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A NÉRON MODEL OF THE UNIVERSAL JACOBIAN

UN MODÈLE DE NÉRON POUR LA JACOBIENNE UNIVERSELLE

Abstract. — Jacobians of degenerating families of curves are well-understood over 1-dimensional bases due to work of Néron and Raynaud; the fundamental tool is the Néron model and its description via the Picard functor. Over higher-dimensional bases Néron models typically do not exist, but in this paper we construct a universal base change $\widetilde{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ after which a Néron model $N_{g,n}/\widetilde{\mathcal{M}}_{g,n}$ of the universal jacobian does exist. This yields a new partial compactification of the moduli space of curves, and of the universal jacobian over it. The map $\widetilde{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is separated and relatively representable. The Néron model $N_{g,n}/\widetilde{\mathcal{M}}_{g,n}$ is separated and has a group law extending that on the jacobian. We show that Caporaso's balanced Picard stack acquires a torsor structure after pullback to a certain open substack of $\widetilde{\mathcal{M}}_{g,n}$.

Résumé. — Les jacobiennes de dégénérescences de courbes au-dessus de bases de dimension 1 sont bien comprises grâce aux travaux de Néron et Raynaud ; l'outil fondamental est le modèle de Néron et sa description à l'aide du foncteur de Picard. En général, sur des bases de dimension supérieure les modèles de Néron n'existent pas, mais dans cet article nous construisons un changement de base $\widetilde{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ qui est universel pour la propriété qu'un modèle de Néron $N_{g,n}/\widetilde{\mathcal{M}}_{g,n}$ de la jacobienne universelle existe. Ceci fournit une nouvelle compactification partielle de l'espace de modules des courbes et de la jacobienne universelle qui vit dessus. Le morphisme $\widetilde{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ est séparé et relativement représentable. Le modèle de Néron $N_{g,n}/\widetilde{\mathcal{M}}_{g,n}$ est séparé et possède une loi de groupe qui étend celle de la jacobienne.

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Nous montrons que le « champ de Picard équilibré » de Caporaso acquiert une structure de torseur après changement de base à un certain sous-champ ouvert de $\widetilde{\mathcal{M}}_{g,n}$.

1. Introduction

1.1. Models of the universal jacobian

The jacobian $\mathcal{J}_{g,n}$ of the universal curve over $\mathcal{M}_{g,n}$ is an abelian scheme, and comes with a natural extension as a semiabelian scheme $\mathcal{J}^0_{g,n}$ over $\overline{\mathcal{M}}_{g,n}$, parametrising line bundles of degree zero on every component of every fibre. This model $\mathcal{J}^0_{g,n}$ is not proper, and sections of $\mathcal{J}_{g,n}$ often fail to extend to $\mathcal{J}^0_{g,n}$, causing many difficulties in defining linear series on stable curves, compactifying cycles on the moduli space, etc. This motivates the search for "better" models of $\mathcal{J}_{g,n}$ over $\overline{\mathcal{M}}_{g,n}$, which has been very extensively studied; the literature is too large to describe in detail here, but highlights include [Cap08, Chi15, Est01, KP19, Mel09]…

1.2. Néron models

Ideally, one would like a model of $\mathcal{J}_{g,n}$ over $\overline{\mathcal{M}}_{g,n}$ which is proper and admits a group structure; properness guarantees that sections extend and intersection theory makes sense, and the group law gives additional structure to the resulting cycles and linear series. However, this is too much to ask for, as such models of abelian varieties generally fail to exist even over Dedekind base schemes. In the 1960s, Néron proved a beautiful theorem showing that, by replacing properness by the weaker condition that sections should extend, very nice models could be constructed of 1-dimensional families of abelian varieties. More precisely, given a Dedekind scheme S, a dense open $U \subseteq S$ and an abelian variety A/U, Néron showed the existence of a *Néron model*; a smooth separated model N/S satisfying the *Néron mapping property*:

given any smooth morphism $T \to S$ and U-morphism $f\colon T \times_S U \to A$, there is a unique $F\colon T \to N$ extending f.

Néron models automatically inherit group laws from their generic fibres, and have been by some margin the most widely-used models of abelian schemes over Dedekind bases in geometry and number theory over the past half-century. However, Néron's work cannot in general be directly applied to the universal jacobian, since $\overline{\mathcal{M}}_{g,n}$ is rarely of dimension 1. The definition of the Néron model makes sense over any base scheme, without restrictions on the dimension, the problem is that Néron's proof of existence fails, and indeed Néron models generally fail to exist (see [Hol19]).

1.3. Summary of main results

The universal jacobian rarely admits a Néron model; in this article, we construct a "universal" stack over $\overline{\mathcal{M}}_{g,n}$ over which a Néron model *does* exist. Slightly more formally, we construct a regular integral stack $\widetilde{\mathcal{M}}_{g,n}$ together with a separated representable map $\widetilde{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ locally of finite presentation, and with the following properties:

(1) $\widetilde{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is an isomorphism over $\mathcal{M}_{g,n}$ (it is birational);
(2) the universal jacobian $\mathcal{J}_{g,n}$ admits a Néron model over $\widetilde{\mathcal{M}}_{g,n}$ (the *universal Néron model*);
(3) If $t\colon T \to \overline{\mathcal{M}}_{g,n}$ is any morphism such that $t^*\mathcal{J}_{g,n}$ admits a Néron model over T, then the map t factors uniquely via $\widetilde{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ (a more precise statement can be found in Section 10).

The map $\widetilde{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is not proper in general; this is forced upon us by condition (2) and the results of [Hol19], unless a Néron model already exists over $\overline{\mathcal{M}}_{g,n}$. In fact it is not in general quasi-compact, essentially because component groups of Néron models of test curves in $\overline{\mathcal{M}}_{g,n}$ can be arbitrarily large. The fibres of $\widetilde{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ are all either single points, or countably infinite disjoint unions of finite-type schemes. In (3), it is not in general true that the pullback of the Néron model over $\widetilde{\mathcal{M}}_{g,n}$ is the Néron model over T. Indeed, this would be too much to ask, since e.g. t might be a ramified cover of its image (in Lemma 12.6 we *do* prove such a property for a restricted class of morphisms t.)

Inside $\widetilde{\mathcal{M}}_{g,n}$ there is an important quasi-compact open substack $\widetilde{\mathcal{M}}^{\leqslant 1}_{g,n}$, which can be defined as the largest open substack over which the pullback of the universal curve is regular (more details are given in Section 12). In Proposition 13.2, we show that Caporaso's balanced Picard stack $\mathcal{P}_{d,g}$ (described in more detail below) admits a torsor structure over the universal Néron model, after base-change to $\widetilde{\mathcal{M}}^{\leqslant 1}_{g,n}$.

1.4. Comparison to results of Caporaso and Chiodo

1.4.1. Caporaso's balanced Picard stack

Given integers d, g such that $\gcd(d-g+1, 2g-2) = 1$, Caporaso constructs in [Cap08] a model $\mathcal{P}_{d,g}/\overline{\mathcal{M}}_g$ for the degree-d jacobian(1) such that, for every test curve $f\colon T \to \overline{\mathcal{M}}_g$ transverse to the boundary of $\overline{\mathcal{M}}_g$, there is an isomorphism of algebraic spaces from $f^*\mathcal{P}_{d,g}$ to the Néron model of $f^*\mathcal{J}_{d,g}$.

Note that (unless $g = 2$) the condition $\gcd(d-g+1, 2g-2) = 1$ excludes the case $d = 0$. It is therefore nonsensical in general to ask $\mathcal{P}_{d,g}$ to have a group structure, but it does make sense to ask whether it has a torsor structure. This turns out not to be the case, as can be verified by an explicit computation in genus 3; the key point is that, given two test curves through the same point in the boundary, the special fibres of their Néron models will be isomorphic as schemes but *not* as group schemes/torsors. In Proposition 13.2 we will show that $\mathcal{P}_{g,d}$ does acquire a canonical torsor structure after pullback to the stack $\widetilde{\mathcal{M}}^{\leqslant 1}_{g,n}$.

1.4.2. Chiodo's results over the stack of twisted curves

As in the present paper, Chiodo [Chi15] replaces $\overline{\mathcal{M}}_g$ by a different stack living over $\overline{\mathcal{M}}_g$, in his case the stack $\overline{\mathcal{M}}^l_g$ of l-twisted curves (in the sense of Abramovich

(1) The coarse moduli space of line bundles of degree d.

and Vistoli). We will not give the definition of the stack of l-twisted curves here, but recall that it comes with a non-separated non-representable forgetful map to $\overline{\mathcal{M}}_g$, which is an isomorphism over $\mathcal{M}_g$. Chiodo constructs a *group* scheme $\mathrm{Pic}^{0,l}_g$ over $\overline{\mathcal{M}}^l_g$, extending the pullback of the universal jacobian. This has the property that, if $f\colon T \to \overline{\mathcal{M}}_g$ is a test curve transverse to the boundary, then there is a lift of f to $\overline{\mathcal{M}}^l_g$ such that the pullback of $\mathrm{Pic}^{0,l}_g$ is the Néron model of the jacobian.

Chiodo's approach is thus similar to ours; he replaces $\overline{\mathcal{M}}_g$ by a larger stack, and then obtains a group scheme model with a property related to Néron models. The key differences are

(1) Chiodo's $\overline{\mathcal{M}}^l_g \to \overline{\mathcal{M}}_g$ is not separated or representable, whereas our $\widetilde{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ has both properties;

(2) Chiodo applies the Néron mapping property along test curves, whereas we extend the mapping property to larger bases.

1.5. Consequences, and relation to more recent work

1.5.1. The double ramification cycle

In [Hol21], a slight variant of the construction of $\widetilde{\mathcal{M}}_{g,n}$ is used to extend the double ramification cycle to the boundary of the moduli space of curves. This slight variation comes about because the double ramification cycle is concerned with extending one particular section of the universal jacobian, whereas a Néron model must extend all sections. In a forthcoming joint work of the author with Johannes Schmitt, we will use this to compute precisely the components and multiplicities of the double ramification cycle in the meromorphic case.

1.5.2. Enriched structures

In [Mai98], Mainó defines an *enriched structure* on a stable curve C/k (with irreducible components $C_1, \ldots, C_r$) to be a collection of line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_r$ on C satisfying

(1) for each i,

$$\mathscr{L}_i|_{C_i} \cong \mathcal{O}_{C_i}\left(-C_i \cap C_i^c\right)$$

$\left(\text{here } C_i^c = \cup_{j\neq i} C_i\right)$ and

$$\mathscr{L}_i|_{C_i^c} \cong \mathcal{O}_{C_i^c}\left(C_i \cap C_i^c\right).$$

(2) $\bigotimes_i \mathscr{L}_i \cong \mathcal{O}_C$.

In [BH16] we construct a moduli stack of enriched structures over $\overline{\mathcal{M}}_{g,n}$, resolve various conjectures of Mainó, and construct an isomorphism between the stack of enriched structures and the open substack $\widetilde{\mathcal{M}}^{\leqslant 1}_{g,n} \hookrightarrow \widetilde{\mathcal{M}}_{g,n}$. Combining with Proposition 13.2, it follows that Caporaso's balanced Picard stack $\mathcal{P}_{d,g}$ acquires a torsor structure over the stack of enriched structures.

1.6. A detailed description of the case $g = 1$, $n = 2$

The definition of our space $\widetilde{\mathcal{M}}_{g,n}$ is via a universal property concerning Néron models, which are themselves defined by another universal property. As such, the reader may suspect our construction to be somewhat abstract and hard to work with. To provide reassurance, we give here a very detailed description of the structure and properties of our stack in the case when $g = 1$, $n = 2$ (the first interesting case), working over a field k for simplicity.

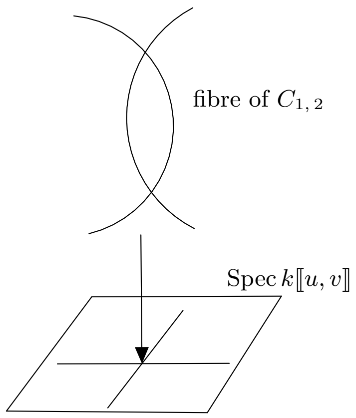

Figure 1.1. Curve over $\overline{\mathcal{M}}_{1,2}$

1.6.1. Coordinates at the non-treelike point

The map $\beta\colon \widetilde{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is an isomorphism not only over $\mathcal{M}_{g,n}$, but in fact over a larger locus, that of "treelike curves"; curves for which the dual graph is a tree with some self-loops attached. The only non treelike point in $\overline{\mathcal{M}}_{1,2}$ is the 2-pointed 2-gon (Figure 1.1), and β is an isomorphism away from that point. The completed local ring at that point is isomorphic to $k[\![u, v]\!]$, and we choose coordinates u and v so that local equations for the singularities are given by $xy = u$ and $x'y' = v$. Equivalently, we can write equations; the universal curve over $k[\![u, v]\!]$ is given by the projectivisation of

$$y^2 = \left((x-1)^2 - u\right)\left((x+1)^2 - v\right).$$

1.6.2. The locus on which the curve is regular

The restriction of $\beta^{\leqslant 1}\colon \widetilde{\mathcal{M}}^{\leqslant 1}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ to $k[\![u, v]\!]$ is very easy to describe. First blow up the closed point of $S \coloneqq \operatorname{Spec} k[\![u, v]\!]$. Then delete the two points where the strict transforms of the coordinate axes meet the exceptional curve, and denote the resulting scheme by $\widetilde{S}/S$ (cf. figure 2). This $\widetilde{S}/S$ is exactly the restriction of $\beta^{\leqslant 1}$ to S. The fibre over the closed point is isomorphic to $\mathbb{G}_m$.

Write $N_{1,2}$ for the universal Néron model over $\widetilde{\mathcal{M}}_{1,2}$, and $N^{\leqslant 1}_{1,2}$ for its restriction to $\widetilde{\mathcal{M}}^{\leqslant 1}_{1,2}$, The group scheme $N^{\leqslant 1}_{1,2}$ over $\widetilde{\mathcal{M}}^{\leqslant 1}_{1,2}$ is also easy to describe. We will abuse notation by writing $C_{1,2}$ for the pullback of $C_{1,2}$ to S, and similarly for $J_{1,2}$. Write U for the complement of the coordinate axes in S (equivalently, the locus where $C_{1,2}$ is smooth over S). Over U we have an Abel–Jacobi isomorphism $\alpha\colon C_{1,2}|_U \to J_{1,2}$ via one of the marked sections $\sigma \in C_{1,2}(S)$. Write $C^{\mathrm{sm}}_{1,2} \subseteq C_{1,2}$ for the locus where $C_{1,2} \to S$ is smooth (note that $C^{\mathrm{sm}}_{1,2}$ is larger than $C_{1,2}|_U$, as it includes smooth points in fibres which are not everywhere smooth). The pullback of $C^{\mathrm{sm}}_{1,2}$ to $\widetilde{\mathcal{M}}^{\leqslant 1}_{1,2}$ is again smooth, and is in fact isomorphic to $N^{\leqslant 1}_{1,2}$. More precisely, applying the Néron mapping property to α yields a map $C^{\mathrm{sm}}_{1,2} \to N^{\leqslant 1}_{1,2}$, which is an isomorphism. This map depends on the choice of the section σ, but the remainder of this discussion does not depend on that choice.

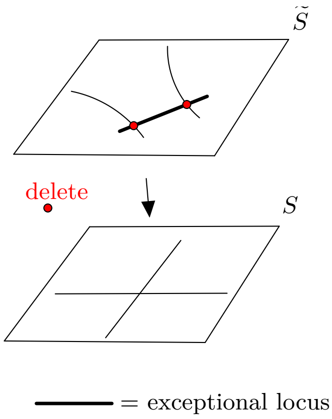

Figure 1.2. $\widetilde{S} \to S$

The above discussion gives the structure of $N^{\leqslant 1}_{1,2}$ as a scheme, but it remains to describe the group structure. The most concrete description in this setting is to check with a computer algebra package that the group structure on $J_{1,2}$ extends to $C^{\mathrm{sm}}_{1,2} \cong N^{\leqslant 1}_{1,2}$. In general, we construct Néron models as quotients of Picard spaces (moduli spaces of line bundles), and the group structure is given by tensor product of line bundles.

