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CENTRAL EXTENSIONS AND BOUNDED COHOMOLOGY

EXTENSIONS CENTRALES ET
COHOMOLOGIE BORNÉE

ABSTRACT. — It was shown by Gersten that a central extension of a finitely generated group is quasi-isometrically trivial provided that its Euler class is bounded. We say that a finitely generated group G satisfies Property QITB (quasi-isometrically trivial implies bounded) if the Euler class of any quasi-isometrically trivial central extension of G is bounded. We exhibit a finitely generated group G which does not satisfy Property QITB. This answers a question by Neumann and Reeves, and provides partial answers to related questions by Wienhard and Blank. We also prove that Property QITB holds for a large class of groups, including amenable groups, right-angled Artin groups, relatively hyperbolic groups with amenable peripheral subgroups, and 3-manifold groups.

Finally, we show that Property QITB holds for every finitely presented group if a conjecture by Gromov on bounded primitives of differential forms holds as well.

RÉSUMÉ. — Gersten a montré qu'une extension centrale d'un groupe de type fini est quasi-isométriquement triviale lorsque sa classe d'Euler est bornée. On dit qu'un groupe G de type fini vérifie la propriété QITB (quasi-isométriquement trivial implique borné) si la classe d'Euler de toute extension centrale quasi-isométriquement triviale de G est bornée. Nous exhibons un groupe de type fini G qui ne satisfait pas la propriété QITB. Cela répond à une question de Neumann et Reeves et fournit des réponses partielles à des questions reliées de Wienhard et Blank. Nous prouvons également que la propriété QITB est satisfaite par une grande classe de groupes, contenant les groupes moyennables, les groupes d'Artin à angle droit, les groupes

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relativement hyperboliques avec des sous-groupes périphériques moyennables et les groupes de 3-variétés.

Enfin, nous montrons que la propriété QITB est valable pour tout groupe de présentation finie si une conjecture de Gromov sur les primitives bornées de formes différentielles est également valable.

1. Introduction

Let

$$1 \longrightarrow Z \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1$$

be a central extension of groups, where G and Z (hence, E) are finitely generated. Any such extension defines a cohomology class $\omega \in H^2(G, Z)$, which will be called the *Euler class* of the extension. It is well known that the Euler class completely determines the isomorphism class of a central extension (see Section 2), and it is natural to investigate which geometric features it encodes.

We say that a class $\omega \in H^2(G, Z)$ is *bounded* if it lies in the image of the comparison map $H_b^2(G, Z) \rightarrow H^2(G, Z)$, i.e. if it can be described by a bounded cocycle (see Section 2 for the precise definition). Following [Ger, Ger92, KL01], we say that the extension of finitely generated groups

$$1 \longrightarrow Z \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1$$

is *quasi-isometrically trivial* if there exists a quasi-isometry

$$f: E \longrightarrow Z \times G$$

such that the following diagram commutes, up to bounded error:

$$\begin{array}{ccc} E & \xrightarrow{\pi} & G \\ \downarrow f & & \downarrow \text{Id} \\ Z \times G & \xrightarrow{\pi_2} & G \end{array}.$$

Here $\pi_2: Z \times G \rightarrow G$ is the projection on the second factor, and, for clarity, the diagram commuting up to bounded error means that given a word metric d_G on G we have $\sup_{h \in E} d_G(\pi(h), \pi_2(f(h))) < +\infty$.

It was first shown by Gersten [Ger92, Ger] that a central extension is quasi-isometrically trivial provided that its Euler class is bounded. In this paper we address the following:

QUESTION 1.1. — *Is the Euler class of a quasi-isometrically trivial extension necessarily bounded?*

Question 1.1 was first asked by Neumann and Reeves in [NR96, NR97] (see also [Why10, Remark 2.6]). Moreover, it turns out to be equivalent to questions on ℓ^∞ -cohomology posed in [Bla14, Wie12] (see Question 1.13 and Proposition 1.14), and related to a Conjecture by Gromov (see Conjecture 1.18 and Corollary 1.20). We provide here a negative answer to Question 1.1:

THEOREM 1.2. — *There exists a quasi-isometrically trivial central extension of a finitely generated group G by \mathbb{Z} whose Euler class is not bounded.⁽¹⁾*

DEFINITION 1.3. — *Let G be a finitely generated group. Then G satisfies Property QITB (“quasi-isometrically trivial \Rightarrow bounded”) if the following condition holds: for every finitely generated abelian group Z , the Euler class of any quasi-isometrically trivial central extension of G by Z is bounded.*

Theorem 1.2 states that there exists a finitely generated group G which does not satisfy Property QITB. Nevertheless, we show that Property QITB holds for large families of groups:

THEOREM 1.4. — *Suppose the finitely generated group G belongs to one of the following families:*

- (1) *amenable groups;*
- (2) *relatively hyperbolic groups with respect to a finite family of amenable peripheral subgroups (in particular, hyperbolic groups);*
- (3) *right-angled Artin groups;*
- (4) *fundamental groups of compact orientable 3-manifolds.*

Then G satisfies Property QITB.

For amenable and right-angled Artin groups we can prove more, at least when we extend by a torsion-free abelian group (meaning that we consider a central extension where the group on the left is torsion-free abelian). We call a central extension of G *virtually trivial* if it pulls back to a trivial central extension on a finite-index subgroup of G , see Definition 5.12. It is easy to check that virtually trivial central extensions are quasi-isometrically trivial.

THEOREM 1.5. — *Let G be a finitely generated amenable group, or a finitely generated right-angled Artin group. Then a central extension of G by a finitely generated torsion-free abelian group is quasi-isometrically trivial if and only if it is virtually trivial.*

We remark that the torsion-freeness assumption cannot be dropped, see Remark 5.16. However, even without that assumption, one can replace “if and only if it is virtually trivial” with “if and only if its Euler class has finite order”, see Theorem 5.15.

Our strategy to prove Theorems 1.4 and 1.5 is as follows. We first show that they hold for amenable groups. Then, for every group G , we introduce the subspace $H_2^{\text{am}}(G, \mathbb{R}) \subseteq H_2(G, \mathbb{R})$ generated by those elements of $H_2(G, \mathbb{R})$ which lie in the image of a map $f_*: H_2(A, \mathbb{R}) \rightarrow H_2(G, \mathbb{R})$, where $f: A \rightarrow G$ is a homomorphism and A is amenable. Whenever $H_2^{\text{am}}(G, \mathbb{R}) = H_2(G, \mathbb{R})$ we are able to prove that any quasi-isometrically trivial extension by a torsion-free abelian group is virtually trivial. In order to prove Property QITB for all the groups listed in the statement of Theorem 1.4, we then observe that such property holds whenever every element in the annihilator of $H_2^{\text{am}}(G, \mathbb{R})$ is bounded.

⁽¹⁾Since the first version of this paper appeared on ArXiv, a finitely presented example of a group with the same property has been found [AM22].

Further examples of groups satisfying Property QITB may be built thanks to the following results:

PROPOSITION 1.6. — *Let G_1, G_2 be groups satisfying Property QITB. Then the direct product $G_1 \times G_2$ satisfies Property QITB.*

PROPOSITION 1.7. — *Let $G = G_1 *_H G_2$ be a transverse amalgamated product, where H is amenable. If G_1, G_2 satisfy Property QITB, then G satisfies Property QITB.*

We refer the reader to Definition 5.18 for the notion of transverse amalgamated product. Free products are particular cases of transverse amalgamated products, hence we get the following:

COROLLARY 1.8. — *If G_1, G_2 satisfy Property QITB, then the free product $G_1 * G_2$ satisfies QITB.*

Using Propositions 1.6 and 1.7 one can prove that the fundamental groups of many higher dimensional graph manifolds defined in [FLS15] satisfy QITB.

Before investigating the relationship between Property QITB and other cohomological properties of groups, we ask here the following question, which shows that, surprisingly enough, the geometry of central extensions seems to still be quite elusive.

QUESTION 1.9. — *If $1 \rightarrow Z \rightarrow E \rightarrow G \rightarrow 1$ is a quasi-isometrically trivial extension, then Z is undistorted in E . There is no apparent reason why the converse of this statement should also hold. Therefore, we ask here the following question: does there exist a non-quasi-isometrically trivial extension $1 \rightarrow Z \rightarrow E \rightarrow G \rightarrow 1$ for which Z is undistorted in E ?*

Weakly bounded cochains

Let A be an abelian group (we will be mainly interested in the cases when either A is finitely generated, or $A = \mathbb{R}$). We denote by $C^*(G, A)$ the bar resolution of G with coefficients in A (see e.g. [Bro82, Chapter III]). Recall that a cochain $\omega \in C^n(G, A)$ (which is a map $\omega: G^n \rightarrow A$) is *bounded* if the set $\omega(G^n)$ is bounded as a subset of A (if A is a finitely generated abelian group, this amounts to asking that $\omega(G^n)$ be finite). Following [NR96, NR97] (where only the case of degree 2 was considered), we say that a cochain $\omega \in C^n(G, A)$ is *weakly bounded* if, for every fixed $(n-1)$ -tuple $(g_2, \dots, g_n) \in G^{n-1}$, the set

$$\omega(G, g_2, \dots, g_n) \subseteq A$$

is bounded.

