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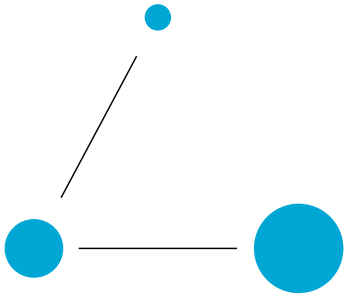

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ROOTS OF KOSTLAN POLYNOMIALS: MOMENTS, STRONG LAW OF LARGE NUMBERS AND CENTRAL LIMIT THEOREM

RACINES DES POLYNÔMES DE KOSTLAN : MOMENTS, LOI FORTE DES GRANDS NOMBRES ET THÉORÈME CENTRAL LIMITE

Abstract. — We study the number of real roots of a Kostlan random polynomial of degree d in one variable. More generally, we consider the counting measure of the set of real roots of such polynomials. We compute the large degree asymptotics of the central moments of these random variables. As a consequence, we obtain a strong Law of Large Numbers and a Central Limit Theorem. In particular, the real roots of a Kostlan polynomial almost surely equidistribute as the degree diverges. Moreover, the fluctuations of their counting measure converge in distribution to the Standard Gaussian White Noise. More generally, our results hold for the real zeros of a random real section in the complex Fubini–Study model.

Keywords: Kostlan polynomials, Complex Fubini–Study model, Kac–Rice formula, Law of Large Numbers, Central Limit Theorem, Method of moments.
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Résumé. — Nous étudions le nombre de racines réelles d'un polynôme de Kostlan de degré d en une variable. Plus généralement, nous nous intéressons à la distribution de la mesure de comptage de l'ensemble des racines réelles d'un tel polynôme. Nous obtenons l'asymptotique des moments centrés de ces variables aléatoires, dans la limite des grands degrés. Nous en déduisons une loi forte des grands nombres et un théorème central limite. En particulier, les racines réelles d'un polynôme de Kostlan s'équidistribuent presque surement lorsque le degré tend vers l'infini. De plus, les fluctuations de la mesure de comptage de cet ensemble aléatoire convergent en distribution vers le bruit blanc gaussien standard. Nos résultats sont valables plus généralement pour les zéros réels d'une section réelle aléatoire dans le modèle dit de Fubini–Study complexe.

Contents

1. Introduction

Kostlan polynomials

A real *Kostlan polynomial* of degree d is a univariate random polynomial of the form

$$\sum_{k=0}^{d} a_k \sqrt{\binom{d}{k}} X^k,$$

where the coefficients $(a_k)_{0 \leqslant k \leqslant d}$ are independent $\mathcal{N}(0,1)$ random variables. Here and in the following we use the standard notation $\mathcal{N}(m, \sigma^2)$ for the real Gaussian distribution of mean m and variance σ^2. These random polynomials are also known as *elliptic polynomials* in the literature (see [ST04] for example). The roots of such a polynomial form a random subset of $\mathbb{R}$ that we denote by Z_d. Kostlan proved (cf. [Kos93]) that for all $d \in \mathbb{N}$, the average number of roots of this random polynomial is $\mathbb{E}[\mathrm{Card}(Z_d)] = d^{\frac{1}{2}}$, where $\mathrm{Card}(Z_d)$ is the cardinality of Z_d. It was later proved by Dalmao (see [Dal15]) that $\mathrm{Var}(\mathrm{Card}(Z_d)) \sim \sigma^2 d^{\frac{1}{2}}$ as $d \to +\infty$, where σ is some explicit positive constant. Dalmao also proved that $\mathrm{Card}(Z_d)$ satisfies a Central Limit Theorem as $d \to +\infty$.

In this paper, we study the higher moments of $\mathrm{Card}(Z_d)$ in the large degree limit. Let $p \in \mathbb{N}$, we denote by $m_p(\mathrm{Card}(Z_d))$ the p^{th} central moment of $\mathrm{Card}(Z_d)$. A consequence of our main result (Theorem 1.12) is that, as $d \to +\infty$, we have:

$$m_p\left(\mathrm{Card}(Z_d)\right) = \mu_p \sigma^p d^{\frac{p}{4}} + o\left(d^{\frac{p}{4}}\right),$$

where σ is the constant appearing in Dalmao's variance estimate, and $(\mu_p)_{p \in \mathbb{N}}$ is the sequence of moments of the standard real Gaussian distribution $\mathcal{N}(0,1)$. This results allows us to prove a strong Law of Large Numbers: $d^{-\frac{1}{2}} \mathrm{Card}(Z_d) \to 1$ almost surely. We also prove that $d^{-\frac{1}{2}} \mathrm{Card}(Z_d)$ concentrates around 1 in probability, faster than any negative power of d. Finally, we recover Dalmao's Central Limit Theorem by the method of moments. The original proof used the Wiener–Ito expansion of the number of roots. In fact, we improve this result by proving a Central Limit Theorem for the counting measure of Z_d (see Theorem 1.9 below).

Equivalently, one can define Z_d as the set of zeros, on the real projective line $\mathbb{R}P^1$, of the homogeneous Kostlan polynomial

$$\sum_{k=0}^{d} a_k \sqrt{\binom{d}{k}} X^k Y^{d-k},$$

where $(a_k)_{0 \leqslant k \leqslant d}$ are independent $\mathcal{N}(0,1)$ variables. In this setting, an homogeneous Kostlan polynomial of degree d is a standard Gaussian vector in the space of real global holomorphic sections of the line bundle $\mathcal{O}(d) \to \mathbb{C}P^1$, equipped with its natural L^2-inner product (see Section 2.1). Recall that $\mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}$, where $\mathcal{O}(1)$ is the hyperplane line bundle over $\mathbb{C}P^1$. Then, Z_d is the real zero set of this random section. In this paper, we study more generally the real zeros of random real sections of positive Hermitian line bundles over real algebraic curves.

Framework and background

Let us introduce our general framework. More details are given in Section 2.1 below. Let $\mathcal{X}$ be a smooth complex projective manifold of dimension 1, that is a smooth compact Riemann surface. Let $\mathcal{E}$ and $\mathcal{L}$ be holomorphic line bundles over $\mathcal{X}$. We assume that $\mathcal{X}$, $\mathcal{E}$ and $\mathcal{L}$ are endowed with compatible real structures and that the real locus of $\mathcal{X}$ is not empty. We denote by M this real locus, which is then a smooth closed (i.e. compact without boundary) submanifold of $\mathcal{X}$ of real dimension 1.

Let $h_{\mathcal{E}}$ and $h_{\mathcal{L}}$ denote Hermitian metrics on $\mathcal{E}$ and $\mathcal{L}$ respectively, that are compatible with the real structures. We assume that $(\mathcal{L}, h_{\mathcal{L}})$ has positive curvature ω, so that $\mathcal{L}$ is ample and ω is a Kähler form on $\mathcal{X}$. The form ω induces a Riemannian metric g on $\mathcal{X}$, hence on M. Let us denote by $|\mathrm{d}V_M|$ the arc-length measure on M defined by g.

For all $d \in \mathbb{N}$, we denote by $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ the space of global real holomorphic sections of $\mathcal{E} \otimes \mathcal{L}^d \to \mathcal{X}$. Let $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, we denote by $Z_s = s^{-1}(0) \cap M$ the real zero set of s. Since s is holomorphic, if $s \neq 0$ its zeros are isolated and Z_s is finite. In this case, we denote by ν_s the counting measure of Z_s, that is $\nu_s = \sum_{x \in Z_s} \delta_x$, where δ_x stands for the unit Dirac mass at x. For any $\phi \in \mathcal{C}^0(M)$, we denote by $\langle \nu_s \, , \phi \rangle = \sum_{x \in Z_s} \phi(x)$. Quantities of the form $\langle \nu_s \, , \phi \rangle$ are called the *linear statistics* of ν_s. Note that $\langle \nu_s \, , \mathbf{1} \rangle = \mathrm{Card}(Z_s)$, where $\mathbf{1}$ is the constant unit function.

For any $d \in \mathbb{N}$, the space $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is finite-dimensional, and its dimension can be computed by the Riemann–Roch Theorem. Moreover, the measure $|\mathrm{d}V_M|$ and the metrics $h_{\mathcal{E}}$ and $h_{\mathcal{L}}$ induce a Euclidean L^2-inner product on this space (see Equation (2.1)). Let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, see Section 2.3. Then, ν_{s_d} is an almost surely well-defined random Radon measure on M. We denote by $Z_d = Z_{s_d}$ and by $\nu_d = \nu_{s_d}$ for simplicity. In this setting, the linear statistics of ν_d were studied in [Anc21, GW16, Let16, LP19], among others. In particular, the exact asymptotics of their expected values and their variances are known.

Theorem 1.1 (Gayet–Welschinger). — *For every $d \in \mathbb{N}$, let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then the following holds as $d \to +\infty$:*

$$\forall\, \phi \in \mathcal{C}^0(M), \qquad \mathbb{E}[\langle \nu_d \, , \phi \rangle] = d^{\frac{1}{2}} \left(\frac{1}{\pi} \int_M \phi \, |\mathrm{d}V_M| \right) + \|\phi\|_\infty \, O\left(d^{-\frac{1}{2}}\right),$$

where the error term $O(d^{-\frac{1}{2}})$ is independent of ϕ, i.e. $d^{-\frac{1}{2}} \mathbb{E}[\nu_d] = \frac{1}{\pi} |\mathrm{d}V_M| + O(d^{-1})$.

Theorem 1.1 is [GW16, Theorem 1.2] with $n = 1$ and $i = 0$. See also [Let16, Theorem 1.3] when $\mathcal{E}$ is not trivial. The case the linear statistics is discussed in [Let16, Section 5.3].

We use the following notation for the central moments of ν_d.

Definition 1.2 (Central moments of ν_d). — *Let $d \in \mathbb{N}$ and let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Let ν_d denote the counting measure of the real zero set of s_d. For all $p \in \mathbb{N}^*$, for all $\phi_1, \ldots, \phi_p \in \mathcal{C}^0(M)$ we denote:*

$$m_p(\nu_d)(\phi_1, \ldots, \phi_p) = \mathbb{E}\left[\prod_{i=1}^{p} \Big(\langle \nu_d \, , \phi_i \rangle - \mathbb{E}[\langle \nu_d \, , \phi_i \rangle] \Big) \right].$$

For all $\phi \in \mathcal{C}^0(M)$, we denote by $m_p(\langle \nu_d\,, \phi\rangle) = m_p(\nu_d)(\phi, \dots, \phi)$ the p^{th} central moment of $\langle \nu_d\,, \phi\rangle$. As above, we denote the p^{th} central moment of $\mathrm{Card}(Z_d)$ *by* $m_p(\mathrm{Card}(Z_d)) = m_p(\nu_d)(\mathbf{1}, \dots, \mathbf{1})$.

Of course, $m_p(\nu_d)$ is only interesting for $p \geqslant 2$. For $p = 2$, the bilinear form $m_2(\nu_d)$ encodes the covariance structure of the linear statistics of ν_d. In particular, for all $\phi \in \mathcal{C}^0(M)$, we have $m_2(\nu_d)(\phi, \phi) = \mathrm{Var}(\langle \nu_d\,, \phi\rangle)$. The large degree asymptotics of $m_2(\nu_d)$ has been studied in [LP19, Theorem 1.6].

THEOREM 1.3 (Letendre–Puchol). — *For all $d \in \mathbb{N}$, let $s_d \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ be a standard Gaussian vector. There exists $\sigma > 0$ such that, for all ϕ_1 and $\phi_2 \in \mathcal{C}^0(M)$, the following holds as $d \to +\infty$:*

$$m_2(\nu_d)(\phi_1, \phi_2) = d^{\frac{1}{2}}\sigma^2 \int_M \phi_1\phi_2 \,|\mathrm{d}V_M| + o\left(d^{\frac{1}{2}}\right).$$

Remarks 1.4. —

- In fact, $\sigma = (\frac{1+\mathcal{I}_{1,1}}{\pi})^{\frac{1}{2}}$, where $\mathcal{I}_{1,1}$ is the constant appearing in [LP19, Theorem 1.6]. Since σ is universal, it is the same as the one appearing in Dalmao's variance estimate [Dal15, Theorem 1.1]. An integral expression of σ is given by [Dal15, Proposition 3.1]. The positivity of σ is non-trivial and is given by [Dal15, Corollary 1.2]. See also [LP19, Theorem 1.8].
- One can understand how the error term $o(d^{\frac{1}{2}})$ in Theorem 1.3 depends on (ϕ_1, ϕ_2) (see [LP19, Theorem 1.6]).

In [Anc21], Ancona derived a two terms asymptotic expansion of the non-central moments of the linear statistics of ν_d. As a consequence, he proved the following (cf. [Anc21, Theorem 0.5]).

THEOREM 1.5 (Ancona). — *For all $d \in \mathbb{N}$, let $s_d \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ be a standard Gaussian vector. Let $p \geqslant 3$, for all $\phi_1, \dots, \phi_p \in \mathcal{C}^0(M)$ we have:*

$$m_p(\nu_d)(\phi_1, \dots, \phi_p) = o\left(d^{\frac{p-1}{2}}\right).$$

Remark 1.6. — In Ancona's paper the line bundle $\mathcal{E}$ is trivial. The results of [Anc21] rely on precise estimates for the Bergman kernel of $\mathcal{L}^d$. These estimates still hold if we replace $\mathcal{L}^d$ by $\mathcal{E} \otimes \mathcal{L}^d$, where $\mathcal{E}$ is any fixed real Hermitian line bundle (see [MM07, Theorem 4.2.1]). Thus, all the results in [Anc21] are still valid for random real sections of $\mathcal{E} \otimes \mathcal{L}^d \to \mathcal{X}$.

1.1. Main results

In this paper, we prove a strong Law of Large Numbers (Theorem 1.7) and a Central Limit Theorem (Theorem 1.9) for the linear statistics of the random measures $(\nu_d)_{d\in\mathbb{N}}$ defined above. These results are deduced from our main result (Theorem 1.12), which gives the precise asymptotics of the central moments $(m_p(\nu_d))_{p\geqslant 3}$ (cf. Definition 1.2), as $d \to +\infty$.

Recall that all the results of the present paper apply when Z_d is the set of real roots of a Kostlan polynomial of degree d. If one considers homogeneous Kostlan

polynomials in two variables, then $Z_d \subset M = \mathbb{R}P^1$. Since $\mathbb{R}P^1$ is obtained from the Euclidean unit circle by identifying antipodal points, it is a circle of length π. In this case, $|\mathrm{d}V_M|$ is the Lebesgue measure on this circle, normalized so that $\mathrm{Vol}(M) = \pi$. If one wants to consider the original Kostlan polynomials in one variable, then $Z_d \subset M = \mathbb{R}$, where $\mathbb{R}$ is seen as a standard affine chart in $\mathbb{R}P^1$. In this case, the measure $|\mathrm{d}V_M|$ admits the density $t \mapsto (1+t^2)^{-1}$ with respect to the Lebesgue of $\mathbb{R}$, and $\mathrm{Vol}(M) = \pi$ once again.

THEOREM 1.7 (Strong Law of Large Numbers). — *Let $\mathcal{X}$ be a real projective curve whose real locus M is non-empty. Let $\mathcal{E} \to \mathcal{X}$ and $\mathcal{L} \to \mathcal{X}$ be real Hermitian line bundles such that $\mathcal{L}$ is positive. Let $(s_d)_{d \geqslant 1}$ be a sequence of standard Gaussian vectors in $\prod_{d \geqslant 1} \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. For any $d \geqslant 1$, let Z_d denote the real zero set of s_d and let ν_d denote the counting measure of Z_d.*

Then, almost surely, $d^{-\frac{1}{2}} \nu_d \xrightarrow[d \to +\infty]{} \frac{1}{\pi} |\mathrm{d}V_M|$ in the weak-$*$ sense. That is, almost surely:

$$\forall\, \phi \in \mathcal{C}^0(M), \qquad d^{-\frac{1}{2}} \langle \nu_d \,, \phi \rangle \xrightarrow[d \to +\infty]{} \frac{1}{\pi} \int_M \phi \, |\mathrm{d}V_M| \,.$$

In particular, almost surely,

$$d^{-\frac{1}{2}} \,\mathrm{Card}(Z_d) \xrightarrow[d \to +\infty]{} \frac{1}{\pi} \,\mathrm{Vol}(M)$$

and

$$\frac{1}{\mathrm{Card}(Z_d)} \nu_d \xrightarrow[d \to +\infty]{\text{weak}-*} \frac{1}{\mathrm{Vol}(M)} \, |\mathrm{d}V_M| \,.$$

DEFINITION 1.8 (Standard Gaussian White Noise). — *The* Standard Gaussian White Noise *on M is a random Gaussian generalized function $W \in \mathcal{D}'(M)$, whose distribution is characterized by the following facts, where $\langle \cdot \,, \cdot \rangle_{(\mathcal{D}', \mathcal{C}^\infty)}$ denotes the usual duality pairing between $\mathcal{D}'(M)$ and $\mathcal{C}^\infty(M)$:*

- *for any $\phi \in \mathcal{C}^\infty(M)$, the variable $\langle W \,, \phi \rangle_{(\mathcal{D}', \mathcal{C}^\infty)}$ is a real centered Gaussian;*
- *for all*

$$\phi_1, \phi_2 \in \mathcal{C}^\infty(M), \mathbb{E}\Big[\langle W \,, \phi_1 \rangle_{(\mathcal{D}', \mathcal{C}^\infty)} \langle W \,, \phi_2 \rangle_{(\mathcal{D}', \mathcal{C}^\infty)} \Big] = \int_M \phi_1 \phi_2 \, |\mathrm{d}V_M| \,.$$

In particular, we have $\langle W \,, \phi \rangle_{(\mathcal{D}', \mathcal{C}^\infty)} \sim \mathcal{N}(0, \|\phi\|_2^2)$, where $\|\phi\|_2$ stands for the L^2-norm of ϕ.

Here and in the following, we avoid using the term "distribution" when talking about elements of $\mathcal{D}'(M)$ and rather use the term "generalized function". This is to avoid any possible confusion with the distribution of a random variable.

THEOREM 1.9 (Central Limit Theorem). — *Let $\mathcal{X}$ be a real projective curve whose real locus M is non-empty. Let $\mathcal{E} \to \mathcal{X}$ and $\mathcal{L} \to \mathcal{X}$ be real Hermitian line bundles such that $\mathcal{L}$ is positive. Let $(s_d)_{d \geqslant 1}$ be a sequence of standard Gaussian vectors in $\prod_{d \geqslant 1} \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. For any $d \geqslant 1$, let Z_d denote the real zero set of s_d and let ν_d denote the counting measure of Z_d.*

Then, the following holds in distribution, in the space $\mathcal{D}'(M)$ of generalized functions on M:

$$\frac{1}{d^{\frac{1}{4}}\sigma}\left(\nu_d - \frac{d^{\frac{1}{2}}}{\pi}\,|\mathrm{d}V_M|\right) \xrightarrow[d\to+\infty]{} W,$$

where W denotes the Standard Gaussian White Noise on M (see Definition 1.8). In particular, for any test-functions $\phi_1, \ldots, \phi_k \in \mathcal{C}^\infty(M)$, the random vector:

$$\frac{1}{d^{\frac{1}{4}}\sigma}\left(\left(\langle \nu_d\,,\phi_1\rangle - \frac{d^{\frac{1}{2}}}{\pi}\int_M \phi_1\,|\mathrm{d}V_M|\right),\ \ldots,\ \left(\langle \nu_d\,,\phi_k\rangle - \frac{d^{\frac{1}{2}}}{\pi}\int_M \phi_k\,|\mathrm{d}V_M|\right)\right)$$

converges in distribution to a centered Gaussian vector in $\mathbb{R}^k$ of variance matrix

$$\left(\int_M \phi_i\phi_j\,|\mathrm{d}V_M|\right)_{1\leqslant i,\,j\leqslant k}.$$

Moreover, for all $\phi \in \mathcal{C}^0(M)$,

$$\frac{1}{d^{\frac{1}{4}}\sigma}\left(\langle \nu_d\,,\phi\rangle - \frac{d^{\frac{1}{2}}}{\pi}\int_M \phi\,|\mathrm{d}V_M|\right) \xrightarrow[d\to+\infty]{} \mathcal{N}\left(0, \|\phi\|_2^2\right)$$

in distribution. In particular,

$$\frac{1}{d^{\frac{1}{4}}\sigma\,\mathrm{Vol}(M)}\left(\mathrm{Card}(Z_d) - \frac{d^{\frac{1}{2}}}{\pi}\,\mathrm{Vol}(M)\right) \xrightarrow[d\to+\infty]{} \mathcal{N}(0,1).$$

Before stating our main result, we need to introduce some additional notations.

Definition 1.10 (Moments of the standard Gaussian). — *For all $p \in \mathbb{N}$, we denote by μ_p the p^{th} moment of the standard real Gaussian distribution. Recall that, for all $p \in \mathbb{N}$, we have $\mu_{2p} = \frac{(2p)!}{2^p p!}$ and $\mu_{2p+1} = 0$.*

Definition 1.11 (Partitions). — *Let A be a finite set, a* partition *of A is a family $\mathcal{I}$ of non-empty disjoint subsets of A such that $\bigsqcup_{I\in\mathcal{I}} I = A$. We denote by $\mathcal{P}_A$ (resp. $\mathcal{P}_k$) the set of partitions of A (resp. $\{1, \ldots, k\}$). A* partition into pairs *of A is a partition $\mathcal{I} \in \mathcal{P}_A$ such that $\mathrm{Card}(I) = 2$ for all $I \in \mathcal{I}$. We denote by $\mathcal{PP}_A$, (resp. $\mathcal{PP}_k$) the set of partitions into pairs of A (resp. $\{1, \ldots, k\}$). Note that $\mathcal{P}_\emptyset = \{\emptyset\} = \mathcal{PP}_\emptyset$.*

Theorem 1.12 (Central moments asymptotics). — *Let $\mathcal{X}$ be a real projective curve whose real locus M is non-empty. Let $\mathcal{E} \to \mathcal{X}$ and $\mathcal{L} \to \mathcal{X}$ be real Hermitian line bundles such that $\mathcal{L}$ is positive. For all $d \in \mathbb{N}$, let $s_d \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$ be a standard Gaussian vector, let Z_d denote the real zero set of s_d and let ν_d denote its counting measure. For all $p \geqslant 3$, for all $\phi_1, \ldots, \phi_p \in \mathcal{C}^0(M)$, the following holds as $d \to +\infty$:*

$$\begin{aligned} m_p(\nu_d)(\phi_1, \ldots, \phi_p) &= \sum_{\mathcal{I}\in\mathcal{PP}_p}\ \prod_{\{i,j\}\in\mathcal{I}} m_2(\nu_d)\,(\phi_i,\phi_j) + \left(\prod_{i=1}^p \|\phi_i\|_\infty\right) O\left(d^{\frac{1}{2}\left\lfloor\frac{p-1}{2}\right\rfloor}(\ln d)^p\right) \\ &= d^{\frac{p}{4}}\sigma^p \sum_{\mathcal{I}\in\mathcal{PP}_p}\ \prod_{\{i,j\}\in\mathcal{I}} \left(\int_M \phi_i\phi_j\,|\mathrm{d}V_M|\right) + o\left(d^{\frac{p}{4}}\right), \end{aligned}$$

where $\lfloor\cdot\rfloor$ denotes the integer part, $\|\cdot\|_\infty$ denotes the sup-norm, σ is the same positive constant as in Theorem 1.3, the set $\mathcal{PP}_p$ is defined by Definition 1.11, and the error term $O(d^{\frac{1}{2}\lfloor\frac{p-1}{2}\rfloor}(\ln d)^p)$ does not depend on $(\phi_1, \dots, \phi_p)$.

In particular, for all $\phi \in \mathcal{C}^0(M)$, we have:

$$m_p\left(\langle \nu_d\,,\phi\rangle\right) = \mu_p \operatorname{Var}(\langle \nu_d\,,\phi\rangle)^{\frac{p}{2}} + \|\phi\|_\infty^p\, O\Big(d^{\frac{1}{2}\lfloor\frac{p-1}{2}\rfloor}(\ln d)^p\Big)$$
$$= \mu_p d^{\frac{p}{4}}\sigma^p\left(\int_M \phi^2\,|\mathrm{d}V_M|\right)^{\frac{p}{2}} + o\left(d^{\frac{p}{4}}\right).$$

Remark 1.13. — If p is odd, then the first term vanishes in the asymptotic expansions of Theorem 1.12. Indeed, in this case $\mathcal{PP}_p = \emptyset$ and $\mu_p = 0$. Hence, if p is odd, for all $\phi_1, \dots, \phi_p \in \mathcal{C}^0(M)$, we have $m_p(\nu_d)(\phi_1, \dots, \phi_p) = O(d^{\frac{p-1}{4}}(\ln d)^p)$. If p is even, we have $m_p(\nu_d)(\phi_1, \dots, \phi_p) = O(d^{\frac{p}{4}})$ for all $\phi_1, \dots, \phi_p \in \mathcal{C}^0(M)$.