A typical test curve in S transversal to the boundary might be defined by the equation $u = \lambda v$ for some $\lambda \in k^\times$. Writing $f\colon T \to \operatorname{Spec} k[\![v,u]\!]$ for the inclusion, we see explicitly that f factors via $\beta^{\leqslant 1}$. The pullback of $N^{\leqslant 1}_{1,2}$ to T is identified with the pullback of $C^{\mathrm{sm}}_{1,2}$ to T. We therefore get a group structure on $f^*C^{\mathrm{sm}}_{1,2}$. By restriction, we also get a group structure on the restriction of $f^*C^{\mathrm{sm}}_{1,2}$ to the closed point of T, in other words on $C^{\mathrm{sm}}_{1,2}|_{u=v=0}$. Note that the latter scheme is independent of the choice of parameter $\lambda \in k^\times$. The crucial point is that the induced group structure on $C^{\mathrm{sm}}_{1,2}|_{u=v=0}$ *does* depend non-trivially on λ. This dependence can be made completely explicit; the scheme $C^{\mathrm{sm}}_{1,2}|_{u=v=0}$ is isomorphic to two copies of $\mathbb{G}_m$ with a marked point 1 on one of them, and the choice of $\lambda \in k^\times$ corresponds to the choice of an element to act as a square root of 1 on the other copy of $\mathbb{G}_m$. This means that it is *not* possible to put a group structure on the whole of $C^{\mathrm{sm}}_{1,2}$; if we want a group structure we must pull back to $\widetilde{\mathcal{M}}^{\leqslant 1}_{1,2}$ first.

1.6.3. Weakly-transversal test curves

For a moment we return to the case of general g and n, to discuss the pullback of the universal Néron model over test curves not transversal to the boundary. We have a smooth separated group algebraic space $N_{g,n}$ over $\widetilde{\mathcal{M}}_{g,n}$ extending $\beta^* J_{g,n}$, the Néron model of $\beta^* J_{g,n}$.

A *non-degenerate test curve* is a map from a trait to $\overline{\mathcal{M}}_{g,n}$ sending the generic point to $\mathcal{M}_{g,n}$. Any non-degenerate test curve $f\colon T \to \overline{\mathcal{M}}_{g,n}$ will factor via β, but we can see already from the 1-dimensional case that we cannot hope that $f^* N_{g,n}$ will be the Néron model of $f^* J_{g,n}$ for all such f. However, we have something nearly as good. We say f is *weakly transversal to the boundary* if the greatest common divisor of the thickness in the singularities of $f^* C_{g,n}$ is 1. This certainly includes the case of test curves transverse to the boundary, since these correspond to all the thicknesses being 1. We then find that for all test curves f which are weakly transversal to the boundary, we have that $f^* N_{g,n}$ is naturally isomorphic *as a group algebraic space* to the Néron model of $f^* J_{g,n}$. We omit the proof of this result; it is a simple generalisation of Lemma 12.6.

1.6.4. The case $g = 1$, $n = 2$ again

We continue in the notation of Section 1.6.1. A typical weakly-transversal test curve might be defined by the equation $u^n = \lambda v^m$ for $\lambda \in k^\times$ and m, n coprime positive integers. These m and n correspond exactly to the thickness of the two singularities in the curve $f^* C_{1,2}$. This means that the Néron model over T must have component group cyclic of order $m + n$.

This tells us that $\widetilde{\mathcal{M}}_{g,n}$ cannot be quasi-compact, since $N_{g,n}$ is a finite-type group space over it, but it has arbitrarily large cyclic groups appearing in the component group. In fact, $\widetilde{\mathcal{M}}_{g,n}$ can be described by iterating the blowup construction from Section 1.6, see Section 11 for more details.

1.6.5. Toric description

In terms of toric varieties, we can see S as (the completion of) the toric variety with fan $\mathbb{Q}^2_{\geqslant 0}$, and $\widetilde{S}^{\leqslant 1}$ as having fan being the union of the rays in $\mathbb{Q}^2_{\geqslant 0}$ through $(0, 1)$, $(1, 1)$ and $(1, 0)$. The fan of $\widetilde{S}$ consists of all rays in $\mathbb{Q}^2_{\geqslant 0}$ having rational slope, as is easily deduced from Section 11. This is (imprecisely) illustrated in Figure 1.3.

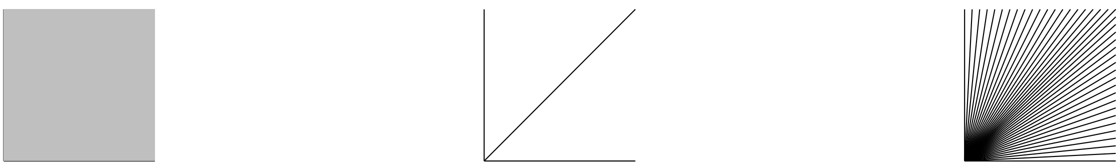

Figure 1.3. Fans of S, $\widetilde{S}^{\leqslant 1}$ and $\widetilde{S}$ respectively.

1.7. Outline of the construction

In [Hol19] we defined the condition of *alignment* on a family of prestable curves, and showed that this condition was closely related to the existence of a Néron model of the jacobian — in particular, if the family is aligned and the total space is regular, then a Néron model exists. In Section 2 we recall the definition, actually introducing a slight variant which will be more convenient for our construction (the difference is discussed in detail in Remark 2.6).

Our ultimate goal is to construct a universal stack over which a Néron model exists. Given a suitable family of prestable curves (it does not need to be the universal family), in Sections 3 to 7 we define and construct a universal stack over which the universal curve becomes aligned. This is carried out locally on the base (we spend quite some time carefully determining how "locally" we need to work), then glueing affine patches.

To show that the universal aligning scheme in fact admits a Néron model, we have to show that the universal aligning scheme is regular, and moreover that the universal curve over it admits a regular model which is still aligned. In Section 8 we show that, if the singularities of C/S are "mild enough", then the universal aligning scheme is indeed regular (in particular, this holds in the universal case). We also show that the construction of the universal aligning morphism can be made slightly more explicit in this situation.

In Section 9 we will show that, again if C/S has mild singularities, it is possible to resolve the singularities of the pulled-back curve over the universal aligning scheme. In Section 10 we will apply this to show that the universal aligning morphism is in fact a universal Néron model admitting morphism.

Section 11 contains a worked example, and in Section 13 we relate our results to some constructions of Caporaso from [Cap08].

In this paper, "algebraic stack" means "algebraic stack in the sense of [Sta13, Tag 026O]".

1.8. Acknowledgements

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2. Definition and basic properties of aligned curves

In this section we briefly recall the definitions from [Hol19]. For us "monoid" means "commutative monoid with zero"'. A graph has finitely many edges and vertices, and is allowed loops, and multiple edges between vertices.

Definition 2.1. — *A* cycle *in a graph is a path which starts and ends at the same point, and otherwise does not repeat any edge or vertex.*

Let G be a finite graph with edge set E and vertex set V. To any subset $E_0 \subseteq E$ we associate the unique subgraph of G with edges E_0 and with no isolated vertices — we will often fail to distinguish between E_0 and the subgraph. We say E_0 is cycle-connected *if for every pair e, e' of distinct edges in E_0 there is at least one cycle $\gamma \subseteq E_0$ such that $e \in \gamma$ and $e' \in \gamma$.*

Lemma 2.2. — *The maximal cycle-connected subsets of E form a partition of E.*

We write $\mathrm{Part}(G)$ for this partition. Before giving the proof we discuss a few examples.

(1) If G is a tree, then the elements of $\mathrm{Part}(G)$ consist of single edges (since there are no cycles).
(2) If G itself is a cycle then $\mathrm{Part}(G)$ has cardinality 1 (all the edges are in the same part).

Proof. — It suffices to show that if two subsets $\mathbb{E}$ and $\mathbb{E}'$ of E are both cycle-connected and both contain an edge e, then the union $\mathbb{E} \cup \mathbb{E}'$ is also cycle-connected. If a and b are distinct edges in $\mathbb{E} \cup \mathbb{E}'$, then let γ_a denote a cycle in $\mathbb{E} \cup \mathbb{E}'$ containing both a and e, and let γ_b be a cycle in $\mathbb{E} \cup \mathbb{E}'$ containing both b and e.

We will construct a cycle in $\mathbb{E} \cup \mathbb{E}'$ containing both a and b by "splicing" b into γ_a. Let p_0 and p_1 be the ends of b (necessarily distinct unless $a = b = e$). Let γ_0 be the shortest sub-path of γ_b which starts at p_0, does not contain b, and which meets γ_a (say at a point q_0). We define γ_1 and q_1 similarly: γ_1 is the shortest sub-path of γ_b which starts at p_1, does not contain b, and which meets γ_a, say at q_1. Let γ' denote the sub-path of γ_a which goes between q_0 and q_1 and which contains a. Then the union of γ', γ_0, γ_1 and b is a cycle containing a and b. ☐

We can also describe this partition in terms of 2-vertex-connected[(2)] subgraphs. For each loop l, $\{l\} \in \mathrm{Part}(G)$. For each maximal 2-vertex-connected subgraph H of the graph obtained from G by deleting loops, $\mathrm{Part}(G) \ni \mathrm{edges}(H)$ (cf. [Hol19, remark after Lemma 5.11]).

Definition 2.3. — *Let L be a monoid, G a graph, and ℓ a function assigning to each edge of G an element of L (we call (G, ℓ) a graph labelled by L). We say $P \in \mathrm{Part}(G)$ is* aligned *(with respect to ℓ) if there exist $l \in L$ and positive integers $n(e)$ for each edge $e \in P$, such that for all $e \in P$ the relation*

$$\ell(e) = l^{n(e)}$$

holds in L. We say G is aligned *(with respect to ℓ) if every $P \in \mathrm{Part}(G)$ is aligned.*

For example, if G is a tree (or if L is free on one generator) then (G, ℓ) is automatically aligned. On the other hand, if $L = \mathbb{N}^2$ and G is a 2-gon with edges labelled by $(0, 1)$ and $(1, 0)$ then (G, ℓ) is not aligned.

Now we turn to prestable curves, by which we mean proper, flat, finitely presented curves whose geometric fibres are connected and have at worst ordinary double point

(2) i.e. connected and remains connected after removing any one vertex.

singularities. These were called “semistable curves” in [Hol19]. We recall from [Hol19, Propositions 2.5 and 2.10] a result on the local structure of such curves:

Proposition 2.4. — *Let S be a locally noetherian scheme, C/S a prestable curve, s a geometric point of S, and c a geometric point of C lying over s. Then there exists an element α in the maximal ideal of the étale local ring $\mathcal{O}^{\mathrm{et}}_{S,s}$ and an isomorphism of complete local rings*

$$\frac{\widehat{\mathcal{O}^{\mathrm{et}}_{S,s}}[\![x,y]\!]}{(xy-\alpha)} \to \widehat{\mathcal{O}^{\mathrm{et}}_{C,c}}.$$

The element α is not in general unique, but, the ideal $\alpha\mathcal{O}^{\mathrm{et}}_{S,s} \lhd \mathcal{O}^{\mathrm{et}}_{S,s}$ is unique. We call it the singular ideal *of c. If C/S is smooth over a schematically-dense open of S then the singular ideal is never a zero-divisor.*

Definition 2.5. — *Let S be a locally noetherian scheme and C/S a prestable curve. Let $s \in S$ be a geometric point, and write Γ for the dual graph*[(3)] *of the fibre C_s. Let $\mathrm{Prin}(\mathcal{O}^{\mathrm{et}}_{S,s})$ be the monoid of principal ideals of $\mathcal{O}^{\mathrm{et}}_{S,s}$. We label Γ by elements of $\mathrm{Prin}(\mathcal{O}^{\mathrm{et}}_{S,s})$ by assigning to an edge $e \in \Gamma$ the singular ideal in $\mathrm{Prin}(\mathcal{O}^{\mathrm{et}}_{S,s})$ of the singular point of C_s associated to e. We denote this element by $\ell(e) \in \mathrm{Prin}(\mathcal{O}^{\mathrm{et}}_{S,s})$.*

We say C/S is aligned at s *if and only if this labelled graph is aligned. We say C/S* is aligned *if it is aligned at s for every geometric point $s \in S$.*

For example, a generically-smooth curve over a Dedekind base is always aligned (since the monoids of principal ideals are free on one generator), and curves of compact type are always aligned since the graphs contain no cycles. More detailed examples of aligned and non-aligned curves can be found in [Hol19, § 2.3].

Remark 2.6. — In [Hol19] we used a slightly different definition of alignment, namely we said a 2-vertex-connected graph (H, ℓ) was aligned if for all pairs of edges e, e' there existed positive integers n, n' such that $\ell(e)^n = \ell(e')^{n'}$. We said a graph G was aligned if every 2-vertex-connected subgraph was. When we want to distinguish between these notions we will call the version from [Hol19] “irregularly aligned” and the new version in this paper “regularly aligned”.

There are two differences between these definitions. One is superficial, the use of 2-vertex-connected subgraphs instead of the partition from Lemma 2.2. This makes no actual difference to the definition since the partition in Lemma 2.2 breaks the graph into maximal 2-vertex-connected subgraphs and loops, and the alignment condition is vacuous for loops. The other difference is more substantial, where we switch from imposing relations between labels to imposing that the labels admit a common multiplicative generator. If C/S is regularly aligned then it is irregularly aligned, and if S has factorial étale local rings (for example if S is regular) then the converse holds. We are mainly interested in the case where S is regular, so this is not of great importance. In this paper, we will construct a universal regularly-aligning morphism, and a Néron model over it. A universal irregularly-aligning morphism also exists, but the universal object is not in general regular, and so we cannot construct a Néron model over it (probably one does not exist).

(3) Defined as in for example [Liu02, 10.3.17].

Remark 2.7. — The notion of alignment is fppf-local on the target, i.e. it is preserved under flat base-change and satisfies fppf descent. We sketch a proof of this fact: let $R \to R'$ be a faithfully flat ring map, and let $r_1, r_2 \in R$ and $a, u_1, u_2 \in R'$ with u_i units and $r_i = u_i a^{n_i}$ for some $n_i \in \mathbb{Z}_{>0}$. Suppose the map

$$R \to R\left[t, u_1^{\pm 1}, u_2^{\pm 1}\right] / (r_1 - u_1 t^{n_1}, r_2 - u_2 t^{n_2})$$

becomes an isomorphism after base-change to R', then by faithful flatness it was an isomorphism.

Definition 2.8. — *Let S be a locally noetherian algebraic stack, and C/S a prestable curve. Let $S' \to S$ be a smooth cover by a scheme. We say that C/S is aligned if and only if $C \times_S S' \to S'$ is aligned. This makes sense by Remark 2.7.*

Remark 2.9. — We will make frequent use of the Picard spaces of our semistable curves. Let C/S be a prestable curve, then we write $\mathrm{Pic}_{C/S}$ for the sheafified relative Picard functor of C/S as in for example [BLR90, Chapter 8]; it is relatively representable by an algebraic space. We write $\mathrm{Pic}^0_{C/S}$ for the fibrewise connected component of identity (an open subgroup space), which coincides with the subspace of line bundles having degree 0 on every component of every geometric fibre. We write $\mathrm{Pic}^{[0]}_{C/S}$ for the open subgroup space of $\mathrm{Pic}_{C/S}$ corresponding to line bundles which have total degree 0 on every fibre. If C/S has connected geometric fibres then $\mathrm{Pic}^0_{C/S} \hookrightarrow \mathrm{Pic}^{[0]}_{C/S}$ is an isomorphism.