Bounded cochains provide a subcomplex $C_b^*(G, A)$ of $C^*(G, A)$ (while weakly bounded cochains do not). The cohomology of $C_b^*(G, A)$ is denoted by $H_b^*(G, A)$. The inclusion of bounded cochains into ordinary cochains induces the *comparison map*

$$c^*: H_b^*(G, A) \rightarrow H^*(G, A) .$$

We say that a class $\alpha \in H^*(G, A)$ is *bounded* if it may be represented by a bounded cocycle, i.e. if it lies in the image of the comparison map c^* , and *weakly bounded* if it may be represented by a weakly bounded cocycle.

Suppose now that A is finitely generated. As mentioned above, Gersten proved that a central extension is quasi-isometrically trivial provided it may be described by a bounded cocycle. Neumann and Reeves then observed that a central extension of a finitely generated group is quasi-isometrically trivial if and only if its Euler class is weakly bounded (see Corollary 2.5). Therefore, Theorem 1.2 implies the following:

COROLLARY 1.10. — *There exist a finitely generated group G and a class $\alpha \in H^2(G, \mathbb{Z})$ such that α is weakly bounded, but not bounded.*

Moreover, Theorem 1.4 and Propositions 1.6 and 1.7 imply that weak boundedness and boundedness are indeed equivalent (in degree 2) for a large class of groups.

ℓ^∞ -cohomology

A key ingredient in our proof of Theorem 1.2 is the fact that weakly bounded classes may be characterized in terms of the ℓ^∞ -cohomology $H_{(\infty)}^*(G, A)$ of G , which was first defined by Gersten in [Ger, Ger92], and further studied e.g. in [Bla14, BNW12a, Ger98, Mil22, Min99, Min00].

The ℓ^∞ -cohomology of a group G in degree $\leq n$ was originally defined via the cellular cohomology complex of an Eilenberg–MacLane space X for G , under the assumption that X has a finite n -skeleton. It was observed by Wienhard [Wie12, Section 5] (see also [Bla14, Section 6.3]) that ℓ^∞ -cohomology may be defined in purely algebraic terms (i.e. without referring to any cellular complex providing a model for G , hence without restricting to groups admitting models with finite skeleta). Recently, Milizia [Mil22] extended Gersten’s definition of the ℓ^∞ -cohomology $H_{(\infty)}^*(X, A)$ of a cellular complex X to avoid any assumption on the skeleta of X . He also proved that $H_{(\infty)}^*(X, A)$ is canonically isomorphic to $H_{(\infty)}^*(G, A)$ whenever X is an Eilenberg–MacLane space for G . This approach has the advantage to provide a topological interpretation of $H_{(\infty)}^*(G, A)$ also when G is not finitely presented (or even finitely generated). This topological interpretation will prove very useful in our proof of Theorem 1.2.

The ℓ^∞ -cohomology of a group comes with a natural map

$$\iota^*: H^*(G, A) \rightarrow H_{(\infty)}^*(G, A).$$

In Section 3 we provide a direct proof of the following:

PROPOSITION 1.11. — *Let $\alpha \in H^n(G, A)$, $n \in \mathbb{N}$. Then α is weakly bounded if and only if $\iota^n(\alpha) = 0$.*

As a corollary, we recover the following characterization of quasi-isometrically trivial central extensions, which was proved by Kleiner and Leeb via a different strategy (see also [Why10, Theorem 0.3]):

THEOREM 1.12 ([KL01, Theorem 1.8]). — *Let $\alpha \in H^2(G, \mathbb{Z})$ be the Euler class of a central extension of a finitely generated group. Then $\iota^2(\alpha) = 0$ in $H_{(\infty)}^2(G, \mathbb{Z})$ if and only if the extension is quasi-isometrically trivial.*

Since bounded classes are weakly bounded, Proposition 1.11 implies that the composition

$$(1.1) \quad H_b^n(G, A) \xrightarrow{c^n} H^n(G, A) \xrightarrow{\iota^n} H_{(\infty)}^n(G, A)$$

is the zero map for every $n \in \mathbb{N}$, a result which was already known (at least for $A = \mathbb{R}$) to Gersten (see [Ger98, Proposition 10.3] for the case where G admits an Eilenberg–MacLane model with a finite n -skeleton, and [Wie12] or [Bla14] for the general case). The following question was posed (for $A = \mathbb{R}$) by Wienhard in [Wie12, Question 8] and by Blank in [Bla14, Question 6.3.10]:

QUESTION 1.13. — *When is the sequence (1.1) exact?*

The results proved in this paper partially answer this question. Indeed, since the kernel of ι^n coincides with the space of weakly bounded classes, the sequence (1.1) is exact if and only if every weakly bounded n -class is bounded. For a fixed group G , it is not difficult to show that this condition holds for $A = \mathbb{Z}$ if and only if it holds for every A , as A varies in the class of finitely generated abelian groups (see Proposition 2.6). Therefore, we have the following:

PROPOSITION 1.14. — *The sequence*

$$H_b^2(G, \mathbb{Z}) \xrightarrow{c^n} H^2(G, \mathbb{Z}) \xrightarrow{\iota^n} H_{(\infty)}^2(G, \mathbb{Z})$$

is exact if and only if the group G satisfies Property QITB.

Some subtleties arise when comparing the case with integral coefficients with the case with real coefficients. By Lemma 2.8 and Proposition 3.1, if the sequence

$$(1.2) \quad H_b^n(G, \mathbb{R}) \xrightarrow{c^n} H^n(G, \mathbb{R}) \xrightarrow{\iota^n} H_{(\infty)}^n(G, \mathbb{R})$$

is exact, then the sequence

$$(1.3) \quad H_b^n(G, \mathbb{Z}) \xrightarrow{c^n} H^n(G, \mathbb{Z}) \xrightarrow{\iota^n} H_{(\infty)}^n(G, \mathbb{Z})$$

is also exact. As a corollary of Theorem 1.2 we then have the following:

COROLLARY 1.15. — *Let $n = 2$. There exists a finitely generated group G for which the sequences (1.2) and (1.3) are not exact.*

On the contrary, the same sequences are exact for all the groups listed in the statement of Theorem 1.4 (see Section 5).

The question whether the exactness of sequence (1.3) implies the exactness of (1.2) seems quite tricky: it would hold, for example, provided that the answer to Question 1.16–(2) were positive.

QUESTION 1.16. — *Let us say that a class $\alpha \in H^n(G, \mathbb{R})$ is integral if it belongs to the image of $H^n(G, \mathbb{Z})$ via the change of coefficients homomorphism. As a consequence of Lemma 2.8, an integral class $\alpha \in H^n(G, \mathbb{R})$ is (weakly) bounded if and only if it is the image of a (weakly) bounded class in $H^n(G, \mathbb{Z})$ under the change of coefficients homomorphism.*

- (1) *Does every bounded class in $H^n(G, \mathbb{R})$ belong to the \mathbb{R} -linear subspace spanned by integral bounded classes? Or, equivalently, does the image of $H_b^n(G, \mathbb{Z})$ in $H^n(G, \mathbb{R})$ span (over \mathbb{R}) the space of bounded classes in $H^n(G, \mathbb{R})$?*
- (2) *Does every weakly bounded class in $H^n(G, \mathbb{R})$ belong to the \mathbb{R} -linear subspace spanned by weakly bounded integral classes?*

Our example of a finitely generated group which does not satisfy Property QITB is not finitely presented. In Section 3 we show that in higher degrees it is possible (and much easier) to find finitely presented groups supporting weakly bounded classes that are not bounded:

PROPOSITION 1.17. — *For every $n \geq 3$, there exists a finitely presented group for which the sequences (1.2) and (1.3) are not exact.*

As discussed in the following subsection, the situation in degree 2 seems to be very different.

Property QITB and bounded differential forms

In [Gro93], Gromov proposed the following:

CONJECTURE 1.18 ([Gro93, p. 93]). — *Let V be a closed Riemannian manifold, and let $\alpha \in H^2(V, \mathbb{R})$. Then α is \tilde{d} -bounded if and only if it is bounded.⁽²⁾*

Recall from [Gro91, Gro93] that a class $\alpha \in H^2(V, \mathbb{R})$ is \tilde{d} -bounded if the following holds: if $\omega \in \Omega^2(V)$ is a closed differential form representing α via the de Rham isomorphism, and $\tilde{\omega} \in \Omega^2(\tilde{V})$ is the lift of ω to the universal covering \tilde{V} of V , then $\tilde{\omega} = d\varphi$ for some $\varphi \in \Omega^1(\tilde{V})$ such that $\sup_{x \in \tilde{V}} |\varphi_x| < +\infty$. Moreover, α is bounded if it lies in the image of the comparison map between the singular bounded cohomology of V and the usual singular cohomology of V , i.e. if it admits a representative c in the singular chain complex such that $c(\sigma)$ is uniformly bounded as σ varies among all the singular simplices in V .

The study of the growth of primitives in non-compact manifolds was initiated by Sullivan [Sul76], Gromov [Gro81] and Brooks [Bro81] and has then been proved to be closely related to coarse invariants of (fundamental) groups (see e.g. [NS10, Žuk00]). We refer the reader e.g. [Sik01] for a brief account on the topic, and for a self-contained proof of the fact that bounded classes are \tilde{d} -bounded, and to [BI07, CHL104, Wie12] for further developments of the theory.

In Section 3 we prove the following:

⁽²⁾In view of the finitely presented example mentioned in the previous footnote, and Corollary 1.20, this conjecture is now known to be false.

THEOREM 1.19. — *Let V be a closed Riemannian manifold. Then V satisfies the statement of Conjecture 1.18 if and only if the sequence*

$$H_b^2(\pi_1(V), \mathbb{R}) \xrightarrow{c^n} H^2(\pi_1(V), \mathbb{R}) \xrightarrow{\iota^n} H_{(\infty)}^2(\pi_1(V), \mathbb{R})$$

is exact (and, if this condition holds, then $\pi_1(V)$ satisfies QITB).