Other interesting corollaries of Theorem 1.12 include the following.

Corollary 1.14 (Concentration in probability). — *In the setting of Theorem 1.12, let $(\varepsilon_d)_{d\geqslant 1}$ denote a sequence of positive numbers and let $\phi \in \mathcal{C}^0(M)$. Then, for all $p \in \mathbb{N}^*$, as $d \to +\infty$, we have:*

$$\mathbb{P}\left(d^{-\frac{1}{2}}\,|\langle \nu_d\,,\phi\rangle - \mathbb{E}[\langle \nu_d\,,\phi\rangle]| > \varepsilon_d\right) = O\left((d^{\frac{1}{4}}\varepsilon_d)^{-2p}\right).$$

In particular, for all $p \in \mathbb{N}^*$, as $d \to +\infty$, we have:

$$\mathbb{P}\left(d^{-\frac{1}{2}}\,|\mathrm{Card}(Z_d) - \mathbb{E}[\mathrm{Card}(Z_d)]| > \varepsilon_d\right) = O\left((d^{\frac{1}{4}}\varepsilon_d)^{-2p}\right).$$

Corollary 1.14 and Theorem 1.1 imply that $\mathbb{P}(\mathrm{Card}(Z_d) > \sqrt{d}C) = O(d^{-\frac{p}{2}})$ for any $p \in \mathbb{N}^*$ and $C > \frac{1}{\pi}\operatorname{Vol}(M)$. In the same spirit, [GW11, Theorem 2] proves that there exists $D > 0$ such that

$$\mathbb{P}\left(\mathrm{Card}(Z_d) > \sqrt{d}\varepsilon_d\right) = O\left(\frac{\sqrt{d}}{\varepsilon_d}e^{-D\varepsilon_d^2}\right)$$

for any positive sequence $(\varepsilon_d)_{d\geqslant 1}$ such that $\varepsilon_d \xrightarrow[d\to+\infty]{} +\infty$. In the other direction, the following corollary bounds the probability that Z_d is empty.

Corollary 1.15 (Hole probability). — *In the setting of Theorem 1.12, let U be a non-empty open subset of M. Then, for all $p \in \mathbb{N}^*$, as $d \to +\infty$, we have $\mathbb{P}(Z_d \cap U = \emptyset) = O(d^{-\frac{p}{2}})$.*

About the proofs

The proof of Theorem 1.12 relies on several ingredients. Some of them are classical, such as Kac–Rice formulas and estimates for the Bergman kernel of $\mathcal{E}\otimes\mathcal{L}^d$, other are new, such as the key combinatorial argument that we develop in Section 3.

Kac–Rice formulas are a classical tool in the study of the number of real roots of random polynomials (see [AT07, AW09] for example). More generally, they allow to express the moments of local quantities associated with the level sets of a Gaussian process, such as their volume or their Euler characteristic, only in terms of the

correlation function of the process. Even if these formulas are well-known, it is the first time, to the best of our knowledge, that they are used to compute the exact asymptotics of central moments of order greater than 3. The Kac–Rice formulas we use in this paper were proved in [Anc21]. We recall them in Proposition 2.23. They allow us to write the p^{th} central moment $m_p(\nu_d)(\phi_1, \dots, \phi_p)$ as the integral over M^p of $\phi : (x_1, \dots, x_p) \mapsto \prod_{i=1}^p \phi_i(x_i)$ times some density function $\mathcal{D}_d^p$. Here we are cheating a bit: the random set Z_d being almost surely discrete, the Kac–Rice formulas yield the so-called factorial moments of ν_d instead of the usual moments. This issue is usual (compare [AT07, AW09]), and it will not trouble us much since the central moments can be written as linear combinations of the factorial ones. For the purpose of this sketch of proof, let us pretend that we have indeed:

$$m_p(\nu_d)\,(\phi_1, \dots, \phi_p) = \int_{M^p} \phi \mathcal{D}_d^p \,|\mathrm{d}V_M|^p\,.$$

This simplified situation is enough to understand the main ideas of the proof. The correct statement is given in Lemma 3.5 below.

The density $\mathcal{D}_d^p$ is a polynomial in the Kac–Rice densities $(\mathcal{R}_d^k)_{1\leqslant k\leqslant p}$ appearing in Definition 2.21. As such, it only depends on the correlation function of the Gaussian process $(s_d(x))_{x\in M}$, which is the Bergman kernel of $\mathcal{E}\otimes\mathcal{L}^d$. This kernel admits a universal local scaling limit at scale $d^{-\frac{1}{2}}$, which is exponentially decreasing (cf. [MM07, MM15]). In [Anc21], the author used these Bergman kernel asymptotics and Olver multispaces (see [Olv01]) to prove estimates for the $(\mathcal{R}_d^k)_{1\leqslant k\leqslant p}$ in the large d limit. These key estimates are recalled in Propositions 2.24 and 2.26 below. They allow us to show that $\mathcal{D}_d^p(x) = O(d^{\frac{p}{2}})$, uniformly in $x \in M^p$. Moreover, we show that $\mathcal{D}_d^p(x) = O(d^{\frac{p}{4}-1})$, uniformly in $x \in M^p$ such that one of the components of x is far from the others. By this we mean that $x = (x_1, \dots, x_p)$ and there exists $i \in \{1, \dots, p\}$ such that, for all $j \neq i$, the distance from x_i to x_j is larger than $b_p \frac{\ln d}{\sqrt{d}}$, where $b_p > 0$ is some well-chosen constant.

In order to understand the integral of $\phi\mathcal{D}_d^p$, we split M^p as follows. For any point $x = (x_1, \dots, x_p) \in M^p$, we define a graph (see Definition 3.7) whose vertices are the integers $\{1, \dots, p\}$, with an edge between i and j if and only if $i \neq j$ and the distance from x_i to x_j is less than $b_p \frac{\ln d}{\sqrt{d}}$. The connected components of this graph yield a partition $\mathcal{I}(x) \in \mathcal{P}_p$ (see Definition 3.8) encoding how the $(x_i)_{1\leqslant i\leqslant p}$ are clustered in M, at scale $d^{-\frac{1}{2}}$. An example of this construction is represented on Figure 3.1 in Section 3.2. Denoting by $M_{\mathcal{I}}^p = \{x \in M^p \mid \mathcal{I}(x) = \mathcal{I}\}$, we have:

$$m_p(\nu_d)(\phi_1, \dots, \phi_p) = \sum_{\mathcal{I}\in\mathcal{P}_p} \int_{M_{\mathcal{I}}^p} \phi\mathcal{D}_d^p \,|\mathrm{d}V_M|^p\,.$$

Thanks to our estimates on $\mathcal{D}_d^p$, we show that if $\mathcal{I}$ contains a singleton then the integral over $M_{\mathcal{I}}^p$ is $O(d^{\frac{p}{4}-1})$. Hence it contributes only an error term in Theorem 1.12. Moreover, denoting by $|\mathcal{I}|$ the cardinality of $\mathcal{I}$ (i.e. the number of clusters), the volume of $M_{\mathcal{I}}^p$ is $O(d^{\frac{|\mathcal{I}|-p}{2}}(\ln d)^p)$. Hence, the integral over $M_{\mathcal{I}}^p$ is $O(d^{\frac{|\mathcal{I}|}{2}}(\ln d)^p)$. If $|\mathcal{I}| < \frac{p}{2}$, this is also an error term in Theorem 1.12. Thus, the main contribution in $m_p(\nu_d)(\phi_1, \dots, \phi_p)$ comes from the integral of $\phi\mathcal{D}_d^p$ over the pieces $M_{\mathcal{I}}^p$ indexed by partitions $\mathcal{I} \in \mathcal{P}_p$ without singletons and such that $|\mathcal{I}| \geqslant \frac{p}{2}$. These are exactly the

partitions into pairs of $\{1, \dots, p\}$. Finally, if $\mathcal{I} \in \mathcal{PP}_p$, we prove the contribution of the integral over $M^p_{\mathcal{I}}$ to be equal to the product of covariances $\prod_{\{i,j\}\in\mathcal{I}} m_2(\nu_d)(\phi_i, \phi_j)$, up to an error term. When all the test-functions $(\phi_i)_{1\leqslant i\leqslant p}$ are equal, the p^{th} moment μ_p of the standard Gaussian distribution appears as the cardinality of the set of partitions of $\{1, \dots, p\}$ into pairs.

Concerning the corollaries, Corollaries 1.14 and 1.15 follow from Theorem 1.12 and Markov's Inequality for the $2p^{\text{th}}$ moment. The strong Law of Large Numbers (Theorem 1.7) is deduced from Theorem 1.12 for $p = 6$ by a Borel–Cantelli type argument. The Central Limit Theorem (Theorem 1.9) for the linear statistics is obtained by the method of moments. The functional version of this Central Limit Theorem is then obtained by the Lévy–Fernique Theorem (cf. [Fer67]), which is an extension of Lévy's Continuity Theorem adapted to generalized random processes.

Higher dimension

In this paper, we are concerned with the real roots of a random polynomial (or a random section) in an ambient space of dimension 1. There is a natural higher dimensional analogue of this problem. Namely, one can consider the common zero set $Z_d \subset \mathbb{R}P^n$ of r independent real Kostlan polynomials in $n+1$ variables, where $r \in \{1, \dots, n\}$. More generally, we consider the real zero set Z_d of a random real section of $\mathcal{E}\otimes\mathcal{L}^d \to \mathcal{X}$ in the complex Fubini–Study model, where $\mathcal{X}$ is a real projective manifold of complex dimension n whose real locus M is non-empty, $\mathcal{L}$ is a positive line bundle as above, and $\mathcal{E}$ is a rank r real Hermitian bundle with $1 \leqslant r \leqslant n$. Then, for d large enough, Z_d is almost surely a smooth closed submanifold of codimension r in the smooth closed n-dimensional manifold M. In this setting, M is equipped with a natural Riemannian metric that induces a volume measure $|\mathrm{d}V_M|$ on M and a volume measure ν_d on Z_d. As in the 1-dimensional case, ν_d is an almost surely well-defined random Radon measure on M. In this higher dimensional setting, we have the following analogues of Theorem 1.1 and 1.3 (see [Let16, Let19, LP19]):

$$\forall\, \phi \in \mathcal{C}^0(M), \mathbb{E}[\langle \nu_d\,, \phi\rangle] = d^{\frac{r}{2}} \frac{\mathrm{Vol}(\mathbb{S}^{n-r})}{\mathrm{Vol}(\mathbb{S}^n)} \int_M \phi\, |\mathrm{d}V_M| + \|\phi\|_\infty\, O\left(d^{\frac{r}{2}-1}\right),$$

$$\forall\, \phi_1, \phi_2 \in \mathcal{C}^0(M), m_2(\nu_d)(\phi_1, \phi_2) = d^{r-\frac{n}{2}} \sigma^2_{n,\,r} \int_M \phi_1\phi_2\, |\mathrm{d}V_M| + o\left(d^{r-\frac{n}{2}}\right),$$

where $\sigma_{n,r} > 0$ is a universal constant depending only on n and r. In [LP19], Letendre and Puchol proved some analogues of Corollaries 1.14 and 1.15 for any n and r. They also showed that the strong Law of Large Numbers (Theorem 1.7) holds if $n \geqslant 3$.

Most of the proof of Theorem 1.12 is valid in any dimension and codimension. In fact, the combinatorics are simpler when $r < n$. The only things we are missing, in order to prove the analogue of Theorem 1.12 for any n and r, are higher dimensional versions of Propositions 2.24 and 2.26. The proofs of these propositions (see [Anc21]) rely on the compactness of Olver multispaces, which holds when $n = 1$ but fails for $n > 1$. This seems to be only a technical obstacle and the authors are currently working toward the following.

Conjecture 1.16. — *Let s_d be a random section in the complex Fubini–Study model in dimension n and codimension $r \in \{1, \ldots, n\}$. Let ν_d denote the volume measure of the real zero set of s_d. For all $p \geqslant 3$, for all $\phi_1, \ldots, \phi_p \in \mathcal{C}^0(M)$, the following holds as $d \to +\infty$:*

$$m_p(\nu_d)(\phi_1, \ldots, \phi_p) = d^{\frac{p}{2}\left(r-\frac{n}{2}\right)}\sigma_{n,r}^p \sum_{\mathcal{I} \in \mathcal{PP}_p} \prod_{\{i,j\} \in \mathcal{I}} \left(\int_M \phi_i\phi_j \left|\mathrm{d}V_M\right|\right) + o\left(d^{\frac{p}{2}\left(r-\frac{n}{2}\right)}\right).$$

In particular, for all $\phi \in \mathcal{C}^0(M)$,

$$m_p(\langle \nu_d\,, \phi\rangle) = \mu_p d^{\frac{p}{2}\left(r-\frac{n}{2}\right)}\sigma_{n,r}^p \left(\int_M \phi^2 \left|\mathrm{d}V_M\right|\right)^{\frac{p}{2}} + o\left(d^{\frac{p}{2}\left(r-\frac{n}{2}\right)}\right).$$

Proving this conjecture for $n = 2$ and $p = 4$ is enough to prove that the strong Law of Large Numbers (Theorem 1.7) holds for $n = 2$, which is the only missing case. This conjecture also implies the Central Limit Theorem (Theorem 1.9) in dimension n and codimension r, with the same proof as the one given in Section 4.2. Note that a Central Limit Theorem for the volume of the common zero set of r Kostlan polynomials in $\mathbb{R}P^n$ was proved in [AADL21, AADL18].

1.2. Other related works

The complex roots of complex Kostlan polynomials have been studied in relation with Physics in [BBL92]. More generally, complex zeros of random holomorphic sections of positive line bundles over projective manifolds were studied in [SZ99] and some subsequent papers by the same authors. In [SZ99], they computed the asymptotics of the expected current of integration over the complex zeros of such a random section, and proved a Law of Large Numbers similar to Theorem 1.7. In [SZ10], they obtained a variance estimate for this random current, and proved that it satisfies a Central Limit Theorem. This last paper extends the results of [ST04] for the complex roots of a family of random polynomials, including elliptic ones. In [BSZ00], Bleher, Shiffman and Zelditch studied the p-points zero correlation function associated with random holomorphic sections. These functions are the Kac–Rice densities for the non-central p^{th} moment of the linear statistics in the complex case. The results of [Anc21] are also valid in the 1-dimensional complex case, see [Anc21, Section 5]. Thus, Theorem 1.12 can be extended to the complex case, with the same proof.

In Corollaries 1.14 and 1.15, we deduce from Theorem 1.12 some concentration in probability, faster than any negative power of d. However, our results are not precise enough to prove that this concentration is exponentially fast in d. In order to obtain such a large deviation estimate, one would need to investigate how the constants involved in the error terms in Theorem 1.12 grow with p. Some large deviations estimates are known for the complex roots of random polynomials. As far as real roots are concerned, the only result of this kind we are aware of is [BDFZ20].

The Central Limit Theorem (Theorem 1.9) was already known for the roots of Kostlan polynomials, see [Dal15]. In the wider context of random real geometry, Central Limit Theorems are known in several settings, see [AADL21, AADL18,

Ros19] and the references therein. The proofs of all these results rely on Wiener chaos techniques developed by Kratz–Leòn [KL01]. Our proof of Theorem 1.9 follows a different path, relying on the method of moments, for two reasons. First, to the best of our knowledge, Wiener chaos techniques are not available for random sections of line bundles. Second, these techniques are particularly convenient when dealing with random models with slowly decaying correlations, for example Random Waves models. In these cases, one of the first chaotic components, usually the second or the fourth, is asymptotically dominating. Hence, one can reduce the study of, say, the number of zeros to that of its dominating chaotic component, which is easier to handle. In the complex Fubini–Study setting we are considering in this paper, the correlations are exponentially decreasing, so that all chaotic components of the number of zeros have the same order of magnitude as the degree goes to infinity. In order to use Wiener chaoses to prove a Central Limit Theorem in our setting, one would have to study the joint asymptotic behavior of all the chaotic components, which seems more difficult than our method.

For real zeros in ambient dimension $n = 1$, Nazarov and Sodin [NS16] proved a strong Law of Large Numbers, as $R \to +\infty$, for the number of zeros of a Gaussian process lying in the interval $[-R, R]$. Finally, to the best of our knowledge, Theorem 1.12 gives the first precise estimate for the central moments of the number of real zeros of a family of random processes.

Organization of the paper

This paper is organized as follows. In Section 2 we introduce the object of our study and recall some useful previous results. More precisely, we introduce our geometric framework in Section 2.1. In Section 2.2 we introduce various notations that will allow us to make sense of the combinatorics involved in our problem. The random measures we study are defined in Section 2.3. Finally, we state the Kac–Rice formulas for higher moments in Section 2.4 and we recall several results from [Anc21] concerning the density functions appearing in these formulas. In Section 3, we prove our main result, that is the moments estimates of Theorem 1.12. Section 4 is concerned with the proofs of the corollaries of Theorem 1.12. We prove the Law of Large Numbers (Theorem 1.7) in Section 4.1, the Central Limit Theorem (Theorem 1.9) in Section 4.2 and the remaining corollaries (Corollaries 1.14 and 1.15) in Section 4.3.

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2. Framework and background

We start this section by defining precisely the geometric setting in which we work. This is the purpose of Section 2.1. In Section 2.2, we introduce the counting measures

that we study and explain how they can be split in terms that are easier to handle. Section 2.3 is dedicated to the definition of the model of random sections we consider. Finally, we recall the Kac–Rice formulas we need in Section 2.4, as well as several useful estimates for the density functions appearing in these formulas.

2.1. Geometric setting

In this section, we introduce our geometric framework, which is the same as that of [Anc21, GW16, Let16, Let19, LP19]. See also [BSZ00, SZ99, SZ10], where the authors work in a related complex setting. A classical reference for some of the material of this section is [GH94].

- Let $(\mathcal{X}, c_{\mathcal{X}})$ be a smooth real projective curve, that is a smooth complex projective manifold $\mathcal{X}$ of complex dimension 1, equipped with an anti-holomorphic involution $c_{\mathcal{X}}$. We denote by M the real locus of $\mathcal{X}$, that is the set of fixed points of $c_{\mathcal{X}}$. Throughout the paper, we assume that M is not empty. In this case, M is a smooth compact submanifold of $\mathcal{X}$ of real dimension 1 without boundary, that is the disjoint union of a finite number of circles.
- Let $(\mathcal{E}, c_{\mathcal{E}})$ and $(\mathcal{L}, c_{\mathcal{L}})$ be two real holomorphic line bundles over $(\mathcal{X}, c_{\mathcal{X}})$. Denoting by $\pi_{\mathcal{E}}$ (resp. $\pi_{\mathcal{L}}$) the bundle projection, this means that $\mathcal{E} \to \mathcal{X}$ (resp. $\mathcal{L} \to \mathcal{X}$) is an holomorphic line bundle such that $\pi_{\mathcal{E}} \circ c_{\mathcal{E}} = c_{\mathcal{X}} \circ \pi_{\mathcal{E}}$ (resp. $\pi_{\mathcal{L}} \circ c_{\mathcal{L}} = c_{\mathcal{X}} \circ \pi_{\mathcal{L}}$) and that $c_{\mathcal{E}}$ (resp. $c_{\mathcal{L}}$) is anti-holomorphic and fiberwise $\mathbb{C}$-anti-linear. For any $d \in \mathbb{N}$, we denote by $c_d = c_{\mathcal{E}} \otimes c_{\mathcal{L}^d}$. Then, $(\mathcal{E} \otimes \mathcal{L}^d, c_d)$ is a real holomorphic line bundle over $(\mathcal{X}, c_{\mathcal{X}})$.
- We equip $\mathcal{E}$ with a real Hermitian metric $h_{\mathcal{E}}$. That is $(\mathcal{E}, h_{\mathcal{E}})$ is an Hermitian line bundle, and moreover $c_{\mathcal{E}}^* h_{\mathcal{E}} = \overline{h_{\mathcal{E}}}$. Similarly, let $h_{\mathcal{L}}$ denote a real Hermitian metric on $\mathcal{L}$. For all $d \in \mathbb{N}$, we denote by $h_d = h_{\mathcal{E}} \otimes h_{\mathcal{L}}^d$, which defines a real Hermitian metric on $\mathcal{E} \otimes \mathcal{L}^d \to \mathcal{X}$. We assume that $(\mathcal{L}, h_{\mathcal{L}})$ is positive, in the sense that its curvature form ω is a Kähler form. Recall that ω is locally defined as $\frac{1}{2i}\partial\overline{\partial} \ln(h_{\mathcal{L}}(\zeta, \zeta))$, where ζ is any local holomorphic frame of $\mathcal{L}$. The Kähler structure defines a Riemannian metric $g = \omega(\cdot, i\cdot)$ on $\mathcal{X}$, hence on M. The Riemannian volume form on $\mathcal{X}$ associated with g is simply ω. We denote by $|\mathrm{d}V_M|$ the arc-length measure on M associated with g. For all $k \in \mathbb{N}^*$, we denote by $|\mathrm{d}V_M|^k$ the product measure on M^k.
- For any $d \in \mathbb{N}$, we denote by $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ the space of global holomorphic sections of $\mathcal{E} \otimes \mathcal{L}^d$. This is a complex vector space of complex dimension N_d. By the Riemann–Roch Theorem, N_d is finite and diverges to infinity as $d \to +\infty$. We denote by:
$$\mathbb{R}H^0\left(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d\right) = \left\{ s \in H^0\left(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d\right) \middle| s \circ c_{\mathcal{X}} = c_d \circ s \right\}$$
the space of global real holomorphic sections of $\mathcal{E} \to \mathcal{L}^d$, which is a real vector space of real dimension N_d. Let $x \in M$, the fiber $(\mathcal{E} \otimes \mathcal{L}^d)_x$ is a complex line equipped with a $\mathbb{C}$-anti-linear involution $c_d(x)$. We denote by $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x$ the set of fixed points of $c_d(x)$, which is a real line. Then, $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d) \to M$ is a real line bundle and, for any $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, the restriction of s to M is a smooth section of $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d) \to M$.

- The volume form ω and the Hermitian metric h_d induce an Hermitian L^2-inner product $\langle \cdot\,, \cdot \rangle$ on $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. It is defined by:

$$\forall\, s_1, s_2 \in H^0\left(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d\right), \qquad \langle s_1\,, s_2 \rangle = \int_{\mathcal{X}} h_d(s_1, s_2)\omega. \tag{2.1}$$

  The restriction of this inner product to $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is a Euclidean inner product.
- For any section $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, we denote by $Z_s = s^{-1}(0) \cap M$ its real zero set. Since s is holomorphic, if $s \neq 0$ its zeros are isolated. In this case, Z_s is finite by compactness of M, and we denote by $\nu_s = \sum_{x \in Z_s} \delta_x$, where δ_x stands for the unit Dirac mass at $x \in M$. The measure ν_s is called the *counting measure* of Z_s. It is a Radon measure, that is a continuous linear form on the space $(\mathcal{C}^0(M), \|\cdot\|_\infty)$ of continuous functions equipped with the sup-norm. It acts on continuous functions by: for all $\phi \in \mathcal{C}^0(M)$, $\langle \nu_s\,, \phi \rangle = \sum_{x \in Z_s} \phi(x)$.

Example 2.1 (Kostlan scalar product). — We conclude this section by giving an example of our geometric setting. We consider $\mathcal{X} = \mathbb{C}P^1$, equipped with the conjugation induced by the one in $\mathbb{C}^2$. Its real locus is $M = \mathbb{R}P^1$. We take $\mathcal{E}$ to be trivial and $\mathcal{L}$ to be the dual of the tautological line bundle $\{(v, x) \in \mathbb{C}^2 \times \mathbb{C}P^1 | v \in x\}$, that is $\mathcal{L} = \mathcal{O}(1)$. Both $\mathcal{E}$ and $\mathcal{L}$ are canonically real Hermitian line bundle over $\mathbb{C}P^1$, and the curvature of $\mathcal{L}$ is the Fubini–Study form, normalized so that $\mathrm{Vol}(\mathbb{C}P^1) = \pi$. The corresponding Riemannian metric on $\mathbb{R}P^1$ is the quotient of the metric on the Euclidean unit circle, so that the length of $\mathbb{R}P^1$ is π.

