume of $\mathbb{P}(F')$ controls a true volume of $\mathbb{P}(F_U)$.

Proposition 3.1. — *Let X be an n-dimensional compact complex manifold and let $f : Y \to X$ be a smooth proper map of relative dimension $m > 0$. For $0 \leqslant p < m$ let η_p, (η_{p+1}), be a (p,p)-form, (respectively a $(p+1,p+1)$-form), on Y which are strictly positive on the fibres of $Y \to X$ and let ω_{n-1}, (ω_n), be a strictly positive $(n-1,n-1)$-form, (respectively a strictly positive $2n$-form), on X. Then there exists some positive constant C such that the form*

$$\Omega := Cf^*(\omega_n) \wedge \eta_p + f^*(\omega_{n-1}) \wedge \eta_{p+1}$$

is positive on Y and strictly positive on any complex tangent $(n+p)$-plane whose projection on X is at least $(n-1)$-dimensional.

Proof. — Let $y \in Y$ be a point and $P \subset T_yY$ be a complex $(n+p)$-plane tangent at y. If f_*P is at most $(n-2)$-dimensional, then it is clear that both forms $f^*(\omega_n) \wedge \eta_p$ and $f^*(\omega_{n-1}) \wedge \eta_{p+1}$ vanish on P. If f_*P is $(n-1)$-dimensional, then $f^*(\omega_n) \wedge \eta_p$ vanishes on P and $f^*(\omega_{n-1}) \wedge \eta_{p+1}$ is strictly positive on P. So we are left with the case when f_*P is n-dimensional. In this case $f^*(\omega_n) \wedge \eta_p$ is positive on P but $f^*(\omega_{n-1}) \wedge \eta_{p+1}$ not necessarily so because of possible negativity of η_{p+1} on horizontal directions.

Since X is compact the problem is local on the base X, so we may suppose that X is a polydisc in $\mathbb{C}^n$. Since f is proper the problem is also local on the fibres, so locally around y we may suppose that f is the projection from a polydisc $\Delta^n \times \Delta^m$ to Δ^n. Let $w_1, \ldots, w_n$ and $z_1, \ldots, z_m$ and $\omega := \sum_{j=1}^n i\mathrm{d}w_j \wedge \mathrm{d}\bar{w}_j$, $\eta := \sum_{j=1}^m i\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$ be the coordinate functions and the standard hermitian forms on Δ^n and Δ^m respectively. We set $v_j := \frac{\partial}{\partial z_j}$, for $j = 1, \ldots, m$ and $h_k := \frac{\partial}{\partial w_k}$, for $k = 1, \ldots, n$. We may choose the coordinates $w_1, \ldots, w_n$ so that at $f(y)$ we have $\omega_{n-1} = \omega^{n-1}$. Moreover we can compare the forms ω_n and ω^n and for our purposes we may suppose that $\omega_n = c_1\omega^n$ for a positive constant c_1. We will further suppose for simplicity that our tangent $(n+p)$-plane P is generated by the vectors $h_1+v'_1, \dots, h_n+v'_n, v_1, \dots, v_p$, where $v'_1, \dots, v'_n$ are vertical tangent vectors for f. For the next computation we shall identify $\bigwedge_{\mathbb{R}}^{q,q}(T_y^*Y)$ to the space of hermitian forms on $\bigwedge^q(T_yY)$, cf. [BM14, Proposition 10.5.4]. We will also write $\widehat{h}_j$ for $h_1 \wedge \dots h_{j-1} \wedge h_{j+1} \wedge \dots \wedge h_n$. With these conventions we get

$$\begin{aligned}
&\Omega\Big((h_1+v'_1)\wedge\cdots\wedge(h_n+v'_n) \\
&\qquad\qquad \wedge v_1\wedge\cdots\wedge v_p, (h_1+v'_1)\wedge\cdots\wedge(h_n+v'_n)\wedge v_1\wedge\cdots\wedge v_p\Big) \\
&= C\omega_n\left(h_1\wedge\cdots\wedge h_n, h_1\wedge\cdots\wedge h_n\right)\eta_p\left(v_1\wedge\cdots\wedge v_p, v_1\wedge\cdots\wedge v_p\right) + \\
&\quad \sum_{j=1}^n \omega^{n-1}\left(\widehat{h}_j,\widehat{h}_j\right)\eta_{p+1}\Big(v_1\wedge\cdots\wedge v_p\wedge\left(h_j+v'_j\right), v_1\wedge\cdots\wedge v_p\wedge\left(h_j+v'_j\right)\Big) \\
&= Cc_1\eta_p\left(v_1\wedge\cdots\wedge v_p, v_1\wedge\cdots\wedge v_p\right) \\
&\qquad\qquad + \sum_{j=1}^n \eta_{p+1}\left(v_1\wedge\cdots\wedge v_p\wedge\left(h_j+v'_j\right), v_1\wedge\cdots\wedge v_p\wedge\left(h_j+v'_j\right)\right).
\end{aligned}$$

We clearly have a positive lower bound on $\eta_p(v_1 \wedge \dots \wedge v_p, v_1 \wedge \dots \wedge v_p)$ and it remains to estimate $\eta_{p+1}(v_1 \wedge \dots \wedge v_p \wedge (h_j+v'_j), v_1 \wedge \dots \wedge v_p \wedge (h_j+v'_j))$. But in the decomposition $\eta_{p+1}(v_1 \wedge \dots \wedge v_p \wedge (h_j+v'_j), v_1 \wedge \dots \wedge v_p \wedge (h_j+v'_j))$ $= \eta_{p+1}(v_1 \wedge \dots \wedge v_p \wedge h_j, v_1 \wedge \dots \wedge v_p \wedge h_j) + 2\Re e(\eta_{p+1}(v_1 \wedge \dots \wedge v_p \wedge h_j, v_1 \wedge \dots \wedge v_p \wedge v'_j)) + \eta_{p+1}(v_1 \wedge \dots \wedge v_p \wedge v'_j, v_1 \wedge \dots \wedge v_p \wedge v'_j)$, the first term is clearly bounded from below and also the sum of the following two terms, since the third term is positive and quadratic and the second one is linear in v'_j. So a uniform constant C satisfying our requirements may be found. ☐

Proposition 3.2. — *Let X be an irreducible compact complex space of dimension n, let G be a coherent sheaf of rank $m+1>1$ on X and let η be a positive $(1,1)$-form on the fibres of $\mathbb{P}(G) \to X$. Let further ω_{n-1}, (ω_n), be a strictly positive $(n-1,n-1)$-form, (respectively a strictly positive $2n$-form), on X and denote by Z the irreducible component of $\mathbb{P}(G)$ covering X and by $f: Z \to X$ the projection map. Then there exist some positive constant C such that the form*

$$\Omega := Cf^*(\omega_n) \wedge \eta^p + f^*(\omega_{n-1}) \wedge \eta^{p+1}$$

is positive on Z and strictly positive on any complex tangent $(n+p)$-plane whose projection on X is at least $(n-1)$-dimensional.

Proof. — The point is to show that we can adapt the argument of Proposition 3.1 to the given situation and produce the desired constant C even though the morphism $f: Z \to X$ is no longer supposed to be smooth.

There will be no restriction of generality if we suppose that the form η on $\mathbb{P}(G)$ appears via a partition of unity $(\pi_i)_i$ subordinated to an open cover $(U_i)_i$ of X using local embeddings $\mathbb{P}(G)_{U_i} \subset \mathbb{P}(H_i)$ and vertical positive forms η_i on $\mathbb{P}(H_i)$, where H_i are free $\mathcal{O}_{U_i}$-modules of finite rank covering G_{U_i} and η_i are vertical Fubini–Study $(1,1)$-forms on $\mathbb{P}(H_i)$. The fact that the forms η_i are defined on $\mathbb{P}(H_i)$ allows us to use the inequalities obtained in the proof of Proposition 3.1 over compact subsets K_i contained in the interior sets of $\operatorname{Supp}(\phi_i)$. Indeed, the morphisms $\mathbb{P}(H_i) \to U_i$

are smooth and there exist positive constants C_i such that on vertical p-planes P tangent to $\mathbb{P}(G)_{K_i}$ comparison formulas $\eta_{|P} \leqslant C_i \eta_{i|P}$ hold. □

Proposition 3.3. — *Let X be a complex space and let U be a relatively compact open subset of X. Let further $H \to G$ be an epimorphism of coherent sheaves on X. Then any hermitian metric $\bar{g}$ on G admits a hermitian lift to H which is strictly positive over U.*

Proof. — One can see that a hermitian lift g of $\bar{g}$ exists using the definition of the metrics which are C^∞ sections in the corresponding sheaves $G^{1,1}$ and $H^{1,1}$, see [Bin83, (4.2)]. The fact that this lift may be changed into one which is strictly positive over U follows from [Bin83, Lemma 4.4]. □

For the convenience of the reader we include next the computation of some easy integrals on $\mathbb{C}^n$ to be used later.

Lemma 3.4. — *Let $(z_1, \ldots z_n)$ be the standard coordinate functions on $\mathbb{C}^n$, put*

$$z_0 = 1, z := (1, z_1, \ldots, z_n) \text{ and } |z|^2 := \sum_{j=0}^{n} |z_j|^2 = 1 + \sum_{j=1}^{n} |z_j|^2 .$$

Let further ω be the restriction to $\mathbb{C}^n$ of the (standard) Fubini–Study metric on $\mathbb{P}^n$, where $\mathbb{C}^n$ is seen as a chart domain of $\mathbb{P}^n$, i.e. $\omega = \frac{i}{2\pi}\partial\bar{\partial}\log|z|^2$, and

$$I_{j\bar{k}} := \int_{(\mathbb{C}^*)^n} \frac{z_j \bar{z}_k}{|z|^2}\omega^n, \text{ for } j, k \in \{0, \ldots, n\}.$$

Then $I_{j\bar{k}}$ equals 0 for $j \neq k$ and $\frac{1}{n+1}$ for $j = k$.

