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BRUHAT–TITS THEORY FROM BERKOVICH'S POINT OF VIEW. ANALYTIC FILTRATIONS

POINT DE VUE DE BERKOVICH SUR L'IMMEUBLE ET FILTRATIONS

Abstract. — We define filtrations by affinoid groups, in the Berkovich analytification of a connected reductive group, related to Moy–Prasad filtrations. They are parametrized by a cone, whose basis is the Bruhat–Tits building and whose vertex is the neutral element, via the notions of Shilov boundary and holomorphically convex envelope.

Résumé. — Nous définissons des groupes affinoïdes, dans l'analytifié d'un groupe réductif connexe, étroitement liés aux filtrations de Moy–Prasad. Ils sont paramétrés par un cône, dont la base est l'immeuble de Bruhat–Tits et dont le sommet est l'élément neutre, grâce aux notions de bord de Shilov et d'enveloppe convexe holomorphe.

1. Introduction

Let G be a connected reductive group over a discretely valued complete non-Archimedean field with a perfect residue field k. Berkovich [Ber90, Chapter 5] (split

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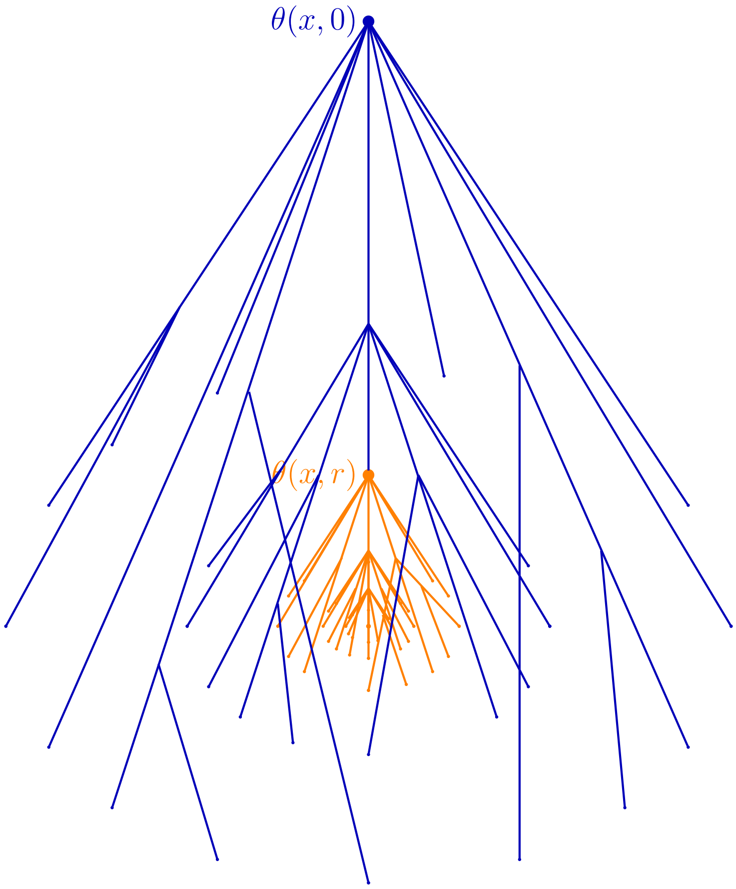
case) and Rémy–Thuillier–Werner [RTW10] (general case) showed that there is a canonical embedding of the Bruhat–Tits building of G into the Berkovich analytification G^{an}. This embedding and related ideas form what is called "*Berkovich's point of view on Bruhat–Tits buildings*". In this article, we observe that Berkovich's point of view allows to define and parametrize natural k-analytic filtrations related to Moy–Prasad filtrations [MP94] [MP96]. Let x be a point in the reduced Bruhat–Tits building of G over k. The group $G(k)$ acts on the reduced and enlarged buildings. This action is compatible with the canonical projection from the enlarged building to the reduced one. Let us consider the stabilizer of a preimage of x in the enlarged building, this is a compact open subgroup of $G(k)$ independent of the preimage. One idea of Berkovich's point of view is to construct a k-affinoid group G_x that realizes this stabilizer. The space G_x is equipped with a partial order and has a maximal point: its Shilov Boundary denoted $\theta(x)$. The space G_x can be recovered from $\theta(x)$ taking the holomorphically convex envelope. The preceding constructions and results for general connected reductive groups are done in [RTW10]. Now our filtrations are some k-affinoid subgroups $\{G_{x,r}\}_{r\in\mathbb{R}_{\geqslant 0}}$ of G_x satisfying also that the Shilov boundary of $G_{x,r}$ is a singleton $\theta(x,r)$, and that the holomorphically convex envelope of $\theta(x,r)$ is $G_{x,r}$.



The heuristic picture represents G_x and its filtrations. Pictorially, the stabilizer of x is the set of lower extremal points, G_x is the whole picture and $G_{x,r}$ is the orange part. The k-points of $G_{x,r}$ are the orange lower extremal points. The neutral element is the central lower point. For $r > r'$ we have

$$G_{x,r} \underset{\neq}{\subset} G_{x,r'}.$$

When r goes to $+\infty$, $\theta(x,r)$ goes to the neutral element. The set $\{\theta(x,r) \mid r \geqslant 0\}$ is a line segment joining $\theta(x)$ to the neutral element.

Our construction of $G_{x,r}$ use dilatations and Néron blowups [MRR20], generic fibers of formal completions of schemes over ring of integers and affinoid descent [RTW10]. We also define similar filtrations for the Lie algebra.

We show that in tame situations $G_{x,r}(k)$ is the corresponding Moy–Prasad group for $r > 0$. We prove that the map

$$\theta : \mathrm{BT}(G,k) \times \mathbb{R}_{\geqslant 0} \to G^{an}$$

is continuous and injective. This gives birth to a topological cone in G^{an}. We also compute some examples.

Let us now describe the structure of the document and of our construction. A posteriori, the formal definition of our filtrations is as follows. Given a point x in the Bruhat–Tits $\mathrm{BT}(G,k)$ of G and a positive real number r, we choose a k-affinoid extension K/k such that G is split, the image $\iota_{K/k}(x)$ of x in $\mathrm{BT}(G,k)$ is special and r is in $\mathrm{ord}(K)$. We consider the canonical Demazure group scheme $\mathfrak{G}$ over K° attached to $\iota_{K/k}(x)$. It is a split reductive group whose generic fiber is $G \times_k K$. We then consider the congruence subgroup $\mathfrak{G}_r$ of $\mathfrak{G}$, whose definition is given in Section 2. Next, we consider the K-affinoid analytification (i.e the generic fiber of the formal completion along the special fiber) $\widehat{\mathfrak{G}_r}_{\ \eta}$ of $\mathfrak{G}_r$, this is a K-affinoid group. Then $G_{x,r}$ is defined as $\mathrm{pr}_{K/k} \widehat{\mathfrak{G}_r}_{\ \eta}$ where $\mathrm{pr}_{K/k}$ is the canonical projection from $(G \times_k K)^{an}$ to G^{an}. We prove that it is a k-affinoid group, independent of the choice of K. In order to prove that we apply Rémy–Thuillier–Werner's descent theorem, see Section 3 for the statement of this theorem. Applying this descent theorem requires to prove a certain identity:

$$\mathrm{pr}_{K/k}^{-1}\mathrm{pr}_{K/k}\left(\widehat{\mathfrak{G}_r}_{\ \eta}\right) = \widehat{\mathfrak{G}_r}_{\ \eta} \ (*).$$

Proving this identity is a guideline for many statements of this paper and is not a formal consequence of the definition given above. That is why our construction is done step by step. The first step (Section 4) deals with split groups. In this step, $(*)$ is proved using explicit computations together with the notion of peaked point. For non split groups, we reduce in Section 6 to the split case choosing a finite Galois extension L/k that splits G. During this step, we need to show that objects defined at the first step are stable under $\mathrm{Gal}(L/k)$. In this occasion, we before define and study in Section 5 filtrations for k-affinoid groups H that are analytification of Demazure models after a finite Galois base change, here $(*)$ is obtained using Galois stability. Filtrations of Lie algebras are considered in Section 7. In Section 8 we compare our filtrations with Moy–Prasad ones and in Section 9 we define the cone. Sections 10 and 11 contain examples.

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Notation and prerequisites

Let G be a connected reductive group scheme over a complete non-Archimedean field k. We assume here that k is discretely valued with a perfect residue field. This implies that functoriality of buildings holds [RTW10, § 1.3.4]. So for any non-Archimedean extension K/k we have a canonical map between (reduced) Bruhat–Tits buildings $\iota_{K/k} : \mathrm{BT}(G,k) \to \mathrm{BT}(G,K)$. We fix a uniformizer π_k of k. We denote by $|.|_K$ the norm on a non-Archimedean extension K of k and ord the extension to K of the additive valuation on k such that $\mathrm{ord}(\pi_k) = 1$. We assume $|.|_K = e^{-\mathrm{ord}(.)}$. We sometimes use the notation $|\ |$ instead of $|\ |_K$. Moreover, we put $K^\circ = \{x \in K \mid \ |x|_K \leqslant 1\}$, $K^{\circ\circ} = \{x \in K |\ |x|_K < 1\}$ and $\widetilde{K} = K^\circ / K^{\circ\circ}$. We assume that the reader is familiar with reductive group schemes, Bruhat–Tits theory and Berkovich spaces. One can read the necessary material in [RTW10, § 1]. If X is a Berkovich k-analytic space and K is a non-Archimedean extension, we denote by $X \widehat{\otimes} K$ the analytic base change to K. If X is an affine scheme, then $\mathcal{O}(X)$ denotes its coordinate ring. If G is an affine group scheme over a ring, the ring $\mathcal{O}(X)$ is canonically an Hopf algebra denoted $\mathrm{Hopf}(X)$.

2. Schematic congruence subgroups

In this section we recall some results about congruence subgroups for group schemes. Congruence subgroups are built using dilatations and Néron blowups [MRR20]. Given

$$Z \underset{closed}{\subset} D \underset{closed}{\subset} X$$

schemes such that D is locally principal, the dilatation of Z in X along D is a scheme $\mathrm{Bl}_Z^D X$ defined in [MRR20]. Now assume that $X = \mathfrak{G}$ is a smooth group scheme over K° where K is a non-Archimedean field, $D = \mathfrak{G} \times_{K^\circ} K^\circ/\boldsymbol{\pi}$ and $Z = e$ is the neutral section of the group scheme D, here $\boldsymbol{\pi}$ is a principal ideal in K°. In this case $\mathrm{Bl}_Z^D \mathfrak{G}$ is a group scheme over K° called the $\boldsymbol{\pi}$-congruence subgroup of $\mathfrak{G}$ and is denoted $\mathfrak{G}_{\boldsymbol{\pi}}$. Let π be a generator of $\boldsymbol{\pi}$.

Proposition 2.1 ([MRR20]). —

(1) *Assume* $\mathfrak{G} = \mathrm{Spec}(\mathfrak{A})$ *and let* $J \subset \mathfrak{A}$ *be the augmentation ideal of the Hopf algebra* $\mathfrak{A}$*. Then*
$$\mathfrak{G}_{\boldsymbol{\pi}} = \mathrm{Spec}\left(\mathfrak{A}\left[\boldsymbol{\pi}^{-1}J\right]\right)$$
where $\mathfrak{A}[\boldsymbol{\pi}^{-1}J]$ *is the ring generated by* $\mathfrak{A}$ *and* $\boldsymbol{\pi}^{-1}J = \{\pi^{-1}j | j \in J\}$ *inside* $\mathfrak{A} \otimes_{K^\circ} K$.

(2) *Assume* $\boldsymbol{\pi} \neq K^\circ$*, then the scheme* $\mathfrak{G}_{\boldsymbol{\pi}} \times_{K^\circ} \widetilde{K}$ *is a vector group over* $\widetilde{K}$*, in particular it is irreducible.*

(3) *We have* $\mathfrak{G}_{\boldsymbol{\pi}}(K^\circ) = \ker(\mathfrak{G}(K^\circ) \to \mathfrak{G}(K^\circ/\boldsymbol{\pi}))$.