3. Universal aligning morphisms

Definition 3.1. — *Let S be a locally noetherian and reduced algebraic stack, and C/S a prestable curve. Let $U \hookrightarrow S$ be the largest open over which C is smooth.*

(1) *We say C/S is* generically smooth *if U is dense in S.*
(2) *Let $f \colon T \to S$ be a morphism of algebraic stacks. We say that f is* non-degenerate *if T is reduced and $U \times_S T$ is dense in T.*
(3) *Let $f \colon T \to S$ be a non-degenerate morphism of locally-noetherian stacks. We say f is* aligning *if the prestable curve $C \times_S T \to T$ is aligned in the sense of Definition 2.8.*
(4) *The category of aligning morphisms is defined as the full sub-2-category of stacks over S whose objects are aligning morphisms.*
(5) *A* universal aligning morphism *is a terminal object in the 2-category of aligning morphisms over S.*

Remark 3.2. —

(1) From now until the end of Section 7 we will be working to construct the universal aligning morphism. After that, we will show that it is actually the universal Néron model admitting morphism.
(2) We will show that the universal aligning morphism is in fact a separated algebraic space locally of finite type over S.

4. Quasisplit curves

In order to construct the universal aligning morphism by descent, we will make use of the notion of a "quasisplit" prestable curve. This differs from (though is somewhat similar to) the notion of a split prestable curve in [Jon96]; neither is stronger than the other.

DEFINITION 4.1. — *Let S be a scheme, and let C/S be a prestable curve. We write* $\mathrm{Sing}(C/S)$ *for the closed subscheme of C where $C \to S$ is not smooth (more precisely, it is the closed subscheme cut out by the first fitting ideal of the sheaf of relative 1-forms of C/S). We say C/S is* quasisplit *if the following two conditions hold:*

(4.1.1) *the morphism* $\mathrm{Sing}(C/S) \to S$ *is an immersion Zariski-locally on the source (for example, a disjoint union of closed immersions);*

(4.1.2) *for every field-valued fibre C_k of C/S, every irreducible component of C_k is geometrically irreducible.*

Remark 4.2. — Given a quasisplit prestable curve C/S and a geometric point $\bar{s}$ of S with image $s \in S$, the graph $\Gamma_{\bar{s}}$ depends only on s and not on $\bar{s}$. As such, it makes sense to talk about the dual graph of C_s for $s \in S$ a point. Moreover, the labels on such a graph, which a-priori live in the étale local ring at $\bar{s}$, are easily seen to live in the henselisation of the Zariski local ring, by condition 4.1.2. Applying condition 4.1.1, we see that the labels in fact live in the Zariski local ring at s. For the remainder of this paper, we will use this without further comment.

LEMMA 4.3. — *Let S be a locally noetherian scheme and C/S a prestable curve. Then there exists an étale cover $f\colon S' \to S$ such that $f^*C \to S'$ is quasisplit.*

Proof. — By [Sta13, Tag 04GL] and standard reductions, we deduce the existence of an étale cover over which the non-smooth locus is a disjoint union of closed immersions. Since the smooth locus admits sections étale locally, we can arrange a section through every irreducible component of every fibre; these irreducible components are then automatically geometrically irreducible. ☐

5. Specialisation maps between labelled graphs for quasisplit curves

DEFINITION 5.1. — *A morphism of graphs sends vertices to vertices, and sends edges to either edges or vertices (thinking of the latter as "contracting an edge"), such that the obvious compatibility conditions hold.*

A morphism of edge-labelled graphs

$$\phi\colon (\Gamma, \ell\colon \mathrm{edges}(\Gamma) \to L) \to (\Gamma', \ell'\colon \mathrm{edges}(\Gamma') \to L')$$

is a pair consisting of a morphism of graphs $\Gamma \to \Gamma'$ and a morphism of monoids from L to L' such that the labellings on non-contracted edges match up. An isomorphism is a morphism with a two-sided inverse.

Let S be a locally noetherian scheme, and let C/S be a quasisplit prestable curve. Let s, $\zeta \in S$ be two points such that ζ specialises to s (i.e. $s \in \overline{\{\zeta\}}$). Write Γ_s and Γ_ζ for the corresponding labelled graphs — recall from Remark 4.2 that this makes sense without choosing separable closures of the residue fields, and moreover that the labels of Γ_s and Γ_ζ may be taken to lie in the Zariski local rings at s and ζ respectively. Write

$$\mathrm{sp}\colon \mathcal{O}_{S,s} \hookrightarrow \mathcal{O}_{S,\zeta} \tag{5.1}$$

for the canonical (injective) map, which induces a map from principal ideals of $\mathcal{O}_{S,s}$ to principal ideals of $\mathcal{O}_{S,\zeta}$.

We will define a map of labelled graphs

$$\phi\colon \Gamma_s \to \Gamma_\zeta,$$

writing ϕ_V for the map on vertices and ϕ_E for the map on edges (the map on monoids is that induced by (5.1)). First we define the map on vertices. Let $v \in V_\zeta$ be a vertex of Γ_ζ. Then v corresponds to an irreducible component v of the fibre over ζ. Let $\mathscr{V}$ denote the Zariski closure of this component in C. Then $\mathscr{V} \times_S s$ is a union of irreducible components of C_s, call them $v_1, \cdots, v_n$. We define $\phi_V(v_i) = v$ for $1 \leqslant i \leqslant n$. To obtain a well-defined map $\phi_V\colon V_s \to V_\zeta$ we need to check that each irreducible component of C_s arises in this way from exactly one vertex of Γ_ζ. That every irreducible component of C_s arises from at least one component of C_ζ follows from the flatness of C/S. For the uniqueness, note that if the closures of two irreducible components of C_ζ both contain some irreducible component v of C_s then this would contradict the smoothness of $C \to S$ at the generic point of v.

Next we define the map on edges. Write $Z = \overline{\{\zeta\}} \subseteq S$ for the closure of ζ. Let $e \in E_s$ be an edge of Γ_s. Then there are exactly two possibilities[4]:

(Case 1) there exists an open neighbourhood $s \in Z^0 \subseteq Z$ and a unique section $\tilde{e}\colon Z^0 \to \mathrm{Sing}(C_{Z^0}/Z^0)$ such that $(\tilde{e})_s = e$. Then define $\phi_E(e) = (\tilde{e})_\zeta$;

(Case 2) Case 1 does not hold *and* (writing v_1, v_2 for the endpoints of e) we have that $\phi_V(v_1) = \phi_V(v_2)$. Then map e to $\phi_V(v_1)$.

Locally around e, the curve C_Z is the spectrum of $\mathcal{O}_Z[x,y]/(xy - l'(e))$ where $l'(e) \in \mathcal{O}_Z$ is a representative of $l(e)$. From this local structure we see that case 2 holds if and only if the label $l(e)$ becomes a unit at ζ; in other words, if and only if $l'(e)$ is non-zero in $\mathcal{O}_Z$.

Definition 5.2. — *We call the map $\Gamma_s \to \Gamma_\zeta$ constructed just above the* specialisation map.

A more intuitive description of the specialisation morphism $\phi\colon \Gamma_s \to \Gamma_\zeta$ on labelled graphs may be given as follows: starting with Γ_s, first replace each label by its image under sp. Then contract every edge whose label is a unit. This is exactly the labelled

[4] To see this, note first that a section in $C_{Z_0}(Z_0)$ as in Case 1 is unique if it exists. Suppose we have $\phi_V(v_1) \neq \phi_V(v_2)$. Then v_1 is contained in some irreducible component T_1 of C_Z, and v_2 is in a component T_2, with $T_1 \neq T_2$. In this situation observe that $e \in T_1 \cap T_2$, and (by considering the local structure of the singularities of a quasisplit prestable curve) we find that $T_1 \cap T_2$ is locally on Z a union of sections, so Case 1 must hold.

graph Γ_ζ. Given that such a simple description is available, why did we give the long-winded definition above? Essentially this is because it is otherwise not a-priori clear that the labelled graph resulting from this simple description is (naturally) isomorphic to the labelled graph Γ_ζ.

6. Controlled curves

Definition 6.1. — *Let C/S be a prestable curve. A point $\mathfrak{s} \in S$ is called a* controlling point *for C/S if all of the following conditions hold:*

(1) *S is affine and noetherian, and C/S is quasisplit;*
(2) *for each edge e of $\Gamma_\mathfrak{s}$, the intersection of $\ell(e)$ with $\mathcal{O}_S(S)$ inside the local ring $\mathcal{O}_{S,\mathfrak{s}}$ is a principal ideal;*
(3) *for every point s in S, there exists a point $\eta_s \in S$ such that*
   (a) *both s and $\mathfrak{s}$ are in the closure of η_s;*
   (b) *the specialisation map $\Gamma_s \to \Gamma_{\eta_s}$ is an isomorphism on the underlying graphs.*

The intuition behind this definition is that for every point $s \in S$, the labelled graph Γ_s should be obtained from the labelled graph $\Gamma_\mathfrak{s}$ by a suitable specialisation map. However, this cannot be exactly true, as in general $\mathfrak{s}$ will not lie in the closure of s. The specialisation is then mediated by the point η_s, which we can think of as the generic point of the stratum containing s, where S is stratified by the labels of edges.

By (2), we can think of the labels of $\Gamma_\mathfrak{s}$ as being principal ideals in $\mathcal{O}_S(S)$. We will generally reserve lowercase fraktur letters $\mathfrak{s}$ and $\mathfrak{t}$ for controlling points.

Definition 6.2. — *Let C/S be a prestable curve over a locally noetherian scheme. Let τ be in {smooth, étale}. A* controlled τ-cover *of C/S consists of a collection $(S_i, \mathfrak{s}_i)_{i \in I}$ of pointed schemes and a map $\bigsqcup_{i\in I} S_i \to S$ such that*

(1) *$\bigsqcup_{i\in I} S_i \to S$ is a cover in the τ-topology;*
(2) *for each i, the point $\mathfrak{s}_i$ is a controlling point for $C \times_S S_i$.*

Lemma 6.3. — *Let C/S be a quasisplit curve over a locally noetherian scheme, and $\mathfrak{s} \in S$ a point. Then there exists an open neighbourhood V of $\mathfrak{s}$ in S such that $\mathfrak{s}$ is a controlling point for C_V/V.*

Proof. — Shrinking S, we may assume that S is affine and that every label on the graph $\Gamma_\mathfrak{s}$ is generated by an element of $\mathcal{O}_S(S)$ (i.e. these locally-principal ideals are principal). Then delete from S every irreducible component that does not contain $\mathfrak{s}$.

Let Σ denote the smallest collection of (reduced) closed subsets of S which is closed under:

- pairwise intersections;
- taking irreducible components;

and which contains the image in S of $\mathrm{Sing}(C/S)$. Note that Σ is finite; each time we form intersections or take irreducible components we add in only finitely many new closed subsets. Since every infinite tree has an infinite branch, if Σ were infinite then we would find an infinite strictly decreasing chain of closed subsets, contradicting the fact that S is noetherian.

Now set

$$Z \coloneqq \bigcup \{\sigma \in \Sigma \,|\, \mathfrak{s} \notin \sigma\},$$

the union of all elements of Σ which don't contain $\mathfrak{s}$. Note this is a closed subset since Σ is finite. Let V denote the complement of Z in S.

Now let $s \in V$ be any point. Suppose first that Γ_s consists of a single vertex and no edges - this is equivalent to saying that s is not contained in any element of Σ, or to saying that C_s/s is smooth. Now the locus where C_V/V is smooth is open in V and is non-empty (since it contains s). Observe that every irreducible component of V contains $\mathfrak{s}$ by construction. Let η_s be the generic point of an irreducible component of V containing both s and $\mathfrak{s}$; then C_{η_s}/η_s is smooth and both $\mathfrak{s}$ and s are specialisations of η_s.

Suppose now that C_s/s is not smooth. Let $\sigma \in \Sigma$ be the smallest element of Σ which contains s. It is clear that σ must be irreducible, and that σ contains $\mathfrak{s}$. Let η_s be the generic point of σ. Then both $\mathfrak{s}$ and s are contained in the closure of η_s. It remains to see that the specialisation map

$$\mathrm{sp}\colon \Gamma_s \to \Gamma_{\eta_s}$$

is an isomorphism on the underlying graphs. This is equivalent to checking that no label of Γ_s is mapped to a unit in $\mathcal{O}_{S,\eta_s}$. Well, any label l on an edge of Γ_s which becomes a unit at η_s will cut out a proper closed subscheme of σ containing s, but this is impossible by the definition of σ. ☐

Combining Lemmas 4.3 and 6.3 yields

Lemma 6.4. — *Let C/S be a prestable curve over a locally noetherian base. Then there exists an étale controlling cover for C/S.*

7. Construction of universal aligning morphisms

7.1. The case of controlled curves

Throughout this section we fix a prestable curve C/S and a controlling point $\mathfrak{s} \in S$. The universal aligning morphism over S will be built by glueing together (infinitely many) affine patches. These affine patches will be indexed by *thickness functions*:

Definition 7.1. — *Let E denote the set of edges of the graph $\Gamma_{\mathfrak{s}}$. A* thickness function *is a function*

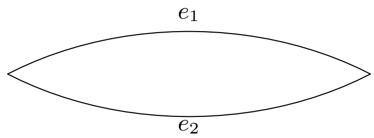
$$M\colon E \to \mathbb{Z}_{\geqslant 0}$$

satisfying the following condition:

Let Γ_M be the graph obtained from $\Gamma_{\mathfrak{s}}$ by contracting every edge e such that $M(e) = 0$. We then require that for each set $\mathbb{E} \in \mathrm{Part}(\Gamma_M)$, we have $\gcd(M(\mathbb{E})) = 1$.

This definition makes sense for any graph, but we will only apply it to the graph over a controlling point. We now give three examples of controlled curves and the possible thickness functions.

Example 7.2. — Let $S = \operatorname{Spec} \mathbb{C}[\![u,v]\!]$, and let $\mathfrak{s}$ be the closed point. Assume $\mathfrak{s}$ is a controlling point, and suppose the graph over $\mathfrak{s}$ is



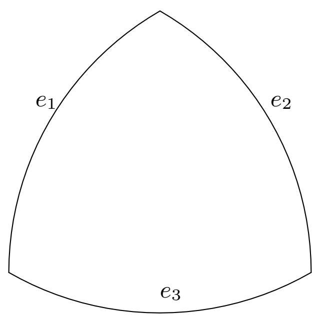
with $\ell(e_1) = (u)$, $\ell(e_2) = (v)$. Then thickness functions are exactly those functions which send the two edges to non-negative coprime integers, e.g.

$$M(e_1) = 2, \qquad M(e_2) = 3$$

or

$$M(e_1) = 0, \qquad M(e_2) = 1.$$

Example 7.3. — Let $S = \operatorname{Spec} \mathbb{C}[\![u,v,w]\!]$, and let $\mathfrak{s}$ be the closed point. Assume $\mathfrak{s}$ is a controlling point, and suppose the graph over $\mathfrak{s}$ is



with $\ell(e_1) = (u)$, $\ell(e_2) = (v)$ and $\ell(e_3) = (w)$. Then again the thickness functions are exactly those functions which send the three edges to non-negative integers with no common factor, e.g.

$$M(e_1) = 2, \qquad M(e_2) = 3, \qquad M(e_3) = 5$$

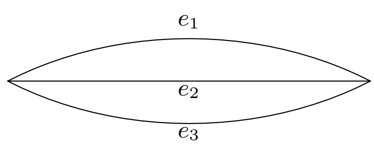
or

$$M(e_1) = 0, \qquad M(e_2) = 2, \qquad M(e_3) = 3$$

or

$$M(e_1) = 0, \qquad M(e_2) = 0, \qquad M(e_3) = 1.$$

Example 7.4. — Let $S = \operatorname{Spec} \mathbb{C}[\![u,v,w]\!]$, and let $\mathfrak{s}$ be the closed point. Assume $\mathfrak{s}$ is a controlling point, and suppose the graph over $\mathfrak{s}$ is



with $\ell(e_1) = (u)$, $\ell(e_2) = (v)$ and $\ell(e_3) = (w)$. Then every function M which sends the three edges to *positive* integers with no common factor is a thickness function, but if M sends one of the e_i to 0 then the other two e_j become loops in the graph Γ_M obtained by contracting e_i, and so must be sent to 0 or 1. Thickness functions are exactly those sending the e_i to positive coprime integers, and those sending (e_1, e_2, e_3) to some permutation of $(0, 1, 1)$, $(0, 0, 1)$ or $(0, 0, 0)$.

