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TRIANGULATIONS OF UNIFORM SUBQUADRATIC GROWTH ARE QUASI-TREES

LES TRIANGULATIONS À CROISSANCE
UNIFORMÉMENT SOUS-QUADRATIQUE SONT
DES QUASI-ARBRES

ABSTRACT. — It is known that for every $\alpha \geq 1$ there is a planar triangulation in which every ball of radius r has size $\Theta(r^\alpha)$. We prove that for $\alpha < 2$ every such triangulation is quasi-isometric to a tree. The result extends to Riemannian 2-manifolds of finite genus, and to large-scale-simply-connected graphs. We also prove that every planar triangulation of asymptotic dimension 1 is quasi-isometric to a tree.

RÉSUMÉ. — On sait que pour tout $\alpha \geq 1$ il existe une triangulation plane dans laquelle toute boule de rayon r a une taille $\Theta(r^\alpha)$. Nous prouvons que pour $\alpha < 2$ une telle triangulation est quasi-isométrique à un arbre. Le résultat s'étend aux 2-variétés riemanniennes de genre fini, et aux graphes simplement connexes à grande échelle. Nous prouvons également que toute triangulation plane de dimension asymptotique 1 est quasi-isométrique à un arbre.

Keywords: planar triangulation, 2-manifold, uniform volume growth, quasi-tree, asymptotic dimension.

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1. Introduction

Motivated by an observation of physicists that in certain planar triangulations the size of the ball of radius r is of order r^4 [ADJ97, Ang03], it was proved in [BS01] that for every $\alpha \geq 1$ there is a triangulation of the plane in which every metric ball $B_v(r)$ of radius r has size $|B_v(r)| = \Theta(r^\alpha)$ independently of the choice of the centre v of the ball. The constructions of [BS01] are quasi-isometric to trees. Our first result is that for $\alpha < 2$, this is not a coincidence, i.e. every such triangulation must be quasi-isometric to a tree. A *planar triangulation* is a connected plane graph every edge of which is contained in two facial triangles —see Section 2.1 for more detailed definitions.

THEOREM 1.1. — *Let G be a planar triangulation. Then either G is quasi-isometric to a tree, or for every $r \in \mathbb{N}$ there is a vertex v such that $|B_v(r)| > r^2$.*

For every $\alpha > 2$ on the other hand, a construction of Ebrahimnejad & Lee [EL21] yields —after minor modifications— planar triangulations that are not quasi-isometric to a tree and satisfy $|B_v(r)| = \Theta(r^\alpha)$ uniformly in v . Combining these constructions with Theorem 1.1, and recalling that the triangular lattice has quadratic growth rate, we deduce that there is a planar triangulation with uniform volume growth $\Theta(r^\alpha)$ which is not quasi-isometric to a tree if and only if $\alpha \geq 2$. (Such triangulations must still have relatively small cutsets at all scales [BP11].)

The trees constructed in [BS01] can easily be modified into 2-manifolds with uniform volume growth $\Theta(r^\alpha)$ for any $\alpha > 1$. We will also prove the following continuous analogue of Theorem 1.1:

COROLLARY 1.2. — *Let M be a connected, complete, Riemannian 2-manifold of finite genus, with uniform subquadratic volume (i.e. area) growth. Then M is quasi-isometric to a tree.*

Our next result is a variant of Theorem 1.1 in which planarity is replaced by the requirement that G be *large-scale-simply-connected (LSSC)*. The *cycle space* $\mathcal{C}(G)$ of a graph G is its first simplicial homology group over the 2-element field \mathbb{Z}_2 .

THEOREM 1.3. — *Let G be a graph, and suppose $\mathcal{C}(G)$ is generated by cycles of length at most k . Then either G is quasi-isometric to a tree, or for every $r \in \mathbb{N}$ there is a vertex v such that $|B_v(r)| \geq c(k)r^2$.*

Here the $c(k)$ are universal constants. We remark that Theorem 1.1 follows from Theorem 1.3 when G is finite or 1-ended, because the cycle space is generated by the facial triangles in this case, but requires arguments specific to the planar case when G has more ends. (A planar triangulation can have any number of ends up to the cardinality of the continuum; the duals of the graphs in [Geo17] provide some examples. Likewise, M need not be of finite type in Corollary 1.2, since we are not imposing a restriction on the number of punctures; for example M could be homeomorphic to the Cantor sphere.) The LSSC condition cannot be relaxed in Theorem 1.3, even if we impose a strong additional condition like planarity, as we show with an example in Section 3.

Fujiwara & Whyte proved that every LSSC graph G of asymptotic dimension 1 is a quasi-tree [FW07]⁽¹⁾. (The converse is well-known: every unbounded quasi-tree has asymptotic dimension 1.) We prove that if G is a planar triangulation, then the LSSC condition can be dropped:

THEOREM 1.4. — *Let G be an infinite planar triangulation. Then either $\text{asdim}(G) = 2$, or G is quasi-isometric to a tree (in which case $\text{asdim}(G) = 1$).*

This uses a recent result that every planar graph has asymptotic dimension at most 2 [BBE⁺20, JL22]. Theorem 1.4 cannot be extended to planar graphs with arbitrarily long facial cycles as shown by [FW07, Example 2.4.]. We can replace asdim by the Assouad-Nagata dimension ANdim throughout the above discussion. In particular, we deduce that $\text{asdim}(G) = \text{ANdim}(G)$ when G is a planar triangulation. We present an example in Section 3.1 showing that this equality does not hold for all planar graphs.

