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A RADIUS 1 IRREDUCIBILITY CRITERION FOR LATTICES IN PRODUCTS OF TREES

UN CRITÈRE D'IRRÉDUCTIBILITÉ DE RAYON 1 POUR LES RÉSEAUX DANS LES PRODUITS D'ARBRES

Abstract. — Let T_1, T_2 be regular trees of degrees $d_1, d_2 \geqslant 3$. Let also $\Gamma \leqslant \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ be a group acting freely and transitively on $VT_1 \times VT_2$. For $i = 1$ and 2, assume that the local action of Γ on T_i is 2-transitive; if moreover $d_i \geqslant 7$, assume that the local action contains $\mathrm{Alt}(d_i)$. We show that Γ is irreducible, unless (d_1, d_2) belongs to an explicit small set of exceptional values. This yields an irreducibility criterion for Γ that can be checked purely in terms of its local action on a ball of radius 1 in T_1 and T_2. Under the same hypotheses, we show moreover that if Γ is irreducible, then it is hereditarily just-infinite, provided the local action on T_i is not the affine group $\mathbf{F}_5 \rtimes \mathbf{F}_5^*$. The proof of irreducibility relies, in several ways, on the Classification of the Finite Simple Groups.

Résumé. — Soient T_1, T_2 des arbres réguliers de degrés $d_1, d_2 \geqslant 3$ et $\Gamma \leqslant \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ un groupe agissant librement et transitivement sur $VT_1 \times VT_2$. Pour $i = 1$ et 2, on suppose que l'action locale de Γ sur T_i est 2-transitive; si en outre $d_i \geqslant 7$, on suppose également que

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cette action locale contient le groupe $\mathrm{Alt}(d_i)$. Nous montrons que Γ est irréductible, à moins que le couple (d_1, d_2) n'appartienne à une courte liste explicite de valeurs exceptionnelles. Ce résultat donne lieu à un critère d'irréductibilité pour Γ qui s'exprime en termes de son action locale sur les boules de rayon 1 de T_1 et T_2. Sous les mêmes hypothèses, en supposant en outre que l'action locale de Γ sur T_i ne soit pas l'action 2-transitive naturelle du groupe affine $\mathbf{F}_5 \rtimes \mathbf{F}_5^*$, nous montrons aussi que si Γ est irréductible, alors il est héréditairement juste infini. La démonstration du critère d'irréductibilité repose, de plusieurs façons, sur la Classification des Groupes Simples Finis.

1. Introduction

The study of lattices in products of trees was pioneered by D. Wise [Wis96] and M. Burger and S. Mozes [BM97, BM00b]. Their seminal works revealed that the class of finitely generated groups admitting a Cayley graph that is isomorphic to the Cartesian product of two trees is very rich: it contains not only products of virtually free groups, but also certain S-arithmetic groups and some finitely presented virtually simple groups, among many others. Such groups are called *BMW-groups* and form a special class of lattices in products of trees. An introduction to this fascinating subject may be consulted in [Cap19, § 4]. The goal of this paper is to present a sufficient condition, that is straightforward to check in practise, ensuring that a BMW-group is irreducible.

Let T_1, T_2 be locally finite trees and $\Gamma \leqslant \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ be a group acting with finite stabilizers and finitely many orbits. Equivalently Γ is a discrete subgroup of $\mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ acting cocompactly on $T_1 \times T_2$. Such a group Γ is called a *cocompact lattice* in the product $T_1 \times T_2$. Since we only consider cocompact lattices in this paper, the adjective *cocompact* will henceforth be omitted. We say that Γ is *reducible* if it contains a finite index subgroup of the form $K_1 \times K_2$, where $K_i \leqslant \Gamma$ acts trivially on T_{3-i} and freely and cocompactly on T_i for $i = 1$ and 2. Otherwise Γ is called *irreducible*. Determining whether a given lattice is reducible is a crucial basic question, and there is no known algorithm deciding if that property holds in full generality. Burger and Mozes observed however that the irreducibility of Γ can be tested in an efficient algorithmic way under an extra hypothesis on the *local action* of Γ on T_1 or T_2. We recall that, given a group G acting on a graph X by automorphisms, the *local action of level n* of G at a vertex v is the action of the stabilizer G_v on the n-ball around v. The G_v-action on the 1-sphere around v is called the *local action* for short. We say that the local action of G has a property $\mathcal{P}$ (e.g. is transitive, primitive, 2-transitive, etc.) if the local action of G at every vertex has property $\mathcal{P}$. If G is vertex-transitive, then the local actions of G at all vertices are pairwise isomorphic; in that case, the corresponding abstract permutation group is called the *local action of G on X*. An important fact due to Burger and Mozes [BM00a, § 3.3], [BM00b, § 5] is that, if $d_i \geqslant 6$, if G is vertex-transitive on T_i and if the local action of Γ on T_i contains $\mathrm{Alt}(d_i)$ for $i = 1$ or 2, then the irreducibility of Γ can be tested by considering the local action of level 2 of Γ on T_i. More generally, using the work of V. Trofimov and R. Weiss [TW95], one can show that, for all $d_i \geqslant 3$, if the local action of Γ on T_i is 2-transitive, then the irreducibility of Γ can be tested by considering the local action of level 7 of Γ on T_i (see [Cap19, Corollary 4.12]).

An action of a group on a set is called *regular* if it is free and transitive. In the present paper, we focus on the special class of lattices in $T_1 \times T_2$ formed by the groups $\Gamma \leqslant \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ acting regularly on $VT_1 \times VT_2$. In that case the tree T_i is regular of degree d_i. Following [Cap19, § 4], such a group Γ is called a *BMW-group of degree* (d_1, d_2). When this is the case, the group Γ has a generating set S such that the Cayley graph of (Γ, S) is the Cartesian product $T_1 \times T_2$. This paper was initiated by the following observation, which follows easily from the aforementioned work of Trofimov–Weiss. It shows that, in principle, when the local action on *both* tree factors is 2-transitive, then the lattice Γ is "almost always" irreducible.

Theorem 1.1. — *Let $d_1 \geqslant d_2 \geqslant 3$, let T_1, T_2 be regular trees of degrees d_1, d_2 and let $\Gamma \leqslant \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ be a group acting regularly on the vertices of $T_1 \times T_2$. Assume that for $i = 1, 2$, the local action F_i of Γ on T_i is 2-transitive. If*

$$d_1 \geqslant (d_2!)\Big((d_2 - 1)!\Big)^{\frac{d_2\left((d_2-1)^5-1\right)}{d_2-2}},$$

then Γ is irreducible.

A similar idea is used by C. H. Li in his proof of [Li05, Theorem 1.1].

The condition that the Γ-action on $VT_1 \times VT_2$ be free is essential in Theorem 1.1. Indeed, given any $d \geqslant 3$, let W_d be the free product of d copies of the cyclic group of order 2. Then $\mathrm{Sym}(d)$ acts by automorphisms on W_d by permuting the d generators of order 2. Therefore, for all $d_1 \geqslant d_2 \geqslant 3$, the direct product

$$\Gamma = \Big(W_{d_1} \rtimes \mathrm{Sym}\,(d_1)\Big) \times \Big(W_{d_2} \rtimes \mathrm{Sym}\,(d_2)\Big)$$

is an obviously reducible lattice in $T_1 \times T_2$, where T_i is the regular tree of degree d_i. Its local action on T_i is $\mathrm{Sym}(d_i)$. Moreover Γ acts transitively, but not freely, on $VT_1 \times VT_2$, showing that the hypothesis of freeness of the Γ-action cannot be removed in Theorem 1.1.

Under extra assumptions on the local action, the bound contained in Theorem 1.1 can be vastly improved. This is illustrated by the following result, where $\mathbf{C}_n$ denotes the cyclic group of order n.

Theorem 1.2. — *Let $d_1 \geqslant d_2 \geqslant 3$, let T_1, T_2 be regular trees of degrees d_1, d_2 and let $\Gamma \leqslant \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ be a group acting regularly on the vertices of $T_1 \times T_2$. Assume that for $i = 1, 2$, the local action F_i of Γ on T_i is 2-transitive. Assume moreover that if $d_i \geqslant 7$, then $F_i \geqslant \mathrm{Alt}(d_i)$. Then Γ is irreducible provided none of the following conditions holds:*

(i) *$d_2 = 3$, and*

$$d_1 \in \Big\{23, 24, 47\Big\}.$$

(ii) *$d_2 = 4$, and*

$$d_1 \in \Big\{6n \mid n \geqslant 2 \text{ divides } 972\Big\} \cup \Big\{12n - 1 \mid n \text{ divides } 972\Big\}.$$

(iii) *$d_2 = 5$, $F_2 \cong \mathbf{C}_5 \rtimes \mathbf{C}_4$, and*

$$d_1 \in \Big\{10, 19, 20, 39, 40, 79\Big\}.$$

(iv) $d_2 = 5$, $\mathrm{soc}(F_2) \cong \mathrm{Alt}(5)$, *and*

$$d_1 \in \Big\{30n \mid n \geqslant 2 \textit{ divides } 768\Big\} \cup \Big\{60n - 1 \mid n \textit{ divides } 768\Big\}.$$

(v) $d_2 = 6$, $\mathrm{soc}(F_2) \cong \mathrm{Alt}(5)$, *and*

$$d_1 \in \Big\{30n \mid n \geqslant 2 \textit{ divides } 200\Big\} \cup \Big\{60n - 1 \mid n \textit{ divides } 200\Big\}.$$

(vi) $d_2 \geqslant 6$, *and*

$$d_1 \in \left\{\frac{d_2!}{2} - 1, \frac{d_2!}{2}, d_2! - 1, \frac{d_2!(d_2-1)!}{4} - 1, \frac{d_2!(d_2-1)!}{4}, \frac{d_2!(d_2-1)!}{2} - 1, \frac{d_2!(d_2-1)!}{2}, d_2!(d_2-1)! - 1\right\}.$$

The *socle* of a finite group F, denoted by $\mathrm{soc}(F)$, is the subgroup generated by all the minimal normal subgroups of F. While contemplating the list of possible exceptions in small degree in Theorem 1.2, it is useful to keep in mind the list of finite 2-transitive groups of degree $\leqslant 6$, which is recalled in Table 2.1 below.

Theorem 1.2 provides in particular an irreducibility criterion for a BMW-group Γ of degree (d_1, d_2): if the pair (d_1, d_2) is not one of the exceptions from the list (i)–(vi) in the theorem, then Γ is irreducible provided its local action on T_i is 2-transitive and, in case $d_i \geqslant 7$, if it also contains $\mathrm{Alt}(d_i)$, for $i = 1$ and 2. That criterion depends only on the local actions of level 1, and is thus considerably easier to use in practise than the other criteria mentioned above.

Notice the contrast between the bound on d_1 in Theorem 1.1 and the range of values for (d_1, d_2) in Theorem 1.2. Examples of reducible lattices Γ as in Theorem 1.2 with $(d_1, d_2) = (23, 3)$, $(24, 3)$, $(47, 3)$, $(11663, 4)$, $(19, 5)$, $(39, 5)$ and $(79, 5)$ will be highlighted, relying on the work of Xu–Fang–Wang–Xu [XFWX05], Li–Lu [LL09], M. Conder [Con09] and Ling–Lou [LL16, LL17], see Proposition 2.3 below. In particular Theorem 1.2 is sharp in the case $d_2 = 3$. It would be very interesting to determine which of the exceptional values occurring in Theorem 1.2 are indeed realized by actual examples of reducible lattices (for $4 \leqslant d_2 \leqslant 6$, not all values of d_1 appearing in the theorem can be realized, see Remark 2.12), or at least whether infinitely many values of (d_1, d_2) with $d_2 \geqslant 6$ can occur. As we shall see in Section 2.2, this is a question in finite group theory. The examples found for the small values of d_2 provide evidence for a positive answer to the latter question.

We point out the following immediate consequence.

Corollary 1.3. — *Let T_1, T_2 be regular trees of degrees $d_1, d_2 \in \{3, 4, 5, 6\}$ and let $\Gamma \leqslant \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ be a group acting regularly on the vertices of $T_1 \times T_2$. Assume that for $i = 1, 2$, the local action of Γ on T_i is 2-transitive. Then Γ is irreducible.*

The hypothesis of the regularity of the Γ-action on $VT_1 \times VT_2$ is essential: indeed, there are examples of reducible lattices $\Gamma \leqslant \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ acting with 2 orbits of vertices and locally 2-transitive actions at every vertex of both factors, with $(d_1, d_2) = (3, 4)$, see Remark 2.4 below.

Using the basic covering theory of graphs (see Proposition 2.2), one shows that Theorem 1.2 is equivalent to a statement on finite groups acting on graphs, namely Theorem 3.1 below. The proof of the latter statement relies heavily, and in several ways, on the Classification of the Finite Simple groups. Particularly relevant is the classification, due to Liebeck–Praeger–Saxl [LPS00, Corollary 5], of all pairs (G, M) consisting of a finite almost simple group G and a subgroup $M \leqslant G$ whose order involves all primes dividing the order of G (see Section 2.7 below). It is moreover closely related to the well studied field of finite groups admitting an s-arc transitive Cayley graph (see [LX14] and references therein).

Combining the work of Burger–Mozes [BM00a], Bader–Shalom [BS06] and V. Trofimov [Tro07], we will show that if a lattice Γ satisfies the hypotheses of Theorem 1.2 and if it is irreducible, then it is *hereditarily just-infinite*, i.e. Γ is infinite, and every proper quotient of every finite index subgroup of Γ is finite.

Corollary 1.4. — *Retain the hypotheses of Theorem 1.2 and assume that Γ is irreducible. Assume moreover that if $d_i = 5$ then $F_i \not\cong \mathbf{C}_5 \rtimes \mathbf{C}_4$ for $i = 1$ and 2. Then Γ is hereditarily just-infinite.*

See Corollary 4.4 for a more general statement.

Numerous explicit examples of BMW-groups of small degree satisfying the hypotheses of Theorem 1.2 and Corollary 1.4 are given in [Cap19, § 4], [Rad17] and [Rat04].

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2. Preliminaries

2.1. Groups acting on graphs and local action

For graphs and trees, we use the terminology and notation of [BL01, § 2.1]. A graph X consists of a set of vertices VX, a set of oriented edges EX, two maps $\partial_0, \partial_1 \colon EX \to V$ representing the endpoints of edges, and an orientation reversing map $EX \to EX : e \mapsto \bar{e}$ satisfying $\partial_i \bar{e} = \partial_{1-i} e$ and $\bar{\bar{e}} = e \neq \bar{e}$. For $x \in VX$ we set $E(x) = \{e \in EX \mid \partial_0(e) = x\}$. A *geometric edge* of X is a pair $\{e, \bar{e}\}$ with $e \in EX$.

Let now G be group acting on a graph X by automorphisms. We denote by G_x the stabilizer of an element $x \in VX \cup EX$. For $x \in VX$ and $m \geqslant 0$, we also denote by $G_x^{[m]}$ the subgroup of G fixing all vertices $y \in VX$ at distance $d(x, y) \leqslant m$. The quotient group $G_x/G_x^{[1]}$, viewed as a permutation group on $E(x)$, is the *local action* of G at x. More generally, the group $G_x/G_x^{[m]}$, viewed as a permutation group on the m-ball around x, is called the *local action of level* m of G at x.

An *edge inversion* is an element $g \in G$ such that $ge = \bar{e}$ for some $e \in EX$. If G acts without edge inversion, then we can form the quotient graph $G\backslash X$, see [BL01, § 2.2]. We say that the G-action on X is *free* if G acts freely on VX and freely on the set of geometric edges. Equivalently, the G-action on X is free if G acts freely on VX and has no edge inversion.