Proof. —

(1) By [MRR20, Remark 2.2], the ring of $\mathfrak{G}_{\boldsymbol{\pi}}$ is $\mathfrak{A}[\frac{I}{\pi}]$ where I is the ideal in $\mathfrak{A}$ that defines the closed subscheme e and $\mathfrak{A}[\frac{I}{\pi}]$ is the $\mathfrak{A}$-subalgebra of $\mathfrak{A}[\pi^{-1}]$ generated by fraction i/π with $i \in I$. It is clear that $\mathfrak{A}[\pi^{-1}] = \mathfrak{A} \otimes_{K^\circ} K$ and that $I = J + \pi\mathfrak{A}$. We deduce that $\mathfrak{A}[\frac{I}{\pi}] = \mathfrak{A}[\boldsymbol{\pi}^{-1}J]$.

(2) We have $\mathfrak{G}_{\boldsymbol{\pi}} \times_{K^\circ} K^\circ/\boldsymbol{\pi} = \mathfrak{G}_{\boldsymbol{\pi}} \times_D Z$, by [MRR20, Proposition 2.9] this is a vector group over $K^\circ/\boldsymbol{\pi}$. We have
$$\mathfrak{G}_{\boldsymbol{\pi}} \times_{K^\circ} \widetilde{K} = (\mathfrak{G}_{\boldsymbol{\pi}} \times_{K^\circ} K^\circ/\boldsymbol{\pi}) \times_{K^\circ/\boldsymbol{\pi}} \widetilde{K},$$
which is a vector group over $\widetilde{K}$.

(3) This is [MRR20, Lemma 4.1]. □

Let $G = \mathrm{Spec}(A)$ be an affine k-group scheme of finite type. Let K/k be a Galois extension and $\mathfrak{A}$ be a flat sub-Hopf-K°-algebra of finite type of the Hopf K-algebra $A_K = A \otimes_k K$ such that $\mathfrak{A} \otimes_{K^\circ} K = A_K$. In this situation, we say that $\mathfrak{G} = \mathrm{Spec}(\mathfrak{A})$ is $\mathrm{Gal}(K/k)$-stable if $\mathfrak{A}$ is $\mathrm{Gal}(K/k)$-stable in $A \otimes_k K$. The following Proposition shows that Galois stability is preserved under the operation of taking congruence subgroups.

Lemma 2.2. — *Assume that* $\mathfrak{G}$ *is* $\mathrm{Gal}(K/k)$*-stable. Then for any principal ideal* $\boldsymbol{\pi} \subset K^\circ$*, the congruence subgroup* $\mathfrak{G}_{\boldsymbol{\pi}}$ *is* $\mathrm{Gal}(K/k)$*-stable.*

Proof. — Let $\varepsilon_{\mathfrak{A}} : \mathfrak{A} \to K^\circ$ be the augmentation, $J = \ker(\varepsilon_{\mathfrak{A}})$. Let us remark that $\varepsilon_{\mathfrak{A}}$ is the restriction to $\mathfrak{A}$ of the augmentation $\varepsilon_A \otimes \mathrm{Id} : A \otimes_k K \to K$ of A_K. So $J = \ker(\varepsilon_A \otimes \mathrm{Id}) \cap \mathfrak{A}$. The set $\ker(\varepsilon_A \otimes \mathrm{Id})$ is $\mathrm{Gal}(K/k)$-stable, and $\mathfrak{A}$ is stable by hypothesis, so J is $\mathrm{Gal}(K/k)$-stable as the intersection of two $\mathrm{Gal}(K/k)$-stable subsets of $A \otimes_k K$. By Proposition 2.1, the ring of $\mathfrak{G}_{\boldsymbol{\pi}}$ is $\mathfrak{A}[\boldsymbol{\pi}^{-1}J] \subset A \otimes_k K$ and so it is $\mathrm{Gal}(K/k)$-stable. □

3. Berkovich k-analytic spaces

In this section k is a non-Archimedean field. References for Berkovich analytic spaces are [Ber90] and [Ber93]. To each scheme X of finite type over k, Berkovich [Ber90, § 3.4] associated a k-analytic space X^{an} such that for any non-Archimedean field K/k, there is a bijection $X^{an}(K) \simeq X(K)$.

Proposition 3.1 ([Ber90, § 3.4.2]). — *If $X = \mathrm{Spec}(A)$, where A is a finitely generated ring over k, then the underlying topological space X^{an} coincides with the set of all multiplicative seminorms on A whose restriction to k is the norm on k. A point x in X^{an} is also denoted $|\ |_x$.*

If $x \in X^{an}$, we define

$$\mathrm{Hol}(x) := \{y \in X^{an} |\ |f|_y \leqslant |f|_x\ \forall\ f \in A\}.$$

The following definition/proposition is extracted from Rémy–Thuillier–Werner's work [RTW10, § 1.2.4] [Thu05, § 2.1.1] (see also [Ber90, § 5.3.2]).

Definition 3.2 (Analytification of k°-schemes). — *Let $\mathfrak{A}$ be a flat topologically finitely presented k°-algebra whose spectrum $\mathcal{M}(\mathcal{A})$ we denote $\mathfrak{X}$. Let $X = \mathrm{Spec}(\mathfrak{A} \otimes_{k^\circ} k)$ be the generic fiber of $\mathfrak{X}$. The map*

$$|\ |_{\mathfrak{A}} : \mathfrak{A} \otimes_{k^\circ} k \to \mathbb{R}_{\geqslant 0}, a \mapsto \inf \left\{ |\lambda| \,\middle|\, \lambda \in k^\times \ \text{ and } \ a \in \lambda (\mathfrak{A} \otimes 1) \right\}$$

is a norm on $\mathfrak{A} \otimes_{k^\circ} k$. The Banach algebra $\mathcal{A} = \overline{\mathfrak{A} \otimes_{k^\circ} k}^{|\ |_{\mathfrak{A}}}$ obtained by completion is a strictly k-affinoid algebra whose spectrum is denoted by $\widehat{\mathfrak{X}}_\eta$ and is called the generic fiber of the formal completion of $\mathfrak{X}$ along its special fiber. This affinoid space is naturally an affinoid domain in X^{an} (whose points are multiplicative seminorms on $\mathfrak{A} \otimes_{k^\circ} k$ which are bounded with respect to the seminorm $|.|_{\mathfrak{A}}$). Moreover, there is a reduction map $\tau : \widehat{\mathfrak{X}}_\eta \to \mathfrak{X} \times_{k^\circ} \widetilde{k}$ defined as follows: a point x in $\widehat{\mathfrak{X}}_\eta$ gives a sequence of ring homomorphisms:

$$\mathfrak{A} \to \mathcal{H}(x)^\circ \to \widetilde{\mathcal{H}(x)}$$

whose kernel $\tau(x)$ defines a prime ideal of $\mathfrak{A} \otimes_{k^\circ} \widetilde{k}$, i.e a point in $\mathfrak{X} \times_{k^\circ} \widetilde{k}$. If the scheme $\mathfrak{X}$ is integrally closed in its generic fiber — in particular if $\mathfrak{X}$ is smooth — then τ is the reduction map of Berkovich (see [Ber90, § 2.4]*). And so the Shilov Boundary of $\widehat{\mathfrak{X}}_\eta$ is in bijection with the irreducible components of the special fiber $\mathfrak{X} \times_{k^\circ} \widetilde{k}$. Moreover, the spectral norm ρ on $\mathcal{A}$ is equal to $|\ |_{\mathfrak{A}}$ if and only if the algebra $\mathfrak{A} \otimes_{k^\circ} k$ is reduced* [Thu05, Proposition 2.1.1].

Corollary 3.3. — *Let $\mathfrak{X} = \mathrm{Spec}(\mathfrak{A})$ be a smooth k°-scheme with irreducible special fiber. Let X be the generic fiber of $\mathfrak{X}$. Then*

(1) *$\widehat{\mathfrak{X}}_\eta$ is a strictly k-affinoid domain of X^{an},*
(2) *the Shilov boundary of $\widehat{\mathfrak{X}}_\eta$ is a singleton equal to $|\ |_{\mathfrak{A}}$,*
(3) *$\widehat{\mathfrak{X}}_\eta$ is the holomorphically convex envelope of $|\ |_{\mathfrak{A}} \in X^{an}$.*

Proof. —

(1) This is contained in Proposition 3.2.
(2) Since the special fiber of $\mathfrak{X}$ is irreducible, the Shilov boundary of $\widehat{\mathfrak{X}}_\eta$ is a singleton by Proposition 3.2. The algebra $\mathfrak{A} \otimes_{k^\circ} \widetilde{k}$ is reduced since $\mathfrak{X}$ is smooth, thus by Proposition 3.2, $|\ |_{\mathfrak{A}}$ is the spectral norm. This implies that $\mathrm{Shi}(\widehat{\mathfrak{X}}_\eta) = |\ |_{\mathfrak{A}}$.
(3) Put $A = \mathfrak{A} \otimes_{k^\circ} k$. Recall that the holomorphically convex envelope of $|\ |_{\mathfrak{A}}$ is

$$\mathrm{Hol}\left(|\ |_{\mathfrak{A}}\right) = \{x \in G^{an} | |f|_x \leqslant |f|_{\mathfrak{A}} \quad \forall\ f \in A\}.$$

By Proposition 3.2 the k-affinoid algebra $\mathcal{A}$ of $\widehat{\mathfrak{X}}_\eta$ is the completion of A relatively to the norm $|\ |_{\mathfrak{A}}$. Let i denote the natural corresponding injective k-algebras morphism $A \to \mathcal{A}$. The inclusion $\widehat{\mathfrak{X}}_\eta \subset G^{an}$ is given by

$$\begin{aligned}\iota : \mathcal{M}(\mathcal{A}) &\to G^{an}\\ |\ |_x &\mapsto |\ |_x \circ i\ .\end{aligned}$$

Since $\mathcal{M}(\mathcal{A})$ is the set of all multiplicative seminorms on $\mathcal{A}$ bounded by $|\ |_{\mathfrak{A}}$, $\iota(\mathcal{M}(\mathcal{A}))$ is contained in the holomorphically convex envelope of $|\ |_{\mathfrak{A}}$. Reciprocally, let $x \in \mathrm{Hol}(|\ |_{\mathfrak{A}})$, $x = |\ |_x$ is a multiplicative seminorm $A \to \mathbb{R}_{\geqslant 0}$ such that $|f|_x \leqslant |f|_{\mathfrak{A}}\ \forall\ f \in A$. Since $\mathcal{A}$ is the completion of A relatively to $|\ |_{\mathfrak{A}}$, $|\ |_x$ induces a multiplicative seminorm on $\mathcal{A}$ bounded by $|\ |_{\mathfrak{A}}$. This ends the proof of Corollary 3.3. □

If K/k is an affinoid extension, and X is a k-analytic space, we denote by $\mathrm{pr}_{K/k}$ the canonical surjective map $X\widehat{\otimes}K \to X$. If K/k is a finite Galois extension and X is a k-analytic space, the group $\mathrm{Gal}(K/k)$ acts on $X\widehat{\otimes}K$ and $\mathrm{pr}_{K/k}$ induces an isomorphism $(X\widehat{\otimes}K)/\mathrm{Gal}(K/k) \simeq X$. This implies that if D_K is a subset of $X\widehat{\otimes}K$ then D_K is $\mathrm{Gal}(K/k)$-stable if and only if $\mathrm{pr}_{K/k}^{-1}(\mathrm{pr}_{K/k}(D_K)) = D_K$. We now state a descent theorem, due to Rémy–Thuillier–Werner.

Theorem 3.4 ([RTW10, Appendix A]). — *Let X be a k-affinoid space. Let K be a k-affinoid extension. Let D be a subset of X, then D is a k-affinoid domain of X if and only if the subset $\mathrm{pr}_{K/k}^{-1}(D)$ is a K-affinoid domain in $X\widehat{\otimes}K$.*

Corollary 3.5. — *Let X be a k-affinoid space. Let K/k be a finite Galois extension. Let D_K be a $\mathrm{Gal}(K/k)$-stable K-affinoid domain of $X\widehat{\otimes}K$, put $D = \mathrm{pr}_{K/k}(D_K)$. Then D is a k-affinoid domain of X.*

Proof. — Since D_K is $\mathrm{Gal}(K/k)$-stable, $\mathrm{pr}_{K/k}^{-1}(\mathrm{pr}_{K/k}(D_K)) = D_K$, so $\mathrm{pr}_{K/k}^{-1}(D)$ is K-affinoid, so by Theorem 3.4, D is k-affinoid. □

We now show that Galois stability is preserved by taking the generic fiber of the formal completion along the special fiber.