Since the class of fundamental groups of compact Riemannian manifolds coincides with the class of finitely presented groups, we obtain the following:

COROLLARY 1.20. — *If Conjecture 1.18 holds and G is a finitely presented group, then the sequence*

$$H_b^2(G, \mathbb{R}) \xrightarrow{c^2} H^2(G, \mathbb{R}) \xrightarrow{\iota^2} H_{(\infty)}^2(G, \mathbb{R})$$

is exact. In particular, Gromov's Conjecture would imply that every finitely presented group satisfies QITB. In fact, Gromov's conjecture would be equivalent to the fact that every finitely presented group satisfies QITB provided that the answer to Question 1.16–(2) were affirmative for every finitely presented group.

Note however that Gromov himself stated in [Gro93] that “the evidence in favour of the conjecture is rather limited and it would be safe to make some extra assumption on $\pi_1(V)$ ”.

Plan of the paper

In Section 2 we introduce bounded and weakly bounded cochains, and we prove that a central extension is quasi-isometrically trivial if and only if its Euler class is weakly bounded. In Section 3 we introduce Gersten's ℓ^∞ -cohomology, we prove the characterization of weakly bounded cochains described in Proposition 1.11 (which allows us to recover Theorem 1.12 by Kleiner and Leeb), and we prove Proposition 1.17 and Theorem 1.19. Section 4 is devoted to the proof of Theorem 1.2, i.e. to the construction of a finitely generated group admitting a quasi-isometrically trivial extension with an unbounded Euler class. In Section 5 we construct examples of groups satisfying Property QITB, and we prove Theorems 1.4 and 1.5, and Propositions 1.6 and 1.7.

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2. Preliminaries

Quasi-isometries

Let us briefly recall the definition of quasi-isometry. If (X, d) , (Y, d') are metric spaces, a map $f: X \rightarrow Y$ is a quasi-isometric embedding if there exist constants $k \geq 1$, $c \geq 0$ such that

$$\frac{d(x_1, x_2)}{k} - c \leq d(f(x_1), f(x_2)) \leq k \cdot d(x_1, x_2) + c$$

for every $x_1, x_2 \in X$. A map $g: Y \rightarrow X$ is a *quasi-inverse* of f if it is a quasi-isometric embedding, and the maps $g \circ f: X \rightarrow X$, $f \circ g: Y \rightarrow Y$ are uniformly close to the identity of X and Y , respectively. A quasi-isometry is a quasi-isometric embedding that admits a quasi-inverse.

If G is a finitely generated group and S is a finite symmetric generating set for G (symmetric meaning that $s \in S$ if and only if $s^{-1} \in S$), then the *Cayley* graph $C_S(G)$ of G with respect to S is the graph having G as set of vertices and $G \times S$ as set of edges, where the edge (g, s) joins g with gs . The graph $C_S(G)$ is endowed with a path metric for which every edge is isometric to a segment of unitary length. It is well known that, if S, S' are finite generating sets for G , then the identity of G extends to a quasi-isometry between $C_S(G)$ and $C_{S'}(G)$. Thus, one can define the quasi-isometry type of G as the quasi-isometry type of any of its Cayley graphs.

Notation

If $g \in G$ then we denote by $\|g\|_S$ the distance in $C_S(G)$ between g and the identity of the group (i.e. the minimal number of factors needed to describe g as a product of elements of S and their inverses).

(Weakly) bounded classes

Let A be an abelian group (as in the introduction, we assume that either A is finitely generated, or $A = \mathbb{R}$). Recall from the introduction that a class $\alpha \in H^*(G, A)$ is *bounded* if it may be represented by a bounded cocycle, and *weakly bounded* if it may be represented by a weakly bounded cocycle.

A 2-cocycle $\omega \in C^2(G, A)$ is *normalized* if $\omega(1, G) = \omega(G, 1) = 0$, where 1 is the identity of G . It is well known that every cohomology class may be represented by a normalized cocycle.

LEMMA 2.1. — *Let $S = \{x_1, \dots, x_n\}$ be a symmetric set of generators of G , and let $\omega \in C^2(G, A)$ be a normalized cocycle. Suppose that $|\omega(g, x_i)| \leq C$ for every $g \in G$, $i = 1, \dots, n$. Then*

$$|\omega(G, g)| \leq 2C\|g\|_S \quad \text{for every } g \in G.$$

In particular, ω is weakly bounded.

Proof. — Let g be an element of G . We will prove by induction on $\|g\|_S$ that

$$|\omega(G, g)| \leq 2C\|g\|_S .$$

The case $\|g\|_S = 0$ follows from the fact that ω is normalized. Assuming the inequality for all elements of length $j - 1$, if $\|g\| = j$ and $g = g'x_i$ with $\|g'\|_S = j - 1$, then for every $h \in G$ we have (by using the cocycle relation):

$$\begin{aligned} |\omega(h, g)| &= |\omega(h, g') - \omega(g', x_i) + \omega(hg', x_i)| \\ &\leq |\omega(h, g')| + |\omega(g', x_i)| + |\omega(hg', x_i)| \\ &\leq 2C\|g'\|_S + C + C = 2C\|g\|_S . \end{aligned}$$

□

Remark 2.2. — Our terminology slightly differs from Neumann and Reeves'. In fact, our weakly bounded cocycles correspond to *right* weakly bounded cocycles in the terminology introduced in [NR96, NR97], where a cocycle $\omega \in Z^2(G, A)$ is called *left weakly bounded* if $\omega(g, G)$ is a bounded subset of A for every $g \in G$, and *weakly bounded* if it is both right weakly bounded and left weakly bounded. It is not difficult to show that a class in $H^2(G, A)$ admits a left weakly bounded representative if and only if it admits a right weakly bounded representative. In fact, it turns out that left weak boundedness, right weak boundedness and weak boundedness are equivalent for elements of $H^2(G, A)$, by [NR97, Theorem 4.1].

The Euler class of a central extension

Let us now consider a central extension

$$1 \longrightarrow Z \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1 ,$$

and let $s: G \rightarrow E$ be a section of $\pi: E \rightarrow G$. For $g_1, g_2 \in G$, the element $s(g_1)s(g_2)s(g_1g_2)^{-1}$ lies in the kernel of π , hence in the image of i . Up to identifying $i(Z)$ with Z we may thus define the cochain $\omega \in C^2(G, Z)$ given by

$$\omega_s(g_1, g_2) = s(g_1)s(g_2)s(g_1g_2)^{-1} \in Z .$$

Let us recall the following well-known facts (see e.g. [Bro82, Chapter 4]):

- (1) The cochain ω_s is a cocycle;
- (2) If s' is another section of π , then $\omega_{s'}$ is cobordant to ω_s ; therefore, the class $[\omega_s] \in H^2(G, Z)$ does not depend on the choice of s , and will be called the *Euler class* of the extension;
- (3) If ω' is any representative of the Euler class, then there exists a section $s': G \rightarrow E$ such that $\omega' = \omega_{s'}$;
- (4) The cocycle ω_s is normalized if and only if $s(1) = 1$ (in this case, we say that s is normalized too).

Two central extensions are isomorphic if they are described by the rows of a commutative diagram as follows:

$$\begin{array}{ccccccc} 1 & \longrightarrow & Z & \xrightarrow{i} & E & \xrightarrow{\pi} & G \longrightarrow 1 \\ & & \downarrow \text{Id} & & \downarrow h & & \downarrow \text{Id} \\ 1 & \longrightarrow & Z & \xrightarrow{i} & E' & \xrightarrow{\pi} & G \longrightarrow 1, \end{array}$$

where h is a homomorphism (hence, an isomorphism). It is readily seen that isomorphic central extensions share the same Euler class. In fact, it readily follows from the facts listed above that, via the Euler class, the module $H^2(G, Z)$ classifies central extensions of G by Z up to isomorphism.

The Euler class of a quasi-isometrically trivial central extension

The following characterization of quasi-isometrically trivial extensions was proved by Kleiner and Leeb [KL01] in the case when Z is a finitely-generated torsion free abelian group (i.e. $Z \cong \mathbb{Z}^n$). However, the proof in [KL01] works verbatim in the general case.

PROPOSITION 2.3 ([KL01, Proposition 8.3]). — *Let*

$$1 \longrightarrow Z \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1$$

be a central extension. Then the following conditions are equivalent:

- (1) *The extension is quasi-isometrically trivial.*
- (2) *The projection π admits a Lipschitz section $s: G \rightarrow E$.*

It is almost tautological that the 2-cocycle ω associated to a section $s: G \rightarrow E$ is bounded if and only if s is a *quasihomomorphism* in the sense of Kapovich and Fujiwara [FK16]. Therefore, a group G satisfies Property QITB if and only if the existence of a Lipschitz section for a central extension of G implies the existence of a quasihomomorphic section for the same extension. We refer the reader to [Heu20] for a discussion of (not necessarily central) extensions with bounded Euler class in terms of quasihomomorphisms.

The following Lemma 2.4 and its immediate Corollary 2.5 play a fundamental role in our study of quasi-isometrically trivial central extensions. They are stated in [NR96, Section 4]. For the sake of completeness, we provide here a proof of Lemma 2.4.

LEMMA 2.4. — *Let $s: G \rightarrow E$ be a normalized section for the central extension*

$$1 \longrightarrow Z \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1,$$

and let $\omega \in C^2(G, Z)$ be the associated 2-cocycle. Then s is Lipschitz if and only if ω is weakly bounded.