Lemma 2.1. — *Let X be a connected graph and $G \leqslant \mathrm{Aut}(X)$ be a group of automorphisms. Given a normal subgroup N of G acting freely on X, the kernel of the G-action on the quotient graph $N\backslash X$ coincides with N.*

Proof. — Let $g \in G$ act trivially on the quotient graph $N\backslash X$ and let $x \in VX$. Since $gN(x) = N(x)$, there exists $n \in N$ with $gn(x) = x$. Let now y be any vertex of X fixed by $h = gn$, and let e be an oriented edge with $\partial_0 e = y$. Then e and $h(e)$ belong to the same N-orbit since g acts trivially on $N\backslash X$. Since $\partial_0 e = y = \partial_0 h(e)$, any element of N mapping e to $h(e)$ fixes y. Since N acts freely, we deduce that $h(e) = e$. Thus h fixes all edges emanating from y, hence also all the neighbours of y. Since the graph X is connected, this implies that $h = gn$ acts trivially on X. Thus $g \in N$ as required. □

2.2. A reduction to finite group theory

The following basic result from the covering theory of graphs allows one to go back and forth between reducible lattices in products of trees and finite groups acting on products of graphs, without affecting the local actions.

Proposition 2.2. — *Let T_1, T_2 be regular trees of degree d_1, d_2 and $\Gamma \leqslant \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ be a group acting transitively on the vertices of the Cartesian product $T_1 \times T_2$. For $i = 1, 2$, let F_i denote the local action of Γ on T_i, let K_i be the projection on $\mathrm{Aut}(T_i)$ of the kernel of the Γ-action on T_{3-i}. Assume that K_i acts freely on T_i (as defined in Section 2.1). Then for $i = 1$ and 2, we have:*

- (i) *the group $G = \Gamma/K_1 \times K_2$ acts transitively on the Cartesian product $VX_1 \times VX_2$, where X_i is the quotient graph $K_i\backslash T_i$,*
- (ii) *X_i is of degree d_i and the local action of G on X_i is isomorphic to F_i,*
- (iii) *the G-action on X_i is faithful,*
- (iv) *if the Γ-action on $VT_1 \times VT_2$ is free, then so is the G-action on $VX_1 \times VX_2$.*

Conversely, let X_1, X_2 be regular graphs of degree d_1, d_2 and $G \leqslant \mathrm{Aut}(X_1) \times \mathrm{Aut}(X_2)$ be a group acting with m orbits (resp. acting regularly) on the vertices of the Cartesian product $X_1 \times X_2$. Assume that the local action of G at every vertex of X_i is isomorphic to the permutation group F_i. Then there is a group $\Gamma \leqslant \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ acting with m orbits (resp. acting regularly) on $VT_1 \times VT_2$, where T_i is the regular tree of degree d_i, such that:

- (v) *The local action of Γ at every vertex of T_i is isomorphic to F_i,*

(vi) Γ *contains a normal subgroup of the form* $K_1 \times K_2$ *such that the quotient group* $\Gamma/K_1 \times K_2$ *is isomorphic to* G*,*

(vii) K_i *is the fundamental group of the graph* X_i *acting by covering transformations on the tree* T_i*.*

Proof. — For the first part, notice that since K_i acts freely on T_i, the quotient graph $X_i = K_i \backslash T_i$ is well defined. Recall that, by definition, the vertex set of X_i consists of the K_i-orbits in VT_i, and the oriented edges of X_i are defined as the K_i-orbits of oriented edges in T_i, so that the projection map $VT_i \to VX_i$ is a morphism of graphs. Since K_i acts freely, the quotient map $T_i \to X_i$ can also be viewed as a covering map in the classical sense from topology. Assertions (i)–(iv) now follow from the basic covering theory of graphs (for which we refer to [Bas93] and [Ser77]), together with Lemma 2.1.

The converse is also a standard application of the covering theory of graphs. □

Given Proposition 2.2, the following result is an easy consequence of known results on s-arc transitive Cayley graphs due to Li–Lu [LL09], Xu–Fang–Wang–Xu [XFWX05] and M. Conder [Con09]. We denote by T_n the regular tree of degree n.

Proposition 2.3. —

(i) *There exists a reducible lattice* $\Gamma_{3,23} \leqslant \mathrm{Aut}(T_3) \times \mathrm{Aut}(T_{23})$ *acting regularly on the vertices of* $T_3 \times T_{23}$*, whose local action on* T_3 *(resp.* T_{23}*) is* $\mathrm{Sym}(3)$ *(resp.* $\mathrm{Sym}(23)$*).*

(ii) *There exists a reducible lattice* $\Gamma_{3,24} \leqslant \mathrm{Aut}(T_3) \times \mathrm{Aut}(T_{24})$ *acting regularly on the vertices of* $T_3 \times T_{24}$*, whose local action on* T_3 *(resp.* T_{24}*) is* $\mathrm{Sym}(3)$ *(resp.* $\mathrm{Sym}(24)$*).*

(iii) *There exists a reducible lattice* $\Gamma_{3,47} \leqslant \mathrm{Aut}(T_3) \times \mathrm{Aut}(T_{47})$ *acting regularly on the vertices of* $T_3 \times T_{47}$*, whose local action on* T_3 *(resp.* T_{47}*) is* $\mathrm{Sym}(3)$ *(resp.* $\mathrm{Alt}(47)$*).*

(iv) *There exists a reducible lattice* $\Gamma_{4,11663} \leqslant \mathrm{Aut}(T_4) \times \mathrm{Aut}(T_{11663})$ *acting regularly on the vertices of* $T_4 \times T_{11663}$*, whose local action on* T_4 *(resp.* T_{11663}*) is* $\mathrm{Sym}(4)$ *(resp.* $\mathrm{Alt}(11663)$*).*

(v) *There exists a reducible lattices* $\Gamma_{5,n} \leqslant \mathrm{Aut}(T_5) \times \mathrm{Aut}(T_n)$ *acting regularly on the vertices of* $T_5 \times T_n$*, for* $n = 19, 39$ *and* 79*, whose local action on* T_5 *is* $\mathbf{C}_5 \rtimes \mathbf{C}_4$*, and whose respective local action on* T_{19}*,* T_{39} *and* T_{79} *is* $\mathrm{Sym}(19)$*,* $\mathrm{Alt}(39)$ *and* $\mathrm{Alt}(79)$*.*

Proof. — By [LL09, Theorem 1.1], there is a 3-regular graph Y which is a Cayley graph of the group $B = \mathrm{Sym}(23)$, whose full automorphism group G is isomorphic to $\mathrm{Sym}(24)$, and such that the local action of G on Y is $\mathrm{Sym}(3)$. Let A be the stabilizer in G of a vertex $y \in VY$. Hence $|A| = 24$, $A \cap B = \{1\}$ and $G = AB$. Let moreover X be the complete graph on 24 vertices, on which G acts faithfully by automorphisms. Let $x \in VX$ be the vertex fixed by B. Since $G = AB$ and $A \cap B = \{1\}$, it follows that the diagonal G-action on the vertex set of $X \times Y$ is free and transitive. The assertion (i) thus follows from Proposition 2.2.

For (ii), we use a similar argument, using a degree 3 Cayley graph Y of an index 2 subgroup of $\mathrm{Sym}(23) \times \mathrm{Sym}(24)$ appearing in [Con09, Theorem 2.1(d)]. We define X to be the complete bipartite graph $\mathbf{K}_{24,24}$ in this case.

The proof of (iii), (iv) and (v) are also similar. For (iii) and (iv), one uses a degree 3 Cayley graph of Alt(47) appearing in [Con09, Theorem 2.1 (e)] (such a graph was first constructed in [XFWX05]) and a degree 4 Cayley graph of Alt(11663) discussed in [Con09, § 3]. For (v) and $n = 39, 79$, one uses the graph from [LL16] and [LL17, Theorem 1.1 (2)] respectively. For (v) and $n = 19$, an example was constructed by M. Giudici using MAGMA. ☐

Remark 2.4. — Using the converse part of Proposition 2.2, one can also construct reducible lattices Γ in regular trees of smaller degrees with 2-transitive local actions.

For example, consider the group $G = \mathrm{Sym}(4) \times \mathbf{C}_2$. It acts locally 2-transitively on the bipartite graph X with 2 vertices and 4 geometric edges, as well as on the graph Y which is the 1-skeleton of the cube. The diagonal action of G on $X \times Y$ has 2 orbits of vertices, and the vertex-stabilizers are non-trivial (they are isomorphic to Sym(3)). Invoking Proposition 2.2, we obtain a locally 2-transitive reducible lattice $\Gamma \leqslant \mathrm{Aut}(T_4) \times \mathrm{Aut}(T_3)$ acting with 2 orbits of vertices, since $\widetilde{X} \cong T_4$ and $\widetilde{Y} \cong T_3$.

Another example is constructed similarly using the group $G = \mathrm{Sym}(5)$, that acts locally 2-transitively on the complete graph $X = \mathbf{K}_5$, as well as on the Petersen graph Y. The diagonal action of G on $X \times Y$ has 2 orbits of vertices (the stabilizers of vertices in the corresponding orbits are respectively of order 4 and 6). This yields a locally 2-transitive reducible lattice $\Gamma \leqslant \mathrm{Aut}(T_4) \times \mathrm{Aut}(T_3)$ acting with 2 orbits of vertices, since $\widetilde{\mathbf{K}_5} \cong T_4$ and $\widetilde{Y} \cong T_3$.

2.3. Locally 2-transitive actions

Recall that a permutation group $G \leqslant \mathrm{Sym}(\Omega)$ is *quasi-primitive* if every non-trivial normal subgroup of G acts transitively on Ω.

LEMMA 2.5 ([BM00a, Lemma 1.4.2]). — *Let X be a connected graph, let $G \leqslant \mathrm{Aut}(X)$ be a group whose local action is quasi-primitive and let $N \leqslant G$ be a normal subgroup of G. Set*

$$VX'(N) = \{x \in VX | N_x \text{ acts transitively on } E(x)\},$$
$$VX''(N) = \{x \in VX | N_x \text{ acts trivially on } E(x)\}.$$

Then one of the following assertions holds:

(i) *$VX''(N) = X$ and N acts freely on VX.*
(ii) *$VX'(N) = X$ and N acts transitively on the set of geometric edges of X. In particular N has at most 2 orbits of vertices.*
(iii) *$VX = VX'(N) \cup VX''(N)$ is a G-invariant bipartition of X, and for any $x'' \in VX''(N)$, the 1-ball $B(x'', 1)$ around x'' is a strict fundamental domain for the N-action on X.*

The case (ii) splits into two subcases, according to whether N is transitive on VX. In particular, we deduce the following when G is vertex-transitive.

Corollary 2.6. — *Let X be a connected graph, let $G \leqslant \mathrm{Aut}(X)$ be a vertex-transitive group whose local action is quasi-primitive. For any normal subgroup $N \leqslant G$, one of the following assertions holds:*

(i) *N acts freely on VX.*
(ii) *N is transitive on VX and on the set of geometric edges.*
(iii) *N has exactly two orbits on VX, which form a G-invariant bipartition of X, and N is transitive on the set of geometric edges.*

Proof. — Since N is normal and G is vertex-transitive, the N_x-action on $E(x)$ is isomorphic to the N_y-action on $E(y)$ for any two vertices $x, y \in VX$. Thus only the cases (i) or (ii) from Lemma 2.5 can occur. In the second case, observe that if N is not transitive on VX, then no element of N can map a vertex to a neighbour, because N_x acts transitively on $E(x)$ for all $x \in VX$. Thus the N-orbits form a G-invariant partition of VX such that no two element of a given class are adjacent. Since N is transitive on the set of geometric edges, it has at most 2 orbits of vertices. The desired assertion follows. □

In case $N \leqslant G$ is a normal subgroup acting non-freely on VX, we have the following.

Corollary 2.7. — *Let X be a connected graph, let $G \leqslant \mathrm{Aut}(X)$ be a vertex-transitive group whose local action is quasi-primitive. Let $N \leqslant G$ be a normal subgroup. Assume that neither N nor $\mathrm{C}_G(N)$ acts freely on VX. Then either $|VX| \leqslant 2$, or X is complete bipartite and N acts regularly on the set of geometric edges.*

Proof. — Let $M = \mathrm{C}_G(N)$. Since N is normal in G, so is M. In view of the hypotheses, both M and N satisfy the conclusion (ii) in Lemma 2.5.

We now invoke Corollary 2.6 for M.

If M is transitive on VX, then $N_x = N_y$ for any pair of vertices $x, y \in VX$. Since N_x is also transitive on $E(x)$ it follows that $|VX| \leqslant 2$.

If M is not transitive on VX, then X is bipartite and M has two orbits on VX. Let $x \neq y$ be adjacent vertices.

The M_y-action on $E(y)$ is transitive by Lemma 2.5. Thus, for all neighbours x' of y, we have $N_x = N_{x'}$. Since N_x is transitive on $E(x)$, we see that N_x-orbit of y coincides with the set of neighbours of x. Since $N_x = N_{x'}$ and $N_{x'}$ is transitive on $E(x')$, we deduce that x and x' have the same set of neighbours. Similarly, any neighbour y' of x has the same set of neighbours as y. Since X is connected, this implies that X is a complete bipartite graph. Given a geometric edge $\{x, y\}$, the stabilizer $N_{x,y}$ is trivial since it commutes with M, which is transitive on the geometric edges by Corollary 2.6. The conclusion follows since N is transitive on the set of geometric edges by Corollary 2.6. □

We also record the following information about the case where the local action of G is 2-transitive and $N \leqslant G$ is a normal subgroup acting freely on VX but not freely on geometric edges.

Lemma 2.8. — *Let X be a connected graph, let $G \leqslant \mathrm{Aut}(X)$ be a group whose local action is 2-transitive, and let $N \leqslant G$ be a normal subgroup of G acting freely on VX but non-freely on the set of geometric edges of X. Let $x \in VX$. Then:*

(i) *For each* $e \in E(x)$*, there is a unique element* $s_e \in N$ *with* $s_e(e) = \bar{e}$ *and* $s_e^2 = 1$.
(ii) N *acts regularly on* VX.
(iii) N *is generated by the set* $\{s_e \mid e \in E(x)\}$.
(iv) $G_x^{[1]} = \{1\}$.
(v) $G \cong N \rtimes G_x$ *and* $\mathrm{C}_{G_x}(N) = \{1\}$.
(vi) $\mathrm{Z}(G) \leqslant N$.

Proof. — The hypotheses on N imply the existence of an edge $f \in EX$ and an element $s \in N$ with $s(f) = \bar{f}$. Since N is free on VX we have $s^2 = 1$. Let $x = \partial_0(f)$. For each $e \in E(x)$ there is $g \in G_x$ with $g(f) = e$. Set $s_e = gsg^{-1} \in N$. Thus we have proved Assertion (i) for some vertex x, and the assertion will follow for all vertices as soon as we show that N is vertex-transitive. The group $\langle s_e \mid e \in E(x)\rangle$ contains an element mapping x to each of its neighbours. Since X is connected, it follows that the latter group is transitive on VX. Thus N is transitive and Assertions (i), (ii) and (iii) follow since N acts freely on VX by hypothesis. Moreover (v) is a consequence of (ii). Finally, observe that an element $g \in G_x^{[1]}$ fixes each $e \in E(x)$, and thus centralizes s_e. Thus $g \in \mathrm{C}_{G_x}(N)$ by (iii). Thus $g = 1$ by (v), and (iv) holds.

Let $Z = \mathrm{Z}(G)$ be the center of G. Its image under the projection $G \to G/N \cong G_x$ is a central subgroup of G_x. The group G_x acts 2-transitively on $E(x)$, and that action is faithful by (iv). It follows that $\mathrm{Z}(G_x) = \{1\}$. Hence $Z \leqslant N$ and (vi) holds. □

2.4. Vertex stabilizers of locally 2-transitive actions

The following important result due to V. Trofimov and R. Weiss provides very precise information about vertex-stabilizers for proper vertex-transitive locally 2-transitive actions of discrete groups on locally finite graphs. It plays a crucial role in our considerations.