Proposition 3.6. — *Let K/k be a finite Galois extension. Let $X = \mathrm{Spec}(A)$ be an affine k-scheme of finite type and let $\mathfrak{X} = \mathrm{Spec}(\mathfrak{A})$ be a smooth K°-scheme of finite type such that $\mathfrak{X} \times_{K^\circ} K = X \times_k K$ and such that $\mathfrak{X} \times_{K^\circ} \widetilde{K}$ is irreducible. Assume that $\mathfrak{A}$ is a $\mathrm{Gal}(K/k)$-stable subalgebra of $A \otimes_k K$. Then the generic fiber of the formal completion of $\mathfrak{X}$ along its special fiber is a $\mathrm{Gal}(K/k)$-stable K-affinoid domain $\widehat{\mathfrak{X}}_\eta$ of $X^{an}\widehat{\otimes}K$.*

Proof. — Let $|\ |_x \in \widehat{\mathfrak{X}}_\eta \subset X^{an}\widehat{\otimes}K$, it is a seminorm on $A \otimes_k K$ bounded by $|\ |_{\mathfrak{A}}$. Let $\gamma \in \mathrm{Gal}(K/k)$, we need to show that $|\ |_x.\gamma$ stay in $\widehat{\mathfrak{X}}_\eta$. Let $f \in A \otimes_k K$, then $(|\ |_x.\gamma)(f) = |\gamma.f|_x$. By definition of $|\ |_x$, we have $|\gamma.f|_x \leqslant |\gamma.f|_{\mathfrak{A}}$. Since $\mathfrak{A}$ is $\mathrm{Gal}(K/k)$ stable in $A \otimes_k K$, we have $\gamma.\mathfrak{A} = \mathfrak{A}$ for all $\gamma \in \mathrm{Gal}(K/k)$ and we deduce the following.

$$\begin{aligned}|\gamma.f|_{\mathfrak{A}} &= \inf_{\lambda\in K^\times}\{|\lambda||\gamma.f\in\mathfrak{A}\subset\mathfrak{A}\otimes_{K^\circ}K\}\\ &= \inf_{\lambda\in K^\times}\left\{|\lambda|\Big|f\in\gamma^{-1}\mathfrak{A}\subset\mathfrak{A}\otimes_{K^\circ}K\right\}\\ &= \inf_{\lambda\in K^\times}\{|\lambda||f\in\mathfrak{A}\subset\mathfrak{A}\otimes_{K^\circ}K\}\\ &= |f|_{\mathfrak{A}}\end{aligned}$$

Consequently, we have $(|\ |_x.\gamma)(f) = |\gamma.f|_x \leqslant |\gamma.f|_{\mathfrak{A}} = |f|_{\mathfrak{A}}$. Thus $|\ |_x.\gamma \leqslant |\ |_{\mathfrak{A}}$, and so $(|\ |_x.\gamma) \in \widehat{\mathfrak{X}}_\eta$ by Corollary 3.3. The proof ends here. □

4. The split case

Let $G = \mathrm{Spec}(\mathrm{Hopf}(G))$ be a split connected reductive group scheme over a non-Archimedean discretely valued field k with a perfect residue field. Let $\mathrm{BT}(G,k)$ be the reduced Bruhat–Tits building of G. Let x be a special point in $\mathrm{BT}(G,k)$ and $r \in \mathbb{R}_{\geqslant 0}$. Since G is split and x is special, the canonical scheme $\mathfrak{G}$ attached to x by Bruhat–Tits is a Demazure (i.e. split, reductive and connected) k°-group scheme, as remarked in [RTW10, Page 19]. The scheme $\mathfrak{G}$ is smooth. We fix a k-affinoid extension K such that the real number r is contained in $\mathrm{ord}(K)$. Let $\boldsymbol{\pi}_r \subset K^\circ$ be the unique ideal of K° generated by elements π_r with $\mathrm{ord}(\pi_r) = r$. We now consider the $\boldsymbol{\pi}_r$-congruence subgroup $\mathfrak{G}_{\boldsymbol{\pi}_r}$ of $\mathfrak{G} \times_{k^\circ} K^\circ$, we also denote it as $\mathfrak{G}_r$. We have identifications

$$\mathfrak{G}_r \times_{K^\circ} K = G \times_k K$$

and

$$\mathrm{Hopf}(\mathfrak{G}_r) \otimes_{K^\circ} K = \mathrm{Hopf}(G) \otimes_k K.$$

We now consider $\widehat{\mathfrak{G}_r}_{\eta}$, the generic fiber of the formal completion of $\mathfrak{G}_r$ along its special fiber. Since $\mathfrak{G}_r$ is smooth by [MRR20, Theorem 3.2], the Shilov boundary of the K-affinoid group $\widehat{\mathfrak{G}_r}_{\eta}$ is in bijection with the set of generic points of irreducible components of the special fiber of $\mathfrak{G}_r$ (cf Proposition 3.2 and Corollary 3.3). So by Proposition 2.1, it is a singleton, let us denote it by x_r. Moreover by Corollary 3.3, the holomorphically convex envelope $\mathrm{Hol}(x_r)$ of x_r is $\widehat{\mathfrak{G}_r}_{\eta}$. We want an explicit formula for x_r. Let $\mathfrak{T}$ be a maximal k°-split torus of $\mathfrak{G}$ and Φ be the corresponding set of roots. Let $\mathfrak{B}$ be a Borel subgroup such that $\mathfrak{T}$ is a Levi subgroup of $\mathfrak{B}$. Let Φ^-, Φ^+ be the corresponding sets of negative and positive roots. For each $\alpha \in \Phi$, we have a canonical k°-root subgroup $\mathfrak{U}_\alpha \subset \mathfrak{G}$. Choose an ordering on Φ^-, Φ^+, then the multiplication morphism of k°-schemes

$$\prod_{\alpha\in\Phi^-}\mathfrak{U}_\alpha \times_{k^\circ} \mathfrak{T} \times_{k^\circ} \prod_{\alpha\in\Phi^+}\mathfrak{U}_\alpha \to \mathfrak{G} \tag{4.1}$$

is an open immersion. Its image, which does not depend on the choice of the ordering, is denoted $\underline{\Omega}$ and is called the open cell of $\mathfrak{G}$. Taking generic fibers, we obtain similar objects for G. The objects

$$\begin{aligned} T &:= \mathfrak{T} \times_{k^\circ} k \\ U_\alpha &:= \mathfrak{U}_\alpha \times_{k^\circ} k \\ B &:= \mathfrak{B} \times_{k^\circ} k \end{aligned}$$

are respectively a maximal split torus, a root subgroup, and a Borel subgroup of $G = \mathfrak{G} \times_{k^\circ} k$. We can identify canonically Φ with the set of roots associated to G, T. Moreover (4.1) induces an open immersion

$$\prod_{\alpha \in \Phi^-} U_\alpha \times_k T \times_k \prod_{\alpha \in \Phi^+} U_\alpha \to G$$

whose image, independent of the ordering, is denoted Ω and is called the open cell of G. We can identify Ω and $\underline{\Omega} \times_{k^\circ} k$. The open cell Ω is affine and the open immersion $\Omega \to G$ corresponds to an injective morphism of Hopf algebras from $\mathcal{O}(G)$ to $\mathcal{O}(\Omega)$ (see [Ber90, line 24 page 103]). The torus $\mathfrak{T}$ is split, so is isomorphic to $(\mathbb{G}_m/k^\circ)^s$ for some integer s. Fix an isomorphism

$$\mathfrak{T} \simeq \operatorname{Spec}\left(k^\circ\left[X_1, \ldots, X_s, Y_1, \ldots, Y_s\right] / \left(X_i Y_i = 1 \text{ for } 1 \leqslant i \leqslant s\right)\right).$$

Fix an integral Chevalley basis of $\operatorname{Lie}(\mathfrak{G}, k^\circ)$, it induces, for each root $\alpha \in \Phi$, a k°-isomorphism $\mathfrak{U}_\alpha \simeq \mathbb{G}_a$, where $\mathbb{G}_a$ is the additive group over k°. Thus we have fixed an isomorphism $\mathfrak{U}_\alpha \simeq \operatorname{Spec}(k^\circ[Z_\alpha])$, i.e. we have fixed an isomorphism $\mathcal{O}(\mathfrak{U}_\alpha) \simeq k^\circ[Z_\alpha]$, for any root α. Since

$$\Omega = \prod_{\alpha \in \Phi^-} U_\alpha \times_k T \times_k \prod_{\alpha \in \Phi^+} U_\alpha,$$

we obtain

$$\mathcal{O}(\Omega) = \bigotimes_{\alpha \in \Phi^-} \mathcal{O}(U_\alpha) \otimes_k \mathcal{O}(T) \otimes_k \bigotimes_{\alpha \in \Phi^+} \mathcal{O}(U_\alpha).$$

The torus T is equal to $\mathfrak{T} \times_{k^\circ} k$. The previously fixed isomorphism

$$\mathfrak{T} \simeq \operatorname{Spec}\left(k^\circ\left[X_1, \ldots, X_s, Y_1, \ldots, Y_s\right] / \left(X_i Y_i = 1 \;\text{ for }\; 1 \leqslant i \leqslant s\right)\right)$$

induces a similar isomorphism over k for T. The set

$$\left\{X^{\mathrm{k}} Y^l \,\middle|\, \mathrm{k}, l \in \mathbb{N}; \mathrm{k} \neq 0 \Rightarrow l = 0\right\}$$

is a basis of the k-vector space $k[X, Y]/XY - 1$. We need an other basis of $\mathcal{O}(\mathbb{G}_m)$, “centered at unity”. The set

$$\left\{(X-1)^{\mathrm{k}}(Y-1)^l \,\middle|\, \mathrm{k}, l \in \mathbb{N}; \mathrm{k} \neq 0 \Rightarrow l = 0\right\}$$

is a basis of the k-vector space $k[X, Y]/XY - 1$. The previously fixed isomorphisms $\{\mathcal{O}(\mathfrak{U}_\alpha) \simeq k^\circ[Z_\alpha]\}_{\alpha \in \Phi}$ induce isomorphisms $\{\mathcal{O}(U_\alpha) \simeq k[Z_\alpha]\}$. We identify the corresponding objects. The set $\{{Z_\alpha}^{m_\alpha} \mid m_\alpha \in \mathbb{N}\}$ is a basis of the k-vector space $\mathcal{O}(U_\alpha)$. These considerations allow us to fix an isomorphism

$$\begin{aligned}\mathcal{O}(\Omega) &\simeq \left(\bigotimes_{\alpha \in \Phi^-} k\left[Z_\alpha\right]\right) \otimes_k \left(\bigotimes_{i=1}^{s} k\left[X_i, Y_i\right]/X_iY_i - 1\right) \otimes_k \left(\bigotimes_{\alpha \in \Phi^+} k\left[Z_\alpha\right]\right)\\ &\simeq k\left[X_1, \ldots, X_s, Y_1, \ldots, Y_s, \{Z_\alpha\}_{\alpha \in \Phi}\right] / \left(X_iY_i - 1, 1 \leqslant i \leqslant s\right).\end{aligned}$$

Moreover the set

$$\left\{\prod_{i=1}^{s} \left(X_i - 1\right)^{k_i} \left(Y_i - 1\right)^{l_i} \prod_{\alpha \in \Phi} {Z_\alpha}^{m_\alpha} \,\middle|\, k_i, l_i, m_\alpha \in \mathbb{N}; \forall\, 1 \leqslant i \leqslant s, k_i \neq 0 \Rightarrow l_i = 0\right\}$$

is a k-basis of the k-vector space $\mathcal{O}(\Omega)$. So given $f \in \mathcal{O}(\Omega)$, f can be written uniquely as

$$f = \sum_{k_1, \ldots, k_s, l_1, \ldots, l_s, m_\alpha \alpha \in \Phi} a_{k_1 \ldots k_s l_1 \ldots l_s, m_\alpha \alpha \in \Phi} \prod_{i=1}^{s} \left(X_i - 1\right)^{k_i} \left(Y_i - 1\right)^{l_i} \prod_{\alpha \in \Phi} Z_\alpha^{m_\alpha}.$$