It is a consequence of Gromov's theorem [Gro81] that there is no group of volume growth of order $\Theta(r^\alpha)$ for non-integer α . De la Harpe [dlH00] asks for an elementary proof of this fact. Theorem 1.3 easily implies that there is no finitely presented group of superlinear but subquadratic growth, see Corollary 5.1. Alternative elementary proofs of this fact, covering the infinitely presented case as well, are provided in [IS87, Jus71, WvdD84]. Some elementary proofs of the fact that every group of linear growth is quasi-isometric to \mathbb{Z} have been provided by Bill Thurston⁽²⁾.

2. Preliminaries

2.1. Basic definitions

We use standard graph-theoretic terminology following e.g. [Die05]. A *plane graph* is a subset of \mathbb{R}^2 that is homeomorphic to a graph when the latter is viewed as an 1-complex. In other words, a plane graph is a planar graph endowed with a fixed embedding in \mathbb{R}^2 . A *face* of a plane graph G is a component of $\mathbb{R}^2 \setminus G$. A *facial triangle* of G is a cycle consisting of three edges that bounds a face. A *planar triangulation* is a connected plane graph every edge of which is contained in two facial triangles. For example, the 1-skeleton of every triangulation of \mathbb{R}^2 in the topological sense is a planar triangulation.

The *edge space* $\mathcal{E}(G)$ of a graph $G = (V, E)$ is the vector space \mathbb{F}_2^E over the 2-element field \mathbb{F}_2 , where vector addition amounts to symmetric difference. The *cycle space* $\mathcal{C}(G)$ is the subspace of $\mathcal{E}(G)$ generated by the edge-sets of cycles.

The *graph distance* $d(x, y)$ between two vertices $x, y \in V$ is the minimum number of edges in an x - y path in G .

⁽¹⁾For Cayley graphs this was also proved in [Gen08].

⁽²⁾Is there a simple proof that a group of linear growth is quasi isometric to \mathbb{Z} ?

2.2. Manning's theorem

A *quasi-isometry* between graphs $G = (V, E)$ and $H = (V', E')$ is a map $f : V \rightarrow V'$ such that the following hold for fixed constants $M \geq 1, A \geq 0$:

- (i) $M^{-1}d(x, y) - A \leq d(f(x), f(y)) \leq Md(x, y) + A$ for every $x, y \in V$, and
- (ii) for every $z \in V'$ there is $x \in V$ such that $d(z, f(x)) \leq A$.

Here $d(\cdot, \cdot)$ stands for the graph distance in the corresponding graph G or H . We say that G and H are *quasi-isometric*, if such a map f exists. Quasi-isometries between arbitrary metric spaces are defined analogously.

We now recall Manning's [Man05] characterization of the graphs allowing a quasi-isometry to a tree, which graphs we will call *quasi-trees*:

DEFINITION 2.1. — *A graph $G = (V, E)$ has the δ -Bottleneck Property (BP), if for all $x, y \in V$ there is a "midpoint" $m = m(x, y)$ with $d(x, m) = d(y, m) = d(x, y)/2$, such that any path from x to y meets the ball $B_m(\delta)$. (We allow m to lie in the middle of an edge.)*

THEOREM 2.2 ([Man05]). — *A graph G is quasi-isometric to a tree if and only if it satisfies the δ -(BP) for some $\delta > 0$.*

This theorem generalises to arbitrary *length spaces*, which will allow us to apply it to manifolds. A metric space (X, d) is a *length space*, if for every $x, y \in X$ and $\epsilon > 0$ there is an x - y arc in X of length less than $d(x, y) + \epsilon$. The definition of the δ -(BP) for a length space is similar, we just let m be an approximate midpoint.

THEOREM 2.3. — *A length space (X, d) is quasi-isometric to a tree if and only if it satisfies the δ -(BP) for some $\delta > 0$.*

3. Results for Triangulations and Graphs

For a graph G and a subgraph $H \subseteq G$, we define the boundary $\partial H := \{v \in V(H) \mid \text{there is } vw \in E(G) \text{ with } w \notin V(H)\}$. The following lemma will be used in the proof of Theorem 1.1.