In this setting, $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ (resp. $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$) is the space of homogeneous polynomials of degree d in two variables with complex (resp. real) coefficients. If $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is such a polynomial, then Z_s is the set of its roots in $\mathbb{R}P^1$. Finally, up to a factor $(d+1)\pi$ which is irrelevant to us, the inner product of Equation (2.1) is defined by:

$$\langle P\,, Q \rangle = \frac{1}{\pi^2 d!} \int_{\mathbb{C}^2} P(z)\overline{Q(z)} e^{-\|z\|^2}\, \mathrm{d}z,$$

for any homogeneous polynomials P and Q of degree d in two variables. In particular, the family

$$\left\{ \sqrt{\binom{d}{k}} X^k Y^{d-k} \;\middle|\; 0 \leqslant k \leqslant d \right\}$$

is an orthonormal basis of $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ for this inner product.

2.2. Partitions, products and diagonal inclusions

In this section, we introduce some notations that will be useful throughout the paper, in particular to sort out the combinatorics involved in the proof of Theorem 1.12 (see Section 3). In all this section, M denotes a smooth manifold.

Notation 2.2. — Let A be a finite set.

- We denote by $\mathrm{Card}(A)$ or by $|A|$ the cardinality of A.
- We denote by M^A the Cartesian product of $|A|$ copies of M, indexed by the elements of A.
- A generic element of M^A is denoted by $\underline{x}_A = (x_a)_{a\in A}$. If $B \subset A$ we denote by $\underline{x}_B = (x_a)_{a\in B}$.
- Let $(\phi_a)_{a\in A}$ be continuous functions on M, we denote by $\phi_A = \boxtimes_{a\in A}\phi_a$ the function on M^A defined by: $\phi_A(\underline{x}_A) = \prod_{a\in A}\phi_a(x_a)$, for all $\underline{x}_A = (x_a)_{a\in A}$.

If A is clear from the context or of the form $\{1, \dots, k\}$ with $k \in \mathbb{N}^*$, we use the simpler notations x for $\underline{x}_A$ and ϕ for ϕ_A.

Recall the we defined the set $\mathcal{P}_A$ (resp. $\mathcal{P}_k$) of partitions of a finite set A (resp. of $\{1, \dots, k\}$) in the introduction, see Definition 1.11.

Definition 2.3 (Diagonals). — *Let A be a finite set, we denote by Δ_A the* large diagonal *of M^A, that is:*

$$\Delta_A = \left\{(x_a)_{a\in A} \in M^A \;\middle|\; \exists\, a, b \in A \text{ such that } a \neq b \text{ and } x_a = x_b\right\}.$$

Moreover, for all $\mathcal{I} \in \mathcal{P}_A$, we denote by

$$\Delta_{A,\mathcal{I}} = \left\{(x_a)_{a\in A} \in M^A \;\middle|\; \forall\, a, b \in A, (x_a = x_b \iff \exists\, I \in \mathcal{I}, a \in I \text{ and } b \in I)\right\}.$$

If $A = \{1, \dots, k\}$, we use the simpler notations Δ_k for Δ_A, and $\Delta_{k,\mathcal{I}}$ for $\Delta_{A,\mathcal{I}}$.

Definition 2.4 (Diagonal inclusions). — *Let A be a finite set and let $\mathcal{I} \in \mathcal{P}_A$. The* diagonal inclusion *$\iota_{\mathcal{I}} : M^{\mathcal{I}} \to M^A$ is the function defined, for all $\underline{y}_{\mathcal{I}} = (y_I)_{I\in\mathcal{I}} \in M^{\mathcal{I}}$, by $\iota_{\mathcal{I}}(\underline{y}_{\mathcal{I}}) = (x_a)_{a\in A}$, where for all $I \in \mathcal{I}$, for all $a \in I$, we set $x_a = y_I$.*

Remark 2.5. — With the previous definitions, we have $M^A = \bigsqcup_{\mathcal{I}\in\mathcal{P}_A}\Delta_{A,\mathcal{I}}$ and $\Delta_A = \bigsqcup_{\mathcal{I}\in\mathcal{P}_A\setminus\{\mathcal{I}_0(A)\}}\Delta_{A,\mathcal{I}}$, where we denoted by $\mathcal{I}_0(A) = \{\{a\}|a \in A\}$. Moreover, $\iota_{\mathcal{I}}$ is a smooth diffeomorphism from $M^{\mathcal{I}} \setminus \Delta_{\mathcal{I}}$ onto $\Delta_{A,\mathcal{I}} \subset M^A$. Note that $\Delta_{A,\mathcal{I}_0(A)}$ is the configuration space $M^A \setminus \Delta_A$ of $|A|$ distinct points in M. In the following, we avoid using the notation $\Delta_{A,\mathcal{I}_0(A)}$ and use $M^A \setminus \Delta_A$ instead.

Let us now go back to the setting of Section 2.1, in which M is the real locus of the projective manifold $\mathcal{X}$. Let $d \in \mathbb{N}$ and let $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d) \setminus \{0\}$. In Section 2.1, we defined the counting measure ν_s of the real zero set Z_s of s. More generally, for any finite set A, we can define the counting measures of the sets $Z_s^A = \{(x_a)_{a\in A} \in M^A | \ \forall\ a \in A, x_a \in Z_s\}$ and $Z_s^A \setminus \Delta_A$. The latter is especially interesting for us, since this is the one that appears in the Kac–Rice formulas, see Proposition 2.23 below.

Definition 2.6 (Counting measures). — *Let $d \in \mathbb{N}$ and let A be a finite set. For any non-zero $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$, we denote by:*

$$\nu_s^A = \sum_{x\in Z_s^A}\delta_x \qquad \text{and} \qquad \widetilde{\nu}_s^A = \sum_{x\in Z_s^A\setminus\Delta_A}\delta_x,$$

where δ_x is the unit Dirac mass at $x = (x_a)_{a\in A}$ and Δ_A is defined by Definition 2.3. These measures are the counting measures of Z_s^A and $Z_s^A \setminus \Delta_A$ respectively. Both ν_s^A

and $\widetilde{\nu}_s^A$ are Radon measure on M^A. They act on $\mathcal{C}^0(M^A)$ by: for any $\phi \in \mathcal{C}^0(M^A)$,

$$\left\langle \nu_s^A , \phi \right\rangle = \sum_{x \in Z_s^A} \phi(x) \qquad \text{and} \qquad \left\langle \widetilde{\nu}_s^A , \phi \right\rangle = \sum_{x \in Z_s^A \setminus \Delta_A} \phi(x).$$

As usual, if $A = \{1, \dots, k\}$, we denote ν_s^k for ν_s^A and $\widetilde{\nu}_s^k$ for $\widetilde{\nu}_s^A$.

We have seen in Remark 2.5 that M^A splits as the disjoint union of the diagonals $\Delta_{A,\mathcal{I}}$, with $\mathcal{I} \in \mathcal{P}_A$. Taking the intersection with Z_s^A yields a splitting of this set. Using the diagonal inclusions of Definition 2.4, this can be expressed in terms of counting measures as follows.

Lemma 2.7. — *Let $d \in \mathbb{N}$ and let A be a finite set. Then, for any section $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \setminus \{0\}$, we have:*

$$\nu_s^A = \sum_{\mathcal{I} \in \mathcal{P}_A} (\iota_{\mathcal{I}})_* \left(\widetilde{\nu}_s^{\mathcal{I}} \right).$$

Proof. — Recall that $M^A = \bigsqcup_{\mathcal{I} \in \mathcal{P}_A} \Delta_{A,\mathcal{I}}$. Then, we have:

$$\nu_s^A = \sum_{x \in Z_s^A} \delta_x = \sum_{\mathcal{I} \in \mathcal{P}_A} \left(\sum_{x \in Z_s^A \cap \Delta_{A,\mathcal{I}}} \delta_x \right).$$

Let $\mathcal{I} \in \mathcal{P}_A$, recall that $\iota_{\mathcal{I}}$ defines a smooth diffeomorphism from $M^{\mathcal{I}} \setminus \Delta_{\mathcal{I}}$ onto $\Delta_{A,\mathcal{I}}$. Moreover, $\iota_{\mathcal{I}}(Z_s^{\mathcal{I}} \setminus \Delta_{\mathcal{I}}) = Z_s^A \cap \Delta_{A,\mathcal{I}}$ (see Definition 2.3 and 2.4). Hence,

$$\sum_{x \in Z_s^A \cap \Delta_{A,\mathcal{I}}} \delta_x = \sum_{y \in Z_s^{\mathcal{I}} \setminus \Delta_{\mathcal{I}}} \delta_{\iota_{\mathcal{I}}(y)} = \sum_{y \in Z_s^{\mathcal{I}} \setminus \Delta_{\mathcal{I}}} (\iota_{\mathcal{I}})_* \delta_y = (\iota_{\mathcal{I}})_* \widetilde{\nu}_s^{\mathcal{I}}. \qquad \square$$

2.3. Zeros of random real sections

Let us now introduce the main object of our study: the random Radon measure ν_d encoding the real zeros of a random real section $s_d \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. The model of random real sections we study is often referred to as the complex Fubini–Study model. It was introduced in this generality by Gayet and Welschinger in [GW11]. This model is the real counterpart of the model of random holomorphic sections studied by Shiffman and Zelditch in [SZ99] and subsequent articles.

Definition 2.8 (Gaussian vectors). — *Let $(V, \langle \cdot , \cdot \rangle)$ be a Euclidean space of dimension N, and let Λ denote a positive self-adjoint operator on V. Recall that a random vector X in V is said to be a* centered Gaussian *with* variance operator Λ *if its distribution admits the following density with respect to the normalized Lebesgue measure:*

$$v \mapsto \frac{1}{(2\pi)^{\frac{N}{2}} \det(\Lambda)^{\frac{1}{2}}} \exp\left(-\frac{1}{2} \left\langle v , \Lambda^{-1} v \right\rangle \right).$$

We denote by $X \sim \mathcal{N}(0, \Lambda)$ the fact that X follows this distribution. If $X \sim \mathcal{N}(0, \mathrm{Id})$, where Id *is the identity of V, we say that X is a* standard Gaussian vector *in V.*

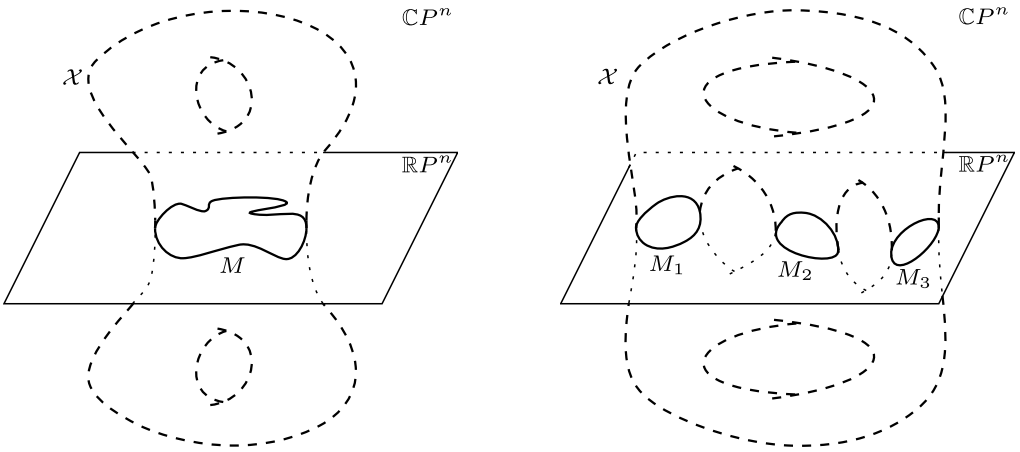
Remark 2.9. — Recall that if $(e_1, \dots, e_N)$ is an orthonormal basis of V, a random vector $X \in V$ is a standard Gaussian vector if and only if $X = \sum_{i=1}^N a_i e_i$ where the $(a_i)_{1 \leqslant i \leqslant N}$ are independent identically distributed $\mathcal{N}(0,1)$ real random variables.

In the setting of Section 2.1, for any $d \in \mathbb{N}$, the space $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is endowed with the Euclidean inner product $\langle \cdot, \cdot \rangle$ defined by Equation (2.1). We denote by s_d a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Almost surely $s_d \neq 0$, hence its real zero set and the associated counting measure are well-defined. For simplicity, we denote by $Z_d = Z_{s_d}$ and by $\nu_d = \nu_{s_d}$. Similarly, for any finite set A, we denote by $\nu_d^A = \nu_{s_d}^A$ and by $\tilde{\nu}_d^A = \tilde{\nu}_{s_d}^A$ (see Definition 2.6).

Example 2.10 (Kostlan polynomials). — In the setting of Example 2.1, we have $M = \mathbb{R}P^1$, and $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is the set of real homogeneous polynomials of degree d in two variables endowed with the Kostlan inner product. In this case, the standard Gaussian vector $s_d \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is the homogeneous Kostlan polynomial of degree d defined in Section 1.

Note that the rotations of the circle $M = \mathbb{R}P^1$ are induced by orthogonal transformations of $\mathbb{R}^2$. The group $O_2(\mathbb{R})$ acts on the space of homogeneous polynomials in two variables of degree d by composition on the right. Example 2.1 shows that the Kostlan inner product is invariant under this action, hence so is the distribution of s_d. In other terms, the random process $(s_d(x))_{x \in M}$ is stationary.

Example 2.11. — Let us give another example showing that the complex Fubini–Study model goes beyond Kostlan polynomials. Let $\mathcal{X}$ be a Riemann surface embedded in $\mathbb{C}P^n$. We assume that $\mathcal{X}$ is real, in the sense that it is stable under complex conjugation in $\mathbb{C}P^n$. Note that, in our setting, we can always realize $\mathcal{X}$ in such a way for some n large enough, by Kodaira's Embedding Theorem. Then $M = \mathcal{X} \cap \mathbb{R}P^n$ is the disjoint union of at most $\mathfrak{g}+1$ smooth circles in $\mathbb{R}P^n$, where $\mathfrak{g}$ is the genus of $\mathcal{X}$. Figure 2.1 below shows two examples of this geometric situation.



(a) M is a circle even though $\mathcal{X}$ has genus 2.

(b) $\mathcal{X}$ is connected but M has three connected components M_1, M_2 and M_3.

Figure 2.1. Two examples of a real Riemann surface $\mathcal{X}$ embedded in $\mathbb{C}P^n$ and its real locus M.

We choose $\mathcal{E}$ to be trivial and $\mathcal{L}$ to be the restriction to $\mathcal{X}$ of the hyperplane line bundle $\mathcal{O}(1) \to \mathbb{C}P^n$. Then, the elements of $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ are restrictions to $\mathcal{X}$ of homogeneous polynomials of degree d in $n+1$ variables with real coefficients. This space is equipped with the inner product (2.1). In Equation (2.1), the Kähler form ω is the restriction $\mathcal{X}$ of the Fubini–Study form on $\mathbb{C}P^n$. However, the domain of integration is $\mathcal{X}$, so that (2.1) is not the analogue in $n+1$ variables of the Kostlan inner product of Example 2.1.

- On Figure 2.1a, the curve M is connected hence diffeomorphic to $\mathbb{R}P^1$. However, $\mathcal{E} \otimes \mathcal{L}^d \to \mathcal{X}$ is not an avatar of $\mathcal{O}(d) \to \mathbb{C}P^1$ for some $d \geqslant 1$, since $\mathcal{X} \not\simeq \mathbb{C}P^1$. In particular, the previous construction gives a random homogeneous polynomial s_d on $M \simeq \mathbb{R}P^1$, which is Gaussian and centered but is not a Kostlan polynomial of some degree. Unlike what happens in Example 2.10, rotations in M are not obtained as restriction of isometries of $\mathcal{X}$ (generically, the only isometries of $\mathcal{X}$ are the identity and the complex conjugation). Thus, there is no reason why the process $(s_d(x))_{x \in M}$ should be stationary.
- On Figure 2.1b, the real locus M had several connected components while $\mathcal{X}$ is connected. Since the inner product (2.1) is defined by integrating on the whole complex locus $\mathcal{X}$, the values of s_d in different connected components of M are a priori correlated.

Lemma 2.12 (Boundedness of linear statistics). — *Let $d \geqslant 1$ and $s_d \sim \mathcal{N}(0, \mathrm{Id})$ in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. For all $\phi \in \mathcal{C}^0(M)$, the random variable $\langle \nu_d , \phi \rangle$ is bounded. In particular, it admits finite moments of any order.*

Proof. — We have $|\langle \nu_d , \phi \rangle| = |\sum_{x \in Z_d} \phi(x)| \leqslant \|\phi\|_\infty \operatorname{Card}(Z_d)$. Hence it is enough to prove that $\operatorname{Card}(Z_d)$ is a bounded random variable.

The cardinality of Z_d is the number of zeros of s_d in M, which is smaller than the number of zeros of s_d in $\mathcal{X}$. Now, almost surely, $s_d \neq 0$, and the complex zero set of s_d defines a divisor which is Poincaré-dual to the first Chern class of $\mathcal{E} \otimes \mathcal{L}^d$ (see [GH94, pp. 136 and 141]). Hence, almost surely:

$$\operatorname{Card}(Z_d) \leqslant \operatorname{Card}\{x \in \mathcal{X} \mid s_d(x) = 0\} \leqslant \int_{\mathcal{X}} c_1\left(\mathcal{E} \otimes \mathcal{L}^d\right) = d \cdot \deg(\mathcal{L}) + \deg(\mathcal{E}). \quad \square$$

Remark 2.13. — In the case of polynomials, the proof is clearer: the number of real roots of a non-zero polynomial of degree d is bounded by the number of its complex roots, which is at most d.

2.4. Kac–Rice formulas and density functions

In this section, we recall some important facts about Kac–Rice formulas. These formulas are classical tools in the study of moments of local quantities such as the cardinality, or more generally the volume, of the zero set of a smooth Gaussian process. Classical references for this material are [AT07, AW09]. With a more geometric point of view, the following formulas were proved and used in [Anc21, GW16, LP19], see also [Let16]. In the same spirit, Lerario and Stecconi derived a Kac–Rice formula for sections of fiber bundles, see [LS19, Theorem 23, point 6].

Remark 2.14. — In some of the papers we refer to in this section, the line bundle $\mathcal{E}$ is taken to be trivial. That is the authors considers random real sections of $\mathcal{L}^d$ instead of $\mathcal{E}\otimes\mathcal{L}^d$. As we already explained (see Remark 1.6), the proofs of the results we cite below rely on asymptotics for the Bergman kernel of $\mathcal{E}\otimes\mathcal{L}^d$, as $d\to+\infty$. These asymptotics do not depend on $\mathcal{E}$, see [MM07, Theorem 4.2.1] and [MM15, Theorem 1]. Hence the results established in the case of a trivial line bundle $\mathcal{E}$ can be extended to the case of a general $\mathcal{E}$ without modifying the proofs.

Definition 2.15 (Jacobian). — *Let V and V' be two Euclidean spaces of dimension N and N' respectively. Let $L : V \to V'$ be a linear map and let L^* denote its adjoint. We denote by $\left|\det{}^{\perp}(L)\right| = \det(LL^*)^{\frac{1}{2}}$ the* Jacobian *of L.*

Remark 2.16. — If $N' \leqslant N$, an equivalent definition of the Jacobian is the following: $|\det{}^{\perp}(L)|$ is the absolute value of the determinant of the matrix of L, restricted to $\ker(L)^{\perp}$, in orthonormal bases. Note that, in any case, $|\det{}^{\perp}(L)| > 0$ if and only if L is surjective.

Let us now consider the geometric setting of Section 2.1.

Definition 2.17 (Evaluation map). — *Let $k \in \mathbb{N}^*$, for any $d \in \mathbb{N}$ and any $x = (x_1, \ldots, x_k) \in M^k$, we denote by:*

$$\mathrm{ev}_x^d : \mathbb{R}H^0\left(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d\right) \longrightarrow \bigoplus_{i=1}^{k} \mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_{x_i}$$

the evaluation map *at x, defined by $s \mapsto (s(x_1), \ldots, s(x_k))$.*

Lemma 2.18 ([Anc21, Proposition 2.11]). — *Let $k \in \mathbb{N}^*$, there exists $d_k \in \mathbb{N}$ such that, for all $d \geqslant d_k$, for all $x \in M^k \setminus \Delta_k$, the evaluation map ev_x^d is surjective (i.e. $\det{}^{\perp}(\mathrm{ev}_x^d) > 0$).*

Remark 2.19. — The dimension N_d of $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$ does not depend on k, while the dimension of the target space of ev_x^d equals k. In particular, for any d, the linear map ev_x^d can only be surjective if $k \leqslant N_d$. This shows that $d_k \xrightarrow[k\to+\infty]{} +\infty$. Moreover, $(d_k)_{k\geqslant 1}$ is non-decreasing.

Remark 2.20. — If $s_d \sim \mathcal{N}(0, \mathrm{Id})$ in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$, then the random vector $\mathrm{ev}_x^d(s_d) = (s_d(x_1), \ldots, s_d(x_k))$ is a centered Gaussian in $\bigoplus_{i=1}^k \mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_{x_i}$ whose variance operator is $\mathrm{ev}_x^d(\mathrm{ev}_x^d)^*$. If $d \geqslant d_k$, then for all $x \in M^k \setminus \Delta_k$, this variance operator is positive and $\mathrm{ev}_x^d(s_d)$ is non-degenerate.

Definition 2.21 (Kac–Rice density). — *Let $k \in \mathbb{N}^*$, for any $d \geqslant d_k$ (see Lemma 2.18), we define the* density function *$\mathcal{R}_d^k : M^k \setminus \Delta_k \to \mathbb{R}$ as follows:*

$$\forall\, x = (x_1, \ldots, x_k) \in M^k \setminus \Delta_k, \qquad \mathcal{R}_d^k(x) = (2\pi)^{-\frac{k}{2}} \frac{\mathbb{E}\left[\prod_{i=1}^{k} \|\nabla_{x_i} s_d\| \,\middle|\, \mathrm{ev}_x^d(s_d) = 0\right]}{|\det{}^{\perp}(\mathrm{ev}_x^d)|}.$$

Here, ∇ is any connection on $\mathcal{E}\otimes\mathcal{L}^d$ and

$$\mathbb{E}\left[\prod_{i=1}^{k} \|\nabla_{x_i} s_d\| \,\middle|\, \mathrm{ev}_x^d(s_d) = 0\right]$$

stands for the conditional expectation of $\prod_{i=1}^k \|\nabla_{x_i} s_d\|$ *given that* $\mathrm{ev}_x^d(s_d) = 0$.

Remark 2.22. — Recall that if s is a section and $s(x) = 0$, then the derivative $\nabla_x s$ does not depend on the choice of the connection ∇. In the conditional expectation appearing at the numerator in Definition 2.21, we only consider derivatives of s_d at the points $x_1, \dots, x_k$, under the condition that s_d vanishes at these points. Hence $\mathcal{R}_d^k$ does not depend on the choice of ∇.

We can now state the Kac–Rice formula we are interested in, see [Anc21, Propositions 2.5 and 2.9]. See also [LP19, Theorem 5.5] in the case $k = 2$. Recall that $\widetilde{\nu}_d^k$ was defined by Definition 2.6 and is the counting measure of $(s_d^{-1}(0) \cap M)^k \setminus \Delta_k$.

Proposition 2.23 (Kac–Rice formula). — *Let* $k \in \mathbb{N}^*$ *and let* $d \geqslant d_k$. *Let* $s_d \sim \mathcal{N}(0, \mathrm{Id})$ *in* $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, *for any* $\phi \in \mathcal{C}^0(M^k)$, *we have:*

$$\mathbb{E}\Big[\big\langle \widetilde{\nu}_d^k, \phi \big\rangle\Big] = \int_{x \in M^k} \phi(x) \mathcal{R}_d^k(x) \, |\mathrm{d}V_M|^k .$$

where $\mathcal{R}_d^k$ *is the density function defined by Definition 2.21.*

Let $k \geqslant 2$ and $d \in \mathbb{N}$. If $x \in \Delta_k$, then the evaluation map ev_x^d can not be surjective. Hence, the continuous map $x \mapsto |\det^{\perp}(\mathrm{ev}_x^d)|$ from M^k to $\mathbb{R}$ vanishes on Δ_k, and one would expect $\mathcal{R}_d^k$ to be singular along the diagonal. Yet, Ancona showed that one can extend continuously $\mathcal{R}_d^k$ to the whole of M^k and that the extension vanishes on Δ_k, see [Anc21, Theorem 4.7]. Moreover, he showed that $d^{-\frac{k}{2}} \mathcal{R}_d^k$ is uniformly bounded on $M^k \setminus \Delta_k$ as $d \to +\infty$. We will use this last fact repeatedly in the proof of Theorem 1.12, see Section 3 below.