In order to simplify formulas, let us introduce some notation. We denote a parameter $(k_1, \ldots, k_s, l_1, \ldots, l_s, \{m_\alpha\}_{\alpha \in \Phi})$ with $k_i, l_i, m_\alpha \in \mathbb{N}$ and $k_i \neq 0 \Rightarrow l_i = 0$, as appearing above, by the symbol u. Let U the set of all such parameters u. Moreover, the element $\prod_{i=1}^{s}(X_i - 1)^{k_i}(Y_i - 1)^{l_i} \prod_{\alpha \in \Phi} Z_\alpha^{m_\alpha}$ is denoted by the symbol $((X-1)(Y-1)Z)^u$. With these notations, an element $f \in \mathcal{O}(\Omega)$ is written uniquely as

$$f = \sum_{u \in U} a_u \Big((X-1)(Y-1)Z\Big)^u.$$

Lemma 4.1. — *The point* $\mathrm{pr}_{K/k}(x_r)$ *belongs to* Ω^{an} *and corresponds to the norm*

$$\begin{aligned}\mathcal{O}(\Omega) &\to \mathbb{R}_{\geqslant 0}\\ \sum_{u \in U} a_u \Big((X-1)(Y-1)Z\Big)^u &\mapsto \max_{u \in U} |a_u| e^{-r|u|}.\end{aligned}$$

Proof. — Since G is split and $\mathrm{pr}_{K/k}(x_r)$ is the restriction of x_r to $\mathcal{O}(G)$, we can assume $K = k$. By [RTW10, § 1.2.4], x_r is the unique point in $\widehat{\mathfrak{G}_r}_{\,\eta}$ sent to the generic point of $\mathfrak{G}_r \times_{k^\circ} \widetilde{k}$ by the reduction map. Let σ denote the generic point of $\mathfrak{G}_r \times_{k^\circ} \widetilde{k}$. Let $\underline{\Omega}_r$ be

$$\prod_{\alpha \in \Phi^-} \mathfrak{U}_{\alpha,r} \times_{k^\circ} \mathfrak{T}_r \times_{k^\circ} \prod_{\alpha \in \Phi^+} \mathfrak{U}_{\alpha,r}.$$

The multiplication map on $\mathfrak{G}_r$ induces an open immersion $\underline{\Omega}_r \to \mathfrak{G}_r$. The special fiber $\underline{\Omega}_r \times_{k^\circ} \widetilde{k}$ is open in $\mathfrak{G}_r \times_{k^\circ} \widetilde{k}$ (and non empty), consequently σ is contained in $\underline{\Omega}_r \times_{k^\circ} \widetilde{k}$. The commutative diagram

$$\begin{array}{ccc}\widehat{\underline{\Omega}_r}_{\,\eta} & \xrightarrow{\pi} & \underline{\Omega}_r \times_{k^\circ} \widetilde{k} \ni \sigma\\ \downarrow & & \downarrow\\ \widehat{\mathfrak{G}_r}_{\,\eta} & \longrightarrow & \mathfrak{G}_r \times_{k^\circ} \widetilde{k}\end{array}$$

whose vertical arrows are inclusions shows that $\mathrm{Shi}(\widehat{\mathfrak{G}_{r}}_{\eta}) = \pi^{-1}(\sigma)$. So $\mathrm{Shi}(\widehat{\mathfrak{G}_{r}}_{\eta}) = \mathrm{Shi}(\widehat{\underline{\Omega}_{r}}_{\eta})$. By [RTW10, § 1.2.4] the only point in $\mathrm{Shi}(\widehat{\underline{\Omega}_{r}}_{\eta})$ is the norm $|\ |_{\mathcal{O}(\underline{\Omega}_r)}$ on $\mathcal{O}(\Omega)$ given by

$$|f|_{\mathcal{O}(\underline{\Omega}_r)} = \inf\left\{|\lambda|\,|\lambda \in k \quad \text{and} \quad f \in \lambda\left(\mathcal{O}\left(\underline{\Omega}_r\right) \otimes 1\right)\right\}.$$

Let us describe $\mathcal{O}(\underline{\Omega}_r)$ explicitly. Let us fix an element $\pi_r \in k^\circ$ such that $\mathrm{ord}(\pi_r) = r$. We have

$$\mathcal{O}\left(\underline{\Omega}_r\right) = \bigotimes_{\alpha \in \Phi^-} \mathcal{O}\left(\mathfrak{U}_{\alpha,r}\right) \otimes_{k^\circ} \mathcal{O}\left(\mathfrak{T}_r\right) \otimes_{k^\circ} \bigotimes_{\alpha \in \Phi^+} \mathcal{O}\left(\mathfrak{U}_{\alpha,r}\right).$$

By Proposition 2.1, we have

$$\mathcal{O}\left(\mathfrak{U}_{\alpha,r}\right) = k^\circ\left[{\pi_r}^{-1} Z_\alpha\right] \subset k\left[Z_\alpha\right]$$

and

$$\mathcal{O}\left(\mathfrak{T}_r\right) = k^\circ\left[{\pi_r}^{-1}\left(X_1 - 1\right), \ldots, {\pi_r}^{-1}\left(X_s - 1\right), {\pi_r}^{-1}\left(Y_1 - 1\right), \ldots, {\pi_r}^{-1}\left(Y_s - 1\right)\right].$$

Finally, we get the formula

$$\mathcal{O}\left(\underline{\Omega}_r\right) = k^\circ\left[\left\{{\pi_r}^{-1} Z_\alpha\right\}_{\alpha \in \Phi}, \left\{{\pi_r}^{-1}\left(X_i - 1\right), {\pi_r}^{-1}\left(Y_i - 1\right)\right\}_{1 \leqslant i \leqslant s}\right] \subset \mathcal{O}(\Omega).$$

For $f \in \mathcal{O}(\Omega)$, write $f = \sum_{u \in U} a_u((X-1)(Y-1)Z)^u$. Now we obtain

$$\begin{aligned}
|f|_{\mathcal{O}(\underline{\Omega}_r)} &= \inf\left\{|\lambda|\,|\lambda \in k \quad \text{and} \quad f \in \lambda\left(\mathcal{O}\left(\underline{\Omega}_r\right) \otimes 1\right)\right\} \\
&= \inf\left\{|\lambda| \,\middle|\, \lambda \in k \quad \text{and} \quad a_u \in \lambda\left(\pi_r^{-1}\right)^{|u|} k^\circ \ \ \forall\, u \in U\right\} \\
&= \inf\left\{|\lambda| \,\middle|\, \lambda \in k \quad \text{and} \quad |a_u| \leqslant |\lambda||\pi_r^{-1}|^{|u|} \ \ \forall\, u \in U\right\} \\
&= \inf\left\{|\lambda| \,\middle|\, \lambda \in k \quad \text{and} \quad |a_u||\pi_r|^{|u|} \leqslant |\lambda| \ \ \forall\, u \in U\right\} \\
&= \max_{u \in U} |a_u|\,|\pi_r|^{|u|} \\
&= \max_{u \in U} |a_u| e^{-r|u|}
\end{aligned}$$

This ends the proof of Lemma 4.1. □

Now let us fix a point y in $A(G,T,k)$. We choose an affinoid extension E/k such that firstly the point $\iota_{E/k}(y)$ is a special point in the building $\mathrm{BT}(G,E)$ and secondly the real number r is contained in $\mathrm{ord}(E)$, it is easy to see that such an extension exists using [RTW10, Proposition 1.6]. Since $\iota_{E/k}(x)$ is also a special point, there exists $t \in T(E)$ such that $t.x = y$. Let $\mathfrak{G}_y$ be the canonical K°-Demazure scheme attached to $\iota_{E/k}(y)$. Let $y_r \in (G \times_k K)^{an}$ be the unique point in the Shilov boundary of $\widehat{\mathfrak{G}_{y,r}}_{\eta}$, and let $\theta(y,r)$ be the image of y_r under the canonical projection $(G \times_k K)^{an} \to G^{an}$. Let us use the point x to identify the apartment $A(G,T,k)$ with $V(T) = \mathrm{Hom}_{Ab}(X^*(T), \mathbb{R})$.

Proposition 4.2. — *The point $\theta(y,r)$ belongs to Ω^{an} and corresponds to the norm*

$$\begin{aligned} \mathcal{O}(\Omega) &\to \mathbb{R}_{\geqslant 0} \\ \sum_{u\in U} a_u\Big((X-1)(Y-1)Z\Big)^u &\mapsto \max_{u\in U}|a_u|e^{-r|u|}\prod_{\alpha\in\Phi} e^{m_\alpha <y,\alpha>} \end{aligned}$$

where $< .,. >$ is the map $V(T)\times X^*(T)\to\mathbb{R}$, $(y,\alpha)\mapsto <y,\alpha> = y(\alpha)$.

Proof. — Here again, we can assume $E=k$. Let $t\in T(k)$ such that $y=t.x$. The element t normalizes the root group U_α and conjugation by t induces an automorphism of U_α which is just the homothety of ratio $\alpha(t)\in k^\times$. If we read it through the isomorphisms $\mathrm{Spec}(k[Z_\alpha])\simeq U_\alpha$, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}\Big(\mathcal{O}(T)\Big[\{Z_\alpha\}_{\alpha\in\Phi}\Big]\Big) & \longrightarrow & \Omega \\ \Big\downarrow \tau & & \Big\downarrow int(t) \\ \mathrm{Spec}\Big(\mathcal{O}(T)\Big[\{Z_\alpha\}_{\alpha\in\Phi}\Big]\Big) & \longrightarrow & \Omega \end{array}$$

where τ is induced by the $\mathcal{O}(T)$-automorphism τ^* of $\mathcal{O}(T)[\{Z_\alpha\}_{\alpha\in\Phi}$ mapping Z_α to $\alpha(t)Z_\alpha$ for any $\alpha\in\Phi$. It follows that $\theta(t.x,r)$ is the point of G^{an} defined by the multiplicative norm on $\mathcal{O}(\Omega)$ mapping $f=\sum_{u\in U}a_u((X-1)(Y-1)Z_\alpha)^u$ to

$$\begin{aligned} |\tau^*(f)|_{\theta(x,r)} &= \left|\sum_{u\in U}\left(a_u\prod_{\alpha\in\Phi}\alpha(t)^{m_\alpha}\right)\Big((X-1)(Y-1)Z_\alpha\Big)^u\right|_{\theta(x,r)} \\ &= \max_{u\in U}|a_u|e^{-r|u|}\prod_{\alpha\in\Phi}|\alpha(t)|^{m_\alpha} \\ &= \max_{u\in U}|a_u|e^{-r|u|}\prod_{\alpha\in\Phi}e^{m_\alpha<y,\alpha>}. \end{aligned}$$

□

Proposition 4.3. — *The point $\theta(y,r)\in\Omega^{an}$ is peaked (in the sense of* [Ber90, § 5]*).*

Remark 4.4. — The point $\theta(y,r)$ is peaked as a point in Ω^{an} and also as a point in G^{an} by [PP15, Lemma 2.2.3].

Proof. — Let K/k be a non-Archimedean extension. We have to show that the norm $||.|| := |.|_{\theta(y,r)}\otimes|.|_K$ on the algebra $\mathcal{O}(\Omega)\otimes_k K$ is multiplicative. Recall that $\|.\|$ is the norm defined as $\|f\| = \inf\max_i |g_i|_{\theta(y,r)}|\lambda_i|_K$ where the infimum is taken over all representatives $f=\sum_i g_i\otimes\lambda_i$. The set $\{((X-1)(Y-1)Z)^u\otimes 1|u\in U\}$ is a K-basis of $\mathcal{O}(\Omega)\otimes_k K$. Let $f\in\mathcal{O}(\Omega)\otimes_k K$ and let $\{a_u^K\}_{u\in U}$ be the coordinates of f in the previous basis i.e. such that $f=\sum_{u\in U}((X-1)(Y-1)Z)^u\otimes a_u^K$. By definition of $\|.\|$, we have

$$\|f\|\leqslant\max_{u\in U}\left|\Big((X-1)(Y-1)Z\Big)^u\right|_{\theta(y,r)}\left|a_u^K\right| = \max_{u\in U}\left|a_u^K\right|e^{-r|u|}\prod_{\alpha\in\Phi}e^{m_\alpha<y,\alpha>}.$$

Now let $f=\sum_{i=1}^N(\sum_{u\in U}a_u^i((X-1)(Y-1)Z)^u)\otimes\lambda_i$ be an other representative of f. We have $f=\sum_{u\in U}((X-1)(Y-1)Z)^u\otimes(\sum_{i=1}^N a_u^i\lambda_i)$ and so for all $u\in U$, $\sum_{i=1}^N a_u^i\lambda_i=a_u^K$ and $\max_{i=1}^N|a_u^i\lambda_i|\geqslant|a_u^K|$. Let $u\in U$, we have

$$\begin{aligned}
&\max_{i=1}^N\left|\sum_{u\in U}a_u^i\Big((X-1)(Y-1)Z\Big)^u\right|_{\theta(y,r)}|\lambda_i|_K\\
&=\max_{i=1}^N\left(\max_{u\in U}\left|a_u^i\right|e^{-r|u|}\prod_{\alpha\in\Phi}e^{m_\alpha<y,\alpha>}\right)|\lambda_i|_K\\
&\geqslant\max_{i=1}^N\left(\left|a_u^i\right|e^{-r|u|}\prod_{\alpha\in\Phi}e^{m_\alpha<y,\alpha>}\right)|\lambda_i|_K\\
&\geqslant\left|a_u^K\right|_K e^{-r|u|}\prod_{\alpha\in\Phi}e^{m_\alpha<y,\alpha>}.
\end{aligned}$$

We deduce that

$$\|f\|\geqslant\max_{u\in U}\left|a_u^K\right|_K e^{-r|u|}\prod_{\alpha\in\Phi}e^{m_\alpha<y,\alpha>}.$$

Consequently

$$\|f\|=\max_{u\in U}\left|a_u^K\right|_K e^{-r|u|}\prod_{\alpha\in\Phi}e^{m_\alpha<y,\alpha>}.$$

So $\|.\|$ is the norm on $\mathcal{O}(\Omega\times_k K)$ given by the "same formula" as the norm $\theta(y,r)$ on $\mathcal{O}(\Omega)$, so $\|.\|$ is in particular multiplicative. □

Let us put $G_{y,r}=\mathrm{pr}_{E/k}(\widehat{\mathfrak{G}_{y,r}}_{\eta})$.