Proof. — Suppose s is Lipschitz. Let $S = \{x_1, \dots, x_n\}$ be a symmetric set of generators for G , $S' = \{z_1, \dots, z_k\}$ a symmetric set of generators for Z , and let \bar{S} be the set of generators for E given by $\{s(x_i), i(z_j) \mid i = 1, \dots, n, j = 1, \dots, k\}$. We also denote by d_G, d_Z and d_E the word metrics on G, Z and E induced by S, S' and \bar{S} , respectively. Since d_E -balls of finite radius only contain a finite number of elements of $i(Z)$, in order to show that ω is weakly bounded it suffices to show that, for every $h \in G$, the value of

$$\|\omega(g, h)\|_{\bar{S}} = \|s(gh)^{-1}s(g)s(h)\|_{\bar{S}}$$

is uniformly bounded as g varies in G . But, if k is a Lipschitz constant for s , then

$$\begin{aligned} \|s(gh)^{-1}s(g)s(h)\|_{\bar{S}} &\leq \|s(gh)^{-1}s(g)\|_{\bar{S}} + \|s(h)\|_{\bar{S}} = d_E(s(gh), s(g)) + \|s(h)\|_{\bar{S}} \\ &\leq kd_G(gh, g) + \|s(h)\|_{\bar{S}} = kd_G(h, 1) + \|s(h)\|_{\bar{S}}, \end{aligned}$$

which is independent of g , as required.

On the other hand, suppose that there exists $C \geq 0$ such that $\|\omega(G, x_i)\|_{S'} \leq C$ for every $i = 1, \dots, n$. Let g be an element of G . We show by induction on $\|g\|_S$ that

$$\|s(g)\|_{\bar{S}} \leq (1 + C)\|g\|_S.$$

The case $\|g\|_S = 0$ follows from the fact that s is normalized. Let us assume the above inequality for all $g' \in G$ with $\|g'\|_S \leq j - 1$, and suppose $\|g\|_S = j$. Then $g = g'x_i$ for some $g' \in G$ with $\|g'\|_S = j - 1$ and some $x_i \in S$. Now

$$\begin{aligned} \|s(g)\|_{\bar{S}} &= \|s(g')s(x_i)i(-\omega(g', x_i))\|_{\bar{S}} \\ &\leq \|s(g')\|_{\bar{S}} + \|s(x_i)\|_{\bar{S}} + \|i(-\omega(g', x_i))\|_{\bar{S}} \\ &\leq (1 + C)\|g'\|_S + 1 + \|\omega(G, x_i)\|_{S'} \\ &\leq (1 + C)(\|g\|_S - 1) + 1 + C = (1 + C)\|g\|_S. \end{aligned}$$

By Lemma 2.1 we now have

$$\|\omega(G, x)\|_{S'} \leq 2C\|x\|_S \quad \text{for every } x \in G.$$

Take $g, h \in G$ and suppose $d_G(g, h) = k$. Then $g = hx$, where $\|x\|_S = k$. We have

$$s(g) = s(hx) = s(h)s(x)i(-\omega(h, x)),$$

hence

$$\begin{aligned} d_E(s(g), s(h)) &= d_E(s(h)s(x)i(-\omega(h, x)), s(h)) = \|s(x)i(-\omega(h, x))\|_{\bar{S}} \\ &\leq \|s(x)\|_{\bar{S}} + \|i(\omega(h, x))\|_{\bar{S}} \leq \|s(x)\|_{\bar{S}} + \|\omega(h, x)\|_{S'} \\ &\leq (1 + C)\|x\|_S + 2C\|x\|_S = (1 + 3C)d_G(g, h). \end{aligned}$$

This concludes the proof of Lemma 2.4. \square

COROLLARY 2.5. — *A central extension of G by Z is quasi-isometrically trivial if and only if its Euler class is a weakly bounded element of $H^2(G, Z)$.*

The following proposition shows that, in order to detect Property QITB, it is sufficient to deal with extensions of G by \mathbb{Z} .

PROPOSITION 2.6. — *Let G be a finitely generated group. Then every weakly bounded class in $H^2(G, \mathbb{Z})$ is bounded if and only if every weakly bounded class in $H^2(G, Z)$ is bounded for every finitely generated abelian group Z .*

Proof. — The “if” part of the statement is obvious. Let then Z be a finitely generated abelian group, and assume that every weakly bounded class in $H^2(G, Z)$ is bounded. Let us consider a weakly bounded cocycle $\omega \in C^2(G, Z)$. We have $Z \cong \mathbb{Z}^k \oplus F$, where F is a finite abelian group, hence we may consider ω as a map

$$\omega: G^2 \rightarrow \mathbb{Z}^k \oplus F, \quad \omega = (\omega_1, \dots, \omega_k, \omega_F),$$

where $\omega_i(g_1, g_2)$ (resp. $\omega_F(g_1, g_2)$) is the i^{th} component of the projection of $\omega(g_1, g_2)$ onto \mathbb{Z}^k (resp. the projection of $\omega(g_1, g_2)$ onto F). Since ω is weakly bounded, every ω_i is also weakly bounded. Under our assumptions, this implies that ω_i is cobordant to a bounded cocycle $\omega'_i \in C^2(G, \mathbb{Z})$ for every $i = 1, \dots, k$. This easily implies that the cocycle ω is cobordant to the cocycle $\omega' = (\omega'_1, \dots, \omega'_k, \omega_F)$ in $C^2(G, Z)$. But ω' is bounded, and this concludes the proof. \square

Putting together Proposition 2.6 and Corollary 2.5 we get the following:

COROLLARY 2.7. — *Let G be a finitely generated group. Then the following conditions are equivalent:*

- (1) *G satisfies Property QITB.*
- (2) *Every weakly bounded class in $H^2(G, \mathbb{Z})$ is bounded.*

Integral vs. real coefficients

Bounded cohomology with real coefficients is better understood than bounded cohomology with integral coefficients (for example, for amenable groups the real bounded cohomology vanishes, while bounded cohomology with integral coefficients may be non-trivial). Therefore, before proceeding with our investigation of Property QITB, we first point out the following results, which sometimes will allow us to work with real coefficients, rather than with integral ones.

LEMMA 2.8. — *Let $\alpha \in H^n(G, \mathbb{Z})$, and denote by $\alpha_{\mathbb{R}}$ the image of α in $H^n(G, \mathbb{R})$ under the change of coefficients map. Then:*

- (1) *α is weakly bounded if and only if $\alpha_{\mathbb{R}}$ is weakly bounded;*
- (2) *α is bounded if and only if $\alpha_{\mathbb{R}}$ is bounded.*

Proof. — Statement (2) is proved in [Min01, Theorem 15] and in [Fri17, Proposition 2.18], and the very same argument also implies (1). \square

COROLLARY 2.9. — *For every $n \in \mathbb{N}$, if every weakly bounded class in $H^n(G, \mathbb{R})$ is bounded, then every weakly bounded class in $H^n(G, \mathbb{Z})$ is bounded and G satisfies Property QITB.*

Proof. — Let us denote by $\psi: H^n(G, \mathbb{Z}) \rightarrow H^n(G, \mathbb{R})$ the change of coefficients map, and let $\beta \in H^n(G, \mathbb{Z})$ be weakly bounded. Then $\psi(\beta) \in H^n(G, \mathbb{R})$ is weakly bounded, hence bounded by hypothesis. Lemma 2.8 now implies that β is bounded, as desired. The last assertion now follows from Corollary 2.7. \square

3. ℓ^∞ -cohomology

As mentioned in the introduction, weakly bounded classes may be characterized in terms of the so-called ℓ^∞ -cohomology of G , which we are now going to define.

Let A be either a finitely generated abelian group, or the field of real numbers, and let $\ell^\infty(G, A)$ be the module of bounded functions from G to A (here in the case where A is finitely generated abelian, we say that a function is bounded if and only if it takes finitely many values). We can endow $\ell^\infty(G, A)$ with the structure of a left G -module via the left action defined by

$$(g \cdot f)(h) = f(g^{-1}h), \quad f \in \ell^\infty(G, A), \quad g, h \in G.$$

We then denote by $C_{(\infty)}^*(G, A)$ the cochain complex $C^*(G, \ell^\infty(G, A))$, and we define the ℓ^∞ -cohomology $H_{(\infty)}^*(G, A)$ of G as the cohomology of the complex $C_{(\infty)}^*(G, A)$ (we refer the reader e.g. to [Bro82] for the definition of cohomology with twisted coefficients).

If we consider A as a trivial G -module, then we can equivariantly embed A into the submodule of $\ell^\infty(G, A)$ given by the constant maps. This map induces a chain map $\iota^*: C^*(G, A) \rightarrow C_{(\infty)}^*(G, A)$, which defines in turn a map

$$\iota^*: H^*(G, A) \rightarrow H_{(\infty)}^*(G, A).$$

We are now ready to prove Proposition 1.11 from the introduction, which we recall here for the convenience of the reader. As mentioned in the introduction, Corollary 2.5 and Proposition 1.11 allow us to recover Kleiner and Leeb's characterization via ℓ^∞ -cohomology of quasi-isometrically trivial central extensions.