Theorem 2.9. — *Let* $G \leqslant \mathrm{Aut}(X)$ *be a vertex-transitive automorphism group of a connected locally finite graph* X*. Let* (v, w) *be an edge of* X*. Suppose that the local action is 2-transitive, and that the stabilizer* G_v *is finite. Then:*

(i) (Trofimov–Weiss [TW95, Theorem 1.4]) *We have*

$$G_v^{[5]} \cap G_w^{[5]} = \{1\}.$$

In particular $G_v^{[6]} = \{1\}$.

(ii) (Trofimov–Weiss [TW95, Theorem 1.3 and 2.3]) *If* $G_v^{[1]} \cap G_w^{[1]} \neq \{1\}$ *(e.g. if* $G_v^{[2]} \neq \{1\}$*), then the local action at* v *contains a normal subgroup isomorphic to* $\mathrm{PSL}_n(\mathbf{F}_q)$ *in its natural action on the points of the* $n-1$*-dimensional projective space over the finite field* $\mathbf{F}_q$ *of order* q*. Moreover* $G_v^{[1]} \cap G_w^{[1]}$ *is a* p*-group, where* p *is the characteristic of* $\mathbf{F}_q$.

(iii) (R. Weiss [Wei79, Theorem 1.1 and 1.4]) *If the local action at* v *contains a normal subgroup isomorphic to* $\mathrm{PSL}_2(\mathbf{F}_q)$ *in its natural action on the points of the projective line over a finite field* $\mathbf{F}_q$*, then there is* $s \in \{2, 3, 4, 5, 7\}$ *such that for any geodesic segment* $(v_1, v_2, \ldots, v_s)$ *of length* $s - 1$*, we have*

$$G_{v_1}^{[1]} \cap G_{v_2}^{[1]} \cap G_{v_3} \cap \cdots \cap G_{v_s} = \{1\}.$$

Moreover if $\mathrm{char}(\mathbf{F}_q) \geqslant 5$ *then* $s \leqslant 4$*, and if* $\mathrm{char}(\mathbf{F}_q) = 2$ *then* $s \leqslant 5$.

2.5. The 2-transitive groups of degree $\leqslant 6$

In the proof of Theorem 1.2, we will encounter several case-by-case discussions depending notably on the list of 2-transitive groups of small degree. For the reader's convenience, that list is recalled in Table 2.1.

Degree d	$G \leqslant \mathrm{Sym}(d)$	$\lvert G\rvert$
3	$\mathrm{Sym}(3) \cong \mathbf{C}_3 \rtimes \mathbf{C}_2 \cong \mathbf{F}_3 \rtimes \mathbf{F}_3^*$	6
4	$\mathrm{Alt}(4) \cong \mathrm{PSL}_2(\mathbf{F}_3) \cong \mathbf{F}_4 \rtimes \mathbf{F}_4^*$	12
4	$\mathrm{Sym}(4) \cong \mathrm{PGL}_2(\mathbf{F}_3)$	24
5	$\mathbf{C}_5 \rtimes \mathbf{C}_4 \cong \mathbf{F}_5 \rtimes \mathbf{F}_5^*$	20
5	$\mathrm{Alt}(5) \cong \mathrm{PSL}_2(\mathbf{F}_4)$	60
5	$\mathrm{Sym}(5) \cong \mathrm{P\Gamma L}_2(\mathbf{F}_4)$	120
6	$\mathrm{Alt}(5) \cong \mathrm{PSL}_2(\mathbf{F}_5)$	60
6	$\mathrm{Sym}(5) \cong \mathrm{PGL}_2(\mathbf{F}_5)$	120
6	$\mathrm{Alt}(6)$	360
6	$\mathrm{Sym}(6)$	720

Table 2.1. 2-transitive groups of degree $\leqslant 6$

Keeping that list in mind, we now present two consequences of Theorem 2.9 needed for the proof of Theorem 1.2. The following one should be compared with [BM00a, Lemma 3.5.1].

Corollary 2.10. — *Let* $G \leqslant \mathrm{Aut}(X)$ *be a vertex-transitive automorphism group of a connected locally finite graph* X *of degree* d *with finite vertex-stabilizers. Suppose that the local action* $F \leqslant \mathrm{Sym}(d)$ *is* 2*-transitive. Suppose moreover that at least one of the following conditions holds:*

(a) *the point stabilizer* F_p *is almost simple.*

(b) F *is sharply* 2*-transitive and* $d \leqslant 5$.

Then for $v \in VX$*, we have* $G_v^{[2]} = \{1\}$*, and the group* $G_v^{[1]}$ *is isomorphic to a normal subgroup of* F_p*. Furthermore, if (a) holds and if* $G_v^{[1]} \neq \{1\}$ *then* $G_v^{[1]}$ *is almost simple with socle isomorphic to* $\mathrm{soc}(F_p)$.

Proof. — Let $w \in VX$ be adjacent of v. Each of the conditions (a) and (b) implies that $G_v^{[1]} \cap G_w^{[1]} = \{1\}$ by Theorem 2.9 (ii) (see Table 2.1). The groups $G_v^{[1]}$ and $G_w^{[1]}$ are both normal subgroups of $G_{v,w}$, and the quotient $G_{v,w}/G_w^{[1]}$ is isomorphic to F_p. The image of $G_v^{[1]} \leqslant G_{v,w}$ under the projection $G_{v,w} \to G_{v,w}/G_w^{[1]}$ is injective (since $G_v^{[1]} \cap G_w^{[1]} = \{1\}$) and isomorphic to a normal subgroup of the group F_p. The required conclusions follow. □

The various possible exceptions appearing in Theorem 1.2 find their roots in the following result.

Corollary 2.11. — *Let $G \leqslant \mathrm{Aut}(X)$ be a vertex-transitive automorphism group of a connected locally finite graph X of degree d with finite vertex-stabilizers. Suppose that the local action $F \leqslant \mathrm{Sym}(d)$ is 2-transitive, and moreover that $F \geqslant \mathrm{Alt}(d)$ if $d \geqslant 7$. Let $x \in VX$. Then one of the following assertions holds:*

(i) $d \geqslant 6$ *and* $|G_x| \in \{\frac{d!}{2}, d!, \frac{d!(d-1)!}{4}, \frac{d!(d-1)!}{2}, d!(d-1)!\}$.
(ii) $d = 3$ *and* $|G_x| \in \{6n \mid n \text{ divides } 2^3\}$.
(iii) $d = 4$ *and* $|G_x| \in \{12n \mid n \text{ divides } 2^2 \cdot 3^5\}$.
(iv) $d = 5$ *and* $F = \mathbf{C}_5 \rtimes \mathbf{C}_4$, *then* $|G_x| \in \{20, 40, 80\}$.
(v) $d = 5$ *and* $\mathrm{soc}(F) = \mathrm{Alt}(5)$ *and* $|G_x| \in \{60n \mid n \text{ divides } 2^8 \cdot 3\}$.
(vi) $d = 6$, $\mathrm{soc}(G) = \mathrm{PSL}_2(\mathbf{F}_5)$ *and* $|G_x| \in \{60n \mid n \text{ divides } 2^3 \cdot 5^2\}$.

Proof. — If $F = \mathrm{Alt}(d)$ or $\mathrm{Sym}(d)$ with $d \geqslant 6$, we must have (i) by Corollary 2.10. Similarly, if $d = 5$ and $F = \mathbf{C}_5 \rtimes \mathbf{C}_4$ then $|G_x| \in \{20, 40, 80\}$ by Corollary 2.10. In the remaining cases, we apply Theorem 2.9 (iii) using the list in Table 2.1. ☐

Remark 2.12. — The structure of G_x in the case where $d \leqslant 6$ can be described more precisely, see [Wei79, Theorems (1.2) and (1.3)]. Those results could be used to sharpen slightly the range of values appearing in Corollary 2.11, and hence also those in Theorem 1.2; we will not perform that sharpening here.

2.6. Finite simple $\{2, 3, 5\}$-groups

The following result is a consequence of the CFSG.

Proposition 2.13 ([HL00, Theorem III (1) and Table 1]). — *Let S be a non-abelian finite simple group such that the only prime divisors of $|S|$ are 2, 3 and 5. Then S is isomorphic to* $\mathrm{Alt}(5)$, $\mathrm{Alt}(6)$ *or* $\mathrm{PSp}_4(3) \cong U_4(2)$, *respectively of order* $2^2 \cdot 3 \cdot 5$, $2^3 \cdot 3^2 \cdot 5$ *and* $2^6 \cdot 3^4 \cdot 5$.

2.7. Subgroups of a finite simple group involving all its primes

Given a finite set X, we denote by $\pi(X)$ the set of prime divisors of $|X|$. The following important result will be crucial to our purposes.

Theorem 2.14 (Liebeck–Praeger–Saxl [LPS00, Corollary 5]). — *Let G be a finite almost simple group with socle N. Let $M \leqslant G$ be a subgroup not containing N such that $\pi(M) \supseteq \pi(N)$. Then the possibilities for N and M are all listed in* [LPS00, Table 10.7].

The following consequence, that can be extracted from the list given by Liebeck–Praeger–Saxl, will be sufficient for us. (Extra caution is needed in view of the exceptional isomorphisms between small finite simple groups.)

Corollary 2.15. — *Retain the assumptions of Theorem 2.14 and suppose in addition that $M \cap N$ has a composition factor isomorphic to* $\mathrm{Alt}(d)$ *for some $d \geqslant 5$. Then either there exist positive integers $k \leqslant c$ such that $N = \mathrm{Alt}(c)$ and $\mathrm{Alt}(k) \lhd M \leqslant \mathrm{Sym}(k) \times \mathrm{Sym}(c-k)$ and $k \geqslant p$ for all primes $p \leqslant c$, or the pair $(N, M \cap N)$ is one of the exceptions listed in Table 2.2 (see* [LPS00] *for the notation).*

	N	$\|N\|$	$M \cap N$
(1)	$\mathrm{Alt}(6)$	$2^3 \cdot 3^2 \cdot 5$	$L_2(5) \cong \mathrm{Alt}(5)$
(2)	$\mathrm{U}_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	$\mathrm{Alt}(7)$
(3)	$\mathrm{U}_4(2)$	$2^6 \cdot 3^4 \cdot 5$	$M \cap N \leqslant 2^4.\mathrm{Alt}(5), \mathrm{Sym}(6)$
(4)	$\mathrm{U}_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	$\mathrm{Alt}(7)$
(5)	$\mathrm{PSp}_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	$\mathrm{Alt}(7)$
(6)	$\mathrm{Sp}_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	$\mathrm{Alt}(7), \mathrm{Sym}(7), \mathrm{Alt}(8), \mathrm{Sym}(8)$
(7)	$\mathrm{P}\Omega_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	$M \cap N \leqslant P_1, P_3, P_4, \mathrm{Alt}(9)$

Table 2.2. The exceptional pairs $(N, M \cap N)$ in Corollary 2.15

2.8. On subgroups of direct products of simple groups

The following fact is an easy corollary of an important consequence of the Classification of the Finite Simple Groups, namely the fact the every automorphism of a non-abelian finite simple group centralizes a non-trivial subgroup (see [KS04, § 9.5.3]). Note however that we shall need that result only in the case of the alternating groups.

Proposition 2.16. — *Let S be a non-abelian finite simple group. Let $G = S_1 \times S_2$ be the direct product of two groups isomorphic to S, and let $A_1, A_2 \leqslant G$ be subgroups of G that are also isomorphic to S. If $A_1 \cap A_2 = \{1\}$, then there is $i \in \{1, 2\}$ such that $A_i = S_1 \times \{1\}$ or $A_i = \{1\} \times S_2$.*

Proof. — Assume that $A_i \neq S_1 \times \{1\}$ and $A_i \neq \{1\} \times S_2$ for $i = 1$ and 2. Then by Goursat's Lemma, for $i = 1, 2$ there exists an isomorphism $\varphi_i \colon S_1 \to S_2$ such that $A_i = \{(x, \varphi_i(x)) \mid x \in S_1)\}$. Since $A_1 \cap A_2 = \{1\}$, it follows that $\varphi_1^{-1}\varphi_2$ is an automorphism of S_1 whose only fixed point is the trivial element. By [KS04, § 9.5.3] (see also the announcement in [Gor82, Theorem 1.48]), the group S_1 must be solvable, contradicting the hypotheses. □

3. Finite groups with locally 2-transitive actions on product graphs

The goal of this section is to prove Theorem 1.2. It will be deduced as a corollary to the following result. See Section 3.10.

Theorem 3.1. — *Let X_1, X_2 be finite connected regular graphs of degree $d_1 \geqslant d_2 \geqslant 3$ and let $G \leqslant \mathrm{Aut}(X_1) \times \mathrm{Aut}(X_2)$ be a group acting freely and transitively on the vertices of the Cartesian product $X_1 \times X_2$. For $i = 1$ and 2, we assume that the G-action on X_i is faithful and that its local action F_i is locally 2-transitive; we assume moreover that if $d_i \geqslant 7$ then $F_i \geqslant \mathrm{Alt}(d_i)$. Then X_1 is the complete graph $\mathbf{K}_{d_1+1}$ or the complete bipartite graph $\mathbf{K}_{d_1, d_1}$. Moreover one of the conditions* (i)–(vi) *listed in Theorem 1.2 is satisfied.*

The proof occupies the rest of this section.

3.1. The standing hypotheses and notation

We fix the notation and assumptions adopted throughout. For $i=1,2$, let $d_i \geqslant 3$ and $F_i \leqslant \mathrm{Sym}(d_i)$ be a 2-transitive permutation group. Let $\mathcal{E}(F_1,F_2)$ be the collection of triples (X_1,X_2,G) satisfying the following conditions:

(**Hyp1**) X_i is a connected d_i-regular graph for $i=1$ and 2.
(**Hyp2**) $G \leqslant \mathrm{Aut}(X_1)\times\mathrm{Aut}(X_2)$ is a finite group.
(**Hyp3**) G acts transitively on $VX_1 \times VX_2$.
(**Hyp4**) The G-action on X_i is faithful for $i=1$ and 2.
(**Hyp5**) The local action of G on X_i is isomorphic to F_i for $i=1$ and 2.

We further denote by $\mathcal{F}(F_1,F_2)$ the subcollection consisting of those triples $(X_1,X_2,G)\in\mathcal{E}(F_1,F_2)$ satisfying in addition:

(**Hyp6**) G acts freely on $VX_1\times VX_2$.

Lemma 3.2. — *Let $(X_1,X_2,G)\in\mathcal{E}(F_1,F_2)$, let $x_1\in VX_1$ and $x_2\in VX_2$. Then:*

(i) *G_{x_1} acts transitively on VX_2 and G_{x_2} acts transitively on VX_1.*
(ii) *$G=G_{x_1}G_{x_2}$.*

If in addition $(X_1,X_2,G)\in\mathcal{F}(F_1,F_2)$, then:

(iii) *G_{x_1} acts freely on VX_2 and G_{x_2} acts freely on VX_1.*
(iv) *$G_{x_1}\cap G_{x_2}=\{1\}$.*

Proof. — Assertion (i) is immediate from (**Hyp2**) and (**Hyp3**); assertion (ii) follows from (i), while (iii) and (iv) are equally straightforward. □

Thus, if $(X_1,X_2,G)\in\mathcal{F}(F_1,F_2)$, we may see X_1 as a Cayley graph of G_{x_2} and vice-versa, unless $|VX_1|\leqslant 2$ (resp. $|VX_2|\leqslant 2$). The latter inequality is never satisfied in our setting: indeed, since $|VX_i|=|G_{x_{3-i}}|$ and since $G_{x_{3-i}}$ has a 2-transitive action on a set of cardinality $d_{3-i}\geqslant 3$, we have $|VX_i|\geqslant 3$.

3.2. Proof of Theorem 1.1

As mentioned in the introduction, Theorem 1.1 is a straightforward consequence of Theorem 2.9. Let us record the proof. We first need the following result.