LEMMA 3.1. — *Let G be a planar triangulation, and let H be a finite connected subgraph of G . Suppose two vertices $x, y \in V(H)$ are connected by a path P in $(G \setminus H) \cup \{x, y\}$. Then x, y are connected by a path in ∂H .*

Proof. — Since H is connected, there is an x - y path Q in H . Then $C := P \cup Q$ is a cycle (Figure 3.1). Pick one of the two sides A of C —i.e. one of the two components into which C separates the plane by the Jordan curve theorem— and let \mathcal{T} denote the set of all facial triangles of G lying in $A \cup C$ and having all their three vertices in H . Let K be the element of $\mathcal{C}(G)$ defined by the sum $E(C) + \bigoplus_{T \in \mathcal{T}} E(T)$, where $E(T)$ denotes the edge-set of T . (In the example of Figure 3.1, K happens to be a cycle $P \cup P'$, but in general K will be more complicated if H has "holes".) Notice that $E(P) \subset K$. Since every element of the cycle space of a finite graph can be written as a disjoint union of edge-sets of cycles [Die05, Proposition 1.9.2.], and no

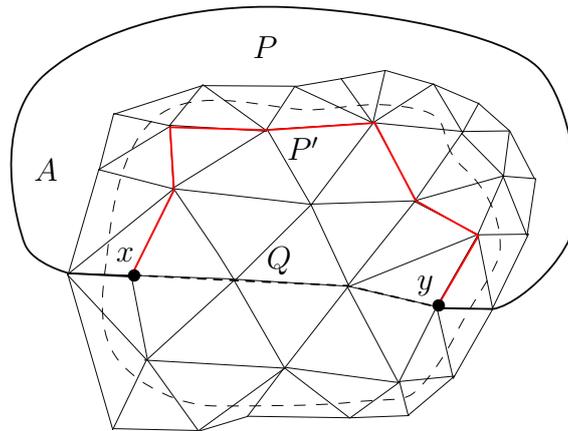


Figure 3.1. The situation in the proof of Lemma 3.1. The subgraph H is enclosed by the dashed curve, and \mathcal{T} comprises those triangles enclosed between the bold dashed path Q and the path P' depicted in red, if colour is shown.

internal vertex of P is incident with an element of \mathcal{T} , there is a cycle C' containing P such that $E(C') \subset K$. We claim that $P' := C' \setminus P \subseteq \partial H$. Since P' is an x - y path by definition, this claim proves our statement.

To establish the claim, we will show the stronger $V(K) \setminus P \subseteq \partial H$, where $V(K)$ denotes the set of vertices incident with an edge of K . To see this, let $e = uv$ be an edge in K , and let T_1, T_2 be the two facial triangles containing e . Notice that $u, v \in V(H)$ by the definitions, so it remains to check that at least one other vertex of $T_1 \cup T_2$ is not in H .

There are two cases. If $e \in E(C)$, then it cannot be that both T_1, T_2 are in \mathcal{T} , and therefore none of them is in \mathcal{T} since $e \in K$. In this case the vertex w of $T_1 \cup T_2$ inside A is not in H since uvw is not in \mathcal{T} . The other case is where $e \notin E(C)$, and therefore e lies in A . Then exactly one of T_1, T_2 must be in \mathcal{T} for e to be in K , which means that the other T_i contains a vertex not in H as desired. \square

We now prove our first main theorem, by repeatedly applying the above Lemma 3.1 to the “bottlenecks” of Definition 2.1:

Proof of Theorem 1.1. — Suppose G is not quasi-isometric to a tree. Given $r \in \mathbb{N}$, Manning’s Theorem 2.2 says that G does not have the r -(BP). Thus we can find two vertices $p, q \in V(G)$ and a p - q geodesic Γ , such that letting m be the midpoint of Γ , the ball $B_m(r)$ does not separate p from q . In particular, p, q lie outside $B_m(r)$. Let P be a p - q path outside $B_m(r)$.

For every $i \in [1, r]$, we claim that there is a path P_i in $\partial B_m(i)$ joining the two points p_i, q_i of $\Gamma \cap \partial B_m(i)$. Indeed, $(P \cup \Gamma) \setminus B_m(i-1)$ contains a p_i - q_i path in $(G \setminus B_m(i)) \cup \{p_i, q_i\}$, and so our claim follows from Lemma 3.1, applied with $H = B_m(i)$.

Notice that the $P_i, i \in [1, r]$ are pairwise disjoint, they are contained in $B_m(r)$, and $|P_i| \geq d(p_i, q_i) = 2i$. Thus $|B_m(r)| \geq \sum_{i \in [1, r]} 2i > r^2$. \square

Theorem 1.3 can be proved along the same lines, except that instead of Lemma 3.1 we use the following observation of Timar [Tim07]. We say that G is k -SC (k -simply-connected), if $\mathcal{C}(G)$ is generated by a set of cycles each of length at most k .

LEMMA 3.2 ([Tim07, Theorem 5.1.]). — *Let G be a k -SC graph, and let H be a finite connected subgraph of G . Suppose two vertices $x, y \in V(H)$ are connected by a path P in $(G \setminus H) \cup \{x, y\}$. Then x, y are connected by a path each vertex of which is at distance at most $k/2$ from ∂H .*

(Timar’s formulation is different but equivalent: it says that if Π is a minimal cut separating two vertices u, v of G , and Π_1, Π_2 is a proper bipartition of Π , then there are vertices $x_i \in \Pi_i$ with $d(x_1, x_2) \leq k/2$. To deduce Lemma 3.2 from this formulation, let u be any vertex of H , let v be any vertex of P , and let Π be the set of edges of G between H and the component of $G \setminus H$ meeting P . The above can be rephrased as saying that any two edges in Π can be connected by a sequence of paths, each starting and ending in Π , and each of length at most $k/2$.)