Proposition 2.24 ([Anc21, Theorem 3.1]). — *For any* $k \in \mathbb{N}^*$, *there exists a constant* $C_k > 0$ *such that, for all* d *large enough, for all* $x \in M^k \setminus \Delta_k$, *we have* $d^{-\frac{k}{2}} \mathcal{R}_d^k(x) \leqslant C_k$.

Let $d \geqslant 1$ and let $s_d \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ be a standard Gaussian. A fundamental idea in our problem is that the values (and more generally the k-jets) of s_d at two points x and $y \in M$ are "quasi-independent" if x and y are far from one another, at scale $d^{-\frac{1}{2}}$. More precisely, $(s_d(x))_{x \in M}$ defines a Gaussian process with values in $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)$ whose correlation kernel is the Bergman kernel E_d of $\mathcal{E} \otimes \mathcal{L}^d$. Recall that E_d is the integral kernel of the orthogonal projection from the space of square integrable sections of $\mathcal{E} \otimes \mathcal{L}^d$ onto $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, for the inner product defined by Equation (2.1). In particular, for any $x, y \in \mathcal{X}$, we have $E_d(x, y) \in (\mathcal{E} \otimes \mathcal{L}^d)_x \otimes (\mathcal{E} \otimes \mathcal{L}^d)_y^*$. For our purpose, it is more convenient to consider the normalized Bergman kernel

$$e_d : (x, y) \mapsto E_d(x, x)^{-\frac{1}{2}} E_d(x, y) E_d(y, y)^{-\frac{1}{2}}.$$

For any $x \in \mathcal{X}$, the map $E_d(x, x)$ is an endomorphism of the 1-dimensional space $(\mathcal{E} \otimes \mathcal{L}^d)_x$, hence can be seen as a scalar. Note that e_d is the correlation kernel of the normalized process $(E_d(x, x)^{-\frac{1}{2}} s_d(x))_{x \in M}$, which has unit variance and the same zero set as $(s_d(x))_{x \in M}$.

The normalized Bergman kernel e_d admits a universal local scaling limit at scale $d^{-\frac{1}{2}}$ around any point of $\mathcal{X}$ (cf. [MM07, Theorem 4.2.1]). Moreover, the correlations are exponentially decreasing at scale $d^{-\frac{1}{2}}$, in the sense that there exists $C > 0$ such

that $\|e_d(x,y)\| = O(\exp(-C\sqrt{d}\rho_g(x,y)))$ as $d \to +\infty$, uniformly in (x,y), where ρ_g is the geodesic distance. Similar estimates hold for the derivatives of e_d with the same constant C. We refer to [MM15, Theorem 1] for a precise statement. These facts were extensively used in [Anc21, Let19, LP19] and we refer to these papers for a more detailed discussion of how estimates for the Bergman kernel are used in the context of random real algebraic geometry. An important consequence of these estimates that we use in the present paper is Proposition 2.26 below.

Definition 2.25. — *Let $p \in \mathbb{N}$, we denote by $b_p = \frac{1}{C}(1+\frac{p}{4})$, where $C > 0$ is the same as above. That is C is the constant appearing in the exponential in* [MM15, Theorem 1, Equation (1.3)].

Proposition 2.26 ([Anc21, Proposition 3.2]). — *Let $p \geqslant 2$. Recall that ρ_g denotes the geodesic distance and that b_p is defined by Definition 2.25. The following holds uniformly for all $k \in \{2, \dots, p\}$, for all A and $B \subset \{1, \dots, k\}$ disjoint such that $A \sqcup B = \{1, \dots, k\}$, for all $x \in M^k \setminus \Delta_k$ such that for all $a \in A$ and $b \in B$ we have $\rho_g(x_a, x_b) \geqslant b_p \frac{\ln d}{\sqrt{d}}$:*

$$\mathcal{R}_d^k(x) = \mathcal{R}_d^{|A|}(\underline{x}_A)\mathcal{R}_d^{|B|}(\underline{x}_B) + O\left(d^{\frac{k}{2}-\frac{p}{4}-1}\right).$$

Here we used the notations defined in Section 2.2 (see Notation 2.2), and by convention $\mathcal{R}_d^0 = 1$ for all d.

Proof. — Proposition 2.26 is the same as [Anc21, Proposition 3.2] but for two small points. We refer to [Anc21] for the core of the proof of this proposition. Here, let us just explain what the differences are between Proposition 2.26 and [Anc21, Proposition 3.2], and how these differences affect the proof.

- Here we consider a line bundle of the form $\mathcal{E} \otimes \mathcal{L}^d$, while in [Anc21] the author only considers $\mathcal{L}^d$, which corresponds to the case where $\mathcal{E}$ is trivial. In the proof of [Anc21, Proposition 3.2], the geometry of line bundle $\mathcal{L}^d$ only appears through the leading term in the asymptotics of its Bergman kernel, as $d \to +\infty$. More precisely, the necessary estimates are those of [MM07, Theorem 4.2.1] and [MM15, Theorem 1]. One can check that, if we replace $\mathcal{L}^d$ by $\mathcal{E} \otimes \mathcal{L}^d$ in the estimates of [MM07, MM15], we obtain estimates of the same form for the Bergman kernel of $\mathcal{E} \otimes \mathcal{L}^d$. Thus, the addition of the line bundle $\mathcal{E}$ does not affect the proof, except in a typographical way.
- In the statement of [Anc21, Proposition 3.2], the constant b_p is replaced by $\frac{1}{C} = b_0$ and the error term is replaced by

  $$O\left(d^{\frac{k}{2}-1}\right).$$

  Let us explain how using b_p instead of b_0 yields an error term of the form $O(d^{\frac{k}{2}-\frac{p}{4}-1})$. The error term in Proposition 2.26 and [Anc21, Proposition 3.2] comes from the off-diagonal decay estimate of [MM15, Theorem 1]. More precisely, in both cases, the term:

  $$\left|\mathcal{R}_d^k(x) - \mathcal{R}_d^{|A|}(\underline{x}_A)\mathcal{R}_d^{|B|}(\underline{x}_B)\right|$$

is bounded from above by some constant (that does not depend on x and d) times the $\mathcal{C}^k$-norm of the normalized Bergman kernel e_d of $\mathcal{E} \otimes \mathcal{L}^d$ restricted to $\{(x,y) \in M^2 \mid \rho_g(x,y) \geqslant L\}$, where

$$L = \min_{(a,b) \in A \times B} \rho_g(x_a, x_b).$$

By [MM15, Theorem 1], this term is $O(d^{\frac{k}{2}} e^{-C\sqrt{d}L})$. In the setting of [Anc21, Proposition 3.2], we have $L \geqslant \frac{1}{C} \frac{\ln d}{\sqrt{d}}$, which gives an error term of the form

$$O\left(d^{\frac{k}{2}-1}\right).$$

In Proposition 2.26, we have $L \geqslant \frac{1}{C}(1+\frac{p}{4})\frac{\ln d}{\sqrt{d}}$, so that the error term is indeed

$$O\left(d^{\frac{k}{2}-\frac{p}{4}-1}\right). \qquad \square$$

Remarks 2.27. — We conclude this section with some comments for the reader who might be interested in the proof of [Anc21, Proposition 3.2].

- In [Anc21], estimates for the Bergman kernel (i.e. the correlation function of the random process under study) are obtained using peak sections. This alternative method yields the necessary estimates without having to use the results of [MM07, MM15].
- [Anc21, the proof of Proposition 3.2] is written for $k = 2$ and $|A| = 1 = |B|$, for clarity of exposition. The extension to $k > 2$ is non-trivial, and the proof for $k > 2$ requires in fact the full power of the techniques developed in [Anc21, Section 4]. More recently, the authors developed similar techniques for smooth stationary Gaussian processes in $\mathbb{R}$, see [AL21, Theorem 1.14].

3. Asymptotics of the central moments

The goal of this section is to prove Theorem 1.12. In Section 3.1 we derive an integral expression of the central moments we want to estimate, see Lemma 3.5 below. Then, in Section 3.2, we define a decomposition of the manifolds M^A, where M is as in Section 2.1 and A is a finite set. In Sections 3.3, 3.4 and 3.5, we compute the contributions of the various pieces of the decomposition defined in Section 3.2 to the asymptotics of the integrals appearing in Lemma 3.5. Finally, we conclude the proof of Theorem 1.12 in Section 3.6.

3.1. An integral expression of the central moments

The purpose of this section is to derive a tractable integral expression for the central moments of the form $m_p(\nu_d)(\phi_1, \ldots, \phi_p)$, defined by Definition 1.2 and appearing in Theorem 1.12. This is done in Lemma 3.5 below. Before stating and proving this lemma, we introduce several concepts and notations that will be useful to deal with the combinatorics of this section and the following ones.

Recall that we already defined the set $\mathcal{P}_A$ of partitions of a finite set A (see Definition 1.11). The next concept we need to introduce is that of induced partition.

Definition 3.1 (Induced partition). — *Let B be a finite set, let $A \subset B$ and let $\mathcal{I} \in \mathcal{P}_B$. The partition $\mathcal{I}$ defines a partition $\mathcal{I}_A = \{I \cap A | I \in \mathcal{I}, I \cap A \neq \emptyset\}$ of A, which is called the partition of A* induced *by $\mathcal{I}$.*

In this paper, we will only consider the case where $\mathcal{I} \in \mathcal{P}_B$ and A is the reunion of some of the elements of $\mathcal{I}$, that is: for any $I \in \mathcal{I}$, either $I \subset A$ or $I \cap A = \emptyset$. This condition is equivalent to $\mathcal{I}_A \subset \mathcal{I}$, and in this case we have $\mathcal{I}_A = \{I \in \mathcal{I} \mid I \subset A\}$.

Another useful notion is the following order relation on partitions.

Definition 3.2 (Order on partitions). — *Let A be a finite set and let $\mathcal{I}, \mathcal{J} \in \mathcal{P}_A$. We denote by $\mathcal{J} \leqslant \mathcal{I}$ and we say that $\mathcal{J}$ is* finer *than $\mathcal{I}$ if $\mathcal{J}$ is obtained by subdividing the elements of $\mathcal{I}$. That is, for any $J \in \mathcal{J}$ there exists $I \in \mathcal{I}$ such that $J \subset I$. If $\mathcal{J} \leqslant \mathcal{I}$ and $\mathcal{J} \neq \mathcal{I}$, we say that $\mathcal{J}$ is* strictly finer *than $\mathcal{I}$, which we denote by $\mathcal{J} < \mathcal{I}$.*

One can check the following facts. Given a finite set A, the relation $\leqslant$ defines a partial order on $\mathcal{P}_A$ such that $\mathcal{I} \mapsto |\mathcal{I}|$ is decreasing. The partially ordered set $(\mathcal{P}_A, \leqslant)$ admits a unique maximum equal to $\{A\}$. It also admits a unique minimum equal to $\{\{a\} | a \in A\}$, that we denote by $\mathcal{I}_0(A)$ as in Remark 2.5. If A is of the form $\{1, \ldots, k\}$, we use the simpler notations $\mathcal{I}_0(k) = \mathcal{I}_0(A)$. Finally, note that $\mathcal{J} \leqslant \mathcal{I}$ if and only if $\mathcal{J} = \bigsqcup_{I \in \mathcal{I}} \mathcal{J}_I$.

The last thing we need to define is the notion of subset adapted to a partition. A subset A of a finite set B is adapted to the partition $\mathcal{I} \in \mathcal{P}_B$ if $B \setminus A$ is a union of singletons of $\mathcal{I}$. That is, A is adapted to $\mathcal{I}$ if and only if $\mathcal{I} = \mathcal{I}_A \sqcup \{\{i\} \mid i \in B \setminus A\}$, or equivalently if and only if $\mathcal{I} \leqslant \{A\} \sqcup \mathcal{I}_0(B \setminus A)$. Here is the formal definition.

Definition 3.3 (Subset adapted to a partition). — *Let B be a finite set and let $\mathcal{I} \in \mathcal{P}_B$, we denote by:*

$$\mathcal{S}_{\mathcal{I}} = \{A \subset B | \forall\ I \in \mathcal{I},\ \text{if } |I| \geqslant 2,\ \text{then } I \subset A\}\,.$$

A subset $A \in \mathcal{S}_{\mathcal{I}}$ is said to be adapted *to $\mathcal{I}$.*

The set $\mathcal{S}_{\mathcal{I}}$ will appear as an index set in the integral expression derived in Lemma 3.5 below. The only thing we need to know about it is the following natural result.

Lemma 3.4. — *Let B be any finite set, then the map $(A, \mathcal{I}) \mapsto (A, \mathcal{I}_A)$ defines a bijection from $\{(A, \mathcal{I}) \mid \mathcal{I} \in \mathcal{P}_B, A \in \mathcal{S}_{\mathcal{I}}\}$ to $\{(A, \mathcal{J}) \mid A \subset B, \mathcal{J} \in \mathcal{P}_A\}$.*

Proof. — The inverse map is given by $(A, \mathcal{J}) \mapsto (A, \mathcal{J} \sqcup \{\{i\} \mid i \in B \setminus A\})$. □

We can now derive the integral expression of the central moments we are looking for. We will make use of the various notations introduced in Section 2.2.

Lemma 3.5. — *Let $p \geqslant 2$ and let $\phi_1, \ldots, \phi_p \in \mathcal{C}^0(M)$. Let $d_p \in \mathbb{N}$ be given by Lemma 2.18, for all $d \geqslant d_p$ we have:*

$$m_p(\nu_d)(\phi_1, \ldots, \phi_p) = \sum_{\mathcal{I} \in \mathcal{P}_p} \int_{\underline{x}_{\mathcal{I}} \in M^{\mathcal{I}}} \iota_{\mathcal{I}}^* \phi\,(\underline{x}_{\mathcal{I}})\ \mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}})\,|\mathrm{d}V_M|^{|\mathcal{I}|}\,,$$

where, for any finite set B, any $\mathcal{I} \in \mathcal{P}_B$ and any $\underline{x}_{\mathcal{I}} = (x_I)_{I \in \mathcal{I}} \in M^{\mathcal{I}}$,

$$\mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}}) = \sum_{A \in \mathcal{S}_{\mathcal{I}}} (-1)^{|B|-|A|} \mathcal{R}_d^{|\mathcal{I}_A|}\left(\underline{x}_{\mathcal{I}_A}\right) \prod_{i \notin A} \mathcal{R}_d^1\left(x_{\{i\}}\right).$$

Here, $\mathcal{S}_{\mathcal{I}}$ is the one we defined in Definition 3.3.

Proof. — By definition, $m_p(\nu_d)(\phi_1, \dots, \phi_p)$ is the expectation of a linear combination of integrals over sets of the form Z_d^k, with $k \in \{1, \dots, p\}$. We subdivide each Z_d^k according to the various diagonals in M^k. The corresponding splitting of the counting measure ν_d^k of Z_d^k was computed in Lemma 2.7. We can then apply the Kac–Rice formula to each term. The result follows by carefully reordering the terms.

We start from the definition of $m_p(\nu_d)$, see Definition 1.2. Developing the product, we get:

$$\begin{aligned} m_p(\nu_d)(\phi_1, \dots, \phi_p) &= \sum_{A \subset \{1,\dots,p\}} (-1)^{p-|A|} \mathbb{E}\left[\prod_{i \in A} \langle \nu_d \,,\, \phi_i \rangle\right] \prod_{i \notin A} \mathbb{E}[\langle \nu_d \,,\, \phi_i \rangle] \\ &= \sum_{A \subset \{1,\dots,p\}} (-1)^{p-|A|} \mathbb{E}\left[\left\langle \nu_d^A \,,\, \phi_A \right\rangle\right] \prod_{i \notin A} \mathbb{E}[\langle \nu_d \,,\, \phi_i \rangle] \\ &= \sum_{A \subset \{1,\dots,p\}} \sum_{\mathcal{I} \in \mathcal{P}_A} (-1)^{p-|A|} \mathbb{E}\left[\left\langle \widetilde{\nu}_d^{\mathcal{I}} \,,\, \iota_{\mathcal{I}}^* \phi_A \right\rangle\right] \prod_{i \notin A} \mathbb{E}[\langle \nu_d \,,\, \phi_i \rangle] \,, \end{aligned}$$

where the last equality comes from Lemma 2.7. By the Kac–Rice formulas of Proposition 2.23, this equals:

$$\sum_{A \subset \{1,\dots,p\}} \sum_{\mathcal{I} \in \mathcal{P}_A} (-1)^{p-|A|} \left(\int_{M^{\mathcal{I}}} (\iota_{\mathcal{I}}^* \phi_A) \, \mathcal{R}_d^{|\mathcal{I}|} \, |\mathrm{d}V_M|^{|\mathcal{I}|} \right) \prod_{i \notin A} \left(\int_M \phi_i \mathcal{R}_d^1 \, |\mathrm{d}V_M| \right).$$

By Lemma 3.4, we can exchange the two sums and obtain the following:

$$\begin{aligned} &\sum_{\mathcal{I} \in \mathcal{P}_p} \sum_{A \in \mathcal{S}_{\mathcal{I}}} (-1)^{p-|A|} \left(\int_{M^{\mathcal{I}_A}} \left(\iota_{\mathcal{I}_A}^* \phi_A \right) \mathcal{R}_d^{|\mathcal{I}_A|} \, |\mathrm{d}V_M|^{|\mathcal{I}_A|} \right) \prod_{i \notin A} \left(\int_M \phi_i \mathcal{R}_d^1 \, |\mathrm{d}V_M| \right) \\ &\qquad = \sum_{\mathcal{I} \in \mathcal{P}_p} \sum_{A \in \mathcal{S}_{\mathcal{I}}} (-1)^{p-|A|} \int_{\underline{x}_{\mathcal{I}} \in M^{\mathcal{I}}} \iota_{\mathcal{I}}^* \phi \left(\underline{x}_{\mathcal{I}} \right) \mathcal{R}_d^{|\mathcal{I}_A|} \left(\underline{x}_{\mathcal{I}_A} \right) \prod_{i \notin A} \mathcal{R}_d^1 \left(x_{\{i\}} \right) |\mathrm{d}V_M|^{|\mathcal{I}|} \\ &\qquad = \sum_{\mathcal{I} \in \mathcal{P}_p} \int_{\underline{x}_{\mathcal{I}} \in M^{\mathcal{I}}} \iota_{\mathcal{I}}^* \phi \left(\underline{x}_{\mathcal{I}} \right) \, \mathcal{D}_d^{\mathcal{I}} \left(\underline{x}_{\mathcal{I}} \right) |\mathrm{d}V_M|^{|\mathcal{I}|} \,, \end{aligned}$$

which concludes the proof. □

Before going further, let us try to give some insight into what is going to happen. Our strategy is to compute the large d asymptotics of the terms of the form

$$\int_{\underline{x}_{\mathcal{I}} \in M^{\mathcal{I}}} \iota_{\mathcal{I}}^* \phi \left(\underline{x}_{\mathcal{I}} \right) \mathcal{D}_d^{\mathcal{I}} (\underline{x}_{\mathcal{I}}) \, |\mathrm{d}V_M|^{|\mathcal{I}|} \tag{3.1}$$

appearing in Lemma 3.5. In the case where p is even, which is a bit simpler to describe, many of these terms will contribute a leading term of order $d^{\frac{p}{4}}$, and the others will only contribute a smaller order term. Terms of the form (3.1) can all be dealt with in the same way, hence we will not make any formal difference between them in the course of the proof. Note however, that the term indexed by $\mathcal{I}_0(p)$ is simpler to understand, as shown in Example 3.6 below. This term is the one we considered in our sketch of proof in Section 1. At each step in the following, we advise the reader to first understand what happens for this term, before looking at the general case.

Example 3.6 (Term indexed by $\mathcal{I}_0(p)$). — Recall that $\mathcal{I}_0(p) = \{\{i\} \mid 1 \leqslant i \leqslant p\}$. Identifying $\mathcal{I}_0(p)$ with $\{1, \ldots, p\}$ through the canonical bijection $i \mapsto \{i\}$, the map $\iota_{\mathcal{I}_0(p)} : M^{\mathcal{I}_0(p)} \to M^p$ is the identity of M^p, see Definition 2.4. Hence, the term indexed by $\mathcal{I}_0(p)$ in Lemma 3.5 is:

$$\int_{(x_1, \ldots, x_p) \in M^p} \left(\prod_{i=1}^{p} \phi_i(x_i) \right) \mathcal{D}_d^{\mathcal{I}_0(p)}(x_1, \ldots, x_p) \, |\mathrm{d}V_M|^p .$$

Then, $\mathcal{S}_{\mathcal{I}_0(p)}$ is just the set of all subsets of $\{1, \ldots, p\}$, see Definition 3.3. Furthermore, for any $A \subset \{1, \ldots, p\}$ the induced partition $\mathcal{I}_0(p)_A$ equals $\{\{i\} \mid i \in A\}$, hence is canonically in bijection with A. Finally, we obtain that:

$$\mathcal{D}_d^{\mathcal{I}_0(p)} : (x_1, \ldots, x_p) \longmapsto \sum_{A \subset \{1, \ldots, p\}} (-1)^{p-|A|} \mathcal{R}_d^{|A|}(\underline{x}_A) \prod_{i \notin A} \mathcal{R}_d^1(x_i).$$

3.2. Cutting M^A into pieces

In this section, we define a way to cut the Cartesian product M^A into disjoint pieces, for any finite subset A. The upshot is to use this decomposition of M^A with $A = \mathcal{I} \in \mathcal{P}_p$, in order to obtain the large d asymptotics of integrals of the form (3.1). For this to work, we need to define a splitting of M^A that depends on the parameters $p \in \mathbb{N}$ and $d \geqslant 1$.

Definition 3.7 (Clustering graph). — *Let A be a finite set and let $p \in \mathbb{N}$. For all $d \geqslant 1$, for all $x = (x_a)_{a \in A} \in M^A$, we define a graph $G_d^p(x)$ as follows.*

- *The vertices of $G_d^p(x)$ are the elements of A.*
- *Two vertices a and $b \in A$ are joined by an edge of $G_d^p(x)$ if and only if $\rho_g(x_a, x_b) \leqslant b_p \frac{\ln d}{\sqrt{d}}$ and $a \neq b$, where ρ_g is the geodesic distance in M and b_p is the constant defined by Definition 2.25.*

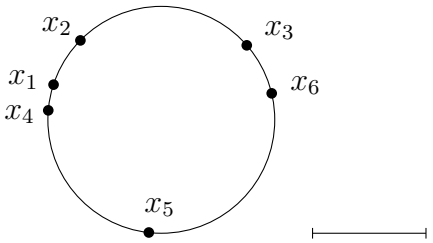
The clustering graph $G_d^p(x)$ contains more information than we need. What we are actually interested in is the partition of A given by the connected components of $G_d^p(x)$. This partition, defined below, encodes how the components of x are clustered in M at scale $b_p \frac{\ln d}{\sqrt{d}}$. An example of this construction is given on Figure 3.1 below, for $A = \{1, \ldots, 6\}$ and M a circle.

Definition 3.8 (Clustering partition). — *Let A be a finite set and let $p \in \mathbb{N}$. For all $d \geqslant 1$, for all $x \in M^A$, we denote by $\mathcal{I}_d^p(x)$ the partition of A defined by the connected components of $G_d^p(x)$. That is a and $b \in A$ belong to the same element of $\mathcal{I}_d^p(x)$ if and only if they are in the same connected component of $G_d^p(x)$.*

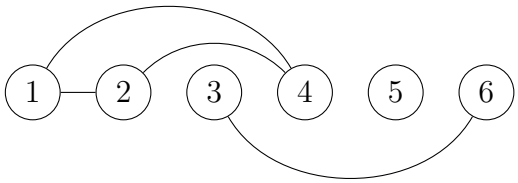
Given $p \in \mathbb{N}$ and $d \geqslant 1$, we divide the points of M^A according to their clustering partition at scale $b_p \frac{\ln d}{\sqrt{d}}$. This yields a splitting of M^A into pieces indexed by $\mathcal{P}_A$.