Proof. — We start with the case $j = k$. It is readily shown that $\int_{(\mathbb{C}^*)^n} \omega^n = 1$, cf. [Dem07]. Thus $\sum_{j=0}^n I_{j\bar{j}} = 1$. On $(\mathbb{C}^*)^n$ we use the coordinate change f, defined by $z_1 = \frac{1}{w_1}$, $z_j = \frac{w_j}{w_1}$ for $j \notin \{0, 1\}$, to get

$$\begin{aligned} |z|^2 = \frac{|w|^2}{|w_1|^2}, f^*\omega &= f^*\left(\frac{i}{2\pi}\partial\bar{\partial}\log\left(1 + \sum_{j=1}^{n}|z_j|^2\right)\right) \\ &= \frac{i}{2\pi}\partial\bar{\partial}\log\left(\frac{1}{|w_1|^2}\left(1 + \sum_{j=1}^{n}|w_j|^2\right)\right) \\ &= \frac{i}{2\pi}\partial\bar{\partial}\log|w|^2 - \frac{i}{2\pi}\partial\bar{\partial}\log|w_1|^2 = \frac{i}{2\pi}\partial\bar{\partial}\log|w|^2 =: \omega_w \end{aligned}$$

and

$$I_{0\bar{0}} = \int_{(\mathbb{C}^*)^n} \frac{1}{|z|^2}\omega^n = \int_{(\mathbb{C}^*)^n} \frac{|w_1|^2}{|w|^2}\omega_w^n = I_{1\bar{1}},$$

hence $I_{j\bar{j}} = \frac{1}{n+1}$ for all $j \in \{0, \ldots, n\}$.

We now look at the case $j \neq k$. The same substitution as in the case $j = k$ shows that $I_{0\bar{1}} = I_{1\bar{0}}$ and a similar computation also gives $I_{1\bar{k}} = I_{0\bar{k}}$ for all $k \notin \{0, 1\}$. By symmetry all $I_{j\bar{k}}$ are thus equal when $j \neq k$. Finally the orientation reversing coordinate change $z_1 = \bar{w}_1$, $z_j = w_j$ for $j \notin \{0, 1\}$ leads to $I_{0\bar{1}} = -I_{1\bar{0}} = -I_{0\bar{1}}$, whence our assertion. □

For any hermitian holomorphic line bundle (L,h) on a complex manifold we will denote by R^L its curvature form with respect to the Chern connection of (L,h) and by $c_1(L,h) := \frac{i}{2\pi}R^L$ the associated Chern form. We have $R^L = -\partial\bar{\partial}\log|s|^2$, where s is any (non-vanishing) holomorphic local section of L. If L is the associated determinant line bundle of a holomorphic hermitian bundle (E,h) and if $(h_{j\bar{k}}) = (h(s_j,s_k))$ is the matrix of h with respect to a local holomorphic frame $s=(s_1,\ldots,s_r)$ of E, then $R^L = -\partial\bar{\partial}\log(\det(h_{j\bar{k}}))$.

The next Lemma 3.5 serves in establishing a local comparison formula between (pseudo-)volumes of projective subbundles in the following situation. To a pure rank one quotient F of a locally free sheaf V of rank $r+1$ over a reduced complex space X we associate a pure quotient F' of rank r of $\bigwedge^r V$ in a canonical way via the cokernel of the morphism $\bigwedge^r \mathrm{Ker}(V\to F)\to\bigwedge^r V$. (Recall that a coherent sheaf F is called *pure of dimension* d if F as well as all its non-trivial coherent subsheaves are of dimension d.) When X and V are endowed with hermitian metrics we get corresponding (pseudo-)volume forms on $\mathbb{P}(F)$ and $\mathbb{P}(F')$. We shall see that the corresponding (pseudo-)volume of $\mathbb{P}(F)$ computed over any open relatively compact subset U of X controls the volume of $\mathbb{P}(F')$ over the same subset U. Since we are interested in volumes of irreducible complex spaces it is enough to integrate volume forms over dense Zariski open subsets. Therefore we may and will assume that X is smooth and that F is locally free over X.

Let ω_{n-1},ω_n be positive forms on X of bidegrees $(n-1,n-1)$ and (n,n) respectively. For simplicity of notation we will denote also by ω_{n-1} and ω_n the pullbacks of these forms to $\mathbb{P}(F)$ and $\mathbb{P}(F')$. We will denote by η_V the relative Fubini–Study $(1,1)$-form on $\mathbb{P}(V)$ with respect to a hermitian metric h on V and we will use the same symbol for its restriction to $\mathbb{P}(F)$, and likewise for the induced relative Fubini–Study $(1,1)$-form $\eta_{\bigwedge}$ on $\mathbb{P}(\bigwedge^r V)$ and for its restriction to $\mathbb{P}(F')$. The (pseudo-)volume forms we consider on $\mathbb{P}(F)$ and on $\mathbb{P}(F')$ have each two components, namely ω_n and $\eta_V\wedge\omega_{n-1}$ on $\mathbb{P}(F)$, and $\eta_{\bigwedge}^{r-1}\wedge\omega_n$ and $\eta_{\bigwedge}^{r}\wedge\omega_{n-1}$ on $\mathbb{P}(F')$.

The comparison of ω_n on $\mathbb{P}(F)$ to

$$\eta_{\bigwedge}^{r-1}\wedge\omega_n \;\text{ on }\; \mathbb{P}(F')$$

is immediately done since their fibre integrals with respect to the projections $\mathbb{P}(F)\to X$ and $\mathbb{P}(F')\to X$ are both equal to ω_n by the projection formula for fibre integrals, [Sto71]. In the sequel we denote fibre integrals by $\int_{Y\to X}$. By the projection formula again we get

$$\int_{\mathbb{P}(F)\to X}(\eta_V\wedge\omega_{n-1}) = \left(\int_{\mathbb{P}(F)\to X}\eta_V\right)\wedge\omega_{n-1}$$

and

$$\int_{\mathbb{P}(F')\to X}\left(\eta_{\bigwedge}^{r}\wedge\omega_{n-1}\right) = \left(\int_{\mathbb{P}(F')\to X}\eta_{\bigwedge}^{r}\right)\wedge\omega_{n-1}$$

so we are left with the task of comparing $\int_{\mathbb{P}(F)\to X}\eta_V$ to $\int_{\mathbb{P}(F')\to X}\eta_{\bigwedge}^{r}$. The result is

Lemma 3.5. — *In the above set-up we have*

$$\int_{\mathbb{P}(F)\to X}\eta_V=\int_{\mathbb{P}(F')\to X}\eta^r_{\Lambda}-(r-1)\frac{i}{2\pi}R^{\det V}.$$

Proof. — Any $(1,1)$-form on X is determined by its restriction to the smooth curves on X. Moreover fibre integration is compatible with restriction, cf. [Sto71], so we may assume that X is smooth one-dimensional. The property we want to prove is local on the base so we will consider a point x of X around which V is trivial and endowed with a hermitian metric h^V. Let $s=(s_0,s_1,\,\dots,\,s_r)$ be a normal holomorphic local frame of (V,h^V) at x, i.e. such that with respect to it we have $h^V_{i\bar{j}}(x)=\delta_{ij}$ and $(\mathrm{d}h^V_{i\bar{j}})(x)=0$ for all $i,j\in\{0,\,\dots,\,r\}$, cf. [Kob87, Proposition 1.4.20].

We start by computing the $(1,1)$-form η_V on $\mathbb{P}(F)$. We may consider $\mathbb{P}(F)$ as a subspace of $\mathbb{P}(V)$ but also as a subspace of $\mathbb{P}_{sub}(V^*)$ using the canonical identification $\mathbb{P}(V)\cong\mathbb{P}_{sub}(V^*)$. For our metric computation we will prefer the latter point of view. Here $\mathbb{P}_{sub}(V^*)$ is the projective bundle parameterizing 1-dimensional subspaces in the fibers of V^*. The form η_V on $\mathbb{P}_{sub}(V^*)$ is the curvature of the tautological quotient $\mathcal{O}(1)$ on $\mathbb{P}_{sub}(V^*)$ and is computed as follows, see also [Kob87, proof of Theorem 3.6.17], [MM07, Section 5.1.1], [GH78] or [Dem07]. The metric h^V on V induces a metric h^{V^*} on V^*, a further metric $h^{\mathcal{O}(-1)}$ on the tautological subbundle $\mathcal{O}(-1)$ on $\mathbb{P}_{sub}(V^*)$ and a metric $h^{\mathcal{O}(1)}$ on the dual line bundle $\mathcal{O}(1)$. A holomorphic section v in V defines a fibrewise linear functional on V^* by $f\mapsto(f,v)$, hence a section σ_v in $\mathcal{O}(1)$ on $\mathbb{P}_{sub}(V^*)$. If $f(x)\neq 0$ one gets around the point $[f(x)]\in\mathbb{P}_{sub}(V^*)_x$

$$|\sigma_v([f])|^2_{h^{\mathcal{O}(1)}}=\frac{|(f,v)|^2}{|f|^2_{h^{V^*}}}$$

and if v doesn't vanish at $f(x)$

$$\eta_V=\frac{i}{2\pi}R^{\mathcal{O}(1)}=-\frac{i}{2\pi}\partial\bar{\partial}\log|\sigma_v([f])|^2_{h^{\mathcal{O}(1)}},$$

where $R^{\mathcal{O}(1)}$ denotes the curvature of the Chern connection of $\mathcal{O}(1)$ on $\mathbb{P}_{sub}(V^*)$. The chosen holomorphic frame of V trivializes V giving $V\cong X\times V_x\cong X\times\mathbb{C}^{r+1}$. Let $e_j=s_j(x)$ and denote by $(e^*_j)_j$ the dual base of $(e_j)_j$. Denote by z a local coordinate function on X at x. With respect to the above trivialization of V the form η_V decomposes as a sum of a vertical component $\eta_{V,\,vert}$, which is the corresponding Fubini–Study form in that fiber, a horizontal component $\eta_{V,\,hor}$, and a mixed component, which is immediately seen to vanish at all points lying over x by the normality of the chosen frame s. At the point $(x,(1:0:\,\dots\,:0))$ a direct computation shows that

$$\eta_{V,\,hor}=\frac{i}{2\pi}\partial\bar{\partial}\log h^{V^*}_{0,\bar{0}}=\frac{i}{2\pi\left(h^{V^*}_{0,\bar{0}}\right)^2}\partial\bar{\partial}h^{V^*}_{0,\bar{0}}=\frac{i}{2\pi}\frac{\partial^2h^{V^*}_{0,\bar{0}}}{\partial z\partial\bar{z}}\mathrm{d}z\mathrm{d}\bar{z}.$$