Now that we have thickness functions to index the affine patches of our universal aligning morphism, we will start to explicitly construct the affine patches themselves. We first set up some notation which we will use repeatedly for the remainder of this paper.

Setup 7.5. —

(1) *C/S is a prestable curve with controlling point $\mathfrak{s}$;*
(2) *We write $R = \mathcal{O}_S(S)$;*
(3) *$U \hookrightarrow S$ is the largest open in S over which C is smooth, and we assume U is dense in S;*
(4) *M is a thickness function, and Γ_M is the graph obtained from $\Gamma_{\mathfrak{s}}$ by contracting every edge e such that $M(e) = 0$;*

The next definition is crucial, as it gives us the building blocks for the affine patches of our universal aligning morphism.

Definition 7.6. — *Notation as in Setup 7.5, and let $\mathbb{E} \in \mathrm{Part}(\Gamma_M)$. Write $M(e) = m_e$, and write $\ell(e) \in R$ for some generator of the label of an edge e. Let $n_\star \colon \mathbb{E} \to \mathbb{Z}$ be a function such that $\sum_{e \in \mathbb{E}} n_e m_e = 1$ — this is possible by the coprimality condition. Define*

$$R'_{\mathbb{E}} = \frac{R\left[a, u_e^{\pm 1} : e \in \mathbb{E}\right]}{\left(\ell(e) - a^{m_e} u_e : e \in \mathbb{E},\ 1 - \prod_{e \in \mathbb{E}} u_e^{n_e}\right)}.$$

This depends on the n_e (so the notation is not good). However, this dependence will turn out not to matter (see Proposition 7.14). To simplify the notation, we will assume that a choice of n_e has been made once-and-for-all for every collection of m_e.

Remark 7.7. — For example, suppose we are in the setup of Section 1.6, so $S = \operatorname{Spec} R$ with $R = k[\![u, v]\!]$. The graph over the origin is a 2-gon, so the partition is $\{\{e_1, e_2\}\}$. Suppose we take the thickness function taking the value 1 on both edges. Then we can take $n_1 = 1$ and $n_2 = 0$, and the ring $R'_{\mathbb{E}}$ is given by

$$R'_{\mathbb{E}} = \frac{R\left[a, u_1^{\pm 1}, u_2^{\pm 1}\right]}{(u - au_1, v - au_2, 1 - u_1)} \cong \frac{R\left[a, u_2^{\pm 1}\right]}{(u - a, v - au_2)} \cong \frac{R\left[u_2^{\pm 1}\right]}{(v - uu_2)}$$

which is just an affine patch of the blowup of R at the origin, with a point deleted.

Lemma 7.8. — *Let $\mathbb{A}^{\mathbb{E}} = \operatorname{Spec} \mathbb{Z}[x_e : e \in \mathbb{E}]$. There is a natural map from S to $\mathbb{A}^{\mathbb{E}}$, corresponding to the ring homomorphism sending x_e to $\ell(e)$. Then $R'_{\mathbb{E}}$ is the coordinate ring of the pullback to S of the toric variety over $\mathbb{A}^{\mathbb{E}}$ whose fan is the ray in $\mathbb{Q}^{\mathbb{E}}_{\geqslant 0}$ through the point $(m_e : e \in \mathbb{E})$.*

Proof. — This is almost obvious from the definition, the only point where we must take care is that toric varieties (in the sense of [Ful93]) are by definition normal, so a-priori the toric variety associated to the ray spanned by the m_e is the normalisation of the spectrum of

$$\frac{\mathbb{Z}\left[x_e : e \in E\right]\left[a, u_e^{\pm 1} : e \in \mathbb{E}\right]}{\left(x_e - a^{m_e} u_e : e \in \mathbb{E},\ 1 - \prod_{e \in \mathbb{E}} u_e^{n_e}\right)}.$$

But the coprimality of the m_e guarantees that this ring is in fact already normal. □

This toric interpretation will be very useful in Section 8.1 when we come to show that these patches are non-singular.

Lemma 7.9. — *We have an R-algebra map*

$$\begin{aligned} = \phi\colon R'_{\mathbb{E}} &\to \mathcal{O}_U(U) \\ a &\mapsto \prod_{e \in \mathbb{E}} \ell(e)^{n_e} \\ u_e &\mapsto \ell(e)\phi(a)^{-m_e}. \end{aligned}$$

We call the element $a \in R'_{\mathbb{E}}$ the *aligning element* of $R'_{\mathbb{E}}$.

Proof. — Note that $\mathcal{O}_U(U)$ is a localisation of R and that each $\ell(e)$ is a unit in U. To check that the map is well-defined we need to show that it sends $1 - \prod_{e \in \mathbb{E}} u_e^{n_e}$ and each of the $\ell(e) - a^{m_e} u_e$ to zero. The latter is easy, the former requires a small calculation:

$$\begin{aligned} \phi\left(1 - \prod_{e \in \mathbb{E}} u_e^{n_e}\right) &= 1 - \prod_{e \in \mathbb{E}} \left(\ell(e)^{n_e} \phi(a)^{-m_e n_e}\right) \\ &= 1 - \left(\prod_{e \in \mathbb{E}} \ell(e)^{n_e}\right) \phi(a)^{\sum_{e \in \mathbb{E}} -m_e n_e} \\ &= 1 - \phi(a)\phi(a)^{-1} = 0. \end{aligned}$$

□

Definition 7.10. — *Notation as in Setup 7.5. Define*

$$R'_M = \bigotimes_{\mathbb{E} \in \mathrm{Part}(\Gamma_M)} R'_{\mathbb{E}}.$$

By Lemma 7.9 and the universal property of the tensor product we get a natural map

$$R'_M \to \mathcal{O}_U(U),$$

and we define R_M to be the image of R'_M under the above map. Set $S_M = \operatorname{Spec} R_M$ (in other words, S_M is the closure of the image of U in $\operatorname{Spec} R'_M$ under the given map). Write $a_{\mathbb{E}}$ for the image in R_M of the aligning element in $R'_{\mathbb{E}}$, then define a sequence $a := (a_{\mathbb{E}})_{\mathbb{E} \in \mathrm{Part}(\Gamma_M)}$ — this will be used in Lemmas 8.10 and 8.2.

Remark 7.11. — In the above definition, the procedure of "take the product, then take the image of the result in $\mathcal{O}_U(U)$' is essentially a way to "splice together" all the rings of relations $R'_{\mathbb{E}}$ without worrying about how the relations they capture are connected to each other. In general this image R_M is not very explicit (for example, it is not immediately apparent how to write down generators and relations for it, given the same for the $R_{\mathbb{E}}$). On the other hand, if the singularities of C/S are "mild enough" (for example, in the universal case) then we find that $R_M = R'_M$, see Remark 8.7 and Lemma 8.10.

Definition 7.12. — *Notation as in Setup 7.5. Let $f\colon T \to S$ be a non-degenerate morphism. We say f is* M-aligning *if for all $\mathbb{E} \in \mathrm{Part}(\Gamma_M)$ there exists $a_{\mathbb{E}} \in \mathcal{O}_T(T)$ such that for all $e \in \mathbb{E}$ we have $f^*\ell(e) = (a_{\mathbb{E}})^{M(e)}$ as ideals in $\mathcal{O}_T(T)$. We define the universal M-aligning morphism to S to be a terminal object in the category of M-aligning morphisms to S.*

Lemma 7.13. — *Notation as in Setup 7.5. Let $f\colon T \to S$ be an aligning morphism. Then there exists an étale cover $\bigsqcup_{i \in I} T_i \to T$, and for each i a thickness function M_i, such that each $T_i \to S$ is M_i-aligning.*

Proof. — By Lemma 4.3 we may assume T is quasisplit. Let $\mathfrak{t} \in T$ be any point. Shrinking T, we may assume by Lemma 6.3 that $\mathfrak{t}$ is a controlling point for f^*C/T. We are done if we can find a thickness function M such that, after possibly shrinking T further, we have that C_T/T is M-aligning.

Let L_{units} denote the set of labels of edges of $\Gamma_{\mathfrak{s}}$ which are units at $f(\mathfrak{t}) \in S$ — this is exactly the same as the set of labels of edges of $\Gamma_{\mathfrak{s}}$ which are contracted by the natural map to $\Gamma_{f(\mathfrak{t})}$ (cf. Definition 5.2). Because $\mathfrak{t}$ is controlling for f^*C/T, we see that L_{units} is also exactly the set of labels of edges of $\Gamma_{\mathfrak{s}}$ which pull back to units on T (recalling that we can think of the labels of $\Gamma_{\mathfrak{s}}$ as principal ideals in R). The thickness function M we will construct will take the value 0 exactly on edges in L_{units}.

Because f is aligning, we know that for each set $\mathbb{E} \in \mathrm{Part}(\Gamma_{\mathfrak{t}})$ there exists a principal ideal $a_{\mathbb{E}} \lhd \mathcal{O}_{T,\mathfrak{t}}$ such that every edge in $\mathbb{E}$ is labelled by some power of $a_{\mathbb{E}}$ - write $\ell(e) = a_{\mathbb{E}}^{m_e}$. Moreover, replacing $a_{\mathbb{E}}$ by some positive power, we may assume for each $\mathbb{E}$ that

$$\gcd\{m_e : e \in \mathbb{E}\} = 1.$$

Define $M(e) = m_e$ if $\ell(e) \notin L_{\text{units}}$, and $M(e) = 0$ otherwise.

Now shrinking T we may assume that each $a_{\mathbb{E}}$ stays a principal ideal in $\mathcal{O}_T(T)$, and moreover that for every $\mathbb{E}$ and every $e \in \mathbb{E}$, the relation $\ell(e) = a_{\mathbb{E}}^{M(e)}$ holds globally on T. ☐

Proposition 7.14. — *Notation as in Setup 7.5. The natural map $S_M \to S$ is a universal M-aligning morphism.*

Proof. — First we need to check that $f_S\colon S_M \to S$ is M-aligning. Fix $\mathbb{E} \in \mathrm{Part}(\Gamma_M)$, and let $a_{\mathbb{E}}$ be as in Definition 7.10. Then it is easy to see from the construction in Definition 7.6 that for every $e \in \mathbb{E}$ we have $f_S^*\ell(e) = (a_{\mathbb{E}})^{M(e)}$, so the definition of M-alignment is satisfied.

Now we need to show that any other M-aligning morphism $f\colon T \to S$ factors uniquely via $S_M \to S$. The uniqueness of a factorisation of f via S_M is clear as f is non-degenerate and $S_M \to S$ is affine and hence separated. For the existence, it suffices to construct a factorisation $\phi\colon T \to \operatorname{Spec} R'_M$; indeed, then $\phi(f^{-1}U) \subseteq S_M$, and $f^{-1}U$ is by assumption dense in T (which is reduced), hence ϕ factors via S_M.

Fix $\mathbb{E} \in \operatorname{Part}(\Gamma_M)$; by the universal property of the fibre product it is enough to construct a factorisation of f via $\operatorname{Spec} R'_{\mathbb{E}}$.

Write $R[1/\ell]$ for the ring obtained from R by inverting the labels of all edges of $\Gamma_{\mathfrak{s}}$. Then the canonical map $\mathcal{O}_T(T) \to \mathcal{O}_T(T) \otimes_R R[1/\ell]$ is injective since T is non-degenerate. Hence we can talk about whether various products of labels with integer exponents lie in $\mathcal{O}_T(T)$, even though $\mathcal{O}_T(T)$ may not be a domain.

By definition of f being M-aligning, there is an element $t \in \mathcal{O}_T(T)$ such that for all $e \in \mathbb{E}$, we have $f^*\ell(e) = (t)^{M(e)}$ as ideals in $\mathcal{O}_T(T)$. Choose integers n_e for $e \in \mathbb{E}$ such that $\sum_{e\in\mathbb{E}} M(e)n_e = 1$. Choose also for each e a generator $\ell(e) \in R$ of the label of e. Note that $\prod_{e\in\mathbb{E}} f^*\ell(e)^{n_e} \in \mathcal{O}_T(T)$, indeed it differs from t by multiplication by some unit. Similarly, $f^*\ell(e)/t^{M(e)}$ is a unit in $\mathcal{O}_T(T)$ for each e. We define an R-algebra map

$$\begin{aligned} R'_{\mathbb{E}} &\to \mathcal{O}_T(T)\\ a &\mapsto t\\ u_e &\mapsto f^*\ell(e)/t^{M(e)} \end{aligned}$$

which is well-defined by a verification similar to that in the proof of Lemma 7.9. This yields a map $T \to \operatorname{Spec} R'_{\mathbb{E}}$ as required. □

For a given thickness function M we have now constructed a universal M-aligning morphism $S_M \to S$. Our goal in the remainder of this section will be to glue together these S_M as M runs over different thickness functions, to construct a universal aligning morphism to S. The next definition and lemma will show how to glue.

Definition 7.15. — *Notation as in Setup 7.5. Let M, N be two thickness functions. Define*

$$\delta_M = \bigcup \Big\{ \mathbb{E} \in \operatorname{Part}(\Gamma_M) \ : \exists\, e \in \mathbb{E} \text{ with } M(e) \neq N(e) \Big\}$$

and

$$\delta_N = \bigcup \Big\{ \mathbb{E} \in \operatorname{Part}(\Gamma_N) \ : \exists\, e \in \mathbb{E} \text{ with } M(e) \neq N(e) \Big\}.$$

The set of edges of Γ_M is naturally a subset of the edges of Γ, and similarly for Γ_N. We then set $\delta_{M,N} = \delta_M \cup \delta_N$, with the union taken inside the set of edges of Γ. Then define

$$S_{M,N} = \operatorname{Spec}\left(R_M \left[\ell(e)^{-1} : e \in \delta_{M,N} \right] \right).$$

Here we are writing $R_M[\ell(e)^{-1} : e \in \delta_{M,N}]$ for the localisation of R_M at the principal ideals $\ell(e)$ as e runs over the edges in $\delta_{M,N}$. Formally these labels are principal ideals rather than elements of R, but we can effectively "invert" such a principal ideal by choosing a generator and inverting that. We see that $S_{M,N}$ is an open subscheme of S_M.

It is clear that $S_{M,N} \to S$ is M-aligning, since it factors via $S_M \to S$. Part of the point of the definition is that we also have:

Lemma 7.16. — *In the above notation:*

1. *The morphism $S_{M,N} \to S$ is N-aligning;*
2. *The induced map $S_{M,N} \to S_N$ factors via $S_{N,M} \to S_N$.*

Proof. —

(1) Let $\mathbb{E} \in \operatorname{Part}(\Gamma_N)$. We want to show that there exists an element $a \in R$ such that for all $e \in \mathbb{E}$, we have $\ell(e) = (a)^{N(e)}$. Well, if $M|_{\mathbb{E}} \neq N|_{\mathbb{E}}$ then $\mathbb{E} \subseteq \delta_{M,N}$ so we may take $a = 1$. On the other hand, if $M|_{\mathbb{E}} = N|_{\mathbb{E}}$ then there exists $\mathbb{E}_M \in \operatorname{Part}(\Gamma_M)$ such that $\mathbb{E} \subseteq \mathbb{E}_M$, and we may take the same aligning element a as works for $\mathbb{E}_M$.

