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LOCAL-GLOBAL PRINCIPLES FOR HOMOGENEOUS SPACES OF REDUCTIVE GROUPS OVER GLOBAL FUNCTION FIELDS

PRINCIPE LOCAL-GLOBAL POUR LES ESPACES HOMOGÈNES DE GROUPES RÉDUCTIFS SUR LES CORPS GLOBAUX DE FONCTIONS

Abstract. — Let X be a homogeneous space of a reductive group with reductive stabilizers, defined over a global field of positive characteristic. Using duality theorems for complexes of tori, we study cohomological obstructions to various arithmetic properties.

Résumé. — Soit X un espace homogène d'un groupe réductif à stabilisateurs réductifs, défini sur un corps global de caractéristique positive. À l'aide de théorèmes de dualité pour les complexes de tores, on étudie les obstructions cohomologiques naturelles à différentes propriétés arithmétiques.

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1. Introduction

Let K be a global field of characteristic $p \geqslant 0$ and let $\mathbf{A}_K$ denote the ring of adèles of K. Let G be a reductive group over K, and X be a homogeneous space of G. We are interested in rational points on X, and more precisely, on various local-global principles associated to X: does X satisfy the Hasse principle, i.e. does $X(\mathbf{A}_K) \neq \emptyset$ imply $X(K) \neq \emptyset$? If not, can we explain the failure using the so-called Brauer–Manin obstruction to the Hasse principle? Assuming that $X(K) \neq \emptyset$, can we estimate the size of $X(K)$ by studying the so-called weak and strong approximation on X (with a Brauer–Manin obstruction if necessary), i.e. the closure of the set $X(K)$ in the topological space $X(\mathbf{A}_K^S)$, where S is a (not necessarily finite) set of places of K and $\mathbf{A}_K^S$ is the ring of S-adèles (with no components in S)?

The answer to those questions is known in the case where K is a number field, provided that the geometric stabilizers of points in X are connected (see [Bor96] and [BD13]).

In the case of a global field of positive characteristic, the answer is known for semisimple simply connected groups (thanks to works by Harder, Kneser, Chernousov, Platonov, Prasad), but the general case is essentially open (see [Ros21a, Theorem 1.9] for some related results). In this paper, we deal with these questions when both G and the geometric stabilizers are smooth, connected and *reductive*.

Several new ingredients are needed to obtain our results:

- To show that the Brauer–Manin obstruction to the Hasse principle is the only one (see Theorem 2.5 for the precise result), one has to use Poitou–Tate duality for complexes of tori in positive characteristic (proven in [DH20]) and a (non-straightforward) compatibility result between Brauer–Manin and Poitou–Tate pairings.
- The statement on weak approximation (Theorem 4.2) relies on some part of Poitou–Tate exact sequence (which is established in [DH20]) for a certain complex of tori, and on abelianization techniques (namely Lemma 4.3). Beforehand we define in section 3 a new abelianization map associated to a homogeneous space $X = G/H$ as above, and prove a rather intricate compatibility formula (Theorem 3.7).
- Theorem 5.8 presents the obstruction to strong approximation. As in the number field case (settled in [BD13]), it is related to the Brauer–Manin pairing, but there are two important differences. The first one is that there is an additional term in the exact sequence describing the obstruction, which reflects the fact that the global reciprocity map of class field theory is not surjective in positive characteristic. The second difference is that a fibration method like in loc. cit. would probably not work here (see for instance Section 5 in loc. cit.). Therefore, one should again rely on abelianization techniques (in particular the compatibility formula of Theorem 3.7 in its full generality, and not only for elements of the algebraic Brauer group $\mathrm{Br}_1 X$). An important role is also played by duality theorems for complex of tori, some of them extending results of [DH20].

Notation and conventions

In the whole article (except in Section 3, where k is an arbitrary field), we consider a finite field k and a projective, smooth and irreducible k-curve E. We set $K = k(E)$, which is a global function field of characteristic p, and fix a separable closure K^s of K. The absolute Galois group $\mathrm{Gal}(K^s/K)$ of K is denoted by Γ_K. Denote by Ω_K the set of all places of K; for every $v \in \Omega_K$, we will identify the Brauer group $\mathrm{Br}\, K_v$ of the completion K_v to $\mathbf{Q}/\mathbf{Z}$ thanks to local class field theory. For every K-variety X, we set $\bar{X} = X \times_K K^s$. The (cohomological) Brauer group of X is denoted $\mathrm{Br}\, X$, and we set

$$\mathrm{Br}_1 X := \ker\left[\mathrm{Br}\, X \to \mathrm{Br}\, \bar{X}\right].$$

We still denote by $\mathrm{Br}\, K$ the image of $\mathrm{Br}\, K$ in $\mathrm{Br}\, X$, even though the map $\mathrm{Br}\, K \to \mathrm{Br}\, X$ is not necessarily injective if X has no rational point. Notation like $H^i(K,C)$ for a commutative K-group scheme C (resp. a bounded complex of commutative K-group schemes) always denotes fppf cohomology (resp. fppf hypercohomology) of C. It coincides with étale (=Galois) cohomology when C (resp. every group scheme occurring in C) is smooth. For every finite set of places S of K, we set

$$Ш^i_S(K,C) := \ker\left[H^i(K,C) \to \prod_{v\notin S} H^i\left(K_v,C\right)\right].$$

$$Ш^i(K,C) = Ш^i_\emptyset(K,C); \quad Ш^i_\omega(K,C) = \varinjlim_{S} Ш^i_S(K,C),$$

where the direct limit runs over all finite subsets S of Ω_K. The Pontryagin dual A^D of a topological group A is the group of continuous homomorphism from A to $\mathbf{Q}/\mathbf{Z}$ (if topology is not specified, we assume that A is discrete).

Let G be a reductive group (always meaning: smooth connected reductive) over K. Let G^{ss} denote the derived subgroup of G and G^{sc} the simply connected cover of G^{ss}, together with the obvious morphism $\rho : G^{\mathrm{sc}} \to G$. Set $G^{\mathrm{tor}} = G/G^{\mathrm{ss}}$ (it is the maximal toric quotient of G). Let $T^{\mathrm{sc}} \subset G^{\mathrm{sc}}$ and $T_G \subset G$ be maximal tori such that $\rho(T^{\mathrm{sc}}) \subset T_G$. Let C_G be the complex $C_G := [T^{\mathrm{sc}} \xrightarrow{\rho} T_G]$, with T_G in degree 0. Following Borovoi (cf. [Bor98] in characteristic zero), we have a natural map of Galois (hyper)cohomology sets:

$$\mathrm{ab}^1_G : H^1(K,G) \to H^1_{\mathrm{ab}}(K,G) := H^1\left(K, C_G\right),$$

which is functorial in K. There is an exact sequence of K-group schemes

$$1 \to \mu_G \to G^{\mathrm{sc}} \to G^{\mathrm{ss}} \to 1,$$

where μ_G is a finite K-group scheme of multiplicative type. It induces an exact triangle

$$\mu_G[1] \to C_G \to G^{\mathrm{tor}} \to \mu_G[2].$$

We denote by Z_G the center of a reductive group G.

2. Hasse principle for homogeneous spaces

We start with extending a well-known result on the abelianization maps to the positive characteristic case.

Proposition 2.1. — *Let F be either a global field with no real place, or a non-archimedean local field. Then there exists a natural exact sequence of groups*

$$G^{\mathrm{sc}}(F) \to G(F) \xrightarrow{\mathrm{ab}^0_G} H^0\left(F, C_G\right) \to 1$$

and the map $\mathrm{ab}^1_G : H^1(F,G) \to H^1(F,C_G)$ is a bijection.

Proof. — If the characteristic of K is zero, see [Bor98, Section 5]. We now assume that the characteristic of F is positive.

By construction, one has a short exact sequence of groups and pointed sets

$$G^{\mathrm{sc}}(F) \xrightarrow{\rho_*} G(F) \xrightarrow{\mathrm{ab}^0_G} H^0(F,C_G) \to H^1\left(F,G^{\mathrm{sc}}\right) \xrightarrow{\rho_*} H^1(F,G) \xrightarrow{\mathrm{ab}^1_G} H^1(F,C_G)\,.$$

In the global case, following [Har75, Satz A], the set $H^1(F,G^{\mathrm{sc}})$ is trivial, hence the map ab^1_G has trivial kernel and the first sequence in the statement is exact. A twisting argument implies that the map ab^1_G is even injective. For the local case, the injectivity (and exactness of the first sequence) follows from [BT87, Theorem 4.7(ii)].

The proof of the surjectivity is an adaptation of the proofs of [Bor98, Theorems 5.4 and 5.7], except that the existence of anisotropic maximal tori over local fields of positive characteristic is provided by [DeB06, Lemma 2.4.1] (see also [Ros21b, Proposition 4.4]). □

We are interested in the Hasse principle for homogeneous spaces under G, with *reductive* (recall that by definition this includes smoothness and connectedness) geometric stabilizers. Following Raynaud (see [Ray70, Definition VI.1.1 and Proposition VI.1.2]), if G is a smooth group scheme over K, a homogeneous space of G is a smooth K-scheme X with an action of G, such that for any $x \in X(K^s)$ (such an x exists since X is smooth), the stabilizer of x in $\bar{G} := G \times_K K^s$ is a finite type (over K^s) subgroup scheme H_x of $\bar{G}$ and $\bar{X}$ is isomorphic to the quotient $\bar{G}/H_x$.

Definition 2.2. — *Let X be a homogeneous space of a reductive group G with reductive K^s-stabilizer $\bar{H} = H_x$. Let L_X be the K-kernel defined by X and $\sigma_X \in H^2(K,L_X)$ be the Springer class, that is the class of the gerbe associated to X (cf.* [FSS98, § 5.2.] *or* [Bor93, § 7.7] *in characteristic zero). By assumption, the K^s-stabilizers are reductive, hence following loc. cit., there exists a K-torus T, which is a K-form of $\bar{H}^{\mathrm{tor}}$, and a natural map of marked sets*

$$\mathrm{ab}^2_X : H^2\left(K, L_X\right) \to H^2(K,T)\,,$$

which is functorial in K. On the other hand we have the class $\eta_X \in H^1(K,[T \to G^{\mathrm{tor}}])$ constructed by Borovoi in [Bor99] *(where characteristic zero is assumed, but not used in the definition of η_X). There is an exact sequence in Galois (hyper)-cohomology*

$$\begin{aligned} H^1(K,T) \to H^1\left(K, G^{\mathrm{tor}}\right) &\to H^1\left(K, \left[T \to G^{\mathrm{tor}}\right]\right) \\ \to H^2(K,T) \quad &\to H^2\left(K, G^{\mathrm{tor}}\right). \end{aligned} \tag{2.1}$$

Proposition 2.3. — *The image of η_X in $H^2(K,T)$ is $\mathrm{ab}^2_X(\sigma_X)$.*

Proof. — The method is similar to the one used in the proof of [BvH12, Theorem 9.6]. We start with the case when G itself is a torus. Then T is a subtorus of G and $[T\to G]$ is quasi-isomorphic to the quotient group G/T. Moreover, X is a principal homogeneous space of G/T with class $[X]\in H^1(K,G/T)\simeq H^1(K,[T\to G])$ corresponding to η_X. As $\sigma_X\in H^2(K,T)$ is (by definition) just the image of $[X]$ by the coboundary map $H^1(K,G/T)\to H^2(K,T)$, the result holds in this case.

Assume now that G is an arbitrary reductive group but satisfies the additional hypothesis:

$$(*)\qquad \text{The canonical morphism } T\to G^{\mathrm{tor}} \text{ is injective.}$$

Then $Y:=X/G^{\mathrm{ss}}$ is a homogenous space of G^{tor} with stabilizer (defined over K in this case) T. The Springer class $\sigma_Y\in H^2(K,T)$ is (by construction) just $\mathrm{ab}^2_X(\sigma_X)$, and it is also the image of $\eta_Y\in H^1(K,[T\to G^{\mathrm{tor}}])$ in $H^2(K,T)$ by the first case. But $\eta_Y=\eta_X$ by functoriality of the class η_X ([Bor99, § 1.6]), whence the result when $(*)$ is satisfied.

We now deal with the general case. By [BvH12, Proposition 9.9] (whose proof is identical in characteristic p thanks to the assumption that G and $\bar H$ are reductive), there exists a homogeneous space Z of a K-group $F=G\times P$, where P is a quasi-trivial torus, such that: the homogeneous space Z satisfies $(*)$ and there is a K-morphism $\pi:Z\to X$, compatible with the respective actions of F, G (via the projection $F\to G$), which makes Z an X-torsor under P. In particular the G-homogeneous space Z still has geometric stabilizer $\bar H$, with $L_Z=L_X$ and $\sigma_Z=\sigma_X$. Since Proposition 2.3 holds for Z (because its satisfies $(*)$), it also holds for X by functoriality of the class η_X and commutativity of the diagrams

$$\begin{tikzcd}
H^2(K,L_Z) \arrow[r, "\mathrm{ab}^2_Z"] \arrow[d] & H^2(K,T) \arrow[d, "="] \\
H^2(K,L_X) \arrow[r, "\mathrm{ab}^2_X"] & H^2(K,T)
\end{tikzcd}$$

$$\begin{tikzcd}
H^1(K,[T\to F^{\mathrm{tor}}]) \arrow[r] \arrow[d] & H^2(K,T) \arrow[d, "="] \\
H^1(K,[T\to G^{\mathrm{tor}}]) \arrow[r] & H^2(K,T)
\end{tikzcd}$$ □

Remark 2.4. — The previous proposition holds over an arbitrary field. Recall also that the existence of a K-point on X implies that the class σ_X is neutral, as well as the vanishing of η_X.

Theorem 2.5. — *Let G be a reductive group over K. Then the Brauer-Manin obstruction to the Hasse principle is the only one for homogeneous spaces of G with reductive geometric stabilizers. More precisely, such a torsor X has a rational point if and only if X has an adelic point orthogonal to the subgroup*

$$\text{Б}(X):=\ker\left[(\operatorname{Br}X/\operatorname{Br}K)\to\prod_{v\in\Omega_K}(\operatorname{Br}X_{K_v}/\operatorname{Br}K_v)\right]$$

for the Brauer–Manin pairing:

$$X(\mathbf{A}_K) \times (\operatorname{Br} X / \operatorname{Br} K) \to \mathbf{Q}/\mathbf{Z}, \quad (P_v) \mapsto \langle \alpha, (P_v) \rangle_{BM} := \sum_{v \in \Omega_K} \alpha(P_v). \tag{2.2}$$

Recall that by global class field theory, the sum $\sum_{v \in \Omega_K} \alpha(P_v)$ is zero for every element $\alpha \in \operatorname{Im}[\operatorname{Br} K \to \operatorname{Br} X]$, hence the Brauer–Manin pairing (2.2) is well defined. Also, for $\alpha \in Б(X)$, the element $\langle \alpha, (P_v) \rangle_{BM}$ is independent of the choice of $(P_v) \in X(\mathbf{A}_K)$, because the localisation $\alpha_v \in \operatorname{Br} X_{K_v}$ of α is a constant element (i.e. it comes from $\operatorname{Br} K_v$) for every place v.

Before starting the proof of Theorem 2.5, we prove the following (well known) lemma, for which we didn't find an appropriate reference.

Lemma 2.6. — *Let k be a field and*

$$1 \to G_1 \to G_2 \to G_3 \to 1$$

be an exact sequence of smooth connected linear algebraic groups over k.

Then G_2 is reductive if and only if G_1 and G_3 are reductive.

Proof. — Without loss of generality, one can assume that k is algebraically closed. We denote by $R_u(G)$ the unipotent radical of G.

- Assume G_1 and G_3 are reductive. The image of $R_u(G_2)$ inside G_3 is contained in $R_u(G_3)$, hence it is trivial. Therefore, $R_u(G_2) \subset G_1$, and $R_u(G_2)$ is unipotent connected and normal in G_1, hence $R_u(G_2)$ is trivial. So G_2 is reductive.
- Assume that G_2 is reductive. Since $R_u(G_1)$ is a characteristic subgroup of G_1, it is normal in G_2, hence $R_u(G_1) \subset R_u(G_2)$, hence $R_u(G_1)$ is trivial, so G_1 is reductive. To conclude the proof, one uses the non-trivial classical fact that a quotient of a reductive group is reductive (see [Bor91, Corollary 14.11]).

□

Proof of Theorem 2.5. — Fix a point $x \in X(K^s)$ and let $\bar{H} = H_x$ be the stabilizer of x in $\bar{G}$. Up to replacing G by a flasque resolution ([CT08, Proposition 3.1])

$$1 \to S \to G' \to G \to 1,$$

where S is a flasque torus (which is central in G') and G' is a quasi-trivial group (that is: extension of a quasi-trivial torus by a semisimple simply connected group), we can assume that the group G itself is quasitrivial. Indeed X is also a homogeneous space of G' such that the K^s-stabilizer of x is reductive by Lemma 2.6 (the stabilizer is an extension of H_x by S_{K^s}). In particular $\operatorname{Pic} \bar{G} = 0$ and the group of characters $\widehat{G^{\mathrm{tor}}}$ of G^{tor} is a permutation Galois module ([CT08, Proposition 2.2]).