Lemma 3.3. — *Let T_1,T_2 be regular trees of degree $d_1,d_2\geqslant 3$ and let $\Gamma\leqslant \mathrm{Aut}(T_1)\times\mathrm{Aut}(T_2)$ be a group acting regularly on the vertices of $T_1\times T_2$. Assume that the local action of Γ on T_i is 2-transitive for $i=1,2$. For $i=1,2$, let K_i be the projection on $\mathrm{Aut}(T_i)$ of the kernel of the Γ-action on T_{3-i}. Then K_i acts freely on T_i.*

Proof. — Since Γ acts freely on $VT_1\times VT_2$, it follows that K_i acts freely on VT_i. We need to show that K_i does not invert any edge of T_i. If K_i contains an edge inversion, then K_i acts regularly on VT_i by Lemma 2.8 (ii). Let $v\in VT_{3-i}$. Let K_i' be the kernel of the Γ-action on T_{3-i}, so that $K_i'\cong K_i$ and K_i' acts regularly on VT_i. Clearly $K_i'\leqslant\Gamma_v$. Since Γ acts regularly on $VT_1\times VT_2$, it follows that Γ_v is regular on VT_i, so that $K_i'=\Gamma_v$. Since that equality holds for all $v\in VT_{3-i}$, it follows that Γ_v acts trivially on T_{3-i}. This contradicts the hypothesis that Γ_v is 2-transitive on $E(v)$. □

The example following Theorem 1.1 in the introduction shows that Lemma 3.3 may fail if the Γ-action on $VT_1 \times VT_2$ is not free.

Proof of Theorem 1.1. — Retain the notation of Lemma 3.3 and assume that $d_1 \geqslant d_2$ and that Γ is reducible. We must show that $d_1 < M$, where

$$M = (d_2!)\,((d_2-1)!)^{\frac{d_2\left((d_2-1)^5-1\right)}{d_2-2}}.$$

The reducibility of Γ ensures that the quotient $\Gamma/K_1 \times K_2$ is finite. Moreover by Lemma 3.3, we may invoke Proposition 2.2, which ensures that the set $\mathcal{F}(F_1, F_2)$ is non-empty, where F_1, F_2 denote the local actions of Γ on T_1, T_2. Let $(X_1, X_2, G) \in \mathcal{F}(F_1, F_2)$, let $x \in VX_2$ and $v \in VT_2$. In view of Theorem 2.9, an upper bound on the order of $|G_x|$ is provided by the order $\mathrm{Aut}(T_2)_v/\mathrm{Aut}(T_2)_v^{[6]}$. The latter group is isomorphic to the iterated permutational wreath product

$$\mathrm{Sym}\,(d_2-1) \wr \mathrm{Sym}\,(d_2-1) \wr \mathrm{Sym}\,(d_2-1) \wr \mathrm{Sym}\,(d_2-1) \wr \mathrm{Sym}\,(d_2-1) \wr \mathrm{Sym}\,(d_2)\,,$$

whose order is M. In particular X_1 is a d_1-regular graph of order bounded above by that number. Since $\mathrm{Aut}(X_1)$ is locally 2-transitive, we have $|VX_1| \leqslant 2$ or $|VX_1| \geqslant d_1 + 1$. The former case is impossible, since it would imply that $|G_{x_2}| = 2$ by Lemma 3.2, contradicting that G is locally 2-transitive on the graph X_2 whose degree is $d_2 \geqslant 3$. Thus we obtain $d_1 + 1 \leqslant M$, which is the required bound. □

Remark 3.4. — The bound obtained in the proof above can directly be sharpened by exploiting Theorem 2.9 in a more precise way. We will do this in the proof of Theorem 3.1.

3.3. Assuming that N acts freely on VX_1 and on VX_2

From now on, we choose a member $(X_1, X_2, G) \in \mathcal{F}(F_1, F_2)$. We also fix $N \neq \{1\}$ a minimal normal subgroup of G. Thus N is characteristically simple. Hence it is isomorphic to the k^{th} direct power of a finite simple group S. We also fix $x_1 \in VX_1$ and $x_2 \in VX_2$.

LEMMA 3.5. — *Assume that $N \cong S^k$ acts freely on both VX_1 and VX_2, but not freely on the set of geometric edges of X_i for $i = 1$ or 2. Then:*

(i) $|N| = |G_{x_{3-i}}|$.
(ii) $G_{x_{3-i}}$ *is isomorphic to a subgroup of* G_{x_i}.
(iii) N *and* S *are not abelian.*
(iv) $d_1, d_2 \geqslant 5$.
(v) $|F_i|$ *has at least* 3 *prime divisors.*

Proof. — Lemma 2.8 applies to the N-action on X_i. It follows that $G \cong N \rtimes G_{x_i}$. In view of Lemma 3.2, the assertion (i) follows. Since the N-action on VX_{3-i} is free, the projection of $G_{x_{3-i}}$ under the projection $G \to G/N \cong G_{x_i}$ maps $G_{x_{3-i}}$ injectively onto a subgroup of G_{x_i}. This proves (ii).

If $N \cong S^k$ were abelian (or equivalently if S were abelian), then the order of N would be a power of 2 since N is generated by involutions (see Lemma 2.8 (i) and (iii)). Thus $G_{x_{3-i}}$ is a 2-group by (i). A 2-group does not admit a 2-transitive action on a set containing more than two elements. This is a contradiction since $G_{x_{3-i}}$ is 2-transitive on a set of cardinality $d_{3-i} \geqslant 3$. This proves (iii). If $d_1 \leqslant 4$ or $d_2 \leqslant 4$, then the set of prime divisors of $|G_{x_1}|$ or $|G_{x_2}|$ would be contained in $\{2, 3\}$. Hence the same would apply to $|N|$ by (i) and (ii). Thus N would be solvable by Burnside's theorem, hence abelian since N is characteristically simple. This contradicts (iii). Thus (iv) holds. If $|F_i|$ has at most 2 prime divisors, then the same holds for $|G_{x_i}|$, hence also $|G_{x_{3-i}}|$ by (ii), hence $|G|$ by Lemma 3.2. Thus G is solvable by Burnside's theorem. The minimal normal subgroup N must thus be abelian, contradicting (iii). This proves (v). □

LEMMA 3.6. — *For $j = 1$ and 2, we assume that $F_j \geqslant \mathrm{Alt}(d_j)$ if $d_j \geqslant 7$. If N acts freely on VX_1 and on VX_2, then it acts freely on X_1 and on X_2.*

Proof. — Suppose for a contradiction that N does not act freely on X_i for some $i \in \{1, 2\}$. This means that N does not act freely on the set of geometric edges of X_i. By Lemma 3.5 (iv), we have $d_i \geqslant 5$. Moreover F_i has at least 3 distinct prime divisors by Lemma 3.5 (v). In particular $F_i \not\cong \mathbf{C}_5 \rtimes \mathbf{C}_4$. By the hypothesis made on F_i, we deduce that F_i is not solvable (see Table 2.1). The hypotheses imply that the socle of F_i is isomorphic to $\mathrm{Alt}(d_i)$, or to $\mathrm{Alt}(5)$ if $d_i = 6$. Note moreover that, since $G_{x_i}^{[1]} = \{1\}$ by Lemma 2.8 (iv), we have $G_{x_i} \cong F_i$.

Recall that $N \cong S^k$, where S is a finite simple group. We distinguish two cases.

Suppose first that $C_G(N) = \{1\}$. Then the G-conjugation action on N yields an injective homomorphism of $G_{x_i} \cong G/N$ into $\mathrm{Out}(N)$. By the Krull–Remak–Schmidt theorem (see [Rob96, Theorem 3.3.8]), the outer automorphism group $\mathrm{Out}(N)$ is isomorphic to the wreath product $\mathrm{Out}(S) \wr \mathrm{Sym}(k)$. The group $\mathrm{Out}(S)$ is solvable by the Schreier conjecture. Since G_{x_i} is not solvable, we deduce that $k \geqslant 5$.

Since $|N| = |G_{x_{3-i}}|$ by Lemma 3.5 (i), we infer that the order of $G_{x_{3-i}}$ is a k^{th} power. We have $d_{3-i} \geqslant 5$ by Lemma 3.5 (iv). Let p be a prime with $d_{3-i}/2 < p < d_{3-i}$. Using Corollary 2.11, we see that p divides $|G_{x_{3-i}}|$, but p^4 does not. Therefore $|G_{x_{3-i}}|$ cannot be a k^{th} power with $k \geqslant 4$, and we have reached a contradiction. This finishes the proof in the case where $C_G(N) = \{1\}$.

Assume now that $C_G(N)$ is non-trivial. Let $M \neq \{1\}$ be a minimal normal subgroup of G contained in $C_G(N)$. Thus MN is a normal subgroup of G. Since N is minimal normal in G and non-abelian by Lemma 3.5 (iii), we have $M \cap N = \{1\}$. In view of Lemma 2.8 (v), we deduce that $MN \cap G_{x_i} \neq \{1\}$. Since $G_{x_i} \cong F_i$ is almost simple with socle $\mathrm{Alt}(d_i)$ or $\mathrm{Alt}(5)$ if $d_i = 6$, we deduce that $G_{x_i}^+ := G_{x_i} \cap MN$ contains the socle of G_{x_i}, and has index 1 or 2 in G_{x_i}. Lemma 2.8 (v) also implies that $M \cap G_{x_i} = \{1\}$, so that the natural homomorphism $MN \to N$ yields an injective homomorphism $G_{x_i}^+ \to N$. By Lemma 3.5 (i) and (ii), we also have $|N| \leqslant |G_{x_i}|$, hence $|N| \leqslant 2|G_{x_i}^+|$. It follows $G_{x_i}^+$ is isomorphic to a subgroup of index at most 2 in N. Since N is characteristically simple and non-abelian, we deduce that $G_{x_i}^+ \cong N$. In particular N, hence also $G_{x_i}^+$, is simple. This implies that $G_{x_i}^+$ is the socle of G_{x_i}.

Since $G \cong N \rtimes G_{x_i}$ and $M \cap N = \{1\}$, the projection map $G \to G_{x_i}$ yields an injective homomorphism of M into G_{x_i}, whose image is normal since M is normal in G. Since G_{x_i} is almost simple with socle $G_{x_i}^+$, while M is characteristically simple, it follows that $G_{x_i}^+ \cong M$. Using Lemma 3.5 (i) and (ii), we obtain that

$G_{x_{3-i}} \cong M \cong N \cong G^+_{x_i}$, which is $\mathrm{Alt}(d_i)$ or $\mathrm{Alt}(5)$. Since MN has index at most 2 in G by Corollary 2.6, we have $G_{x_{3-i}} \leqslant MN$. Since the G-action on $VX_1 \times VX_2$ is free, we have $G^+_{x_i} \cap G_{x_{3-i}} = \{1\}$. Applying Proposition 2.16 to the group $MN \cong M \times N$, we infer that one of the two groups $G^+_{x_i}$ or $G_{x_{3-i}}$ coincides with one of the two simple factors of MN. Both of those factors are normal subgroups of G. It finally follows that $G^+_{x_i}$ or $G_{x_{3-i}}$ is normal in G. Hence $G^+_{x_i}$ acts trivially on VX_i or $G_{x_{3-i}}$ acts trivially on VX_{3-i}. This implies that $|VX_i| \leqslant 2$ or $|VX_{3-i}| \leqslant 2$, which contradicts Lemma 3.2 since $d_1, d_2 \geqslant 3$. □

3.4. Assuming that N acts freely on VX_i but not on VX_{3-i}

Lemma 3.7. — *Let $i \in \{1,2\}$. Assume that N acts freely on VX_i and non-freely on VX_{3-i}. Then $G^+_{x_i} := G_{x_i} \cap NG_{x_{3-i}}$ has index 1 or 2 in G_{x_i} and $|G^+_{x_i}| = |N : N_{x_{3-i}}|$ divides $|G_{x_{3-i}} : N_{x_{3-i}}|$. Furthermore we have $|G_{x_i}| < |G_{x_{3-i}}|$.*

Proof. — The N-orbits on VX_i define a G-invariant partition into subsets of size $|N|$. Since $G_{x_{3-i}}$ acts regularly on VX_i by Lemma 3.2, we infer that $|N|$ divides $|G_{x_{3-i}}|$. We have $|G_{x_{3-i}}| = |G_{x_{3-i}} : N_{x_{3-i}}||N_{x_{3-i}}|$ and $|N| = |N : N_{x_{3-i}}||N_{x_{3-i}}|$. It follows that $|N : N_{x_{3-i}}|$ divides $|G_{x_{3-i}} : N_{x_{3-i}}|$.

By Lemma 3.2, the group G_{x_i} acts regularly on VX_{3-i}. The hypotheses imply that N has at most two orbits on VX_{3-i} by Corollary 2.6, so that $NG_{x_{3-i}}$ has index 1 or 2 in G. Accordingly, the group $G^+_{x_i} = G_{x_i} \cap NG_{x_{3-i}}$ has index 1 or 2 in G_{x_i}, and we have $|G^+_{x_i}| = |N : N_{x_{3-i}}|$.

Since $N_{x_{3-i}}$ is transitive on $E(x_{3-i})$ (see Lemma 2.5 and Corollary 2.6), we have $|N_{x_{3-i}}| \geqslant d_{3-i} \geqslant 3$. Thus $|G_{x_i}| \leqslant 2|G^+_{x_i}| \leqslant 2|G_{x_{3-i}} : N_{x_{3-i}}| \leqslant 2|G_{x_{3-i}}|/3 < |G_{x_{3-i}}|$. □

3.5. A minimality condition

We shall now consider $(X_1, X_2, G) \in \mathcal{F}(F_1, F_2)$ satisfying the following:

(**Min**) For all $(X'_1, X'_2, G') \in \mathcal{F}(F_1, F_2)$, we have $|VX'_1 \times VX'_2| \geqslant |VX_1 \times VX_2|$.

Lemma 3.8. — *Let $(X_1, X_2, G) \in \mathcal{F}(F_1, F_2)$ satisfy* (**Min**). *Then there is $i \in \{1,2\}$ such that the N-action on the graph X_i is not free.*

Proof. — If the N-action on X_i were free for $i = 1$ and 2, then the triple

$$(N\backslash X_1, N\backslash X_2, G/N)$$

would belong to $\mathcal{F}(F_1, F_2)$ by Lemma 2.1 (see also Proposition 2.2), which would contradict the hypothesis (**Min**). □

Lemma 3.9. — *Let $(X_1, X_2, G) \in \mathcal{F}(F_1, F_2)$ satisfy* (**Min**). *Assume moreover that there is $i \in \{1,2\}$ such that N does not act freely on VX_i. Then $\mathrm{C}_G(N) = \{1\}$. In particular N is not abelian. Moreover $\max\{d_1, d_2\} \geqslant 5$.*

Proof. — Assume that $\mathrm{C}_G(N) \neq \{1\}$. Let $M \neq \{1\}$ be a minimal normal subgroup of G contained in $\mathrm{C}_G(N)$. We aim at finding a contradiction.

Suppose first that M acts freely on both VX_1 and VX_2. By Lemma 3.8, the M-action cannot be free on the set of geometric edges of both X_1 and X_2.

Assume that M is not free on X_i. Then, by Lemma 2.8, the M-action is regular on VX_i. Since $[M, N] = \{1\}$, the group N_{x_i} fixes pointwise the M-orbit of x_i. Thus N_{x_i} acts trivially on VX_i. On the other hand N has at most two orbits on VX_i by Corollary 2.6, since the N-action on VX_i is not free by hypothesis. It follows that $|VX_i| \leqslant 2$, so that $|G_{x_{3-i}}| \leqslant 2$ by Lemma 3.2, which is absurd.