Proof of Theorem 1.3. — Given $r \in \mathbb{N}$, define p, q, m as in the proof of Theorem 1.1. For every $i \in [1, r]$, we can apply Lemma 3.2 with $H = B_m(i)$ to obtain a path P_i at distance at most $k/2$ from $\partial B_m(i)$ joining the two points p_i, q_i of $P \cap \partial B_m(i)$. Let $P'_j = P_{(k+1)j}, j \in [1, r/(k+2)]$, and notice that the P'_j are pairwise disjoint, they are contained in $B_m(r)$, and $|P'_j| \geq d(p_{(k+1)j}, q_{(k+1)j}) = 2(k+1)j$. Thus

$$|B_m(r)| \geq \sum_{j \in [1, r/(k+2)]} 2(k+1)j > (k+1) \frac{r^2}{(k+2)^2}.$$

□

We finish this section with an example showing that the condition of large-scale-simple-connectedness cannot be relaxed in Theorem 1.3. Let T be a tree of uniform volume growth $\Theta(r^{1.5})$, as provided in [BS01]. Let T' be a copy of T . Choose an infinite sequence $\{v_i\}_{i \in \mathbb{N}}$ of leaves of T , and identify each v_i with its copy v'_i in T' to obtain a planar graph G . Notice that the distance between v_i and v_j is the same in each of the three graphs T, T' and G . It follows that for every $r \in \mathbb{N}$ and $v \in V(T)$, we have $B_v^G(r) \subseteq B_v^T(r) \cup B_{v'}^{T'}(r)$, i.e. a ball of a given radius in G is at most twice as large as a ball of the same radius in T . By choosing the v_i appropriately, e.g. so that $d(v_i, v_j) \geq 2^i$ for every $j < i$, we can ensure that G is not quasi-isometric to a tree.

3.1. The asymptotic dimension of planar triangulations

We now prove Theorem 1.4 using some of the ideas of the proof of Theorem 1.1.

We recall one of the standard definitions of asymptotic dimension $\text{asdim}(X)$ of a metric space (X, d) from e.g. [JL22]. The reader can think of X as being a graph endowed with its graph distance d . We include this definition for the purpose of fixing notation, the reader can consult other sources to gain more intuition.

Let \mathcal{U} be a collection of subsets of (X, d) —usually a cover. For $s > 0$, we say that \mathcal{U} is s -disjoint, if $d(U, U') := \inf \{d(x, x') \mid x \in U, x' \in U'\} \geq s$ whenever $U, U' \in \mathcal{U}$

are distinct. More generally, we say that \mathcal{U} is $(n + 1, s)$ -disjoint, if $\mathcal{U} = \bigcup_{i=1}^{n+1} \mathcal{U}_i$ for subcollections \mathcal{U}_i each of which is s -disjoint. The indices $1, \dots, n + 1$ are the *colours* of \mathcal{U} . We define the *asymptotic dimension* $\text{asdim}(X)$ as the smallest n such that for every $s \in \mathbb{R}_+$ there is an $(n + 1, s)$ -disjoint cover \mathcal{U} of X with $\sup_{U \in \mathcal{U}} \text{diam}(U) < \infty$. Here, the diameter $\text{diam}(U)$ is measured with respect to the metric d of X . We say that \mathcal{U} is D -bounded for any $D > \sup_{U \in \mathcal{U}} \text{diam}(U)$.

The *Assouad-Nagata dimension* $\text{ANdim}(X)$ is defined by putting a stronger restriction on the diameters: we define $\text{ANdim}(X)$ to be the smallest n for which there is $c \in \mathbb{R}$ such that for every $s \in \mathbb{R}_+$ there is an $(n + 1, s)$ -disjoint and cs -bounded cover \mathcal{U} of X .

Proof of Theorem 1.4. — It has been proved that every planar graph has asymptotic dimension at most 2 [BBE⁺20, JL22]. Thus it only remains to show that $\text{asdim}(G) = 1$ implies that G is quasi-isometric to a tree when G is a planar triangulation. We will prove the following stronger statement, where d denotes the graph distance of G :

If G admits a 2-colouring $c : V(G) \rightarrow \{0, 1\}$ such that for some $r \in \mathbb{N}$,
 (3.1) every monochromatic connected subgraph of G has diameter less than r with respect to d , then G is a quasi-tree.

To see that (3.1) is satisfied when $\text{asdim}(G) = 1$, let \mathcal{U} be a $(2, 2)$ -disjoint and r -bounded cover of G , consisting of subcollections $\mathcal{U}_1, \mathcal{U}_2$, and set $c(v) = i$ if $v \in \bigcup \mathcal{U}_i$. As an exercise, the reader could try to check that the triangular lattice T (i.e. the planar triangulation with all vertices having degree 6) does not admit a 2-colouring as in (3.1), and deduce that $\text{asdim}(T) > 1$.