(2) The map $S_{N,M} \to S_N$ is a localisation; more precisely, we invert exactly those $\ell(e)$ for which $e \in \delta_{N,M}$. For a morphism from a scheme T to S_N to factor via the open immersion $S_{N,M} \to S_N$ is exactly equivalent to checking that each of these $\ell(e)$ maps to a unit in $\mathcal{O}_T(T)$ (this is the universal property of localisation). We apply this principle to the morphism $S_{M,N} \to S_N$; we need to check that each $\ell(e)$ for $e \in \delta_{N,M}$ maps to a unit in $S_{M,N}$, but this is immediate since $\delta_{M,N} = \delta_{N,M}$, and since these labels all come by pullback from S.

□

From this lemma and symmetry we obtain a canonical isomorphism $S_{M,N} \xrightarrow{\sim} S_{N,M}$ for all pairs of thickness functions M, N. We can now define the universal aligning morphism (in Lemma 7.19 we will verify that it actually satisfies the universal property).

Definition 7.17. — *Notation as in Setup 7.5. Define $\beta\colon \widetilde{S} \to S$ to be the result of glueing together all the S_N as N runs over all thickness functions, along the open subschemes $S_{N,N'}$.*

For this glueing to make sense we must check a cocycle condition, see [Sta13, Tag 01JC]; the first part is immediate from the definition of the $\delta_{M,N}$, and the second follows from the fact that these patches are all reduced and map birationally to S.

Before verifying that $\widetilde{S} \to S$ has the universal property of the universal aligning morphism, we verify that it is separated.

Proposition 7.18. — *The map $\beta\colon \widetilde{S} \to S$ is separated.*

Proof. — The map $\beta\colon \widetilde{S} \to S$ is quasi-separated because $\widetilde{S}$ is locally noetherian, so it is enough to check the valuative criterion for separatedness. Let V denote the spectrum of a valuation ring, with generic point η and closed point v. Let $f, g\colon V \to \widetilde{S}$ be morphisms which agree on η and such that the composites with the canonical map $\widetilde{S} \to S$ agree — in particular, $f^*\ell(e) = g^*\ell(e)$ for every edge e of $\Gamma_{\mathfrak{s}}$. We will show that $f = g$, verifying the valuative criterion.

First, the S_N form an open cover of S as N runs over thickness functions, so there exist thickness functions M and N such that f factors via $S_M \to S$ and g factors via $S_N \to S$. If $M = N$ then the result is clear since $S_M \to S$ is affine and hence separated. Thus we may as well assume that $M \neq N$.

We will show that both f and g factor via $S_{M,N}$, which is affine over S and so $f = g$. To do this, we need to show that for every edge $e \in \delta_{M,N}$ we have that $f^*\ell(e)$ is a unit on V. Well, fix some $e_0 \in \delta_{M,N}$. Then (perhaps switching M and N) we may assume that there exists $\mathbb{E} \in \operatorname{Part}(\Gamma_M)$ such that $e \in \mathbb{E}$ and such that $N|_{\mathbb{E}} \neq M|_{\mathbb{E}}$ (and note that M does not vanish on any edge in $\mathbb{E}$). Since f is M-aligning (it factors via S_M), we know there exists $a \in \mathcal{O}_V(V)$ such that for all $e \in \mathbb{E}$, we have $f^*\ell(e) = (a)^{M(e)}$.

Observe that $f^*\ell(e_0) \neq 0$, otherwise we cannot have $f(\eta) \in S_{M,N}$, which contradicts the fact that f and g agree on η. Since $M(e_0) \neq 0$, this tells us that $a \neq 0$.

We now divide into two cases:

(Case 1) There exists $e' \in \mathbb{E}$ such that $N(e') = 0$. Then $g^*\ell(e')$ is a unit, so $a^{M(e')}$ is a unit, so a is a unit, so all the $f^*\ell(e)$ for $e \in \mathbb{E}$ are units as required.

(Case 2) N does not vanish on any $e \in \mathbb{E}$. Then there exists $\mathbb{E}_N \in \operatorname{Part}(\Gamma_N)$ such that $\mathbb{E} \subseteq \mathbb{E}_N$, so there exists $b \in \mathcal{O}_V(V)$ such that for all $e \in \mathbb{E}_N$ we have $g^*\ell(e) = (b)^{N(e)}$, so certainly the same holds for $\mathbb{E}$. Now since $M|_{\mathbb{E}} \neq N|_{\mathbb{E}}$ and M does not vanish on $\mathbb{E}$, there exist integers c_e for $e \in \mathbb{E}$ such that

$$d := \sum_{e \in \mathbb{E}} c_e M(e) \neq 0 \text{ and } \sum_{e \in \mathbb{E}} c_e N(e) = 0.$$

Moreover, a similar argument to that above tells us that $b \neq 0$. It is enough to show that a is a unit on V.

Pick a generator $\ell(e)$ of the principal ideal $\ell(e)$ for each $e \in \mathbb{E}$. Then we find that, up to multiplication by units on V, we have

$$\prod_{e \in \mathbb{E}} f^*\ell(e)^{c_e} = \prod_{e \in \mathbb{E}} a^{c_e M(e)} = a^d$$

and

$$\prod_{e \in \mathbb{E}} g^*\ell(e)^{c_e} = \prod_{e \in \mathbb{E}} b^{c_e N(e)} = 1.$$

The left hand sides are equal, so a^d is a unit on V, so a is a unit on V and we are done. □

Lemma 7.19. — *Notation as in Setup 7.5. Then $\beta\colon \widetilde{S} \to S$ is a universal aligning morphism for C/S. The map β is locally of finite presentation.*

Proof. — Let $f\colon T \to S$ be aligning. The uniqueness of a factorisation of f via β holds because f is non-degenerate and β is separated by Proposition 7.18. For existence, by Lemma 7.13 we can choose an étale cover $T' := \bigsqcup_i T_i \to Y$ and for each i a thickness function M_i such that $T_i \to S$ is M_i-aligning. By Lemma 7.14 each $T_i \to S$ factors via $S_{M_i} \to S$; in particular, the disjoint union $T'' \to T$ factors (uniquely) via the $\widetilde{S} \to S$. We need to descend this to a morphism $T \to \widetilde{S}$, which comes down to verifying that two morphisms $T' \times_T T' \to \widetilde{S}$ coincide. But this is immediate from the uniqueness observed above.

Local finite presentation holds because the S_M are clearly of finite presentation. □

7.2. Universal aligning morphisms: the general case

Theorem 7.20. — *Let C/S be a generically smooth prestable curve over a reduced locally noetherian algebraic stack. Then a universal aligning morphism for C/S exists, and is a separated algebraic space locally of finite presentation over S.*

Proof. — The stack S admits a smooth cover by a scheme (which is necessarily reduced and locally noetherian). A universal aligning morphism will descend along a smooth cover (as will the property of being an algebraic space) since it is defined by a universal property, and descent for algebraic spaces is effective. As such, it is enough to consider the case where S is a scheme. Similarly, by Lemma 6.4 we can assume that C/S has a controlling point $\mathfrak{s}$. The existence then follows from Lemma 7.19. Separatedness and local finite presentation follow by étale descent of those properties, and the same lemma. □

8. Regularity and normal crossings

In the previous section we built a universal aligning morphism for a generically smooth prestable curve. Next we will show that the universal aligning morphism is in fact the universal Néron model admitting morphism. While this will be formalised later, we present here the basic outline of the argument. Let $T \to S$ be a non-degenerate morphism and C/S a prestable curve. We know from [Hol19] that if the jacobian of C_T/T admits a Néron model then C_T/T is aligned, so $T \to S$ will factor uniquely via the universal aligning morphism $\widetilde{S} \to S$.

This is *not* enough, as we still need to show the existence of a Néron model over $\widetilde{S}$. To apply the results of [Hol19] to guarantee the existence of a Néron model for the jacobian of $C_{\tilde{S}}/\widetilde{S}$, we need that $C_{\tilde{S}}/\widetilde{S}$ is aligned *and* that $C_{\tilde{S}}$ is regular. The alignment is immediate from the defining property of $\widetilde{S}$, but regularity generally fails. We have two main tasks:

(1) Show that $\widetilde{S}$ is regular (otherwise we have little hope of making $C_{\tilde{S}}$ regular);
(2) Show that we can resolve the singularities of $C_{\tilde{S}}$ in such a way that $C_{\tilde{S}}$ remains aligned.

For both of these steps we will need to impose some more constraints on the singularities of C/S. Here we will consider two variants; one a "toric" version where we assume S comes with the structure of a smooth toric variety, and one an "absolute" version where we assume only that the non-smooth locus of C lies over a normal crossings divisor in S (these will be described more precisely in their respective sections). The absolute cases subsumes the toric one, so logically we are free to only treat the former. On the other hand, the proofs are very much simpler in the toric case, and this case is also sufficient to treat our application to the moduli space of curves. We treat also the absolute version since it seems interesting for example to apply these results to integral models of elliptic fibrations coming from K3 surfaces; but we recommend the reader to stick to the proof in the toric case, at least the first time around.

8.1. The toric case

In this section we work over a fixed regular base scheme Λ; for example, $\Lambda = \operatorname{Spec}\mathbb{C}$ or $\operatorname{Spec}\mathbb{Z}$. Notation as in Setup 7.5. Recalling that E is the set of edges of the graph

Γ, we set $\mathbb{A}^E = \Lambda[x_e : e \in E]$. There is then a natural map $\phi\colon S \to \mathbb{A}^E$ corresponding to mapping x_e to $\ell(e)$ (cf. Lemma 7.8). We say C/S is *toric* if the map ϕ is smooth.

Lemma 8.1. — *Fix* $\mathbb{E} \in \operatorname{Part}(\Gamma_M)$. *Then the scheme* $\operatorname{Spec} R'_{\mathbb{E}}$ *from Definition 7.6 is smooth over* Λ.

Proof. — From [Ful93, § 2.1] one deduces that an affine toric variety coming from a cone σ is smooth over Λ if and only if σ can be generated by a subset of some lattice basis. In this case our cone is a ray (Lemma 7.8), so this is automatic. Then $R'_{\mathbb{E}}$ is smooth over Λ, since the induced map to the affine toric variety associated to the ray is also smooth. □

Showing S_M to be regular needs slightly more work, as we take a fibred product and then a Zariski closure. However, the fibred product is not hard to handle, and we will see that the resulting scheme is smooth, so taking the closure is not necessary in this case.

For each $\mathbb{E} \in \operatorname{Part}(\Gamma_M)$ we have a vector $m_{\mathbb{E}} \coloneqq (m_e : e \in \mathbb{E}) \in \mathbb{Z}^{\mathbb{E}}_{\geqslant 0}$, whose span is the cone of the toric variety associated to $R'_{\mathbb{E}}$. Write $\mathbf{m}_{\mathbb{E}}$ for the unique vector in $\mathbb{Z}^{E}_{\geqslant 0}$ which projects to $m_{\mathbb{E}} \in \mathbb{Z}^{\mathbb{E}}_{\geqslant 0}$, and to the zero vector in every other coordinate direction. Then the span σ'_M of the vectors $\mathbf{m}_{\mathbb{E}} : \mathbb{E} \in \operatorname{Part}(\Gamma_M)$ is the cone associated to the ring R'_M. As in Lemma 7.8, this is almost obvious from the construction, but we should verify that the resulting monoid is saturated; this is so because its projection to each of the $\mathbb{Z}^{\mathbb{E}}_{\geqslant 0}$ is saturated. We see thus that $\operatorname{Spec} R'_M$ comes with a smooth map to a normal toric variety. In particular, the open orbit is dense, and so we see $R_M = R'_M$.

Lemma 8.2. — *The scheme* S_M *is smooth over* Λ, *and the sequence* $(a_{\mathbb{E}})_{\mathbb{E} \in \operatorname{Part}(\Gamma_M)}$ *from Definition 7.10 is a relative normal crossings divisor.*

Proof. — In Lemma 8.1 we verified that each $m_{\mathbb{E}}$ can be extended to a lattice basis for $\mathbb{Z}^{\mathbb{E}}_{\geqslant 0}$, write $B_{\mathbb{E}}$ for such a basis. Lifting each element of $B_{\mathbb{E}}$ to $\mathbb{Z}^{E}_{\geqslant 0}$ by filling in the other coordinates with zeros, and repeating for each element of the partition $\operatorname{Part}(\Gamma_M)$ yields a lattice basis for $\mathbb{Z}^{E}_{\geqslant 0}$ containing all the $\mathbf{a}_{\mathbb{E}}$. This verifies that S_M is smooth over Λ. The elements of the sequence $a_{\mathbb{E}}$ correspond to the vectors $\mathbf{a}_{\mathbb{E}}$; together these form a subset of a lattice basis and span the cone of S_M, hence they correspond to a relative normal crossings divisor. □

8.2. The absolute case

Definition 8.3 (Normal crossings singularities). — *Notation as in Setup 7.5. We say* C/S *has* normal crossings singularities *if the sequence* $(\ell(e) : e \in \Gamma_{\mathfrak{s}})$ *has normal crossings — in other words, if for every set* $J \subseteq \operatorname{edges}(\Gamma_{\mathfrak{s}})$, *we have that the closed subscheme*

$$V\left(\ell(e) : e \in J\right) \subseteq S$$

is regular and has codimension $\#J$ *in* S *at every point in that subscheme.*

Definition 8.4 (étale normal crossings singularities). — *Let* C/S *be a prestable curve over a locally noetherian scheme. We say* C/S *has* étale normal crossings singularities *if for every geometric point* $\bar{s}$ *of* S, *and for every subset* $J \subseteq \operatorname{edges}(\Gamma_{\bar{s}})$, *the closed subscheme*

$$V(\ell(e) : e \in J) \subseteq \operatorname{Spec} \mathcal{O}^{et}_{S,\bar{s}}$$

is regular and has codimension $\# J$.

Note that having étale normal crossings singularities is smooth-local on the target, and so makes sense when S is an algebraic stack. The Deligne–Mumford–Knudsen moduli stack of stable curves (over a regular base) is an example of a family of curves with étale normal crossings singularities.

Lemma 8.5. — *Let* C/S *be a prestable curve over an excellent scheme with étale normal crossings singularities. Then there exists an étale controlled cover* $\bigsqcup_{i\in I} S_i \to S$ *(with controlling points* $\mathfrak{s}_i \in S_i$*) such that the pullback of* C *to each* S_i *has normal crossings singularities.*

Proof. — This is clear by Lemma 6.3, the finiteness of the sets of edges, and the openness of the regular locus in an excellent scheme. ☐

Suppose C/S has étale normal crossings singularities. In particular, by taking the indexing set J to be empty, we deduce that S must be regular. It is then easy to check that C is also regular, by looking at the local equations at the non-smooth points. On the other hand, C can be regular without having étale normal crossings singularities — for example if S is a trait and C has multiple non-smooth points. Finally, note that the universal stable curve $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ has étale normal crossings singularities.

In the next lemma we will show some regularity properties for rings of the form we considered in Definition 7.6 while constructing the universal aligning morphism.

Lemma 8.6. — *Let* R *be a ring, and* $x_1, \cdots, x_d \in R$ *a collection of elements such that for all* $J \subseteq \{1, \cdots, d\}$, *the quotient*

$$R/\left(x_j : j \in J\right)$$

is regular. Let $m_1, \cdots, m_d$ *be non-negative integers with* $\gcd(m_1, \cdots, m_d) = 1$, *and let* $n_1, \cdots, n_d$ *be integers such that* $\sum_{i=1}^d m_i n_i = 1$. *Define*

$$R' = \frac{R\left[a, u_1^{\pm 1}, \cdots, u_d^{\pm 1}\right]}{\left(x_i - a^{m_i} u_i \,:\, (1 \leqslant i \leqslant d), 1 - \prod_i u_i^{n_i}\right)}.$$

Then R' *and* R'/a *are regular.*

Proof. — It is easy to check that R'/a is regular; it is even smooth over $R/(x_i : m_i > 0)$, which is regular by assumption. We need to show R' itself is regular; this will take more care, since it is not in general smooth over its image — it resembles an affine patch of a blowup.