Since $H^1(K, G^{\mathrm{tor}}) = 0$ (the torus G^{tor} being quasi-trivial), we have by exact sequence (2.1) that the canonical map $H^1(K, [T \to G^{\mathrm{tor}}]) \to H^2(K, T)$ is injective. In addition, the class η_X (viewed as an element of $H^2(K, T)$) is just $\mathrm{ab}^2_X(\sigma_X)$ by Proposition 2.3.

Assume that $X(\mathbf{A}_K) \neq \emptyset$. Then the class σ_X is neutral at every place v of K, which implies that $\eta_X = \mathrm{ab}^2_X(\sigma_X) \in Ш^2(K, T)$. The Brauer–Manin pairing defines a morphism $\beta_X : Б(X) \to \mathbf{Q}/\mathbf{Z}$ (recall that if $\alpha \in Б(X)$, then $\langle \alpha, (P_v) \rangle_{BM} \in \mathbf{Q}/\mathbf{Z}$ is independent of $(P_v) \in X(\mathbf{A}_K)$).

By [BDH13, § 4], there is a complex $\mathrm{UPic}(X)$ (up to a shift this is the complex $\mathrm{UPic}\,\bar{X}$ of [BvH12]; we will recover this complex in Section 3) such that $\mathrm{Br}_1 X/\operatorname{Br} K$ is isomorphic to $H^1(K, \mathrm{UPic}(X))$ (this is valid over any field K such that $H^3(K, \mathbf{G}_m) = 0$). Moreover by [BvH12, Theorem 5.8], (whose proof is still valid in characteristic p thanks to the additional assumption that G and H are reductive), the complex $\mathrm{UPic}(X)$ is quasi-isomorphic to $[\widehat{G^{\mathrm{tor}}} \to \widehat{T}]$ (recall that $\operatorname{Pic}\bar{G} = 0$). Therefore, the exact sequence of complexes of Galois modules

$$0 \to \widehat{T} \to \left[\widehat{G^{\mathrm{tor}}} \to \widehat{T}\right] \to \widehat{G^{\mathrm{tor}}}[1] \to 0$$

induces an exact triangle in the derived category

$$\widehat{G^{\mathrm{tor}}} \to \widehat{T} \xrightarrow{\lambda} \mathrm{UPic}(X) \to \widehat{G^{\mathrm{tor}}}[1]\,. \tag{2.3}$$

Since $\widehat{G^{\mathrm{tor}}}$ is a permutation Galois module, $H^1(F, \widehat{G^{\mathrm{tor}}}) = 0$ for any field extension F/K, and $Ш^2(K, \widehat{G^{\mathrm{tor}}}) = 0$ by Shapiro's lemma and Čebotarev's Theorem. Hence the long exact sequence associated to the triangle (2.3) induces an isomorphism of abelian groups

$$\psi_X := \lambda_* : Ш^1\left(K, \widehat{T}\right) \xrightarrow{\sim} Б(X).$$

By [DH20, Theorem 5.2], there is a Poitou–Tate perfect duality of finite groups

$$\langle , \rangle_{PT} :\ Ш^2(K,T) \times Ш^1\left(K, \widehat{T}\right) \to \mathbf{Q}/\mathbf{Z}\,. \tag{2.4}$$

This pairing and the class $\eta_X = \mathrm{ab}^2(\sigma_X)$ define a morphism $\alpha_X : Ш^1(K, \widehat{T}) \to \mathbf{Q}/\mathbf{Z}$.

Lemma 2.7. — *The following diagram*

$$\begin{tikzcd} Ш^1\left(\widehat{T}\right) \arrow[d, "\psi_X"'] \arrow[dr, "\alpha_X"] & \\ Б(X) \arrow[r, "\beta_X"] & \mathbf{Q}/\mathbf{Z} \end{tikzcd}$$

is commutative (up to sign).

Proof. — By [BvH12, Theorem 9.6] (whose proof is identical in characteristic p thanks to our additional assumptions), the class $\eta_X \in H^1(K, [T \to G^{\mathrm{tor}}]) \hookrightarrow H^2(K,T)$ coincides (up to a sign) with the element of $\mathrm{Ext}^2(\mathrm{UPic}(X), \mathbf{G}_m)$ given by the map $w : \mathrm{UPic}(X) \to \mathbf{G}_m[2]$ of exact sequence (1) in [HS13]. Therefore the class η_X viewed in $Ш^2(T) \subset H^2(K,T) = \mathrm{Ext}^2(\widehat{T}, \mathbf{G}_m)$ is just $\partial(\lambda)$, where $\partial : \mathrm{Hom}_{\Gamma_K}(\widehat{T}, \mathrm{UPic}(X)) \to H^2(K,T)$ is the map defined in exact sequence (2) of [HS13]. Now Theorem 3.5. of loc. cit. (whose proof in characteristic p is identical) shows that for every $a \in Ш^1(\widehat{T})$, we have

$$\langle \eta_X, a\rangle_{PT} = \langle \partial(\lambda), a\rangle_{PT} = \beta_X(\lambda_*(a)),$$

or in other words:

$$\alpha_X(a) = \beta_X(\psi_X(a)).$$

This concludes the proof of the Lemma 2.7. □

We can now finish the proof of Theorem 2.5. Assuming that $X(\mathbf{A}_K)^{Б(X)} \neq \emptyset$, we have $\beta_X = 0$, hence $\alpha_X = 0$ by the lemma. The exactness of the pairing (2.4) and Proposition 2.3 imply that $\mathrm{ab}^2_X(\sigma_X) = \eta_X = 0$. Now, the analogue of [Bor93], Proposition 6.5 for global fields of positive characteristic (the proof of which is the same, except that [Bor93, Lemma 5.7] is replaced by Proposition 2.1) implies that the map ab^2_X has "trivial kernel", hence η_X is neutral. As a consequence, there exists a G-equivariant morphism $P \to X$ defined over K, where P is a K-torsor under G. Since G is quasi-trivial, $H^1(K,G)$ is trivial (see [Har75, Satz A]), hence $P(K) \neq \emptyset$, therefore $X(K) \neq \emptyset$. □

3. Abelianization of homogeneous spaces and a compatibility formula

Let H be a reductive subgroup of a reductive group G over an arbitrary field k. Set $X = G/H$ and denote by $e \in X(k)$ the image of the neutral element of G in X. Consider the complex of Galois modules (or of commutative smooth k-group schemes) defined by

$$C_X := [T_{H^{\mathrm{sc}}} \to T_H \oplus T_{G^{\mathrm{sc}}} \to T_G]\,,$$

with T_G in degree 0. In other words, we have $C_X := \mathrm{Cone}(C_H \to C_G)$. Denote by ab^0_X (see [Dem13, § 2.6], where the characteristic zero assumption is not needed thanks to our additional assumptions that G and H are smooth and reductive) the abelianization map

$$\mathrm{ab}^0_X : X(k) \to H^0_{\mathrm{ab}}(k,X) := H^0\left(k, C_X\right).$$

Following [Dem11c], we set

$$\mathrm{Br}_1(X,G) = \ker\left[\mathrm{Br}\, X \to \mathrm{Br}\, \bar{G}\right],$$

which is a subgroup of $\mathrm{Br}\, X$ containing $\mathrm{Br}_1 X$. We also consider the subgroup $\mathrm{Br}_{1,e}(X,G)$ of $\mathrm{Br}_1(X,G)$ consisting of those elements α such that $\alpha(e) = 0$. Quotienting by $\mathrm{Br}\, k$ induces an isomorphism $\mathrm{Br}_{1,e}(X,G) \simeq \mathrm{Br}_1(X,G)/\mathrm{Br}\, k$. We use a similar notation for $\mathrm{Br}_{1,e} X \simeq \mathrm{Br}_1 X/\mathrm{Br}\, k$.

The goal of this section is to prove (see Theorem 3.7 below) that there is a natural isomorphism

$$\phi_X : H^1\left(k, \widehat{C}_X\right) \xrightarrow{\sim} \mathrm{Br}_{1,e}(X,G)\,,$$

and that the following diagram

$$\begin{tikzcd} X(k) \times \mathrm{Br}_{1,e}(X,G) \arrow[r, "\mathrm{ev}"] \arrow[d, "\mathrm{ab}'_X"'] & \mathrm{Br}(k) \arrow[d, "="] \\ H^0(k, C_X) \times H^1\left(k, \widehat{C}_X\right) \arrow[u, shift left=8ex, "\phi_X"'] \arrow[r, "\cup"] & \mathrm{Br}(k) \end{tikzcd}$$

is commutative ($\widehat{C}_X$ denotes the dual complex of C_X: it is the cone of the morphism of complexes

$$\left[\widehat{T}_G \to \widehat{T}_{G^{\mathrm{sc}}}\right] \to \left[\widehat{T}_H \to \widehat{T}_{H^{\mathrm{sc}}}\right]$$

where $\widehat{T}_H$ is in degree 0). Here ab'_X is an abelianization map, which we will define in section 3.1 by changing a little the map ab^0_X. In [Dem20], Demeio proves such a compatibility in characteristic zero for ab^0_X with a long and impressive cocycle computation. It is quite likely that the maps ab^0_X and our new map ab'_X actually coincide, but we did not succeed in proving this. The modified map ab'_X seems more suitable to check the required compatibility.

3.1. Abelianization map over a field

From now on, let k be a field, X a smooth geometrically integral k-variety and $\pi : Y \to X$ a torsor under a reductive k-group H. We fix a point $y_0 \in Y(k)$ and let $x_0 := \pi(y_0) \in X(k)$. Following [Dem11c], we define Z to be Y/Z_H, z_0 to be the image of y_0 and $\mathrm{UPic}'(\pi)$ to be the following complex of Galois modules:

$$\mathrm{UPic}'(\pi) := \left[\bar{k}(Z)^\times \to \mathrm{Div}\left(\bar{Z}\right) \to \mathrm{Pic}'\left(\bar{Z}/\bar{X}\right)\right] ,$$

with $\bar{k}(Z)^\times$ in degree -1 (here $\mathrm{Pic}'(\bar{Z}/\bar{X})$ is the relative Picard group of $\bar{Z}$ over $\bar{X}$). We also define

$$\mathrm{UPic}(\pi) := \left[\bar{k}(Z)^\times/\bar{k}^\times \to \mathrm{Div}\left(\bar{Z}\right) \to \mathrm{Pic}'\left(\bar{Z}/\bar{X}\right)\right] ,$$

with the obvious natural exact sequence of complexes:

$$0 \to \bar{k}^\times[1] \to \mathrm{UPic}'(\pi) \to \mathrm{UPic}(\pi) \to 0\,.$$

We will sometimes need the following pointed version of those complexes, which are canonically quasi-isomorphic to the previous ones:

$$\mathrm{UPic}'(\pi)_0 := \left[\bar{k}(Z)^\times_0 \to \mathrm{Div}\left(\bar{Z}\right)_0 \to \mathrm{Pic}'\left(\bar{Z}/\bar{X}\right)\right]$$

and

$$\mathrm{UPic}(\pi)_0 := \left[\bar{k}(Z)^\times_{0,1} \to \mathrm{Div}\left(\bar{Z}\right)_0 \to \mathrm{Pic}'\left(\bar{Z}/\bar{X}\right)\right] ,$$

where $\bar{k}(Z)^\times_0$ (resp. $\bar{k}(Z)^\times_{0,1}$) denotes the subgroup of rational functions defined at z_0 (resp. taking the value 1 at z_0) and $\mathrm{Div}(\bar{Z})_0$ is the group of divisors D such that z_0 is not contained in the support of D. We also have the classical complexes $\mathrm{UPic}(Z) = [\bar{k}(Z)^*/\bar{k}^* \to \mathrm{Div}\,\bar{Z}]$, $\mathrm{UPic}'(Z)$ etc., corresponding to the case when $\pi : Z \to Z$ is the identity map (we already encountered UPic in the proof of Theorem 2.5).

The construction of $\mathrm{UPic}'(\pi)$ is contravariant in π (see for instance [Dem11c, Proposition 2.2]), and for any $x \in X(k)$, the natural morphism

$$\bar{k}^\times[1] \to \mathrm{UPic}'\left(\pi_x : Z_x \to \mathrm{Spec}\,k\right)$$

is a quasi-isomorphism, where Z_x denotes the fiber of Z at x.

Therefore, one gets a well-defined specialization map

$$\mathrm{ab}''_\pi : X(k) \to \mathrm{Hom}_k\left(\mathrm{UPic}'(\pi), \bar{k}^\times[1]\right) ,$$

where Hom_k denotes the set of morphisms in the derived category of bounded complexes of Galois modules.

In addition, when a point $y_0 \in Y(k)$ is given, one gets a natural splitting $\mathrm{UPic}(\pi) \to \mathrm{UPic}'(\pi)$, hence it defines the required map

$$\mathrm{ab}'_\pi : X(k) \to \mathrm{Hom}_k\left(\mathrm{UPic}(\pi), \bar{k}^\times[1]\right) .$$

When $Y = X$ and $\pi = \mathrm{id}_X$, we denote ab'_π by ab'_X.

We want to think about ab'_π as an abelianization map for the set of rational points of X, which is a replacement of ab^0_X when $X = G/H$.

We now study the dévissage of $\mathrm{UPic}(\pi)$ in terms of Y and H :

Lemma 3.1. — *There is a natural exact triangle*

$$\mathrm{UPic}(\pi) \to \mathrm{UPic}(Y) \to \mathrm{UPic}(H) \to \mathrm{UPic}(\pi)[1] .$$

Proof. — This is essentially the proof of [Dem11c, Corollary 3.3]. Let $H' = H/Z_H$ be the quotient of H by its center. Since $\mathrm{UPic}(\pi)_0[1]$ is the cone of $\mathrm{UPic}(Z)_0 \to \mathrm{UPic}(H')_0$, the following commutative diagram of complexes

$$\begin{tikzcd}
\mathrm{UPic}(Z)_0 \arrow[r] \arrow[d] & \mathrm{UPic}(H')_0 \arrow[d] \\
\mathrm{UPic}(Y)_0 \arrow[r, "\varphi"] & \mathrm{UPic}(H)_0
\end{tikzcd}$$

induces a canonical morphism of complexes

$$\alpha : \mathrm{UPic}(\pi)_0[1] \to \mathrm{cone}(\varphi) ,$$

such that the following diagram of exact triangles is commutative:

$$\begin{tikzcd}
\mathrm{UPic}(Z)_0 \arrow[r] \arrow[d] & \mathrm{UPic}(H')_0 \arrow[r] \arrow[d] & \mathrm{UPic}(\pi)_0[1] \arrow[r] \arrow[d, "\alpha"] & \mathrm{UPic}(Z)_0[1] \arrow[d] \\
\mathrm{UPic}(Y)_0 \arrow[r, "\varphi"] & \mathrm{UPic}(H)_0 \arrow[r] & \mathrm{cone}(\varphi) \arrow[r] & \mathrm{UPic}(Y)_0[1] .
\end{tikzcd}$$

Let us now prove that α is a quasi-isomorphism. Since the complexes $\mathrm{UPic}(\pi)_0[1]$ and $\mathrm{cone}(\varphi)$ are concentrated in degrees -2 to 0, we only compute the cohomology corresponding to those degrees: one has a commutative diagram of long exact sequences

(3.1)

$$\begin{tikzcd}
0 \arrow[r] & H^{-2}\left(\mathrm{UPic}(\pi)[1]\right) \arrow[r] \arrow[d, "\alpha^{-2}"] & U\left(\bar{Z}\right) \arrow[r] \arrow[d] & \widehat{H'} \arrow[r] \arrow[d] & H^{-1}\left(\mathrm{UPic}(\pi)[1]\right) \arrow[r, dashed] \arrow[d, "\alpha^{-1}"] & {} \\
0 \arrow[r] & H^{-2}(\mathrm{cone}(\varphi)) \arrow[r] & U\left(\bar{Y}\right) \arrow[r] & \widehat{H} \arrow[r, "\beta"] & H^{-1}(\mathrm{cone}(\varphi)) \arrow[r, dashed] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, dashed] & \mathrm{Pic}\left(\bar{Z}\right) \arrow[r] \arrow[d] & \mathrm{Pic}\left(\bar{H'}\right) \arrow[r] \arrow[d] & H^0\left(\mathrm{UPic}(\pi)[1]\right) \arrow[r] \arrow[d, "\alpha^0"] & 0 \\
{} \arrow[r, dashed] & \mathrm{Pic}\left(\bar{Y}\right) \arrow[r] & \mathrm{Pic}\left(\bar{H}\right) \arrow[r] & H^0(\mathrm{cone}(\varphi)) \arrow[r] & 0 .
\end{tikzcd}$$

The proof of [Dem11c, Corollary 3.3] ensures that α^{-2} and α^0 are isomorphisms.