PROPOSITION 3.1. — *Let $\alpha \in H^n(G, A)$, $n \geq 2$. Then α is weakly bounded if and only if $\iota^n(\alpha) = 0$.*

Proof. — Let $\omega \in C^n(G, A)$ be a representative of α , and suppose $\iota^n(\alpha) = 0$. This means that there exists a cochain $\varphi \in C_{(\infty)}^{n-1}(G, A)$, i.e. a map $\varphi: G^{n-1} \rightarrow \ell^\infty(G, A)$, such that, for every $g_1, \dots, g_n, h \in G$,

$$\begin{aligned} \omega(g_1, \dots, g_n) &= (\delta\varphi)(g_1, \dots, g_n)(h) \\ &= g_1 \cdot (\varphi(g_2, \dots, g_n))(h) + \sum_{i=1}^{n-1} (-1)^i \varphi(g_1, \dots, g_i g_{i+1}, \dots, g_n)(h) \\ &\quad + (-1)^n \varphi(g_1, \dots, g_{n-1})(h) \\ &= \varphi(g_2, \dots, g_n)(g_1^{-1}h) + \sum_{i=1}^{n-1} (-1)^i \varphi(g_1, \dots, g_i g_{i+1}, \dots, g_n)(h) \\ &\quad + (-1)^n \varphi(g_1, \dots, g_{n-1})(h). \end{aligned}$$

By setting $h = 1$, we obtain

$$\begin{aligned} \omega(g_1, \dots, g_n) &= \varphi(g_2, \dots, g_n)(g_1^{-1}) + \sum_{i=1}^{n-1} (-1)^i \varphi(g_1, \dots, g_i g_{i+1}, \dots, g_n)(1) \\ &\quad + (-1)^n \varphi(g_1, \dots, g_{n-1})(1). \end{aligned}$$

Therefore, if we set

$$f \in C^{n-1}(G, A), \quad f(g_1, \dots, g_{n-1}) = -\varphi(g_1, \dots, g_{n-1})(1),$$

then

$$\begin{aligned} |(\omega + \delta f)(g_1, \dots, g_n)| &= |\varphi(g_2, \dots, g_n)(g_1^{-1}) - \varphi(g_2, \dots, g_n)(1)| \\ &\leq 2\|\varphi(g_2, \dots, g_n)\|_\infty. \end{aligned}$$

Hence

$$|(\omega + \delta f)(G, g_2, \dots, g_n)| \leq 2\|\varphi(g_2, \dots, g_n)\|_\infty < +\infty,$$

and α is weakly bounded.

Suppose now that $\omega \in C^n(G, A)$ is a weakly bounded representative of α , and set

$$\varphi: G^{n-1} \rightarrow \ell^\infty(G, A), \quad \varphi(g_1, \dots, g_{n-1})(h) = \omega(h^{-1}, g_1, \dots, g_{n-1})$$

(the fact that φ is well-defined is due to the weak boundedness of ω). Then, for every $g_1, \dots, g_n, h \in G$ we have

$$\begin{aligned} (\delta\varphi)(g_1, \dots, g_n)(h) &= \omega(h^{-1}g_1, g_2, \dots, g_n) + \sum_{i=1}^{n-1} (-1)^i \omega(h^{-1}, g_1, \dots, g_i g_{i+1}, \dots, g_n) \\ &\quad + (-1)^n \omega(h^{-1}, g_1, \dots, g_{n-1}) \\ &= \omega(g_1, \dots, g_n), \end{aligned}$$

where the last equality is due to the fact that ω is a cocycle. Thus $\iota^n(\alpha) = 0$, as desired. \square

Higher degrees

As we explain below, the following result readily implies Proposition 1.17 from the introduction.

PROPOSITION 3.2. — *Let $A = \mathbb{R}$ or \mathbb{Z} , let G be an n -dimensional non-amenable Poincaré duality group, and let $G' = G \times \mathbb{Z}$. Then the sequence*

$$H_b^{n+1}(G', A) \xrightarrow{c^{n+1}} H^{n+1}(G', A) \xrightarrow{\iota^{n+1}} H_{(\infty)}^{n+1}(G', A)$$

is not exact.

Proof. — By Lemma 2.8 and Proposition 3.1, it is sufficient to prove the proposition for $A = \mathbb{Z}$.

We first show that

$$(3.1) \quad H_{(\infty)}^{n+1}(G', \mathbb{Z}) = 0.$$

Since G' is an $(n+1)$ -dimensional Poincaré duality group, we have

$$H_{(\infty)}^{n+1}(G', \mathbb{R}) = H^{n+1}(G', \ell^\infty(G', \mathbb{R})) \cong H_0(G', \ell^\infty(G', \mathbb{R})).$$

It was first observed in [BNW12b] that $H_*(G', \ell^\infty(G', \mathbb{R}))$ is isomorphic to the so-called *uniformly finite homology* of G' (see also [BD15, DL17]). By a fundamental result by Block and Weinberger, amenable groups can be characterized as those groups for which uniformly finite homology does not vanish [BW92] in degree 0. Since G' contains the non-amenable group G as a subgroup, it is itself non-amenable, hence $H_{(\infty)}^{n+1}(G', \mathbb{R}) = 0$. In order to get (3.1) it is now sufficient to recall from [Ger96, Proposition 7.2] that $H_{(\infty)}^{n+1}(G', \mathbb{Z}) \cong H_{(\infty)}^{n+1}(G', \mathbb{R})$. As usual, Gersten proved this fact under the assumption that G' admits a finite n -skeleton. For the general case, observe that the short exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} \rightarrow 0$ of coefficients induces the short exact sequence of G' -modules $0 \rightarrow \ell^\infty(G', \mathbb{Z}) \rightarrow \ell^\infty(G', \mathbb{R}) \rightarrow \ell^\infty(G', \mathbb{R}/\mathbb{Z}) \rightarrow 0$, hence the short exact sequence of complexes

$$(3.2) \quad 0 \rightarrow C_{(\infty)}^*(G', \mathbb{Z}) \rightarrow C_{(\infty)}^*(G', \mathbb{R}) \rightarrow C_{(\infty)}^*(G', \mathbb{R}/\mathbb{Z}) \rightarrow 0.$$

Observe now that $\ell^\infty(G', \mathbb{R}/\mathbb{Z}) \cong \text{Hom}(\mathbb{Z}G', \mathbb{R}/\mathbb{Z})$ as G' -modules, and recall that $\text{Hom}(\mathbb{Z}G', \mathbb{R}/\mathbb{Z})$ is an injective G' -module (see e.g. [Bro82, Proposition 6.1]). Therefore, $H_{(\infty)}^n(G', \mathbb{R}/\mathbb{Z}) = H^n(G', \ell^\infty(G', \mathbb{R}/\mathbb{Z})) = 0$ for every $n \geq 1$. By looking at the long exact sequence in cohomology induced by (3.2) we may now conclude that $H_{(\infty)}^n(G', \mathbb{Z}) \cong H_{(\infty)}^n(G', \mathbb{R})$ for every $n \geq 1$.

Let now $\alpha \in H^{n+1}(G', \mathbb{Z})$ be bounded. The group homomorphism $h: G \times \mathbb{Z} \rightarrow G \times \mathbb{Z}$, $h(g, m) = (g, 2m)$ induces the multiplication by 2 on $H^{n+1}(G', \mathbb{Z})$. Since maps induced by homomorphisms do not increase the seminorm of cohomology classes, this implies that $\|\alpha\|_\infty = 0$. Thus $\langle \alpha, \beta \rangle = 0$ for every $\beta \in H_{n+1}(G', \mathbb{Z})$. Since G' is an $(n+1)$ -Poincaré duality group, this implies in turn that $\alpha = 0$. We have thus shown that both maps in the sequence of the statement (with $A = \mathbb{Z}$) are null. Since $H^{n+1}(G', \mathbb{Z}) \neq 0$, the sequence is not exact. \square

The following corollary implies Proposition 1.17 from the introduction.

COROLLARY 3.3. — *Let $A = \mathbb{Z}$ or \mathbb{R} . For every $n \geq 3$, let $G_n = \Gamma_2 \times \mathbb{Z}^{n-2}$, where Γ_2 is the fundamental group of the closed oriented surface of genus 2. Then the sequence*

$$H_b^n(G_n, A) \xrightarrow{c^n} H^n(G_n, A) \xrightarrow{\iota^n} H_{(\infty)}^n(G_n, A)$$

is not exact.

Proof. — We can apply the previous proposition to the non-amenable $(n-1)$ -dimensional Poincaré duality group $G = \Gamma_2 \times \mathbb{Z}^{n-3}$. \square

Property QITB and Gromov's Conjecture

The strategy described in Proposition 3.2 cannot be implemented in degree 2. Indeed, as stated in the introduction we have the following:

THEOREM 3.4 (Theorem 1.19). — *Let V be a compact Riemannian manifold. Then the following are equivalent:*

- (1) *A class $\alpha \in H^2(V, \mathbb{R})$ is \tilde{d} -bounded if and only if it is bounded.*

(2) *The sequence*

$$H_b^2(\pi_1(V), \mathbb{R}) \xrightarrow{c^2} H^2(\pi_1(V), \mathbb{R}) \xrightarrow{\iota^2} H_{(\infty)}^n(\pi_1(V), \mathbb{R})$$

is exact.

In particular, by Corollary 2.9 and Proposition 3.1, if V satisfies Gromov's Conjecture 1.18, then $\pi_1(V)$ satisfies Property QITB.