Assume now that M is not free on X_{3-i}. Then $|G_{x_i}|$ divides $|G_{x_{3-i}}|$ by Lemma 3.5 (ii). In particular $|G_{x_i}| \leqslant |G_{x_{3-i}}|$, so that the N-action cannot be free on VX_{3-i} by Lemma 3.7. We may thus apply the same argument as in the preceding paragraph to conclude that $|VX_{3-i}| \leqslant 2$, leading to a contradiction.

This shows that the M-action cannot be free on both VX_1 and VX_2.

Assume first that the M-action is not free on VX_i. We may then invoke Corollary 2.7, which ensures that X_i is the complete bipartite graph $\mathbf{K}_{d_i, d_i}$ and that M and N both act regularly on the set of geometric edges of X_i. In particular N_{x_i} acts regularly on $E(x_i)$. It follows that the 2-transitive permutation group F_i has a regular normal subgroup. Thus F_i is of affine type and d_i is a prime power. We also have $|G_{x_{3-i}}| = |VX_i| = 2d_i$. Since $G_{x_{3-i}}$ admits a 2-transitive permutation action on d_{3-i} points, we deduce that $d_{3-i}(d_{3-i} - 1)$ divides $2d_i$. Since d_i is a prime power, we must have $d_{3-i} = 3$ and d_i is a power of 3. Applying Corollary 2.11 to the stabilizer $G_{x_{3-i}}$, which has order $2d_i$, we deduce that $d_i = 3$. Therefore $|G_{x_{3-i}}| = |VX_i| = 6$ and hence $G_{x_{3-i}} \cong \mathrm{Sym}(3)$. Moreover $|G_{x_i}| \in \{6, 12\}$ since $X_i \cong \mathbf{K}_{3,3}$.

Since M and N both act regularly on the set of geometric edges of X_i, we have $M = N \cong \mathbf{C}_3^2$. Notice that N is a 3-Sylow subgroup of G. It has 4 cyclic subgroups of order 3, which are permuted by G. Since N is minimal normal in G, that permutation action must be fixed-point-free. Each involution in G has 0, 2 or 4 fixed points in VX_i, and if some involution fixes 4 vertices, then $G_{x_i}^{[1]}$ is non-trivial. Assume now that $|G_{x_i}| = 6$, so that $G_{x_i} \cong \mathrm{Sym}(3)$ and $G_{x_i}^{[1]} = \{1\}$. Then every involution $\sigma \in G_{x_i}$ has 2 fixed points on VX_i. It follows that its conjugation action on N maps each element on its inverse, and hence it acts trivially on the set of cyclic subgroups of N. This implies that the cyclic subgroup of $G_{x_{3-i}}$ of order 3 is normalized by both G_{x_1} and G_{x_2}. Thus it is normal in G by Lemma 3.2, contradicting the minimality of N. We deduce that $|G_{x_i}| = 12$. Hence X_{3-i} is a 3-regular graph with 12 vertices which is also a Cayley graph for G_{x_i} on which the group G acts locally 2-transitively. Such a graph does not exist by [LL09, Theorem 1.1].

We conclude finally that the M-action is free on VX_i, and non-free on VX_{3-i}. Lemma 3.7 successively implies that $|G_{x_i}| < |G_{x_{3-i}}|$, and that the N-action on VX_{3-i} cannot be free. We may finish the proof by swapping X_1 and X_2 and use the same argument as in the previous paragraph. This confirms that $\mathrm{C}_G(N) = \{1\}$.

If $d_1, d_2 \leqslant 4$, then the only prime divisors of $|G|$ would be 2 and 3, so that G would be solvable and N abelian, a contradiction. □

3.6. Assuming that $|F_i|$ has only two prime divisors

LEMMA 3.10. — *Let* $(X_1, X_2, G) \in \mathcal{F}(F_1, F_2)$ *satisfy* (**Min**). *Assume that there is* $i \in \{1, 2\}$ *such that* $|F_i|$ *has only two primes divisors. Then the* N*-action on* VX_{3-i} *is not free.*

Proof. — The hypothesis on F_i implies that $|G_{x_i}|$ has only two primes divisors. Suppose for a contradiction that N acts freely on VX_{3-i}. Then $|N|$ divides $|VX_{3-i}| = |G_{x_i}|$ by Lemma 3.2. So the characteristically simple group N must be abelian. Hence N acts freely on VX_i by Lemma 3.9, and also freely on the set of geometric edges of X_1 and X_2 by Lemma 3.5 (iii). This contradicts Lemma 3.8. □

3.7. Assuming that $F_1 \cong \mathbf{C}_5 \rtimes \mathbf{C}_4$

LEMMA 3.11. — *Let* $(X_1, X_2, G) \in \mathcal{F}(F_1, F_2)$ *satisfy* (**Min**). *Assume that* $d_1 = 5$, $F_1 \cong \mathbf{C}_5 \rtimes \mathbf{C}_4$. *Then* $d_2 \geqslant 5$.

Proof. — Suppose for a contradiction that $d_2 \leqslant 4$. In particular F_2 is a $\{2, 3\}$-group, and G_{x_2} is a $\{2, 3\}$-group as well. Moreover the hypothesis on F_1 implies that G_{x_1} is a $\{2, 5\}$-group whose order is not divisible by 25. In particular G is a $\{2, 3, 5\}$-group whose order is divisible by 5 but not by 25 in view of Lemma 3.2.

Lemma 3.10 ensures that the N-actions on VX_1 and on VX_2 are both non-free. Thus $\mathrm{C}_G(N) = \{1\}$ by Lemma 3.9; in particular N is not abelian. Thus 5 divides $|N|$, and since 25 does not divide $|G|$, we infer that N is simple non-abelian and that G is almost simple. From Proposition 2.13, we have $N \cong \mathrm{Alt}(5), \mathrm{Alt}(6)$ or $\mathrm{U}_4(2)$. Moreover Lemma 3.2 affords a factorization $G = G_{x_1} G_{x_2}$ of G as a product of two solvable subgroups.

If $N \cong \mathrm{Alt}(5)$, then $|G| = 60$ or 120, while $|G_{x_1}| = 20, 40$ or 80 by Corollary 2.11. Therefore $|G_{x_2}| \leqslant 6$, whence $d_2 = 3$ and $G_{x_2} \cong \mathrm{Sym}(3)$. Hence the graph X_2, which is a Cayley graph for G_{x_1}, contradicts [LL09, Theorem 1.1] in that case.

If $N \cong \mathrm{Alt}(6) \cong \mathrm{PSL}_2(\mathbf{F}_9)$, we invoke [LX14, Proposition 4.1] and deduce that G_{x_2} has a normal 3-Sylow subgroup, whose order is 9. Thus $d_2 = 4$ by Corollary 2.11, and we get a contradiction since a finite group with a normal 3-Sylow subgroup cannot have a quotient isomorphic to $\mathrm{Alt}(4)$ or $\mathrm{Sym}(4)$.

Finally, if $N \cong \mathrm{U}_4(2)$, then [LX14, Proposition 4.1] ensures that G_{x_1} has a normal subgroup isomorphic to $\mathbf{C}_2^4$. Since the only normal 2-group in $G_{x_1}/G_{x_1}^{[1]} \cong \mathbf{C}_5 \rtimes \mathbf{C}_4$ is the trivial one, we deduce that $G_{x_1}^{[1]}$ contains a subgroup isomorphic to $\mathbf{C}_2^4$. By Corollary 2.10, the group $G_{x_1}^{[1]}$ is isomorphic to a subgroup of a point stabilizer in F_1. In particular $|G_{x_1}^{[1]}| \leqslant 4$, a contradiction. □

3.8. If N is not simple then X_1 is a complete bipartite graph

Lemma 3.12. — *Let* $(X_1, X_2, G) \in \mathcal{F}(F_1, F_2)$ *satisfy* (**Min**). *Assume that:*

(1) *There is* $i \in \{1, 2\}$ *such that* N *does not act freely on* VX_i.
(2) *For* $j = 1$ *and* 2*, if* $d_j \geqslant 7$ *then* $F_j \geqslant \mathrm{Alt}(d_j)$.
(3) N *is not simple.*
(4) $d_1 \geqslant d_2$.

Then X_1 *is the complete bipartite graph* $\mathbf{K}_{d_1, d_1}$ *and one of the following conditions holds:*

(i) $d_2 = 3$ *and* $d_1 = 24$.
(ii) $d_2 = 4$, *and* $d_1 \in \left\{6n \mid n \geqslant 2 \text{ divides } 2^2 \cdot 3^5\right\}$.
(iii) $d_2 = 5$, $F_2 \cong \mathbf{C}_5 \rtimes \mathbf{C}_4$ *and* $d_1 \in \left\{10, 20, 40\right\}$.
(iv) $d_2 = 5$, $\mathrm{soc}(F_2) \cong \mathrm{Alt}(5)$ *and* $d_1 \in \left\{30n \mid n \geqslant 2 \text{ divides } 2^8 \cdot 3\right\}$.
(v) $d_2 = 6$, $\mathrm{soc}(F_2) \cong \mathrm{Alt}(5)$, *and* $d_1 \in \left\{30n \mid n \geqslant 2 \text{ divides } 2^3 \cdot 5^2\right\}$.
(vi) $d_2 \geqslant 6$, *and* $d_1 \in \left\{\frac{d_2!}{2}, \frac{d_2!(d_2-1)!}{4}, \frac{d_2!(d_2-1)!}{2}\right\}$.

Proof. — The group N is characteristically simple, so that $N = S_1 \times \cdots \times S_k$, where S_i is isomorphic to a finite simple group S for all i. Moreover S is not abelian and $d_1 \geqslant 5$ by Lemma 3.9. The condition (3) ensures that $k \geqslant 2$. Furthermore we have $C_G(N) = \{1\}$ by Lemma 3.9.

The first step is to establish the following.

Claim. — *There is* $j \in \{1, 2\}$ *such that* $d_j \geqslant 5$, $F_j \not\cong \mathbf{C}_5 \rtimes \mathbf{C}_4$ *and* N *does not act freely on* VX_j.

By Lemma 3.9, we have $d_1 \geqslant 5$. Assume that $j = 1$ does not satisfy the claim. Then either N acts freely on VX_1, or N does not act freely on VX_1 and $F_1 \cong \mathbf{C}_5 \rtimes \mathbf{C}_4$.

If N acts freely on VX_1, then it acts non-freely on VX_2 by (1), and it follows from Lemma 3.7 that $|G_{x_1}|$ has a subgroup of index at most 2 whose order divides $|G_{x_2} : N_{x_2}|$. Since $d_1 \geqslant 5$, it follows that $|G_{x_1}|$, and thus also $|G_{x_2}|$ is divisible by 5. In particular $d_2 \geqslant 5$. Moreover $|N|$ divides $|VX_1|$ which is equal to $|G_{x_2}|$ by Lemma 3.2. Thus $|G_{x_2}|$ has at least 3 prime divisors (because N is not solvable). In particular $F_2 \not\cong \mathbf{C}_5 \rtimes \mathbf{C}_4$. Thus $j = 2$ satisfies the claim in this case.

Assume now that N does not act freely on VX_1 and that $F_1 \cong \mathbf{C}_5 \rtimes \mathbf{C}_4$. Hence $d_2 \leqslant d_1 = 5$ by (4). In view of Lemma 3.11, we have $d_2 = 5$. Moreover N does not act freely on VX_2 by Lemma 3.10. It follows that $\mathrm{soc}(F_2) = \mathrm{Alt}(5)$ since otherwise G_{x_1} and G_{x_2} would both be $\{2, 5\}$-groups, contradicting that N is non-abelian. Thus $j = 2$ satisfies the claim in this case as well. This ends the proof of the claim.

In view of the claim, we may, upon replacing i by $3-i$, strengthen the hypothesis (1) and assume in addition that $d_i \geqslant 5$ and that $F_i \not\cong \mathbf{C}_5 \rtimes \mathbf{C}_4$. In particular $\mathrm{soc}(F_i)$ is simple and 2-transitive.

Assume next that the S_1-action on VX_i is not free. In particular the S_j-action on VX_i is not free for all $j \in \{1, \ldots, k\}$ since the simple factors of N are permuted transitively under the conjugation action of G.

Since $d_i \geqslant 5$ and $\mathrm{soc}(F_i)$ is simple, we know that the socle of $N_v/N_v^{[1]}$ is simple and 2-transitive on $E(v)$ for every vertex $v \in VX_i$. For $j \neq m \in \{1, \dots, k\}$ and any $v \in VX_i$, it follows that if $(S_j)_v$ is non-trivial on $E(v)$ then $(S_m)_v$ is trivial on $E(v)$. We now apply Lemma 2.5 to the normal subgroups S_j and S_m of N. For each of them we get a bipartition of X_i, and the previous observation together with the fact that S_j and S_m are conjugate in G implies that $(S_j)_v$ is non-trivial on $E(v)$ if and only if $(S_m)_v$ is trivial on $E(v)$ for all $v \in VX_i$. Since this holds for all pairs $j \neq m \in \{1, \dots, k\}$, it follows that $k = 2$. Given two adjacent vertices v, w such that $(S_1)_v$ is non-trivial on $E(v)$, we know infer that $(S_1)_v$ fixes all neighbours of w and $(S_2)_w$ fixes all neighbours of v. Using that $(S_1)_v$ is transitive on the neighbours of v (resp. $(S_2)_w$ is transitive on the neighbours of w) we deduce that X_i is the complete bipartite graph K_{d_i,d_i}. Since $d_{3-i} \geqslant 3$ and $G_{x_{3-i}}$ has a 2-transitive action on a set of cardinality d_{3-i}, we obtain

$$2d_{3-i} \leqslant d_{3-i}\,(d_{3-i} - 1) \leqslant \left|G_{x_{3-i}}\right| = |VX_i| = 2d_i.$$

Hence $d_i = \max\{d_1, d_2\} = d_1$. Moreover the equality case $d_1 = d_2$ occurs only if $d_1 = d_2 = 3$, which is impossible since $d_1 \geqslant 5$. It follows that $d_1 > d_2$, hence $i = 1$. So X_1 is the complete bipartite graph $\mathbf{K}_{d_1,d_1}$. Moreover $G_{x_1}/G_{x_1}^{[1]} \cong F_1$ is almost simple, with socle equal to $\mathrm{Alt}(d_1)$ if $d_1 \geqslant 7$.

If $d_2 = 3$, we invoke [LL09, Theorem 1.1] and deduce that $d_1 = 24$.

If $d_2 \geqslant 4$ we use the fact that the order of G_{x_2}, which is equal to $|VX_1| = 2d_1$, is subject to Corollary 2.11. This provides numerical constraints on (d_1, d_2). Those can be slightly strengthened by observing that G_{x_2} acts vertex-transitively on the complete bipartite graph $\mathbf{K}_{d_1,d_1}$, and thus possesses a subgroup of index 2. In particular G_{x_2} cannot be $\mathrm{Alt}(d_2)$ or $\mathrm{Alt}(d_2) \times \mathrm{Alt}(d_2 - 1)$ for all $d_2 \geqslant 4$. The required conditions (i)–(vi) follow.

We assume henceforth that the S_1-action on VX_i is free. In particular $|S|$ divides $|VX_i| = |G_{x_{3-i}}|$, so that $d_{3-i} \geqslant 5$ and $F_{3-i} \not\cong \mathbf{C}_5 \rtimes \mathbf{C}_4$. In particular, if the action of S_1 (hence of N) on VX_{3-i} is not free, then $j = 1$ and 2 both satisfy the claim above, and we may thus argue as in the case already treated.

It remains to consider the case where all simple factors of N act freely on both VX_1 and VX_2, since G permutes transitively the simple factors of N. In particular $|S|$ divides $|VX_j| = |G_{x_{3-j}}|$ for $j = 1$ and 2, hence $d_1 \geqslant d_2 \geqslant 5$ and $F_1 \not\cong \mathbf{C}_5 \rtimes \mathbf{C}_4 \not\cong F_2$. In particular F_1 and F_2 are both almost simple by the hypothesis (2).