Let $p \in \operatorname{Spec} R'$ be any point, and write q for the image of p in $\operatorname{Spec} R$. Localising R at q, we may assume that R is local, with closed point q. Re-ordering the x_i, we may assume that $x_1, \cdots, x_e \in q$ and $x_{e+1}, \cdots, x_d \notin q$ for some $0 \leqslant e \leqslant d$. Writing $D = \dim R$, our normal crossings assumptions imply that there exist $g_1, \cdots, g_{D-e} \in R$ such that

$$q = (x_1, \cdots, x_e, g_1, \cdots, g_{D-e}).$$

Now if $e = 0$ then we will now show that $R \to R'$ is an isomorphism, so the result is clear. Indeed, an inverse to the structure map $R \to R'$ is given by a map $f\colon R' \to R$ sending $a \mapsto \prod_i x_i^{n_i}$ and $u_i \mapsto x_i f(a)^{-m_i}$ (these make sense because $e = 0$ implies that the x_i are units). Checking that this is well-defined works just as in the proof of Lemma 7.9, and it is clear that the composite $R \to R' \to R$ is the identity. We need to check that $R' \to R \to R'$ is also the identity. It clearly sends u_i to u_i, and sends a to $\prod_i x_i^{n_i}$, so we need to check that $a = \prod_i x_i^{n_i}$ in R. Well,

$$\prod_i x_i^{n_i} = \prod_i (a^{m_i} u_i)^{n_i} = a^{\sum_i m_i n_i} \prod_i u_i^{n_i} = a^1 \cdot 1 = a.$$

It remains to treat the case $e \geqslant 1$. It then follows that $m_i = 0$ for every $e < i \leqslant d$, otherwise R'/qR' is empty, contradicting the existence of p. Again reordering, we may assume that $1 \leqslant m_1 \leqslant m_2 \leqslant \cdots \leqslant m_e$. We find that

$$R'/qR' = \frac{R/q\left[a, u_1^{\pm 1}, \cdots, u_d^{\pm 1}\right]}{(a^{m_1}, x_{e+1} - u_{e+1}, \cdots, x_d - u_d, 1 - \prod_{1 \leqslant i \leqslant d} u_i^{n_i})}.$$

We then see that

$$\frac{R'}{((a) + q)R'} = \frac{R/q\left[a, u_1^{\pm 1}, \cdots, u_d^{\pm 1}\right]}{(a, x_{e+1} - u_{e+1}, \cdots, x_d - u_d, 1 - \prod_{1 \leqslant i \leqslant d} u_i^{n_i})}$$

is regular and of dimension $e - 1$. From this we deduce that there exist elements $f_1, \cdots, f_{e-1} \in R'$ such that

$$\begin{aligned} p &= (a, f_1, \cdots, f_{e-1}, x_1, \cdots, x_e, g_1, \cdots, g_{D-e}) \\ &= (a, f_1, \cdots, f_{e-1}, g_1, \cdots, g_{D-e}), \end{aligned}$$

so p can be generated by D elements. Now it is clear that every irreducible component of R' has dimension at least D (count generators and relations), and hence it follows that R' is regular at p and has pure dimension D. □

Remark 8.7. — The reader should probably skip the next two Lemmas (8.8 and 8.9) together with part (2) of Lemma 8.10, at least at a first reading. Part (1) of Lemma 8.10 is all that is needed for the main results of the article, and the proof is quite a bit easier.

The point of Lemmas 8.8 and 8.9 is to allow us to prove part (2) of Lemma 8.10, which is used to make the construction of the universal Néron model admitting morphism more explicit. In Definition 7.10 there is an annoying step where we have to take the closure of the image of U to build the ring R_M from the ring R'_M (the latter being given by a nice presentation). What we get from part (2) of Lemma 8.10 is exactly that we can *skip* this step — the image of U is already dense in $\operatorname{Spec} R'_M$, so taking the closure does nothing (at least in the case of normal-crossings singularities).

Lemma 8.8. — *In the notation of Lemma 8.6, assume also that R is local, with maximal ideal $\mathfrak{m}$. Assume that at least one of the x_i lies in $\mathfrak{m}$. The following are equivalent:*

1. *$R'/\mathfrak{m}$ is connected;*
2. *$R'/\mathfrak{m}$ is non-zero;*
3. *for all $1 \leqslant i \leqslant d$, we have that $(x_i \in R^\times \implies m_i = 0)$.*

Proof. — The implication 1 $\implies$ 2 is from the definition of a connected ring. Suppose 3 fails, so (say) $x_1 \in R^\times$ and $m_1 \neq 0$, so a becomes a unit in $R'/\mathfrak{m}$. We also know some $x_i \notin R^\times$, say $x_d \notin R^\times$. Then in $R'/\mathfrak{m}$ we find that $0 = x_d = a^{m_d} u_d$ is a unit, so $R'/\mathfrak{m} = 0$. This shows 2 $\implies$ 3.

For 3 $\implies$ 1 we must work a little harder. A little more notation: let $k = R/\mathfrak{m}$, and assume that $x_1, \cdots, x_e \notin R^\times$ and that $x_{e+1}, \cdots, x_d \in R^\times$ for some $e > 0$. Then

$$\begin{aligned} R'/\mathfrak{m} &= \frac{k\left[a, u_1^{\pm 1}, \cdots, u_d^{\pm 1}\right]}{\left(u_1 a^{m_1}, \cdots, u_e a^{m_e}, x_{e+1} - u_{e+1}, \cdots, x_d - u_d, 1 - \prod_{i=1}^{d} u_i^{n_i}\right)} \\ &= \frac{k\left[a, u_1^{\pm 1}, \cdots, u_e^{\pm 1}\right]}{\left(a^{m_1}, \cdots, a^{m_e}, 1 - \prod_{i=1}^{e} u_i^{n_i} \prod_{i=e+1}^{d} x_i^{n_i}\right)}. \end{aligned}$$

Quotienting by (a) will not affect whether this ring is connected, so it is enough to show that

$$\frac{k[u_1^{\pm 1}, \cdots, u_e^{\pm 1}]}{\left(1 - \prod_{i=1}^{e} u_i^{n_i} \prod_{i=e+1}^{d} x_i^{n_i}\right)}$$

is connected. We will in fact show that this ring is a domain. Perhaps extending the field k, we can absorb the x_i. Moreover since $\sum_{i=1}^{d} m_i n_i = \sum_{i=1}^{e} m_i n_i = 1$ we know that n_i have no common factor. Perhaps swapping u_i and u_i^{-1} for some i, we may assume all $n_i \geqslant 0$. Moreover, the localisation of a domain is a domain. So it is enough to show the ring

$$\frac{k\left[u_1, \cdots, u_e\right]}{\left(1 - \prod_{i=1}^{e} u_i^{n_i}\right)}$$

is a domain, i.e. we must show $P := 1 - \prod_{i=1}^{e} u_i^{n_i}$ is irreducible.

Without loss of generality assume $n_1 \neq 0$. Think of P as a polynomial in x_1, and for any $i \neq 1$ write $\mathrm{NP}_{x_i}(P)$ for the Newton polygon of P with respect to the valuation coming from x_i. The single edge of the Newton polygon $\mathrm{NP}_{x_i}(P)$ has slope n_i/n_1, so we see that if h is a factor of P then $\deg_{x_1}(h)\frac{n_i}{n_1} \in \mathbb{Z}$, i.e.

$$\left.\frac{n_1}{\gcd(n_1, n_i)}\right| \deg_{x_1}(h).$$

Since the n_i have no common factor, this implies that $\deg_{x_1}(P) = n_1 \mid \deg_{x_1}(h)$, and we are done. $\square$

Lemma 8.9. — *In the notation of Lemma 8.6. Let $U \subseteq \operatorname{Spec} R$ be dense open such that every x_i is a unit on U. Define $\phi\colon U \to \operatorname{Spec} R'$ by*

$$\begin{aligned} R' &\to \mathcal{O}_U(U) \\ a &\mapsto \prod_i x_i^{n_i} \\ u_i &\mapsto x_i \phi(a)^{-m_i} \end{aligned}$$

(cf. Lemma 7.9). Then the image of ϕ is dense in $\operatorname{Spec} R'$.

Proof. — Without loss of generality, we may assume U is given by $U = \operatorname{Spec} R_0$ where $R_0 = R[1/x_i : 1 \leqslant i \leqslant d]$. Set $S = \operatorname{Spec} R$, and $S' = \operatorname{Spec} R'$. First, we want to show that the natural map $U \to S' \times_S U$ is an isomorphism, in other words that $R' \otimes_R R_0 \to R_0$ is an isomorphism. Since not all $m_i = 0$ we find that a becomes a unit in $R' \otimes_R R_0$, and the result then follows by elementary manipulations.

Now let $p \in S' \setminus U$ be a point, with image $q \in S$. So $q \notin U$, so some x_i is contained in q. Localising R at q, the hypotheses are preserved, but now we also have that R is local and R'/q is non-zero (since p exists). By Lemma 8.8 this implies that the closed fibre S'_q is connected. Write $\overline{\phi(U)}$ for the closure of the image of U in S'. I claim that the fibre $\overline{\phi(U)}_q$ is non-empty. Suppose for now that the claim is true. Then p lies in the same connected component of S' as $\phi(U)$. But S' is regular by Lemma 8.6, so every connected component of S' is irreducible, so $p \in \overline{\phi(U)}$ and the lemma is proven.

It remains to verify the claim that $\overline{\phi(U)}_q \neq \emptyset$. For this, let $f\colon T \to S$ be a map from the spectrum of a discrete valuation ring to S sending the closed point to q and such that for all $1 \leqslant i \leqslant d$ we have

$$\operatorname{ord}_T f^* x_i = m_i.$$

This is possible because the non-unit x_i form a regular sequence in R, and because

$$x_i \in R^\times \implies m_i = 0,$$

otherwise the point p could not exist (by Lemma 8.8).

We then find that $\prod_{i=1}^d (f^* x_i)^{n_i}$ is a uniformiser on T, and that for all i the element $f^* x_i / a^{m_i}$ is a unit on T. We can therefore lift the map f to a map $f'\colon T \to S'$ by

$$\begin{aligned} R' &\to \mathcal{O}_T(T) \\ a &\mapsto \prod_i (f^* x_i)^{n_i} \\ u_i &\mapsto f^* x_i a^{-m_i}. \end{aligned} \tag{8.1}$$

The image of the closed point of T under f' lies over q. The image of T under f' is integral, and the image of the generic point lies over U, so the closure of $\phi(U)$ has non-empty fibre over q as required. □

Lemma 8.10. — *Notation as in Setup 7.5, and assume C/S has normal crossings singularities. For each $\mathbb{E} \in \operatorname{Part}(\Gamma_M)$, let $a_{\mathbb{E}}$ denote the image of the aligning element of $R'_{\mathbb{E}}$ in R'_M. Then*

(1) *The sequence* $(a_{\mathbb{E}} : \mathbb{E} \in \mathrm{Part}(\Gamma_M))$ *form a normal-crossings divisor in* R'_M *(in particular they are distinct);*
(2) *We have* $R_M = R'_M$.

Proof. — If $\mathrm{Part}(\Gamma_M)$ has only one element then (1) follows immediately from Lemma 8.6, and (2) from Lemma 8.9. Fortunately the general case can be treated with similar ease, only the notation is worse. Choose an ordering on $\mathrm{Part}(\Gamma_M)$, say $\mathrm{Part}(\Gamma_M) = \{\mathbb{E}_1, \cdots, \mathbb{E}_r\}$. Set $R_0 = R$, and $R_i = \bigotimes_{j=1}^{i} R'_{\mathbb{E}_j}$. We will prove the claim by induction on i. The case $i = 0$ is exactly the assumption that C/S has normal crossings singularities (in particular U is dense in R). Suppose we know the result for some R_i, then claim 1 (respectively claim 2) for R_{i+1} is exactly what we get by applying Lemma 8.6 (respectively Lemma 8.9) to the ring R_i and the sequence $a_{\mathbb{E}_1}, \cdots, a_{\mathbb{E}_i}$, with suitably chosen $m_\star$ and $n_\star$. ☐

Theorem 8.11. — *Notation as in Setup 7.5, and assume C/S has normal crossings singularities. Let $\beta\colon \widetilde{S} \to S$ be the universal aligning morphism. Then $\widetilde{S}$ is regular. Moreover, the set $\{\beta^*\ell(e) : e \in \mathrm{edges}(\Gamma_{\mathfrak{s}})\}$ has normal crossings in $\widetilde{S}$, i.e. for every $J \subseteq \mathrm{edges}(\Gamma_{\mathfrak{s}})$, the underlying reduced subscheme of*

$$V\left(\beta^*\ell(e) : e \in J\right) \subseteq \widetilde{S}$$

is regular.

Proof. — The claim is local on $\widetilde{S}$, so we can fix a thickness function M and check the claim on S_M. The result is then immediate from Lemma 8.10 (or Lemma 8.2 for the toric case, in which case "regular" can be replaced by "smooth over $\mathbb{Z}$"). ☐

9. Resolving singularities over the universal aligning scheme

For the motivation behind the results in this section we refer to the discussion at the start of Section 8. In brief, in order to apply the results of [Hol19] to prove the existence of a Néron model for the jacobian of the universal curve over the universal aligning scheme, we need the curve to have a *regular* aligned model. Alignment is already built in; in this section we prove that we can resolve the singularities to give a regular model, without disturbing the alignment.

Definition 9.1. — *Let S be a scheme, $U \subseteq S$ a dense open, and C/U a smooth proper curve. A* model *for C/U is a proper flat morphism $\bar{C} \to S$ together with an isomorphism $\bar{C} \times_S U \xrightarrow{\sim} C$. This isomorphism will often be suppressed in the notation.*

Lemma 9.2. — *Notation as in Setup 7.5, and assume C/S has normal crossings singularities. Then the pull-back of C_U to $S_M = \operatorname{Spec} R_M$ has a prestable* regular *aligned model.*

Note that C is regular since C/S has normal crossings singularities.

Proof. — Let $f\colon S_M \to S$ be the structure map. Note that S_M is regular by Theorem 8.11. Write $C_0 = C \times_S S_M$, this is an aligned prestable curve. We will resolve the singularities of C_0 by blowing up, taking care to preserve alignment as we do so.

For $i = 0, 1, \cdots$, we define $Z_i \subseteq C_i$ to be the reduced closed subscheme where C_i is not regular, and then define C_{i+1} to be the blowup of C_i at Z_i.

It is enough to show:

(1.1) each C_i is a prestable curve over S_M;
(1.2) some C_N is regular;
(1.3) each C_i is aligned over S_M.

Let $x \in \mathcal{O}_S(S)$ be a label of an edge in $\Gamma_{\mathfrak{s}}$. From part (1) of Lemma 8.10 we see that x is a power of a (unique) irreducible element x', and that $V(x')$ is a regular subscheme of S_M. The curve C_0 is the regular at the corresponding point if and only if $x = x'$, and a local calculation shows that the blowup is again a prestable curve, whose singularities have the same simple structure. Hence each C_i is a prestable curve. This establishes (1.1).

Write P for the (finite) set of generic points of the union of the closed subschemes $V(f^*x)$ as x runs over labels of $\Gamma_{\mathfrak{s}}$. Note that each $\mathfrak{p} \in P$ is a codimension 1 point on the regular scheme S_M, and moreover (again by Theorem 8.11) that the quotient $R/\mathfrak{p}$ is also regular. Write $\operatorname{ord}_{\mathfrak{p}}$ for the corresponding $(\mathbb{Z} \cup \{\infty\})$-valued discrete valuation on the local ring at $\mathfrak{p}$. Given $i \geqslant 0$, define a non-negative integer

$$\delta_i = \sum_{\mathfrak{p} \in P} \sum_{\substack{q \in C_i \\ \text{singular}}} (\operatorname{ord}_{\mathfrak{p}} \ell(q) - 1)\,;$$

here the second sum runs over non-smooth points q in the fibre over $\mathfrak{p}$. Such a point corresponds to an edge of the graph $\Gamma_{\mathfrak{p}}$, and $\ell(q)$ denotes the label of the edge of the dual graph; $\ell(q)$ lives in the étale local ring at $\mathfrak{p}$. It is enough to show:

(2.1) if $\delta_i > 0$ then $\delta_i > \delta_{i+1}$;
(2.2) if $\delta_i = 0$ then C_i is regular;
(2.3) each C_i is aligned (this holds for C_0 by assumption).