To prove (3.1), we assume such c exists, and we claim that G has the R -BP for $R := 10r$ —we are being generous— from which the result follows via Manning’s Theorem 2.2.

For if not, then as before we can find two vertices $p, q \in V(G)$ and a p - q geodesic Γ , such that letting m be the midpoint of Γ , the ball $B_m(R)$ does not separate p from q . Let P be a p - q path outside $B_m(R)$.

A *monochromatic component* is a maximal connected subgraph of G on which c is constant. Let C_0 be the monochromatic component of m in c . By our assumption, C_0 has diameter less than r with respect to d , and in particular it is contained in $B_m(r)$. The idea is to recursively apply Lemma 3.1 starting from C_0 , to obtain longer and longer monochromatic components (in alternating colours) surrounding it, contradicting our assumption that monochromatic connected subgraphs of G have bounded diameters.

To make this precise, let C_0, C_1, \dots, C_k be a longest sequence of monochromatic components (in alternating colours) with all the following properties. The two *parts* of Γ are the two components into which m separates Γ .

- (i) C_i contains a path joining the two parts of Γ for all $i > 0$;
- (ii) C_i separates C_{i-1} from P in the subgraph $\Gamma \cup P$ for all $i > 0$;
- (iii) $C_i \subseteq B_m(9r)$, and
- (iv) $C_i \cap \Gamma \subseteq B_m(r)$.

Since C_0 satisfies these conditions by definition, such a sequence always exists, although it may comprise a single member $C_0 = C_k$. Properties (i) and (ii) together with the finiteness of Γ guarantee that the sequence terminates.

An example figure can be obtained as follows. Draw a sequence of simple closed curves S_0, S_1, \dots, S_k in the plane, nested inside each other, and all surrounding a point m . Triangulate the interior of S_0 , which we think of as C_0 . Also triangulate each of the annuli bounded by S_i and S_{i-1} , which we think of as C_i . Finally, draw a path Γ through m , with both endpoints p, q outside S_k .

Back to our proof, let C'_k be the set of neighbours of C_k . Notice that C'_k is monochromatic, as all its vertices must have the opposite colour of C_k since the latter was a monochromatic component. Let $H := C_k \cup C'_k$. Applying Lemma 3.1, which we can because C_k satisfies (iii), and so H avoids P , we obtain a path $M \subset \partial H \subseteq C'_k$ joining the two parts of Γ . Let C_{k+1} be the monochromatic component containing M , which exists since $M \subseteq C'_k$ is monochromatic. Then C_{k+1} satisfies (i) because it contains M , and it satisfies (ii) because C'_k separates C_k from P in $\Gamma \cup P$. Notice that $d(m, M) \leq r + 1$ because C_k satisfies (iv) and $M \subseteq C'_k$. Therefore, if $C_{k+1} \supseteq M$ violates (iii), then its diameter is at least $8r$ by the triangle inequality, contradicting our assumption. If C_{k+1} violates (iv), then again this contradicts our assumption that $\text{diam}(C_{k+1}) < r$, because C_{k+1} meets both parts of Γ and Γ is a geodesic through m . \square

Remark 3.3. — Our proof is similar to that of Fujiwara & Whyte, who proved that large-scale-simply-connected graphs of asymptotic dimension 1 are quasi-trees [FW07]. In fact our proof can be adapted to yield their result: If G is k -SC, then its k^{th} power G^k is 3-SC. We can apply the above proof to G^k , replacing Lemma 3.1 by Lemma 3.2.

It is a consequence of Theorem 1.4 that when G is a planar triangulation, $\text{asdim}(G)$ equals the Assouad-Nagata dimension $\text{ANdim}(G)$. Indeed, it is well-known that $\text{ANdim}(T) = 1$ for every tree T , and ANdim is invariant under quasi-isometry by definition. We now present an example showing that this property of planar triangulations does not extend to all planar graphs: we construct a planar graph G with $\text{asdim}(G) = 1$ and $\text{ANdim}(G) = 2$.

This graph is constructed as follows. For every $n \in \mathbb{N}$, let H_n be the $n \times n$ square grid, i.e. the Cartesian product of the path of length n with itself. Obtain G_n from H_n by subdividing each edge n times. Let G be the graph obtained from the union of all the G_n after joining one vertex of G_n with one vertex of G_{n+1} with a path of length n for every $n \in \mathbb{N}$; it will not matter which vertex.

We claim that $\text{ANdim}(G) > 1$. For if $\text{ANdim}(G) = 1$, then there is $c \in \mathbb{R}$ such that for every $s \in \mathbb{N}$, there is an $(2, s)$ -disjoint and cs -bounded cover of G as defined in Section 3.1. For every n , we then have a $(2, ns)$ -disjoint and cns -bounded cover \mathcal{U}_n of G . The restriction of \mathcal{U}_n to G_n yields a $(2, s)$ -disjoint and cs -bounded cover \mathcal{U}'_n of H_n , because all distances in G_n are n times larger than corresponding distances in H_n . Using a standard compactness argument we can find a subsequence of the \mathcal{U}'_n converging to a $(2, s)$ -disjoint and cs -bounded cover of the infinite square lattice \mathbb{Z}^2 , contradicting the fact that $\text{ANdim}(\mathbb{Z}^2) = 2$. (A similar idea appears in [BBE⁺20].)