Let us now prove that α^{-1} is an isomorphism (the proof of this fact in [Dem11c] is too sketchy): using the exact sequences $\widehat{Z_H} \to \mathrm{Pic}(\bar{Z}) \to \mathrm{Pic}(\bar{Y})$ and $\widehat{Z_H} \to \mathrm{Pic}(\bar{H}') \to \mathrm{Pic}(\bar{H})$, diagram chasing in (3.1) proves that the result follows from the commutativity of the following diagram:

$$\begin{tikzcd} \mathrm{Pic}\left(\bar{X}\right) & H^{-1}\left(\mathrm{UPic}(\pi)_0[1]\right) \arrow[l, "\sim"'] \arrow[d, "\alpha^{-1}"] \\ \widehat{H} \arrow[u, "\Delta"] \arrow[r, "\beta"] & H^{-1}(\mathrm{cone}(\varphi)) \,, \end{tikzcd}$$

where $\Delta : \widehat{H} \to \mathrm{Pic}(\bar{X})$ is the map defined by $\chi \mapsto \chi_*[\bar{Y}]$ and the isomorphism

$$H^{-1}\left(\mathrm{UPic}(\pi)_0[1]\right) \to \mathrm{Pic}\left(\bar{X}\right)$$

is constructed in [Dem11c]. To prove the required commutativity, given $\chi \in \widehat{H}$, functoriality of the various maps implies that it is sufficient to consider the case $H = \mathbf{G}_m$ (and $H' = 1$, $Z = X$) and $\chi = \mathrm{id} : \mathbf{G}_m \to \mathbf{G}_m$. More precisely, let $D \in \mathrm{Div}(\bar{X})$ with support not containing x_0, let $\pi : Y_D \to \bar{X}$ be the associated $\mathbf{G}_m$-torsor. Let $U \subset \bar{X}$ be the complement of the support of D. By construction, the pullback $Y_{D,U} \to U$ of $Y_D \to \bar{X}$ admits a canonical section s over U, inducing a map $f : Y_{D,U} \to \mathbf{G}_m$ which can be seen as an element $f \in \bar{k}(Y_D)^\times$. Let $y_0 := s(x_0) \in Y_{D,U}(k)$ and let Y_0 be the fiber of π at x_0. Then the restriction of f at Y_0 and the point $y_0 \in Y_0(k)$ induce a morphism $f_0 : \mathbf{G}_m \to \mathbf{G}_m$. Making explicit the maps α^{-1} and β, the required commutativity boils down to the natural equalities $\pi^*(D) = \mathrm{div}(f)$ in $\mathrm{Div}(\bar{Y}_D)$ and $f_0 = \mathrm{id}$. $\square$

We now want to compare the map ab'_π defined earlier with the maps ab^0_H and ab^1_H defined by Borovoi in [Bor98] for reductive k-groups H. We first prove the following

Lemma 3.2. — *With the above notation, we have a commutative diagram (up to a sign) with exact rows*

$$\begin{tikzcd} H(k) \arrow[r, hook] \arrow[d, "\mathrm{ab}^0_H"] & Y(k) \arrow[r, "\pi"] \arrow[d, "\mathrm{ab}'_Y"] & X(k) \arrow[r] \arrow[d, "\mathrm{ab}'_\pi"] & H^1(k,H) \arrow[d, "\mathrm{ab}^1_H"] \\ H^0_{\mathrm{ab}}(k,H) \arrow[r] & \mathrm{Hom}_k\left(\mathrm{UPic}(Y), \bar{k}^\times[1]\right) \arrow[r, "\pi'"] & \mathrm{Hom}_k\left(\mathrm{UPic}(\pi), \bar{k}^\times[1]\right) \arrow[r] & H^1_{\mathrm{ab}}(k,H) \,, \end{tikzcd}$$

where the second line comes from Lemma 3.1 and from the isomorphisms

$$H^i_{\mathrm{ab}}(k,H) = \mathrm{Hom}_k\left(\mathrm{UPic}(H), \bar{k}^\times[i+1]\right)$$

constructed in [BvH09, Theorem 4.8 and Corollary 4.9].

In addition, when $Y = G$ is a reductive k-group, then the map

$$\mathrm{ab}'_G : G(k) \to \mathrm{Hom}_k\left(\mathrm{UPic}(G), \bar{k}^\times[1]\right) = H^0_{\mathrm{ab}}(k,G)$$

coincides with ab^0_G.

Proof. — The commutativity of the central square is a consequence of the functoriality of the map ab'_π with respect to morphisms of torsors.

Let us now prove the commutativity of the left hand side square : using the functoriality of the map ab_π and the fact that the morphism $H \to Y$, given by the action on $y_0 \in Y(k)$, is H-equivariant, it is sufficient to prove that ${\mathrm{ab}^0}_H = {\mathrm{ab}'}_H$, where ${\mathrm{ab}'}_H : H(k) \to \mathrm{Hom}_k(\mathrm{UPic}(H), \bar{k}^\times[1])$ is defined as ${\mathrm{ab}'}_{\mathrm{id}_H}$. There exists a coflasque resolution (see [CT08, Proposition 4.1])

$$1 \to P \to H_1 \to H \to 1\,,$$

where P is a quasi-trivial k-torus and H_1 is an extension of a (coflasque) torus T by a semisimple simply connected k-group H^{ss}. Then functoriality of the map ${\mathrm{ab}'}_H$ and Hilbert 90 imply that it is enough to prove the required compatibility ${\mathrm{ab}'}_T = {\mathrm{ab}^0}_T$ for the k-torus T. By definition, the map

$${\mathrm{ab}'}_T : T(k) \to \mathrm{Hom}_k\left(\widehat{T}[1], \bar{k}^\times[1]\right) = \mathrm{Hom}_k\left(\widehat{T}, \bar{k}^\times\right)$$

is given by $t \mapsto (\chi \mapsto \chi(t))$, and it clearly coincides with the map ${\mathrm{ab}^0}_T : T(k) \to T(k)$, composed with the natural identification $T(k) \xrightarrow{\sim} \mathrm{Hom}_k(\widehat{T}, \bar{k}^\times)$. It concludes the proof of the commutativity of the left hand side square. Note that this also proves the last statement in Lemma 3.2.

Let us now prove the remaining commutativity, concerning the right hand side square. Let $x \in X(k)$ and consider the torsor $\pi_x : Y_x \to \operatorname{Spec} k$ defined as the pullback of π by x (recall that Y_x denotes the fiber of Y at x). We have a commutative diagram

$$\begin{tikzcd}
\mathrm{UPic}(\pi) \arrow[r, "y"] \arrow[d, "\alpha"] & \mathrm{UPic}'(\pi) \arrow[r, "x^*"] \arrow[d, "\alpha'"] & \mathrm{UPic}'(\pi_x) \arrow[d, "\alpha_x"] & \bar{k}^\times[1] \arrow[l] \arrow[d, "="] \\
\mathrm{cone}(\varphi)[-1] \arrow[r, "y"] & \mathrm{cone}(\varphi')[-1] \arrow[r, "x^*"] & \mathrm{cone}(\varphi'_x)[-1] & \bar{k}^\times[1]\,, \arrow[l]
\end{tikzcd}$$

where the vertical maps are quasi-isomorphisms (see the proof of Lemma 3.1). We now prove that the diagram

$$\begin{tikzcd}
\mathrm{cone}(\varphi)[-1] \arrow[r, "y"] & \mathrm{cone}(\varphi')[-1] \arrow[r, "x^*"] & \mathrm{cone}(\varphi'_x)[-1] & \bar{k}^\times[1] \arrow[l] \\
\mathrm{UPic}(H)[-1] \arrow[r, "="] \arrow[u] & \mathrm{UPic}(H)[-1] \arrow[r, "="] \arrow[u] & \mathrm{UPic}(H)[-1] \arrow[u] & \mathrm{UPic}(Y_x)[-1] \arrow[l, "\varphi_x"'] \arrow[u, "\partial_{Y_x}"']
\end{tikzcd}$$

is commutative. The only non-obvious square is the right hand side one, but it comes from the functoriality of the cone in the category of complexes, together with the definition of ∂_{Y_x}.

The two previous commutative diagrams imply that the square

$$\begin{tikzcd}
X(k) \arrow[r] \arrow[d, "{\mathrm{ab}'}_\pi"] & H^1(k, H) \arrow[d, "{\mathrm{ab}'}^1_H"] \\
\mathrm{Hom}_k\left(\mathrm{UPic}(\pi), \bar{k}^\times[1]\right) \arrow[r] & \mathrm{Hom}_k\left(\mathrm{UPic}(H), \bar{k}^\times[2]\right)
\end{tikzcd}$$

commutes, where ${\mathrm{ab}'}^1_H$ maps the cohomology class of a torsor $W \to \operatorname{Spec} k$ under H to the rightmost morphism in the natural exact triangle

$$\bar{k}^\times[1] \to \mathrm{UPic}'(W) \to \mathrm{UPic}(W) \xrightarrow{\partial_W} \bar{k}^\times[2]$$

with the canonical identification $\mathrm{UPic}(W) \xrightarrow{\sim} \mathrm{UPic}(H)$.

We conclude the proof using [BvH09, Theorem 5.5], which proves that ${\mathrm{ab}'}^1_H = -\mathrm{ab}^1_H$. □

We need to prove other properties of the map ab'_π:

Proposition 3.3. — *Let $X = G/H$ be the quotient of a reductive group G by a reductive subgroup H. Let $\pi : G \to X$ be the quotient map (pointed by $e \in G(k)$). Then for all $g \in G(k)$ and $x \in X(k)$,*

$$\mathrm{ab}'_\pi(g \cdot x) = \pi'\left(\mathrm{ab}^0_G(g)\right) + \mathrm{ab}'_\pi(x)\,.$$

In particular, if G is semi-simple and simply connected, we have $\mathrm{ab}'_\pi(g\cdot x) = \mathrm{ab}'_\pi(x)$.

Proof. — Consider $\mathrm{id}_G : G \to G$ as a torsor under the trivial group, and $\mathrm{id} \times\pi : G \times_k G \to G \times_k X$ as a natural torsor under H. Then we have natural morphisms of torsors:

$$\begin{tikzcd} G \arrow[d, "\mathrm{id}_G"] & G \times_k G \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "\mathrm{id}_G \times\pi"] & G \arrow[d, "\pi"] \\ G & G \times_k X \arrow[l, "p_G"'] \arrow[r, "p_X"] & X\,, \end{tikzcd}$$

that induce, by functoriality of $\mathrm{UPic}(\pi)$ and by Lemma 3.1, a commutative diagram in the derived category, where the rows are exact triangles (see [BvH09, Lemma 5.1] for the third vertical map):

$$\begin{tikzcd} \mathrm{UPic}(H)[-1] \arrow[r] \arrow[d, "\sim"] & \mathrm{UPic}(G) \oplus \mathrm{UPic}(\pi) \arrow[r] \arrow[d, "p_G^*+p_X^*"] & \mathrm{UPic}(G) \oplus \mathrm{UPic}(G) \arrow[r] \arrow[d, "\sim"] & \mathrm{UPic}(H) \arrow[d, "\sim"] \\ \mathrm{UPic}(1 \times H)[-1] \arrow[r] & \mathrm{UPic}\left(\mathrm{id}_G \times\pi\right) \arrow[r] & \mathrm{UPic}\left(G \times_k G\right) \arrow[r] & \mathrm{UPic}(1 \times H)\,. \end{tikzcd}$$

The five lemma implies that the morphism $p_G^* + p_X^* : \mathrm{UPic}(G) \oplus \mathrm{UPic}(\pi) \to \mathrm{UPic}(\mathrm{id}_G \times\pi)$ is an isomorphism.

By functoriality of the construction of ab'_π, the morphism of torsors

$$\begin{tikzcd} G \times_k G \arrow[r, "m_G"] \arrow[d, "\mathrm{id}_G \times\pi"] & G \arrow[d, "\pi"] \\ G \times_k X \arrow[r, "m"] & X \end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd}
& \bar{k}^\times[1] & & \bar{k}^\times[1] \arrow[ll, "\times"'] & \oplus & \bar{k}^\times[1] \\
& \mathrm{UPic}\,(\mathrm{id}_G \times \pi) \arrow[u, "{(g,x)}"] & & \mathrm{UPic}(G) \arrow[u, "g"] \arrow[ll, "p_G^*+p_X^*"'] & \oplus & \mathrm{UPic}(\pi) \arrow[u, "x"] \\
\mathrm{UPic}(\pi) \arrow[uur, "g\cdot x"] \arrow[ur, "m^*"] \arrow[urrr, "{(\mathrm{can},\mathrm{id})}"'] & & & & &
\end{tikzcd},$$

which concludes the proof. The non-trivial commutativity in this last diagram is that of the triangle at the bottom, which we explain now: recall that we are given a point $x_0 = \pi(e) \in X(k)$. Consider the following commutative diagram of morphisms of torsors:

$$\begin{tikzcd}
G \arrow[r, "{(\mathrm{id}_G,1)}"] \arrow[d, "\mathrm{id}_G"] & G \times G \arrow[d, "{(\mathrm{id}_G,\pi)}"] & G \arrow[l, "{(1,\mathrm{id}_G)}"'] \arrow[d, "\pi"] \\
G \arrow[r, "\iota_{x_0}"] & G \times X & X \arrow[l, "\iota_1"']
\end{tikzcd},$$

where the bottom horizontal maps are defined by $\iota_{x_0}(g) = (g, x_0)$ and $\iota_1(x) = (1, x)$. If we denote by ϖ_G (resp. ϖ_X) the projection from $\mathrm{UPic}(G) \oplus \mathrm{UPic}(\pi)$ to $\mathrm{UPic}(G)$ (resp. $\mathrm{UPic}(\pi)$), then we deduce from the previous diagram that $\varpi_G = \iota_{x_0}^* \circ (p_G^* + p_X^*)$ and $\varpi_X = \iota_1^* \circ (p_G^* + p_X^*)$. But $m \circ \iota_{x_0} = \pi : G \to X$ and $m \circ \iota_1 = \mathrm{id}_X : X \to X$, therefore we get that the required triangle commutes.

Finally, if G is assumed to be semi-simple and simply connected, then the complex $\mathrm{UPic}(G)$ is quasi-isomorphic to 0, hence the map π' is trivial, which implies the required result. ☐

Remark 3.4. — It is worth noting that the construction of the map ab_π depends on the choice of a k-point $y_0 \in Y(k)$. But one can prove, using the same kind of arguments as in the proof of Lemma 3.3, that the map ab_π depends only on the image x_0 of y_0 in $X(k)$. More precisely, two points $y_0, y_0' \in Y(k)$ such that $\pi(y_0) = \pi(y_0')$ define the same map ab_π, or equivalently, the construction of this map depends only on the choice of a point $x_0 \in \pi(Y(k))$.

Lemma 3.2 and Proposition 3.3 imply the following proposition:

Proposition 3.5. — *Let F be a non-archimedean local field or a global field with no real place. Let $X = G/H$ be the quotient of a reductive group G by a reductive subgroup H.*

Then the map $\mathrm{ab}'_\pi : X(F) \to \mathrm{Hom}_k(\mathrm{UPic}(\pi), \bar{F}^\times[1])$, associated to the torsor $\pi : G \to X$, is surjective, and for all $x, x' \in X(F)$, $\mathrm{ab}'_\pi(x) = \mathrm{ab}'_\pi(x')$ if and only if there exists $g \in G^{\mathrm{sc}}(F)$ such that $x' = g \cdot x$.

Proof. — Consider the following commutative diagram with exact rows (see Lemma 3.2):

$$\begin{tikzcd}
H(F) \arrow[r] \arrow[d, "\mathrm{ab}^0_H"] & G(F) \arrow[r, "\pi"] \arrow[d, "\mathrm{ab}^0_G"] & X(F) \arrow[r] \arrow[d, "\mathrm{ab}'_\pi"] & H^1(F,H) \arrow[r] \arrow[d, "\mathrm{ab}^1_H"] & H^1(F,G) \arrow[d, "\mathrm{ab}^1_G"] \\
H^0_{\mathrm{ab}}(F,H) \arrow[r] & H^0_{\mathrm{ab}}(F,G) \arrow[r, "\pi'"] & \mathrm{Hom}_F\left(\mathrm{UPic}(\pi), \bar{F}^\times[1]\right) \arrow[r] & H^1_{\mathrm{ab}}(F,H) \arrow[r] & H^1_{\mathrm{ab}}(F,G)\,.
\end{tikzcd}$$

Then diagram chasing proves that surjectivity of ab'_π is a consequence of surjectivity of ab^1_H, injectivity of ab^1_G, surjectivity of ab^0_G (those properties follows from Proposition 2.1), and of Proposition 3.3.