The rest of the proof aims at reaching a contradiction, thereby showing that the only possible situation is the one we have just described. We distinguish two cases.

Case 1. $d_1 \leqslant 6$. — Then the only prime divisors of $|G_{x_1}|$ and $|G_{x_2}|$ are 2, 3 and 5. Thus the same holds for $|G|$, whence also $|S|$, by Lemma 3.2. Moreover $\mathrm{soc}(F_1)$ and $\mathrm{soc}(F_2)$ are isomorphic to $A_5 \cong \mathrm{PSL}_2(\mathbf{F}_5)$ (acting on 5 or 6 points) or A_6 (acting on 6 points), see Table 2.1. By Corollary 2.11, this implies that 3^4 does not divide $|G_{x_j}|$ for $j = 1$ and 2. In particular 3^7 does not divide $|G|$ by Lemma 3.2, so that $S \cong \mathrm{Alt}(5)$ or $\mathrm{Alt}(6)$ by Proposition 2.13.

We claim that if N acts freely on VX_1, then $d_1 = d_2 = 6$. Indeed, if the claim fails, then N acts freely on VX_1 and $d_2 = 5$ since $5 \leqslant d_2 \leqslant d_1 \leqslant 6$. The hypothesis (1) implies that N does not act freely on VX_2. By Lemma 3.7, the stabilizer G_{x_1} has a subgroup $G_{x_1}^+$ of index at most 2 whose order divides $|G_{x_2} : N_{x_2}|$. Since $d_2 = 5$, the group $G_{x_2}/G_{x_2}^{[1]}$ is almost simple with socle $\mathrm{Alt}(5)$. Moreover $N_{x_2}/N_{x_2}^{[1]}$ contains the socle of $G_{x_2}/G_{x_2}^{[1]}$ since the N-action on VX_2 is not free. Using Corollary 2.11, we deduce that $|G_{x_2} : N_{x_2}|$ is not divisible by 5, whereas 5 divides $|G_{x_1}^+|$. This is a contradiction.

In view of the claim, we may, upon swapping the indices 1 and 2, assume that the N-action on VX_1 is not free. As observed above, we have $d_j \in \{5, 6\}$ and $\mathrm{soc}(F_j) \in \{\mathrm{Alt}(5), \mathrm{Alt}(6)\}$ for $j = 1, 2$ in the case at hand. We shall now consider three cases successively namely $(d_1, \mathrm{soc}(F_1))) = (6, \mathrm{Alt}(6))$, $(6, \mathrm{Alt}(5))$ or $(5, \mathrm{Alt}(5))$.

If $(d_1, \mathrm{soc}(F_1))) = (6, \mathrm{Alt}(6))$, then $N_{x_1}/N_{x_1}^{[1]}$ contains $\mathrm{Alt}(6)$, so that $S \cong \mathrm{Alt}(6)$. In particular $|N|$, hence also $|G|$, is divisible by 3^{2k}. We have already seen that $|G|$ is not divisible by 3^7, so that $k \leqslant 3$. Corollary 2.10 ensures that $G_{x_1}^{[1]}$ is either trivial, or almost simple with socle isomorphic to $\mathrm{Alt}(5)$. In particular there is no homomorphism $G_{x_1} \to \mathrm{Sym}(3)$ with transitive image. Recall from Corollary 2.6 that N has at most two orbits on VX_1. In particular $|G : G_{x_1}N| \leqslant 2$. Since the conjugation action of G permutes transitively the simple factors of N, the case $k = 3$ is impossible, and we have $k = 2$. Hence $N = S_1 \times S_2 \cong \mathrm{Alt}(6) \times \mathrm{Alt}(6)$ and we know that $N_{x_1}/N_{x_1}^{[1]}$ is almost simple with socle $\mathrm{Alt}(6)$. Considering the projection of N_{x_1} on the simple factors of N, we deduce that image of $N_{x_1}^{[1]}$ under at least one of these projections must be trivial. In other words $N_{x_1}^{[1]}$ is contained in one of the two simple factors of N. We have seen above that all simple factors of N act freely on VX_1. Therefore $N_{x_1}^{[1]} = \{1\}$. Thus $G_{x_1}^{[1]} \cap N = \{1\}$, so that $G_{x_1}^{[1]}$ embeds in the quotient group G/N. Since $C_G(N) = \{1\}$ and $N \cong \mathrm{Alt}(6) \times \mathrm{Alt}(6)$, the quotient G/N embeds in $(\mathrm{Out}(\mathrm{Alt}(6)) \times \mathrm{Out}(\mathrm{Alt}(6))) \rtimes \mathrm{Sym}(2)$, which is a 2-group. On the other hand, by Corollary 2.10, the group $G_{x_1}^{[1]}$ is either trivial or almost simple (with socle $\mathrm{Alt}(5)$). We deduce that $G_{x_1}^{[1]} = \{1\}$. Therefore we have $N_{x_1} \cong \mathrm{Alt}(6)$ and $|G_{x_1} : N_{x_1}| \leqslant 2$, so that the N-action on VX_2 is not free by Lemma 3.7. We now distinguish 3 subcases.

If $(d_2, \mathrm{soc}(F_2)) = (6, \mathrm{Alt}(6))$, then by symmetry we have $N_{x_2} \cong \mathrm{Alt}(6)$, and it then follows from Proposition 2.16 that $N_{x_1} \cap N_{x_2}$ is non-trivial. This is absurd since the G-action on $VX_1 \times VX_2$ is free.

If $(d_2, \mathrm{soc}(F_2)) = (6, \mathrm{Alt}(5))$, then $|G_{x_2}|$ is not divisible by 3^2 in view of Corollary 2.11, and we obtain a contradiction since $|N|$, whence also $|G|$, is divisible by 3^4.

If $(d_2, \mathrm{soc}(F_2)) = (5, \mathrm{Alt}(5))$, we consider the group $H = NG_{x_1}$, which is of index at most 2 in G since the N-action on VX_1 has at most 2 orbits. The local action of H on VX_2 is $\mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$, so $H_{x_2}/H_{x_2}^{[1]} = \mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$. Moreover $|H_{x_2}| = |N : N_{x_1}| = 2^3.3^2.5$, so that $|H_{x_2}^{[1]}| = 3$ or 6. On the other hand, consider a vertex $y_2 \in VX_2$ adjacent to x_2. By Theorem 2.9 (ii), the group $H_{x_2}^{[1]} \cap H_{y_2}^{[1]}$ is a 2-group. Therefore the natural image of $H_{x_2}^{[1]}$ in $H_{y_2}/H_{y_2}^{[1]}$ is non-trivial. Moreover it is isomorphic to a normal subgroup of a point stabilizer in $\mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$. Since the latter groups are 3-transitive, it follows that the order of $H_{x_2}^{[1]}$ is divisible by 4, a contradiction. This finishes the case $(d_1, \mathrm{soc}(F_1)) = (6, \mathrm{Alt}(6))$.

If $(d_1, \mathrm{soc}(F_1)) = (6, \mathrm{Alt}(5))$, then $|G_{x_1}|$ is not divisible by 3^2 in view of Corollary 2.11. It follows that the N-action on VX_2 cannot be free, since otherwise $|N|$

would divide $|G_{x_1}| = |VX_2|$, so the latter would be divisible by $3^k \geqslant 3^2$. We may thus assume that $\mathrm{soc}(F_2) \cong \mathrm{Alt}(5)$, since otherwise $(d_2, \mathrm{soc}(F_2)) = (6, \mathrm{Alt}(6))$ and we may swap X_1 and X_2 and invoke the case that has already been treated. If $d_2 = 6$, then $|G_{x_2}|$ is not divisible by 3^2 by Corollary 2.11, so that $|G|$ is not divisible by 3^3. This yields $k = 2$. If $d_2 = 5$, then $|G_{x_2}|$ is not divisible by 3^3 by Corollary 2.11, so that $|G|$ is not divisible by 3^4. Thus $k \leqslant 3$, but if $k = 3$, then $|N|$ is divisible by 3^3 and $|G/N|$ is divisible by 3 since G permutes transitively the simple factors of N. Since $|G|$ is not divisible by 3^4, we obtain $k = 2$ in all cases. If $S \cong \mathrm{Alt}(6)$, then $|N|$ is divisible dy 3^4, which is impossible. So $S \cong \mathrm{Alt}(5)$ and $N \cong \mathrm{Alt}(5) \times \mathrm{Alt}(5)$. Since the N-action on both VX_1 and VX_2 is non-free, it follows that $N_{x_j}/N_{x_j}^{[1]}$ contains $\mathrm{Alt}(5)$ for $j = 1, 2$. Since the simple factors of N act freely on VX_1 and VX_2, we have $N_{x_j}^{[1]} = \{1\}$. Using again Proposition 2.16, we deduce that $N_{x_1} \cap N_{x_2}$ is non-trivial, a contradiction.

If $(d_1, \mathrm{soc}(F_1)) = (5, \mathrm{Alt}(5))$, then $|G_{x_1}|$ is not divisible by 5^2. It follows that the N-action on VX_2 cannot be free, since otherwise $|N|$ would divide $|G_{x_1}| = |VX_2|$, so the latter would be divisible by $5^k \geqslant 5^2$. Moreover we have $(d_2, \mathrm{soc}(F_2)) = (5, \mathrm{Alt}(5))$, since $d_1 \geqslant d_2 \geqslant 5$. In particular $|G_{x_2}|$ is not divisible by 5^2, hence $k = 2$. We cannot have $S \cong \mathrm{Alt}(5)$, since otherwise we would get the same contradiction as in the previous paragraph. Thus $S \cong \mathrm{Alt}(6)$. Thus $|G_{x_1}^{[1]}|$ and $|G_{x_2}^{[1]}|$ are both divisible by 3 since $|G| = |G_{x_1}||G_{x_2}|$. Thus $|N_{x_1}^{[1]}|$ is divisible by 3 since otherwise $|G/N|$ would be divisible by 3. This is not the case since $C_G(N) = \{1\}$, so that the quotient G/N embeds in $(\mathrm{Out}(\mathrm{Alt}(6)) \times \mathrm{Out}(\mathrm{Alt}(6))) \rtimes \mathrm{Sym}(2)$, which is a 2-group. We now consider the projection of N_{x_1} to each simple factor S_j of N. Since $N_{x_1}/N_{x_1}^{[1]}$ contains $\mathrm{Alt}(5)$ and since the only subgroups of $\mathrm{Alt}(6)$ containing a subnormal subgroup isomorphic to $\mathrm{Alt}(5)$ are $\mathrm{Alt}(5)$ and $\mathrm{Alt}(6)$, we deduce that $N_{x_1}^{[1]}$ is contained in one of the two simple factors of N. This is impossible, since all the simple factors of N act freely on VX_1. This proves that the case $d_1 \leqslant 6$ does not occur.

Case 2. $d_1 \geqslant 7$. — We claim that if N acts freely on VX_1, then $d_1 = d_2$. Indeed, if the claim fails, then N acts freely on VX_1 and $d_1 > d_2$. The hypothesis (1) implies that N does not act freely on VX_2. By Lemma 3.7, the stabilizer G_{x_1} has a subgroup $G_{x_1}^+$ of index at most 2 whose order divides $|G_{x_2} : N_{x_2}|$. By the discussion directly preceding Case (1), the group $G_{x_2}/G_{x_2}^{[1]}$ is almost simple. Moreover $N_{x_2}/N_{x_2}^{[1]}$ contains the socle of $G_{x_2}/G_{x_2}^{[1]}$ since the N-action on VX_2 is not free. Since $d_1 \geqslant 7$, we deduce from the hypothesis (2) that 7 divides $|G_{x_1}^+|$, hence also $|G_{x_2} : N_{x_2}|$. It then follows from Corollary 2.11 that $d_2 \geqslant 8$. We may then invoke Corollary 2.10 to establish that $\frac{d_1!}{4}$ divides $|G_{x_1}^+|$, and that $|G_{x_2} : N_{x_2}|$ divides $\frac{d_2!(d_2-1)!}{d_2!/2} = 2(d_2-1)!$. Since $|G_{x_1}^+|$ divides $|G_{x_2} : N_{x_2}|$, we deduce that $d_1 = d_2 = 8$. The claim follows.

In view of the claim, we may, upon swapping the indices 1 and 2, assume that the N-action on VX_1 is not free.

For $j \in \{1, 2\}$, if the permutation group F_j has almost simple stabilizers, then Corollary 2.10 ensures that

$$\left|G_{x_j}\right| \leqslant d_j!\,(d_j-1)! \leqslant d_1!\,(d_1-1)!.$$

This holds in particular for $j=1$. If the point stabilizers in F_2 are not almost simple, then the discussion directly preceding Case (1) implies that either $(d_2, \mathrm{soc}(F_2)) = (5, \mathrm{Alt}(5))$ or $(d_2, \mathrm{soc}(F_2)) = (6, \mathrm{Alt}(5))$. In all cases, we invoke Corollary 2.11, which respectively yields the following upper bounds:

$$|G_{x_2}| \leqslant 5!4!4^4$$

if $(d_2, \mathrm{soc}(F_2)) = (5, \mathrm{Alt}(5))$, or

$$|G_{x_2}| \leqslant 5!5!5$$

if $(d_2, \mathrm{soc}(F_2)) = (6, \mathrm{Alt}(5))$. In either case, we obtain

$$|G_{x_2}| \leqslant 7!6! \leqslant d_1!\,(d_1-1)!.$$

This proves that $|G| = |G_{x_1}||G_{x_2}| \leqslant d_1!^2(d_1-1)!^2$. On the other hand we know that $N_{x_1}/N_{x_1}^{[1]}$ contains the socle of F_1, which is the alternating group $\mathrm{Alt}(d_1)$ in the case at hand. Considering the projection of N_{x_1} to each of the simple factors of N, we infer that $d_1!/2 = |\mathrm{Alt}(d_1)| \leqslant |S|$. This yields

$$\frac{d_1!^k}{2^k} \leqslant |S|^k = |N| \leqslant |G| \leqslant d_1!^2\,(d_1-1)!^2.$$

We deduce that $k \leqslant 3$. In particular $\mathrm{Out}(N)$ is solvable, so that the N-action on VX_2 is not free since otherwise G_{x_2} would map injectively in $\mathrm{Out}(N)$ since $C_G(N) = \{1\}$, contradicting that G_{x_2} has a non-abelian simple subquotient. Moreover, the group NG_{x_1} has index at most 2 in G by Corollary 2.6, and G permutes transitively the k simple factors of N. Thus, if $k=3$ then G_{x_1} has a transitive action on a 3-point set. However, by Corollary 2.10, the group G_{x_1} does not have any subgroup of index 3. Thus $k=2$. Hence $N = S_1 \times S_2 \cong S \times S$.

The N-action on both VX_1 and VX_2 is non-free, hence each has at most 2 orbits. Recall moreover that the S_1- and S_2-actions on both VX_1 and VX_2 are all free. In particular $|S|$ divides both $|VX_1| = |G_{x_2}|$ and $|VX_2| = |G_{x_1}|$.

Assume that $G_{x_1}^{[1]}$ is non-trivial. Then it is almost simple with socle $\mathrm{Alt}(d_1-1)$ by Corollary 2.10. Therefore so is $N \cap G_{x_1}^{[1]} = N_{x_1}^{[1]}$, since $C_G(N) = \{1\}$ and $\mathrm{Out}(N)$ is solvable. Since both simple factors of N act freely on VX_1, we see that the projection map $N \to S_1$ yields an injective homomorphism of N_{x_1} into S. Since $|S|$ divides $|G_{x_1}|$, we obtain that $\frac{d_1!}{2}\frac{(d_1-1)!}{2}$ divides $|S|$, which in turn divides $d_1!(d_1-1)!$. It follows that the image of N_{x_1} into S has index at most 4. Since S is simple, the image of N_{x_1} into S must be surjective, which is absurd since the normal subgroup $N_{x_1}^{[1]}$ is non-trivial. This proves that $G_{x_1}^{[1]} = \{1\}$.