It remains to check that $\text{asdim}(G) = 1$. Given $s \in \mathbb{R}$, we can construct a $(2, s)$ -disjoint cover $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$ of G as follows. Put all G_n 's with $n \leq 4s$ in one element U of \mathcal{U}_1 . For the larger G_n 's, start with a proper 2-colouring of H_n with colours $\{1, 2\}$, and extend it to G_n as follows. Colour each vertex v of degree 2 with the colour of the nearest vertex v' of degree 4 (inherited from the colouring of H_n) whenever $d(v, v') \leq s$. The yet uncoloured vertices form a family of paths of length at least $2s$ each. Easily, each such path P can be decomposed into an even number of subpaths each of length between s and $2s$. We colour these subpaths in alternating colours, so that each endvertex of P receives the opposite colour of its neighbour outside P . We complete the definition of \mathcal{U} by putting each monochromatic component of colour i in \mathcal{U}_i . It is straightforward to check that \mathcal{U} is a $(2, s)$ -disjoint cover of G , and that $\sup_{U \in \mathcal{U}} \text{diam}(U) < \infty$. Thus $\text{asdim}(G) = 1$ as claimed.

4. Manifolds

In this section we prove continuous analogues of the above results on planar triangulations. We will state our results for Riemannian 2-manifolds, although our proofs apply more generally to any topological 2-manifold endowed with a metric that turns it into a length space, in particular to any Finsler 2-manifold. A 2-manifold is *planar*, if it is homeomorphic to a subspace of the 2-sphere \mathbb{S}^2 . For a subset X of a manifold we define $\text{Area}(X)$ to be its 2-dimensional Hausdorff measure $\mathcal{H}^2(X)$. (This definition generalises the Riemannian area.) As usual we denote by $B_p(r)$ the ball of radius r centered at a point p . The main result of this section is

THEOREM 4.1. — *Let M be a connected, complete, planar, Riemannian 2-manifold. Then either M is quasi-isometric to a tree, or for every $r \in \mathbb{R}_+$ there is $p \in M$ such that $\text{Area}(B_p(r)) > r^2/8$.*

The proof of this follows the lines of the proof of Theorem 1.1. We will need the following lemma which is the analogue of Lemma 3.1, and will play a similar role in our proof.

LEMMA 4.2. — *Let S be a planar 2-manifold, and let $K \subseteq S$ be a connected and compact subspace. Suppose $x, y \in \partial K$ are connected by an arc P in $(S \setminus K) \cup \{x, y\}$. Then for every $\epsilon > 0$, x, y are connected by an arc contained in the ϵ -neighbourhood of ∂K .*

Remark 4.3. — We have to consider a neighbourhood of ∂K here, as we cannot find an x - y arc in ∂K if K is too fractal. For example, K could be a pseudo-arc in $S = \mathbb{R}^2$, in which case $\partial K = K$ contains no non-trivial arc.

Proof of Lemma 4.2. — It is well-known that S admits a triangulation T [AS60, I. 46]. Given $\epsilon > 0$, we can subdivide the triangles of T if necessary to obtain a triangulation T_ϵ of S in which every triangle (i.e. 2-cell) has diameter at most ϵ with respect to the metric of S . Let $G = G(\epsilon)$ denote the 1-skeleton of T_ϵ , and note that G is a planar triangulation by definition.

Let $H \subset G$ denote the subgraph of G consisting of the boundaries of the triangles of T_ϵ intersecting K . Since K is connected, so is H , and since K is compact, H is

finite. Let x', y' be vertices of ∂H in the boundary of triangles Δ_x, Δ_y containing x, y respectively, and let P' be an $x'-y'$ path in G at distance at most ϵ from P , which can be found inside the union of the triangles of T_ϵ intersecting P . Applying Lemma 3.1 we obtain an $x'-y'$ path Q_ϵ in ∂H . Notice that $\partial H \cap K = \emptyset$, and ∂H is contained in the ϵ -neighbourhood of K . Thus $Q_\epsilon \subseteq \partial H$ is contained in the ϵ -neighbourhood of ∂K . We extend Q_ϵ by an $x-x'$ arc in Δ_x and a $y-y'$ arc in Δ_y to obtain the desired $x-y$ arc. \square

In the proof of Theorem 1.1 we used Lemma 3.1 to find a sequence of paths Q_i joining points at distance $2i$, and each contained in the boundary of a ball with a fixed center. We will argue analogously in the proof of Theorem 4.1, and the following lemma will be used to show that the union of the paths that Lemma 4.2 provides has large total area.