Let us now prove the second part of the Proposition: let x, $x' \in X(F)$ such that $\mathrm{ab}'_\pi(x) = \mathrm{ab}'_\pi(x')$. Using the above diagram and the injectivity of ab^1_H (see Proposition 2.1), we get that x and x' have the same image in $H^1(F,H)$. Therefore, there exists $g \in G(F)$ such that $g \cdot x = x'$. Applying Lemma 3.3, we get that $\pi'(\mathrm{ab}^0_G(g)) = 0$, hence by Proposition 2.1, g lifts to $G^{\mathrm{sc}}(F)$, which concludes the proof. $\square$

Let $X = G/H$, with G and H reductive over the field k. Let us now construct a canonical isomorphism $\phi_X : \widehat{C}_X \to \mathrm{UPic}(\pi)$ in the derived category, inspired by [Dem11c, sections 4.1.2 and 4.1.3].

By construction, $\widehat{C}_X$ is the cone of the morphism of complexes

$$\left[\widehat{T}_G \to \widehat{T}_{G^{\mathrm{sc}}}\right] \to \left[\widehat{T}_H \to \widehat{T}_{H^{\mathrm{sc}}}\right]$$

where $\widehat{T}_H$ is in degree 0. Consider the following commutative diagram of complexes, where the vertical maps are either obvious or defined in [BvH09, Section 4]):

$$\begin{tikzcd}
\left[\widehat{T}_G \to \widehat{T}_{G^{\mathrm{sc}}}\right] \arrow[r] \arrow[d] & \left[\widehat{T}_H \to \widehat{T}_{H^{\mathrm{sc}}}\right] \arrow[d] \\
\mathrm{cone}\left(\mathrm{UPic}(T_G)_0 \to \mathrm{UPic}\left(T_{G^{\mathrm{sc}}}\right)_0\right) \arrow[r] & \mathrm{cone}\left(\mathrm{UPic}(T_H)_0 \to \mathrm{UPic}\left(T_{H^{\mathrm{sc}}}\right)_0\right) \\
\mathrm{cone}\left(\mathrm{UPic}(G)_0 \to \mathrm{UPic}\left(G^{\mathrm{sc}}\right)_0\right) \arrow[r] \arrow[u] \arrow[d] & \mathrm{cone}\left(\mathrm{UPic}(H)_0 \to \mathrm{UPic}\left(H^{\mathrm{sc}}\right)_0\right) \arrow[u] \arrow[d] \\
\mathrm{UPic}(G)_0 \arrow[r, "\varphi"] & \mathrm{UPic}(H)_0 \\
\mathrm{UPic}(Z)_0 \arrow[r] \arrow[u] & \mathrm{UPic}(H')_0\,. \arrow[u]
\end{tikzcd} \tag{3.2}$$

All the vertical maps, except the bottom ones, are quasi-isomorphisms. Hence this diagram induces a natural isomorphism $\phi'_X : \widehat{C}_X \to \mathrm{cone}(\varphi)$, which we can compose with the quasi-isomorphism $\alpha : \mathrm{UPic}(\pi)[1] \to \mathrm{cone}(\varphi)$ induced by the two last lines of the above diagram (see the proof of Lemma 3.1), to get a natural isomorphism $\phi_X : \widehat{C}_X \to \mathrm{UPic}(\pi)[1]$.

By construction, this isomorphism fits into the following commutative diagram of exact triangles in the derived category:

$$
\begin{tikzcd}
\widehat{C}_H[-1] \arrow[r] \arrow[d, "\phi_H"] & \widehat{C}_X \arrow[r] \arrow[d, "\phi_X"] & \widehat{C}_G \arrow[r] \arrow[d, "\phi_G"] & \widehat{C}_H \arrow[d, "\phi_H"] \\
\mathrm{UPic}(H)[-1] \arrow[r] & \mathrm{UPic}(\pi) \arrow[r] & \mathrm{UPic}(G) \arrow[r] & \mathrm{UPic}(H)\,,
\end{tikzcd} \tag{3.3}
$$

where the vertical maps are isomorphisms (morphisms ϕ_H and ϕ_G are defined in [BvH09, Section 4], and also in the first four lines of diagram (3.2)).

In addition, [BDH13, Lemma 3.1] implies that the natural morphism $H^0(k, C_X) \to \mathrm{Hom}_k(\widehat{C}_X, \bar{k}^\times[1])$ is an isomorphism. These facts lead to the following:

Definition 3.6. — *We denote by* $\mathrm{ab}'_X : X(k) \to H^0(k, C_X)$ *the composition*

$$X(k) \xrightarrow{\mathrm{ab}'_\pi} \mathrm{Hom}_k\left(\mathrm{UPic}(\pi), \bar{k}^\times[1]\right) \xrightarrow{\widehat{\phi_X}} \mathrm{Hom}_k\left(\widehat{C}_X, \bar{k}^\times[1]\right) \xrightarrow{\sim} H^0\left(k, C_X\right)\,,$$

where $\pi : G \to X$ *is the quotient morphism.*

In particular, Lemma 3.2, Definition 3.6 and diagram (3.3) imply that the following useful diagram is commutative (up to sign):

$$
\begin{tikzcd}
H(k) \arrow[r] \arrow[d, "\mathrm{ab}^0_H"] & G(k) \arrow[r, "\pi"] \arrow[d, "\mathrm{ab}^0_G"] & X(k) \arrow[r] \arrow[d, "\mathrm{ab}'_X"] & H^1(k,H) \arrow[r] \arrow[d, "\mathrm{ab}^1_H"] & H^1(k,G) \arrow[d, "\mathrm{ab}^1_G"] \\
H^0\left(k, C_H\right) \arrow[r] & H^0\left(k, C_G\right) \arrow[r] & H^0\left(k, C_X\right) \arrow[r] & H^1\left(k, C_H\right) \arrow[r] & H^1\left(k, C_G\right)\,,
\end{tikzcd} \tag{3.4}
$$

where the unnamed maps are the natural ones.

3.2. The compatibility result

We can now prove the main compatibility result of this section, which can be seen as a generalization of [Dem11c, Theorem 4.14] and [BDH13, Theorem 6.2]:

Theorem 3.7. — *Let k be a field and G be a reductive group. Let $H \subset G$ be a reductive k-subgroup and $X := G/H$. Let $\pi : G \to X$ be the quotient map (pointed by $e \in G(k)$ and its image $x_0 := \pi(e)$). The canonical isomorphism $\phi_X : \widehat{C}_X \to$ $\mathrm{UPic}(\pi)$ in the derived category induces an isomorphism*

$$\phi_X : H^1\left(k, \widehat{C}_X\right) \xrightarrow{\sim} \mathrm{Br}_{1,e}(X,G)\,,$$

and the following diagram

$$
\begin{tikzcd}
X(k) \times \mathrm{Br}_{1,e}(X,G) \arrow[r, "\mathrm{ev}"] & \mathrm{Br}(k) \arrow[d, "="] \\
H^0(k, C_X) \times H^1\left(k, \widehat{C}_X\right) \arrow[r, "\cup"] \arrow[u, shift right=8, "\phi_X"'] \arrow[from=u, shift left=8, "\mathrm{ab}'_X"'] & \mathrm{Br}(k)
\end{tikzcd}
$$

is commutative, up to a universal sign (independent of all the data).

Proof. — Following [Dem11c], there is a natural morphism $\mathrm{UPic}'(\pi) \to \tau_{\leqslant 2}R\pi_* \mathbf{G}_m[1]$ inducing an isomorphism $H^1(k, \mathrm{UPic}(\pi)) \xrightarrow{\sim} \mathrm{Br}_{1,e}(X, G)$. Together with the isomorphism $\phi_X : \widehat{C}_X \to \mathrm{UPic}(\pi)$, we get the required isomorphism. Let us now prove the commutativity.

For any $x \in X(k)$, one can define a natural splitting $\mathrm{ab}'(x)$ of $\bar{k}^\times \to \tau_{\leqslant 2}R\pi_*\mathbf{G}_m$ induced by x.

We can decompose the diagram above as the composition of the following diagrams:

$$\begin{tikzcd}
X(k) \arrow[d, "\mathrm{ab}'"'] \arrow[ddd, bend right=60, "\mathrm{ab}'_\pi"'] & \times & \mathrm{Br}_{1,e}(X,G) \arrow[r, "\mathrm{ev}"] & \mathrm{Br}(k) \arrow[d, "="] \\
\mathrm{Hom}_k\left(\tau_{\leqslant 2}R\pi_*\mathbf{G}_m[1], \bar{k}^\times[1]\right) \arrow[d] & \times & H^1\left(k, \tau_{\leqslant 2}R\pi_*\mathbf{G}_m[1]\right) \arrow[u] \arrow[r, "\cup"] & \mathrm{Br}(k) \arrow[d, "="] \\
\mathrm{Hom}_k\left(\mathrm{UPic}'(\pi), \bar{k}^\times[1]\right) \arrow[d, "e"'] & \times & H^1\left(k, \mathrm{UPic}'(\pi)\right) \arrow[u] \arrow[r, "\cup"] & \mathrm{Br}(k) \arrow[d, "="] \\
\mathrm{Hom}_k\left(\mathrm{UPic}(\pi), \bar{k}^\times[1]\right) \arrow[d, "\widehat{\phi_X}"'] & \times & H^1\left(k, \mathrm{UPic}(\pi)\right) \arrow[u, "e"'] \arrow[r, "\cup"] & \mathrm{Br}(k) \arrow[d, "="] \\
\mathrm{Hom}_k\left(\widehat{C}_X, \bar{k}^\times[1]\right) \arrow[d, "\sim"'] & \times & H^1\left(k, \widehat{C}_X\right) \arrow[u, "\phi_X"'] \arrow[r, "\cup"] & \mathrm{Br}(k) \arrow[d, "="] \\
H^0(k, C_X) & \times & H^1\left(k, \widehat{C}_X\right) \arrow[u, "="'] \arrow[r, "\cup"] & \mathrm{Br}(k)\,.
\end{tikzcd}$$

By construction, and by functoriality of cup-products, this last diagram is commutative (up to sign). ☐

Remark 3.8. — Let $X = G/H$ as in Definition 3.6 and Theorem 3.7. Up to replacing G by a flasque resolution G_1, one can realise X as the quotient of the quasi-trivial group G_1 by a reductive subgroup H_1. Since $\mathrm{Pic}(\bar{G}_1) = 0$, one gets a natural isomorphism

$$\widehat{C_X} \xrightarrow{\sim} \left[\widehat{G_1} \to \widehat{T_{H_1}} \to \widehat{T_{H_1^{\mathrm{sc}}}}\right],$$

where $\widehat{G_1}$ is a permutation Galois module. Assuming the group G is quasi-trivial will be very useful in the next two sections.

3.3. The abelianization map over an arbitrary base

In this section, we extend the definition of the map ab'_X for homogeneous spaces of reductive group schemes defined over an arbitrary base scheme S. It will be useful in the next sections in order to take integral points into account (case $S = \mathrm{Spec}(\mathcal{O}_v)$).

Let S be an integral regular noetherian scheme and H be a reductive group scheme over S. Let $\pi : Y \to X$ be a torsor under H. For any S-scheme W, let $p_W : W \to S$ denote the structure morphism.

Let $Z := Y/Z_H$ and $\varpi : Z \to X$ be the associated $H' := H/Z_H$-torsor. We define $\mathrm{UPic}'(\pi)$ to be the following complex of étale sheaves over S:

$$\underline{\mathrm{UPic}'}(\pi) := \left[p_{Z*}\underline{K}^{\times}_{Z/X} \to p_{Z*}\underline{\mathrm{Div}}_{Z/X} \to p_{X*}\underline{\mathrm{Pic}}_{Z/X}\right] ,$$

where the sheaves $\underline{K}^{\times}_{Z/X}$ and $\underline{\mathrm{Div}}_{Z/X}$ are defined in [HS13, Appendix A]. By loc. cit., there is a natural exact sequence of étale sheaves over X

$$0 \to (\mathbf{G}_m)_X \to \varpi_*\underline{K}^{\times}_{Z/X} \to \varpi_*\underline{\mathrm{Div}}_{Z/X} \to \underline{\mathrm{Pic}}_{Z/X} \to 0\,.$$

Applying p_{X*}, one gets a natural morphism $(\mathbf{G}_m)_S \to \underline{\mathrm{UPic}'}(\pi)$, and we define $\underline{\mathrm{UPic}}(\pi)$ to be the cone of this morphism, whence an exact triangle

$$(\mathbf{G}_m)_S \to \underline{\mathrm{UPic}'}(\pi) \to \underline{\mathrm{UPic}}(\pi) \to (\mathbf{G}_m)_S[1]\,.$$

Let $y_0 \in Y(S)$ be a section, and let $x_0 := \pi(y_0)$. Denote by $D(S)$ the derived category of bounded complexes of étale sheaves over S. Following the construction in section 3.1, we get a natural map, functorial in S and X,

$$\mathrm{ab}'_\pi : X(S) \to \mathrm{Hom}_{D(S)}\left(\underline{\mathrm{UPic}}(\pi), (\mathbf{G}_m)_S\right)\,.$$

Let now G be a reductive group scheme over S and $H \subset G$ a reductive subgroup scheme. Taking $Y = G$ and $X = G/H$, we can apply the previous constructions. Assume further that H and G admit compatible maximal tori $T_H \subset T_G$. The diagram (3.2), as a diagram of étale sheaves over S, still holds. All vertical morphisms, except the bottom ones, are quasi-isomorphisms of étale sheaves over S, since it is true over any separably closed field (see after diagram (3.2)). Similarly, since the result holds over separably closed fields, diagram (3.2) induces an isomorphism of complexes of étale sheaves $\phi_X : \widehat{C}_X \to \underline{\mathrm{UPic}}(\pi)[1]$.

As a conclusion, composing the map $\mathrm{ab}'_\pi : X(S) \to \mathrm{Hom}_{D(S)}(\underline{\mathrm{UPic}}(\pi), (\mathbf{G}_m)_S)$ with the isomorphism ϕ_X, one gets the required map:

$$\mathrm{ab}'_X : X(S) \to H^0(S, C_X)\,,$$

that is functorial in S and (H, G), and that coincides with the definition of Section 3.1 in the case S is the spectrum of a field.

4. Weak approximation

From now on, the setting is the following: $K = k(E)$ is again the function field of a projective, smooth and irreducible curve E over a finite field k. We consider a reductive linear algebraic group G over K, and X a homogeneous space of G. We assume that $X(K) \neq \emptyset$. Let $e \in X(K)$ and let $H \subset G$ be the stabilizer of e in G. Then $X = G/H$ (with e identified to the image of the neutral element of G in X) and we still suppose that H is a reductive subgroup of G. Set $X(K_\Omega) = \prod_{v \in \Omega_K} X(K_v)$.

We are interested in the closure of $X(K)$ in $X(K_\Omega)$, for the product topology. We define $Б_\omega(X)$ as the subgroup of $\mathrm{Br}_{1,e} X \simeq \mathrm{Br}_1 X/\mathrm{Br}\, K$ consisting of those elements

α such that their localization $\alpha_v \in \mathrm{Br}_{1,e}\, X_{K_v}$ is zero for all but finitely many v. For each finite set of places S of K, we set $K_S = \prod_{v \in S} K_v$ and we define

$$\text{Б}_S(X) := \ker \left[\mathrm{Br}_{1,e}\, X \to \prod_{v \notin S} \mathrm{Br}_{1,e}\, X_{K_v} \right].$$

In particular $\text{Б}(X)$ (cf. Section 2) identifies to $\text{Б}_{\emptyset}(X)$.

Remark 4.1. — Assume that X admits a regular compactification, i.e. there exists a regular proper K-variety X^c and an open immersion $X \to X^c$. Then the group $\text{Б}_\omega(X)$ is exactly the algebraic Brauer group of X^c (see for example [BDH13, Proposition 4.1]).

For all $\alpha \in \text{Б}_\omega(X)$, $(P_v) \in X(K_\Omega)$, the Brauer–Manin pairing:

$$\langle \alpha, (P_v) \rangle_{BM} = \sum_{v \in \Omega_K} \alpha(P_v)$$

is well-defined. For every subgroup B of $\text{Б}_\omega(X)$, we denote by $X(K_\Omega)^B \subset X(K_\Omega)$ the orthogonal of B for the Brauer–Manin pairing.

Theorem 4.2. — *Let G be a reductive group over K and H a reductive subgroup. Then the Brauer–Manin obstruction to weak approximation on $X = G/H$ associated to $\text{Б}_\omega(X)$ is the only one, i.e. $X(K)$ is dense in $X(K_\Omega)^{\text{Б}_\omega(X)}$. More precisely, for any finite set S of places of K, the Brauer–Manin pairing induces a surjective map $X(K_S) \to (\text{Б}_S(X)/\text{Б}(X))^D$, whose kernel is exactly the closure of $X(K)$ inside $X(K_S)$.*

Proof. — Up to replacing G by a flasque resolution, one can assume that G is quasi-trivial. Using an elementary instance of the fibration method, we see that G satisfies weak approximation, since G^{sc} (by [Pra77]) and the quasi-trivial (hence K-rational) torus G^{tor} do satisfy weak approximation. In addition, we have $H^1(K,G) = 1$ by [Har75] and Hilbert's 90 (the latter shows that $H^1(K, G^{\mathrm{tor}}) = 0$ thanks to Shapiro's lemma).