Proof. — We denote by $\Omega_b^*(\tilde{V})$ the space of bounded differential forms on \tilde{V} with bounded differential, and by $H_b^*(\tilde{V})$ the associated cohomology (caveat: $H_b^*(\tilde{V})$ is not at all equal to $H_b^*(\tilde{V}, \mathbb{R})$; indeed, since \tilde{V} is simply connected, $H_b^n(\tilde{V}, \mathbb{R}) = 0$ for every $n \geq 1$ [Gro82, page 40], [Iva87, Theorem 2.4], [FM23, Corollary 4], while we will see below that $H_b^*(\tilde{V})$ is isomorphic to $H_{(\infty)}^*(\tilde{V}, \mathbb{R})$, which is often non-trivial in positive degree).

Observe that, if ω is a k -differential form on V , then the pull-back $\tilde{\omega}$ to \tilde{V} is equivariant with respect to a cocompact action, hence it belongs to $\Omega_b^k(\tilde{V})$. We thus have a map $\psi: H^*(V) \rightarrow H_b^*(\tilde{V})$, where $H^*(V)$ denotes the usual de Rham cohomology of V . Via the identification $H^*(V) \cong H^*(V, \mathbb{R})$ due to de Rham isomorphism, the kernel of ψ coincides with the space of \tilde{d} -bounded classes defined in the introduction, by definition of \tilde{d} -bounded.

Every smooth manifold admits a PL-structure [Whi40, Mun66], hence V is homeomorphic to the geometric realization of a simplicial complex. If we endow the universal covering \tilde{V} with the induced simplicial structure, integration provides a chain map $\Omega_b^*(\tilde{V}) \rightarrow C_{(\infty)}^*(\tilde{V}, \mathbb{R})$, where $C_{(\infty)}^*(\tilde{V}, \mathbb{R})$ denotes here the space of bounded cellular cochains on \tilde{V} . If we denote by $H_{(\infty)}^*(\tilde{V}, \mathbb{R})$ the cohomology of $C_{(\infty)}^*(\tilde{V}, \mathbb{R})$, we thus get a map

$$I^*: H_b^*(\tilde{V}) \rightarrow H_{(\infty)}^*(\tilde{V}, \mathbb{R}) .$$

It is shown in [Mil22, Theorem 1.11] (see also [Min99, Theorem 3.1]) that the map I^* is an isomorphism in every degree. Moreover, by lifting cellular cochains from V to \tilde{V} we get a well-defined map $\iota_V^*: H^*(V, \mathbb{R}) \rightarrow H_{(\infty)}^*(\tilde{V}, \mathbb{R})$.

It readily follows from the explicit description of the de Rham isomorphism that the following diagram is commutative:

$$\begin{array}{ccc} H^*(V) & \xrightarrow{\psi} & H_b^*(\tilde{V}) \\ \downarrow \cong & & \downarrow \cong \\ H^*(V, \mathbb{R}) & \xrightarrow{\iota_V^*} & H_{(\infty)}^*(\tilde{V}, \mathbb{R}) , \end{array}$$

where the vertical arrows correspond to the de Rham isomorphism and to the map I^* . Therefore, a class $\alpha \in H^*(V, \mathbb{R})$ is \tilde{d} -bounded if and only if it belongs to the kernel of ι_V^* , and condition (1) is equivalent to the exactness of the sequence

$$H_b^2(V, \mathbb{R}) \xrightarrow{c_V^2} H^2(V, \mathbb{R}) \xrightarrow{\iota_V^2} H_{(\infty)}^2(\tilde{V}, \mathbb{R}) .$$

Let us now set $G = \pi_1(V)$, and let $f: V \rightarrow X$ be a classifying map, where X is a $K(G, 1)$ -space. We then have canonical identifications $H^*(X, \mathbb{R}) \cong H^*(G, \mathbb{R})$ and $H_b^*(X, \mathbb{R}) \cong H_b^*(G, \mathbb{R})$ (see e.g. [Fri17, Chapter 5]). We may assume that V and X share the same 2-skeleton. In particular, the 2-skeleton of X is finite, and we have a canonical identification $H_{(\infty)}^*(\tilde{X}, \mathbb{R}) = H_{(\infty)}^*(G, \mathbb{R})$ (see e.g. [Bla14, Mil22, Wie12]; indeed, in Gersten's original approach the module $H_{(\infty)}^*(G, \mathbb{R})$ was *defined* via this identification). We thus have the following commutative diagram, where the vertical arrows are all induced by $f: V \rightarrow X$:

$$(3.3) \quad \begin{array}{ccccc} H_b^2(G, \mathbb{R}) & \xrightarrow{c_G^2} & H^2(G, \mathbb{R}) & \xrightarrow{\iota_G^2} & H_{(\infty)}^2(G, \mathbb{R}) \\ \downarrow f_b^* & & \downarrow f^* & & \downarrow f_{(\infty)}^* \\ H_b^2(V, \mathbb{R}) & \xrightarrow{c_V^2} & H^2(V, \mathbb{R}) & \xrightarrow{\iota_V^2} & H_{(\infty)}^2(\tilde{V}, \mathbb{R}) . \end{array}$$

Using that V and X share the same 2-skeleton (and the fact that singular homology may be computed via cellular cochains), it is easy to show that f^* and $f_{(\infty)}^*$ are both injective.

(2) \Rightarrow (1): Let us suppose the top row of the diagram is exact, and take an element $\alpha \in H^2(V, \mathbb{R})$ with $\iota_V^2(\alpha) = 0$. Since the pull-back of α to \tilde{V} is null (as an ordinary cohomology class), α vanishes on every homology class which may be represented by a sphere. As a consequence, there exists $\beta \in H^2(G, \mathbb{R})$ with $f^*(\beta) = \alpha$. Together with the injectivity of $f_{(\infty)}^*$, the fact that $\iota_V^2(\alpha) = 0$ now implies that $\iota_G^2(\beta) = 0$. We thus have $\beta = c_G^2(\beta_b)$ for some $\beta_b \in H_b^2(G, \mathbb{R})$. By the commutativity of the left square of the diagram, this implies that α is bounded, hence condition (2) holds.

(1) \Rightarrow (2). Suppose now that the bottom row of the diagram is exact, and take $\beta \in \ker i_G^2 \subseteq H^2(G, \mathbb{R})$. Then $f^*(\beta)$ lies in the image of c_V^2 . Now a fundamental theorem by Gromov ensures that f_b^* is an isomorphism [Gro82, Corollary (A), p. 40], [Iva87, Theorem 4.3], [FM23, Theorem 3], and this (together with the injectivity of f^*) implies that β is bounded, as desired. \square

4. A group without Property QITB

We are now ready to exhibit a finitely generated group G admitting a quasi-isometrically trivial central extension whose Euler class is not bounded.

Remark 4.1 (The idea of the construction). — We briefly and informally describe the group that we will be studying. Start with a free product \hat{G} of countably many copies of the fundamental group of the genus-2 surface. Any cohomology class of degree 2 on \hat{G} that takes unbounded values on the surfaces cannot be bounded. However, all these classes are weakly bounded, the rough reason being the following. Each such class can be represented by the (infinite) sum ω of pull-backs of 2-cocycles on the copies of the fundamental group of the genus-2 surface, where we take the pull-back under the natural projection to a factor of the free product. Since the genus-2 surface is hyperbolic, these 2-cocycles may be chosen to be bounded. Moreover, for

any pair (g_1, g_2) of elements of \widehat{G} , only a finite number of projections of g_1 and g_2 on the free factors of \widehat{G} are non-trivial. Putting all these facts together it is easy to show that the set $\omega(\widehat{G}, g_1, g_2)$ is bounded, hence every degree-2 class on \widehat{G} is weakly bounded.

The idea is then to “make \widehat{G} finitely generated”. This can be done by adding “stable letters” t_i to \widehat{G} that conjugate the generators of each of the surface groups to the generators of the “next” surface group. In this way, the generators of the first surface group and the stable letters suffice to generate the new group G . Notice that \widehat{G} is the fundamental group of a locally CAT(0) complex obtained by gluing surfaces and squares (we do not need this fact), which is one way to control the (co)homology of G . The idea is that some of the cohomology classes on \widehat{G} that we discussed above should be the pull-back of weakly bounded classes on G , and we believe that it is possible to show this using CAT(0) techniques. However, below we actually give a different description of G that will allow us to use small-cancellation techniques instead. (We will not need the equivalence of the two descriptions.)

4.1. Description of the group

Let us now proceed with the construction. Let G be the finitely generated group described by the following (infinite) presentation \mathcal{P} :

$$\langle a_1, a_2, a_3, a_4, t_1, t_2, t_3, t_4 \mid r_0, r_1, \dots, r_i, \dots \rangle, \\ r_i = [t_1^i a_1 t_1^{-i}, t_2^i a_2 t_2^{-i}] \cdot [t_3^i a_3 t_3^{-i}, t_4^i a_4 t_4^{-i}],$$

where $[w_1, w_2]$ denotes the commutator $w_1 w_2 w_1^{-1} w_2^{-1}$. We denote by F_8 the free group on the generators $a_1, \dots, a_4, t_1, \dots, t_4$, and by N the normal closure of the relations r_i in F_8 , so that $G \cong F_8/N$.

Notice that all relations are products of two commutators, and therefore are best thought of as yielding a closed surface of genus 2 in the corresponding presentation complex. As in the heuristic explanation above, the standard generators of the fundamental groups of any two such surfaces are conjugate, but the conjugating elements are different so one can expect that the whole fundamental groups are not conjugate to each other. Indeed, we will show that the surfaces yield distinct elements in homology.