LEMMA 4.4. — *Let $\{P_t^\epsilon, t \in (0, T), \epsilon \in (0, 1)\}$ be a family of arcs in a metric space (X, d) , and let $p_t^\epsilon, q_t^\epsilon$ denote the two endpoints of P_t^ϵ . Suppose that for some $L > 0$, and every $\epsilon \in (0, 1)$, we have*

- (i) $d(p_t^\epsilon, q_t^\epsilon) \geq L$ for every $t \in (0, T)$, and
- (ii) $d(P_s^\epsilon, P_t^\epsilon) \geq |s - t| - \epsilon$ for every $s, t \in (0, T)$.

Then $\mathcal{H}^2(X) \geq LT/4$.

For example, X could be a rectangle with side lengths L, T , and P_t^ϵ could be the horizontal arc at height t perturbed within distance ϵ .

Proof of Lemma 4.4. — This is a rather straightforward application of the definition of Hausdorff measure. Recall that $\mathcal{H}^2(X)$ is defined as $\sup_{\delta \rightarrow 0} \mathcal{H}_\delta^2(X)$, where $\mathcal{H}_\delta^2(X) := \inf\{\sum \text{diam}(U_i)^2\}$, the infimum ranging over all countable covers $\{U_i\}$ of X satisfying $\text{diam}(U_i) < \delta$ for every i .

Given $\delta > 0$, the idea is to choose a finite subfamily of our arcs $\{P_t^\epsilon\}$ whose pairwise distance is at least some increment c , and within each arc to choose a sequence of points obeying the same increment c , so that any set U_i used to cover X can only contain $O(\text{diam}(U_i)^2/c^2)$ many of these points as $c \rightarrow 0$. Hereby we choose ϵ much smaller than c , e.g. $\epsilon = c^2$. As we have $\Omega(LT/c^2)$ points to cover in total, this provides the desired lower bound on $\mathcal{H}_\delta^2(X)$.

To make this more precise, given $\delta < 1/2$, choose $N \gg 1/\delta$, let

$$t_i := iT\delta/N, i \in 1, 2, \dots, \left\lfloor \frac{N}{\delta} \right\rfloor,$$

and let $P_i := P_{t_i}^{(\delta/N)^2}$. Notice that $d(P_i, P_j) \geq T(\delta/N|i - j| - (\delta/N)^2)$ by (ii), and therefore any set U with $\text{diam}(U) < \delta$ meets at most $2\text{diam}(U)N/\delta$ of the P_i 's. For each i , let $p_i^j, j \in 1, 2, \dots, \left\lfloor \frac{N}{\delta} \right\rfloor$ be a point on P_i at distance $jL\delta/N$ from a fixed endpoint of P_i , which exists by (i). The triangle inequality implies that if $\text{diam}(U) < \delta$ then U meets at most $2\text{diam}(U)N/\delta$ of the p_i^j 's for any fixed i . Combined with the above remark, we deduce that U meets at most $4(\text{diam}(U)N/\delta)^2$ points p_i^j for any i and j . Since we have at least $\frac{N^2}{\delta^2}$ points to cover in total, we deduce

$$\sum_{U \in \mathcal{U}} 4(\text{diam}(U)N/\delta)^2 \geq \frac{N^2}{\delta^2}$$

for every cover \mathcal{U} of X with diameters bounded by δ , and so $\sum_{U \in \mathcal{U}} (\text{diam}(U))^2 \geq 1/4$. Thus $\mathcal{H}_\delta^2(X) \geq 1/4$. \square

We are now ready to prove the main result of this section.

Proof of Theorem 4.1. — We follow the lines of the proof of Theorem 1.1. If M is not quasi-isometric to a tree, then for every $r \in \mathbb{R}$, Theorem 2.3 provides two points $p, q \in M$ and a p - q arc Γ of length arbitrarily close to $d(p, q)$, such that letting m be the midpoint of Γ , the ball $B_m(r)$ does not separate p from q . Let P be a p - q path outside $B_m(r)$. Applying Lemma 4.2 with $K = K(s) := B_m(s)$ for $s \in (r/2, r)$ we obtain a family $\{P_t^\epsilon, t \in (0, r/2), \epsilon \in (0, 1)\}$ of arcs each joining two points of Γ outside $B_m(r/2)$, which are thus at distance at least r from each other. Here we used the standard fact that every closed bounded subspace of a complete, locally compact, length space is compact. Applying Lemma 4.4 to this family we deduce $\mathcal{H}^2(B_m(r)) \geq r/8$. \square

We can now deduce Corollary 1.2 from Theorem 4.1 as follows. We first perform a finite number of surgery operations along non-contractible, rectifiable, loops of M to produce a 2-manifold M' of genus 0 no ball of which has larger area than a ball of the same radius in M . To maintain completeness, we glue a disc of finite area along each boundary component we created, to obtain a complete planar surface M'' . This has a significant influence on the volume growth for small radii only. Theorem 4.1 thus yields that M'' is a quasi-tree, hence so is M since it is quasi-isometric to M'' . (We do not need to worry about losing smoothness when glueing in the aforementioned discs, because as mentioned at the beginning of this section, our proof of Theorem 4.1 only requires M'' to be a length space that is a topological 2-manifold, not necessarily a Riemannian one.)