Let $C := [H^{\mathrm{tor}} \to G^{\mathrm{tor}}]$ and $C_X := \mathrm{Cone}(C_H \to C_G)$ (cf. Section 3). We have the Cartier duals $\widehat{C} = [\widehat{G^{\mathrm{tor}}} \to \widehat{H^{\mathrm{tor}}}]$ and $\widehat{C_X} = \mathrm{Cone}(\widehat{C_G} \to \widehat{C_H})[-1]$.

By construction, we have a natural commutative diagram of exact triangles of complexes:

(4.1)
$$\begin{tikzcd}
\mu_H[1] \arrow[d] & & \mu_H[2] \arrow[r, "\sim"] \arrow[d] & \mu_H[2] \arrow[d] \\
C_H \arrow[r] \arrow[d] & C_G \arrow[r] \arrow[d, "\mathrm{qis}"] & C_X \arrow[r] \arrow[d] & C_H[1] \arrow[d] \\
H^{\mathrm{tor}} \arrow[r] \arrow[d] & G^{\mathrm{tor}} \arrow[r] & C \arrow[r] \arrow[d] & H^{\mathrm{tor}}[1] \arrow[d] \\
\mu_H[2] & & \mu_H[3] \arrow[r, "\sim"] & \mu_H[3]\,.
\end{tikzcd}$$

Since G is assumed to be quasi-trivial, we have $G^{\mathrm{ss}} = G^{\mathrm{sc}}$, hence $\mu_G = 0$ and C_G is quasi-isomorphic to G^{tor}, which is a quasi-trivial torus. Taking Cartier duals, we get an exact triangle:

$$\widehat{C_X} \to \widehat{G^{\mathrm{tor}}}[1] \to \widehat{C_H} \to \widehat{C_X}[1],$$

whence (using $H^1(K, \widehat{G^{\mathrm{tor}}}) = 0$) an exact sequence:

$$0 \to H^0\left(K, \widehat{C_H}\right) \to H^1\left(K, \widehat{C_X}\right) \to H^2\left(K, \widehat{G^{\mathrm{tor}}}\right)$$

and similarly replacing K with a completion K_v. As $\widehat{G^{\mathrm{tor}}}$ is a permutation Galois module, we have $Ш^2_S(K, \widehat{G^{\mathrm{tor}}}) = 0$ by Shapiro's lemma and Čebotarev's Theorem. Thus

$$Ш^0_S\left(K, \widehat{C_H}\right) \xrightarrow{\sim} Ш^1_S\left(K, \widehat{C_X}\right).$$

We also have an exact triangle

$$\widehat{\mu_H}[-2] \to \widehat{C} \to \widehat{C_X} \to \widehat{\mu_H}[-1],$$

which yields an isomorphism

$$Ш^1_S(K, \widehat{C}) \to Ш^1_S(K, \widehat{C_X}).$$

Summing up, we get isomorphisms

$$Ш^0_S\left(K, \widehat{C_H}\right) \xrightarrow{\sim} Ш^1_S\left(K, \widehat{C_X}\right) \xleftarrow{\sim} Ш^1_S\left(K, \widehat{C}\right).$$

As recalled before (cf. proof of Th. 2.5.), the group $\operatorname{Br}_{1,e} X \simeq \operatorname{Br}_1 X/\operatorname{Br} K$ is isomorphic to $H^1(K, \mathrm{UPic}(X)) = H^1(K, \widehat{C})$ (and this is true over any field). Therefore $Ш^1_S(K, \widehat{C})$ (resp. $Ш^1(K, \widehat{C})$) identifies to $Б_S(X)$ (resp. to $Б(X)$), whence a Brauer–Manin pairing:

$$X(K_S) \times Ш^1_S\left(K, \widehat{C}\right) \to \mathbf{Q}/\mathbf{Z}, ((P_v)_{v\in S}, \alpha) \mapsto \sum_{v \in S} \alpha(P_v),$$

which is trivial on $X(K_S) \times Ш^1(K, \widehat{C})$. It induces a map $\mathrm{BM} : X(K_S) \to (Ш^1_S(K, \widehat{C})/Ш^1(K, \widehat{C}))^D$. On the other hand, local duality for complex of tori ([Dem11b, Theorem 3.1]) induces a map

$$\theta_S : \prod_{v\in S} H^1_{\mathrm{ab}}(K_v, H) = \prod_{v \in S} H^1\left(K_v, C_H\right) \to \left(Ш^0_S\left(K, \widehat{C_H}\right)/Ш^0\left(K, \widehat{C_H}\right)\right)^D.$$

We also have a map in the derived category of Galois modules

$$C_X \otimes^{\mathbf{L}} \widehat{C_X} \to \mathbf{G}_m[1],$$

which induces (for each completion K_v) a cup-product pairing

$$H^0\left(K_v, C_X\right) \times H^1\left(K_v, \widehat{C_X}\right) \to H^2\left(K_v, \mathbf{G}_m\right) = \mathbf{Q}/\mathbf{Z}.$$

The latter is compatible (in an obvious sense) with local duality for the complexes C_H, $\widehat{C_H}$ via the maps $C_X \to C_H[1]$ and $\widehat{C_H} \to \widehat{C_X}[1]$. Finally, denote by $\partial_K : X(K) \to H^1(K, H)$ the coboundary map, which (by composing with the abelianization map) yields a morphism of pointed sets $\partial_K^{\mathrm{ab}} : X(K) \to H^1_{\mathrm{ab}}(K, H) = H^1(K, C_H)$. Similarly we have (for every completion K_v) maps $\partial_v^{\mathrm{ab}} : X(K_v) \to H^1(K_v, C_H)$. By Lemma 3.2, the map ∂_v^{ab} can also be obtained by composing the abelianization map $\mathrm{ab}'_{X_{K_v}}$:

$X(K_v) \to H^0(K_v, C_X)$ (defined in Section 3) with the natural map $H^0(K_v, C_X) \to H^1(K_v, C_H)$ (induced by diagram (4.1)).

Lemma 4.3. — *There is a commutative diagram with exact rows:*
(4.2)

$$\begin{tikzcd}
G(K) \arrow[r] \arrow[d] & X(K) \arrow[r, "\partial_K^{\mathrm{ab}}"] \arrow[d] & H^1(K, C_H) \arrow[r] \arrow[d] & 1 \\
G(K_S) \arrow[r] & X(K_S) \arrow[r, "\partial_S^{\mathrm{ab}}=\prod_{v\in S}\partial_v^{\mathrm{ab}}"] \arrow[d, "BM"] & \prod_{v\in S} H^1\left(K_v, C_H\right) \arrow[r] \arrow[d, "\theta_S"] & 1 \\
 & \left(\text{Ш}^1_S\left(K, \widehat{C}\right)/\text{Ш}^1\left(K, \widehat{C}\right)\right)^D \arrow[r, "\sim"] & \left(\text{Ш}^0_S\left(K, \widehat{C_H}\right)/\text{Ш}^0\left(K, \widehat{C_H}\right)\right)^D \arrow[d] & \\
 & & 0 &
\end{tikzcd}.$$

Proof. — The exactness of rows follows from the triviality of $H^1(K, G)$ and $H^1(K_v, G)$ (recall that G is quasi-trivial) combined with Proposition 2.1. The only non-trivial remaining point is the commutativity of the bottom square. By functoriality of the cup-product and Theorem 3.7, there is a commutative diagram :

$$\begin{tikzcd}
X(K_v) \arrow[r, "\mathrm{ab}'_{X_{K_v}}"] \arrow[d] & H^0(K_v, C_X) \arrow[r] \arrow[d] & H^1(K_v, C_H) \arrow[d] \\
\mathrm{Br}_{1,e}(X_{K_v}, G_{K_v})^D \arrow[r, "\left(\Phi_{X_{K_v}}\right)^D"] & H^1\left(K_v, \widehat{C_X}\right)^D \arrow[r] & H^0\left(K_v, \widehat{C}_H\right)^D
\end{tikzcd}$$

where the left vertical map is given by the local evaluation pairing

$$X(K_v) \times \mathrm{Br}_{1,e}\left(X_{K_v}, G_{K_v}\right) \to \mathbf{Q}/\mathbf{Z},\ (P_v, \alpha_v) \mapsto \alpha_v(P_v)$$

and the right vertical map by local duality for the complex C_H. Now let $a \in \text{Ш}^0_S(K, \widehat{C_H})$ with image $b \in \text{Ш}^1_S(K, \widehat{C_X})$ and $c \in \text{Ш}^1_S(K, \widehat{C})$. Set $\alpha = \Phi_X(b) \in \mathrm{Br}_{1,e} X \subset \mathrm{Br}_{1,e}(X, G)$. Let $v \in S$ and $P_v \in X(K_v)$. By the previous diagram, we have

$$\alpha(P_v) = \left(\partial_v^{\mathrm{ab}}(P_v) \cup a_v\right) \in \mathrm{Br}\, K_v = \mathbf{Q}/\mathbf{Z},$$

where $a_v \in H^0(K_v, \widehat{C_H})$ is the localization of a. Hence

$$\left(BM((P_v))\right).c := \sum_{v\in S} \alpha(P_v) = \sum_{v\in S} \left(\partial_v^{\mathrm{ab}}\left(P_v\right) \cup a_v\right) = \left(\theta_S\left(\partial_S^{\mathrm{ab}}(P_v)\right)\right).a,$$

which yields the required commutativity. □

Remark 4.4. — Actually we used the difficult compatibility proven in Theorem 3.7 only for those elements of $H^1(K_v, \widehat{C}_X)$ coming from $H^1(K_v, \widehat{C})$, so [BDH13, Theorem 6.2] (whose proof is much easier) would be sufficient at this stage. However, we will definitely need Theorem 3.7 in its full generality in Section 5.

We resume the proof of Theorem 4.2. Let us now prove that the right hand side column of diagram (4.2) is exact. Consider the following commutative diagram:

$$\begin{tikzcd}
 & \bigoplus_{v\notin S} H^1(K_v, C_H) \arrow[r, "\sim"] \arrow[d] & \left(\prod_{v\notin S} H^0\left(K_v, \widehat{C_H}\right)\right)^D \arrow[d] & \\
H^1(K, C_H) \arrow[r] \arrow[d, "="] & \bigoplus_{v\in\Omega} H^1(K_v, C_H) \arrow[r] \arrow[d, "p"] & \left(H^0\left(K, \widehat{C_H}\right)/\text{Ш}^0\left(K, \widehat{C_H}\right)\right)^D \arrow[r] \arrow[d] & 0 \\
H^1(K, C_H) \arrow[r] & \prod_{v\in S} H^1(K_v, C_H) \arrow[r] \arrow[d] & \left(\text{Ш}^0_S\left(K, \widehat{C_H}\right)/\text{Ш}^0\left(K, \widehat{C_H}\right)\right)^D \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}$$

Using [DH20, Theorem 5.7], the second row is exact. And by construction, the columns are exact. Hence an easy diagram chase implies that the bottom row is exact. Therefore, the right hand side column in (4.2) is exact. In addition, we know that $G(K)$ is dense in $G(K_S)$. Therefore, an easy diagram chase in (4.2), together with the comparison [BDH13, Theorem 6.2], implies that the map

$$X(K_S) \xrightarrow{BM} \left(\text{Ш}^1_S(K,\widehat{C})/\text{Ш}^1(K,\widehat{C})\right)^D = (\text{Б}_S(X)/\text{Б}(X))^D$$

is surjective, and that the inverse image of 0 is exactly the closure of $X(K)$, which concludes the proof. □

Theorem 4.2 can be slightly refined when the homogeneous space X is a reductive group:

Corollary 4.5. — *Let L be a reductive group over K. Let $C_L = [T^{\mathrm{sc}} \to T_L]$ be the complex of tori associated to L. Then there is an exact sequence of groups*

$$1 \to \overline{L(K)} \to L(K_\Omega) \to \text{Ш}^1_\omega\left(K, \widehat{C_L}\right)^D \to \text{Ш}^1(K, C_L) \to 1\,.$$

Proof. — Using a flasque resolution of L (see for instance [CT08, Proposition 3.1]), one can view L as a homogeneous space $L = G/H$, with G quasi-trivial, and H reductive. By Theorem 4.2, there is an exact sequence of pointed sets

$$1 \to \overline{L(K)} \to L(K_\Omega) \xrightarrow{BM} \text{Б}_\omega(L)^D \simeq \text{Ш}^1_\omega\left(K, \widehat{C_L}\right)^D.$$

By Lemma 4.3, the Brauer–Manin map $L(K_\Omega) \to \text{Ш}^1_\omega(K, \widehat{C_L})^D$ is the composition of the abelianization map $\mathrm{ab}_L : L(K_\Omega) \to \prod_{v\in\Omega_K} H^0(K_v, C_L)$ with the map (which is induced by local duality) $\theta : \prod_{v\in\Omega_K} H^0(K_v, C_L) \to \text{Ш}^1_\omega(K, \widehat{C_L})^D$. Therefore the Brauer–Manin map is a morphism of groups. Proposition 2.1 implies that for every completion K_v, the map

$$\mathrm{ab}^v_L : L(K_v) \to H^0(K_v, C_L)$$

is surjective.

It is now sufficient to show that the sequence of abelian groups

$$\prod_{v \in \Omega_K} H^0\left(K_v, C_L\right) \to Ш^1_\omega\left(K, \widehat{C_L}\right)^D \to Ш^1\left(K, C_L\right) \to 0$$

is exact. This is done by observing that by [Dem11b, Theorem 3.1] and [DH20, Theorem 5.2], this sequence is the dual of the exact sequence of discrete abelian groups

$$0 \to Ш^1\left(K, \widehat{C_L}\right) \to Ш^1_\omega\left(K, \widehat{C_L}\right) \to \bigoplus_{v \in \Omega_K} H^1\left(K_v, \widehat{C_L}\right). \qquad \square$$

Remark 4.6. — The analogues of Theorems 2.5 and 4.2 were previously known in the context of number fields: they are proven by Borovoi in [Bor96] via more geometric techniques (namely fibration methods); the case of principal homogeneous spaces is due to Sansuc [San81]. Another approach over an arbitrary global field is to use flasque resolutions, see [Tha13, Theorem 3.9]. In the next section, we will see that the situation is slightly different for strong approximation.

5. Strong approximation

Notation is as in the previous section. We set $A^\star := \mathrm{Hom}(A, \mathbf{Z})$ for every abelian group A. An abelian group is said to be *in the class* $\mathcal{E}$ (cf. [DH20, Definition 3.10]) if it is an extension of a finitely generated group by a profinite group.

Let $C = [T_1 \to T_2]$ be a short complex of K-tori with dual $\widehat{C} = [\widehat{T_2} \to \widehat{T_1}]$. For every $\chi \in H^{-1}(K, \widehat{C})$, denote by $\chi_v \in H^{-1}(K_v, \widehat{C})$ the localization of χ at the place v. The cup-product pairing

$$H^0\left(K_v, C\right) \times H^{-1}\left(K_v, \widehat{C}\right) \to K_v^*,$$

induces a pairing

$$\begin{aligned} H^0(\mathbf{A}_K, C) \times H^{-1}(K, \widehat{C}) &\to \mathbf{Z} \\ ((g_v), \chi) &\mapsto \sum_v v(\chi_v . g_v) \cdot [k(v) : k] \ . \end{aligned} \tag{5.1}$$

Since the degree of a principal divisor on the projective curve E is zero, this pairing is trivial on the subgroup $H^0(K, C) \times H^{-1}(K, \widehat{C})$.

Lemma 5.1. —

(1) *The kernel P of the map $H^0(\mathbf{A}_K, C)/H^0(K, C) \to H^{-1}(K, \widehat{C})^\star$ induced by* (5.1) *is profinite.*

(2) *The canonical morphism*

$$i : H^0\left(\mathbf{A}_K, C\right)_\wedge / H^0(K, C)_\wedge \to \left(H^0\left(\mathbf{A}_K, C\right) / H^0(K, C)\right)_\wedge$$

is an isomorphism.

(3) *The map* $H^0(\mathbf{A}_K,C)\to H^{-1}(K,\widehat{C})^\star$ *is surjective and the diagram*

$$\begin{tikzcd} H^0(\mathbf{A}_K,C)/H^0(K,C) \arrow[r] \arrow[d] & \left(H^0\left(\mathbf{A}_K,C\right)/H^0(K,C)\right)_\wedge \arrow[d] \\ H^{-1}\left(K,\widehat{C}\right)^\star \arrow[r] & H^{-1}\left(K,\widehat{C}\right)^\star_\wedge \end{tikzcd}$$

is cartesian.