Proposition 4.5. —

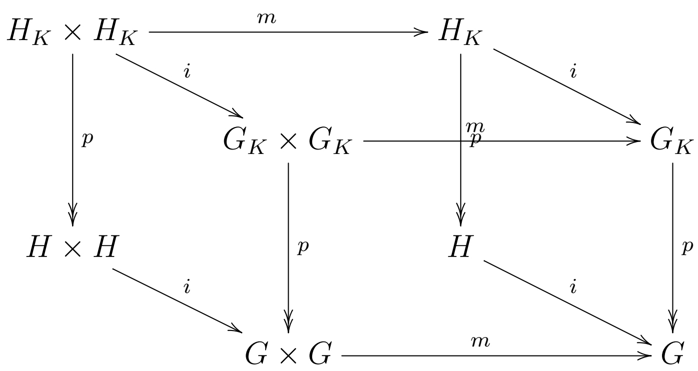
(1) *$G_{y,r}$ is a k-affinoid domain of G^{an}.*
(2) *$G_{y,r}$ is a k-affinoid group.*
(3) *The Shilov boundary of $G_{y,r}$ is a singleton equal to the point $\theta(y,r)$ considered above, moreover $\mathrm{Hol}(\theta(y,r))=G_{y,r}$.*

Proof. —

(1) By Theorem 3.4, it is enough to prove that $\mathrm{pr}_{E/k}^{-1}(G_{y,r})$ is E-affinoid. By Proposition 4.3, the point $\theta(y,r)$ is peaked. So by [Ber90, Corollary 5.2.4] taking holomorphically convex envelope commutes with base change and we have $\mathrm{pr}_{E/k}^{-1}(\mathrm{pr}_{E/k}(\mathrm{Hol}(y_r)))=\mathrm{Hol}(y_r)$. So $\mathrm{pr}_{E/k}^{-1}(G_{y,r})$ is E-affinoid since $\mathrm{Hol}(y_r)=\widehat{\mathfrak{G}_{y,r}}_{\eta}$. This ends the proof of the first assertion.
(2) This is a consequence of the following lemma.

Lemma 4.6. — *Let G be a k-analytic group, let K/k be an affinoid extension, let H_K be a K-affinoid subgroup of $G_K=G\widehat{\otimes}K$. Let $H=\mathrm{pr}_{K/k}(H_K)$, if it is a k-affinoid domain of G then it is a k-affinoid subgroup of G.*

Proof. — Let $m:G\times G\to G$ be the multiplication map and $inv:G\to G$ be the inversion map coming from the analytic group structure on G. We have to show that the restriction and inversion maps factor through H. Consider the following diagram whose four squares are commutative.



Let x be in $H \times H$, it is enough to show that there is y in H such that $m \circ i(x) = i(y)$. Let z in $H_K \times H_K$ such that $p(z) = x$, then

$$m \circ i(x) = m \circ p \circ i(z) = p \circ m \circ i(z) = p \circ i \circ m(z) = i \circ p \circ m(z).$$

So $y = p \circ m(z)$ works. The same argument works for *inv*. □

(3) By [Ber90, Proof of Proposition 2.4.4], the Shilov boundary of $\widehat{\mathfrak{G}_{y,r}}_\eta$ surjects to the Shilov boundary of $G_{y,r}$, so $\mathrm{Shi}(G_{y,r}) = \{\theta(y,r)\}$. By Corollary 3.3, $\mathrm{Hol}(y_r)$ equals $\widehat{\mathfrak{G}_{y,r}}_\eta$. Now since $\theta(y,r)$ is peaked, [Ber90, Corollary 5.2.4] implies that $G_{y,r} = \mathrm{Hol}(\theta(y,r))$. □

5. The rational potentially Demazure case

In this section $G = \mathrm{Spec}(\mathrm{Hopf}(G))$ is a connected reductive group scheme over k. Let $H \subset G^{an}$ be a k-affinoid group such that there exists a finite Galois extension K/k such that $G \times_k K$ is split and $H \widehat{\otimes} K$ is the generic fiber of the formal completion $\widehat{\mathfrak{G}}_\eta$ of a Demazure group scheme $\mathfrak{G}$ over K° satisfying $\mathfrak{G} \times_k K = G$. Let K be such an extension. We call such a H a rational potentially Demazure k-affinoid group in G^{an}. Let $\Gamma \subset \mathbb{Q}_{\geqslant 0}$ be $\mathrm{ord}((k^{sep})^\circ)$ where k^{sep} is a separable closure of k. The set Γ is dense in $\mathbb{R}_{\geqslant 0}$. Fix $r \in \Gamma$. We can assume that $r \in \mathrm{ord}(K)$ and we now fix such a K.

Lemma 5.1. — *There exists a unique K°-Demazure group scheme $\mathfrak{G}$ such that $H \widehat{\otimes} K = \widehat{\mathfrak{G}}_\eta$, moreover it is Galois stable.*

Proof. — Assume $\mathfrak{G} = \mathrm{Spec}(\mathfrak{A})$ and $\mathfrak{G}' = \mathrm{Spec}(\mathfrak{A}')$ are two K°-Demazure group schemes satisfying $H \widehat{\otimes} K = \widehat{\mathfrak{G}}_\eta = \widehat{\mathfrak{G}'}_\eta$. By Proposition 3.2, we have $\mathrm{Shi}(\widehat{\mathfrak{G}'}_\eta) = \mathrm{Shi}(\widehat{\mathfrak{G}}_\eta) = |\ |_{\mathfrak{A}} = |\ |_{\mathfrak{A}'}$. By definition $|\ |_{\mathfrak{A}}$ is a norm on $\mathcal{O}(G \times_k K)$ given by the formula

$$|f|_{\mathfrak{A}} = \inf_{\lambda \in K^\times} \{|\lambda| | f \in \lambda\,(\mathfrak{A} \otimes 1)\}.$$

The valuation of K is discrete, so we have

$$\begin{aligned} f \in \mathfrak{A} \Leftrightarrow 1 &\in \left\{\lambda \in K^\times \,\middle|\, f \in \lambda\left(\mathfrak{A}\otimes 1\right)\right\} \\ &\Leftrightarrow \inf_{\lambda\in K^\times} \left\{|\lambda| \,|f \in \lambda\left(\mathfrak{A}\otimes 1\right)\right\} \leqslant 1 \\ &\Leftrightarrow |f|_{\mathfrak{A}} \leqslant 1. \end{aligned}$$

Similarly we have $f \in \mathfrak{A}' \Leftrightarrow |f|_{\mathfrak{A}'} \leqslant 1$. So finally $f \in \mathfrak{A} \Leftrightarrow f \in \mathfrak{A}'$, as required.

Now let us prove that $\mathfrak{A}$ is Galois stable. Let $\sigma \in \mathrm{Gal}(K/k)$. On one hand, we have

$$\sigma\left(H\widehat{\otimes}K\right) = H\widehat{\otimes}K, \quad \text{so} \quad \sigma\left(\widehat{\mathrm{Spec}\,(\mathfrak{A})}_\eta\right) = \widehat{\mathrm{Spec}\,(\mathfrak{A})}_\eta.$$

On the other hand, we have

$$\widehat{\mathrm{Spec}\,(\sigma(\mathfrak{A}))}_\eta = \sigma\left(\widehat{\mathrm{Spec}(\mathfrak{A})}_\eta\right).$$

So we have

$$\widehat{\mathrm{Spec}\,(\sigma(\mathfrak{A}))}_\eta = \widehat{\mathrm{Spec}\,(\mathfrak{A})}_\eta.$$

Thus by the previous assertion, we have $\sigma(\mathfrak{A}) = \mathfrak{A}$. □

Let $\boldsymbol{\pi}_r$ be an ideal in K° generated by elements π_r such that $\mathrm{ord}(\pi_r) = r$. Let $\widehat{\mathfrak{G}_{\boldsymbol{\pi}_r}}_\eta$ be the generic fiber of the formal completion of $\mathfrak{G}_{\boldsymbol{\pi}_r}$ along its special fiber.

Lemma 5.2. — *The Shilov boundary of* $\widehat{\mathfrak{G}_{\boldsymbol{\pi}_r}}_\eta$ *is a singleton in* $(G \times_k K)^{an}$ *that is stable under the Galois group* $\mathrm{Gal}(K/k)$.

Proof. — The Shilov boundary of $\widehat{\mathfrak{G}_{\boldsymbol{\pi}_r}}_\eta$ is a singleton by Corollary 3.3. By Lemma 5.1 $\mathfrak{G}$ is Galois stable. So by Lemma 2.2 $\mathfrak{G}_{\boldsymbol{\pi}_r}$ is Galois stable. Consequently by Proposition 3.6 $\widehat{\mathfrak{G}_{\boldsymbol{\pi}_r}}_\eta$ is Galois stable and so is its Shilov boundary. □

Let H_r be $\mathrm{pr}_{K/k}\,\widehat{\mathfrak{G}_{\boldsymbol{\pi}_r}}_\eta$.

Proposition 5.3. — *The set* H_r *is a* k*-affinoid group independent of the choice of the extension* K/k *used in order to define it, moreover its Shilov boundary is a singleton* σ_r *and* $\mathrm{Hol}(\sigma_r) = H_r$.

Proof. — This is proved in the same way as Proposition 6.4 (we do not use Proposition 5.3 in order to prove Proposition 6.4). □

Remark 5.4. — A point $x \in \mathrm{BT}(G,k)$ is called rational if there exists a finite Galois extension K/k such that $\iota_{K/k}(x)$ is a special point in $\mathrm{BT}(G,K)$ and G is split over K. The set of rational points is denoted $\underline{\mathrm{BT}(G,k)}$. Using [RTW10, Theorem 2.1 and its proof], we see that each rational point x gives birth canonically to a rational potentially Demazure k-affinoid group G_x ($G_x = \mathrm{pr}_{K/k}\,\widehat{\mathfrak{G}}_\eta$ where $\mathfrak{G}$ is the canonical Demazure K°-group-scheme attached to $\iota_{K/k}(x)$).

The following lemma will be useful in the next section.