Definition 3.9. — *Let A be a finite set, let $\mathcal{I} \in \mathcal{P}_A$, let $p \in \mathbb{N}$ and let $d \geqslant 1$, we define:*

$$M_{\mathcal{I},d}^{A,p} = \left\{ x \in M^A \;\middle|\; \mathcal{I}_d^p(x) = \mathcal{I} \right\}.$$



(a) A configuration $x = (x_1, \ldots, x_6)$ of six distinct points on a circle. The length of the segment is the cut-off distance $b_p \frac{\ln d}{\sqrt{d}}$.



(b) The clustering graph $G_d^p(x)$ associated with x and the cut-off distance represented on the left.

Figure 3.1. Example of a configuration of six points on a curve M. The clustering partition of $x = (x_1, \ldots, x_6)$ is the partition of $\{1, \ldots, 6\}$ according to the connected components of $G_d^p(x)$. Here it equals $\mathcal{I}_d^p(x) = \{\{1, 2, 4\}, \{3, 6\}, \{5\}\}$.

Remarks 3.10. —

- For any $p \in \mathbb{N}$ and $d \geqslant 1$, we have: $M^A = \bigsqcup_{\mathcal{I} \in \mathcal{P}_A} M_{\mathcal{I},d}^{A,p}$.
- Let $\mathcal{I} \in \mathcal{P}_A$ and let $(x_a)_{a \in A} \in M_{\mathcal{I},d}^{A,p}$. For all $I \in \mathcal{I}$, for any a and $b \in I$, the geodesic distance from x_a to x_b satisfies: $\rho_g(x_a, x_b) \leqslant (|I| - 1) b_p \frac{\ln d}{\sqrt{d}} \leqslant |A| b_p \frac{\ln d}{\sqrt{d}}$.
- Recalling Definitions 2.3 and 3.2, if $d > 1$ the set $M_{\mathcal{I},d}^{A,p}$ is a neighborhood of the diagonal $\Delta_{A,\mathcal{I}}$ minus some neighborhood of $\bigsqcup_{\mathcal{J} > \mathcal{I}} \Delta_{A,\mathcal{J}}$. If $d = 1$, we have $M_{\mathcal{I},1}^{A,p} = \Delta_{A,\mathcal{I}}$.

In the course of the proof of Theorem 1.12, we will make use of the following two auxiliary results concerning the sets $M_{\mathcal{I},d}^{A,p}$ defined above.

LEMMA 3.11. — *Let A be a finite set, let $\mathcal{I} \in \mathcal{P}_A$, let $p \in \mathbb{N}$ and let $d \geqslant 1$. Let $B \subset A$ be a union of elements of $\mathcal{I}$, i.e. for all $I \in \mathcal{I}$ we have either $I \subset B$ or $I \cap B = \emptyset$. Then, for all $\underline{x}_A \in M_{\mathcal{I},d}^{A,p}$, we have $\underline{x}_B \in M_{\mathcal{I}_B,d}^{B,p}$.*

Proof. — Let $\underline{x} = (x_a)_{a \in A} \in M_{\mathcal{I},d}^{A,p}$, recall that $\underline{x}_B = (x_a)_{a \in B} \in M^B$. By Definition 3.9, we want to prove that $\mathcal{I}_d^p(\underline{x}_B) = \mathcal{I}_B$, where $\mathcal{I}_B = \{I \in \mathcal{I} \mid I \subset B\}$ is the partition of B induced by $\mathcal{I}$, see Definition 3.1.

Recalling Definitions 3.7 and 3.8, the partition $\mathcal{I}_d^p(\underline{x}_B)$ is defined by the connected components of $G_d^p(\underline{x}_B)$. The graph $G_d^p(\underline{x}_B)$ is a subgraph of $G_d^p(\underline{x}_A)$, obtained by erasing the vertices indexed by $a \in A \setminus B$ and the edges having at least one endpoint in $A \setminus B$. If B was any subset of A, two vertices $a, b \in B$ in the same connected component of $G_d^p(\underline{x}_A)$ could end up in different connected components of $G_d^p(\underline{x}_B)$. In terms of clustering partitions, if $I \in \mathcal{I}_d^p(\underline{x}_A)$, then $I \cap B$ might be a disjoint union of several elements of $\mathcal{I}_d^p(\underline{x}_B)$, neither empty nor equal to I. Our hypothesis that B is a union of elements of $\mathcal{I}$ ensures that this does not happen.

To conclude the proof it is enough to show that, for all $I \in \mathcal{I}$ such that $I \subset B$, we have $I \in \mathcal{I}_d^p(\underline{x}_B)$. Let $I \in \mathcal{I}$ such that $I \subset B$. Since $I \in \mathcal{I} = \mathcal{I}_d^p(\underline{x}_A)$, the elements of I are the vertices of one of the connected components of $G_d^p(\underline{x}_A)$, which is equal

to $G_d^p(\underline{x}_I)$. This connected component is not modified when we erase the vertices in $A \setminus B$ and the corresponding edges from $G_d^p(\underline{x}_A)$. Thus, $G_d^p(\underline{x}_I)$ is still a connected component in $G_d^p(\underline{x}_B)$, and $I \in \mathcal{I}_d^p(\underline{x}_B)$. □

LEMMA 3.12. — *Let A be a finite set, let $\mathcal{I} \in \mathcal{P}_A$, let $p \in \mathbb{N}$ and let $d \geqslant 1$, we have the following inclusion between subsets of M^A:*

$$M^{A,p}_{\mathcal{I},d} \subset \prod_{I \in \mathcal{I}} M^{I,p}_{\{I\},d} \subset \bigsqcup_{\mathcal{J} \geqslant \mathcal{I}} M^{A,p}_{\mathcal{J},d}.$$

Proof. — Let $\underline{x}_A = (x_a)_{a \in A} \in M^{A,p}_{\mathcal{I},d}$. We know that the components of $\underline{x}_A$ are clustered in M according to $\mathcal{I}$. Informally, if we remember that the points indexed by each $I \in \mathcal{I}$ are close together, but we forget about the distances between points in different clusters, then all we can say is that $\underline{x}_A \in \prod_{I \in \mathcal{I}} M^{I,p}_{\{I\},d}$. More formally, let $I \in \mathcal{I}$, then I is a union of elements of $\mathcal{I}$, and $\mathcal{I}_I = \{I\}$. By Lemma 3.11, we have $\underline{x}_I \in M^{I,p}_{\{I\},d}$ for any $I \in \mathcal{I}$. Hence $\underline{x}_A = (\underline{x}_I)_{I \in \mathcal{I}} \in \prod_{I \in \mathcal{I}} M^{I,p}_{\{I\},d}$. This proves the first inclusion in Lemma 3.12.

Let us now prove the second inclusion. Let $\underline{x}_A \in \prod_{I \in \mathcal{I}} M^{I,p}_{\{I\},d} \subset M^A$, we want to recover its clustering partition $\mathcal{I}_d^p(\underline{x}_A)$. We know that the components of $\underline{x}_A$ indexed by a given $I \in \mathcal{I}$ are close together in M, hence belong to the same cluster. However, given I and $J \in \mathcal{I}$, the points indexed by I and those indexed by J might be close enough in M that they belong to the same cluster. Thus, all we can say is that the clusters of $\underline{x}_A$ are indexed by unions of elements of $\mathcal{I}$, that is $\mathcal{I}_d^p(\underline{x}_A) \geqslant \mathcal{I}$. This shows that $\underline{x}_A \in \bigsqcup_{\mathcal{J} \geqslant \mathcal{I}} M^{A,p}_{\mathcal{J},d}$ and concludes the proof of Lemma 3.12. □

We conclude this section by bounding from above the volume of $M^{A,p}_{\mathcal{I},d}$. What we are interested in is an asymptotic upper bound as $d \to +\infty$ with A, $\mathcal{I}$ and p fixed.

LEMMA 3.13 (Volume bound). — *Let A be a finite set, let $\mathcal{I} \in \mathcal{P}_A$ and let $p \in \mathbb{N}$. As $d \to +\infty$, we have:*

$$\mathrm{Vol}\Big(M^{A,p}_{\mathcal{I},d}\Big) = O\left(\left(\frac{\ln d}{\sqrt{d}}\right)^{|A|-|\mathcal{I}|}\right).$$

Proof. — For all $I \in \mathcal{I}$, we fix a preferred element a_I of I. Let $x = (x_a)_{a \in A} \in M^{A,p}_{\mathcal{I},d}$. For all $I \in \mathcal{I}$, for all $a \in I \setminus \{a_I\}$, the point x_a belongs to the geodesic ball of center x_{a_I} and radius $|A| b_p \frac{\ln d}{\sqrt{d}}$. Since M is 1-dimensional, this ball has volume $2|A| b_p \frac{\ln d}{\sqrt{d}}$. Hence,

$$\mathrm{Vol}\Big(M^{A,p}_{\mathcal{I},d}\Big) \leqslant \mathrm{Vol}(M)^{|\mathcal{I}|} \prod_{I \in \mathcal{I}} \left(2\,|A|\, b_p \frac{\ln d}{\sqrt{d}}\right)^{|I|-1} = O\left(\prod_{I \in \mathcal{I}} \left(\frac{\ln d}{\sqrt{d}}\right)^{|I|-1}\right).$$

The result follows from $\sum_{I \in \mathcal{I}} (|I| - 1) = |A| - |\mathcal{I}|$. □

3.3. An upper bound on the contribution of each piece

Let $p \geqslant 2$, given $\mathcal{I} \in \mathcal{P}_p$ and test-functions $(\phi_i)_{1 \leqslant i \leqslant p}$, we want to estimate the integral (3.1) as $d \to +\infty$. This is done by splitting $M^{\mathcal{I}}$ as in Section 3.2 and

estimating the contribution of each piece. One difficulty is that $\mathcal{I} \in \mathcal{P}_p$, and the pieces of our decomposition of $M^{\mathcal{I}}$ are indexed by partitions of $\mathcal{I}$, seen as a finite set. Hence we need to consider partitions of partitions.

Let $\mathcal{J} \in \mathcal{P}_{\mathcal{I}}$, we are interested in the large d asymptotics of:

$$\int_{\underline{x}_{\mathcal{I}} \in M^{\mathcal{I},p}_{\mathcal{J},d}} \iota^*_{\mathcal{I}}\phi\left(\underline{x}_{\mathcal{I}}\right) \mathcal{D}^{\mathcal{I}}_d\left(\underline{x}_{\mathcal{I}}\right) \left|\mathrm{d}V_M\right|^{|\mathcal{I}|}. \tag{3.2}$$

Note that the integral (3.1) is the sum over $\mathcal{J} \in \mathcal{P}_{\mathcal{I}}$ of the terms of the form (3.2). Hence, the central moment $m_p(\nu_d)(\phi_1, \ldots, \phi_p)$ we are interested in is the double sum of these terms over $\mathcal{I} \in \mathcal{P}_p$ and $\mathcal{J} \in \mathcal{P}_{\mathcal{I}}$, see Lemma 3.5. As far as the order of magnitude of the integral (3.2) is concerned, we can think of the test-function $\iota^*_{\mathcal{I}}\phi$ as being constant. This order of magnitude will then result from the competition between two things: the order of the typical values of $\mathcal{D}^{\mathcal{I}}_d$ and the volume of $M^{\mathcal{I},p}_{\mathcal{J},d}$.

In this section, we compute an asymptotic upper bound as $d \to +\infty$ for the contribution of the integral (3.2). This upper bound shows that, if $|\mathcal{J}| < \frac{p}{2}$, then the integral (3.2) only contributes an error term in the estimates of Theorem 1.12. This is not surprising since in this case the volume of $M^{\mathcal{I},p}_{\mathcal{J},d}$ is comparatively small, see Lemma 3.13.

Lemma 3.14. — *Let $p \geqslant 2$ and let $\phi_1, \ldots, \phi_p \in \mathcal{C}^0(M)$. Let $\mathcal{I} \in \mathcal{P}_p$, for all $\mathcal{J} \in \mathcal{P}_{\mathcal{I}}$, we have:*

$$\int_{\underline{x}_{\mathcal{I}} \in M^{\mathcal{I},p}_{\mathcal{J},d}} \iota^*_{\mathcal{I}}\phi\left(\underline{x}_{\mathcal{I}}\right) \mathcal{D}^{\mathcal{I}}_d\left(\underline{x}_{\mathcal{I}}\right) \left|\mathrm{d}V_M\right|^{|\mathcal{I}|} = \left(\prod_{i=1}^{p} \|\phi_i\|_\infty\right) O\Big(d^{\frac{|\mathcal{J}|}{2}} (\ln d)^{|\mathcal{I}|-|\mathcal{J}|}\Big),$$

where the error term does not depend on $(\phi_1, \ldots, \phi_p)$.

Proof. — We have $\|\iota^*_{\mathcal{I}}\phi\|_\infty \leqslant \|\phi\|_\infty \leqslant \prod_{i=1}^p \|\phi_i\|_\infty$. Besides, by the definition of $\mathcal{D}^{\mathcal{I}}_d$ (see Lemma 3.5) and Proposition 2.24, the density $\mathcal{D}^{\mathcal{I}}_d$ is bounded uniformly on $M^{\mathcal{I}}$ by $O(d^{\frac{|\mathcal{I}|}{2}})$. Integrating this estimate over $M^{\mathcal{I},p}_{\mathcal{J},d}$ yields:

$$\left|\int_{\underline{x}_{\mathcal{I}} \in M^{\mathcal{I},p}_{\mathcal{J},d}} \iota^*_{\mathcal{I}}\phi\left(\underline{x}_{\mathcal{I}}\right) \mathcal{D}^{\mathcal{I}}_d\left(\underline{x}_{\mathcal{I}}\right) \left|\mathrm{d}V_M\right|^{|\mathcal{I}|}\right| \leqslant \left(\prod_{i=1}^{p} \|\phi_i\|_\infty\right) O\left(d^{\frac{|\mathcal{I}|}{2}}\right) \mathrm{Vol}\Big(M^{\mathcal{I},p}_{\mathcal{J},d}\Big).$$

Then the result follows from the volume estimate of Lemma 3.13. □

3.4. Contribution of the partitions with an isolated point

In this section, we still work with a fixed $p \geqslant 2$ and some fixed $\mathcal{I} \in \mathcal{P}_p$. Lemma 3.14 seems to say that the main contribution to the integral (3.1) should be given by terms of the form (3.2) where $|\mathcal{J}|$ is large, i.e. $\mathcal{J}$ defines many small clusters. However, by looking more carefully at $\mathcal{D}^{\mathcal{I}}_d$, we can prove that the contribution of $M^{\mathcal{I},p}_{\mathcal{J},d}$ is also small if $\mathcal{J}$ contains an isolated point. In particular, for such a $\mathcal{J}$, the integral (3.2) will only contribute an error term in the estimates of Theorem 1.12. The main result of this section is Corollary 3.17 below.

LEMMA 3.15. — *Let $p \geqslant 2$ and $\mathcal{I} \in \mathcal{P}_p$. We assume that there exists $j \in \{1, \dots, p\}$ such that $\{j\} \in \mathcal{I}$. Then, for all $\mathcal{J} \in \mathcal{P}_{\mathcal{I}}$ such that $\{\{j\}\} \in \mathcal{J}$, we have:*

$$\forall\, \underline{x}_{\mathcal{I}} \in M^{\mathcal{I},p}_{\mathcal{J},d}, \qquad \mathcal{D}^{\mathcal{I}}_d(\underline{x}_{\mathcal{I}}) = O\left(d^{\frac{p}{4}-1}\right),$$

as $d \to +\infty$, uniformly in $\underline{x}_{\mathcal{I}} \in M^{\mathcal{I},p}_{\mathcal{J},d}$.

Remark 3.16. — This statement and its proof are typically easier to understand when $\mathcal{I} = \mathcal{I}_0(p)$. In this case, we can identify $\mathcal{I}$ with $\{1, \dots, p\}$ canonically as in Example 3.6. Then $\mathcal{J} \in \mathcal{P}_p$, and our hypothesis simply means that $\mathcal{J}$ contains a singleton $\{j\}$. The statement of the lemma becomes that: if $x = (x_i)_{1 \leqslant i \leqslant p} \in M^p$ is such that x_j is far from the other components of x, meaning at distance greater than $b_p \frac{\ln d}{\sqrt{d}}$, then

$$\mathcal{D}^{\mathcal{I}_0(p)}_d(x) = O\left(d^{\frac{p}{4}-1}\right).$$

Proof of Lemma 3.15. — Recall that we defined $\mathcal{S}_{\mathcal{I}}$ in Definition 3.3. Since $\{j\} \in \mathcal{I}$, we can split $\mathcal{S}_{\mathcal{I}}$ into the subsets $A \in \mathcal{S}_{\mathcal{I}}$ that contain j, and those that do not. Moreover, $A \mapsto A \sqcup \{j\}$ is a bijection from $\{A \in \mathcal{S}_{\mathcal{I}} \mid j \notin A\}$ to $\{A \in \mathcal{S}_{\mathcal{I}} \mid j \in A\}$.

Let $\underline{x}_{\mathcal{I}} = (x_I)_{I \in \mathcal{I}} \in M^{\mathcal{I},p}_{\mathcal{J},d}$ and let $A \in \mathcal{S}_{\mathcal{I}}$ be such that $j \notin A$. We regroup the terms corresponding to A and $A \sqcup \{j\}$ in the sum defining $\mathcal{D}^{\mathcal{I}}_d(\underline{x}_{\mathcal{I}})$, see Lemma 3.5. Since we have $\mathcal{I}_{A \sqcup \{j\}} = \mathcal{I}_A \sqcup \{\{j\}\}$, we obtain:

$$\begin{aligned}(3.3)\quad & (-1)^{p-|A|}\mathcal{R}^{|\mathcal{I}_A|}_d(\underline{x}_{\mathcal{I}_A}) \prod_{i \notin A} \mathcal{R}^1_d(x_{\{i\}}) \\ & \qquad + (-1)^{p-|A|-1}\mathcal{R}^{|\mathcal{I}_A|+1}_d\left(\underline{x}_{\mathcal{I}_A}, x_{\{j\}}\right) \prod_{i \notin A \sqcup \{j\}} \mathcal{R}^1_d\left(x_{\{i\}}\right) \\ & = (-1)^{p-|A|-1}\left(\mathcal{R}^{|\mathcal{I}_A|+1}_d\left(\underline{x}_{\mathcal{I}_A}, x_{\{j\}}\right) - \mathcal{R}^{|\mathcal{I}_A|}_d\left(\underline{x}_{\mathcal{I}_A}\right)\mathcal{R}^1_d\left(x_{\{j\}}\right)\right) \prod_{i \notin A \sqcup \{j\}} \mathcal{R}^1_d\left(x_{\{i\}}\right),\end{aligned}$$

where we used the fact that $\mathcal{R}^{|\mathcal{I}_A|+1}_d$ is a symmetric function of its arguments, cf. Definition 2.21. Recall that $\mathcal{R}^0_d = 1$ for all $d \geqslant 1$ by convention.

Since $\{\{j\}\} \in \mathcal{J}$, the point $x_{\{j\}}$ is far from the other coordinates of $\underline{x}_{\mathcal{I}}$ by definition of $M^{\mathcal{I},p}_{\mathcal{J},d}$, see Definition 3.9. More precisely, we have $\rho_g(x_I, x_{\{j\}}) \geqslant b_p \frac{\ln d}{\sqrt{d}}$ for all $I \in \mathcal{I} \setminus \{\{j\}\}$. Then, by Proposition 2.26, the term on the right-hand side of Equation (3.3) is $O(d^{\frac{1}{2}(|\mathcal{I}_A|-|A|)+\frac{p}{4}-1})$, uniformly in $\underline{x}_{\mathcal{I}} \in M^{\mathcal{I},p}_{\mathcal{J},d}$. Since $\mathcal{I}_A \in \mathcal{P}_A$, we have $|\mathcal{I}_A| \leqslant |A|$. Hence, the previous term is $O(d^{\frac{p}{4}-1})$. We conclude the proof by summing this estimates over $\{A \in \mathcal{S}_{\mathcal{I}} \mid j \notin A\}$. □

COROLLARY 3.17. — *Let $p \geqslant 2$ and let $\mathcal{I} \in \mathcal{P}_p$. We assume that there exists $j \in \{1, \dots, p\}$ such that $\{j\} \in \mathcal{I}$. Then, for all $\mathcal{J} \in \mathcal{P}_{\mathcal{I}}$ such that $\{\{j\}\} \in \mathcal{J}$, we have:*

$$\begin{aligned}\forall\, \phi_1, \dots, \phi_p \in \mathcal{C}^0(M), \qquad & \int_{\underline{x}_{\mathcal{I}} \in M^{\mathcal{I},p}_{\mathcal{J},d}} \iota^*_{\mathcal{I}}\phi(\underline{x}_{\mathcal{I}})\, \mathcal{D}^{\mathcal{I}}_d(\underline{x}_{\mathcal{I}})\, |\mathrm{d}V_M|^{|\mathcal{I}|} \\ & = \left(\prod_{i=1}^p \|\phi_i\|_\infty\right) O\left(d^{\frac{p}{4}-1}\right),\end{aligned}$$

where the error term does not depend on $(\phi_1, \dots, \phi_p)$.

Proof. — We obtain this corollary by integrating the estimate of Lemma 3.15 over $M^{\mathcal{I},p}_{\mathcal{J},d}$, using the fact that $\|\iota^*_{\mathcal{I}}\phi\|_\infty \leqslant \prod_{i=1}^p \|\phi_i\|_\infty$. □

3.5. Contribution of the partitions into pairs

Let $p \geqslant 2$ and let $\phi_1, \dots, \phi_p$ be test-functions. Recall that $m_p(\nu_d)(\phi_1, \dots, \phi_p)$ is the sum of the integrals (3.2) for $\mathcal{I} \in \mathcal{P}_p$ and $\mathcal{J} \in \mathcal{P}_{\mathcal{I}}$, and that our final goal is to prove Theorem 1.12. This theorem gives the asymptotics of $m_p(\nu_d)(\phi_1, \dots, \phi_p)$ as d goes to infinity, up to an error term of the form

$$O\Big(d^{\frac{1}{2}\left\lfloor\frac{p-1}{2}\right\rfloor}(\ln d)^p\Big).$$

We proved in Lemma 3.14 that the integral (3.2) only contributes an error term if $|\mathcal{J}| < \frac{p}{2}$. Besides, we proved in Corollary 3.17 that the integral (3.2) also contributes an error term if there exists j such that $\{\{j\}\} \in \mathcal{J}$, in particular if $|\mathcal{J}|$ is too large. This last point will be made more precise in Section 3.6 below.

In this section, we study the integrals of the form (3.2) that will contribute to the leading term in the asymptotics of $m_p(\nu_d)(\phi_1, \dots, \phi_p)$. These integrals are indexed by couples of partitions $(\mathcal{I}, \mathcal{J})$ satisfying the following technical condition: the double partitions into pairs. Recall that we denoted by $\mathcal{PP}_A$ the set of partition into pairs of the finite set A, see Definition 1.11, and that we denoted by $\mathcal{I}_S$ the partition of $S \subset A$ induced by a partition $\mathcal{I} \in \mathcal{P}_A$, see Definition 3.1.

Definition 3.18 (Double partition into pairs). — *Let A be a finite set, let $\mathcal{I} \in \mathcal{P}_A$ and let $\mathcal{J} \in \mathcal{P}_{\mathcal{I}}$. We say that a couple $(\mathcal{I}, \mathcal{J})$ is a* double partition into pairs *of A if there exists $S \subset A$ such that:*

(1) $\mathcal{I}_S = \{\{s\} \mid s \in S\}$ *and* $\mathcal{I} \setminus \mathcal{I}_S = \mathcal{I}_{A\setminus S} \in \mathcal{PP}_{A\setminus S}$;
(2) $\mathcal{J}_{\mathcal{I}_S} \in \mathcal{PP}_{\mathcal{I}_S}$ *and* $\mathcal{J} \setminus \mathcal{J}_{\mathcal{I}_S} = \mathcal{J}_{\mathcal{I}\setminus\mathcal{I}_S} = \{\{I\} \mid I \in \mathcal{I} \setminus \mathcal{I}_S\}$.

We denote by $\mathfrak{C}_A$ the set of such double partitions into pairs of A. If $A = \{1, \dots, p\}$ we simply denote $\mathfrak{C}_p = \mathfrak{C}_A$.