4.2. Properties of the group

A key property of the group G is that its presentation \mathcal{P} satisfies the following small cancellation condition:

LEMMA 4.2. — *The presentation \mathcal{P} satisfies the $C'(1/7)$ small cancellation condition. Moreover, no r_i is a proper power in F_8 .*

Proof. — Observe that $|r_i| = 16i + 8$. Moreover, if $i < k$, then the longest common pieces shared by r_i and r_k (or by their cyclically conjugate words) are of the form

$p = t_j^i a_j^{\pm 1} t_j^{-i}$. Therefore, $|p|/|r_i| \leq (2i+1)/(16i+8) = 1/8 < 1/7$, and $|p|/|r_k| < |p|/|r_i| < 1/7$, and this proves the first statement.

In order to show that no r_i is a proper power, first observe that the elements $t_1^i a_1 t_1^{-i}, t_2^i a_2 t_2^{-i}, t_3^i a_3 t_3^{-i}, t_4^i a_4 t_4^{-i}, t_1, t_2, t_3, t_4$ provide a free basis of F_8 (in fact, they obviously generate F_8 , which is Hopfian). Therefore, we have $r_i = [x, y] \cdot [z, t]$ for elements x, y, z, t of a free basis of F_8 . Let us identify elements of F_8 with their reduced form with respect to this basis, and suppose by contradiction $r_i = h^n$ for some $n > 1$. Since the first letter of r_i is x , we have an equality of reduced words $h = x^m w x^{-m'}$ for some $m \geq 1$ (while possibly $m' = 0$) and some word w in x, y, z, t . We first show that $m = m'$. In fact, if $m > m'$ (resp. $m < m'$), then the reduced form of $r_i = h^n$ would contain at least n occurrences of x with a positive (resp. negative) exponent, against the fact that $r_i = [x, y] \cdot [z, t]$. Hence $m = m'$ and $r_i = h^n = x^m w' x^{-m}$, where w' is the reduced form of w^n . In particular, the last letter of the reduced form of r_i should be equal to x^{-1} , against the fact that $r_i = [x, y] \cdot [z, t]$. This concludes the proof. \square

Let us now describe the second cohomology group of G . We denote by X the presentation complex associated to \mathcal{P} , that is, the CW-complex having exactly one vertex, one edge for each generator of \mathcal{P} , and one 2-cell for each relation of \mathcal{P} (with attaching map prescribed by the corresponding relation). By construction we have $\pi_1(X) \cong G$. Moreover, since the relators of the presentation \mathcal{P} are not proper powers in F_8 , and \mathcal{P} satisfies the $C'(1/7)$ -cancellation property, X is aspherical (see e.g. [Ol'91, Theorem 13.3]), and we have a canonical isomorphism $H^2(G, \mathbb{Z}) \cong H_{\text{cell}}^2(X, \mathbb{Z})$.

We compute the (co)homology of X (hence, of G) via cellular (co)chains. Let c_i be the cell corresponding to the relation r_i . By construction, the boundary of c_i is zero, hence c_i is a cycle which defines a class $[c_i] \in H_{\text{cell}}^2(X, \mathbb{Z})$. For every $\alpha \in H_{\text{cell}}^2(X, \mathbb{Z})$ we set $\alpha_i = \langle \alpha, [c_i] \rangle$, where

$$\langle \cdot, \cdot \rangle: H_{\text{cell}}^2(X, \mathbb{Z}) \times H_{\text{cell}}^2(X, \mathbb{Z}) \rightarrow \mathbb{Z}$$

denotes the Kronecker pairing. We then set

$$\psi: H_{\text{cell}}^2(X, \mathbb{Z}) \rightarrow \mathbb{Z}^{\mathbb{N}}, \quad \psi(\alpha) = (\alpha_0, \dots, \alpha_i, \dots).$$

Since $\partial c_i = 0$ for every $i \in \mathbb{N}$, the space $C_{\text{cell}}^2(X, \mathbb{Z})$ contains no non-trivial coboundaries (indeed, for every $\varphi \in C_{\text{cell}}^1(X, \mathbb{Z})$ and every $i \in \mathbb{N}$ we have $\delta\varphi(c_i) = \varphi(\partial c_i) = 0$, hence $\delta\varphi = 0$). On the other hand, since X is 2-dimensional, every \mathbb{Z} -valued map on the set of 2-cells of X is a cocycle. These facts immediately imply the following:

PROPOSITION 4.3. — *The map $\psi: H_{\text{cell}}^2(X, \mathbb{Z}) \rightarrow \mathbb{Z}^{\mathbb{N}}$ is an isomorphism.*

We have already observed that there is a canonical isomorphism $H^2(G, \mathbb{Z}) \cong H_{\text{cell}}^2(X, \mathbb{Z})$. Henceforth, we will denote by the same symbol cohomology classes which correspond to each other under this isomorphism.

PROPOSITION 4.4. — *Let $\alpha \in H^2(G, \mathbb{Z})$ be bounded. Then the sequence $\psi(\alpha) \in \mathbb{Z}^{\mathbb{N}}$ is bounded.*

Proof. — Let us denote by $\alpha_{\mathbb{R}} \in H^2(X, \mathbb{R})$ the image of α under the change of coefficients map and the canonical isomorphism between cellular and singular

homology. The change of coefficients map $H^2(G, \mathbb{Z}) \rightarrow H^2(G, \mathbb{R})$ is obviously norm non-increasing, and the canonical isomorphism $H^2(G, \mathbb{R}) \cong H^2(X, \mathbb{R})$ is isometric (see e.g. [Fri17, Theorem 5.5]). As a consequence, we have $\|\alpha_{\mathbb{R}}\|_{\infty} \leq \|\alpha\|_{\infty}$.

For every $i \in \mathbb{N}$, the relation r_i is the product of two commutators. As a consequence, there exists a map $j_i: \Sigma_2 \rightarrow X$ such that $j_i^*(\alpha_{\mathbb{R}}) = \alpha_i[\Sigma_2]^*$, where Σ_2 is the genus-2 surface, and $[\Sigma_2]^* \in H^2(\Sigma_2, \mathbb{R})$ is the real fundamental coclass of Σ_2 . A standard duality result between bounded cohomology and ℓ^1 -homology implies that the ℓ^{∞} -norm $\|[\Sigma_2]^*\|_{\infty}$ of $[\Sigma_2]^*$ is the inverse of the simplicial volume of the closed surface of genus 2, which is equal to 4 (see e.g. [Fri17, Proposition 7.10 and Section 8.12]). Moreover, group homomorphisms induce norm non-increasing maps on cohomology. Therefore, we have

$$|\alpha_i| = \frac{\|j_i^*(\alpha_{\mathbb{R}})\|_{\infty}}{\|[\Sigma_2]^*\|_{\infty}} = 4 \|j_i^*(\alpha_{\mathbb{R}})\|_{\infty} \leq 4 \|\alpha_{\mathbb{R}}\|_{\infty} \leq 4 \|\alpha\|_{\infty}$$

whence the conclusion. \square

In order to show that the group G does not satisfy Property QITB, we now look for a weakly bounded class $\alpha \in H^2(G, \mathbb{Z})$ such that $\psi(\alpha)$ is not bounded. To this aim we first give the following:

DEFINITION 4.5. — *We say that a class $\alpha \in H^2(G, \mathbb{Z})$ with $\psi(\alpha) = (\alpha_i)_{i \in \mathbb{N}}$ is slow if*

$$\limsup_{i \rightarrow +\infty} \frac{|\alpha_i|}{i} < +\infty .$$

If α is slow, then we set

$$\Lambda(\alpha) = \sup_{i \in \mathbb{N}} \frac{|\alpha_i|}{2i+1} < +\infty .$$

The rest of this section is devoted to the proof of Theorem 4.10, which states that a class in $H^2(G, \mathbb{Z})$ is weakly bounded if and only if it is slow. Since there obviously exist slow classes $\alpha \in H^2(G, \mathbb{Z})$ for which $\psi(\alpha)$ is not bounded (for example, one may take the class α such that $\psi(\alpha) = (1, 2, 3, 4, \dots)$ or such that $\psi(\alpha) = (\log 1, \log 2, \log 3, \dots)$), Propositions 4.4 and Theorem 4.10 imply the existence of weakly bounded classes in $H^2(G, \mathbb{Z})$ which are not bounded, thus showing that G does not satisfy Property QITB.

It is well known that groups admitting a finite $C'(1/7)$ presentation have linear Dehn function, and are therefore word hyperbolic. We have shown above that $H^2(G, \mathbb{Z})$ is not finitely generated, hence our group G does not admit a finite presentation (thus, it cannot be hyperbolic). Nevertheless, we are now going to study a suitably modified Dehn function for the presentation \mathcal{P} . We first define a notion of area which takes the value $2i+1$ on the relation r_i .

DEFINITION 4.6. — *Let w be a word in $N < F_8$. We then set*

$$A(w) = \min \left\{ \sum_{l=1}^k (2i_l + 1) \left| w = \prod_{l=1}^k w_l r_{i_l}^{\pm 1} w_l^{-1}, \quad k, i_l \in \mathbb{N}, \quad w_l \in F_8 \right. \right\} .$$

The proof of the following proposition provides a linear isoperimetric inequality (with respect to our definition of area) for the infinite presentation \mathcal{P} .