Lemma 5.5. —

(1) *The set* $\underline{\mathrm{BT}(G,k)}$ *is dense in* $\mathrm{BT}(G,k)$.
(2) *The set* $\underline{\mathrm{BT}(G,k)} \times \Gamma$ *is dense in* $\mathrm{BT}(G,k) \times \mathbb{R}_{\geqslant 0}$ *for product topologies.*

Proof. — Remark first that if G is split over k, it is obvious that $\underline{\mathrm{BT}(G,k)}$ is dense in $\mathrm{BT}(G,k)$, because for any maximal split torus S over k and any finite extension K/k, the apartment $A(G,S,K)$ is obtained from $A(G,S,k)$ adding regularly $e(K:k)$ times more walls. Let us now prove the proposition. It is enough to show that for any maximal split torus S of G over k, $\underline{A(G,S,k)}$ is dense in $A(G,S,k)$. Let L be a finite Galois extension such that G is split over L. By [BT84, § 4.1.1, § 4.1.2, § 5.1.12], there exists a torus $T \supset S$ defined over k such that $T \times_k L$ is a maximal split torus of $G \times_k L$. There exists a facet F in $A(G,T,L)$ which is $\mathrm{Gal}(L/k)$-stable. The barycentre x of F is $\mathrm{Gal}(L/k)$-stable and so $x \in A(G,S,k)$ (since $A(G,T,L)^{\mathrm{Gal}(L/k)} = A(G,S,k)$). By [Cor20, § 6.3.4, lines 8-9], the point x becomes special over a finite extension K/L. So we have proved that there exists one rational point x in $A(G,S,k)$. Now the set of points $\{g.x \mid g \in S(k^{sep})\}$ consists in a dense subset of $A(G,S,k)$ constituted of rational points. Indeed, let us first show that this set consists in rational points. So let $g \in S(k^{sep})$, there exists a finite extension K/L such that $g \in S(K)$. The point x is special in the building $\mathrm{BT}(G,K)$ (since G is split over L and x is special in the building $\mathrm{BT}(G,L)$), so $g.x$ is special in $\mathrm{BT}(G,K)$. By definition $T(k^{sep})$ acts on $A(G,T,L)$ by translation (the translation vector v associated to $t \in T(k^{sep})$ is given by the formula "$< v, \alpha >= -\mathrm{ord}(\alpha(t)) \quad \forall\ \alpha$", see [BT84, § 4.2.3 (I)]) and for any $g \in S(k^{sep}) \subset T(k^{sep})$, we have $g.x \in A(G,S,k)$, so $g.x$ is a rational point in $\mathrm{BT}(G,k)$. Since $\mathrm{ord}(k^{sep})$ is dense in $\mathbb{R}$, the first assertion follows. The second assertion is a direct consequence of the first one since Γ is dense in $\mathbb{R}_{\geqslant 0}$ for the archimedean topology. □

6. The general case for points in the Bruhat–Tits buildings

Let G be a connected reductive group over k. There exists a finite Galois extension L/k such that $G \times_k L$ is split. Let $(x,r) \in \mathrm{BT}(G,k) \times \mathbb{R}_{\geqslant 0}$. Consider the point $(\iota_{L/k}(x), r) \in \mathrm{BT}(G,L) \times \mathbb{R}_{\geqslant 0}$. Let $\theta_L(\iota_{L/k}(x), r)$ in $(G \times_k L)^{an}$ be the Shilov boundary of the L-affinoid group attached to $(\iota_{L/k}(x), r)$ using the construction of Section 4.

Lemma 6.1. — *The point $\theta_L(\iota_{L/k}(x), r)$ is $\mathrm{Gal}(L/k)$-stable.*

Proof. — Using Lemma 5.2 and Remark 5.4, we see that $\theta_L(\iota_{L/k}(\underline{\mathrm{BT}(G,k)}) \times \Gamma)$ is fixed by $\mathrm{Gal}(L/k)$. Lemma 5.5 implies that $\theta_L(\iota_{L/k}(\mathrm{BT}(G,k), \mathbb{R}_{\geqslant 0})$ is fixed by $\mathrm{Gal}(L/k)$, since θ_L is continuous by the explicit formula given in Proposition 4.2. □

Remark 6.2. — It is possible to prove Lemma 6.1 without using Section 5. Indeed let $S \subset T$ be torus over k such that S is a maximal split torus over k, $x \in A(G,S,k)$, and T is a maximal split torus over K. The point $\iota_{L/k}(x)$ is in $A(G,T,L)$. Moreover, we have an explicit formula for $\theta_L(\iota_{L/k}(x), r)$ (Proposition 4.2) and we can check on the formula that it is $\mathrm{Gal}(L/k)$-stable using the fact that the action of $\mathrm{Gal}(L/k)$ on $X^*(T,L)$ stabilizes $\Phi(G,T,L)$ and associated objects.

So $G_{\iota_{L/k}(x),r}$ is $\mathrm{Gal}(L/k)$-stable and so $\mathrm{pr}_{L/k}^{-1}(\mathrm{pr}_{L/k}(G_{\iota_{L/k}(x),r})) = G_{\iota_{L/k}(x),r}$. So using Theorem 3.4 and Lemma 4.6, we obtain that $\mathrm{pr}_{L/k}(G_{\iota_{L/k}(x),r})$ is a k-affinoid group that we denote by $G_{x,r}$. Moreover, it is easy to see that the Shilov boundary of $G_{x,r}$ is equal to $\mathrm{pr}_{K/l}(\theta_L(\iota_{L/k}(x), r))$, we denote it as $\theta(x,r)$.

Remark 6.3. — For any $(x,r) \in \mathrm{BT}(G,k)\times\mathbb{R}_{\geqslant 0}$, we defined a k-affinoid group $G_{x,r}$ as $\mathrm{pr}_{L/k}\mathrm{pr}_{K/L}(\widehat{\mathfrak{G}_{r\ \eta}})$ where L/k is a finite Galois extension splitting G and K/L is an affinoid extension. Using the compatibility of dilatations under base change [MRR20, Theorem 3.2(6)], it is a formal computation to check that $G_{x,r}$ is well-defined, i.e it does not depend on the choice of extensions L and K used in order to define it. So we see that $G_{x,r} = \mathrm{pr}_{K/k}(\widehat{\mathfrak{G}_{r\ \eta}})$ where K/k is any k-affinoid extension such that G is split over K, $\iota_{K/k}(x)$ is special and $r \in \mathrm{ord}(K)$. Moreover the identity $G_{x,0} = \mathrm{pr}_{K/k}(\widehat{\mathfrak{G}_{\ \eta}})$ shows that $G_{x,0} = G_x$, where G_x is the k-affinoid group defined in [RTW10, Theorem 2.1 and its proof].

Proposition 6.4. —

(1) *The holomorphically convex envelope of $\theta(x,r)$ is $G_{x,r}$.*
(2) *The k-affinoid algebra of $G_{x,r}$ is the completion of $\mathcal{O}(G)$ relatively to the norm $\theta(x,r)$.*

Proof. —

(1) Since $\theta_L(\iota_{L/k}(x),r)$ is $\mathrm{Gal}(L/k)$-stable and $\mathrm{pr}_{L/k}(\theta_L(\iota_{L/k}(x),r)) = \theta(x,r)$, we have $\mathrm{pr}_{L/k}^{-1}(\theta(x,r)) = \theta_L(\iota_{L/k}(x),r)$. So by [MP21, Proposition 4.4], we obtain $\mathrm{Hol}(\theta_L(\iota_{L/k}(x),r)) = \mathrm{pr}_{L/k}^{-1}(\mathrm{Hol}(\theta(x,r)))$, this implies $G_{x,r} = \mathrm{Hol}(\theta(x,r))$ as required.
(2) Let $\mathcal{A}$ be the k-affinoid algebra of $G_{x,r}$. This is reduced and we can assume that its norm equals its spectral norm [Ber90, Proposition 2.1.4] and so equals its unique Shilov boundary point $\theta(x,r)$. Let $\mathcal{A}(x,r)$ be the completion of $\mathcal{O}(G)$ relatively to the norm $\theta(x,r)$. The immersion $G_{x,r} \to G^{an}$ corresponds to an injective morphism of k-algebras $\mathcal{O}(G) \to \mathcal{A}$. This morphism extends to an isometric embedding $i : \mathcal{A}(x,r) \to \mathcal{A}$. Let K/k be an affinoid extension such that we can write $G_{x,r} = \mathrm{pr}_{K/k}(\widehat{\mathfrak{G}_{r\ \eta}})$ (see Remark 6.3). Let $\mathcal{A}_K$ be the K-affinoid algebra of $\widehat{\mathfrak{G}_{r\ \eta}}$. The algebra $\mathcal{A}_K$ is equal to the completion of $\mathcal{O}(G)\otimes_k K$ relatively to the norm $|.|_{\mathfrak{G}_r}$, in particular $\mathcal{O}(G)\otimes_k K$ is dense in $\mathcal{A}_K$. Moreover by definition, $\mathcal{A}\widehat{\otimes}_k K = \mathcal{A}_K$. So $\mathcal{O}(G)\otimes_k K$ is dense in both $\mathcal{A}\widehat{\otimes}_k K$ and $\mathcal{A}(x,r)\widehat{\otimes}_k K$. So $i\widehat{\otimes}\mathrm{Id}_K : \mathcal{A}(x,r)\widehat{\otimes}_k K \to \mathcal{A}\widehat{\otimes}_k K$ is an isomorphism of Banach algebras, and so $\mathcal{A}(x,r) = \mathcal{A}$ by [RTW10, Appendix A, Lemma A.5].

□

Proposition 6.5. — *Let $g \in G(k)$, then $G_{g.x,r} = gG_{x,r}g^{-1}$ and $\theta(g.x,r) = g\theta(x,r)g^{-1}$.*

Proof. — The assertions are equivalent by Proposition 6.4. Let us prove the first one. Choose a k-affinoid extension K/k such that we can write $G_{x,r} = \mathrm{pr}_{K/k}(\widehat{\mathfrak{G}_{x,r\ \eta}})$. The sequence of equalities

$$\begin{aligned} gG_{x,r}g^{-1} &= g\ \mathrm{pr}_{K/k}\left(\widehat{\mathfrak{G}_{x,r}}_{\ \eta}\right)g^{-1} \\ &= \mathrm{pr}_{K/k}\left(g\ \widehat{\mathfrak{G}_{x,r}}_{\ \eta}\ g^{-1}\right) \\ &= \mathrm{pr}_{K/k}\left(\widehat{g\mathfrak{G}_{x,r}g^{-1}}_{\ \eta}\right) \\ &= \mathrm{pr}_{K/k}\left(\widehat{\mathfrak{G}_{g.x,r}}_{\ \eta}\right) \\ &= G_{g.x,r} \end{aligned}$$

ends the proof of Proposition 6.5. □

We end this section with the following result.

Proposition 6.6. — *Assume G splits over a tamely ramified extension. Then for all $(x,r) \in \mathrm{BT}(G,k) \times \mathbb{R}_{\geqslant 0}$, the point $\theta(x,r) \in G^{an}$ is peaked.*

Proof. — We use notation of the beginning of this section. The point $\theta_L(\iota_{L/k}(x),r)$ is peaked by Proposition 4.3. Now [RTW10, Lemma A.10] and [DFN15, Part II, Chapter on Buildings, Section 6 about erratum] end the proof. □

Remark 6.7. — If G does not split over a tamely ramified extension, then in general $\theta(x,r)$ is not peaked (see Proposition 11.2 for a counter-example.)

7. Filtration of Lie algebras

Let $\mathfrak{G}$ be the Lie algebra of G and $\mathfrak{G}^{an}$ its analytification. Let $(x,r) \in \mathrm{BT}(G,k) \times \mathbb{R}_{\geqslant 0}$. Let K/k be a k-affinoid extension such that we can write $G_{x,r} = \mathrm{pr}_{K/k}(\widehat{\mathfrak{G}_r}_{\ \eta})$ (see Remark 6.3). We define

$$\mathfrak{G}_{x,r} = \mathrm{pr}_{K/k}\left(\widehat{\mathrm{Lie}(\mathfrak{G}_r)}_{\eta}\right).$$

Using similar arguments as for $G_{x,r}$, we see that $\mathfrak{G}_{x,r}$ is a k-affinoid subgroup of $\mathfrak{G}^{an}$ equal to the holomorphically convex envelope of its unique Shilov boundary point. We can also define similar filtrations in the context of rational potentially Demazure k-affinoid groups.

8. Comparison with Moy–Prasad filtrations

Let G be a connected reductive k-group scheme that splits over a tamely ramified extension. Let $G(k)^{MP}_{x,r}$ denote the normalized Moy–Prasad filtration as used in [Yu01]. We will use the following facts

- [Yu15, Corollary 8.8] if G is split and x is special, then Moy–Prasad filtrations are obtained by taking set-theoretic congruence subgroups of the integral points of the attached integral Demazure group $\mathfrak{G}_x$;
- [Kim07, Line 15 Page 278] or [KP21, Section Tamely ramified descent] Moy–Prasad filtrations are compatible relatively to field extensions in the tame case; in order to prove the following proposition.