Let us take some time to comment upon this definition. Note that we will mostly use it with $A = \{1, \dots, p\}$. However, the general case will be useful in Lemmas 3.21 and 3.22 below. Condition (1) in Definition 3.18 simply means that $\mathcal{I}$ only contains singletons and pairs. The singletons are exactly those of the form $\{s\}$ with $s \in S$, and the pairs are formed of elements of $A \setminus S$. Note that S is the union of the singletons in $\mathcal{I}$. In particular, it is uniquely defined by $(\mathcal{I}, \mathcal{J})$. Similarly, Condition (2) means that $\mathcal{J}$ only contains singletons and pairs with some additional constraints. The singletons in $\mathcal{J}$ are exactly of the form $\{I\}$ where $I \notin \mathcal{I}_S$ (i.e. I is itself a pair of elements of $A \setminus S$) and the pairs are of the form $\{\{s\}, \{t\}\}$ with $s, t \in S$ distinct. That is, $\mathcal{J}$ contains only pairs of singletons and singletons of pairs...

Remark 3.19. — Let A be a finite set and let $(\mathcal{I}, \mathcal{J}) \in \mathfrak{C}_A$. Let S be as in Definition 3.18. We have $|S| = |\mathcal{I}_S|$, and since $\mathcal{I}_S$ admits a partition $\mathcal{J}_{\mathcal{I}_S}$ into pairs, this cardinality is even. Similarly, since $\mathcal{I} \setminus \mathcal{I}_S$ is a partition into pairs of $A \setminus S$, we have $|A| = |S| + 2|\mathcal{I} \setminus \mathcal{I}_S|$, so that $|A|$ is even. Thus, $\mathfrak{C}_A$ is empty if $|A|$ is odd.

Example 3.20. — Let A be a finite set of even cardinality, the following are examples of double partitions into pairs showing that $\mathfrak{C}_A$ is not empty.

- Taking $S = \emptyset$, we have $\mathcal{I}_S = \emptyset$ and $\mathcal{I} = \mathcal{I} \setminus \mathcal{I}_S$ can be chosen as any element of $\mathcal{PP}_A$. Then there is only one possibility for $\mathcal{J}$, we must have $\mathcal{J} = \mathcal{J}_{\mathcal{I}\setminus\mathcal{I}_S} = \{\{I\} \mid I \in \mathcal{I}\}$.
- Taking $S = A$, we must have $\mathcal{I} = \mathcal{I}_S = \{\{a\} \mid a \in A\}$. Identifying canonically $\mathcal{I}$ with A, we can choose any $\mathcal{J} \in \mathcal{PP}_{\mathcal{I}} \simeq \mathcal{PP}_A$.

In the remainder of this section, we study the large d asymptotics of integrals of the form (3.2) where $(\mathcal{I}, \mathcal{J}) \in \mathfrak{C}_p$. For this, we need to show that the function $\mathcal{D}_d^{\mathcal{I}}$ defined in Lemma 3.5 factorizes nicely on $M_{\mathcal{J},d}^{\mathcal{I},p}$, up to an error term. The first step is Lemma 3.21 where we factor the contribution from the pairs of singletons in $\mathcal{J}$, i.e the elements of $\mathcal{J}_{\mathcal{I}_S}$.

Lemma 3.21. — *Let A be a finite set. For all $(\mathcal{I}, \mathcal{J}) \in \mathfrak{C}_A$, let $S \subset A$ be as in Definition 3.18. Then, the following holds uniformly for all $\underline{x}_{\mathcal{I}} \in M_{\mathcal{J},d}^{\mathcal{I},p}$:*

$$\mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}}) = \mathcal{D}_d^{\mathcal{I}\setminus\mathcal{I}_S}\left(\underline{x}_{\mathcal{I}\setminus\mathcal{I}_S}\right)\left(\prod_{J\in\mathcal{J}_{\mathcal{I}_S}} \mathcal{D}_d^J(\underline{x}_J)\right) + O\left(d^{\frac{|\mathcal{I}|}{2}-\frac{p}{4}-1}\right).$$

Proof. — Recall that, as discussed in Remark 3.19, the set $\mathfrak{C}_A$ is empty if $|A|$ is odd. Hence Lemma 3.21 is true in this case. If $|A|$ is even, the proof is by induction on (one half of) $|A|$.

Base case. If $|A| = 0$, we have $A = \emptyset$, hence $\mathcal{I} = \emptyset$, hence $\mathcal{J} = \emptyset$. Recalling the definition of $\mathcal{D}_d^{\mathcal{I}}$ in Lemma 3.5 and our convention that $\mathcal{R}_d^0 = 1$, we have $\mathcal{D}_d^{\mathcal{I}} = 1$ and similarly $\mathcal{D}_d^{\mathcal{I}\setminus\mathcal{I}_S} = 1$. The product over $\mathcal{J}_{\mathcal{I}_S}$ is indexed by the empty set, hence also equal to 1. Thus, the result is tautologically true.

Inductive step. Let A be a finite set of even positive cardinality. Let us assume that the result of Lemma 3.21 holds for any finite set B of cardinality $|A| - 2$. Let $(\mathcal{I}, \mathcal{J}) \in \mathfrak{C}_A$ and let S be as in Definition 3.18. Let $J \in \mathcal{J}_{\mathcal{I}_S}$, there exists $s, t \in S$ distinct such that $J = \{\{s\}, \{t\}\}$. The key part of the proof is to show that, the following holds uniformly for all $\underline{x}_{\mathcal{I}} \in M_{\mathcal{J},d}^{\mathcal{I},p}$:

$$\mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}}) = \mathcal{D}_d^{\mathcal{I}\setminus J}\left(\underline{x}_{\mathcal{I}\setminus J}\right)\mathcal{D}_d^J(\underline{x}_J) + O\left(d^{\frac{|\mathcal{I}|}{2}-\frac{p}{4}-1}\right). \tag{3.4}$$

Let us assume for now that Equation (3.4) holds. Let us denote by $B = A \setminus \{s, t\}$. Then we have $\mathcal{I} \setminus J = \mathcal{I} \setminus \{\{s\}, \{t\}\} \in \mathcal{P}_B$ and $\mathcal{J} \setminus \{J\} \in \mathcal{P}_{\mathcal{I}\setminus J}$. One can check that $(\mathcal{I} \setminus J, \mathcal{J} \setminus \{J\}) \in \mathfrak{C}_B$. Indeed, the partition $\mathcal{I} \setminus J$ only contains pairs and singletons, and the union of its singletons is $T = S \setminus \{s, t\}$. Then we have $(\mathcal{I} \setminus J)_T = \mathcal{I}_S \setminus J$, and $(\mathcal{J} \setminus \{J\})_{\mathcal{I}_S\setminus J} = \mathcal{J}_{\mathcal{I}_S} \setminus \{J\}$ is a partition into pairs of $\mathcal{I}_S \setminus J$. Moreover, the pairs in $\mathcal{I} \setminus J$ are the same as the pairs in $\mathcal{I}$. Since $|B| = |A| - 2$, we can apply the induction hypothesis for $(\mathcal{I} \setminus J, \mathcal{J} \setminus \{J\}) \in \mathfrak{C}_B$.

Note that $\mathcal{I} \setminus J$ is the union of the elements of $\mathcal{J} \setminus \{J\}$. Then, by Lemma 3.11, if $\underline{x}_{\mathcal{I}} \in M^{\mathcal{I},p}_{\mathcal{J},d}$ we have $\underline{x}_{\mathcal{I}\setminus J} \in M^{\mathcal{I}\setminus J,p}_{\mathcal{J}\setminus\{J\},d}$. Hence, by induction, the following holds uniformly for all $\underline{x}_{\mathcal{I}} \in M^{\mathcal{I},p}_{\mathcal{J},d}$:

$$\mathcal{D}^{\mathcal{I}\setminus J}_d\left(\underline{x}_{\mathcal{I}\setminus J}\right) = \mathcal{D}^{\mathcal{I}\setminus\mathcal{I}_S}_d\left(\underline{x}_{\mathcal{I}\setminus\mathcal{I}_S}\right)\left(\prod_{K\in\mathcal{J}_{\mathcal{I}_S}\setminus\{J\}} \mathcal{D}^K_d\left(\underline{x}_K\right)\right) + O\left(d^{\frac{|\mathcal{I}|}{2}-\frac{p}{4}-2}\right).$$

We use this relation in Equation (3.4). Since $\mathcal{D}^J_d$ is uniformly $O(d)$ by Proposition 2.24, the conclusion of Lemma 3.21 is satisfied for $(\mathcal{I},\mathcal{J}) \in \mathfrak{C}_A$. This concludes the inductive step.

Proof of Equation (3.4). — In order to complete the proof of Lemma 3.21, we need to prove that Equation (3.4) holds. The proof of this fact is in the same spirit as the proof of Lemma 3.15. We start from the definition of $\mathcal{D}^{\mathcal{I}}_d(\underline{x}_{\mathcal{I}})$, given in Lemma 3.5, and regroup the terms in the sum in fours. For each of these quadruples of summands we apply Proposition 2.26, using the fact that $\underline{x}_{\mathcal{I}} \in M^{\mathcal{I},p}_{\mathcal{J},d}$. This yields the result, keeping in mind the uniform upper bound of Proposition 2.24.

Recall that $\mathcal{D}^{\mathcal{I}}_d$ is defined as a sum indexed by $\mathcal{S}_{\mathcal{I}}$, and that the elements of $\mathcal{S}_{\mathcal{I}}$ are the subsets of $\{1,\dots,p\}$ adapted to $\mathcal{I}$ (see Definition 3.3). Since $\{s\}$ and $\{t\} \in \mathcal{I}$, we can decompose $\mathcal{S}_{\mathcal{I}}$ as the following disjoint union:

$$\mathcal{S}_{\mathcal{I}} = \bigsqcup_{\{A\in\mathcal{S}_{\mathcal{I}} \,|\, s\notin A, t\notin A\}} \{A, A\sqcup\{s\}, A\sqcup\{t\}, A\sqcup\{s,t\}\}.$$

Let $A \in \mathcal{S}_{\mathcal{I}}$ be such that $A \subset \{1,\dots,p\}\setminus\{s,t\}$. Given $\underline{x}_{\mathcal{I}} \in M^{\mathcal{I}}$, we regroup the four terms corresponding to A, $A\sqcup\{s\}$, $A\sqcup\{t\}$ and $A\sqcup\{s,t\}$ in the sum defining $\mathcal{D}^{\mathcal{I}}_d(\underline{x}_{\mathcal{I}})$. Keeping in mind that the Kac–Rice densities $(\mathcal{R}^k_d)_{k\geqslant 1}$ are symmetric functions of their arguments, we obtain:

$$\begin{aligned}(3.5)\quad (-1)^{p-|A|}\Big(&\mathcal{R}^{|\mathcal{I}_A|}_d\left(\underline{x}_{\mathcal{I}_A}\right)\mathcal{R}^1_d\left(x_{\{s\}}\right)\mathcal{R}^1_d\left(x_{\{t\}}\right)\\ &- \mathcal{R}^{|\mathcal{I}_A|+1}_d\left(\underline{x}_{\mathcal{I}_A}, x_{\{s\}}\right)\mathcal{R}^1_d\left(x_{\{t\}}\right) - \mathcal{R}^{|\mathcal{I}_A|+1}_d\left(\underline{x}_{\mathcal{I}_A}, x_{\{t\}}\right)\mathcal{R}^1_d\left(x_{\{s\}}\right)\\ &+ \mathcal{R}^{|\mathcal{I}_A|+2}_d\left(\underline{x}_{\mathcal{I}_A}, x_{\{s\}}, x_{\{t\}}\right)\Big)\prod_{i\notin A\sqcup\{s,t\}}\mathcal{R}^1_d\left(x_{\{i\}}\right)\end{aligned}$$

If $\underline{x}_{\mathcal{I}} \in M^{\mathcal{I},p}_{\mathcal{J},d}$, then for any $I \in \mathcal{I}\setminus\{\{s\},\{t\}\}$ we have $\rho_g(x_I, x_{\{s\}}) \geqslant b_p\frac{\ln d}{\sqrt{d}}$ and $\rho_g(x_I, x_{\{t\}}) \geqslant b_p\frac{\ln d}{\sqrt{d}}$ by definition. Applying Propositions 2.24 and 2.26 in Equation (3.5), we obtain:

$$(-1)^{p-|A|}\left(\prod_{i\notin A\sqcup\{s,t\}}\mathcal{R}^1_d\left(x_{\{i\}}\right)\right)\mathcal{R}^{|\mathcal{I}_A|}_d\left(\underline{x}_{\mathcal{I}_A}\right)\mathcal{D}^J_d\left(\underline{x}_J\right) + O\Big(d^{\frac{|\mathcal{I}_A|+p-|A|}{2}-\frac{p}{4}-1}\Big).$$

Since A is adapted to $\mathcal{I}$, we have $\mathcal{I} = \mathcal{I}_A\sqcup\{\{i\} \mid i\notin A\}$ by Definition 3.3. This implies that $|\mathcal{I}| = |\mathcal{I}_A|+p-|A|$. Hence the error term in the previous equation is $O(d^{\frac{|\mathcal{I}|}{2}-\frac{p}{4}-1})$. Summing these terms over the subsets $A \in \mathcal{S}_{\mathcal{I}}$ such that $A \subset \{1,\dots,p\}\setminus\{s,t\}$, we proved that Equation (3.4) is satisfied uniformly for all $\underline{x}_{\mathcal{I}} \in M^{\mathcal{I},p}_{\mathcal{J},d}$. □

With the same notations as in Lemma 3.21, the second step in our factorization of $\mathcal{D}_d^{\mathcal{I}}$ is to factor the contributions from the singletons in $\mathcal{J}$, that is from the $\{\{I\} \mid I \in \mathcal{I} \setminus \mathcal{I}_S\}$. This is the purpose of the following lemma.

LEMMA 3.22. — *Let A be a finite set, let $(\mathcal{I}, \mathcal{J}) \in \mathfrak{C}_A$ and let $S \subset A$ be as in Definition 3.18. Then, the following holds uniformly for all $\underline{x}_{\mathcal{I}} \in M_{\mathcal{J},d}^{\mathcal{I},p}$:*

$$\mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}}) = \left(\prod_{I \in \mathcal{I} \setminus \mathcal{I}_S} \mathcal{R}_d^1(x_I)\right)\left(\prod_{J \in \mathcal{J}_{\mathcal{I}_S}} \mathcal{D}_d^J(\underline{x}_J)\right) + O\left(d^{\frac{|\mathcal{I}|}{2}-\frac{p}{4}-1}\right).$$

Proof. — Applying first Lemma 3.21, uniformly for all $\underline{x}_{\mathcal{I}} \in M_{\mathcal{J},d}^{\mathcal{I},p}$ we have:

$$\mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}}) = \mathcal{D}_d^{\mathcal{I}\setminus\mathcal{I}_S}\left(\underline{x}_{\mathcal{I}\setminus\mathcal{I}_S}\right)\left(\prod_{J \in \mathcal{J}_{\mathcal{I}_S}} \mathcal{D}_d^J(\underline{x}_J)\right) + O\left(d^{\frac{|\mathcal{I}|}{2}-\frac{p}{4}-1}\right). \tag{3.6}$$

By Condition (1) in Definition 3.18, the partition $\mathcal{I} \setminus \mathcal{I}_S$ is a partition into pairs of $A \setminus S$. Recalling Definition 3.3, we have $\mathcal{S}_{\mathcal{I}\setminus\mathcal{I}_S} = \{A \setminus S\}$ hence $\mathcal{D}_d^{\mathcal{I}\setminus\mathcal{I}_S} = \mathcal{R}_d^{|\mathcal{I}\setminus\mathcal{I}_S|}$.

Let $\underline{x}_{\mathcal{I}} \in M_{\mathcal{J},d}^{\mathcal{I},p}$, by Condition (2) in Definition 3.18, for all $I \in \mathcal{I} \setminus \mathcal{I}_S$ we have $\{I\} \in \mathcal{J}$, hence x_I is far from the other components of $\underline{x}_{\mathcal{I}}$. In particular, for all $I, J \in \mathcal{I} \setminus \mathcal{I}_S$ distinct we have $\rho_g(x_I, x_J) \geqslant b_p \frac{\ln d}{\sqrt{d}}$. Applying Proposition 2.26 several times, we obtain:

$$\mathcal{D}_d^{\mathcal{I}\setminus\mathcal{I}_S}\left(\underline{x}_{\mathcal{I}\setminus\mathcal{I}_S}\right) = \mathcal{R}_d^{|\mathcal{I}\setminus\mathcal{I}_S|}\left(\underline{x}_{\mathcal{I}\setminus\mathcal{I}_S}\right) = \prod_{I \in \mathcal{I}\setminus\mathcal{I}_S} \mathcal{R}_d^1(x_I) + O\left(d^{\frac{|\mathcal{I}\setminus\mathcal{I}_S|}{2}-\frac{p}{4}-1}\right), \tag{3.7}$$

where the error term is obtained by using the uniform bound of Proposition 2.24.

By Proposition 2.24 once again, for any $J \in \mathcal{J}_{\mathcal{I}_S}$ we have $\mathcal{D}_d^J(\underline{x}_J) = O(d)$ uniformly on M^J. Since, $\mathcal{J}_{\mathcal{I}_S}$ is a partition into pairs of $\mathcal{I}_S$, we have $|\mathcal{I}_S| = 2|\mathcal{J}_{\mathcal{I}_S}|$. Hence, we have uniformly:

$$\prod_{J \in \mathcal{J}_{\mathcal{I}_S}} \mathcal{D}_d^J(\underline{x}_J) = O\left(d^{\frac{|\mathcal{I}_S|}{2}}\right). \tag{3.8}$$

Since $|\mathcal{I}| = |\mathcal{I}_S| + |\mathcal{I} \setminus \mathcal{I}_S|$, the result follows from Equations (3.6), (3.7) and (3.8). □

Finally, we derive the asymptotics of the integrals of the form (3.2) for $(\mathcal{I}, \mathcal{J}) \in \mathfrak{C}_p$, which is the main result of this section.

LEMMA 3.23. — *Let $p \geqslant 2$, let $(\mathcal{I}, \mathcal{J}) \in \mathfrak{C}_p$ and let S be as in Definition 3.18. Then, for any $\phi_1, \ldots, \phi_p \in \mathcal{C}^0(M)$, we have:*

$$\int_{\underline{x}_{\mathcal{I}} \in M_{\mathcal{J},d}^{\mathcal{I},p}} \iota_{\mathcal{I}}^* \phi(\underline{x}_{\mathcal{I}})\, \mathcal{D}_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}})\, |\mathrm{d}V_M|^{|\mathcal{I}|} =$$
$$\left(\prod_{\{i,j\} \in \mathcal{I}\setminus\mathcal{I}_S} \int_M \phi_i \phi_j \mathcal{R}_d^1 |\mathrm{d}V_M|\right)\left(\prod_{J \in \mathcal{J}_{\mathcal{I}_S}} \int_{M^J} \phi_J \mathcal{D}_d^J |\mathrm{d}V_M|^2\right)$$
$$+ \left(\prod_{i=1}^p \|\phi_i\|_\infty\right) O\left(d^{\frac{1}{2}\left\lfloor\frac{p-1}{2}\right\rfloor}(\ln d)^p\right),$$

where the constant involved in the

$$O\left(d^{\frac{1}{2}\left\lfloor\frac{p-1}{2}\right\rfloor}(\ln d)^p\right)$$

is independent of $(\phi_1, \ldots, \phi_p)$.

Proof. — The idea of the proof is to use the factorization obtained in Lemma 3.22 for the density function $\mathcal{D}_d^{\mathcal{I}}$. The test-function $\iota_{\mathcal{I}}^*\phi$ also factorizes, hence we can write the integrand $(\iota_{\mathcal{I}}^*\phi)\mathcal{D}_d^{\mathcal{I}}$ as a nice product. The difficulty is that the domain of integration $M_{\mathcal{J},d}^{\mathcal{I},p}$ is not a Cartesian product. To deal with this, we use Lemma 3.12 to show that we can replace $M_{\mathcal{J},d}^{\mathcal{I},p}$ by a product of domains of dimension 1 or 2, up to an error term.

Factorization of the integrand. Recall that, by Condition (1) in Definition 3.18, the partition $\mathcal{I}$ only contains singletons and pairs and S is the union of the singletons of $\mathcal{I}$. Let $\underline{x}_{\mathcal{I}} \in M^{\mathcal{I}}$, we have:

$$\begin{aligned}\iota_{\mathcal{I}}^*\phi\left(\underline{x}_{\mathcal{I}}\right) &= \phi\left(\iota_{\mathcal{I}}\left(\underline{x}_{\mathcal{I}}\right)\right)\\ &= \prod_{I\in\mathcal{I}}\prod_{i\in I}\phi_i(x_I) = \left(\prod_{\{i,j\}=I\in\mathcal{I}\setminus\mathcal{I}_S}\phi_i(x_I)\phi_j(x_I)\right)\left(\prod_{\{i\}\in\mathcal{I}_S}\phi_i\left(x_{\{i\}}\right)\right).\end{aligned}$$

Let $F_d^{\mathcal{I}} : M^{\mathcal{I}} \to \mathbb{R}$ be the function defined by:

$$F_d^{\mathcal{I}}(\underline{x}_{\mathcal{I}}) = \left(\prod_{\{i,j\}=I\in\mathcal{I}\setminus\mathcal{I}_S}\phi_i(x_I)\phi_j(x_I)\mathcal{R}_d^1(x_I)\right)\left(\prod_{J\in\mathcal{J}_{\mathcal{I}_S}}\phi_J\left(\underline{x}_J\right)\mathcal{D}_d^J\left(\underline{x}_J\right)\right) \tag{3.9}$$

for all $\underline{x}_{\mathcal{I}} \in M^{\mathcal{I}}$. Here we use Notation 2.2, hence for any $J = \{\{i\},\{j\}\} \in \mathcal{J}$ we have:

$$\phi_J\left(\underline{x}_J\right) = \phi_{\{i\}}\left(x_{\{i\}}\right)\phi_{\{j\}}\left(x_{\{j\}}\right) = \phi_i\left(x_{\{i\}}\right)\phi_j\left(x_{\{j\}}\right).$$

Note that since $\mathcal{I} \in \mathcal{P}_p$, we have $|\mathcal{I}| \leqslant p$ so that $\frac{|\mathcal{I}|}{2} - \frac{p}{4} - 1 \leqslant \frac{p}{4} - 1 < \frac{1}{2}\left\lfloor\frac{p-1}{2}\right\rfloor$. Applying Lemma 3.22, we obtain:

$$\iota_{\mathcal{I}}^*\phi\left(\underline{x}_{\mathcal{I}}\right)\mathcal{D}_d^{\mathcal{I}}\left(\underline{x}_{\mathcal{I}}\right) = F_d^{\mathcal{I}}\left(\underline{x}_I\right) + \left(\prod_{i=1}^p\|\phi_i\|_\infty\right)o\left(d^{\frac{1}{2}\left\lfloor\frac{p-1}{2}\right\rfloor}\right), \tag{3.10}$$

where the error term does not depend on $\underline{x}_{\mathcal{I}}$ or $(\phi_i)_{1\leqslant i\leqslant p}$. Thus, up to an error term, the quantity we are interested in is the integral of $F_d^{\mathcal{I}}$ over $M_{\mathcal{J},d}^{\mathcal{I},p}$.