We view $\mathbb{P}(F)$ as the image of a section $\sigma:X\to\mathbb{P}_{sub}(V^*)\cong X\times\mathbb{P}^r_{sub}$ passing through the point $(x,[1:0:\,\dots\,:0])\in X\times\mathbb{P}^r_{sub}$ and given by $\sigma(z)=(z,[f(z)])$,

$$f(z)=e^*_0+f_1(z)e^*_1+\,\dots\,f_r(z)e^*_r,\tag{3.1}$$

where the functions f_j are holomorphic and vanish at 0. Then $\int_{\mathbb{P}(F)\to X}\eta_V=\sigma^*\eta_V$, which at x gives

$$\begin{aligned}\sigma^*\eta_V&=\sigma^*\eta_{V,\,hor}+\sigma^*\eta_{V,\,vert}=\frac{i}{2\pi}\partial\bar\partial\log h^{V^*}_{0,\bar 0}+\frac{i}{2\pi}\partial\bar\partial\log\left(1+\sum_{j=1}^r|f_j|^2\right)\\&=\frac{i}{2\pi}\left(\frac{\partial^2 h^{V^*}_{0,\bar 0}}{\partial z\partial\bar z}+\sum_{j=1}^r\left|f_j'(0)\right|^2\right)\mathrm{d}z\mathrm{d}\bar z=\frac{i}{2\pi}\left(-\frac{\partial^2 h^{V}_{0,\bar 0}}{\partial z\partial\bar z}+\sum_{j=1}^r\left|f_j'(0)\right|^2\right)\mathrm{d}z\mathrm{d}\bar z.\end{aligned}$$

We will next compute $\eta_{\bigwedge}$ by working on $\mathbb{P}(V^*)$, which is isomorphic to $\mathbb{P}(\bigwedge^r V)$ via the canonical isometry $\bigwedge^r V\cong V^*\otimes\det V$. We will first compute η_{V^*} and then $\eta_{\bigwedge}=\eta_{V^*}+\frac{i}{2\pi}R^{\det V}=\eta_{V^*,vert}+\eta_{V^*,hor}+\frac{i}{2\pi}R^{\det V}$ at points over x. To do this we will use the isomorphism $\mathbb{P}(V^*)\cong\mathbb{P}_{sub}(V)$. Let $E:=\operatorname{Ker}(V\to F)$. We are interested in the restriction of

$$\eta^r_{\bigwedge}=\eta^r_{V^*,\,vert}+r\eta^{r-1}_{V^*,\,vert}\wedge\eta_{V^*,\,hor}+r\eta^{r-1}_{V^*,\,vert}\wedge\frac{i}{2\pi}R^{\det V}$$

to the subbundle $\mathbb{P}_{sub}(E)$ of $\mathbb{P}_{sub}(V)$ and in its fiber integral with respect to $\mathbb{P}_{sub}(E)\to X$.

We compute the integrals of the three terms appearing on the right handside of the above expression of $\eta^r_{\bigwedge}$. The third integral gives

$$\frac{ri}{2\pi}R^{\det V}=-\frac{ri}{2\pi}\sum_{j=0}^r\partial_z\bar\partial_z h^V_{j,\bar j}.$$

Over x with respect to our fixed frame s the subspace E_x of V_x is given by the condition $e_0^*=0$. Consider the standard chart on $\mathbb{P}(E_x)$ corresponding to $e_1^*=1$ and set $w=(1,w_2,\,\dots,\,w_r)$, $w_1=1$, as "coordinate functions" similarly to Lemma 3.4. Then at such a point we get

$$\begin{aligned}\eta_{V^*,\,hor}&=-\frac{i}{2\pi}\partial_z\bar\partial_z\log\left(\frac{1}{|w|^2_{h^V}}\right)=\frac{i}{2\pi}\partial_z\bar\partial_z\log\left(\sum_{j=1}^r\sum_{k=1}^r h^V_{j\,\bar k}w_j\bar w_k\right)\\&=\frac{i}{2\pi}\frac{\sum_{j=1}^r\sum_{k=1}^r w_j\bar w_k\partial_z\bar\partial_z h^V_{j\,\bar k}}{|w|^2}.\end{aligned}$$

Hence taking Lemma 3.4 into account the second integral over the fibres of $\mathbb{P}_{sub}(E)$ gives at x

$$\frac{i}{2\pi}\sum_{j=1}^r\partial_z\bar\partial_z h^V_{j\bar j}=-\frac{i}{2\pi}R^{\det V}-\frac{i}{2\pi}\partial_z\bar\partial_z h_{0\bar 0}\,.$$

In order to compute the first fibre integral we first parameterize $\mathbb{P}_{sub}(E)$ using the map

$$\begin{aligned}(3.2)\quad \tau: X\times\mathbb{P}^{r-1}\to X\times\mathbb{P}^r,\ (x,[v_1:\,\dots\,:v_r])&\\ \mapsto\left(z,\left[-\sum_{j=1}^r v_jf_j(z):v_1:\,\dots\,:v_r\right]\right),&\end{aligned}$$

where the functions f_j are those appearing in formula (3.1). The pull-back of $\eta_{V^*,vert}$ to $X \times \mathbb{P}^{r-1}$ over x through this map equals

$$\tau^*\eta_{V^*,vert} = \frac{i}{2\pi}\partial\bar{\partial}\log\left(\left|\sum_{j=1}^{r} f_j(z)v_j\right|^2 + |v_1|^2 + \ldots + |v_r|^2\right)$$

and we may as before integrate over the standard chart domain $\mathbb{C}^{r-1}$ where $v_1 = 1$. We set $w_1 = 1$ and let $w_2, \ldots, w_r$ be the standard coordinate functions on $\mathbb{C}^{r-1}$ similarly to Lemma 3.4. We also write $\mathrm{d}w_1 = 0$. Then we get

$$\begin{aligned}
\tau^*\eta_{V^*,vert} &= \frac{i}{2\pi}\partial\bar{\partial}\log\left(\left|\sum_{j=1}^{r} f_j(z)w_j\right|^2 + |w|^2\right) \\
&= \frac{i}{2\pi}\partial\frac{\left(\sum_{j=1}^{r} f_j(z)w_j\right)\left(\left(\sum_{j=1}^{r} \bar{f}_j'(z)\bar{w}_j\right)\mathrm{d}\bar{z} + \sum_{j=1}^{r}\bar{f}_j(z)\mathrm{d}\bar{w}_j\right) + \bar{\partial}|w|^2}{\left|\sum_{j=1}^{r} f_j(z)w_j\right|^2 + |w|^2} \\
&= \frac{i}{2\pi}\left(\frac{\left|\sum_{j=1}^{r} f_j'(z)w_j\right|^2 \mathrm{d}z\mathrm{d}\bar{z} + \partial\bar{\partial}|w|^2}{|w|^2} - \frac{\partial|w|^2 \wedge \bar{\partial}|w|^2}{|w|^4}\right) \\
&= \frac{i}{2\pi}\frac{\left|\sum_{j=1}^{r} f_j'(z)w_j\right|^2 \mathrm{d}z \wedge \bar{z}}{|w|^2} + \omega_{FS}
\end{aligned}$$

and

$$\tau^*\eta^r_{V^*,vert} = \frac{ri}{2\pi}\frac{\left|\sum_{j=1}^{r} f_j'(z)w_j\right|^2 \mathrm{d}z\mathrm{d}\bar{z}}{|w|^2} \wedge \omega_{FS}^{r-1}$$

which integrated in the fiber over x gives

$$\frac{i}{2\pi}\sum_{j=1}^{r}\left|f_j'(0)\right|^2 \mathrm{d}z\mathrm{d}\bar{z}$$

by use of Lemma 3.4.

Summing the three integrals up we get the desired statement. □

We next change slightly the set-up of Lemma 3.5 and specialize to the case when the metric h on V is trivial. To compute a true volume of $\mathbb{P}(F)$ it is enough to know the fibre integrals of higher powers of η_V and wedge them with appropriate powers of a Kähler form ω of a hermitian metric on X. In order to do this we start by computing the form

$$\alpha := \int_{\mathbb{P}(F)\to X} \eta_V.$$

Lemma 3.6. — *In the above set-up if the metric h on V is trivial, we have*

$$\alpha := \int_{\mathbb{P}(F)\to X} \eta_V = \int_{\mathbb{P}(F')\to X} \eta^r_{\bigwedge}$$

and for all positive integers p:

$$\int_{\mathbb{P}(F)\to X} \eta_V^p = \alpha^p \qquad \text{and} \qquad \int_{\mathbb{P}(F')\to X} \eta_{\bigwedge}^{r+p} = 0.$$

Proof. — The first assertion is a direct consequence of Lemma 3.5. For the second one we use the same approach and notations as in the proof of Lemma 3.5 with the difference that the metric h is supposed to be trivial and that the dimension of X is some arbitrary positive integer n. Then using again a section $\sigma : X \to \mathbb{P}_{sub}(V^*) \cong X \times \mathbb{P}^r_{sub}$ passing through the point $(x, [1 : 0 : \ldots : 0]) \in X \times \mathbb{P}^r_{sub}$ and given by $\sigma(z) = (z, [f(z)])$, with f as in equation (3.1) to parameterize $\mathbb{P}(F)$ locally at x and putting $f_0(z) = 1$ we get

$$\sigma^*(\eta_V)^p = (\sigma^*(\eta_V))^p = \left(\frac{i}{2\pi}\partial\bar{\partial}\log\left(\sum_{j=0}^r |f_j|^2\right)\right)^p.$$