We first show item (2.1). Let i be such that $\delta_i > 0$. Let $\mathfrak{p} \in P$, and let $q \in C_i$ be a non-smooth point lying over $\mathfrak{p}$, with $\operatorname{ord}_{\mathfrak{p}} \ell(q) = t$. Let p be a generator for $\mathfrak{p}$. The completed local ring on C_i at q is given by

$$\widehat{\mathcal{O}}_{C_i,q} \cong \frac{\widehat{\mathcal{O}}_{S_M,\mathfrak{p}}[\![u,v]\!]}{(uv - p^t)}.$$

Assume $t \geqslant 2$ (this holds for some $\mathfrak{p}$ and q, otherwise $\delta_i = 0$). The blowup of $\widehat{\mathcal{O}}_{C_i,q}$ at $q = (u, v, p)$ has three affine patches:

(2.1.1) "$u_i = 1$', given by:

$$\frac{\widehat{\mathcal{O}}_{S_M,\mathfrak{p}}[\![u,v]\!][v_1,p_1]}{(v_1 - u^{t-2}p_1^t, v - uv_1, p - up_1)}$$

which is regular (a calculation, or cf. [Jon96, § 3.4]);

(2.1.2) "$v_1 = 1$", which is also regular by symmetry;

(2.1.3) "$p_1 = 1$", given by

$$\frac{\widehat{\mathcal{O}}_{S_M,\mathfrak{p}}[\![u,v]\!][u_1,v_1]}{(u_1v_1 - p^{t-2}, u - pu_1, v - pv_1)}.$$

This patch is regular if $t = 2$. If $t > 2$ then the patch is regular except at $q' := (u_1, v_1, p)$. In the latter case we see that $\operatorname{ord}_\mathfrak{p} q' = t - 2$; it has dropped by 2.

As such, we see that at most one non-regular point q' of C_{i+1} maps to q, and if q' exists we have $\operatorname{ord}_\mathfrak{p} q' = \operatorname{ord}_\mathfrak{p} q - 2$. This shows that $\delta_{i+1} < \delta_i$.

Next we show item (2.2). Let i be such that $\delta_i = 0$. Let $c \in C_i$ lie over $s \in S_M$. We will show C_i is regular at c. If $C_i \to S_M$ is smooth at c we are done, so assume this is not the case. Then the completed local ring of C_i at c is given by

$$\frac{\widehat{\mathcal{O}}_{S_M,s}[\![u,v]\!]}{(uv - \ell_c)}$$

where $\ell_c \in \mathcal{O}_{S,s}$ is an element which (by definition) generates the label of the graph Γ_s at the edge e_c corresponding to the point c.

Let $\mathfrak{p} \in P$ be such that $\ell_c \in \mathfrak{p}$ (such $\mathfrak{p}$ exists by construction). The specialisation map

$$\Gamma_s \to \Gamma_\mathfrak{p}$$

does not contract e_c; rather it sends it to an edge labelled by the ideal generated by $\mathrm{sp}(\ell_c)$, where

$$\mathrm{sp}\colon \mathcal{O}_{S_M,s} \to \mathcal{O}_{S_M,\mathfrak{p}}$$

is the specialisation map. By our assumption that $\delta_i = 0$, it follows that $(\mathrm{sp}\,\ell_c) = \mathfrak{p}$, so the closed subscheme $V(\ell_c) \subseteq \operatorname{Spec} \mathcal{O}_{S_M,s}$ is regular.

Write $\dim \mathcal{O}_{S_M,s} = d$. By the above regularity statement, we can find elements $g_1, \cdots, g_{d-1}$ such that

$$\mathfrak{m}_s = (\ell_c, g_1, \cdots, g_{d-1}).$$

Hence the ideal corresponding to c can be generated by

$$(u, v, \ell_c, g_1, \cdots, g_{d-1}) = (u, v, g_1, \cdots, g_{d-1}),$$

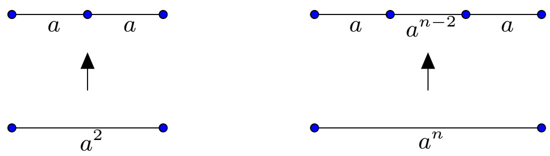
in other words it can be generated by $d + 1$ elements. Since $\dim_c C_M = d + 1$, this proves that C_M is regular at c.

Finally, we show item (2.3). We proceed by induction on i. For $i = 0$ the result is one of our starting assumptions. Let $i \geqslant 1$, and assume the result for $i - 1$. Let $s \in S_M$ be any point, and for each j let Γ_s^j be the graph of C_j over s. Then the labelled graph Γ_s^i can be constructed from the labelled graph Γ_s^{i-1} by the following recipe:

(1) for each edge e such that $\ell(e) = a^2$ for some irreducible element $a \in \mathcal{O}^{\mathrm{et}}_{S_M,s}$, replace e by two edges both with label a;

(2) for each edge e such that $\ell(e) = a^n$ for some irreducible element $a \in \mathcal{O}^{\mathrm{et}}_{S_M,s}$ and integer $n > 2$, replace e by three edges with labels a, a^{n-2} and a in that order.

In pictures:



Our induction hypothesis is that the graph Γ_{i-1} is aligned; equivalently, for every part $P \in \mathrm{Part}(\Gamma_{i-1})$ there exists $\ell(P) \in \mathcal{O}_{S_M, s}$ and positive integers $n(e)$ for $e \in P$, such that for all $e \in P$ we have $\ell(e) = \ell(P)^{n(e)}$. By the same argument as in the proof of item (1.1) we may and do assume that $\ell(P)$ is irreducible.

We need to show that Γ_i is aligned, so let $P' \in \mathrm{Part}(\Gamma_i)$. Then either P' consists of a single edge (in which case there is nothing to check), or P' is obtained by subdivision from some part $P \in \mathrm{Part}(\Gamma_{i-1})$. In that case, the element $\ell(P)$ will satisfy the conditions of alignment for P'. Hence Γ_i is aligned as required. $\square$

Remark 9.3. — The author is grateful to Giulio Orecchia for pointing out that the above construction of a regular model is an entirely canonical procedure, and commutes with smooth base-change. Indeed, the non-regular locus is stable under smooth base change, and the blowups are stable under flat base change. Combining this observation with étale descent, we obtain a canonical desingularisation $\widetilde{C}_{g,n}$ of the universal stable curve over the universal Néron-model-admitting-morphism $\widetilde{\mathcal{M}}_{g,n}$ (see Corollary 10.5).

10. Existence of Néron models over universal aligning schemes

Definition 10.1. — *Let C/S be a generically smooth prestable curve over a regular algebraic stack, and write $U \subseteq S$ for the largest open substack over which C is smooth. Write J for the jacobian of C_U/U; this is an abelian scheme over U. A* Néron-model-admitting morphism *for C/S is a morphism $f\colon T \to S$ of algebraic stacks such that:*

(10.1.1) *T is regular;*
(10.1.2) *$U \times_S T$ is dense in T;*
(10.1.3) *f^*J admits a Néron model over T.*

We define the category of Néron-model-admitting morphisms as the full sub-2-category of the 2-category of stacks over S whose objects are Néron-model-admitting morphisms.

A universal Néron-model-admitting morphism *for C/S is a terminal object in the 2-category[5] of Néron-model-admitting morphisms; it is unique up to isomorphism unique up to unique 2-isomorphism if it exists. We write $\beta\colon \widetilde{S} \to S$ for a universal Néron-model-admitting morphism, and $\widetilde{N}/\widetilde{S}$ for the Néron model of J.*

[5] Since this is a 2-category, care must be taken with terminal objects. However, since our categories are fibred in groupoids there is no ambiguity in the definition of terminal object. All our proofs will be by descent from the case of schemes, so this issue will also not arise in the proofs.

Remark 10.2. — Note that $\widetilde{N}$ is a smooth, separated group algebraic space over $\widetilde{S}$ (if the latter exists).

Theorem 10.3. — *Let S be an algebraic stack locally of finite type over an excellent scheme. Let C/S be a prestable curve such that C/S has étale normal crossings singularities (in particular, C is generically smooth; write $U \hookrightarrow S$ for the largest open over which C is smooth). Let $\beta\colon \widetilde{S} \to S$ denote the universal aligning morphism (note β is an isomorphism over U). Then $\widetilde{S}$ is a universal Néron-model-admitting morphism for C/S, and moreover the Néron model $\widetilde{N}$ is of finite type over $\widetilde{S}$, and its fibrewise-connected-component-of-identity is semi-abelian.*

Note that the universal aligning morphism exists as an algebraic space over S by Theorem 7.20. Before giving the proof we briefly recall the key theorem of [Hol19] on which we depend:

Theorem 10.4. — *Let S be a regular algebraic scheme, and $U \hookrightarrow S$ a dense open subscheme. Let C/S be a prestable curve, smooth over U. We have:*

1. *if the jacobian J/U admits a Néron model over S then C/S is aligned;*
2. *if C is regular and C/S is aligned then the jacobian J/U admits a Néron model over S.*

Moreover, this Néron model is of finite type over S, and its fibrewise-identity-component is a semi-abelian scheme.

Proof. — This is essentially [Hol19, Theorem 1.2], but with some small modifications and additions:

1. Our notion of alignment is slightly different from that in [Hol19], but when the base is regular these notions coincide; see Remark 2.6;
2. That the fibrewise-identity-component is a semi-abelian scheme follows from the third point in [Hol19, Remark 6.3];
3. That the Néron model is of finite type is the main result of [Hol17]. □

Proof of Theorem 10.3. — Let $f\colon S' \to S$ be a smooth cover by a scheme, then f^*C/S' has étale normal crossings singularities, and S' is excellent because it is locally of finite type over an excellent scheme. So by Lemma 8.5 we find that S has a smooth cover by schemes S_i for i in some indexing set I, such that each $C_{S_i} \to S_i$ has a controlling point $\mathfrak{s}_i$ and normal crossings singularities. Formation of the universal aligning morphism $\widetilde{S} \to S$ commutes with smooth base-change[(6)], so by Theorem 8.11 we find that $\widetilde{S}$ is regular, and by Lemma 9.2 we find that $\widetilde{S}$ has a smooth cover by schemes $S_{i,M}$ such that on each $S_{i,M}$ the curve $C_{S_{i,M}}$ has a prestable regular aligned model.

By item (2) of Theorem 10.4 we then know that the jacobian of C_U/U has a Néron model over each $S_{i,M}$ (changing the model of the curve does not change the jacobian over U, so the existence of a regular aligned model implies that the jacobian admits a Néron model). By the same result we know that this Néron model is of finite type and has semi-abelian fibrewise-connected-component-of-identity. Néron models

(6) Even fppf base-change, since alignment is fppf local (see Remark 2.7).

descend along smooth covers by [Hol19, Lemma 6.1], so a Néron model exists over $\widetilde{S}$ (and the properties mentioned also descend). This shows that $\beta\colon \widetilde{S} \to S$ is a Néron model admitting morphism.

Finally, we need to show that any other Néron-model-admitting morphism factors via $\widetilde{S}$. Let $f\colon T \to S$ be a Néron-model-admitting morphism. Since T is reduced and β is separated, a factorisation of f via β will descend along a smooth cover of T; as such, (and using that a scheme smooth over a regular base is regular) we may assume T is a scheme. Then by item (1) of Theorem 10.4 and the fact that f^*C_U has a Néron model over T, we know that f^*C/T is aligned, and hence f factors uniquely via the universal aligning morphism as required.

From now on, we work relative to a fixed base scheme Λ which we assume to be regular and excellent — the basic examples to keep in mind are $\operatorname{Spec}\mathbb{Z}$ and the spectrum of a field. Let g, n be non-negative integers such that $2g-2+n>0$. We write $\mathcal{M}_{g,n}$ for the moduli stack of smooth proper connected n-pointed curves of genus g over Λ, and $\overline{\mathcal{M}}_{g,n}$ for its Deligne–Mumford–Knudsen compactification. By [Knu83, Theorem 2.7] we know that $\overline{\mathcal{M}}_{g,n}$ is a smooth proper Deligne–Mumford stack over Λ, and the boundary $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ is a divisor with normal crossings relative to Λ in the sense of [DM69, Definition 1.8]. We write $J_{g,n}$ for the jacobian of the universal curve $\overline{\mathcal{M}}_{g,n+1} \times_{\overline{\mathcal{M}}_{g,n}} \mathcal{M}_{g,n}$ over $\mathcal{M}_{g,n}$; this is an abelian scheme over $\mathcal{M}_{g,n}$.

A straightforward application of the previous theorem (together with Lemma 8.2 for the smoothness) now yields our main result about the Néron model of the universal jacobian.

Corollary 10.5. — *A universal Néron-model-admitting morphism for $\overline{\mathcal{M}}_{g,n}$ exists and is an algebraic space locally of finite type over $\overline{\mathcal{M}}_{g,n}$, and is smooth over Λ. The Néron model over it is of finite type, and its fibrewise-connected-component-of-identity is semi-abelian.*

Proposition 10.6. — *These objects also satisfy the following properties:*

(1) *the morphism $\beta\colon \widetilde{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is separated and locally of finite presentation;*
(2) *the morphism $\beta\colon \widetilde{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ satisfies the valuative criterion for properness for morphisms from traits to $\overline{\mathcal{M}}_{g,n}$ which map the generic point of the trait to $\mathcal{M}_{g,n}$.*

Proof. —

(1) Follows from the same properties for the universal aligning scheme (Theorem 7.20) and the fact that these properties descend along fppf morphisms.
(2) Immediate from the definition of the universal Néron-model-admitting scheme because Néron models always exist over traits (alternatively, because generically smooth prestable curves over traits are always aligned).

11. A worked example

We return to the example of a non-aligned prestable curve considered in Section 1.6. We compute its universal aligning scheme, and describe the Néron model of its jacobian over the universal aligning scheme.

Construct a stable 2-pointed curve over $\mathbb{C}$ by glueing two copies of $\mathbb{P}^1_{\mathbb{C}}$ at $(0:1)$ and $(1:0)$, and marking the point $(1:1)$ on each copy. Then define C/S to be the universal deformation as a 2-pointed stable curve. Choose coordinates such that $S = \operatorname{Spec}\mathbb{C}[\![x,y]\!]$, and C is smooth over the open subset $U = D(xy)$. Call the sections p and q.

Now the graph over the closed point of S is a 2-gon, with one edge labelled by (x) and the other by (y). The graph over the generic point of $(x=0)$ is a 1-gon with edge labelled by (y), and similarly the graph over the generic point of $(y=0)$ is a 1-gon with edge labelled by (x). All other fibres are smooth. In particular, C/S is aligned except at the closed point, which is a controlling point.

11.1. The universal aligning morphism

We now describe the universal aligning morphism. We will not follow through the construction given in Definition 7.17, but will instead give a more geometric picture via a sequence of blowups of S.

Set $S_0 = S$, and let D_0 and E_0 be the divisors given by $x=0$ and $y=0$ respectively. Let Z_0 denote the locus where $D_0 \cup E_0$ is singular (i.e. the closed point of S). Let $U_0 = S_0 \setminus Z_0$.