Proof. —

(1) Up to replacing C by a quasi-isomorphic complex, one can assume that T_1 is quasi-trivial. This yields a commutative diagram with exact rows and surjective left vertical map:

$$\begin{tikzcd} T_1(\mathbf{A}_K) \arrow[r] \arrow[d] & T_2(\mathbf{A}_K) \arrow[r] \arrow[d] & H^0(\mathbf{A}_K,C) \arrow[r] \arrow[d] & 0 \\ \widehat{T}_1(K)^\star \arrow[r] & \widehat{T}_2(K)^\star \arrow[r] & H^{-1}(K,\widehat{C})^\star & . \end{tikzcd}$$

The snake lemma now implies that the kernel of $H^0(\mathbf{A}_K,C)/H^0(K,C)\to H^{-1}(K,\widehat{C})^\star$ is a quotient of $\ker\left(T_2(\mathbf{A}_K)/T_2(K)\to\widehat{T_2}(K)^\star\right)$ by a closed subgroup, and this kernel is profinite by [Ros21b, Proposition 5.7.5]. Hence the group $\ker\left(H^0(\mathbf{A}_K,C)/H^0(K,C)\to H^{-1}(K,\widehat{C})^\star\right)$ is profinite.

(2) The exact sequence

$$H^0(K,C)\to H^0\left(\mathbf{A}_K,C\right)\to H^0\left(\mathbf{A}_K,C\right)/H^0(K,C)\to 0$$

and [DH20, Lemma 3.12 (a)] show that i is surjective. Since $H^{-1}(K,\widehat{C})^\star$ is a lattice, (1) shows that $H^0(\mathbf{A}_K,C)/H^0(K,C)$ is in the class $\mathcal{E}$. In particular, the canonical map $H^0(\mathbf{A}_K,C)/H^0(K,C)\to(H^0(\mathbf{A}_K,C)/H^0(K,C))_\wedge$ is injective, which shows that P injects into $(H^0(\mathbf{A}_K,C)/H^0(K,C))_\wedge$ as well as in $H^0(\mathbf{A}_K,C)_\wedge/H^0(K,C)_\wedge$. The commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & P \arrow[r] \arrow[d, "="] & H^0\left(\mathbf{A}_K,C\right)/H^0(K,C) \arrow[r] \arrow[d] & H^{-1}\left(K,\widehat{C}\right)^\star \arrow[d] \\ 0 \arrow[r] & P \arrow[r] \arrow[d, "="] & H^0(\mathbf{A}_K,C)_\wedge/H^0(K,C)_\wedge \arrow[r] \arrow[d, "i"] & H^{-1}(K,\widehat{C})^\star_\wedge \arrow[d] \\ 0 \arrow[r] & P \arrow[r] & \left(H^0\left(\mathbf{A}_K,C\right)/H^0(K,C)\right)_\wedge \arrow[r] & H^{-1}\left(K,\widehat{C}\right)^\star_\wedge \end{tikzcd} \tag{5.2}$$

has exact first line by definition, and exact third line because it is obtained by completing the first line and $H^{-1}(K,\widehat{C})^\star$ is a lattice, so [DH20, Lemma 3.12 (b)] applies. To prove the injectivity of i, it is sufficient (by diagram chasing) to show that the second line is exact as well. Let $\pi: H^0(\mathbf{A}_K,C)\to H^0(\mathbf{A}_K,C)/H^0(K,C)$ be the projection. The exact sequence

$$0\to\pi^{-1}(P)\to H^0\left(\mathbf{A}_K,C\right)\to H^{-1}\left(K,\widehat{C}\right)^\star$$

induces (again by [DH20, Lemma 3.12 (b)] an exact sequence

$$\pi^{-1}(P)_\wedge \to H^0(\mathbf{A}_K, C)_\wedge \to H^{-1}\left(K, \widehat{C}\right)^\star_\wedge,$$

whence an exact sequence

$$\pi^{-1}(P)_\wedge/H^0(K,C)_\wedge \xrightarrow{j} H^0(\mathbf{A}_K, C)_\wedge/H^0(K,C)_\wedge \to H^{-1}(K,\widehat{C})^\star_\wedge.$$

But the exact sequence

$$H^0(K,C) \to \pi^{-1}(P) \to P \to 0$$

induces (as P is profinite) a surjective map $u : \pi^{-1}(P)_\wedge/H^0(K,C)_\wedge \to P$ and j factorizes through u, hence the second line of the diagram is exact, as required.

(3) If $C = [0 \to \mathbf{G}_m]$, the required surjectivity is obviously true. If $C = [0 \to T]$, then one can find a resolution $0 \to R \to Q \to T \to 0$ (e.g. a flasque resolution) of T by K-tori R and Q, with Q quasi-trivial. It induces an injective map of lattices $\widehat{T}(K) \to \widehat{Q}(K)$, hence a surjective map $\widehat{Q}(K)^\star \to \widehat{T}(K)^\star$; thus one can reduce to the case of a quasi-trivial torus. By Shapiro's Lemma, this case reduces to the known case of $\mathbf{G}_m$. Hence the map $H^0(\mathbf{A}_K, C) \to H^{-1}(K,\widehat{C})^\star$ is surjective as soon as $C = [0 \to T]$ for any torus T. Let $C = [T_1 \to T_2]$ be an arbitrary complex of tori. Then the map $\widehat{T_2}(K)^\star \to H^{-1}(K,\widehat{C})^\star$ is surjective (again by injectivity of $H^{-1}(K,\widehat{C}) \to \widehat{C}(K)$), hence the surjectivity of $H^0(\mathbf{A}_K, C) \to H^{-1}(K,\widehat{C})^\star$ follows from that of $T_2(\mathbf{A}_K) \to \widehat{T_2}(K)^\star$. Since the diagram (5.2) is commutative with exact rows, the second point follows.

□

Proposition 5.2. —

(1) *There is an exact sequence*

$$\begin{aligned} H^0(K,C) \to H^0\left(\mathbf{A}_K, C\right) &\to \left(H^0\left(\mathbf{A}_K, C\right)/H^0(K,C)\right)_\wedge \\ &\xrightarrow{\partial} H^{-1}\left(K,\widehat{C}\right)^\star_\wedge/H^{-1}\left(K,\widehat{C}\right)^\star \to 0, \end{aligned} \tag{5.3}$$

where the morphism ∂ is given (after completion) by pairing (5.1) *for C.*

(2) *There is an exact sequence*

$$\begin{aligned} H^0(K,C) \to H^0\left(\mathbf{A}_K, C\right) \to \left(H^1\left(K,\widehat{C}\right)/Ш^1\left(K,\widehat{C}\right)\right)^D& \\ \xrightarrow{\partial_C} H^{-1}\left(K,\widehat{C}\right)^\star_\wedge/H^{-1}\left(K,\widehat{C}\right)^\star \to 0.& \end{aligned}$$

Observe that for $C = \mathbf{G}_m$, we have $\widehat{C} = \mathbf{Z}[1]$ and (2) is just the classical exact sequence of global class field theory.

Proof. — By [DH20, Theorems 5.7 and 5.10], there is a commutative diagram with exact rows:

$$\begin{tikzcd} H^0(K,C) \arrow[r] \arrow[d] & H^0\left(\mathbf{A}_K,C\right) \arrow[r] \arrow[d] & \left(H^1\left(K,\widehat{C}\right)/\text{Ш}^1\left(K,\widehat{C}\right)\right)^D \arrow[d] & \\ H^0(K,C)_\wedge \arrow[r] & H^0\left(\mathbf{A}_K,C\right)_\wedge \arrow[r] & \left(H^1\left(K,\widehat{C}\right)/\text{Ш}^1\left(K,\widehat{C}\right)\right)^D \arrow[r] & 0, \end{tikzcd} \tag{5.4}$$

whence exactness of the sequence

$$H^0(K,C) \to H^0\left(\mathbf{A}_K,C\right) \to \left(H^0\left(\mathbf{A}_K,C\right)/H^0(K,C)\right)_\wedge$$

thanks to Lemma 5.1 (2). The exactness of sequence (5.3) now follows from Lemma 5.1 (3), and part (2) of the proposition follows from its part (1) and diagram (5.4). □

We now consider a homogenous space $X = G/H$ with G and H reductive and we assume further that G is quasi-trivial. As in the previous sections, we define $C = [H^{\text{tor}} \to G^{\text{tor}}]$ and $C_X = \text{Cone}(C_H \to C_G) = [T_{H^{\text{sc}}} \to T_H \to G^{\text{tor}}]$. We denote by $\widehat{C}$ and $\widehat{C_X}$ their respective duals (cf. Section 4). We have the analogue of the pairing (5.1) with C (resp. $\widehat{C}$) replaced by C_X (resp. $\widehat{C_X}$), and again the pairing is trivial on $H^0(K,C_X) \times H^{-1}(K,\widehat{C_X})$.

We observe that $H^{-1}(K,\widehat{C}) = H^{-1}(K,\widehat{C_X})$ (and similarly over every completion K_v of K) thanks to the exact triangle

$$\widehat{\mu_H}[-2] \to \widehat{C} \to \widehat{C_X} \to \widehat{\mu_H}[-1].$$

For every finite and non-empty set of places S of K, we set

$$\text{Br}_S\, X = \ker\left[\text{Br}_e X \to \prod_{v\in S} \text{Br}\, X_{K_v}\right]$$

(not to be confused with the groups $Б_S(X)$ of Section 4) and $\text{Br}_S(X,G) := \text{Br}_S\, X \cap \text{Br}_{1,e}(X,G)$, $\text{Br}_{1,S} X := \text{Br}_S\, X \cap \text{Br}_{1,e} X$. We will also use a smooth model $\mathcal{G}$ (resp. $\mathcal{H}$, $\mathcal{X} = \mathcal{G}/\mathcal{H}$) of G (resp. H, X) over some non-empty Zariski open subset $U \neq E$ of the curve E. Shrinking U if necessary, we can assume that $\mathcal{H}$ and $\mathcal{G}$ admit compatible maximal tori $T_{\mathcal{H}} \subset T_{\mathcal{G}}$. We have the corresponding complexes $C_{\mathcal{H}}$, $\mathcal{C} = [\mathcal{H}^{\text{tor}} \to \mathcal{G}^{\text{tor}}]$, $C_{\mathcal{X}}$ defined over U. For every bounded complex $\mathcal{F}$ of flat commutative finite type group schemes over U, the compact support hypercohomology groups $H^i_c(U,\mathcal{F})$ are defined as in [DH20, § 2] and we set

$$D^i(U,\mathcal{F}) := \ker\left[H^i(U,\mathcal{F}) \to \bigoplus_{v\notin U} H^i\left(K_v,F\right)\right] = \text{Im}\left[H^i_c(U,\mathcal{F}) \to H^i(U,\mathcal{F})\right],$$

where F is the generic fibre of $\mathcal{F}$ over K. For $v \in U$, we denote by $H^i_{\text{nr}}(K_v,C_X)$ the image of $H^i(\mathcal{O}_v,C_{\mathcal{X}})$ in $H^i(K_v,C_X)$ and if S is a finite set of places that does not meet U, we set

$$\mathcal{P}^i_S(U,C_X) := \prod_{v\notin U, v\notin S} H^i\left(K_v,C_X\right) \times \prod_{v\in U} H^i_{\text{nr}}\left(K_v,C_X\right)$$

(by convention, $\mathcal{P}^i_\emptyset(U,C_X)$ is denoted by $\mathcal{P}^i(U,C_X)$).

Lemma 5.3. — *For every place $v\in U$, the abelianization map $\mathrm{ab}'_{\mathcal{X}_v}:\mathcal{X}(\mathcal{O}_v)\to H^0(\mathcal{O}_v,C_{\mathcal{X}})$ is surjective.*

Proof. — Using the same method as in [Dem13, Theorem 2.18], it is sufficient (via the version of Lemma 3.2 over $\mathcal{O}_v$) to show that $H^1_{\mathrm{ab}}(\mathcal{O}_v,\mathcal{H})=0$ and $\mathrm{ab}^0:\mathcal{G}(\mathcal{O}_v)\to H^0_{\mathrm{ab}}(\mathcal{O}_v,\mathcal{G})$ is surjective. The nullity of $H^1_{\mathrm{ab}}(\mathcal{O}_v,\mathcal{H})$ follows by devissage from the fact that for an $\mathcal{O}_v$-torus $\mathcal{T}$, we have $H^1(\mathcal{O}_v,\mathcal{T})=H^2(\mathcal{O}_v,\mathcal{T})=0$ (see for example [HS05, proof of Theorem 2.10]). Finally $H^1(\mathcal{O}_v,\mathcal{G}^{\mathrm{sc}})\simeq H^1(\mathbf{F}_v,\widetilde{G^{\mathrm{sc}}})$ is trivial by Lang's Theorem (here $\widetilde{G^{\mathrm{sc}}}$ is the reduction mod. v of the reductive group scheme $\mathcal{G}^{\mathrm{sc}}$; it is a connected linear group scheme over the residue field $\mathbf{F}_v$ of the curve E at v). This implies that the abelianization map $\mathcal{G}(\mathcal{O}_v)\to H^0_{\mathrm{ab}}(\mathcal{O}_v,\mathcal{G})$ is surjective because by definition of the abelianization map, there is an exact sequence

$$\mathcal{G}(\mathcal{O}_v)\to H^0_{\mathrm{ab}}(\mathcal{O}_v,\mathcal{G})\to H^1\left(\mathcal{O}_v,\mathcal{G}^{\mathrm{sc}}\right).$$

□

We need now to extend a few duality results of [DH20] to the three-term complex C_X :

Proposition 5.4. —

(1) *The group $H^1_c(U,C_{\mathcal{X}})$ is in the class $\mathcal{E}$ and $H^0(U,C_{\mathcal{X}})$ is of finite type.*
(2) *The groups $D^0(U,C_{\mathcal{X}})$ and $D^1(U,\widehat{C}_{\mathcal{X}})$ are finite.*
(3) *The group $H^2_c(U,\widehat{C}_{\mathcal{X}})$ is the dual of the discrete group $H^0(U,C_{\mathcal{X}})$ and*

$$H^1_c(U,C_{\mathcal{X}})_\wedge\simeq H^1\left(U,\widehat{C}_{\mathcal{X}}\right)^D.$$

Proof. —

(1) Consider the exact triangle

$$\mu_H[2]\to C_{\mathcal{X}}\to\mathcal{C}\to\mu_H[3], \tag{5.5}$$

which implies that $H^1_c(U,C_{\mathcal{X}})$ is an extension of $H^1_c(U,\mathcal{C})$ by the finite (cf. [DH19, Theorem 1.1]) group $H^3_c(U,\mu_H)$. Since $H^1_c(U,\mathcal{C})$ is in $\mathcal{E}$ ([DH20, Proposition 3.13]), so is $H^1_c(U,C_{\mathcal{X}})$.

As $H^3(U,\mu_H)=0$ and $H^2(U,\mu_H)$ is finite ([DH19, Theorem 1.1 and Corollary 4.9]), we also get that $H^0(U,C_{\mathcal{X}})$ is an extension of $H^0(U,\mathcal{C})$ (which is of finite type by [DH20, Proposition 3.6. (b)]) by a finite group, hence it is also of finite type.

(2) The finiteness of $D^0(U,C_{\mathcal{X}})$ follows from that of $D^0(U,\mathcal{C})$ (see [DH20, Lemma 4.15]) and that of $H^2(U,\mu_H)$ ([DH19, Corollary 4.9]) thanks to the commutative diagram with exact rows:

$$\begin{tikzcd}
H^2(U,\mu_H) \arrow[r] & H^0(U,C_{\mathcal{X}}) \arrow[r] \arrow[d] & H^0(U,\mathcal{C}) \arrow[d] \\
 & \bigoplus_{v\notin U} H^0(K_v,C_X) \arrow[r] & \bigoplus_{v\notin U} H^0(K_v,C).
\end{tikzcd}$$

Similarly $D^1(U,\widehat{C}_{\mathcal{X}})$ contains $D^1(U,\widehat{\mathcal{C}})$ as a finite index subgroup thanks to exact triangle

$$\widehat{\mu_H}[-2] \to \widehat{\mathcal{C}} \to \widehat{C_{\mathcal{X}}} \to \widehat{\mu_H}[-1],$$

and $D^1(U,\widehat{\mathcal{C}})$ is finite by [DH20, Theorem 5.2].