Thus

$$\alpha^p = \left(\frac{i}{2\pi}\partial\bar{\partial}\log\left(\sum_{j=0}^r |f_j|^2\right)\right)^p = \int_{\mathbb{P}(F)\to X} \eta_V^p.$$

Finally, using the parameterization (3.2) we obtain over x

$$\tau^*\eta_{\bigwedge} = \frac{i}{2\pi}\frac{\partial_z\left(\sum_{j=1}^r w_j f_j\right) \wedge \overline{\partial_z\left(\sum_{j=1}^r w_j f_j\right)}}{|w|^2} + \omega_{FS},$$

whence for non-negative p the formula

$$\int_{X\times\mathbb{P}^{r-1}\to X} \left(\tau^*\eta_{\bigwedge}\right)^{r+p} =$$

$$\binom{r+p}{p+1}\int_{X\times\mathbb{P}^{r-1}\to X}\left(\frac{i}{2\pi}\frac{\partial_z\left(\sum_{j=1}^r w_j f_j\right) \wedge \overline{\partial_z\left(\sum_{j=1}^r w_j f_j\right)}}{|w|^2}\right)^{p+1} \wedge \omega_{FS}^{r-1},$$

whose integrand vanishes for $p > 0$. $\square$

To complete the passage of volume computation to $\mathbb{P}(F)$ back from $\mathbb{P}(F')$ it will be therefore enough to bound integrals of the type $\int_X \alpha^p \wedge \omega^{n-p}$ in terms of $\int_X \alpha\wedge\omega^{n-1}$ and $\int_X \omega^n$. This is done using inequalities of Hodge Index type, cf. [Dem93, Remark 5.3]. For this approach it is important to note that the form α is positive on X.

4. Degrees of quotient sheaves and volumes

This section can be seen as part of the proof of the Key Lemma (Lemma 5.7). In it we relate degrees of coherent quotient sheaves of a given coherent sheaf G on a reduced irreducible compact analytic space X to (pseudo-) volumes of projectivized bundles. Let n be the dimension of X. We will fix degree functions $\deg_n$, $\deg_{n-1}$, coming from differential forms ω_n, ω_{n-1}, as in Section 2.3. For simplicity we assume that ω_n, ω_{n-1} are the (n,n)-, respectively the $(n-1,n-1)$-, components of d-closed forms, but the arguments can be easily adapted to the case when X is embedded in some complex manifold X' and ω_n, ω_{n-1} are restrictions of $\partial\bar{\partial}$-closed forms on X'. Unlike in [Tom16] we choose in this paper to reduce ourselves to quotients of rank one. This is done by taking top exterior powers. We start with a pure n-dimensional quotient F of rank r of G and perform on it the following operations which correspond to four steps of the proof of Lemma 5.7:

(1) Go from F to a rank one coherent sheaf $F_2 := (\bigwedge^r F)_{pure}$ on X. Here we denote by E_{pure} the pure d-dimensional part of a d-dimensional coherent sheaf E. Show that $\deg_{n-1} F_2$, $\deg_n F_2$ are uniformly controlled by $\deg_{n-1} F$, $\deg_n F$, in a way depending on G and X only.
(2) Go from F_2 to the irreducible component P_2 of $\mathbb{P}(F_2)$ covering the base X. Show that the pseudo-volume of P_2 defined as in Proposition 3.1 is controlled by $\deg_{n-1} F_2$, $\deg_n F_2$.
(3) Go from P_2 to an injective morphism $F_2 \to F_1$ into a coherent sheaf F_1 of rank one on X.
(4) Recover F from the morphism $\bigwedge^r G \to F_1$.

Step 1. — By property (5) of Proposition 2.4 the n-dimensional degree of a coherent sheaf F on X equals $\operatorname{rank}(F)\operatorname{vol}_{\omega_n}(X)$. If F is a quotient of G this quantity is completely controlled by X and by G. Thus $\deg_n F_2 = \operatorname{vol}_{\omega_n}(X) \leqslant \deg_n F$. In order to obtain control over $\deg_{n-1} F_2$ too, we first normalize X and then desingularize. Let $f : Y \to X$ be the normalization map, let $g : Z \to Y$ be a desingularization of Y and put $h := f \circ g$. Let F_2 be the image of the morphism $\bigwedge^r G \to (\bigwedge^r F)_{pure}$. It is also the pure part of the image of the morphism $\bigwedge^r G \to \bigwedge^r F$.

The following Lemma implies that the pseudo-degree $\deg_{f^*\omega_{n-1}} f^*F$ is bounded in terms of $\deg_{n-1} F$.

Lemma 4.1. — *Let $f : Y \to X$ be a finite morphism, F be a pure coherent sheaf on X, which is a quotient of a (pure) coherent sheaf G and let $G'' := \operatorname{Coker}(G \to f_*f^*G)$. Fix a closed positive $(n-1,n-1)$-form Ω on X, where $n = \dim X$. Then*

$$\deg_{n-1,\, f^*\Omega}(f^*F) \leqslant \deg_{n-1,\,\Omega}(F) + \deg_{n-1,\,\Omega}(G'') .$$

Proof. — Set $F'' := \operatorname{Coker}(F \to f_*f^*F)$. By the projection formula and Grothendieck–Riemann–Roch we have

$$\begin{aligned}\deg_{n-1,\, f^*\Omega}(f^*F) &= \tau_{n-1}(f^*F)\, f^*\Omega \;= f_*(\tau_{n-1}(f^*F))\,\Omega \\ &= \tau_{n-1}(f_*f^*F)\,\Omega \;= \deg_{n-1,\,\Omega}(f_*f^*F)\end{aligned}$$

and the desired inequality follows from the commutative diagram with exact rows and columns below

$$\begin{tikzcd} 0 \arrow[r] & G \arrow[r] \arrow[d] & f_*f^*G \arrow[r] \arrow[d] & G'' \arrow[r] \arrow[d] & 0 \quad \square \\ 0 \arrow[r] & F \arrow[r] \arrow[d] & f_*f^*F \arrow[r] \arrow[d] & F'' \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd} \tag{4.1}$$

By applying the projection formula, Grothendieck–Riemann–Roch and the fact that for a coherent sheaf E on Y the sheaves E and g_*g^*E may differ only in codimension larger than one we get

$$\deg_{g^*f^*\omega_{n-1}} h^*F = \deg_{f^*\omega_{n-1}} g_*g^*f^*F = \deg_{f^*\omega_{n-1}} f^*F.$$

If E is a coherent sheaf of rank r on the non-singular space Z and Ω is a closed $(n-1, n-1)$-form on Z, we have

$$\deg_\Omega E \geqslant \deg_\Omega E_{pure} = \deg_\Omega \bigwedge^r (E_{pure}) = \deg_\Omega (\bigwedge^r E)_{pure},$$

the last equality holding due to the fact that the sheaves $\bigwedge^r(E_{pure})$ and $(\bigwedge^r E)_{pure}$ may differ only in codimension larger than one. In particular we get

$$\deg_{f^*\omega_{n-1}} f^*F = \deg_{h^*\omega_{n-1}} h^*F \geqslant \deg_{h^*\omega_{n-1}} (\bigwedge^r h^*F)_{pure}.$$

Consider now the following commutative diagrams

$$\begin{tikzcd} h^*\bigwedge^r G \arrow[r, "\cong"] & \bigwedge^r h^*G \arrow[r, two heads] & \bigwedge^r h^*F \arrow[r, "\cong"] & h^*\bigwedge^r F \arrow[r, two heads] \arrow[d] & (\bigwedge^r h^*F)_{pure} \\ & & & h^*F_2 \arrow[ur, two heads] & \end{tikzcd} \tag{4.2}$$

$$\begin{tikzcd} \bigwedge^r G \arrow[r, two heads] \arrow[d] & \bigwedge^r F \arrow[r, two heads] \arrow[d] & F_2 \arrow[d] & \\ h_*h^*\bigwedge^r G \arrow[r] & h_*h^*\bigwedge^r F \arrow[r] & h_*h^*F_2 \arrow[r] & h_*\left((\bigwedge^r h^*F)_{pure}\right) \end{tikzcd} \tag{4.3}$$

and note that $(\bigwedge^r h^*F)_{pure}$ is isomorphic to $(h^*F_2)_{pure}$. After applying g_* we get an exact sequence

$$0 \to g_*\left(\mathrm{Tors}\,(h^*F_2)\right) \to g_*h^*F_2 \to g_*\left(\left(\bigwedge^r h^*F\right)_{pure}\right) \to R^1g_*\left(\mathrm{Tors}\,(h^*F_2)\right)$$

so the morphism $g_*h^*F_2 \to g_*((\bigwedge^r h^*F)_{pure})$ is an isomorphism in codimension one and it will remain so after application of f_*. Moreover the natural morphism $F_2 \to h_*h^*F_2$ is injective since F_2 is pure and thus

$$\deg_{\omega_{n-1}} F_2 \leqslant \deg_{\omega_{n-1}} h_* h^* F_2 = \deg_{\omega_{n-1}} h_* \left(\left(\bigwedge^r h^* F \right)_{pure} \right)$$
$$= \deg_{h^*\omega_{n-1}} \left(\bigwedge^r h^* F \right)_{pure} \leqslant \deg_{f^*\omega_{n-1}} f^* F$$

and the last term is controlled in terms of $\deg_{\omega_{n-1}} F$, G and X.