Now set S_1 to be the blowup of S_0 at Z_0, and let D_1 and E_1 be the pullbacks of D_0 and E_0 to S_1. Let Z_1 be the locus where $D_1 \cup E_1$ is singular, and let $U_1 = S_1 \setminus Z_1$. Let $\phi_1 \colon U_0 \to U_1$ be the unique S-morphism (an open immersion). We proceed like this, inductively defining an infinite chain of blowups $S_{i+1} \to S_i$ and open immersions $\phi_i \colon U_i \to U_{i+1}$. Then the universal aligning morphism $\widetilde{S} \to S$ is the colimit of the open immersions ϕ_i. The morphism $\widetilde{S} \to S$ is separated, locally of finite type, is an isomorphism outside the closed point of S, but is not quasi-compact. More precisely, the closed fibre is an infinite union of copies of $\mathbb{G}_m$, which can be indexed by pairs of positive coprime integers. We assign D_0 the label $(0,1)$ and E_0 the label $(1,0)$, and then the exceptional curve of the blowup of components labelled (a,b) and (c,d) is given label $(a+c, b+d)$. So the exceptional curve of the first blowup is labelled $(1,1)$, and two exceptional of the second blowup are labelled $(1,2)$ and $(2,1)$, etc (cf. the Farey sequence). Note that $\widetilde{S}$ is integral and is smooth over $\mathbb{C}$.

11.2. The Néron model

We now describe the Néron model of the jacobian of C_U/U over the universal aligning scheme $\widetilde{S}$. Marking the section p, we find that C_U/U is an elliptic curve, so is canonically isomorphic to its jacobian. Over S, the fibrewise-connected component of p in C^{sm} is isomorphic to $\operatorname{Pic}^0_{C/S}$, and its pullback to $\widetilde{S}$ is the "identity component" of the Néron model. Let E be one of the copies of $\mathbb{G}_m$ lying over the origin in S, indexed by the pair (a, b) of coprime positive integers. Then the component group is constant over E, and is a cyclic group of order $a + b$.

12. Bounded-thickness substacks of $\widetilde{\mathcal{M}}_{g,n}$

In some applications it can be useful to consider only a part of the stack $\widetilde{\mathcal{M}}_{g,n}$ — in particular, in applications where quasi-compactness is important. In this section we define certain open substacks of $\widetilde{\mathcal{M}}_{g,n}$ with good universal properties.

We write Γ' for the graph obtained from Γ by deleting all loops.

Definition 12.1. — *Let S be a scheme, C/S a prestable curve, and $s \in S$ a geometric point. Write $G_1, \cdots, G_n$ for the elements of the partition* $\operatorname{Part}(\Gamma'_s)$ *of the graph Γ'_s. Let $e \geqslant 0$ be an integer. We say C/S is e-*strongly aligned *at s if there exists a sequence $a_1, \cdots, a_n$ of non-zero elements of $\mathcal{O}^{et}_{S,s}$ such that*

1. *for every subset $J \subseteq \{1, \cdots, n\}$, the subscheme*
$$V\left(a_j : j \in J\right) \subseteq \operatorname{Spec} \mathcal{O}^{et}_{S,s}$$
is regular (this is a weak version of having normal-crossings singularities);
2. *for each i and each edge c of G_i, there exists $0 \leqslant r \leqslant e$ such that*
$$\ell(c) = (a_i)^r.$$

*We say C/S is e-*strongly aligned *if it is e-strongly aligned at s for all geometric points s of S.*

Remark 12.2. — If C/S is 1-strongly aligned then C is regular; if c is a non-smooth point lying over s then $\ell(c) = (a_i)$ for some i, and a_i must be contained in the maximal ideal of $\mathcal{O}^{et}_{S,s}$ but not in its square. Then C/S is isomorphic étale-locally at the non-smooth point c to
$$\frac{\mathcal{O}^{et}_{S,s}[x, y]}{(xy - a_i)}.$$
Writing $\mathfrak{m}$ for the maximal ideal at the origin of $\mathcal{O}^{et}_{S,s}[x, y]$ we see that $xy - a_i \in \mathfrak{m}$ and that $xy - a_i \notin \mathfrak{m}^2$, so C is regular at c by [Liu02, Corollary 4.2.12].

Definition 12.3. — *Let g, n be non-negative integers such that $2g - 2 + n > 0$. Let $e \geqslant 0$ be an integer. Choose an étale cover $\bigsqcup_{i \in I} S_i \to \overline{\mathcal{M}}_{g,n}$ by a scheme such that on each S_i the curve has a controlling point $\mathfrak{s}_i$. For each $i \in I$, write $\mathbb{M}_i^{\leqslant e}$ for the set of thickness functions on $C_{\mathfrak{s}_i}$ which take values in $\{0, \cdots, e-1, e\} \subseteq \mathbb{Z}_{\geqslant 0}$. Now define $\widetilde{S}_i^{\leqslant e}$ to be the open subscheme of $\widetilde{S}_i$ covered by the S_M as M runs over $\mathbb{M}_i^{\leqslant e}$. Define $\widetilde{\mathcal{M}}_{g,n}^{\leqslant e}$ to be the image of $\bigsqcup_{i \in I} \widetilde{S}_i^{\leqslant e}$ under the natural étale map $\bigsqcup_{i \in I} \widetilde{S}_i \to \widetilde{\mathcal{M}}_{g,n}$.*

Remark 12.4. —

1. A-priori the definition of $\widetilde{\mathcal{M}}_{g,n}^{\leqslant e}$ depends on the choice of cover $\bigsqcup_{i \in I} \widetilde{S}_i \to \overline{\mathcal{M}}_{g,n}$, but we will see in Lemma 12.5 that this is not the case;

(2) Note that each $\widetilde{S}_i^{\leqslant e} \to S_i$ is quasi-compact, since $\mathbb{M}_i^{\leqslant e}$ is finite. By étale descent of quasi-compactness, the same holds for $\widetilde{\mathcal{M}}_{g,n}^{\leqslant e} \to \overline{\mathcal{M}}_{g,n}$.

Lemma 12.5. — *Fix integers g, n, e as above. Then the category of e-strongly aligned stable n-pointed curves of genus g has a terminal object. This terminal object has a natural map to $\widetilde{\mathcal{M}}_{g,n}$ (since e-strongly aligned implies aligned), and this map is an open immersion, whose image is exactly $\widetilde{\mathcal{M}}_{g,n}^{\leqslant e}$.*

Proof. — In this proof we retain the notation from Definition 12.3 (in particular the cover $\bigsqcup_{i\in I} \widetilde{S}_i \to \overline{\mathcal{M}}_{g,n}$).

The property of being e-strongly aligned is local in the étale topology, since the definition concerns only the étale local rings of S. Observe that the pullback of the canonical stable curve over S_i to $\widetilde{S}_i^{\leqslant e}$ is e-strongly aligned, by definition of $\widetilde{S}_i^{\leqslant e}$ and the fact that the universal curve over $\overline{\mathcal{M}}_{g,n}$ (and hence over S_i) has étale normal-crossings singularities. Combining these observations, we see that the pullback of the universal stable curve over $\overline{\mathcal{M}}_{g,n}$ to $\widetilde{\mathcal{M}}_{g,n}^{\leqslant e}$ is itself e-strongly aligned.

Let $f\colon T \to \overline{\mathcal{M}}_{g,n}$ be a non-degenerate morphism, with resulting stable curve C/T. Suppose that C is e-strongly aligned. In particular, C/T is aligned, and so f factors (uniquely) via $\widetilde{\mathcal{M}}_{g,n}$. If we can show that f in fact factors via $\widetilde{\mathcal{M}}_{g,n}^{\leqslant e}$, then we are done.

Since being e-strongly-aligned is étale local, we may assume that f factors via the étale cover $\bigsqcup_{i\in I} \widetilde{S}_i \to \overline{\mathcal{M}}_{g,n}$. Then it is clear from the construction of the $S_i^{\leqslant e}$ that f factors via $\bigsqcup_{i\in I} \widetilde{S}_i^{\leqslant e}$ and hence via $\widetilde{\mathcal{M}}_{g,n}^{\leqslant e}$, as required.

Finally, we observe that the universal Néron model has a particularly nice universal property for 1-strongly-aligned curves:

Lemma 12.6. — *Let C/S be stable, 1-strongly aligned and smooth over a dense open. Let $f\colon S \to \widetilde{\mathcal{M}}_{g,n}$ be the tautological map. Write $N_{g,n}$ for the Néron model over $\widetilde{\mathcal{M}}_{g,n}$. Then $f^*N_{g,n}$ is the Néron model of the jacobian of C.*

Proof. — A stable curve which is 1-strongly-aligned is necessarily regular by Remark 12.2, hence the pullback of the universal stable curve $C_{g,n}$ to $\widetilde{\mathcal{M}}_{g,n}$ is itself regular, so coincides with its desingularisation $\tilde{C}_{g,n}$ (see Remark 9.3).

As usual we write $\operatorname{Pic}^{[0]}_{C/S}$ for the total-degree-zero part of the relative Picard space of C/S (cf. Remark 2.9), and we find

$$f^* \operatorname{Pic}^{[0]}_{\tilde{C}_{g,n}/\widetilde{\mathcal{M}}_{g,n}} = \operatorname{Pic}^{[0]}_{C/S}.$$

If we write $E_{C/S}$ for the closure of the unit section in $\operatorname{Pic}^{[0]}_{C/S}$ and $\widetilde{E}$ for the closure of the unit section in $\operatorname{Pic}^{[0]}_{C_{g,n}/\widetilde{\mathcal{M}}_{g,n}}$ we find that $f^*\widetilde{E} = E_{C/S}$ since $\widetilde{E}$ is flat over $\widetilde{\mathcal{M}}_{g,n}$. Hence by [Hol19, Theorem 6.2] we find

$$f^*N_{g,n} = f^*\left(\frac{\operatorname{Pic}^{[0]}_{\tilde{C}_{g,n}/\widetilde{\mathcal{M}}_{g,n}}}{\widetilde{E}}\right) = \frac{\operatorname{Pic}^{[0]}_{C/S}}{E_{C/S}},$$

the latter being the Néron model of the jacobian of C/S, again using that C is regular since it is 1-strongly-aligned.

13. Relation to the stack $\mathcal{P}_{d,g}$ of Caporaso

Let $g \geqslant 3$, and let d be an integer such that $\gcd(d-g+1, 2g-2) = 1$. Caporaso [Cap08] constructs a smooth morphism of stacks $p\colon \mathcal{P}_{d,g} \to \overline{\mathcal{M}}_g$ and an isomorphism from $\mathcal{P}_{d,g} \times_{\overline{\mathcal{M}}_g} \mathcal{M}_g$ to the degree-d Picard scheme of the universal curve over $\overline{\mathcal{M}}_g$. This morphism p is relatively representable by schemes, and satisfies the following property:

> *given a trait B with generic point η and a regular stable curve $X \to B$, write $f\colon B \to \overline{\mathcal{M}}_g$ for the moduli map. Then $f^*\mathcal{P}_{d,g}$ is the Néron model of* $\operatorname{Pic}^d_{X_\eta/\eta}$.

More concisely, we might say that $\mathcal{P}_{d,g}$ gives a partial compactification of the degree-d universal jacobian J_g which has a good universal property for test curves B in $\overline{\mathcal{M}}_g$ which meet the boundary with "low multiplicity" (this is equivalent to the given stable curve X/B being regular). In contrast, in this paper we construct a partial compactification N_g of the universal jacobian J_g, with a natural group structure, and which has a good universal property for *all* test curves to (even aligned morphisms to) $\overline{\mathcal{M}}_g$, but at the price of "blowing up the boundary" of $\overline{\mathcal{M}}_g$. In the remainder of this section, we will make the comparison more precise.

Note that the condition $\gcd(d-g+1, 2g-2) = 1$ precludes the possibility that $d = 0$ (unless $g = 2$), and so to compare Caporaso's construction to $\widetilde{\mathcal{M}}_g$ we must consider Néron models of degree-d parts of the jacobian for $d \neq 0$. These are not group schemes, but the Néron mapping property still makes sense, so we define a Néron model to be a smooth separated model satisfying the Néron mapping property. This presents no new difficulties:

Theorem 13.1. — *Let S be a regular separated stack, and C/S an aligned prestable curve, smooth over a dense open $U \hookrightarrow S$. Assume that C is regular. Fix $d \in \mathbb{Z}$, and let J_d/U denote the degree-d jacobian of C over U. Then J_d admits a Néron model over S.*

Proof. — The closure of the unit section in $\operatorname{Pic}_{C/S}$ coincides with the closure of the unit section in $\operatorname{Pic}^{[0]}_{C/S}$, since the latter is open and closed in $\operatorname{Pic}_{C/S}$. In particular, the closure of the unit section in $\operatorname{Pic}_{C/S}$ is flat over S, and so the quotient N of $\operatorname{Pic}_{C/S}$ by the closure of the unit section exists as an algebraic space over S. It is easily verified that N is the Néron model of $\operatorname{Pic}_{C_U/U}$ — the proof of the main theorem of [Hol19] carries over almost verbatim. Then the closure of J_d inside N is exactly the Néron model of J_d that we seek.

The next proposition shows that the restriction of the Néron model of the universal jacobian to the open substack $\widetilde{\mathcal{M}}_g^{\leqslant 1}$ is given by the pullback of Caporaso's construction.

Proposition 13.2. — *Let $g \geqslant 3$ and d be integers such that* $\gcd(d-g+1, 2g-2) = 1$. *Write $\widetilde{\mathcal{P}}$ for the pullback of $\mathcal{P}_{d,g}$ to $\widetilde{\mathcal{M}}_g^{\leqslant 1}$, and write N_g for the Néron model of J_g over $\widetilde{\mathcal{M}}_g^{\leqslant 1}$. Then the canonical map $h\colon \widetilde{\mathcal{P}} \to N_g$ given by the Néron mapping property is an isomorphism.*

Proof. — It is enough to check the map h is an isomorphism on every geometric fibre over $\widetilde{\mathcal{M}}_g^{\leqslant 1}$. Let p be a geometric point of $\widetilde{\mathcal{M}}_g^{\leqslant 1}$. Then there exists a trait T with geometric closed point t, and a morphism $g\colon T \to \overline{\mathcal{M}}_g$ such that

(1) the given stable curve $X \to T$ is regular;
(2) the map g factors via a map $\widetilde{g}\colon T \to \widetilde{\mathcal{M}}_g^{\leqslant 1}$;
(3) this factorisation $\widetilde{g}$ maps t to p.

Such a T can be constructed by choosing a trait in $\widetilde{\mathcal{M}}_g^{\leqslant 1}$ through p which is transversal to the boundary divisor of the universal curve over $\widetilde{\mathcal{M}}_g^{\leqslant 1}$. This is certainly possible as the boundary has normal crossings, and one then takes g to be the composite of the inclusion of this trait with the structure map $\widetilde{\mathcal{M}}_g \to \overline{\mathcal{M}}_g$ We need to verify that it satisfies these three conditions:

(1) Write $C_{g,n}$ for the pullback of the universal stable curve over $\widetilde{\mathcal{M}}_g^{\leqslant 1}$. Then $C_{g,n}$ is regular by Remark 12.2, and since T meets the boundary transversally we see that the pullback over T is also regular.
(2 & 3) These are clear by construction and the valuative criterion for separatedness applied to $\widetilde{\mathcal{M}}_g^{\leqslant 1} \to \overline{\mathcal{M}}_g$.

Now since X is regular, we find (as in the proof of Theorem 13.1) that $\widetilde{g}^*N_g$ is the Néron model of the jacobian of the generic fibre of $X \to T$, and the same holds for $\widetilde{g}^*\widetilde{\mathcal{P}} = g^*\mathcal{P}$. In particular, this shows that the fibres of N_g and of $\widetilde{\mathcal{P}}$ over p are isomorphic. Moreover, the given map between them is an isomorphism; this is true because it is so over the generic point of T (apply the uniqueness part of the Néron mapping property).

In particular, this shows that, after pullback along the morphism $\widetilde{\mathcal{M}}_g^{\leqslant 1} \to \overline{\mathcal{M}}_g$, the stack $\mathcal{P}_{d,g}$ admits a natural torsor structure extending that over $\mathcal{M}_g$.

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