Invoking again that N_{x_1} maps injectively to S and that $|S|$ divides $|G_{x_1}| \in \{d_1!, d_1!/2\}$, we now deduce that $S \cong \mathrm{Alt}(d_1) \cong N_{x_1}$. Since N has at most 2 orbits on VX_1, we deduce that $|G_{x_2}| = |VX_1| \in \{|N : N_{x_1}|, 2|N : N_{x_1}|\} = \{d_1!/2, d_1!\}$. In particular $d_2 \geqslant 7$. We may thus apply the same arguments for G_{x_2} as for G_{x_1} in the previous paragraph to establish that $G_{x_2}^{[1]} = \{1\}$ and that $N_{x_2} \cong \mathrm{Alt}(d_1) \cong N_{x_1}$. Since $N = S_1 \times S_2 \cong \mathrm{Alt}(d_1) \times \mathrm{Alt}(d_1)$, we deduce from Proposition 2.16 that $N_{x_1} \cap N_{x_2}$ is non-trivial. Therefore the G-action on $VX_1 \times VX_2$ is not free. This final contradiction finishes the proof of Lemma 3.12. □

3.9. If N is simple then X_1 is a complete graph

LEMMA 3.13. — *Let* $(X_1, X_2, G) \in \mathcal{F}(F_1, F_2)$ *satisfy* (**Min**). *Assume that:*

(1) *There is* $i \in \{1,2\}$ *such that* N *does not act freely on* VX_i.
(2) *For* $j = 1$ *and* 2, *if* $d_j \geqslant 7$ *then* $F_j \geqslant \mathrm{Alt}(d_j)$.
(3) N *is simple.*
(4) $d_1 \geqslant d_2$.

Then X_1 *is the complete graph* $\mathbf{K}_{d_1+1}$, *and one of the following conditions holds:*

(i) $d_2 = 3$, *and* $d_1 \in \Big\{23, 47\Big\}$.
(ii) $d_2 = 4$, *and* $d_1 \in \Big\{12n - 1 \mid n \geqslant 2 \text{ divides } 2^2 \cdot 3^5\Big\}$.
(iii) $d_2 = 5$, $F_2 \cong \mathbf{C}_5 \rtimes \mathbf{C}_4$ *and* $d_1 \in \Big\{19, 39, 79\Big\}$.
(iv) $d_2 = 5$, $\mathrm{soc}(F_2) \cong \mathrm{Alt}(5)$ *and* $d_1 \in \Big\{60n - 1 \mid n \text{ divides } 2^8 \cdot 3\Big\}$.
(v) $d_2 = 6$, $\mathrm{soc}(F_2) \cong \mathrm{Alt}(5)$, *and* $d_1 \in \Big\{60n - 1 \mid n \text{ divides } 2^3 \cdot 5^2\Big\}$.
(vi) $d_2 \geqslant 6$, *and* $d_1 \in \Big\{\frac{d_2!}{2} - 1, d_2! - 1, \frac{d_2!(d_2-1)!}{4} - 1, \frac{d_2!(d_2-1)!}{2} - 1, d_2!(d_2-1)! - 1\Big\}$.

Proof. — We know that N is non-abelian and that $d_1 \geqslant 5$ by Lemma 3.9. That lemma ensures that $\mathrm{C}_G(N) = \{1\}$, so that G is almost simple with socle N.

If $d_1 = 5$ and $F_1 \cong \mathbf{C}_5 \rtimes \mathbf{C}_4$, then $d_2 = 5$ by Lemma 3.11. In that case $F_2 \not\cong \mathbf{C}_5 \rtimes \mathbf{C}_4$ since otherwise G would be a $\{2,5\}$-group by Lemma 3.2, hence solvable, a contradiction. Therefore, upon replacing (X_1, X_2, G) by (X_2, X_1, G) in the case $d_1 = d_2 = 5$, we may assume without loss of generality that F_1 is almost simple. In particular G_{x_1} is not solvable, hence $N \cap G_{x_1} \neq \{1\}$ since $C_G(N) = \{1\}$ and $\mathrm{Out}(N)$ is solvable. Thus N does not act freely on VX_1.

Since $d_1 \geqslant d_2$, the hypothesis (2) implies that $\pi(G_{x_2}) \subseteq \pi(G_{x_1})$ with the notation of Section 2.7, so that $\pi(G) = \pi(G_{x_1})$. Moreover N is not contained in G_{x_1}, since G acts faithfully on X_1. Thus all the hypotheses of Corollary 2.15 are satisfied.

We shall now consider successively the seven exceptional cases of Corollary 2.15 displayed in Table 2.2 and show that none of them occurs. An observation that we shall used repeatedly is the following. Table 2.2 provides us with the possible values of the index $|N : N_{x_1}|$. We know moreover that N has at most two orbits on VX_1 (by Corollary 2.6) and the G_{x_2} acts regularly on VX_1 (by Lemma 3.2). Thus $|VX_1| = |G_{x_2}|$ equals $|N : N_{x_1}|$ or $2|N : N_{x_1}|$. This can be confronted with Corollary 2.11, which provides independent constraints that the number $|G_{x_2}|$ must satisfy.

The numbering of the cases below is chosen according to the numbering of the rows in Table 2.2.

Case 1. $N = \mathrm{Alt}(6)$ *and* $N_{x_1} = \mathrm{PSL}_2(\mathbf{F}_5)$. — By Corollary 2.15, we have $|N : N_{x_1}| = 6$. Hence $|G_{x_2}| = |VX_1| \in \{6, 12\}$, so that $d_2 \leqslant 4$. From [LL09, Theorem 1.1], we deduce that $d_2 \neq 3$. Thus $|VX_1| = 12$, and $d_2 = 4$. Hence $G_{x_2} \cong \mathrm{Alt}(4)$, so G_{x_2} does not have any subgroup of index 2. But N acts with two orbits on VX_1, so that NG_{x_1} is an index 2 subgroup of G, and $G_{x_2} \cap NG_{x_1}$ is an index 2 subgroup of G_{x_2} by Lemma 3.2. This is a contradiction.

Case 2. $N = \mathrm{U}_3(5)$ *and* $N_{x_1} = \mathrm{Alt}(7)$. — By Corollary 2.15, we have $|N : N_{x_1}| = 2 \cdot 5^2$. Hence $|G_{x_2}| = |VX_1| \in \{50, 100\}$. This is impossible by Corollary 2.11.

Case 3. $N = \mathrm{U}_4(2)$ *and* $N_{x_1} \leqslant 2^4.\mathrm{Alt}(5)$ *or* $N_{x_1} \leqslant \mathrm{Sym}(6)$. — Then the only primes dividing $|N : N_{x_1}|$ are 2 and 3, so that G_{x_2} is a $\{2,3\}$-group. In particular it is solvable, and $d_2 \in \{3,4\}$. We may thus invoke [LX14, Theorem 1.1]; it follows that the triple (G, G_{x_2}, G_{x_1}) must be as in row 10 or 11 of [LX14, Table 1.2]. In the former case we have $N_{x_1} = 2^4.\mathrm{Alt}(5)$, so that $|G_{x_2}| = 27$ or 54. This is impossible by Corollary 2.11. Thus the triple (G, G_{x_2}, G_{x_1}) is as in row 11 of [LX14, Table 1.2], and $N_{x_1} = \mathrm{Alt}(5), \mathrm{Sym}(5), \mathrm{Alt}(6)$ or $\mathrm{Sym}(6)$. If N is transitive on VX_1, then the hypotheses of [LX14, Lemma 8.30] are satisfied and we get a contradiction. Thus $|G : NG_{x_1}| = 2$. Since $\mathrm{Out}(N)$ is of order 2, we deduce that $N = NG_{x_1}$. In particular $G_{x_1} \leqslant N$ and N_{x_2} is of index 2 in G_{x_2}. Moreover, the information provided by [LX14, Table 1.2] ensures that N_{x_2} is a subgroup of $3_+^{1+2} : 2.\mathrm{Alt}(4)$, which is a parabolic subgroup of $\mathrm{PSp}_4(3) \cong \mathrm{U}_4(2)$. Observe that the natural action of $\mathrm{Alt}(4)$ on the Heisenberg group 3_+^{1+2} does not preserve any subgroup of order 3^2; therefore the largest power of 3 dividing $|N_{x_2}|$ (and hence also G_{x_2}) cannot be 3^3. On the other hand, we have $|N_{x_2}| = |N : N_{x_1}|$. We deduce that $N_{x_1} = G_{x_1}$ can neither be $\mathrm{Alt}(5)$ nor $\mathrm{Sym}(5)$, so that it is $\mathrm{Alt}(6)$ or $\mathrm{Sym}(6)$. The latter possibility is excluded because $\mathrm{Sym}(6)$ is maximal in N, and the factorization $N = N_{x_1}N_{x_2}$ would then contradict [LPS10, Theorem 1.1]. Thus $N_{x_1} = G_{x_1} = \mathrm{Alt}(6)$. It follows that $|N_{x_2}| = 2^3 \cdot 3^2$, hence $N_{x_2} \cong 3 : 2.\mathrm{Alt}(4)$. It follows that N is locally 2-transitive on X_2 with local action at every vertex isomorphic to $\mathrm{Alt}(4)$ by Lemma 2.5. It follows that the point stabilizers in $N_v/N_v^{[1]}$ are cyclic of order 3 for all $v \in VX_2$, so that $N_{x_2}^{[1]}$ is a 3-group. This contradicts that $N_{x_2} \cong 3 : 2.\mathrm{Alt}(4)$.

Case 4. $N = \mathrm{U}_4(3)$ *and* $N_{x_1} = \mathrm{Alt}(7)$. — Then $|N : N_{x_1}| = 2^4 \cdot 3^4$. Thus G_{x_2} is a $\{2,3\}$-group, hence solvable. We may thus invoke [LX14, Theorem 1.1], which yields a contradiction.

Case 5. $N = \mathrm{PSp}_4(7)$ *and* $N_{x_1} = \mathrm{Alt}(7)$. — Then $|N : N_{x_1}| = 2^5 \cdot 5 \cdot 7^3$, so that $|G_{x_2}| \in \{|N : N_{x_1}|, 2|N : N_{x_1}|\}$ violates Corollary 2.11.

Case 6. $N = \mathrm{Sp}_6(2)$ *and* $N_{x_1} = \mathrm{Alt}(7), \mathrm{Sym}(7), \mathrm{Alt}(8)$ *or* $\mathrm{Sym}(8)$. — Here again G_{x_2} is a $\{2,3\}$-group, hence solvable. Since $\mathrm{Out}(N)$ is trivial in this case, we have $G = N$ so that the hypotheses of [LX14, Lemma 8.30] are satisfied. The latter result yields a contradiction.

Case 7. $N = \mathrm{P\Omega}_8^+(2)$ *and* $N_{x_1} \leqslant P_1, P_3, P_4$ *or* $N_{x_1} \leqslant \mathrm{Alt}(9)$. — Then G_{x_2} is a $\{2,3,5\}$-group whose order is divisible by 30, so that $d_2 \in \{5,6\}$ and $F_2 \not\cong \mathbf{C}_5 \rtimes \mathbf{C}_4$.

Let us first consider the case where N_{x_1} is contained in a parabolic subgroup P_k. Then the socle of $N_{x_1}/N_{x_1}^{[1]}$ must be isomorphic to the Levi factor of P_k, which is $\mathrm{SL}_4(\mathbf{F}_2) \cong \mathrm{Alt}(8)$. It follows that $|N : N_{x_1}|$ is divisible by $3^3 \cdot 5$, but $|N : N_{x_1}|$ is not divisible by 25. This contradicts Corollary 2.11 for the $|G_{x_2}|$.

We now assume that $N_{x_1} \leqslant \mathrm{Alt}(9)$. If N_{x_1} is a proper subgroup of $\mathrm{Alt}(9)$, the same numerical considerations as in the case $N_{x_1} \leqslant P_k$ yield a contradiction. It follows that $N_{x_1} = \mathrm{Alt}(9)$, so $|G_{x_2}| = 2^a \cdot 3 \cdot 5$ with $a = 6$ or 7. Using Corollary 2.11, we infer that $d_2 = 5$, so $F_2 = \mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$ because $F_2 \not\cong \mathbf{C}_5 \rtimes \mathbf{C}_4$. Notice that N_{x_1} is a maximal subgroup of N in the case at hand. Therefore G_{x_1} is a maximal subgroup of the almost simple group NG_{x_1}. Denoting $G_{x_2}^+ = G_{x_2} \cap NG_{x_1}$, the factorization $G = G_{x_1}G_{x_2}$ yields a factorization $NG_{x_1} = G_{x_1}G_{x_2}^+$ since N has at most two orbits on VX_1. We may then invoke [LPS10, Theorem 1.1], which ensures that $G_{x_2}^+ = 2^4.\mathrm{Alt}(5)$. In particular $|G_{x_2}|/60 = 2^4$ or 2^5. Recall that $\mathrm{Alt}(5) \cong \Omega_4^-(2)$. A more precise look at the structure of $G_{x_2}^+$ afforded by that factorization reveals that $G_{x_2}^+ \cong \mathbf{F}_2^4 \rtimes \Omega_4^-(2)$, where $\mathbf{F}_2^4$ is the standard $\Omega_4^-(2)$-module: that information can be extracted from [Bau07, Examples (h) and (i) and Lemma 10.7].

On the other hand, Corollary 2.11 yields $d_2 = 5$, and we may invoke [Wei79, Theorem (1.2)] to elucidate the structure of G_{x_2}. Given the possible values for the order of G_{x_2}, we must have $s = 4$ in the notation of [Wei79, Theorem (1.2)], so that the latter result yields an embedding of G_{x_2} as a subgroup of $\mathbf{F}_4^2 \rtimes \mathrm{P\Gamma L}_2(\mathbf{F}_4)$ containing $\mathbf{F}_4^2 \rtimes \mathrm{PSL}_2(\mathbf{F}_4)$, where the action of $\mathrm{Alt}(5) \cong \mathrm{PSL}_2(\mathbf{F}_4)$ on $\mathbf{F}_4^2$ is the standard one. That embedding must map $G_{x_2}^+$ isomorphically onto $\mathbf{F}_4^2 \rtimes \mathrm{PSL}_2(\mathbf{F}_4)$. This is a contradiction, because the groups $\mathbf{F}_2^4 \rtimes \Omega_4^-(2)$ and $\mathbf{F}_4^2 \rtimes \mathrm{PSL}_2(\mathbf{F}_4)$ are not isomorphic (the two corresponding modules of $\mathrm{Alt}(5) \cong \Omega_4^-(2) \cong \mathrm{PSL}_2(\mathbf{F}_4)$ are not isomorphic).

Since all the seven exceptional cases of Corollary 2.15 are excluded, we deduce from the latter result that $N \cong \mathrm{Alt}(c)$ and $\mathrm{Alt}(k) \lhd N_{x_1} \leqslant \mathrm{Sym}(k) \times \mathrm{Sym}(c-k)$, where $k \leqslant c$ are integers such that $p \leqslant k$ for every prime $p \leqslant c$. Moreover $c \geqslant 5$ because $d_1 \geqslant 5$, and the case $c = 5$ is excluded since it would imply that $N \leqslant G_{x_1}$.

If $c = 6$, then $N_{x_1} = \mathrm{Alt}(5)$ and $|G_{x_2}| = 6$ or 12, and we obtain a contradiction with the same arguments as in Case (1) above. We assume henceforth that

$$c \geqslant 7.$$

Hence $G = \mathrm{Alt}(c)$ or $\mathrm{Sym}(c)$. Using the existence of a prime p with $\frac{c+1}{2} < p \leqslant c$ (see [WW80, 1.1] for a more general fact), we deduce from Corollary 2.10 that $G_{x_1}^{[1]} = \{1\}$, so that $G_{x_1} = \mathrm{Alt}(d_1)$ or $\mathrm{Sym}(d_1)$.