Step 2. — Let P_2 be the irreducible component of $\mathbb{P}(F_2)$ which covers X. The projection $p : P_2 \to X$ is bimeromorphic so $\int_{P_2} p^*\omega_n = \int_X \omega_n = \deg_{\omega_n} F_2$, which settles the ω_n-component of the pseudo-metric on P_2. For the ω_{n-1}-component it will be enough to show that $\deg_{p^*\omega_{n-1}} \mathcal{O}_{P_2}(1)$ is controlled by $\deg_{\omega_{n-1}} F_2$. We consider the cartesian square

$$\begin{array}{ccc} P_{2,Y} & \xrightarrow{f'} & P_2 \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{f} & X, \end{array} \tag{4.4}$$

where f is as before the normalization map. Now

$$\deg_{p^*\omega_{n-1}} \mathcal{O}_{P_2}(1) \leqslant \deg_{p^*\omega_{n-1}} f'_* f'^* \mathcal{O}_{P_2}(1) = \deg_{f'^*p^*\omega_{n-1}} f'^* \mathcal{O}_{P_2}(1)$$

and $f'^*\mathcal{O}_{P_2}(1)$ is a quotient of $f'^*p^*F_2 = p'^*f^*F_2$ so

$$\deg_{f'^*p^*\omega_{n-1}} f'^*\mathcal{O}_{P_2}(1) \leqslant \deg_{f'^*p^*\omega_{n-1}} f'^*p^*F_2$$
$$= \deg_{f^*\omega_{n-1}} p'_* f'^* p^* F_2 = \deg_{f^*\omega_{n-1}} p'_* p'^* f^* F_2 = \deg_{f^*\omega_{n-1}} f^* F_2$$

and the last term is controlled by $\deg_{\omega_{n-1}} F_2$ by Lemma 4.1.

Step 3. — Under the previous notations set $F_1 := p_*\mathcal{O}_{P_2}(1)$. The desired injective morphism is given by the composition of the natural morphisms $F_2 \to p_*p^*F_2 \to p_*\mathcal{O}_{P_2}(1) = F_1$.

Step 4. — The recovery of F is a consequence of the following Lemmata.

Lemma 4.2. — *(Pure quotients are determined by their restriction to an open dense subset.) Let G be a coherent sheaf on a pure n-dimensional space X and let $U \subset X$ be a dense Zariski open subspace. A pure d-dimensional quotient $G \to F$ of G is then completely determined by its restriction to U.*

Proof. — Suppose that two quotients $G \to F_1$, $G \to F_2$ with F_1, F_2 pure n-dimensional have the same restriction to U. Set $E_j := \mathrm{Ker}(G \to F_j)$, $1 \leqslant j \leqslant 2$, and $E := E_1 \cap E_2$. From the diagrams

$$\begin{tikzcd}
 & & & C_j \arrow[d, hook] & \\
0 \arrow[r] & E \arrow[r] \arrow[d, hook] & G \arrow[r] \arrow[d] & G/E \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & E_j \arrow[r] \arrow[d, two heads] & G \arrow[r] & F_j \arrow[r] & 0 \\
 & C_j & & &
\end{tikzcd}$$

for $j \in \{1,2\}$ we see that the quotients $G \to F_j$ both appear as the saturation of $G \to G/E$. □

Lemma 4.3. — *(determinant trick) Let $A := \mathcal{O}_{X,x}$ be the local ring of an analytic space of pure dimension d and let F be a pure A-module of dimension n and of rank r over each of its associated points. If F appears as a quotient $G \to F$ of an A-module of finite type G, then F can be completely recovered from the composition of morphisms*

$$G \to \mathrm{Hom}\left(\bigwedge^{r-1} G, \bigwedge^{r} G\right) \to \mathrm{Hom}\left(\bigwedge^{r-1} G, \left(\bigwedge^{r} F\right)_{pure}\right),$$

where the first morphism $\phi : G \to \mathrm{Hom}(\bigwedge^{r-1} G, \bigwedge^r G)$ is given by taking exterior product, $g \mapsto (g_1 \wedge \ldots \wedge g_{r-1} \mapsto g_1 \wedge \ldots \wedge g_{r-1} \wedge g)$, and the second one $\psi : \mathrm{Hom}(\bigwedge^{r-1} G, \bigwedge^r G) \to \mathrm{Hom}(\bigwedge^{r-1} G, (\bigwedge^r F)_{pure})$ is composition with the natural morphism $\bigwedge^r G \to (\bigwedge^r F)_{pure}$. More precisely, one has $F \cong \mathrm{Im}(\psi \circ \phi)$.

Proof. — We will show that $\ker(\psi \circ \phi) = \ker(G \to F)$.

Set $E := \mathrm{Ker}(G \to F)$. It is clear that $E \subset \mathrm{Ker}(\psi \circ \phi)$. We then get a commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & & & C \arrow[d, hook] & \\
0 \arrow[r] & E \arrow[r] \arrow[d, hook] & G \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \mathrm{Ker}(\psi \circ \phi) \arrow[r] \arrow[d, two heads] & G \arrow[r] & \mathrm{Im}(\psi \circ \phi) \arrow[r] & 0 \\
 & C & & &
\end{tikzcd}$$

Since the property we want to show is valid at the generic point of each associated component of F (where F is free of rank r) it follows that $C := \mathrm{Coker}(E \to \mathrm{Ker}(\psi \circ \phi))$ is at most $n-1$ dimensional and we conclude by the purity of F. □

5. Bounded sets of coherent sheaves

Starting from this section all complex spaces will be supposed to be Hausdorff and second countable, i.e. allowing a countable base for their topology. In particular they will be σ-compact, cf. [Bre97, Theorem 12.12]. If X is an analytic space over S with projection morphism $p : X \to S$ and F is a coherent sheaf on X and if $T \to S$ is a morphism, we will write as usual $X_T := X \times_S T$ and F_T for the base change. The projections $X_T \to T$ will be denoted by p_T. For a *germ of an analytic set around a compact set* K in the sense of [BS77, VII.2 (b)] we will use the notation $\widetilde{K}$. When speaking of morphisms or sheaves defined on a germ $\widetilde{K}$ of an analytic space around a compact set K we mean of course that such objects are defined on some analytic space containing K and representing the germ $\widetilde{K}$.

If X is a scheme of finite type over a noetherian scheme S, then according to [Gro61] one can roughly say that a set of isomorphism classes of coherent sheaves on the fibres of X/S is *bounded* if its elements are among the fibres of an algebraic family of coherent sheaves parametrized by a scheme S' of finite type over S. It is not directly clear how to formulate a definition of boundedness in a complex geometrical set-up, which would be also effective in proving properness statements. In [Tom16] we gave such a formulation in the relative case but used it in an absolute setting essentially. An equivalent way to formulate that definition in the absolute case is the following. A minor change made here as compared to [Tom16] is that we no longer ask that the compact set K be a semi-analytic Stein compact set. One can reduce oneself to a situation where this property is satisfied and this is important in some of the arguments, but it is not necessary to include it in the definition.

Definition 5.1. — *Let X be an analytic space and let $\mathfrak{E}$ be a set of isomorphism classes of coherent sheaves on X. We say that the set $\mathfrak{E}$ is* bounded *if there exist a germ $\widetilde{K}$ of an analytic space around a compact set K and a coherent sheaf $\widetilde{E}$ on $X \times \widetilde{K}$ such that $\mathfrak{E}$ is contained in the set of isomorphism classes of fibers of $\widetilde{E}$ over points of K, or, in other words, if there exist an analytic space S', a compact subset $K \subset S'$ and a coherent sheaf E on $X \times S'$ such that $\mathfrak{E}$ is contained in the set of isomorphism classes of fibers of E over points of K.*

For schemes defined over $\mathbb{C}$ this definition recovers Grothendieck's definition, [Tom16, Remark 3.3].

In the relative case it will be convenient to give a definition of boundedness which is less restrictive than the one proposed in [Tom16].

Definition 5.2. — *Let X be an analytic space over an analytic space S and $\mathfrak{E}$ a set of isomorphism classes of coherent sheaves on the fibres X_s of $X \to S$. We say that the set $\mathfrak{E}$ is* (relatively) bounded (over S) *if there exist an at most countable disjoint union $\widetilde{K} := \coprod \widetilde{K}_i$ of germs of analytic spaces around compact sets over S with $\coprod K_i$ proper over S and a coherent sheaf $\widetilde{E}$ on $X_{\widetilde{K}}$ such that $\mathfrak{E}$ is contained in the set of isomorphism classes of fibers of $\widetilde{E}$ over points of $\coprod K_i$.*

Warning: If X and S are complex spaces and E is a coherent sheaf on $X \times S$ then the isomorphism classes of its fibres E_s over S form a relatively bounded set over S

by our Definition 5.2, but the same classes when seen on X in the absolute setting may form an unbounded set according to Definition 5.1. An example is provided by the family given by $(\mathcal{O}_{\mathbb{P}^1}(n))_{n\in\mathbb{Z}}$ over $\mathbb{P}^1 \times \mathbb{Z}$. Another example, this time over an irreducible base S, is the Picard line bundle over $X \times \mathbb{C}^*$, where X is an Inoue surface; in this case $\mathrm{Pic}(X) \cong \mathbb{C}^*$.

The following Remark gives a way of rephrasing Definition 5.2.

Remark 5.3. — A set $\mathfrak{E}$ of isomorphism classes of coherent sheaves on the fibres of $X \to S$ is (relatively) bounded (over S) if and only if for each compact subset L of S there exist a germ $\widetilde{K}$ of analytic space around a compact set K over S and a coherent sheaf $\widetilde{E}$ on $X_{\widetilde{K}}$ such that the elements of $\mathfrak{E}$ in each fiber X_s of $X \to S$ with $s \in L$ are contained in the set of isomorphism classes of fibers of $\widetilde{E}$ over points of $\widetilde{K}$ covering s.

In the above definition it is clear that $\mathfrak{E}$ can be viewed as set of sheaves defined on (some of) the fibers of $X_{\widetilde{K}} \to \widetilde{K}$. (We will be loose on the above terminology and often say "sheaves" instead of "isomorphism classes of coherent sheaves".)