Changing the domain of integration. By Lemma 3.12, we have:

$$M_{\mathcal{J},d}^{\mathcal{I},p} \subset \prod_{J\in\mathcal{J}} M_{\{J\},d}^{J,p} \subset \bigsqcup_{\mathcal{K}\geqslant\mathcal{J}} M_{\mathcal{K},d}^{\mathcal{I},p},$$

hence

$$\prod_{J\in\mathcal{J}} M_{\{J\},d}^{J,p} = M_{\mathcal{J},d}^{\mathcal{I},p} \sqcup \bigsqcup_{\mathcal{K}>\mathcal{J}}\left(M_{\mathcal{K},d}^{\mathcal{I},p} \cap \prod_{J\in\mathcal{J}} M_{\{J\},d}^{J,p}\right). \tag{3.11}$$

We want to prove that the integral of $F_d^{\mathcal{I}}$ over $M_{\mathcal{J},d}^{\mathcal{I},p}$ is equal to its integral over $\prod_{J\in\mathcal{J}} M_{\{J\},d}^{J,p}$, up to an error term. Let $\mathcal{K}\in\mathcal{P}_{\mathcal{I}}$ be such that $\mathcal{K}>\mathcal{J}$. We can bound the integral of $F_d^{\mathcal{I}}$ over

$$\Omega_{\mathcal{K}} = M_{\mathcal{K},d}^{\mathcal{I},p} \cap \prod_{J\in\mathcal{J}} M_{\{J\},d}^{J,p},$$

using the same method as in the proof of Lemma 3.14. We have:

$$\left|\int_{\underline{x}_{\mathcal{I}}\in\Omega_{\mathcal{K}}} F_d^{\mathcal{I}}\left(\underline{x}_{\mathcal{I}}\right)\left|\mathrm{d}V_M\right|^{|\mathcal{I}|}\right| \leqslant \int_{\underline{x}_{\mathcal{I}}\in\Omega_{\mathcal{K}}} \left|F_d^{\mathcal{I}}\left(\underline{x}_{\mathcal{I}}\right)\right|\left|\mathrm{d}V_M\right|^{|\mathcal{I}|} \leqslant \int_{\underline{x}_{\mathcal{I}}\in M_{\mathcal{K},d}^{\mathcal{I},p}} \left|F_d^{\mathcal{I}}\left(\underline{x}_{\mathcal{I}}\right)\right|\left|\mathrm{d}V_M\right|^{|\mathcal{I}|}.$$

Using Proposition 2.24 in Equation (3.9), we prove that the function $F_d^{\mathcal{I}}$ is bounded uniformly over $M^{\mathcal{I}}$ by $(\prod_{i=1}^p \|\phi_i\|_\infty)O(d^{\frac{|\mathcal{I}|}{2}})$. Then, by Lemma 3.13, we have:

$$\begin{aligned}(3.12)\quad \left|\int_{\underline{x}_{\mathcal{I}}\in\Omega_{\mathcal{K}}} F(\underline{x}_{\mathcal{I}})\left|\mathrm{d}V_M\right|^{|\mathcal{I}|}\right| &\leqslant \left\|F_d^{\mathcal{I}}\right\|_\infty \mathrm{Vol}\Big(M_{\mathcal{K},d}^{\mathcal{I},p}\Big)\\ &= \left(\prod_{i=1}^p \|\phi_i\|_\infty\right) O\Big(d^{\frac{|\mathcal{K}|}{2}}(\ln d)^{|\mathcal{I}|-|\mathcal{K}|}\Big).\end{aligned}$$

Recall that $\mathcal{K}>\mathcal{J}$, so that $|\mathcal{K}|<|\mathcal{J}|$, as discussed after Definition 3.2. Since $(\mathcal{I},\mathcal{J})\in\mathfrak{C}_p$, by Definition 3.18 we have:

$$|\mathcal{J}| = \frac{1}{2}|\mathcal{I}_S| + |\mathcal{I}\setminus\mathcal{I}_S| = \frac{1}{2}|S| + \frac{1}{2}(p-|S|) = \frac{p}{2}.$$

Thus, we have $|\mathcal{K}|\leqslant|\mathcal{J}|-1=\frac{p}{2}-1\leqslant\left\lfloor\frac{p-1}{2}\right\rfloor$. On the other hand, $|\mathcal{I}|-|\mathcal{K}|\leqslant|\mathcal{I}|\leqslant p$. Finally, using Equations (3.11) and (3.12), we obtain:

$$\begin{aligned}(3.13)\quad &\int_{M_{\mathcal{J},d}^{\mathcal{I},p}} F_d^{\mathcal{I}}\left|\mathrm{d}V_M\right|^{|\mathcal{I}|}\\ &\qquad= \int_{\prod_{J\in\mathcal{J}} M_{\{J\},d}^{J,p}} F_d^{\mathcal{I}}\left|\mathrm{d}V_M\right|^{|\mathcal{I}|} + \left(\prod_{i=1}^p \|\phi_i\|_\infty\right) O\Big(d^{\frac{1}{2}\left\lfloor\frac{p-1}{2}\right\rfloor}(\ln d)^p\Big).\end{aligned}$$

By Condition (2) in Definition 3.18, if $J\in\mathcal{J}\setminus\mathcal{J}_{\mathcal{I}_S}$ then $J=\{I\}$ for some $I\in\mathcal{I}\setminus\mathcal{I}_S$, so that $M_{\{J\},d}^{J,p}=M$. Bearing this in mind, Equations (3.9), (3.10) and (3.13) yield:

$$\begin{aligned}(3.14)\quad &\int_{\underline{x}_{\mathcal{I}}\in M_{\mathcal{J},d}^{\mathcal{I},p}} \iota_{\mathcal{I}}^*\phi\left(\underline{x}_{\mathcal{I}}\right)\mathcal{D}_d^{\mathcal{I}}\left(\underline{x}_{\mathcal{I}}\right)\left|\mathrm{d}V_M\right|^{|\mathcal{I}|} =\\ &\qquad\left(\prod_{\{i,j\}\in\mathcal{I}\setminus\mathcal{I}_S}\int_M \phi_i\phi_j\mathcal{R}_d^1\left|\mathrm{d}V_M\right|\right)\left(\prod_{J\in\mathcal{J}_{\mathcal{I}_S}}\int_{M_{\{J\},d}^{J,p}}\phi_J\mathcal{D}_d^J\left|\mathrm{d}V_M\right|^2\right)\\ &\qquad\qquad+\left(\prod_{i=1}^p\|\phi_i\|_\infty\right)O\Big(d^{\frac{1}{2}\left\lfloor\frac{p-1}{2}\right\rfloor}(\ln d)^p\Big).\end{aligned}$$

Conclusion of the proof. In order to conclude the proof, we need to replace the integral over $M_{\{J\},d}^{J,p}$ by an integral over M^J in Equation (3.14), for all $J\in\mathcal{J}_{\mathcal{I}_S}$.

Let $J\in\mathcal{J}_{\mathcal{I}_S}$, there exists i and $j\in S$ such that $J=\{\{i\},\{j\}\}$. Then, $\mathcal{P}_J$ has exactly two elements: $\{J\}$ and $\mathcal{I}_0(J)=\{\{\{i\}\},\{\{j\}\}\}$. By Proposition 2.26 and the

definition of $\mathcal{D}_d^J$ (see Lemma 3.5), we have the following uniform estimate for all $\underline{x}_J \in M^{J,p}_{\mathcal{I}_0(J),d}$:

$$\mathcal{D}_d^J(\underline{x}_J) = \mathcal{R}_d^2\left(x_{\{i\}}, x_{\{j\}}\right) - \mathcal{R}_d^1\left(x_{\{i\}}\right)\mathcal{R}_d^1\left(x_{\{j\}}\right) = O\left(d^{-\frac{p}{4}}\right).$$

Since $M^J = M^{J,p}_{\{J\},d} \sqcup M^{J,p}_{\mathcal{I}_0(J),d}$, we obtain:

$$\int_{M^{J,p}_{\{J\},d}} \phi_J \mathcal{D}_d^J \left|\mathrm{d}V_M\right|^2 = \int_{M^J} \phi_J \mathcal{D}_d^J \left|\mathrm{d}V_M\right|^2 + \|\phi_J\|_\infty O\left(d^{-\frac{p}{4}}\right).$$

Then, using one last time the upper bound of Proposition 2.24, the leading term on the right-hand side of Equation (3.14) equals:

$$\left(\prod_{\{i,j\}\in\mathcal{I}\setminus\mathcal{I}_S}\int_M \phi_i\phi_j\mathcal{R}_d^1\left|\mathrm{d}V_M\right|\right)\left(\prod_{J\in\mathcal{J}_{\mathcal{I}_S}}\int_{M^J}\phi_J\mathcal{D}_d^J\left|\mathrm{d}V_M\right|^2\right) + \left(\prod_{i=1}^p\|\phi_i\|_\infty\right)O\left(d^{\frac{|\mathcal{I}\setminus\mathcal{I}_S|}{2}+|\mathcal{J}_{\mathcal{I}_S}|-1-\frac{p}{4}}\right).$$

The conclusion follows using that $(\mathcal{I},\mathcal{J}) \in \mathfrak{C}_p$. Indeed, by Definition 3.18, we have:

$$\frac{|\mathcal{I}\setminus\mathcal{I}_S|}{2} + |\mathcal{J}_{\mathcal{I}_S}| - 1 - \frac{p}{4} = \frac{p-|S|}{4} + \frac{|\mathcal{I}_S|}{2} - \frac{p}{4} - 1 = \frac{|S|}{4} - 1 \leqslant \frac{p}{4} - 1 < \frac{1}{2}\left\lfloor\frac{p-1}{2}\right\rfloor. \quad \square$$

3.6. Proof of Theorem 1.12

In this section, we conclude the proof of our main result, that is the moments estimates of Theorem 1.12. The key argument is that the leading term in the asymptotics of the $m_p(\nu_d)(\phi_1, \ldots, \phi_p)$ comes from integrals of the form (3.2), where $(\mathcal{I},\mathcal{J}) \in \mathfrak{C}_p$ is one of the double partitions into pairs studied in Section 3.5. The integrals of the form (3.2) with $(\mathcal{I},\mathcal{J}) \notin \mathfrak{C}_p$ only contribute an error term. This last fact is proved in the following lemma.

Lemma 3.24. — *Let $p \geqslant 2$, let $\mathcal{I} \in \mathcal{P}_p$ and let $\mathcal{J} \in \mathcal{P}_{\mathcal{I}}$ be such that $(\mathcal{I},\mathcal{J}) \notin \mathfrak{C}_p$. Then, for all $\phi_1, \ldots, \phi_p \in \mathcal{C}^0(M)$ we have:*

$$\int_{\underline{x}_{\mathcal{I}} \in M^{\mathcal{I},p}_{\mathcal{J},d}} \iota_{\mathcal{I}}^*\phi\left(\underline{x}_{\mathcal{I}}\right)\mathcal{D}_d^{\mathcal{I}}\left(\underline{x}_{\mathcal{I}}\right)\left|\mathrm{d}V_M\right|^{|\mathcal{I}|} = \left(\prod_{i=1}^p\|\phi_i\|_\infty\right)O\left(d^{\frac{1}{2}\left\lfloor\frac{p-1}{2}\right\rfloor}(\ln d)^p\right),$$

where the error term does not depend on $(\phi_i)_{1\leqslant i\leqslant p}$.

Proof. — Since $\frac{p}{4} - 1 = \frac{1}{2}(\frac{p-2}{2} - 1) < \frac{1}{2}\left\lfloor\frac{p-1}{2}\right\rfloor$, we already know that the result holds if there exists $i \in \{1, \ldots, p\}$ such that $\{i\} \in \mathcal{I}$ and $\{\{i\}\} \in \mathcal{J}$, by Corollary 3.17. Assuming that this is not the case, we will show in Lemma 3.25 below that: if $\mathcal{I} \in \mathcal{P}_p$ and $\mathcal{J} \in \mathcal{P}_{\mathcal{I}}$, then $|\mathcal{J}| \leqslant \frac{p}{2}$, and equality holds if and only if $(\mathcal{I},\mathcal{J}) \in \mathfrak{C}_p$. Since we assumed that $(\mathcal{I},\mathcal{J}) \notin \mathfrak{C}_p$, we have $|\mathcal{J}| < \frac{p}{2}$. Moreover, since $|\mathcal{J}|$ is an integer, we have in fact $|\mathcal{J}| \leqslant \left\lfloor\frac{p-1}{2}\right\rfloor$. The result follows from Lemma 3.14, since $|\mathcal{I}| \leqslant p$. $\square$

Lemma 3.25. — *Let $p \geqslant 2$, let $\mathcal{I} \in \mathcal{P}_p$ and let $\mathcal{J} \in \mathcal{P}_{\mathcal{I}}$. We assume that, for all $i \in \{1, \ldots, p\}$, if $\{i\} \in \mathcal{I}$ then $\{\{i\}\} \notin \mathcal{J}$. Then $|\mathcal{J}| \leqslant \frac{p}{2}$. Moreover, $|\mathcal{J}| = \frac{p}{2}$ if and only if $(\mathcal{I},\mathcal{J}) \in \mathfrak{C}_p$.*

Proof. — We will use notations consistent with those introduced in Definition 3.18. Let $\mathcal{I}_S = \{I \in \mathcal{I}, |I| = 1\}$ denote the set of singletons in $\mathcal{I}$. We set $S = \bigsqcup_{I \in \mathcal{I}_S} I$, so that $\mathcal{I}_S$ is indeed the partition of S induced by $\mathcal{I}$. Then $\mathcal{I} \setminus \mathcal{I}_S$ is the partition of $\{1, \ldots, p\} \setminus S$ induced by $\mathcal{I}$, and its elements have cardinality at least 2. Since $\{1, \ldots, p\} = \bigsqcup_{I \in \mathcal{I}_S} I \sqcup \bigsqcup_{I \in \mathcal{I} \setminus \mathcal{I}_S} I$, we have:

$$|\mathcal{I}_S| + 2\,|\mathcal{I} \setminus \mathcal{I}_S| \leqslant p. \tag{3.15}$$

Moreover, equality holds in Equation (3.15) if and only if $\mathcal{I} \setminus \mathcal{I}_S$ is a partition into pairs, that is if and only if Condition (1) of Definition 3.18 is satisfied.

Let us denote by $\mathcal{I}' = \{I \in \mathcal{I} \mid \{I\} \in \mathcal{J}\}$. Under our assumptions, if $I \in \mathcal{I}'$ it can not be a singleton, i.e. $\mathcal{I}' \subset \mathcal{I} \setminus \mathcal{I}_S$. Hence, we have:

$$|\mathcal{I}'| \leqslant |\mathcal{I} \setminus \mathcal{I}_S|\,, \tag{3.16}$$

and equality holds in Equation (3.16) if and only if $\mathcal{I}' = \mathcal{I} \setminus \mathcal{I}_S$.

By definition of $\mathcal{I}'$, we have $\mathcal{J}_{\mathcal{I}'} = \{\{I\} \mid I \in \mathcal{I}'\} \subset \mathcal{J}$. Moreover, $\mathcal{J}_{\mathcal{I} \setminus \mathcal{I}'} = \mathcal{J} \setminus \mathcal{J}_{\mathcal{I}'}$ is partition of $\mathcal{I} \setminus \mathcal{I}'$ in subsets of cardinality at least 2. This yields:

$$|\mathcal{J}| = |\mathcal{J}_{\mathcal{I}'}| + |\mathcal{J} \setminus \mathcal{J}_{\mathcal{I}'}| = |\mathcal{I}'| + \left|\mathcal{J}_{\mathcal{I} \setminus \mathcal{I}'}\right| \leqslant |\mathcal{I}'| + \frac{1}{2}\left(|\mathcal{I}| - |\mathcal{I}'|\right) = \frac{1}{2}\left(|\mathcal{I}| + |\mathcal{I}'|\right). \tag{3.17}$$

Moreover, equality holds in Equation (3.17) if and only if $\mathcal{J}_{\mathcal{I} \setminus \mathcal{I}'}$ is a partition into pairs of $\mathcal{I} \setminus \mathcal{I}'$.

By Equations (3.15), (3.16) and (3.17), we have:

$$|\mathcal{J}| \leqslant \frac{1}{2}\left(|\mathcal{I}| + |\mathcal{I}'|\right) \leqslant \frac{1}{2}\left(|\mathcal{I}| + |\mathcal{I} \setminus \mathcal{I}_S|\right) = \frac{1}{2}\left(|\mathcal{I}_S| + 2\,|\mathcal{I} \setminus \mathcal{I}_S|\right) \leqslant \frac{p}{2}. \tag{3.18}$$

Moreover, we have $|\mathcal{J}| = \frac{p}{2}$ if and only if equality holds in (3.15), (3.16) and (3.17). Equality in (3.15) means that Condition (1) in Definition 3.18 is satisfied. If equality holds in (3.16) then $\mathcal{I}' = \mathcal{I} \setminus \mathcal{I}_S$. If in addition equality holds in Equation (3.17), we have that $\mathcal{J}_{\mathcal{I} \setminus \mathcal{I}_S} = \mathcal{J}_{\mathcal{I}'} = \{\{I\} \mid I \in \mathcal{I} \setminus \mathcal{I}_S\} \subset \mathcal{J}$ and that $\mathcal{J}_{\mathcal{I} \setminus \mathcal{I}'} = \mathcal{J}_{\mathcal{I}_S}$ is a partition of $\mathcal{I}_S$ into pairs. Hence, Condition (2) in Definition 3.18 is satisfied. Finally, if $|\mathcal{J}| = \frac{p}{2}$ then $(\mathcal{I}, \mathcal{J}) \in \mathfrak{C}_p$. Conversely, if $(\mathcal{I}, \mathcal{J}) \in \mathfrak{C}_p$, equality holds in Equations (3.15), (3.16) and (3.17) and $|\mathcal{J}| = \frac{p}{2}$. □

We conclude this section with the proof of Theorem 1.12.

Proof of Theorem 1.12. — Let $p \geqslant 3$ and $\phi_1, \ldots, \phi_p \in \mathcal{C}^0(M)$. Let $d \in \mathbb{N}$ be large enough, let $s_d \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ be a standard Gaussian vector and let ν_d denote the counting measure of the real zero set of s_d.

By Lemma 3.5 and the fact that $M^{\mathcal{I}} = \bigsqcup_{\mathcal{J} \in \mathcal{P}_{\mathcal{I}}} M^{\mathcal{I},p}_{\mathcal{J},d}$ for all $\mathcal{I} \in \mathcal{P}_p$, we have:

$$m_p(\nu_d)(\phi_1, \ldots, \phi_p) = \sum_{\mathcal{I} \in \mathcal{P}_p} \sum_{\mathcal{J} \in \mathcal{P}_{\mathcal{I}}} \int_{\underline{x}_{\mathcal{I}} \in M^{\mathcal{I},p}_{\mathcal{J},d}} \iota^*_{\mathcal{I}} \phi\left(\underline{x}_{\mathcal{I}}\right) \mathcal{D}^{\mathcal{I}}_d\left(\underline{x}_{\mathcal{I}}\right) \left|\mathrm{d}V_M\right|^{|\mathcal{I}|}.$$

By Lemma 3.24, up to an error term of the form

$$\left(\prod_{i=1}^{p} \|\phi_i\|_\infty\right) O\left(d^{\frac{1}{2}\left\lfloor \frac{p-1}{2} \right\rfloor} (\ln d)^p\right)$$

we need only consider the terms in this double sum indexed by $(\mathcal{I},\mathcal{J})\in\mathfrak{C}_p$. The expression of these terms is given by Lemma 3.23. Thus, $m_p(\nu_d)(\phi_1,\dots,\phi_p)$ equals:

$$\sum_{(\mathcal{I},\mathcal{J})\in\mathfrak{C}_p}\left(\prod_{\{i,j\}\in\mathcal{I}\setminus\mathcal{I}_S}\int_M\phi_i\phi_j\mathcal{R}^1_d\,|\mathrm{d}V_M|\right)\left(\prod_{J\in\mathcal{J}_{\mathcal{I}_S}}\int_{M^J}\phi_J\mathcal{D}^J_d\,|\mathrm{d}V_M|^2\right)+\left(\prod_{i=1}^p\|\phi_i\|_\infty\right)O\Big(d^{\frac12\left\lfloor\frac{p-1}{2}\right\rfloor}(\ln d)^p\Big)\,, \tag{3.19}$$

where we used the same notations as in Definition 3.18. In particular, recall that $S\subset\{1,\dots,p\}$ is the union of the singletons appearing in $\mathcal{I}$.

Let $(\mathcal{I},\mathcal{J})\in\mathfrak{C}_p$ and let us denote by $S\subset\{1,\dots,p\}$ the union of the singletons in $\mathcal{I}$. Since $\mathcal{I}_S=\{\{i\}\mid i\in S\}$, there is a canonical bijection $i\mapsto\{i\}$ from S to $\mathcal{I}_S$. This first bijection induces a second one between $\mathcal{PP}_S$ and $\mathcal{PP}_{\mathcal{I}_S}$. Using this second bijection, $\mathcal{J}_{\mathcal{I}_S}\in\mathcal{PP}_{\mathcal{I}_S}$ corresponds to a partition into pairs of S, that we abusively denote by $\mathcal{J}_S$ in the following. Explicitly, we have $\mathcal{J}_S=\{\{i,j\}\mid\{\{i\},\{j\}\}\in\mathcal{J}_{\mathcal{I}_S}\}\in\mathcal{PP}_S$.

Let $J=\{\{i\},\{j\}\}\in\mathcal{J}_{\mathcal{I}_S}$ and let $I=\{i,j\}$ denote the corresponding pair in $\mathcal{J}_S$. Using Notation 2.2, we have: $\phi_J=\phi_{\{i\}}\boxtimes\phi_{\{j\}}=\phi_i\boxtimes\phi_j=\phi_I$. Moreover, under the canonically identification $M^J\simeq M^I$, for all $(x,y)\in M^I$ we have:

$$\mathcal{D}^J_d(x,y)=\mathcal{R}^2_d(x,y)-\mathcal{R}^1_d(x)\mathcal{R}^1_d(y)=\mathcal{D}^I_d(x,y).$$

Hence

$$\int_{M^J}\phi_J\mathcal{D}^J_d\,|\mathrm{d}V_M|^2=\int_{M^I}\phi_I\mathcal{D}^I_d\,|\mathrm{d}V_M|^2\,,$$

and the leading term on the right-hand side of Equation (3.19) equals:

$$\sum_{(\mathcal{I},\mathcal{J})\in\mathfrak{C}_p}\left(\prod_{\{i,j\}\in\mathcal{I}\setminus\mathcal{I}_S}\int_M\phi_i\phi_j\mathcal{R}^1_d\,|\mathrm{d}V_M|\right)\left(\prod_{I\in\mathcal{J}_S}\int_{M^I}\phi_I\mathcal{D}^I_d\,|\mathrm{d}V_M|^2\right). \tag{3.20}$$

We want to rewrite (3.20) as a sum over $\{(\Pi,\Sigma)\mid\Pi\in\mathcal{PP}_p,\Sigma\subset\Pi\}$, by a change of variable. To this end, we define a bijection from $\mathfrak{C}_p$ to $\{(\Pi,\Sigma)\mid\Pi\in\mathcal{PP}_p,\Sigma\subset\Pi\}$.

- Let $\Phi:\mathfrak{C}_p\to\{(\Pi,\Sigma)\mid\Pi\in\mathcal{PP}_p,\Sigma\subset\Pi\}$ be the map defined by:
$$\Phi:(\mathcal{I},\mathcal{J})\mapsto((\mathcal{I}\setminus\mathcal{I}_S)\sqcup\mathcal{J}_S,\mathcal{J}_S)\,.$$
For any $(\mathcal{I},\mathcal{J})\in\mathfrak{C}_p$, we have $\mathcal{I}\setminus\mathcal{I}_S\in\mathcal{PP}_{\{1,\dots,p\}\setminus S}$ and $\mathcal{J}_S\in\mathcal{PP}_S$. Hence, $(\mathcal{I}\setminus\mathcal{I}_S)\sqcup\mathcal{J}_S$ is indeed a partition of $\{1,\dots,p\}$ into pairs that contains $\mathcal{J}_S$.
- We now define a map $\Psi:\{(\Pi,\Sigma)\mid\Pi\in\mathcal{PP}_p,\Sigma\subset\Pi\}\to\mathfrak{C}_p$ as follows. Let $\Pi\in\mathcal{PP}_p$ and $\Sigma\subset\Pi$, we denote by $S=\sqcup_{I\in\Sigma}I$. We define a partition $\mathcal{I}=(\Pi\setminus\Sigma)\sqcup\{\{i\}\mid i\in S\}\in\mathcal{P}_p$, so that $\mathcal{I}_{\{1,\dots,p\}\setminus S}=\mathcal{I}\setminus\mathcal{I}_S=\Pi\setminus\Sigma$ is partition into pairs of $\{1,\dots,p\}\setminus S$ and $\mathcal{I}_S=\{\{i\}\mid i\in S\}$. Then we define $\mathcal{J}\in\mathcal{P}_{\mathcal{I}}$ by:
$$\mathcal{J}=\Big\{\{I\}\;\Big|\;I\in\mathcal{I}\setminus\mathcal{I}_S\Big\}\sqcup\Big\{\{\{i\},\{j\}\}\;\Big|\;\{i,j\}\in\Sigma\Big\}\,.$$
Then $\mathcal{J}_{\mathcal{I}\setminus\mathcal{I}_S}=\{\{I\}\mid I\in\mathcal{I}\setminus\mathcal{I}_S\}=\mathcal{J}\setminus\mathcal{J}_{\mathcal{I}_S}$ and $\mathcal{J}_{\mathcal{I}_S}$ is a partition into pairs of $\mathcal{I}_S$. Hence $(\mathcal{I},\mathcal{J})\in\mathfrak{C}_p$, and we set $\Psi(\Pi,\Sigma)=(\mathcal{I},\mathcal{J})$. Note that we have $\mathcal{J}_S=\Sigma$.