Proposition 8.1. — *Assume that we can choose a finite and tamely ramified extension K/k in order to define (as in Remark 6.3) the k-affinoid group $G_{x,r}$ and assume $r > 0$, then $G_{x,r}(k) = G(k)_{x,r}^{MP}$.*

Proof. — Let K/k be a finite tamely ramified extension such that we can write $G_{x,r} = \mathrm{pr}_{K/k}(\widehat{\mathfrak{G}_{r\ \eta}})$. The following equalities

$$\begin{aligned} G_{x,r}(k) &= G_{x,r}(K) \cap G(k) \\ &= \widehat{\mathfrak{G}_{r\ \eta}}(K) \cap G(k) \\ &= \mathfrak{G}_r(K^\circ) \cap G(k) \\ &= G(K)_{x,r}^{MP} \cap G(k) \\ &= G(k)_{x,r}^{MP} \end{aligned}$$

ends the proof. □

9. A cone in G^{an}

In the previous section, we constructed for each pair $(x, r) \subset \mathrm{BT}(G, k) \times \mathbb{R}_{\geqslant 0}$ a k-affinoid group $G_{x,r}$ whose Shilov boundary is a singleton $\theta(x, r)$. If $r' > r$, it is easy to see that $G_{x,r'} \underset{\neq}{\subset} G_{x,r}$. We now introduce a map θ.

Definition 9.1. — *Let θ be the map*

$$\begin{aligned} \theta : \mathrm{BT}(G, k) \times \mathbb{R}_{\geqslant 0} &\to G^{an} \\ (x, r) &\mapsto \theta(x, r). \end{aligned}$$

Theorem 9.2. —

(1) *The map θ is $G(k)$-equivariant relatively to the actions $g.(x, r) = (g.x, r)$ and $g.p = gpg^{-1}$, for $g \in G(k)$, $x \in \mathrm{BT}(G, k)$, $r \in \mathbb{R}_{\geqslant 0}$, and $p \in G^{an}$.*
(2) *For any finite extension k'/k, the diagram*

$$\begin{array}{ccc} \mathrm{BT}(G, k') \times \mathbb{R}_{\geqslant 0} & \xrightarrow{\theta'} & (G \times_k k')^{an} \\ \uparrow{\scriptstyle \iota_{k'/k} \times \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{pr}_{K/k}} \\ \mathrm{BT}(G, k) \times \mathbb{R}_{\geqslant 0} & \xrightarrow{\theta} & G^{an} \end{array}$$

is commutative. Here θ' is defined as θ. In other words, it is the map sending $(x, r) \in \mathrm{BT}(G, k') \times \mathbb{R}_{\geqslant 0}$ to $\mathrm{Shi}(\mathrm{pr}_{K/k'}(\widehat{\mathfrak{G}_{r\ \eta}}))$ where K/k' is an affinoid extension such that G is split over K, $\iota_{K/k'}(x)$ is special and $r \in \mathrm{ord}(K)$, moreover $\mathfrak{G}_r$ is the $\boldsymbol{\pi}_r$-congruence subgroup of the canonical Demazure K°-group scheme attached to $\iota_{K/k'}(x)$.
(3) *The map θ is continuous and injective.*

Proof. —

(1) This is a reformulation of Proposition 6.5.
(2) This is a direct consequence of definitions.

(3) Let us first assume that G is split over k. It is enough to prove that the map $A(G,T,k)\times\mathbb{R}_{\geqslant 0}\to G^{an}$ is continuous and injective for all apartments $A(G,T,k)$. This is a direct consequence of the formula of Proposition 4.2. In general, we choose a finite Galois extension such that G is split over L and conclude directly using the commutative diagram of the second assertion of this theorem. □

The set $\theta(\mathrm{BT}(G,k),\mathbb{R}_{\geqslant 0})\cup e_{G^{an}}$ is a topological cone in G^{an} whose basis is the Bruhat–Tits building and whose vertex is the neutral element $e_{G^{an}}$.

10. Picture for the split torus of rank one

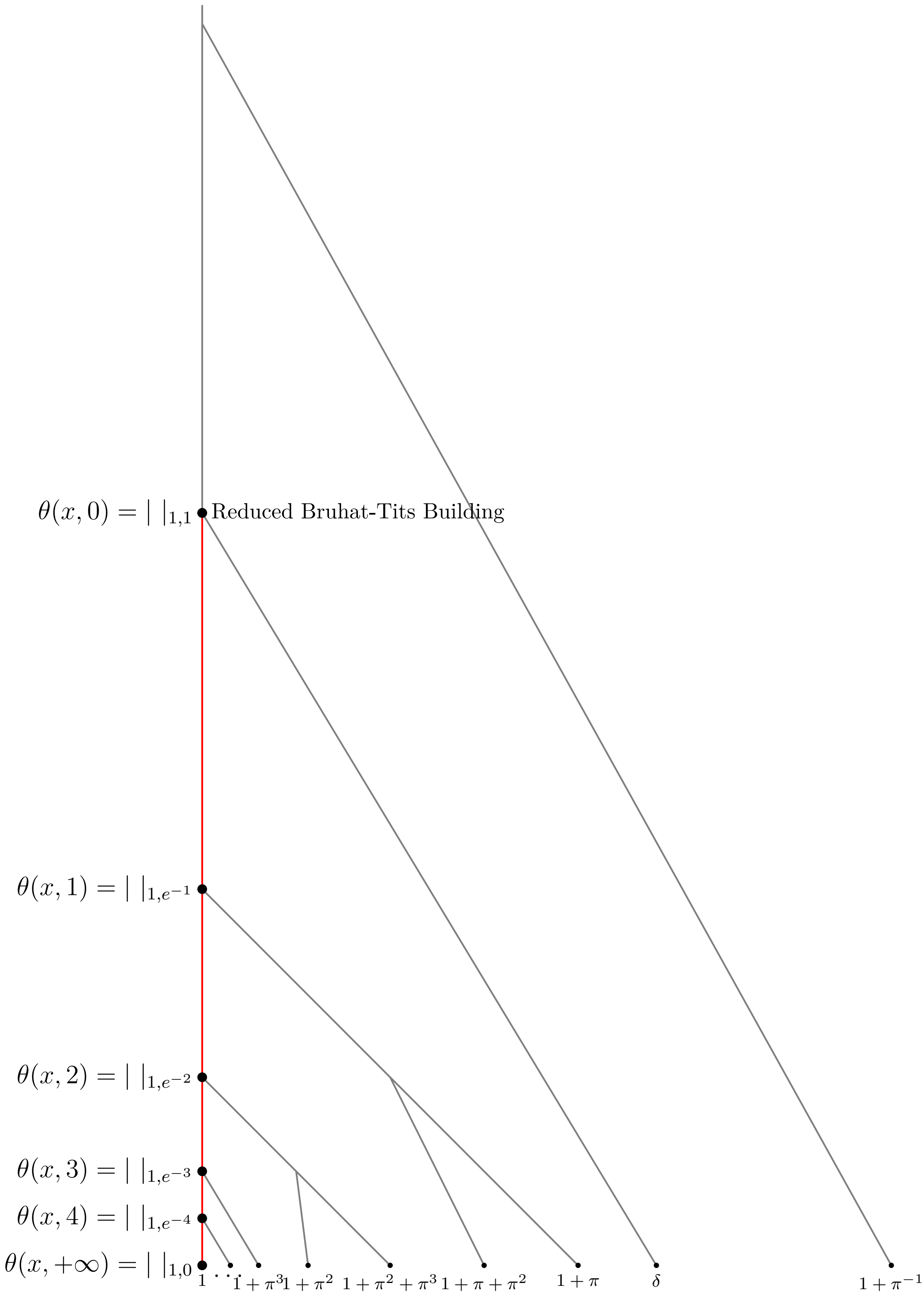
Let G be $\mathrm{Spec}(k[X,Y]/XY-1)$, it is a split torus of rank one over k. The reduced Bruhat–Tits building of G is a singleton $\{x\}$. The point x is special and G is split over k. The grosse cellule of G is G. Let $r\geqslant 0$ and choose a k-affinoid extension K/k such that $r\in\mathrm{ord}(K)$. Let $\mathfrak{G}$ be the K°-Demazure group scheme attached to $\iota_{K/k}(x)$. It is equal to $\mathrm{Spec}(K^{\circ}[X,Y]/XY-1)$. By definition $G_{x,r}$ is equal to $\mathrm{pr}_{K/k}(\widehat{\mathfrak{G}_r}_{\ \eta})$. The ring $\mathrm{Hopf}(\mathfrak{G}_r)$ is equal to $K^{\circ}[{\pi_r}^{-1}(X-1),{\pi_r}^{-1}(Y-1)]\subset K[X,Y]/XY-1$. Writing $f\in K[X,Y]/XY-1$ as $\sum_{(k_1,k_2)\,\in U} a_{k_1k_2}(X-1)^{k_1}(Y-1)^{k_2}$ (U is the set of parameters for the basis of $K[X,Y]/XY-1$ centered at unity), the norm $|\ |_{\mathrm{Hopf}(\mathfrak{G}_r)}$ is explicitly given by the map

$$\begin{aligned}K[X,Y]/XY-1 &\to \mathbb{R}_{\geqslant 0}\\ f &\mapsto \max_{(k_1,k_2)\,\in U}|a_{k_1k_2}|\,e^{-r(k_1+k_2)}\ .\end{aligned}$$

The Shilov boundary of $\widehat{\mathfrak{G}_r}_{\ \eta}$ is $|\ |_{\mathrm{Hopf}(\mathfrak{G}_r)}$. The Shilov boundary $\theta(x,r)$ of $\mathrm{pr}_{K/k}$ $(\widehat{\mathfrak{G}}_{r\eta})$ is $|\ |_{\mathrm{Hopf}(\mathfrak{G}_r)}$ restricted to the k-algebra $\mathrm{Hopf}(G)$. The point $\theta(x,r)\in G^{an}$ is thus equal to the norm on $k[X,Y]/XY-1$ which map

$$\sum_{(k_1,k_2)\,\in U} a_{k_1k_2}(X-1)^{k_1}(Y-1)^{k_2}\quad\text{to}\quad \max_{(k_1,k_2)\,\in U}|a_{k_1k_2}|\,e^{-r(k_1+k_2)}.$$

It corresponds via the embedding $G^{an}\to(\mathbb{A}^1_k)^{an}\setminus 0$ to the norm usually denoted $|\ |_{1,e^{-r}}$ inside $(\mathbb{A}^1_k)^{an}$. We have the picture


$\theta(x,0) = |\ |_{1,1}$
Reduced Bruhat-Tits Building
$\theta(x,1) = |\ |_{1,e^{-1}}$
$\theta(x,2) = |\ |_{1,e^{-2}}$
$\theta(x,3) = |\ |_{1,e^{-3}}$
$\theta(x,4) = |\ |_{1,e^{-4}}$
$\theta(x,+\infty) = |\ |_{1,0}$
1
$1+\pi^3$
$1+\pi^2$
$1+\pi^2+\pi^3$
$1+\pi+\pi^2$
$1+\pi$
δ
$1+\pi^{-1}$

giving some points (of course it is not exhaustive) of G^{an} inside $(\mathbb{A}_k^1)^{an}$. Here δ is an element in $(k^\circ)^\times\backslash 1+k^{\circ\circ}$. The point $\theta(x,0)$ is mapped to the so-called Gauss point, and corresponds to the reduced Bruhat–Tits building. When $r \geqslant 0$ is increasing the point $\theta(x,r)$ is getting closer to 1, the neutral element of G^{an}. The holomorphically convex envelope $G_{x,r}$ of $\theta(x,r)$ should be thought as all the points under (attainable by going only down) $\theta(x,r)$ and the k-rational points of $G_{x,r}$ as certain lower extremities. In this situation the cone is the red line.