As in [Tom16] one can prove the following basic properties of bounded sets of sheaves. We will not reproduce the similar proof here

Proposition 5.4. — *Let X be an analytic space over an analytic space S and let $\mathfrak{E}$, $\mathfrak{E}'$ be two bounded sets of isomorphism classes of sheaves on the fibers of $X \to S$. Suppose that there exists a closed subset L of X proper over S such that the supports of the sheaves from $\mathfrak{E}$ or from $\mathfrak{E}'$ are contained in L. Then the following sets are bounded as well:*

1. *The sets of kernels, cokernels and images of sheaf homomorphisms $F \to F'$, when the isomorphism classes of F and F' belong to $\mathfrak{E}$ and $\mathfrak{E}'$ respectively.*
2. *The set of isomorphism classes of extensions of F by F', for F and F' as above.*
3. *The set of isomorphism classes of tensor products $F \otimes F'$, for F and F' again as above.*

Our main result is the following boundedness criterion.

Theorem 5.5. — *Let S be an analytic space and X an analytic space proper over S. Suppose that X/S is endowed with a relative degree system*

$$\left(\deg_{d,\,s},\, \ldots,\, \deg_{0,\,s}\right)_{s\,\in\, S}$$

coming from differential forms. Let $\mathfrak{F}$ be a set of isomorphism classes of coherent sheaves of dimension at most d on the fibres of X/S. For each $s \in S$ we denote by $\mathfrak{F}_s$ the subset of $\mathfrak{F}$ containing the sheaves on the fibre X_s of X/S. Then the set $\mathfrak{F}$ is bounded if and only if the following conditions are fulfilled:

1. *There exists a bounded set $\mathfrak{E}$ of isomorphism classes of coherent sheaves on the fibres of X/S such that each element of $\mathfrak{F}$ is a quotient of an element of $\mathfrak{E}$.*
2. *There exists a system $\delta = (\delta_d, \ldots, \delta_0)$ of continuous functions on S such that for all $s \in S$, for all $j \in \{0, \ldots, d\}$ and for all $F \in \mathfrak{F}_s$ one has*

$$\deg_{j,\,s}(F) \leqslant \delta_j(s).$$

In general we shall call a set $\mathfrak{F}$ of isomorphism classes of coherent sheaves on the fibers of $X \to S$ *dominated* if it satisfies the first condition of the criterion.

The "only if" is quickly dealt with as follows. The first assertion is clear. For the second one choose an at most countable disjoint union $\widetilde{K} := \coprod_{i\in\mathbb{N}} \widetilde{K}_i$ of germs of analytic spaces around compact sets over S with $\coprod K_i$ proper over S and a coherent sheaf $\widetilde{F}$ on $X_{\tilde{K}}$ such that $\mathfrak{F}$ is contained in the set of isomorphism classes of fibers of $\widetilde{F}$ over points of $\coprod K_i$ as in Definition 5.2. By noetherian induction and flattening we reduce ourselves to the case where $\widetilde{F}$ is flat over $\widetilde{K}$. The degree functions $\deg_j$ of this sheaf will be continuous on $\widetilde{K}$ and therefore attain there maxima $m_{j,n}$ on $\cup_{i=0}^n K_i$. Take now an exhaustive sequence $(L_k)_{k\in\mathbb{N}}$ of compact subsets of S, i.e. such that $\cup_k L_k = S$ and $L_k \subset \mathring{L}_{k+1}$, $\forall\, k \in \mathbb{N}$. Then there exists an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that the image of K_i in S does not meet L_k as soon as $i \geqslant n_k$. It suffices then to construct continuous functions δ_j on S such that $\delta_j|_{L_k} \geqslant m_{j,n_k}$ for all $k \in \mathbb{N}$.

The proof of the "if" part follows the same strategy as in [Gro61] or in [Tom16] to use devissages of arbitrary coherent sheaves into pure sheaves and noetherian induction to reduce the difficulty to a simpler boundedness statement on pure sheaves which works under assumptions of domination and bound of degrees in only one dimension. This will be the content of Lemma 5.7. We will not rewrite this part of the proof since it works in our context exactly as in [Tom16]. It is when dealing with pure sheaves in Lemma 5.7 that the difference between the projective case and the Kähler case appears. Whereas Grothendieck uses projections on linear subspaces in [Gro61], we reduce the problem to the control of volumes of certain analytic cycles in appropriate cycle spaces and conclude by Bishop's theorem. In [Tom16] we worked out this volume control globally by dealing with the cohomology classes of the metrics and the cycles involved. Here we will use the local volume estimates of Section 3 which allow us to deal with the more general set-up we are considering. Recall that by Bishop's theorem the subset of the relative analytic cycle space $C_{X/S,d}$ parameterizing cycles in the fibers of X/S whose volumes are bounded by some function $\delta : S \to \mathbb{R}$ is proper over S, cf. [BM14].

The same argument proves the following more general result for which we introduce more notation: we denote by $N_r(F)$ the sheaf of sections of a coherent sheaf F whose support have dimension less than r and set $F_{(r)} := F/N_r$. (If F is of dimension d then $F_{(d)}$ is pure of dimension d.)

Theorem 5.6. — *Suppose that X/S is a proper relative analytic space endowed with a relative degree system $(\deg_{d,s}, \ldots, \deg_{d',s})_{s\in S}$ coming from differential forms with $d > d'$, $\mathfrak{F}$ is a dominated set of isomorphism classes of coherent sheaves of dimension at most d on the fibres of X/S and $\delta = (\delta_d, \ldots, \delta_{d'})$ is a system of continuous functions on S such that for all $s \in S$, for all $j \in \{d', \ldots, d\}$ and for all $F \in \mathfrak{F}_s$ one has*

$$\deg_{j,s}(F) \leqslant \delta_j(s).$$

Then

(1) *the set of isomorphism classes of sheaves $\{F_{(d'+1)} \mid F \in \mathfrak{F}\}$ on the fibers of X/S is bounded and*

(2) *there exists a system of continuous functions* $\delta' = (\delta'_d, \ldots, \delta'_{d'})$ *such that for all* $s \in S$*, for all* $j \in \{d', \ldots, d\}$ *and for all* $F \in \mathfrak{F}_s$ *one has*

$$\deg_{j,s}(F) \geqslant \delta'_j(s).$$

Lemma 5.7 (Key Lemma). — *Let* X/S *be an equidimensional proper relative analytic space of relative dimension* d *with irreducible general fibers and let*

$$\left(\deg_{d,s}, \deg_{d-1,s}\right)_{s \in S}$$

be a relative degree system coming from differential forms. Let $\delta_{d-1} : S \to \mathbb{R}$ *be some continuous function. If* $\mathfrak{F}$ *is a dominated set of classes of pure* d*-dimensional sheaves on the fibers of* X/S *such that for all* $s \in S$ *and for all* $F \in \mathfrak{F}_s$ *one has* $\deg_{d-1,s}(F) \leqslant \delta(s)$*, then* $\mathfrak{F}$ *is bounded.*

The proof follows exactly the same strategy as the proof of [Tom16, Lemma 4.3] but replaces the global volume estimates by the local volume estimates performed in Section 3. We recall its main line.

Since $\mathfrak{F}$ is dominated we may suppose, after some base change possibly, that a coherent sheaf G on X exists such that all sheaves F in $\mathfrak{F}$ may be realized as quotients of G. By using a flattening stratification for G this implies a bound on $\deg_{d,s} F$ by some continuous function $\delta_d : S \to \mathbb{R}$ on S. The statement to prove is local on S and locally we will have a bound on the generic ranks of the sheaves $F \in \mathfrak{F}$. We may suppose that this generic rank is constant equal to r, say. Then we produce the subspaces P_2 of $\mathbb{P}(\bigwedge^r G)$ of bounded pseudo-volume as in Section 4. The estimates of Section 3 imply then a bound on a true volume function on these subspaces. Hence by Bishop's Theorem they represent points in a subset in the relative cycle space $C_{\mathbb{P}(\bigwedge^r G)/S}$ which is proper over S. We go by base change to this relative cycle space and we produce an analytic family of sheaves of the type F_1 described in Section 4 by pushing forward the restriction of the sheaf $\mathcal{O}_{\mathbb{P}(\bigwedge^r G)}(1)$. The final step of Section 4 through Lemma 4.3 tells us now how we can recover the sheaves $F \in \mathfrak{F}$ from this family.

6. Corollaries

In this section we present some consequences of Theorem 5.5.

6.1. Properness of the Douady space

We give here a number of applications of our criterion to the properness of irreducible or connected components of the Douady space partially extending results of Fujiki in [Fuj78, Fuj84].

Corollary 6.1. — *Let* X *be a proper analytic space over an analytic space* S*, endowed with a relative degree system* $(\deg_{d,s}, \ldots, \deg_{0,s})_{s \in S}$ *coming from differential forms and let* E *be a coherent sheaf on* X*. Further let* $\delta = (\delta_d, \ldots, \delta_0)$ *be a system of continuous functions on* S *and consider the subset* $D_{E/X/S, d, \leqslant \delta} \subset D_{E/X/S, d}$

of the relative Douady space $D_{E/X/S,d}$ of quotients of E of dimension at most d given by those quotients with degrees bounded by the functions δ_j. Then $D_{E/X/S,d,\leqslant\delta}$ is closed in $D_{E/X/S,d}$ with respect to the standard topology and proper over S. If moreover the functions δ_j are locally constant on S as well as the degree functions on flat families (which means that the degree system is strong), then $D_{E/X/S,d,\leqslant\delta}$ is an analytic subset of $D_{E/X/S,d}$, especially in this case the connected components of the Douady space $D_{E/X/S,d}$ are proper over S.

Proof. — The proof is a straightforward generalization of the absolute case as presented in [Tom16, Corollary 5.3]. For the convenience of the reader we present its line. The fact that $D_{E/X/S,d,\leqslant\delta}$ is closed in $D_{E/X/S,d}$ follows from the continuity of the degree functions in flat families and the final statement is easy. We thus only need to check the fact that $D_{E/X/S,d,\leqslant\delta}$ is proper over S.