By construction, Φ is a bijection from $\mathfrak{C}_p$ to $\{(\Pi, \Sigma) \mid \Pi \in \mathcal{PP}_p, \Sigma \subset \Pi\}$ such that $\Psi = \Phi^{-1}$. Using this bijection to change variables, the sum written in Equation (3.20) becomes:

$$\sum_{\Pi \in \mathcal{PP}_p} \sum_{\Sigma \subset \Pi} \left(\prod_{\{i,j\} \in \Pi \setminus \Sigma} \int_M \phi_i \phi_j \mathcal{R}_d^1 \left| \mathrm{d}V_M \right| \right) \left(\prod_{I \in \Sigma} \int_{M^I} \phi_I \mathcal{D}_d^I \left| \mathrm{d}V_M \right|^2 \right). \tag{3.21}$$

On the other hand, by Lemma 3.5 applied with $p = 2$, for any i and $j \in \{1, \ldots, p\}$ distinct we have:

$$m_2(\nu_d)(\phi_i, \phi_j) = \int_{M^I} \phi_I \mathcal{D}_d^I \left| \mathrm{d}V_M \right|^2 + \int_M \phi_i \phi_j \mathcal{R}_d^1 \left| \mathrm{d}V_M \right|,$$

where we denoted $I = \{i, j\}$. Thus,

$$\sum_{\Pi \in \mathcal{PP}_p} \prod_{\{i,j\} \in \Pi} m_2(\nu_d)(\phi_i, \phi_j)$$

is equal to the term in Equation (3.21), hence is the leading term in Equation (3.19). This proves the first claim in Theorem 1.12. We obtain the second expression of $m_p(\nu_d)(\phi_1, \ldots, \phi_p)$ in Theorem 1.12 by replacing $m_2(\nu_d)$ by its asymptotics, computed in Theorem 1.3.

Let $\phi \in \mathcal{C}^0(M)$. Using what we just proved with $\phi_i = \phi$ for all $i \in \{1, \ldots, p\}$, we have:

$$\begin{aligned} m_p\left(\langle \nu_d, \phi \rangle\right) &= m_p(\nu_d)(\phi, \ldots, \phi) \\ &= \mathrm{Card}\left(\mathcal{PP}_p\right) \left(m_2(\nu_d)(\phi, \phi)\right)^{\frac{p}{2}} + \|\phi\|_\infty^p \, O\left(d^{\frac{1}{2}\left\lfloor \frac{p-1}{2} \right\rfloor} (\ln d)^p\right). \end{aligned}$$

By definition, $m_2(\nu_d)(\phi, \phi) = \mathrm{Var}(\langle \nu_d \phi \rangle)$. Besides, the cardinality of $\mathcal{PP}_p$ is equal to $2^{-\frac{p}{2}} (\frac{p}{2})^{-1} p!$ if p is even and to 0 otherwise. In both cases $\mathrm{Card}(\mathcal{PP}_p) = \mu_p$ (see Definition 1.10). This proves the first expression of $m_p(\langle \nu_d, \phi \rangle)$ stated in Theorem 1.12. We obtain the second expression of $m_p(\langle \nu_d, \phi \rangle)$ by using the expression of $\mathrm{Var}(\langle \nu_d, \phi \rangle)$ given by Theorem 1.3. □

4. Proof of the corollaries of Theorem 1.12

In this section, we prove the corollaries of Theorem 1.12. The strong Law of Large Numbers (Theorem 1.7) is proved in Section 4.1. The Central Limit Theorem (Theorem 1.9) is proved in Section 4.2. Finally, we prove Corollaries 1.14 and 1.15 in Section 4.3.

4.1. Proof of the strong Law of Large Numbers (Theorem 1.7)

The purpose of this section is to prove the strong Law of Large Numbers (Theorem 1.7). This result follows from the moments estimates of Theorem 1.12 by a Borel–Cantelli type argument, using the separability of $\mathcal{C}^0(M)$. This method was already used in [Let19, LP19, SZ99], for example.

We follow the notation of Section 2. In particular, recall that $(s_d)_{d\geqslant 0}$ is a sequence of random vectors such, for all $d\geqslant 0$, $s_d\sim\mathcal{N}(0,\mathrm{Id})$ in $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$. Then, Z_d is the real zero set of s_d and ν_d is the counting measure of Z_d.

Proof of Theorem 1.7. — We first consider the case of one test-function $\phi\in\mathcal{C}^0(M)$. We have:

$$\mathbb{E}\left[\sum_{d\geqslant 1}\left(\frac{\langle\nu_d\,,\phi\rangle-\mathbb{E}[\langle\nu_d\,,\phi\rangle]}{d^{\frac{1}{2}}}\right)^6\right]=\sum_{d\geqslant 1}d^{-3}m_6(\nu_d)(\phi,\,\ldots,\,\phi).$$

By Theorem 1.12 this sum is finite. Indeed, $m_6(\nu_d)(\phi,\,\ldots,\,\phi)=O(d^{\frac{3}{2}})$. Then, almost surely, we have:

$$\sum_{d\geqslant 1}\left(\frac{\langle\nu_d\,,\phi\rangle-\mathbb{E}[\langle\nu_d\,,\phi\rangle]}{d^{\frac{1}{2}}}\right)^6<+\infty,$$

hence $d^{-\frac{1}{2}}(\langle\nu_d,\phi\rangle-\mathbb{E}[\langle\nu_d,\phi\rangle])\xrightarrow[d\to+\infty]{}0$. Thus, for all $\phi\in\mathcal{C}^0(M)$, we have almost surely:

$$d^{-\frac{1}{2}}\,\langle\nu_d\,,\phi\rangle=d^{-\frac{1}{2}}\mathbb{E}[\langle\nu_d\,,\phi\rangle]+o(1)=\frac{1}{\pi}\int_M\phi\,|\mathrm{d}V_M|+o(1).$$

Applying this the constant unit function $\phi=\mathbf{1}$ yields:

$$d^{-\frac{1}{2}}\,\mathrm{Card}(Z_d)\xrightarrow[d\to+\infty]{}\frac{1}{\pi}\,\mathrm{Vol}(M)$$

almost surely.

Now, recall that $(\mathcal{C}^0(M),\|\cdot\|_\infty)$ is separable. Let $(\phi_k)_{k\geqslant 0}$ denote a dense sequence in this space such that $\phi_0=\mathbf{1}$. Almost surely, we have

$$d^{-\frac{1}{2}}\,\langle\nu_d\,,\phi_k\rangle\xrightarrow[d\to+\infty]{}\frac{1}{\pi}\int_M\phi_k\,|\mathrm{d}V_M|\ \text{ for all } k\geqslant 0.$$

Let $\underline{s}=(s_d)_{d\geqslant 1}\in\prod_{d\geqslant 1}\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ denote a fixed sequence belonging to the probability 1 event on which this condition holds. For every $\phi\in\mathcal{C}^0(M)$ and $k\in\mathbb{N}$, we have:

$$\begin{aligned}
&\left|d^{-\frac{1}{2}}\,\langle\nu_d\,,\phi\rangle-\frac{1}{\pi}\int_M\phi\,|\mathrm{d}V_M|\right|\\
&\quad\leqslant d^{-\frac{1}{2}}\,|\langle\nu_d\,,\phi-\phi_k\rangle|+\left|d^{-\frac{1}{2}}\,\langle\nu_d\,,\phi_k\rangle-\frac{1}{\pi}\int_M\phi_k\,|\mathrm{d}V_M|\right|+\frac{1}{\pi}\int_M|\phi-\phi_k|\,|\mathrm{d}V_M|\\
&\quad\leqslant\|\phi-\phi_k\|_\infty\left(d^{-\frac{1}{2}}\,\mathrm{Card}(Z_d)+\frac{\mathrm{Vol}(M)}{\pi}\right)+\left|d^{-\frac{1}{2}}\,\langle\nu_d\,,\phi_k\rangle-\frac{1}{\pi}\int_M\phi_k\,|\mathrm{d}V_M|\right|.
\end{aligned}$$

Since $\phi_0=\mathbf{1}$, the sequence $(d^{-\frac{1}{2}}\,\mathrm{Card}(Z_d))_{d\geqslant 1}$ converges, hence is bounded by some $K>0$. Let $\varepsilon>0$, there exists $k\in\mathbb{N}$ such that $\|\phi-\phi_k\|_\infty<\varepsilon$. Then, for every d large enough, we have:

$$\left|d^{-\frac{1}{2}}\,\langle\nu_d\,,\phi_k\rangle-\frac{1}{\pi}\int_M\phi_k\,|\mathrm{d}V_M|\right|<\varepsilon,$$

hence

$$\left|d^{-\frac{1}{2}}\,\langle\nu_d\,,\phi\rangle-\frac{1}{\pi}\int_M\phi\,|\mathrm{d}V_M|\right|<\varepsilon\left(1+K+\frac{\mathrm{Vol}(M)}{\pi}\right).$$

Thus, for all $\phi \in \mathcal{C}^0(M)$, we have $d^{-\frac{1}{2}} \langle \nu_d \, , \phi \rangle \xrightarrow[d \to +\infty]{} \frac{1}{\pi} \int_M \phi \, |\mathrm{d}V_M|$, which concludes the proof. □

4.2. Proof of the Central Limit Theorem (Theorem 1.9)

In this section, we prove the Central Limit Theorem (Theorem 1.9). The result follows from Theorem 1.12 by the method of moments, see [Bil95, Chapter 30]. This method allows to prove the Central Limit Theorem for any fixed continuous test-function. Then, we apply the Lévy–Fernique Theorem [Fer67, Theorem III.6.5], which is the analogue of Lévy's Continuity Theorem for random generalized functions. Using this theorem, we prove the convergence in distribution to the Standard Gaussian White Noise in $\mathcal{D}'(M)$, the space of generalized functions on M.

Proof of Theorem 1.9. —

Central Limit Theorem for a fixed test-function. Let $\phi \in \mathcal{C}^0(M) \setminus \{0\}$. We define a sequence (X_d) of centered and normalized real random variables by:

$$X_d = \frac{\langle \nu_d \, , \phi \rangle - \mathbb{E}[\langle \nu_d \, , \phi \rangle]}{\mathrm{Var}(\langle \nu_d \, , \phi \rangle)^{\frac{1}{2}}}.$$

Note that since $\phi \neq 0$, by Theorem 1.3, $\mathrm{Var}(\langle \nu_d, \phi \rangle) = m_2(\nu_d)(\phi, \phi)$ is positive for d large enough. Hence X_d is well-defined for d large enough, and we want to prove that $X_d \xrightarrow[d \to +\infty]{} \mathcal{N}(0,1)$ in distribution. By Theorem 1.3 and 1.12, for any integer $p \geqslant 3$, we have:

$$\mathbb{E}[X_d^p] = \frac{m_p(\nu_d)(\phi, \ldots, \phi)}{\mathrm{Var}(\langle \nu_d \, , \phi \rangle)^{\frac{p}{2}}} \xrightarrow[d \to +\infty]{} \mu_p,$$

where μ_p is the p^{th} moment of an $\mathcal{N}(0,1)$ variable (recall Definition 1.10). By the Theorem of Moments (cf. [Bil95, Theorem 30.2]), this implies

$$X_d \xrightarrow[d \to +\infty]{} \mathcal{N}(0,1)$$

in distribution. Replacing the expectation and the variance of $\langle \nu_d \, , \phi \rangle$ by their asymptotics (Theorem 1.1 and Theorem 1.3), we get:

$$X_d = \frac{1}{d^{\frac{1}{4}} \sigma} \left(\int_M \phi^2 \, |\mathrm{d}V_M| \right)^{-\frac{1}{2}} \left(\langle \nu_d \, , \phi \rangle - \frac{d^{\frac{1}{2}}}{\pi} \int_M \phi \, |\mathrm{d}V_M| \right) + o(1).$$

Hence,

$$\frac{1}{d^{\frac{1}{4}} \sigma} \left(\langle \nu_d \, , \phi \rangle - \frac{d^{\frac{1}{2}}}{\pi} \int_M \phi \, |\mathrm{d}V_M| \right) \xrightarrow[d \to +\infty]{} \mathcal{N}(0, \|\phi\|_2^2),$$

in distribution, where $\|\phi\|_2^2 = \int_M \phi^2 \, |\mathrm{d}V_M|$. Of course, this also holds for $\phi = 0$. Taking $\phi = \mathbf{1}$ yields the Central Limit Theorem for the cardinality of Z_d.

Central Limit Theorem in $\mathcal{D}'(M)$. Let us now consider the random measure

$$T_d = \left(d^{\frac{1}{4}}\sigma\right)^{-1}\left(\nu_d - d^{\frac{1}{2}}\,|\mathrm{d}V_M|\right).$$

For all $d \geqslant 1$, T_d defines a random generalized function. Recall that the characteristic functional of this random generalized function is the map

$$\chi_d : \phi \mapsto \mathbb{E}\Big[\exp\left(i\,\langle T_d\,,\phi\rangle_{(\mathcal{D}',\mathcal{C}^\infty)}\right)\Big] \text{ from } \mathcal{C}^\infty(M) \text{ to } \mathbb{C}.$$

Let $\phi \in \mathcal{C}^\infty(M)$, we just proved that

$$\langle T_d\,,\phi\rangle_{(\mathcal{D}',\mathcal{C}^\infty)} = \langle T_d\,,\phi\rangle \xrightarrow[d\to+\infty]{} \mathcal{N}\left(0, \|\phi\|_2^2\right).$$

Hence,

$$\chi_d(\phi) = \mathbb{E}\Big[e^{i\langle T_d\,,\phi\rangle}\Big] \xrightarrow[d\to+\infty]{} e^{-\frac{1}{2}\|\phi\|_2^2},$$

where we recognized the characteristic function of $\langle T_d\,,\phi\rangle$ evaluated at 1. Thus, the sequence (χ_d) converges pointwise to $\chi : \phi \mapsto e^{-\frac{1}{2}\|\phi\|_2^2}$, which is the characteristic functional of the Standard Gaussian White Noise W (recall Definition 1.8). Then, by the Lévy–Fernique Theorem (cf. [Fer67, Theorem III.6.5]), we have $T_d \xrightarrow[d\to+\infty]{} W$ in distribution, as random variables in $\mathcal{D}'(M)$.

Central Limit Theorem for a family of smooth test-functions. Let $\phi_1, \ldots, \phi_k \in \mathcal{C}^\infty(M)$, by the Continuous Mapping Theorem, we have:

$$\left(\langle T_d\,,\phi_1\rangle_{(\mathcal{D}',\mathcal{C}^\infty)}, \ldots, \langle T_d\,,\phi_k\rangle_{(\mathcal{D}',\mathcal{C}^\infty)}\right) \xrightarrow[d\to+\infty]{} \left(\langle W\,,\phi_1\rangle_{(\mathcal{D}',\mathcal{C}^\infty)}, \ldots, \langle W\,,\phi_k\rangle_{(\mathcal{D}',\infty)}\right)$$

in distribution, as random vectors in $\mathbb{R}^k$. This yields the joint Central Limit Theorem for a family of smooth test-functions. □

4.3. Proof of Corollaries 1.14 and 1.15

We first prove Corollary 1.14, that is the concentration in probability. This is just an application of Markov's Inequality.

Proof of Corollary 1.14. — Let $(\varepsilon_d)_{d\geqslant 1}$ be a positive sequence and let $p \in \mathbb{N}^*$. By Markov's inequality for the $2p^{\text{th}}$ moment, we have:

$$\mathbb{P}\left(d^{-\frac{1}{2}}\,|\langle\nu_d\,,\phi\rangle - \mathbb{E}[\langle\nu_d\,,\phi\rangle]| > \varepsilon_d\right) \leqslant \varepsilon_d^{-2p} d^{-p} m_{2p}(\nu_d)(\phi, \ldots, \phi).$$

By Theorem 1.12, the term on the right-hand side is

$$O\Big((d^{\frac{1}{4}}\varepsilon_d)^{-2p}\Big).$$

As usual, one obtains the statement about $\mathrm{Card}(Z_d)$ by specializing the result for $\phi = \mathbf{1}$. □

The proof of Corollary 1.15, that is of the Hole probability, uses Corollary 1.14 with the right test-function. It is similar to the proof of [Let19, Corollary 1.10].

Proof of Corollary 1.15. — Let U be a non-empty open subset of M, and let $\phi_U : M \to \mathbb{R}$ be a continuous function such that for all $x \in M$, $\phi_U(x) > 0$ if $x \in U$, and $\phi_U(x) = 0$ otherwise. Let $\varepsilon > 0$ be such that $\varepsilon < \frac{1}{\pi} \int_M \phi_U \, |\mathrm{d}V_M|$. By Theorem 1.1, for all d large enough we have $d^{-\frac{1}{2}} \mathbb{E}[\langle Z_d \, , \phi_U \rangle] > \varepsilon$. For a degree d such that this condition holds, we have:

$$\mathbb{P}\left(Z_d \cap U = \emptyset\right) = \mathbb{P}\left(\langle Z_d \, , \phi_U \rangle = 0\right) \leqslant \mathbb{P}\left(d^{-\frac{1}{2}} \left| \langle Z_d \, , \phi_U \rangle - \mathbb{E}[\langle Z_d \, , \phi_U \rangle] \right| > \varepsilon\right).$$

By Corollary 1.14, the right-hand side is $O(d^{-\frac{p}{2}})$ for all $p \in \mathbb{N}^*$. □

BIBLIOGRAPHY

[AADL18] Diego Armentano, Jean-Marc Azaïs, Federico Dalmao, and José R. León, *Central Limit Theorem for the volume of the zero set of Kostlan Shub Smale random polynomial systems*, `https://arxiv.org/abs/1808.02967v1`, 2018. ↑1669, 1670

[AADL21] ———, *Central Limit Theorem for the number of real roots of Kostlan Shub Smale random polynomial systems*, Am. J. Math. **143** (2021), no. 4, 1011–1042. ↑1669, 1670

[AL21] Michele Ancona and Thomas Letendre, *Zeros of smooth stationary Gaussian processes*, Electron. J. Probab. **26** (2021), article no. 68. ↑1680

[Anc21] Michele Ancona, *Random sections of line bundles over real Riemann surfaces*, Int. Math. Res. Not. **2021** (2021), no. 9, 7004–7059. ↑1662, 1663, 1667, 1668, 1669, 1670, 1671, 1676, 1677, 1678, 1679, 1680

[AT07] Robert J. Adler and Jonathan E. Taylor, *Random fields and geometry*, 1st ed., Springer Monographs in Mathematics, Springer, 2007. ↑1666, 1667, 1676

[AW09] Jean-Marc Azaïs and Mario Wschebor, *Level sets and extrema of random processes and fields*, 1st ed., John Wiley & Sons, 2009. ↑1666, 1667, 1676

[BBL92] Evgeny B. Bogomolny, Oriol Bohigas, and Patricio Leboeuf, *Distribution of roots of random polynomials*, Phys. Rev. Lett. **68** (1992), no. 18, 2726–2729. ↑1669

[BDFZ20] Ridhipratim Basu, Amir Dembo, Naomi Feldheim, and Ofer Zeitouni, *Exponential concentration of zeroes of stationary Gaussian processes*, Int. Math. Res. Not. (2020), no. 23, 9769–9796. ↑1669

[Bil95] Patrick Billingsley, *Probability and measure*, 3rd ed., Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons, 1995. ↑1699

[BSZ00] Pavel Bleher, Bernard Shiffman, and Steve Zelditch, *Universality and scaling of correlations between zeros on complex manifolds*, Invent. Math. **142** (2000), no. 2, 351–395. ↑1669, 1671

[Dal15] Federico Dalmao, *Asymptotic variance and CLT for the number of zeros of Kostlan–Shub–Smale random polynomials*, C. R. Math. Acad. Sci. Paris **353** (2015), no. 12, 1141–1145. ↑1661, 1663, 1669

[Fer67] Xavier Fernique, *Processus linéaires, processus généralisés*, Ann. Inst. Fourier **17** (1967), no. 1, 1–92. ↑1668, 1699, 1700

[GH94] Philip Griffiths and Joseph Harris, *Principles of algebraic geometry*, 2nd ed., Wiley Classics Library, John Wiley & Sons, 1994, reprint of the 1978 original. ↑1671, 1676

[GW11] Damien Gayet and Jean-Yves Welschinger, *Exponential rarefaction of real curves with many components*, Publ. Math., Inst. Hautes Étud. Sci. (2011), no. 113, 69–96. ↑1666, 1674

[GW16] ———, *Betti numbers of random real hypersurfaces and determinants of random symmetric matrices*, J. Eur. Math. Soc. **18** (2016), no. 4, 733–772. ↑1662, 1671, 1676

[KL01] Marie F. Kratz and José R. León, *Central limit theorems for level functionals of stationary Gaussian processes and fields*, J. Theor. Probab. **14** (2001), no. 3, 639–672. ↑1670

[Kos93] Eric Kostlan, *On the distribution of roots of random polynomials*, From topology to computation: Proceedings of the Smalefest. Papers presented at the conference "From topology to computation: Unity and diversity in the mathematical sciences" held at the University of California at Berkeley, USA, August 5-9, 1990 in honor of Stephen Smale's 60th birthday (M. W. Hirsch et al., eds.), Springer, 1993, pp. 419–431. ↑1661

[Let16] Thomas Letendre, *Expected volume and Euler characteristic of random submanifolds*, J. Funct. Anal. **270** (2016), no. 8, 3047–3110. ↑1662, 1668, 1671, 1676

[Let19] ———, *Variance of the volume of random real algebraic submanifolds*, Trans. Am. Math. Soc. **371** (2019), no. 6, 4129—4192. ↑1668, 1671, 1679, 1697, 1700

[LP19] Thomas Letendre and Martin Puchol, *Variance of the volume of random real algebraic submanifolds II*, Indiana Univ. Math. J. **68** (2019), no. 6, 1649–1720. ↑1662, 1663, 1668, 1671, 1676, 1678, 1679, 1697

[LS19] Antonio Lerario and Michele Stecconi, *Maximal and typical topology of real polynomial singularities*, `https://arxiv.org/abs/1906.04444`, to be published at the *Annales de l'Institut Fourier*, 2019. ↑1676

[MM07] Xianan Ma and George Marinescu, *Holomorphic Morse inequalities and Bergman kernels*, 1st ed., Progress in Mathematics, vol. 254, Birkhäuser, 2007. ↑1663, 1667, 1677, 1678, 1679, 1680

[MM15] ———, *Exponential estimate for the asymptotics of Bergman kernels*, Math. Ann. **362** (2015), no. 3–4, 1327–1347. ↑1667, 1677, 1679, 1680

[NS16] Fedor L. Nazarov and Mikhail Sodin, *Asymptotic laws for the spatial distribution and the number of connected components of zero sets of Gaussian random functions*, J. Math. Phys. Anal. Geom. **12** (2016), no. 3, 205–278. ↑1670

[Olv01] Peter J. Olver, *Geometric foundations of numerical algorithms and symmetry*, Appl. Algebra Eng. Commun. Comput. **11** (2001), no. 5, 417–436. ↑1667

[Ros19] Maurizia Rossi, *Random nodal lengths and Wiener chaos*, Probabilistic methods in geometry, topology and spectral theory. CRM workshops on probabilistic methods in spectral geometry and PDE, August 22–26, 2016, and on probabilistic methods in topology, November 14–18, 2016, Centre de Recherches Mathématiques, Université de Montréal, Montréal, Quebec, Canada, Contemporary Mathematics, vol. 739, American Mathematical Society; Centre de Recherches Mathématiques (CRM), Montréal, 2019, pp. 155–169. ↑1670

[ST04] Mikhail Sodin and Boris Tsirelson, *Random complex zeroes I: Asymptotic normality*, Isr. J. Math. **144** (2004), 125–149. ↑1661, 1669

[SZ99] Bernard Shiffman and Steve Zelditch, *Distribution of zeros of random and quantum chaotic sections of positive line bundles*, Commun. Math. Phys. **200** (1999), no. 3, 661–683. ↑1669, 1671, 1674, 1697

[SZ10] ———, *Number variance of random zeros on complex manifolds. II: Smooth statistics*, Pure Appl. Math. Q. **6** (2010), no. 4, 1145–1167. ↑1669, 1671

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