We remark that the completeness assumption cannot be dropped in Theorem 4.1: one can construct a planar manifold quasi-isometric to the graph obtained from an 1-way infinite path by replacing its i th edge with a cycle of length i . Such a planar graph is not a quasi-tree, and it has linear growth.

5. Growth of groups

As a corollary of Theorem 1.3, we obtain an alternative proof that there is no finitely presented group of superlinear but subquadratic growth without using Gromov’s theorem. For this we do not need to use Manning’s theorem:

COROLLARY 5.1. — *There is no k -SC vertex-transitive graph G of superlinear but subquadratic growth for any $k \in \mathbb{N}$.*

(Sketch). — If G has the δ -(BP) for some δ , we can use a standard compactness argument to obtain a double-ray R and a ball S of radius δ that separates the two ends of R . It follows that G has at least 2 ends, and so G has either linear (if 2-ended [IS88, Theorem 2.8 & Proposition 3.1]) or exponential growth (if infinitely-ended [Hal64]).

If G does not have the δ -(BP) then we apply Theorem 1.3 (omitting the first two sentences of the proof of Theorem 1.1 that invoke Theorem 2.2). \square

6. Questions

As mentioned in the introduction, for every $\alpha > 1$ there is an (1-ended) triangulation T_α of the plane \mathbb{R}^2 with uniform volume growth of order $\Theta(r^\alpha)$ [BS01]. The cartesian product $T_\alpha \times \mathbb{Z}$ can be thought of as the 1-skeleton of a 3-complex, which we can triangulate to obtain a 3-dimensional simplicial complex C_α , in which each copy of a triangle of T_α appears in the boundary of two tetrahedra. Note that C_α has uniform volume growth of order $\Theta(r^{\alpha+1})$, and it is homeomorphic to \mathbb{R}^3 . We are interested in the structure of 3-dimensional simplicial complexes homeomorphic to \mathbb{R}^3 (or other 3-manifolds) that have uniform volume growth of order $\Theta(r^\beta)$ for $\beta < 3$. The following question is an attempt to extend Theorem 1.1 to three dimensions.

PROBLEM 6.1. — *Let X be a simplicial complex homeomorphic to \mathbb{R}^3 , with uniform volume growth r^α for some $\alpha < 3$. Must X be quasi-isometric with a contractible simplicial 2-complex?*

The above construction C_α with $\alpha \in (1, 2)$ provides some examples. Since T_α is quasi-isometric to a tree, C_α is quasi-isometric to the cartesian product of a tree with the 2-way infinite path, which is contractible (and even collapsible, a stronger condition we could ask for in Problem 6.1; see e.g. [AF15] for the definition).

We remark that in Theorems 1.1 and 1.3 the vertex v has to depend on r , in other words, we cannot drop the uniformity of the subquadratic growth if we want to obtain a quasi-tree. For example, let T denote the triangular lattice in \mathbb{R}^2 , and let $X = x_0x_1\dots$ and $Y = y_0y_1\dots$ be two geodesics emanating from a common vertex $x_0 = y_0$, such that $d(x_i, y_i) = \Theta(\sqrt{i})$. Cut T along $X \cup Y$, throw away the larger piece, and identify x_i with y_i in the smaller piece. The resulting planar triangulation can be visualised as a parabolic cone. It has subquadratic volume growth, but not uniformly so, and it is not quasi-isometric with a tree.

Still, for a unimodular random graph we can ask if a statement similar to Theorems 1.1 and 1.3 holds under a subquadratic expected volume growth condition:

PROBLEM 6.2. — *Let (G, o) be a unimodular random graph⁽³⁾ which is k -SC for some k , such that $c_1r < \mathbb{E}(|B_o(r)|) < c_2r^2$ for some constants $c_1, c_2 > 0$. Must (G, o) be almost surely 1-ended? Must it have an infinite sequence of nested, bounded, cut sets separating o from infinity? Must every scaling limit of such a graph be a tree?*

Let G be a planar triangulation of uniform subquadratic volume growth. By Theorem 1.1 G is quasi-isometric with a tree T , and since G must have bounded degrees, it is easy to see that we can choose T to be of bounded degree too. It is a well-known theorem of Lyons [Lyo90] that trees of polynomial growth have no percolation phase transition, i.e. their p_c equals 1. It is also well known that the property $p_c = 1$ is invariant under quasi-isometries between bounded degree graphs [LP16, Theorem 7.15], and so is the exponent of the volume growth. Thus we deduce $p_c(G) = 1$. This leads to the following question:

PROBLEM 6.3. — *Is it true that every (G, o) as in Problem 6.2 satisfies $p_c = 1$ almost surely?*

⁽³⁾See [AL07] for definitions.

Consider a unimodular random graph (G, o) of super-linear but sub-quadratic expected volume growth. We do not impose the k -SC condition, so apart from the aforementioned quasi-trees, examples can be obtained from the sequence of graphs defining the Sierpinski gasket by sampling uniform random roots. In all the examples we know of, simple random walk is subdiffusive, see e.g. [Bar98, Proposition 8.11]. Is this the case for every such (G, o) ?

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