By Theorem 5.5 the set $\mathfrak{F}$ of quotients of E parametrized by $D_{E/X/S,d,\leqslant\delta}$ is bounded over S. Thus there exist an at most countable disjoint union $\widetilde{K} := \coprod \widetilde{K}_i$ of germs an analytic spaces around compact sets over S with $\coprod K_i$ proper over S and a coherent sheaf $\widetilde{F}$ on $X_{\tilde{K}}$ such that $\mathfrak{F}$ is contained in the set of isomorphism classes of fibres of $\widetilde{F}$ over points of $\coprod K_i$. Consider now the contravariant functor which to a complex space T over $\widetilde{K}$ associates the set $\mathrm{Hom}_{X_T}(E_T, \widetilde{F}_T)$. By [Fle81, Section 3.2] this functor is represented by a linear space V over $\widetilde{K}$. It is easy to see that for each i there exists a compact subset K_i' of V covering K_i such that up to a multiplicative constant each non-zero morphism in $\mathrm{Hom}(E_t, F_t)$, $t \in K_i$ is represented by some element of K_i', see the construction of the projective variety over $\widetilde{K}$ associated to V in [Fis76, Section 1.9]. The image $\widetilde{F}'$ of the universal morphism over $\widetilde{K}'$ is a quotient of $E_{\tilde{K}'}$. By Hironaka's flattening theorem and noetherian induction we may assume that $\widetilde{F}'$ is flat over $\widetilde{K}'$ and that the set of its fibres over $\coprod K_i'$ contains $\mathfrak{F}$. Thus the image of $\coprod K_i'$ through the morphism $\phi : \widetilde{K}' \to D_{E/X/S}$ provided by the universal property of the relative Douady space contains $D_{E/X/S,d,\leqslant\delta}$. But $\coprod K_i'$ is proper over S, hence its image $\phi(\coprod K_i')$ through ϕ in $D_{E/X/S}$ is also proper over S. As $D_{E/X/S,d,\leqslant\delta}$ is closed in $\phi(\coprod K_i')$ it follows that $D_{E/X/S,d,\leqslant\delta}$ is proper over S as well. □

Our next concern is to say something about irreducible components of the relative Douady space of quotients containing points representing pure quotient sheaves. We shall restrict ourselves here to the case of pure sheaves and of degrees in dimensions d and $d-1$ but analogous statements for more general degree systems may be proved using Theorem 5.6. We start with a remark which works in a very general context.

Remark 6.2. — If X is an analytic space proper over S and if E is a flat family of d-dimensional coherent sheaves on the fibres of X/S, then the set of points $s \in S$ such that E_s is not pure is a closed analytic subset of S. In other words, purity is a Zariski open property in flat families of coherent sheaves.

This may be proved by adapting Maruyama's approach in [Mar96, Proposition 1.13] to our context. It suffices to work in loc. cit. with local resolutions and apply the purity criterion from [Mar96, Lemma 1.12].

Corollary 6.3. — *Let X be a proper analytic space over an analytic space S, endowed with a strong relative degree system $(\deg_{d,\,s},\, \deg_{d-1,\,s})_{s\,\in\,S}$ coming from differential forms and let E be a coherent sheaf on X. Further let m be a real number and consider the union $D'_{E/X/S,=d,\leqslant m}$ of irreducible components of the relative Douady space $D_{E/X/S,d}$ of quotients of E of dimension d with $(d-1)$-dimensional degrees bounded by m and such that each such irreducible component contains a point represented by a pure quotient sheaf. Then $D'_{E/X/S,\,=\,d,\,\leqslant\,m}$ is proper over S.*

Proof. — By Remark 6.2 the subset parameterizing pure quotient sheaves will be Zariski open and (classically) dense in each irreducible component of $D'_{E/X/S,\,=\,d,\,\leqslant\,m}$. We follow the constructions and the arguments of the proof of Corollary 6.3 applying Theorem 5.6 instead of Theorem 5.5 and with the same notations. It can be seen that it applies to our new situation with the single difference that we a priori only know that the image of $\coprod K'_i$ through the morphism $\phi : \widetilde{K}' \to D_{E/X/S}$ provided by the universal property of the relative Douady space contains the subset $D'_{E/X/S,\,=\,d,\,\leqslant\,m,\,pure}$ of $D'_{E/X/S,\,=\,d,\,\leqslant\,m}$ parameterizing pure quotient sheaves. But this subset is dense in $D'_{E/X/S,\,=\,d,\,\leqslant\,m}$. This and the fact that $\coprod K'_i$ is proper over S immediately imply that $\phi(\coprod K'_i)$ contains $D'_{E/X/S,\,=\,d,\,\leqslant\,m}$. But $D'_{E/X/S,\,=\,d,\,\leqslant\,m}$ is closed in $D_{E/X/S,d}$ hence also in $\phi(\coprod K'_i)$ and is thus proper over S. ☐

As easy consequences we deduce the following versions with parameters.

Corollary 6.4. — *Let X be a proper analytic space over an analytic space S, let T be a locally compact separated topological space and let*

$$\left(\deg_{d,\,(s,\,t)},\,\ldots,\,\deg_{0,\,(s,\,t)}\right)_{(s,\,t)\,\in\,S\,\times\,T}$$

be a continuous family of relative degree systems on X/S parameterized by $S \times T$ and coming from differential forms. Let E be a coherent sheaf on X. Further let $\delta = (\delta_d,\,\ldots,\,\delta_0)$ be a system of continuous functions on $S\times T$ and consider the subset $D_{E/X/S,T,\,d,\,\leqslant\,\delta} \subset D_{E/X/S,d} \times T$ of the product of the relative Douady space $D_{E/X/S,d}$ by T of pairs of quotients of E of dimension at most d and points of T, given by those quotients with degrees bounded by the functions δ_j. Then $D_{E/X/S,T,\,d,\,\leqslant\,\delta}$ is closed in $D_{E/X/S,d} \times T$ and proper over $S \times T$.

Proof. — By the continuity of the degree functions it is clear that $D_{E/X/S,T,\,d,\,\leqslant\,\delta}$ is closed in $D_{E/X/S,d} \times T$. The fact that it is proper over $D_{E/X/S,d} \times T$ is a local property over $S\times T$, so we can look at a product of compact subsets L of S and K of T. For $d \geqslant j \geqslant 0$ we define m_j to be the maximum of the function δ_j on $L \times K$ and put $m = (m_d,\,\ldots,\,m_0)$. Then over $L\times K$ we have $D_{E/X/S,T,\,d,\,\leqslant\,\delta} \subset D_{E/X/S,\,d,\,\leqslant\,m} \times T$ and this space is proper over $S \times T$ by Corollary 6.1. ☐

Corollary 6.5. — *Let X be a proper analytic space over an analytic space S, let T be a locally compact separated topological space and let*

$$\left(\deg_{d,\,(s,\,t)}, \deg_{d-1,\,(s,\,t)}\right)_{(s,\,t)\,\in\,S\,\times\,T}$$

be a continuous family of strong relative degree systems on X/S parameterized by $S \times T$ and coming from differential forms. Let E be a coherent sheaf on X. Further let δ be a continuous function on T and consider the subset

$$D'_{E/X/S,T,=d,\leqslant\delta} \quad := \quad \cup_{t\in T}\left(D'_{E/X/S,=d,\leqslant\delta(t)} \times \{t\}\right)$$

of $D_{E/X/S,d} \times T$, where, as in Corollary 6.3, $D'_{E/X/S,=d,\leqslant\delta(t)}$ denotes the union of irreducible components of the relative Douady space $D_{E/X/S,d}$ of quotients of E of dimension d with $(d-1)$-dimensional degrees bounded by $\delta(t)$ and such that each such irreducible component contains a point represented by a pure quotient sheaf. Then $D'_{E/X/S,T,=d,\leqslant\delta}$ is closed in $D_{E/X/S,d} \times T$ and proper over $S \times T$.

Proof. — As before the properties we have to check on $D'_{E/X/S,T,=d,\leqslant\delta}$ are local over $S \times T$ so we look at what happens over a product $K \times L$ of compact subsets L of S and K of T. Set m to be the maximum of the function δ on K. Then over $S \times K$ we have

$$D'_{E/X/S,K,d,\leqslant\delta} = \left(D'_{E/X/S,=d,\leqslant m} \times K\right) \cap D_{E/X/S,K,=d,\leqslant\delta},$$

where

$$D_{E/X/S,K,d,\leqslant\delta} := \cup_{t\in K}\left\{([q : E_s \to F], t) \in \left(D_{E/X/S,d} \times \{t\}\right) \,\middle|\, s \in S,\ \deg_{d-1,(s,t)}(F) \leqslant \delta(t)\right\}$$

is defined in a similar way to the space of Corollary 6.4. Now the space $D'_{E/X/S,=d,\leqslant m} \times K$ is proper over $S \times K$ by Corollary 6.3 and $D_{E/X/S,K,d,\leqslant\delta}$ is closed in $D_{E/X/S,d}$ by Corollary 6.4 whence the desired conclusion. □

6.2. Semistable sheaves

The existence of an absolute Harder–Narasimhan filtration is known for a variety of cases where a suitable stability notion has been introduced; see [And09] for a systematic treatment. We will be concerned here with the existence of a relative Harder–Narasimhan filtration in the following set-up, cf. [HL10, Section 2.3] for the projective algebraic case.