We shall now use the fact that the factorization $G = G_{x_1}G_{x_2}$ must be described by the main results from [WW80].

If $G = \mathrm{Alt}(c)$, we invoke [WW80, Theorem A]. Case III from [WW80, Theorem A] is impossible since $G_{x_1} = \mathrm{Alt}(d_1)$ or $\mathrm{Sym}(d_1)$. Case II is also impossible in view of our hypotheses on F_2 (special care is required in view of the isomorphism $\mathrm{Alt}(6) \cong \mathrm{PSL}_2(\mathbf{F}_9)$; however $\mathrm{PSL}_2(\mathbf{F}_q)$ appears in [WW80, Theorem A, Case II] only for prime powers q congruent to 3 modulo 4). Thus we are in Case I of [WW80, Theorem A]. This yields $G_{x_1} = \mathrm{Alt}(d_1) = \mathrm{Alt}(k)$ and G_{x_2} acts sharply t-transitively on $\{1, \dots, c\}$, where $t = c - k$.

If $G = \mathrm{Sym}(c)$, we invoke [WW80, Theorem S]. Using similar arguments, we obtain $d_1 = k$ and either G_{x_2} or its index 2 subgroup N_{x_2} acts sharply t-transitively on $\{1, \dots, c\}$, where $t = c - k$.

We next claim that $t = 1$. In order to establish this, we assume that $t \geqslant 2$ and discuss the value of d_2. We shall repeatedly use the fact that a sharply t-transitive group on a set of cardinality c is of order $c \cdot (c-1) \dots (c-t+1)$.

If $d_2 = 3$, then Corollary 2.11 yields $c = 3$ or 4, which is absurd.

If $d_2 = 4$, then Corollary 2.11 yields $c = 9$ and $t = 2$. It then follows that G_{x_2} or N_{x_2} is the affine group $\mathbf{F}_9 \rtimes \mathbf{F}_9^*$, which is absurd since $G_{x_2}/G_{x_2}^{[1]}$ is $\mathrm{Alt}(4)$ or $\mathrm{Sym}(4)$.

If $d_2 = 5$, then $F_2 \not\cong \mathbf{C}_5 \rtimes \mathbf{C}_4$ since $c \geqslant 7$. Thus $\mathrm{soc}(F_2) \cong \mathrm{Alt}(5)$ and G_{x_2} is a $\{2, 3, 5\}$-group. Assume now that $G_{x_2}^{[1]} \neq \{1\}$. It then follows from Theorem 2.9 that

$G_{x_2}^{[1]}$ has a non-trivial normal 2-subgroup. Therefore the same holds for G_{x_2}. Since G_{x_2} is 2-transitive on $\{1,\dots,c\}$, it follows that c is a power of 2. In view of [Cam99, Table 7.3], we must have $c = 16$ since $G_{x_2}/G_{x_2}^{[1]}$ is isomorphic to Alt(5) or Sym(5). We deduce that $t = 2$ (since otherwise $|G_{x_2}|$ would be divisible by 7), and we get a contradiction since the only sharply 2-transitive groups on 16 points are solvable. Thus $G_{x_2}^{[1]} = \{1\}$ and $G_{x_2} \cong \mathrm{Alt}(5)$ or Sym(5). Neither of these two groups has a t-transitive action on a set of $c \geqslant 7$ points.

If $d_2 = 6$ and $\mathrm{soc}(F_2) = \mathrm{PSL}_2(\mathbf{F}_5) \cong \mathrm{Alt}(5)$, then G_{x_2} is a again a $\{2,3,5\}$-group. If $G_{x_2}^{[1]} \neq \{1\}$ then Theorem 2.9 ensures that $O_5(G_{x_2}^{[1]})$ is of order 5 or 25. Hence $O_5(G_{x_2})$ is also of order 5 or 25. Since G_{x_2} is t-transitive on $\{1,\dots,c\}$ and $c \geqslant 7$, we obtain $c = 25 = |O_5(G_{x_2})|$. Moreover $t = 2$ since otherwise $|G_{x_2}|$ would be divisible by 23. Therefore $|G_{x_2}/O_5(G_{x_2})| = 24$, which is absurd since $O_5(G_{x_2}) \leqslant G_{x_2}^{[1]}$. This contradiction shows that $G_{x_2}^{[1]} = \{1\}$, so that $G_{x_2} \cong \mathrm{Alt}(5)$ or Sym(5). As before, we arrive at a contradiction since neither of these two groups has a t-transitive action on a set of $c \geqslant 7$ points.

If $d_2 \geqslant 6$ and $\mathrm{soc}(F_2) = \mathrm{Alt}(d_2)$, then $G_{x_2}^{[1]}$ is either trivial or almost simple with socle $\mathrm{Alt}(d_2 - 1)$ by Corollary 2.10. In the latter case G_{x_2} has two commuting normal subgroups of order $\frac{d_2!}{2}$ and $\frac{(d_2-1)!}{2}$ respectively. This prevents G_{x_2} from admitting any faithful 2-transitive action (since both normal subgroups would have to act freely and transitively, contradicting the fact that they have different orders). Hence $G_{x_2}^{[1]} = \{1\}$, so $G_{x_2} = \mathrm{Alt}(d_2)$ or $\mathrm{Sym}(d_2)$. In view of [Cam99, Table 7.4], the only 2-transitive action of the latter is the natural action on d_2 points, unless $d_2 = 6$, in which case there is a 2-transitive action on 10 points via the exceptional isomorphism $\mathrm{Alt}(6) \cong \mathrm{PSL}_2(\mathbf{F}_9)$. In that case we must have $G_{x_2} = \mathrm{Sym}(6)$, $c = 10$ and $t = 3$, so $d_1 = c - t = 7$ and $d_2 = 6$.

In order to exclude that case, we observe that by Lemma 3.2, the 7-regular graph X_1 is a Cayley graph of G_{x_2}. The corresponding generating set of G_{x_2} must thus contain an involution τ (because 7 is odd) that maps x_1 to a neighbouring vertex y_1. Thus τ normalizes G_{x_1,y_1}. Notice that $G_{x_1,y_1} \cong \mathrm{Alt}(6)$ or Sym(6) since $G_{x_1} \cong \mathrm{Alt}(7)$ or Sym(7). Moreover $\langle G_{x_1} \cup \{\tau\}\rangle$ is transitive on VX_1, and is thus the whole group G. Consider that G_{x_1}-action on $\{1,\dots,10\}$ given through the isomorphism $G \cong \mathrm{Alt}(10)$ or Sym(10). Upon reordering we may assume that the largest orbit of G_{x_1} is $\{1,\dots,7\}$ and that G_{x_1,y_1} fixes the point 1. Since we also know that $G_{x_1} \cong \mathrm{Alt}(7)$ or Sym(7), we deduce from [WW80, Theorems A and S] that G_{x_1} acts trivially on $\{8,9,10\}$. Since τ normalizes G_{x_1,y_1} which is isomorphic to Alt(6) or Sym(6), it must stabilize the set $\{2,\dots,7\}$. Thus, the set $\{8,9,10\} \setminus \{\tau(1)\}$, which is of size 2 or 3, is invariant under both G_{x_1} and τ. This contradicts the fact that $G = \langle G_{x_1} \cup \{\tau\}\rangle$.

This finally shows that $t = 1$. Thus $G_{x_1} = \mathrm{Alt}(d_1)$ or $\mathrm{Sym}(d_1)$ and $G = \mathrm{Alt}(d_1 + 1)$ or $\mathrm{Sym}(d_1 + 1)$, and the group G_{x_2} acts regularly on a set of cardinality $d_1 + 1$, so $|VX_1| = |G_{x_2}| = d_1 + 1$. It follows that X_1 is the complete graph $\mathbf{K}_{d_1+1}$. The numerical constraints satisfied by the pair (d_1, d_2) follow from the fact that the order of G_{x_2} is subject to Corollary 2.11. Furthermore, in case $d_2 = 3$, the more precise conclusion that $d_1 \in \{23, 47\}$ follows from [LL09].

It finally remains to exclude the case $(d_1, d_2) = (11, 4)$. In that case $G_{x_2} \cong \mathrm{Alt}(4)$ and $G = \mathrm{Sym}(12)$ or Alt(12), and $G_{x_1} = \mathrm{Sym}(11)$ or Alt(11). For such a triple

(G, G_{x_1}, G_{x_2}), we deduce from Lemma 3.2 that there exists an element $g \in G_{x_1}$ such that:

- $g^{-1} \in G_{x_2} g G_{x_2}$,
- $|G_{x_2} \backslash G_{x_2} g G_{x_2}| = 4$, and
- $G = \langle \{g\} \cup G_{x_2} \rangle$.

Using GAP, we enumerated all elements of Sym(11) and checked that none of them satisfies all of these three conditions. □

3.10. Proofs of Theorems 3.1 and 1.2

We are now ready to finish the proofs of the main results of this paper.

Proof of Theorem 3.1. — Retain the notation introduced in Section 3.1. Under the hypotheses of Theorem 3.1, we have $d_1 \geqslant d_2$ and the permutation group $F_i \leqslant \mathrm{Sym}(d_i)$ is 2-transitive. Moreover F_i contains $\mathrm{Alt}(d_i)$ if $d_i \geqslant 7$. We need to show that if the set $\mathcal{F}(F_1, F_2)$ is non-empty, then (d_1, d_2) satisfy the constraints listed in the statement of the theorem.

Assume that $\mathcal{F}(F_1, F_2)$ is non-empty. We may then choose $(X_1, X_2, G) \in \mathcal{F}(F_1, F_2)$ satisfying (Min). Let N be a minimal normal subgroup of G. Then N does not act freely on both X_1 and X_2 by Lemma 3.8. Moreover there is $i \in \{1, 2\}$ such that N does not act freely on VX_i by Lemma 3.6. If N is simple, then Lemma 3.13 applies, while if N is not simple, we invoke Lemma 3.12. In either case the required conclusion follows. □

Proof of Theorem 1.2. — Let $\Gamma \leqslant \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ and assume that Γ is reducible. We must show that (d_1, d_2) satisfies the required constraints.

For $i = 1, 2$, let K_i be the projection on $\mathrm{Aut}(T_i)$ of the kernel of the Γ-action on T_{3-i}. Then K_i does not contain any edge inversion by Lemma 3.3. We may therefore invoke Proposition 2.2. Since Γ is reducible, it follows that the quotient group $\Gamma/K_1 \times K_2$ is finite, and so is the quotient graph $X_i = K_i \backslash T_i$. The conclusion is now straightforward from Theorem 3.1. □

4. The just-infinite property

In this final section, we assemble the ingredients needed to establish Corollary 1.4.

We recall that a locally compact group is called *topologically simple* if its only closed normal subgroups are the trivial ones.

The following fundamental result of Bader–Shalom generalizes a result of Burger–Mozes [BM00b, Theorem 4.1] concerning certain lattices in products of trees. Although we shall invoke the result in the context of lattices in products of trees, we do need the more general version of Bader–Shalom, whose hypotheses on the structure of the ambient group are more flexible.

Theorem 4.1 (Bader–Shalom [BS06]). — *Let G_1, G_2 be compactly generated locally compact groups and $\Gamma \leqslant G_1 \times G_2$ be a cocompact lattice whose projection to G_1 and G_2 has dense image. Assume that for $i = 1$ and 2, the intersection M_i of* *all non-identity closed normal subgroups of G_i is topologically simple and that the quotient G_i/M_i is compact. Then Γ is hereditarily just-infinite.*

In the context of groups acting on trees, locally compact groups satisfying the conditions appearing in Theorem 4.1 pop up naturally. This is illustrated by the following result of Burger–Mozes and Nebbia.

Theorem 4.2 (Burger–Mozes [BM00a], Nebbia [Neb00]). — *Let T be a locally finite tree, all of whose vertices have degree $\geqslant 3$. Let also $G \leqslant \mathrm{Aut}(T)$ be a closed subgroup. If the G-action on the set of ends ∂T of T is 2-transitive, then G is compactly generated, the intersection M of all non-identity closed normal subgroups of G is topologically simple, and the quotient G/M is compact. Moreover the M-action on ∂T is 2-transitive.*

Proof. — This follows by combining several results from [BM00a] (see also [Neb00] for the case of regular trees). Details can be found in [CDM13, Proposition 2.1 and Theorem 2.2]. □

A fundamental idea of Burger–Mozes is that, given a tree T and a vertex-transitive group $G \leqslant \mathrm{Aut}(T)$, if G is non-discrete and the local action of G on T is a suitable 2-transitive group, then the closure $\overline{G}$ is 2-transitive on ∂T (see [BM00a, § 3.3]), so that $\overline{G}$ is subject to Theorem 4.2. The 2-transitive groups considered by Burger–Mozes are those with almost simple (or quasi-simple) point stabilizers. In particular, their original arguments do not apply to 2-transitive groups of degree $\leqslant 5$. However, a similar local-to-global phenomenon can also be extracted from the work of V. Trofimov. The following result, due to him, applies to numerous 2-transitive local actions whose point stabilizers need not be almost simple.

Theorem 4.3 (V. Trofimov [Tro07, Proposition 3.1]). — *Let X be a locally finite d-regular graph and $G \leqslant \mathrm{Aut}(X)$ be a vertex-transitive group whose local action on X is the 2-transitive group $F \leqslant \mathrm{Sym}(d)$. Assume that every subnormal subgroup S of the point stabilizer F_1, for which the index $|F_1 : \mathrm{N}_{F_1}(S)|$ divides a power of $d-1$, acts transitively on $\{2, \dots, d\}$. If G is non-discrete, then X is a tree and the closure $\overline{G}$ is 2-transitive on the set of ends ∂X.*

Moreover, the above condition is satisfied if $d = q+1$ and F contains a normal subgroup isomorphic to $\mathrm{PSL}_2(\mathbf{F}_q)$, or if $F \geqslant \mathrm{Alt}(d)$.

Proof. — The statement of [Tro07, Proposition 3.1] ensures that X is a tree. Although he does not write it explicitly, Trofimov's proof actually also shows that $\overline{G}$ is 2-transitive on ∂X. The fact that the condition holds in the case $d = q+1$ and $\mathrm{PSL}_2(\mathbf{F}_q) \lhd F$ is explained in [Tro07, Example 3.2]. If $F \geqslant \mathrm{Alt}(d)$ with $d \geqslant 6$, the condition is clearly satisfied since $\mathrm{Alt}(d-1)$ is simple. For $d \leqslant 5$, it follows from the preceding case (see Table 2.1). □

Combining the three theorems above, we obtain the following result.

Corollary 4.4. — *Let $d_1, d_2 \geqslant 3$, let T_1, T_2 be regular trees of degree d_1, d_2 and let $\Gamma \leqslant \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ be a discrete subgroup acting transitively on $VT_1 \times VT_2$. Assume that for $i = 1, 2$, the local action F_i of Γ on T_i satisfies the condition in Theorem 4.3. If Γ is irreducible, then it is hereditarily just-infinite.*

Proof. — Since Γ is irreducible, its projection $p_i\colon \Gamma \to \mathrm{Aut}(T_i)$ has a non-discrete image for $i = 1$ and 2 by [BM00b, Proposition 1.2]. Let $G_i = \overline{p_i(\Gamma)}$. By Theorem 4.3, the G_i-action on ∂T_i is 2-transitive. Thus Theorem 4.2 ensures that G_1 and G_2 satisfy the hypotheses of Theorem 4.1. The conclusion follows. ☐

Corollary 1.4 is an immediate consequence of Corollary 4.4 (see Table 2.1), recalling that a subgroup $\Gamma \leqslant \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ is discrete if and only if the stabilizer $\Gamma_{(v_1, v_2)}$ of a vertex $(v_1, v_2) \in VT_1 \times VT_2$ is finite.

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