11. Computation in a wild torus

In this section $k=\mathbb{Q}_2$. The polynomial X^2-2 does not have any solution in k. Let $\sqrt{2}\in\overline{k}$ be a root of this polynomial and let K be the field $k(\sqrt{2})\subset\overline{k}$. The extension K/k is a wildly ramified Galois extension. We have $[K:k]=e(K:k)=2$. The element $\sqrt{2}$ is a uniformizer of K. The k-vector space K is 2-dimensional and $\{1,\sqrt{2}\}$ is a k-basis. So each element in K can be written as $x+\sqrt{2}y$ with $x,y\in k$. The norm of $x+\sqrt{2}y$ is equal to $(x+\sqrt{2}y)(x-\sqrt{2}y)=x^2-2y^2$. The set of norm 1 elements is an algebraic group. Let us write the Hopf algebra of the corresponding affine k-group scheme G. The Hopf k-algebra of G is $k[X,Y]/X^2-2Y^2-1$, moreover the comultiplication Δ, the antipode τ and the augmentation ε are

$$\begin{aligned}
\Delta:\mathrm{Hopf}(G)&\to\mathrm{Hopf}(G)\otimes\mathrm{Hopf}(G)\\
X&\mapsto X\otimes X+2Y\otimes Y\\
Y&\mapsto X\otimes Y+Y\otimes X\\
\tau:\mathrm{Hopf}(G)&\to\mathrm{Hopf}(G)\\
X&\mapsto X\\
Y&\mapsto -Y\\
\varepsilon:\mathrm{Hopf}(G)&\to k\\
X&\mapsto 1\\
Y&\mapsto 0\ .
\end{aligned}$$

The k-group G is a torus, indeed the equation

$$\begin{aligned}
k[X,Y]/X^2-2Y^2-1\otimes_k K&\simeq K[X,Y]/X^2-2Y^2-1\\
&\simeq K[X,Y]/\left(X+\sqrt{2}Y\right)\left(X-\sqrt{2}Y\right)-1\\
&\simeq K[U,V]/UV-1
\end{aligned}$$

shows that $G\times_k K\simeq\mathbb{G}_m/K$. The reduced Bruhat–Tits building $\mathrm{BT}(G,k)$ is a singleton $\{x\}$. The point x is a special point of $\mathrm{BT}(G,k)$ and $\iota_{K/k}(x)\in\mathrm{BT}(G,K)$ is special for any finite extension K/k. The group G is not split over k, it is split over K. Let us make explicit the group $G_{x,0}$. We need to find an extension such that G is split over it and the image of x over K is special. The field K works. By definition the k-analytic group $G_{x,0}$ is equal to $\mathrm{pr}_{K/k}(\widehat{\mathfrak{G}}_\eta)$, where $\mathfrak{G}$ is the K°-Demazure group scheme attached to $\iota_{K/k}(x)$. In the coordinates U,V, $\mathfrak{G}=\mathrm{Spec}(K^\circ[U,V]/UV-1)$. Thus in the coordinates X, Y, $\mathrm{Hopf}(\mathfrak{G})$ is equal to the K°-subalgebra $K^\circ[X+\sqrt{2}Y, X-\sqrt{2}Y]$ of $K[X,Y]/X^2-2Y^2-1$ generated by $X+\sqrt{2}Y$ and $X-\sqrt{2}Y$. The k-affinoid algebra of $G_{x,0}$ is the completion of $\mathrm{Hopf}(G)$ relatively to the norm $|\ |_{\mathrm{Hopf}(\mathfrak{G})}\ |_{\mathrm{Hopf}(G)}$. So let us make as explicit as possible the norm $|\ |_{\mathrm{Hopf}(\mathfrak{G})}\ |_{\mathrm{Hopf}(G)}$. By definition, we have

$$\begin{aligned} |\ |_{\mathrm{Hopf}(\mathfrak{G})} : \mathrm{Hopf}\left(G\times_k K\right) &\to \mathbb{R}_{\geqslant 0} \\ f &\mapsto \inf_{\lambda\in K^\times}\left\{|\lambda| \middle| f\in\lambda.K^\circ\left[X+\sqrt{2}Y, X-\sqrt{2}Y\right]\right\}. \end{aligned}$$

And so, by restriction

$$\begin{aligned} |\ |_{\mathrm{Hopf}(\mathfrak{G})}|_{\mathrm{Hopf}(G)} : k[X,Y]/X^2-2Y^2-1 &\to \mathbb{R}_{\geqslant 0} \\ f &\mapsto \inf_{\lambda\in K^\times}\left\{|\lambda| \middle| f\in\lambda.K^\circ\left[X+\sqrt{2}Y, X-\sqrt{2}Y\right]\right\}. \end{aligned}$$

We have to complete $k[X,Y]/X^2-2Y^2-1$ relatively to this norm, in order to simplify notation let us put $||\ || = |\ |_{\mathrm{Hopf}(\mathfrak{G})}\ |_{\mathrm{Hopf}(G)}$. Let us compute the value $||X||$. Since $\sqrt{2}X\notin K^\circ[X+\sqrt{2}Y, X-\sqrt{2}Y]$ and $2X=\sqrt{2}^2X\in K^\circ[X+\sqrt{2}Y, X-\sqrt{2}Y]$, we deduce that

$$\|X\| = e.$$

Let us now compute the value $||Y||$. Since $2Y=\sqrt{2}^2Y\notin K^\circ[X-\sqrt{2}Y, X+\sqrt{2}Y]$ and $2\sqrt{2}Y=\sqrt{2}^3Y\in K^\circ[X-\sqrt{2}Y, X+\sqrt{2}Y]$ we deduce that

$$\|Y\| = e^{\frac{3}{2}}.$$

Consider the algebra

$$k\left\{e^{-1}X, \left(e^{\frac{3}{2}}\right)^{-1}Y\right\}/X^2-2Y^2-1\ ,\ \|\ \|$$

where $k\{e^{-1}X, (e^{\frac{3}{2}})^{-1}Y\}$ is the k-algebra

$$\left\{\sum_{k_1,k_2} a_{k_1k_2}X^{k_1}Y^{k_2} \middle| |a_{k_1k_2}|e^{k_1}\left(e^{\frac{3}{2}}\right)^{k_2}\to 0 \text{ as } k_1+k_2\to\infty\right\}\subset k[[X,Y]].$$

We claim that it is the k-affinoid algebra of $G_{x,0}$. We need to check that

$$\left(k\left\{e^{-1}X, \left(e^{\frac{3}{2}}\right)^{-1}Y\right\}/X^2-2Y^2-1\right)\widehat{\otimes}_k K$$

is isomorphic to the K-affinoid algebra of $\widehat{\mathfrak{G}}_\eta$. In the coordinates U, V, the K-affinoid algebra of $\widehat{\mathfrak{G}}_\eta$ is $K\{U,V\}/UV-1$. The K-algebra $(k\{e^{-1}X, (e^{\frac{3}{2}})^{-1}Y\}/X^2-2Y^2-1)\widehat{\otimes}_k K$ is isomorphic to $K\{e^{-1}X, (e^{\frac{3}{2}})^{-1}Y\}/X^2-2Y^2-1)$. The isomorphism previously considered $K[X,Y]/X^2-2Y^2-1\simeq K[U,V]/UV-1$ induces maps

$$K\left\{e^{-1}X,\left(e^{\frac{3}{2}}\right)^{-1}Y\right\}/X^2-2Y^2-1 \leftrightarrow K\{U,V\}/UV-1$$
$$X+\sqrt{2}Y \leftarrow U$$
$$X-\sqrt{2}Y \leftarrow V$$
$$X \mapsto \frac{U+V}{2}$$
$$Y \mapsto \frac{U-V}{2\sqrt{2}}.$$

These maps are mutual inverse K-Banach algebras isometries.

Now we are interested in the question: Is the Shilov boundary of $G_{x,0}$ a peaked norm? In other words, is $\|\ \|$ a peaked norm?

Recall that by definition $\|\ \|\otimes|\ |_K$ is the norm on $k[X,Y]/X^2-2Y^2-1\otimes_k K$ defined by $\|\ \|\otimes|\ |_K(f) = \inf\max_i \|x_i\|.|\lambda_i|_K$ where inf is taken over all representatives $f=\sum_i x_i\otimes\lambda_i$. Let us start with a Lemma.

Lemma 11.1. —

(1) *Each $f\in k[X,Y]/X^2-2Y^2-1\otimes_k K$ can be written uniquely as*

$$f = x\otimes 1 + y\otimes\sqrt{2}.$$

(2) *Let $f\in k[X,Y]/X^2-2Y^2-1\otimes_k K$ and write $f=x\otimes 1+y\otimes\sqrt{2}$ as in the previous assertion. Then $\|\ \|\otimes|\ |_K(f)=\max\{\|x\|,\|y\|.|\sqrt{2}|\}$.*

Proof. — The first assertion is a direct consequence of the fact that $\{1,\sqrt{2}\}$ is a k-basis of K. Now let us prove the second assertion. Let $Z\in k[X,Y]/X^2-2Y^2-1\otimes_k K$. For a representative $R: Z=\sum_i x_i\otimes\alpha_i$, we use the notation $|Z|_R$ for $\max_i\|x_i\||\alpha_i|_K$. So that we have $\|\ \|\otimes|\ |_K(Z)=\inf_R|Z|_R$. Let $R: f=\sum_{i=1}^S x_i\otimes\alpha_i$ be a representative of f. Each α_i can be written as $\alpha_i=a_i+b_i\sqrt{2}$. Since a_i and b_i are in k we have $|a_i|\neq|b_i\sqrt{2}|$. So $|\alpha_i|=\max\{|a_i|,\ |b_i\sqrt{2}|\}$. We have

$$\begin{aligned} f &= \sum_{i=1}^S x_i\otimes\left(a_i+b_i\sqrt{2}\right)\\ &= \left(\sum_{i=1}^S x_ia_i\right)\otimes 1+\left(\sum_{i=1}^S x_ib_i\right)\otimes\sqrt{2}. \end{aligned}$$

Now let us denote by R' this last representative, i.e

$$R': f=\left(\sum_{i=1}^S x_ia_i\right)\otimes 1+\left(\sum_{i=1}^S x_ib_i\right)\otimes\sqrt{2}.$$

We claim that $|f|_R \geqslant |f|_{R'}$. Indeed

$$\begin{aligned}
|f|_R &= \max_{i=1}^{S} \|x_i\|.|\alpha_i| \\
&= \max_{i=1}^{S} \left\{ \|x_i\|. \max\left\{ |a_i| \ ,\ |b_i\sqrt{2}| \right\} \right\} \\
&= \max_{i=1}^{S} \left\{ \max\left\{ \|x_i\|.|a_i| \ ,\ \|x_i\|.|b_i\sqrt{2}| \right\} \right\} \\
&= \max_{i=1}^{S} \left\{ \max\left\{ \|x_i.a_i\| \ ,\ \|x_i.b_i\|.|\sqrt{2}| \right\} \right\} \\
&\geqslant \max\left\{ \left\| \left(\sum_{i=1}^{S} x_i a_i \right) \right\| .|1|, \quad \left\| \left(\sum_{i=1}^{S} x_i b_i \right) \right\| . \left|\sqrt{2}\right| \right\} \\
&= |f|_{R'}.
\end{aligned}$$

This ends the proof of Lemma 11.1, since $x = \sum_{i=1}^{S} x_i a_i$ and $y = \sum_{i=1}^{S} x_i b_i$. ☐

Proposition 11.2. — *Let us see U inside $k[X,Y]/X^2 - 2Y^2 - 1 \otimes_k K$ via the isomorphism above, i.e $U = X \otimes 1 + Y \otimes \sqrt{2}$. Then $\| \ \| \otimes_k | \ |_K(U) = e$ and $| \ |_{\mathrm{Hopf}(\mathfrak{G})}(U) = 1$. Moreover the norm $\| \ \|$ is not universal (or peaked).*

Proof. — First by definition of $| \ |_{\mathrm{Hopf}(\mathfrak{G})}$, we have $| \ |_{\mathrm{Hopf}(\mathfrak{G})}(U) = |1| = e^0 = 1$. Now let us compute $\| \ \| \otimes_k | \ |_K(U)$ using the previous Lemma 11.1. We have $U = X \otimes 1 + Y \otimes \sqrt{2}$. So we have

$$\begin{aligned}
\| \ \| \otimes_k | \ |_K(U) &= \max\left\{ \|X\| \ ,\ \|Y\||\sqrt{2}| \right\} \\
&= \max\left\{ e, e^{\frac{3}{2}}.e^{-\frac{1}{2}} \right\} \\
&= e\,.
\end{aligned}$$

Now if $\| \ \|$ was peaked, then we would have deduced $\| \ \| \otimes_k | \ |_K = | \ |_{\mathrm{Hopf}(\mathfrak{G})}$ by [Ber90, Corollary 5.2.4]; so $\| \ \|$ is not peaked. ☐

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