Let X be an analytic space proper over an irreducible analytic space S. Suppose that X/S is endowed with a strong relative degree system $(\deg_{d,s}, \ldots, \deg_{d',s})_{s\in S}$ coming from differential forms. Then we may define a *slope vector* $\mu(F)$ for any d-dimensional coherent sheaf F on some fibre X_s of X/S in the following way, cf. [Tom20]:

$$\mu(F) := \left(\frac{\deg_d(F)}{\deg_d(F)}, \frac{\deg_{d-1}(F)}{\deg_d(F)}, \ldots, \frac{\deg_{d'}(F)}{\deg_d(F)}\right) \in \mathbb{R}^{d-d'+1}.$$

We shall call such a sheaf F *semistable* if it is pure and if for any non-trivial coherent subsheaf $F' \subset F$ we have $\mu(F') \leqslant \mu(F)$ with respect to the lexicographic order. (As usual we will say that F is *stable* if the above inequality between slope vectors is strict for all proper subsheaves F' of F). As in [HL10, Section 1.3] one checks that any pure d-dimensional sheaf F on some fibre X_s of X/S admits a Harder–Narasimhan filtration i.e. a unique increasing filtration

$$0 = HN_0(F) \subset HN_1(F) \subset \ldots \subset HN_l(F) = F,$$

with semistable factors $HN_i(F)/HN_{i-1}(F)$ for $1 \leqslant i \leqslant l$ and such that

$$\mu\Big(HN_1(F)/HN_0(F)\Big) > \ldots > \mu\Big(HN_l(F)/HN_{l-1}(F)\Big).$$

We will write

$$\mu_{\min}(F) := \mu\Big(HN_l(F)/HN_{l-1}(F)\Big) \text{ and } \mu_{\max}(F) := \mu\Big(HN_1(F)/HN_0(F)\Big).$$

It is easy to see that F is semistable if and only if $\mu_{\min}(F) = \mu(F)$ if and only if $\mu_{\max}(F) = \mu(F)$.

By a *relative Harder–Narasimhan filtration* for a flat family E of d-dimensional coherent sheaves on the fibres of X/S we mean a proper bimeromorphic morphism of irreducible analytic spaces $T \to S$ and a filtration

$$0 = HN_0(E) \subset HN_1(E) \subset \ldots \subset HN_l(E) = E_T$$

such that the factors $HN_i(E)/HN_{i-1}(E)$ are flat over T for $1 \leqslant i \leqslant l$ and which induces the absolute Harder–Narasimhan filtrations fibrewise over some dense Zariski open subset of S.

Corollary 6.6 (Existence of a relative Harder-Narasimhan filtration). — *In the above set-up any flat family E of d-dimensional coherent sheaves on the fibres of X/S with pure general members has a relative Harder–Narasimhan filtration $HN_\bullet(E)$. Moreover this filtration has the following universal property: if $f : T' \to S$ is a bimeromorphic morphism of irreducible analytic spaces and if $F_\bullet$ is a filtration of $E_{T'}$ with flat factors, which coincides fibrewise with the absolute Harder–Narasimhan filtration over general points $s \in S$, then f factorizes over T and $F_\bullet = HN_\bullet(E)_{T'}$.*

Proof. — We start by working over a compact neighbourhood L of a fixed point s_0 of S.

The support cycles of the flat family E over S give rise to a section of the relative Barlet cycle space $C_{X/S,d}$ over S, [BM14, Fog69, Ryd08]. Using [BM14, Proposition IV.7.1.2] it then follows that the set

$$\Big\{\deg_d(C) \,\Big|\, C \in \Big(C_{X/S,\,d}\Big)_s,\, 0 < C \leqslant \operatorname{Supp}(E_s),\, s \in L\Big\} \text{ is finite.}$$

Let r denote its minimum. For a d-dimensional quotient F of some E_s with $s \in L$ we get the implication

$$\frac{\deg_{d-1}(F)}{\deg_d(F)} \leqslant \frac{\deg_{d-1}(E_s)}{\deg_d(E_s)} \Rightarrow \deg_{d-1}(F) \leqslant \frac{r \deg_{d-1}(E_s)}{\deg_d(E_s)} =: m.$$

By Corollary 6.3 the union $D := D'_{E/X/S,\,=d,\,\leqslant m}$ of irreducible components of the relative Douady space $D_{E/X/S,d}$ of quotients of E of dimension d with $(d-1)$-dimensional degrees bounded by m and such that each such irreducible component contains a point represented by a pure quotient sheaf is proper over S. If the map $D_{\mathring{L}} \to \mathring{L}$ is not surjective then one can see easily that the set of points $s \in S$ such that E_s is not semistable is a closed analytic proper subset of S. It follows in particular that semistability is a Zariski open property in flat families of coherent sheaves.

Suppose now that the map $D_{\mathring{L}} \to \mathring{L}$ is surjective. One consequence of the above argument is that $D_{\mathring{L}}$ has only finitely many irreducible components. Let D' denote the union of those irreducible components of D covering L. We know that the slope vectors of quotients parametrized by such a component are constant along the component. In particular we may choose the minimal such slope vector μ_0 among those which appear on irreducible components of D'. Using the properness of $D'_{\mathring{L}} \to \mathring{L}$ and the properties of the absolute Harder–Narasimhan filtration we can see that there is a unique irreducible component $D_{\min}$ of D' which realizes μ_0 as slope vector of the quotients it parametrizes and such that over general points s of L the corresponding quotients correspond to the last quotient of $HN_\bullet(E_s)$. It follows that $D_{\min} \to S$ is a proper modification. Let then F denote the universal quotient of $E_{D_{\min}}$ and $G := \mathrm{Ker}(E_{D_{\min}} \to F)$ the universal kernel. The family G is a flat family of d-dimensional coherent sheaves on the fibres of $X_{D_{\min}}/D_{\min}$ with pure general members. We apply the same procedure to this new family. The process ends after a finite number of steps and gives the desired relative Harder–Narasimhan filtration of E over S.

Using the above construction and the properties of the Douady space and of the absolute Harder–Narasimhan filtration one easily obtains the claimed universal property for the relative Harder–Narasimhan filtration. ☐

One consequence of the above proof is the openness of semistability in flat families of coherent sheaves:

Corollary 6.7. — *Let X/S be a proper flat family of d-dimensional complex spaces endowed with a strong relative degree system $(\deg_{d,s}, \ldots, \deg_{d',s})_{s\in S}$ coming from differential forms and let E be a coherent sheaf on X flat over S of relative dimension d. Then the set of points of S over which the fibers of E are semistable form a Zariski open subset of S.*

Remark 6.8. — The hypothesis of strongness in Corollary 6.7 cannot be relaxed as can be seen in the case when X is a non-Kählerian compact complex surface. As remarked in Section 2.3 a Gauduchon metric on the surface gives a complete degree system on X. However in general semistability with respect to such a degree system is not an open condition in flat families, see [BTT17, Example 2.1]. Note however that Theorem 5.5 holds without a strongness hypothesis.

The next Corollary deals with openness of stability in a different context and is another example of extending to non necessarily projective manifolds statements on semistable sheaves which were previously proved only under projectivity assumptions, cf. [GKP16, Theorem 3.3]. It is an easy consequence of Corollary 6.5.

Corollary 6.9. — *Let X be a compact complex space of pure dimension d and let $(\Omega_t)_{t\in T}$ be a continuous family of strictly positive* d*-closed forms defining $(d-1)$-degree functions on X and parameterized by a locally compact separated topological space T. Suppose moreover that X is also endowed with a volume form allowing to compute d-degrees and that F is a d-dimensional pure sheaf on X which* *is stable with respect to* Ω_{t_0} *for some* $t_0 \in T$. *Then there exists a neighbourhood* V *of* t_0 *in* T *such that* F *is* Ω_t*-stable for all* $t \in V$.

In the particular (semi-Kählerian) case when X is a compact complex manifold and T is an open subset of $H^{d-1,d-1}(X,\mathbb{R})$ one can give an alternative proof using only the absolute case of Corollary 6.3 and an argument as in [GRT16, Lemma 6.7]. One can in this case choose a convex polytope as compact neighbourhood L and note that the degree functions evaluated on a fixed coherent sheaf E are affine linear. Hence for any torsion free quotient E of F the function $f_{F,E}(t) := \mu_t(F) - \mu_t(E)$ is affine linear and therefore attains its maximum on L at some vertex v of L. Now Corollary 6.3 says that for each vertex v only finitely many such functions exist which are non-negative at v, if any. Then the subset of L where all these functions are non-positive is a convex polytope containing t_0 in its interior. This method of proof allows the following slight extension to pseudo-stable sheaves, where "pseudo" means here that the polarizing form with respect to which one considers stability is positive but not necessarily strictly positive.

Corollary 6.10. — *Let* X *be a compact complex space of pure dimension* d *endowed with a volume form, let* Ω_0, Ω_1 *be positive* d*-closed forms of bi-degree* $(d-1,d-1)$ *with* Ω_1 *strictly positive and let* $\Omega_t := (1-t)\Omega_0 + t\Omega_1$ *for* $t \in [0,1]$. *Suppose moreover that* F *is a* d*-dimensional pure sheaf on* X *which is pseudo-stable with respect to* Ω_0. *Then there exists a neighbourhood* V *of* 0 *in* $[0,1]$ *such that* F *is* Ω_t*-stable for all* $t \in V \setminus \{0\}$.

Proof. — In the particular case at hand for any coherent sheaf E on X the function $t \mapsto \deg_{\Omega_t} E$ is affine on $[0,1]$ and the same holds for slope functions. By looking at slopes of coherent subsheaves F' of F it follows that if F is both pseudo-stable with respect to Ω_0 and stable with respect to Ω_1 then it is stable with respect to all Ω_t for $t \in]0,1]$.

We are left with the case when F is not stable with respect to Ω_1. Since the stability condition may be translated into an inequalities for slopes of pure quotients of F we may apply Corollary 6.3 in the absolute case to conclude that there is a finite number of irreducible components of the Douady space $D_{F/X}$ containing destabilizing quotients for F with respect to Ω_1, i.e. pure quotients E with $\mu_{\Omega_1}(E) \leqslant \mu_{\Omega_1}(F)$. Let $E_1, \ldots, E_k$ be a choice of such quotient sheaves, one for each concerned irreducible component of $D_{F/X}$. Then the functions $t \mapsto (\mu_{\Omega_t}(E_j) - \mu_{\Omega_t}(F))$ being affine and positive at 0, they will remain positive on a neighbourhood V of 0 in $[0,1]$. It is now immediate to check that F will be Ω_t-stable for all $t \in V \setminus \{0\}$. □

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