(3) We first show that Artin–Verdier duality induces an isomorphism $H^1_c(U,C_{\mathcal{X}})_\wedge \to H^1(U,\widehat{C}_{\mathcal{X}})^D$. To prove this, we use a devissage given by the triangle (5.5). By [DH20, Theorem 4.11 (b)], the discrete torsion group $H^i(U,\widehat{\mathcal{C}})$ is dual to the profinite group $H^{2-i}_c(U,\mathcal{C})_\wedge$ for $i=1,2$. There is a commutative diagram

$$\begin{tikzcd}
H^0_c(U,\mathcal{C})_\wedge \arrow[r] \arrow[d] & H^3_c(U,\mu_H) \arrow[r] \arrow[d] & H^1_c(U,C_{\mathcal{X}})_\wedge \arrow[r] \arrow[d] & H^1_c(U,\mathcal{C})_\wedge \arrow[r] \arrow[d] & 0 \\
H^2\left(U,\widehat{\mathcal{C}}\right)^D \arrow[r] & H^0(U,\widehat{\mu}_H)^D \arrow[r] & H^1\left(U,\widehat{\mathcal{C}}_{\mathcal{X}}\right)^D \arrow[r] & H^1\left(U,\widehat{\mathcal{C}}\right)^D \arrow[r] & 0
\end{tikzcd}$$

The second line is exact as the dual of an exact sequence of discrete torsion groups. The first line is exact as well thanks to [DH20, Lemma 3.12] because $H^3_c(U,\mu_H)$ is finite and $H^1_c(U,C_{\mathcal{X}})$ belongs to the class $\mathcal{E}$ (hence it injects into its completion). Now Artin–Mazur–Milne duality for μ_H (see [DH19, Theorem 1.1]) and for $\mathcal{C}$ (see [DH20, Theorem 4.11 (b)]) yield that the first, second, and fourth vertical maps are isomorphisms. The five lemma implies that the third one is also an isomorphism, as required.

The argument to prove the isomorphism $H^1_c(U,C_{\mathcal{X}})_\wedge \simeq H^1(U,\widehat{C}_{\mathcal{X}})^D$ is similar, using the commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & H^2_c\left(U,\widehat{\mathcal{C}}\right) \arrow[r] \arrow[d] & H^2_c\left(U,\widehat{\mathcal{C}_{\mathcal{X}}}\right) \arrow[r] \arrow[d] & H^1_c\left(U,\widehat{\mu}_H\right) \arrow[r] \arrow[d] & H^3_c\left(U,\widehat{\mathcal{C}}\right) \arrow[d] \\
0 \arrow[r] & H^0(U,\mathcal{C})^D \arrow[r] & H^0\left(U,\mathcal{C}_{\mathcal{X}}\right)^D \arrow[r] & H^2(U,\mu_H)^D \arrow[r] & H^{-1}\left(U,\mathcal{C}\right)^D .
\end{tikzcd}$$

Indeed the first and fourth vertical maps are isomorphisms by [DH20, Theorem 4.9 (a)], and so is the third by [DH19, Theorem 1.1]. ☐

Proposition 5.5. —

(1) *There is an exact sequence*

$$H^0(K,C_X) \to H^0\left(\mathbf{A}_K,C_X\right) \to H^1\left(K,\widehat{C}_X\right)^D.$$

(2) *There is an exact sequence*

$$H^1\left(K,\widehat{C}_X\right) \to H^1\left(\mathbf{A}_K,\widehat{C}_X\right) \to H^0(K,C_X)^D.$$

(3) *There is a local duality isomorphism*

$$H^0(K_v,C_X)_\wedge \xrightarrow{\sim} H^1\left(K_v,\widehat{C}_X\right)^D,$$

and the orthogonal of $H^1_{\mathrm{nr}}(K_v,\widehat{C}_X)$ *in the local duality is the image of* $H^0_{\mathrm{nr}}(K_v,C_X)$ *in* $H^0(K_v,C_X)_\wedge$.

Proof. —

(1) We extend the proof of the exactness of the second line of Poitou–Tate exact sequence in [DH20, Theorem 5.7]. Following the same method, it is sufficient to extend the first part of [DH20, Lemma 5.6 (a)] to the complex C_X. Namely, it remains to show the exactness of

$$H^0(U,C_{\mathcal{X}}) \to \mathcal{P}^0(U,C_X) \to \left(H^1\left(K,\widehat{C}_X\right)/\text{Ш}^1\left(K,\widehat{C}_X\right)\right)^D.$$

The proof is exactly the same as for C, applying [DH20, Lemma 2.2] to the complex C_X and using the three following facts (proven in Proposition 5.4): for every non-empty Zariski open subset $V \subset U$, the group $H^1_c(V,C_{\mathcal{X}})$ is in the class $\mathcal{E}$, the group $D^0(V,C_{\mathcal{X}})$ is finite, and Artin–Verdier duality induces an isomorphism $H^1_c(V,C_{\mathcal{X}})_\wedge \simeq H^1(V,\widehat{C}_{\mathcal{X}})^D$.

(2) Similarly, it is sufficient to extend the second part of [DH20, Lemma 5.6 (b)], that is to show the exactness of

$$H^1\left(U,\widehat{C}_{\mathcal{X}}\right) \to \mathcal{P}^1\left(U,\widehat{C}_X\right) \to H^0(K,C_X)^D.$$

Applying again [DH20, Lemma 2.2] to $\widehat{C}_X$, one just has to check the following properties:

- for every non-empty Zariski open subset $V \subset U$, we have

$$H^2_c\left(V,\widehat{C}_{\mathcal{X}}\right) \simeq \left(H^0(V,C_{\mathcal{X}})_\wedge\right)^D = \left(H^0(V,C_{\mathcal{X}}\right)^D.$$

  This holds thanks to Proposition 5.4 (the finite type group $H^0(V,C_{\mathcal{X}})$ has same dual as $H^0(V,C_{\mathcal{X}})_\wedge$).
- the group $D^1(U,\widehat{C}_{\mathcal{X}})$ is finite, which is also proven in Proposition 5.4.

(3) Using the exact triangle (5.5) and the vanishing of $H^3(K_v,\mu_H)$ ([Mil06, Propostion III.6.4]), there is a commutative diagram with exact rows (the completed first line remains exact because $H^2(K_v,\mu_H)$ is finite by [Mil06, Exemple III.6.7], and the second line is obtained by dualizing an exact sequence of discrete torsion groups):

$$\begin{tikzcd}
H^{-1}(K_v,C)_\wedge \arrow[r] \arrow[d] & H^2(K_v,\mu_H) \arrow[r] \arrow[d] & H^0(K_v,C_X)_\wedge \arrow[r] \arrow[d] & H^0(K_v,C)_\wedge \arrow[r] \arrow[d] & 0 \\
H^2\left(K_v,\widehat{C}\right)^D \arrow[r] & H^0(K_v,\widehat{\mu}_H)^D \arrow[r] & H^1\left(K_v,\widehat{C}_X\right)^D \arrow[r] & H^1\left(K_v,\widehat{C}\right)^D \arrow[r] & 0\,.
\end{tikzcd}$$

Since the first, second, and fourth vertical map are isomorphisms by [Mil06, Theorem III.6.10] and [Dem11b, Theorem 3.1], so is the third vertical map.

It remains to show that the map

$$H^0(K_v,C_X)_\wedge/H^0_{\mathrm{nr}}(K_v,C_X) \to H^1\left(\mathcal{O}_v,\widehat{C}_{\mathcal{X}}\right)^D$$

is injective. There is a commutative diagram with exact lines:

$$\begin{tikzcd}
 & & H^0\left(\mathcal{O}_v, C_{\mathcal{X}}\right) \arrow[r] \arrow[d] & H^0\left(\mathcal{O}_v, \mathcal{C}\right) \arrow[r] \arrow[d] & 0 \\
H^{-1}(K_v, C)_\wedge \arrow[r] \arrow[d] & H^2\left(K_v, \mu_H\right) \arrow[r] \arrow[d] & H^0(K_v, C_X)_\wedge \arrow[r] \arrow[d] & H^0\left(K_v, C\right)_\wedge \arrow[r] \arrow[d] & 0 \\
H^2\left(\mathcal{O}_v, \widehat{\mathcal{C}}\right)^D \arrow[r] & H^0\left(\mathcal{O}_v, \widehat{\mu}_H\right)^D \arrow[r] & H^1\left(\mathcal{O}_v, \widehat{C}_{\mathcal{X}}\right)^D \arrow[r] & H^1\left(\mathcal{O}_v, \widehat{\mathcal{C}}\right)^D \arrow[r] & 0\,.
\end{tikzcd}$$

The fourth column is exact and the map $H^{-1}(K_v, C)_\wedge \to H^2(\mathcal{O}_v, \widehat{\mathcal{C}})^D$ is surjective ([Dem11b, Theorems 3.1. and 3.3]). Since $\widehat{\mu}_H$ is a finite group scheme, we have $H^0(\mathcal{O}_v, \widehat{\mu}_H) = H^0(K_v, \widehat{\mu}_H)$, hence the map $H^2(K_v, \mu_H) \to H^0(\mathcal{O}_v, \widehat{\mu}_H)^D$ is injective by [Mil06, Theorem III.6.10]. The required result follows by diagram chasing. $\square$

Proposition 5.6. — *There is an exact sequence*

$$\begin{aligned}
H^0\left(K, C_X\right) \to H^0\left(\mathbf{A}_K, C_X\right) \to \left(H^1\left(K, \widehat{C}_X\right)/\text{Ш}^1\left(K, \widehat{C}_X\right)\right)^D \\
\to H^{-1}\left(K, \widehat{C}_X\right)^{\star}_{\wedge}/H^{-1}\left(K, \widehat{C}_X\right)^{\star} \to 0\,,
\end{aligned} \tag{5.6}$$

where the last non-trivial map is defined via the natural map

$$\left(H^1\left(K, \widehat{C}_X\right)/\text{Ш}^1\left(K, \widehat{C}_X\right)\right)^D \to \left(H^1\left(K, \widehat{C}\right)/\text{Ш}^1\left(K, \widehat{C}\right)\right)^D$$

and the map

$$\begin{aligned}
\left(H^1\left(K, \widehat{C}\right)/\text{Ш}^1\left(K, \widehat{C}\right)\right)^D \xrightarrow{\partial_C} H^{-1}\left(K, \widehat{C}\right)^{\star}_{\wedge}/H^{-1}\left(K, \widehat{C}\right)^{\star} \\
\simeq H^{-1}\left(K, \widehat{C_X}\right)^{\star}_{\wedge}/H^{-1}\left(K, \widehat{C_X}\right)^{\star}.
\end{aligned}$$

Proof. — We consider diagram (4.1) comparing C_X and C. Applying cohomology, we get a commutative diagram

$$\begin{tikzcd}
H^2\left(\mathbf{A}_K, \mu_H\right) \arrow[r] \arrow[d] & \left(H^0\left(K, \widehat{\mu_H}\right)\right)^D \arrow[rr] \arrow[d] & & 0 \\
H^0\left(\mathbf{A}_K, C_X\right) \arrow[r] \arrow[d] & \left(H^1\left(K, \widehat{C}_X\right)/\text{Ш}^1\left(K, \widehat{C}_X\right)\right)^D \arrow[r] \arrow[d] & H^{-1}\left(K, \widehat{C}_X\right)^{\star}_{\wedge}/H^{-1}\left(K, \widehat{C}_X\right)^{\star} \arrow[r] \arrow[d, "\simeq"] & 0 \\
H^0\left(\mathbf{A}_K, C\right) \arrow[r] \arrow[d] & \left(H^1\left(K, \widehat{C}\right)/\text{Ш}^1\left(K, \widehat{C}\right)\right)^D \arrow[r] \arrow[d] & H^{-1}\left(K, \widehat{C}\right)^{\star}_{\wedge}/H^{-1}\left(K, \widehat{C}\right)^{\star} \arrow[r] & 0 \\
0 & 0 & &
\end{tikzcd} \tag{5.7}$$

Local duality for μ_H (cf. [Čes15, Proposition 4.10 (b)] for instance) and Proposition 5.2 (2) imply that the first row and the last one are exact. The exact triangles of (4.1) implies that the first column is exact.

The second column is exact, being the dual of the sequence of discrete groups

$$0 \to H^1\left(K, \widehat{C}\right)/\text{Ш}^1\left(K, \widehat{C}\right) \to H^1\left(K, \widehat{C}_X\right)/\text{Ш}^1\left(K, \widehat{C}_X\right) \to H^0\left(K, \widehat{\mu_H}\right),$$

which is exact, thanks to the exact sequence (over K and K_v)

$$0 \to H^1\left(K, \widehat{C}\right) \to H^1\left(K, \widehat{C}_X\right) \to H^0\left(K, \widehat{\mu_H}\right).$$

Now an easy diagram chase implies the exactness of the second line of (5.7). The exactness of (5.6) then follows from Proposition 5.5 (1). □

Proposition 5.7. —

(1) *The following sequence:*

$$H^0\left(K, C_X\right)_\wedge \to H^0\left(\mathbf{A}_K, C_X\right)_\wedge \to \left(H^1\left(K, \widehat{C}_X\right)/\text{Ш}^1\left(K, \widehat{C}_X\right)\right)^D \to 0 \tag{5.8}$$

is exact.

(2) *Let S be a finite set of places that does not meet U. Set*

$$H^1_S\left(K, \widehat{C}_X\right) = \ker\left[H^1\left(K, \widehat{C}_X\right) \to \bigoplus_{v \in S} H^1\left(K_v, \widehat{C}_X\right)\right].$$

Then the sequence

$$H^0(K, C_X)_\wedge \to H^0\left(\mathbf{A}^S_K, C_X\right)_\wedge \to \left(H^1_S\left(K, \widehat{C}_X\right)/\text{Ш}^1\left(K, \widehat{C}_X\right)\right)^D \to 0 \tag{5.9}$$

is exact.

(3) *There is an exact sequence*

$$H^0\left(U, C_{\mathcal{X}}\right)_\wedge \to \mathbf{P}^0_S\left(U, C_{\mathcal{X}}\right)_\wedge \to \left(H^1_S(K, \widehat{C}_X)/\text{Ш}^1\left(K, \widehat{C}_X\right)\right)^D.$$

Proof. —

(1) Since $(H^1(K, \widehat{C}_X)/\text{Ш}^1(K, \widehat{C}_X))^D$ is profinite and

$$H^{-1}\left(K, \widehat{C}_X\right)^\star_\wedge / H^{-1}\left(K, \widehat{C}_X\right)^\star$$

is uniquely divisible, the surjectivity of the map

$$H^0\left(\mathbf{A}_K, C_X\right)_\wedge \to \left(H^1\left(K, \widehat{C}_X\right)/\text{Ш}^1\left(K, \widehat{C}_X\right)\right)^D$$

follows immediately from Proposition 5.6 and [DH20, Lemma 3.12 (b)]. Applying Proposition 5.5 (2) and taking into account that $H^1(K, \widehat{C}_X)$ is torsion, we get an exact sequence

$$0 \to \text{Ш}^1\left(K, \widehat{C}_X\right) \to H^1\left(K, \widehat{C}_X\right) \to H^1\left(\mathbf{A}_K, \widehat{C}_X\right)_{\text{tors}} \to \left(H^0\left(K, C_X\right)_\wedge\right)^D.$$

Indeed $(H^0(K, C_X)_\wedge)^D = (H^0(K, C_X)^D)_{\text{tors}}$ (cf. [DH20, Remark 5.9]). Dualizing this exact sequence now yields the result, thanks to Proposition 5.5 (3).

(2) Dualizing the exact sequence of discrete torsion groups

$$0 \to H^1_S\left(K, \widehat{C}_X\right) \to H^1\left(K, \widehat{C}_X\right) \to \bigoplus_{v \in S} H^1\left(K_v, \widehat{C}_X\right).$$

and taking into account the local duality isomorphism $H^0(K_v, C_X)_\wedge \xrightarrow{\sim} H^1(K_v, \widehat{C}_X)^D$, one gets a commutative diagram

$$\begin{tikzcd}
& \bigoplus_{v \in S} H^0(K_v, C_X)_\wedge \arrow[r, "="] \arrow[d, "i"] & \bigoplus_{v \in S} H^0(K_v, C_X)_\wedge \arrow[d] & \\
H^0(K, C_X)_\wedge \arrow[r] \arrow[d, "="] & H^0(\mathbf{A}_K, C_X)_\wedge \arrow[r] \arrow[d, "p"] & \left(H^1\left(K, \widehat{C}_X\right)/Ш^1\left(K, \widehat{C}_X\right)\right)^D \arrow[r] \arrow[d] & 0 \\
H^0(K, C_X)_\wedge \arrow[r] & H^0\left(\mathbf{A}_K^S, C_X\right)_\wedge \arrow[r] \arrow[d] & \left(H^1_S\left(K, \widehat{C}_X\right)/Ш^1\left(K, \widehat{C}_X\right)\right)^D \arrow[d] & \\
& 0 & 0\,, &
\end{tikzcd}$$

where the second line (by (1)) and the columns are exact (the left one because $H^0(K_v, C_X)$ is in the class $\mathcal{E}$ by [DH20, Proposition 3.13] and exact triangle (5.5); so the sequence remains exact after completion thanks to loc. cit., Lemma 3.12). A simple diagram chase implies that the bottom line is exact, which proves (2).

(3) For $v \in U$ and $i \in \mathbf{Z}$, set

$$H^i_r\left(K_v, \widehat{C}_X\right) = H^i\left(K_v, \widehat{C}_X\right)/H^i_{\mathrm{nr}}\left(K_v, \widehat{C}_X\right)$$

(and similarly for C_X). Consider the commutative diagram:

$$\begin{tikzcd}
H^1_S\left(K, \widehat{C}_X\right) \arrow[r] & \bigoplus_{v \in U} H^1_r\left(K_v, \widehat{C}_X\right) \oplus \bigoplus_{v \notin (S \cup U)} H^1\left(K_v, \widehat{C}_X\right) \arrow[r] & H^0(U, C_{\mathcal{X}})^D \\
H^1_S\left(K, \widehat{C}_X\right) \arrow[r] \arrow[u, "="] & \prod'_{v \notin S} H^1\left(K_v, \widehat{C}_X\right) \arrow[r] \arrow[u, "p"] & H^0(K, C_X)^D \arrow[u] \\
& \prod_{v \in U} H^1_{\mathrm{nr}}\left(K_v, \widehat{C}_X\right) \arrow[r, "="] \arrow[u] & \prod_{v \in U} H^1_{\mathrm{nr}}\left(K_v, \widehat{C}_X\right)\,. \arrow[u]
\end{tikzcd}$$

The middle column is obviously exact. The right column is exact as well, because by Proposition 5.5 (3), it is the dual of the sequence of discrete groups

$$H^0(U, C_{\mathcal{X}}) \to H^0(K, C_X) \to \bigoplus_{v \in U} H^0_r(K_v, C_X),$$

which is exact by [DH20, Proposition 2.1]. It follows immediately from Proposition 5.5 (2) that the second line of the diagram is exact. Since the map p is obviously surjective, a diagram chase shows that the first line of the diagram is exact as well. Dualizing it (and observing that $H^0(U, C_{\mathcal{X}})$ is an abelian group of finite type by Proposition 5.4 (1)) yields (thanks to Proposition 5.5 (3)) the exactness of

$$H^0(U, C_{\mathcal{X}})_\wedge \to \mathbf{P}^0_S(U, C_{\mathcal{X}})_\wedge \to H^1_S\left(K, \widehat{C}_X\right)^D .$$

Since the canonical map $\mathbf{P}^0_S(U, C_{\mathcal{X}})_\wedge \to H^1_S(K, \widehat{C}_X)^D$ factors through

$$\left(H^1_S\left(K, \widehat{C}_X\right)/Ш^1\left(K, \widehat{C}_X\right)\right)^D,$$

the result is proven. □

Recall that for a finite (possibly empty) set of places S of K, we have the Brauer–Manin pairing

$$\mathrm{BM} : X\left(\mathbf{A}_K^S\right) \times \operatorname{Br} X \to \mathbf{Q}/\mathbf{Z}, \quad ((P_v)_{v\notin S}, \alpha) \mapsto \sum_{v\notin S} \alpha(P_v).$$

By global class field theory, elements of $X(K) \subset X(\mathbf{A}_K^S)$ are orthogonal to $\operatorname{Br}_S(X)$ for this pairing; in particular when $S = \emptyset$, elements of $X(K)$ are orthogonal to $\operatorname{Br}_e X$ (hence to $\operatorname{Br} X$). By continuity of the pairing, the same holds for the closure $\overline{X(K)}^S$ of $X(K)$ in $X(\mathbf{A}_K^S)$ for the strong topology. The following theorem gives various converse statements.

Theorem 5.8. — *Let $X = G/H$ be a homogeneous space of a reductive group G with H reductive. Set $U(X) := K[X]^*/K^*$. Assume that G^{sc} satisfies strong approximation outside S_0 (finite set of places).*

(1) *There is a natural exact sequence of pointed topological spaces:*

$$1 \to \rho\left(\overline{G^{\mathrm{sc}}(K)\cdot G^{\mathrm{sc}}\left(K_{S_0}\right)}\right)\cdot X(K) \to X(\mathbf{A}_K) \xrightarrow{BM} (\operatorname{Br}_{1,e}(X,G)/Б(X))^D \xrightarrow{\partial} U(X)^\star{}_\wedge/U(X)^\star \to 1\,.$$

In particular,

$$X(\mathbf{A}_K)^{\operatorname{Br} X} = \rho\left(\overline{G^{\mathrm{sc}}(K)\cdot G^{\mathrm{sc}}\left(K_{S_0}\right)}\right)\cdot X(K) \quad \textit{and} \quad X\left(\mathbf{A}_K^{S_0}\right)^{\operatorname{Br} X} \subset \overline{X(K)}^{S_0}.$$

(2) *If S is a non-empty finite set of places, there is a natural exact sequence of pointed topological spaces:*

$$1 \to \overline{\rho\left(G^{\mathrm{sc}}\left(K_{S_0}\right)\right)\cdot X(K)}^S \to X\left(\mathbf{A}_K^S\right) \xrightarrow{BM} (\operatorname{Br}_S(X,G)/Б(X))^D \to 1\,.$$

In particular, $X(\mathbf{A}_K^S)^{\operatorname{Br}_S X} = \overline{\rho(G^{\mathrm{sc}}(K_{S_0}))\cdot X(K)}^S$.

If $S_0 \subset S$, we get an exact sequence

$$1 \to \overline{X(K)}^S \to X\left(\mathbf{A}_K^S\right) \xrightarrow{BM} (\operatorname{Br}_S(X,G)/Б(X))^D \to 1\,.$$

Proof. — As earlier, one can assume that the group G is quasi-trivial, up to replacing G by a flasque resolution

$$1 \to S \to G' \to G \to 1.$$

Indeed $\operatorname{Pic}\bar{S} = 0$ (since S is a torus), hence $\operatorname{Br}\bar{G} \hookrightarrow \operatorname{Br}\bar{G}'$, which implies $\operatorname{Br}_1(X,G) = \operatorname{Br}_1(X,G')$ by [San81, Proposition 6.10]. Throughout the proof, we use Theorem 3.7 to translate results concerning complexes of tori and cup-products to results concerning homogeneous spaces and Brauer–Manin obstructions.

(1) By [Dem11c, Theorem 4.14], the group $U(X)$ is isomorphic to $H^{-1}(K,\widehat{C_X})$. By Theorems 3.7 and 5.6, there is a commutative diagram with exact second row:

$$\begin{tikzcd}
X(K) \arrow[r] \arrow[d, "\mathrm{ab}'_K"] & X(\mathbf{A}_K) \arrow[r, "\mathrm{BM}"] \arrow[d, "\mathrm{ab}'_{\mathbf{A}_K}"] & (\operatorname{Br}_{1,e}(X,G)/\text{Б}(X))^D \arrow[r] \arrow[d, "\Phi_X^D", "\sim"'] & U(X)^{\star}{}_{\wedge}/U(X)^{\star} \arrow[r] \arrow[d, "\sim"'] & 1 \\
H^0(K,C_X) \arrow[r] & H^0\left(\mathbf{A}_K,C_X\right) \arrow[r] & H^1\left(K,\widehat{C_X}\right)^D \arrow[r] & H^{-1}\left(K,\widehat{C_X}\right)^{\star}{}_{\wedge}/H^{-1}\left(K,\widehat{C_X}\right)^{\star} \arrow[r] & 1
\end{tikzcd}$$

By Proposition 3.5 and Lemma 5.3, the maps ab'_K and $\mathrm{ab}'_{\mathbf{A}_K}$ are surjective. By diagram chasing, the sequence that consists of the last three non-trivial terms on the first line is also exact. Furthermore, every element x of $X(K) \subset X(\mathbf{A}_K)$ satisfies $\mathrm{BM}(x) = 0$ by class field theory, and the same holds for an element of $X(\mathbf{A}_K)$ of the form $(g_v.x)$ with $(g_v) \in \rho(G^{\mathrm{sc}}(\mathbf{A}_K))$ thanks to the commutativity of the diagram and the property $\mathrm{ab}'(g_v.x) = \mathrm{ab}'(x)$ for every place v of K (Proposition 3.3). It remains to show conversely that every $(P_v) \in X(\mathbf{A}_K)$ such that $\mathrm{BM}((P_v)) = 0$ comes from $\rho(\overline{G^{\mathrm{sc}}(K) \cdot G^{\mathrm{sc}}(K_{S_0})}) \cdot X(K)$ The diagram and the surjectivity of ab'_K imply that there exists $x \in X(K)$ such that

$$\mathrm{ab}'_{\mathbf{A}_K}(x) = \mathrm{ab}'_{\mathbf{A}_K}(P_v) \in H^0(K_v, C_X)$$

for every place v. By Proposition 3.5, there exists for each v an element $g_v \in \rho(G^{\mathrm{sc}}(K_v))$ such that $P_v = g_v.x$, and we can assume that $g_v \in \mathcal{G}^{\mathrm{sc}}(\mathcal{O}_v)$ for $v \notin S$, where $S \supset S_0$ is some finite set of places. Since G^{sc} satisfies strong approximation outside S_0, we finally obtain that (P_v) belongs to $\rho\left(\overline{G^{\mathrm{sc}}(K) \cdot G^{\mathrm{sc}}(K_{S_0})}\right) \cdot X(K)$ as required. The two other assertions in 1. follow immediately.

(2) We follow the proof of [Dem13, Theorem 6.1].

Up to shrinking U, we can assume that it does not meet $(S_0 \cup S)$. We consider the following commutative diagram:

(5.10)

$$
\begin{tikzcd}
 & & & \mathcal{P}_S(U,C_H)_\wedge \arrow[rr] \arrow[dd] & & H^1_S\left(K,\widehat{C}_X\right)^D \arrow[dd] \\
 & & \mathcal{P}_S(U,H) \arrow[rr] \arrow[ur] \arrow[dd] & & (\mathrm{Br}_S(H))^D \arrow[ur] \arrow[dd] & \\
 & H^0(U,C_G)_\wedge \arrow[rr] \arrow[dd] & & \mathcal{P}^0_S(U,C_G)_\wedge \arrow[rr] \arrow[dd] & & H^1_S\left(K,\widehat{C}_G\right)^D \arrow[dd] \\
G(U) \arrow[rr] \arrow[ur] \arrow[dd] & & \mathcal{P}^0_S(U,G) \arrow[rr] \arrow[ur] \arrow[dd] & & (\mathrm{Br}_S(G))^D \arrow[ur] \arrow[dd] & \\
 & H^0(U,C_X)_\wedge \arrow[rr] \arrow[dd] & & \mathcal{P}^0_S(U,C_X)_\wedge \arrow[rr] \arrow[dd] & & H^1_S\left(K,\widehat{C}_X\right)^D \\
X(U) \arrow[rr] \arrow[ur] \arrow[dd] & & \mathcal{P}^0_S(U,X) \arrow[rr] \arrow[ur] \arrow[dd] & & (\mathrm{Br}_S(X,G))^D \arrow[ur] & \\
 & H^1(U,C_H) \arrow[rr] & & \mathcal{P}^0_S(U,C_H) & & \\
H^1(U,H) \arrow[rr] \arrow[ur, "\sim"] \arrow[d] & & \mathcal{P}^1_S(U,H) \arrow[ur, "\sim"] & & & \\
1 & & & & & .
\end{tikzcd}
$$

The definition of the maps $H^0(U,C_X)_\wedge \to H^1(U,C_H)$ and $\mathcal{P}^0_S(U,C_X)_\wedge \to \mathcal{P}^1_S(U,C_H)$ is a consequence of [Dem13, Lemma 6.5] (the proof works the same in the function field context). Using this lemma, we see that the columns in this diagram are exact. The surjectivity of $X(U) \to H^1(U,H)$ is a consequence of the vanishing of $H^1(U,G)$ for U sufficiently small (Hilbert 90, together with Nisnevich Theorem as proven in [Gil02, Theorem 5.1]).

In addition, the fifth line of this diagram is exact by Proposition 5.7 (3), and the maps $H^1(U,H) \to H^1(U,C_H)$ and $\mathcal{P}^1_S(U,H) \to \mathcal{P}^1_S(U,C_H)$ are bijections (see [GA12, Theorem 5.5 and Example 5.4 (iii)] and [Gil02, Theorem 5.1]).

Now, the proof of the second point of the Theorem is a diagram chase in diagram (5.10), following exactly the argument in [Dem13, Theorem 6.1]. In particular, one uses the fact that $\mathcal{P}^0_S(U,C_H)_\wedge$ and $\mathcal{P}^0_S(U,C_H)$ (hence $\mathcal{P}^0_S(H)$) have the same image inside $H^1_S(K,\widehat{C}_H)^D$ which is a consequence of the following proposition:

Proposition 5.9. — *Let $\mathcal{D} = [\mathcal{T}_1 \to \mathcal{T}_2]$ be a complex (with $\mathcal{T}_2$ in degree* 0*) of flat, separated and finite type commutative group schemes over the affine curve $E - S$, such that the restriction of $\mathcal{T}_1$ and $\mathcal{T}_2$ to U are tori. Let $D = [T_1 \to T_2]$ be the generic fibre of $\mathcal{D}$.*

(a) *The group $H^0(U,\mathcal{D})\backslash \mathcal{P}^0_S(\underline{U},D)/\prod_{v\notin S} H^0(\mathcal{O}_v,\mathcal{D})$ is finite.*

(b) *The group $\mathcal{P}^0_S(U,\mathcal{D})/H^0(\overline{U},\mathcal{D})$ is compact.*

Proof. —

(a) Let S_1 be the finite set of places of K that are neither in U nor in S. Recall ([Mil06, Corollary 2.3]) that the group $H^1(K_v, T)$ is finite for all places $v \in \Omega_K$. If $\mathcal{T}_1 = 0$, the result follows immediately from the finiteness of the class group of a torus, as proven in [Con12, Example 1.3.2] (recall that $S \neq \emptyset$). In the general case, we have $H^1(\mathcal{O}_v, \mathcal{T}_1) = 0$ for $v \in U$, whence an exact sequence

$$\mathcal{P}^0_S(U, \mathcal{T}_2) \to \mathcal{P}^0_S(U, D) \to \prod_{v \in S_1} H^1(K_v, T_1),$$

which shows in particular that the image of $\mathcal{P}^0_S(U, \mathcal{T}_2)/\prod_{v \notin S} H^0(\mathcal{O}_v, \mathcal{T}_2)$ is of finite index in $\mathcal{P}^0_S(U, \mathcal{D})/\prod_{v \notin S} H^0(\mathcal{O}_v, \mathcal{D})$. Thus one reduces to the already known case $\mathcal{T}_1 = 0$.

(b) follows from (a) and the compactness of $\prod_{v \notin S} H^0(\mathcal{O}_v, \mathcal{D})$. □

To finish the proof of the Theorem 5.8, one deduces the surjectivity of the map

$$X\left(\mathbf{A}_K^S\right) \to \left(\mathrm{Br}_S(X, G)/\text{Б}(X)\right)^D$$

from exact sequence (5.9) and from the surjectivity of $X(\mathbf{A}_K^S) \to H^0(\mathbf{A}_K^S, C_X)$ (which follows from Lemma 5.3 and Proposition 3.5). □

Remark 5.10. —

(1) In the special case when $X = G$ is a reductive group, we get exact sequences of topological groups:

$$1 \to \rho\left(\overline{G^{\mathrm{sc}}(K) \cdot G^{\mathrm{sc}}(K_{S_0})}\right) \cdot G(K) \to G(\mathbf{A}_K) \xrightarrow{BM} \left(\mathrm{Br}_{1,e}\, G/\text{Б}(G)\right)^D \xrightarrow{\partial} \widehat{G}(K)^{\star}{}_{\wedge}/\widehat{G}(K)^{\star} \to 1\,.$$

$$1 \to \overline{\rho\left(G^{\mathrm{sc}}\left(K_{S_0}\right)\right) \cdot G(K)}^S \to G\left(\mathbf{A}_K^S\right) \xrightarrow{BM} \left(\mathrm{Br}_{1,S}\, G/\text{Б}(G)\right)^D \to 1\,.$$

The fact that BM is a group morphism in this situation follows from the same argument as in Corollary 4.5.

(2) In addition, using [DH20, Theorem 5.10], it is straightforward to continue the previous exact sequence as follows, for any non-empty finite set of places S:

$$\begin{tikzcd}
1 \arrow[r] & \overline{\rho\left(G^{\mathrm{sc}}\left(K_{S_0}\right)\right) \cdot G(K)}^S \arrow[r] & G\left(\mathbf{A}_K^S\right) \arrow[r, "BM"] & \left(\mathrm{Br}_{1,S}\, G\right)^D \arrow[d] \\
& \left(\mathrm{Pic}\, G\right)^D & \bigoplus_v H^1\left(K_v, G\right) \arrow[l] & H^1(K, G)\,, \arrow[l]
\end{tikzcd}$$

where the pairing $H^1(K_v, G) \times \mathrm{Pic}(G)$ can be defined via the natural isomorphism between $\mathrm{Pic}(G)$ and the group $\mathrm{Ext}^1_K(G, \mathbf{G}_m)$ of central extensions of G by $\mathbf{G}_m$ (see for instance [CT08, Corollary 5.7 and Proposition 8.2]). One can even extend this exact sequence to a 9-term Poitou–Tate exact sequence involving non-abelian H^2, using [DH20, Theorem 5.10], following the methods of [Dem11a, Theorem 5.1]. See also [Bor98, Theorem 5.16] for a similar statement over a number field.

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