PROPOSITION 4.7. — *For every $w \in N$,*

$$A(w) \leq |w| .$$

Proof. — We prove the statement by induction on $|w|$, the case $|w| = 0$ being obvious. Thus, let $|w| > 0$. By the Greendlinger's Lemma, there exist a subword w_0 of w and a cyclic permutation r'_i of a relation r_i such that w_0 is also an initial subword of r'_i , and $|w_0| > (1 - 3/7)|r_i| = (4/7)|r_i|$. We thus have $r'_i = w_0 v$, $w = u w_0 z$, and $|v| = |r'_i| - |w_0| < (3/7)|r'_i|$. We also have $w = u w_0 z = u r'_i v^{-1} z$, and, since r'_i is conjugate to r_i and w is conjugate to $r'_i v^{-1} z u$,

$$A(w) = A(r'_i v^{-1} z u) \leq 2i + 1 + A(v^{-1} z u) .$$

But

$$|v^{-1} z u| \leq |w| - |w_0| + |v| \leq |w| - (1/7)|r_i| = |w| - (16i + 8)/7 < |w| - 2i - 1 ,$$

hence $A(v^{-1} z u) \leq |w| - 2i - 1$ by our inductive hypothesis, and $A(w) \leq |w|$, as desired. \square

It is proved in [Mil22] that, in the context of cellular cohomology with real coefficients, weakly bounded classes may be characterized as those classes which admit representatives satisfying a linear isoperimetric inequality, in the following sense:

DEFINITION 4.8. — *Let \widetilde{X} be the universal covering of X , endowed with the induced cellular structure. For every $n \in \mathbb{N}$, let us endow the cellular chain module $C_n^{\text{cell}}(\widetilde{X}, \mathbb{R})$ with the ℓ^1 -norm defined by*

$$\left\| \sum_{c \in S_n} a_c \cdot c \right\|_1 = \sum_{c \in S_n} |a_c| ,$$

where S_n is the set of n -cells of \widetilde{X} . Let $\alpha \in H_{\text{cell}}^n(X, \mathbb{R})$. Then α satisfies a linear isoperimetric inequality if it admits a representative $z \in C_{\text{cell}}^n(X, \mathbb{R})$ such that the following condition holds: if $\tilde{z} \in C_{\text{cell}}^n(\widetilde{X}, \mathbb{R})$ is the pull-back of z to \widetilde{X} , then there exists a constant $K \geq 0$ such that

$$|z(c)| \leq K \cdot \|\partial c\|_1 \quad \text{for every } c \in C_n^{\text{cell}}(X, \mathbb{R}) .$$

PROPOSITION 4.9. — *Let G and X be respectively the group and the presentation complex associated to the presentation \mathcal{P} described in Subsection 4.1.*

Let $\alpha \in H^2(G, \mathbb{Z})$ and let $\alpha_{\mathbb{R}} \in H_{\text{cell}}^2(X, \mathbb{R})$ be the corresponding class in the cellular homology of X with real coefficients. Then α is weakly bounded if and only if $\alpha_{\mathbb{R}}$ satisfies a linear isoperimetric inequality.

Proof. — By Lemma 2.8, the class α is weakly bounded if and only if the corresponding class $\alpha_{\mathbb{R}}$ in $H^2(G, \mathbb{R})$ is so. Moreover, Proposition 3.1 shows that the set of weakly bounded classes in $H^2(G, \mathbb{R})$ coincides with the kernel of the map $\iota^2: H^2(G, \mathbb{R}) \rightarrow H_{(\infty)}^*(G, \mathbb{R})$ described in Section 3. Now the conclusion follows from [Mil22, Theorem 1.8], together with the fact (also proved in [Mil22]) that the

following diagram is commutative:

$$\begin{array}{ccc} H^*(G, \mathbb{R}) & \xrightarrow{\iota^*} & H^*_{(\infty)}(G, \mathbb{R}) \\ \downarrow \cong & & \downarrow \cong \\ H^*_{\text{cell}}(X, \mathbb{R}) & \xrightarrow{\iota_X^*} & H^*_{(\infty)}(\widetilde{X}, \mathbb{R}) . \end{array}$$

□

As explained above, the following theorem concludes the proof of Theorem 1.2 from the introduction.

THEOREM 4.10. — *Let $\alpha \in H^2(G, \mathbb{Z})$. Then α is weakly bounded if and only if it is slow.*

Proof. — By Proposition 4.9, we need to show that the class $\alpha_{\mathbb{R}} \in H^2_{\text{cell}}(X, \mathbb{R})$ corresponding to α satisfies a linear isoperimetric inequality if and only if $\Lambda(\alpha) < +\infty$.

We have already observed that there are no coboundaries in $C^2_{\text{cell}}(X, \mathbb{R})$, hence the class $\alpha_{\mathbb{R}}$ admits a unique representative in $C^2_{\text{cell}}(X, \mathbb{R})$, namely the function (still denoted by $\alpha_{\mathbb{R}}$) which takes the value α_i on the cell c_i . We denote by $\tilde{\alpha}_{\mathbb{R}}$ the pull-back of $\alpha_{\mathbb{R}}$ to \widetilde{X} . Moreover, for every $i \in \mathbb{N}$ we fix a lift \tilde{c}_i of c_i to \widetilde{X} .

Using that the relation r_i has length $16i + 8$, it is not difficult to show that, if \tilde{c}_i is a lift of c_i to \widetilde{X} , then $\|\partial\tilde{c}_i\|_1 = 16i + 8$. Therefore, if $\alpha_{\mathbb{R}}$ satisfies a linear isoperimetric inequality, then there exists $K \geq 0$ such that

$$|\alpha_i| = |\alpha_{\mathbb{R}}(c_i)| = |\tilde{\alpha}_{\mathbb{R}}(\tilde{c}_i)| \leq K \cdot (16i + 8) ,$$

hence $\Lambda(\alpha) < +\infty$.

Let us now suppose that α is slow, so that there exists $\Lambda \geq 0$ such that $|\alpha_i| \leq \Lambda(2i + 1)$ for every $i \in \mathbb{N}$. Let $\tilde{c} \in C^2_{\text{cell}}(\widetilde{X}, \mathbb{R})$ be any chain. By [AG99, Theorem 3.3], we have $\partial\tilde{c} = \sum_{h=1}^k a_h \gamma_h$, where $a_h \in \mathbb{R}$ and γ_h is (the integral chain corresponding to) a simple closed loop in the 1-skeleton of \widetilde{X} . Moreover, the γ_h are *coherent* (according to the terminology of [AG99]), which is equivalent to the fact that $\|\partial\tilde{c}\|_1 = \sum_{h=1}^k |a_h| \cdot \|\gamma_h\|_1$. Simple closed loops in the 1-skeleton of \widetilde{X} correspond to words in $N < F_8$, while (conjugates of) relations in \mathcal{P} correspond to (translates of) cells \tilde{c}_i in \widetilde{X} . Under this correspondence, Proposition 4.7 may be restated as follows: for every $h = 1, \dots, k$, we have $\gamma_h = \partial b_h$, where

$$b_h = \sum_{j=1}^l g_j \cdot \tilde{c}_{i_j} ,$$

for some g_1, \dots, g_l in $G = \pi_1(X) \cong \text{Aut}(\widetilde{X})$, and $\sum_{j=1}^l (2i_j + 1) \leq \|\gamma_h\|_1$. As a consequence,

$$|\tilde{\alpha}_{\mathbb{R}}(b_h)| = \left| \sum_{j=1}^l \tilde{\alpha}_{\mathbb{R}}(g_j \cdot \tilde{c}_{i_j}) \right| = \left| \sum_{j=1}^l \alpha_{\mathbb{R}}(c_{i_j}) \right| \leq \sum_{j=1}^l \Lambda(2i_j + 1) \leq \Lambda \|\gamma_h\|_1 .$$

By construction, we have $\partial\tilde{c} = \sum_{h=1}^k a_h \partial b_h$, thus the chain $\tilde{c} - \sum_{h=1}^k a_h b_h$ is a cycle, hence a boundary (recall that X is aspherical, hence \widetilde{X} is contractible). But \widetilde{X} is a

2-dimensional cellular complex, thus $\tilde{c} = \sum_{h=1}^k a_h b_h$ and

$$|\tilde{\alpha}_{\mathbb{R}}(\tilde{c})| = \left| \tilde{\alpha}_{\mathbb{R}} \left(\sum_{h=1}^k a_h b_h \right) \right| \leq \sum_{h=1}^k |a_h| \cdot |\tilde{\alpha}_{\mathbb{R}}(b_h)| \leq \sum_{h=1}^k \Lambda |a_h| \cdot \|\gamma_h\|_1 = \Lambda \|\partial \tilde{c}\|_1 .$$

This shows that $\alpha_{\mathbb{R}}$ satisfies a linear isoperimetric inequality, and concludes the proof of Theorem 4.10. \square

5. Groups with Property QITB

The main goal of this section is to prove Theorem 1.4. We prove (1) in Corollary 5.2, (2) in Theorem 5.10, (3) follows from Corollary 5.9 together with Remark 5.6, and (4) is Theorem 5.22.

Amenable groups

We first prove that amenable groups satisfy Property QITB.

PROPOSITION 5.1. — *Let G be an amenable group, and let $\alpha \in H^n(G, \mathbb{R})$ be weakly bounded. Then $\alpha = 0$.*

Proof. — Let μ be a right-invariant mean on $\ell^\infty(G, \mathbb{R})$. Let $\omega \in C^n(G, \mathbb{R})$ be a weakly bounded representative of α , and define $f \in C^{n-1}(G, \mathbb{R})$ as follows. For every $(g_2, \dots, g_n) \in G^{n-1}$, the function

$$h_{(g_2, \dots, g_n)} : G \rightarrow \mathbb{R}, \quad h_{(g_2, \dots, g_n)}(g_1) = \omega(g_1, g_2, \dots, g_n)$$

is bounded. We may thus set

$$f(g_2, \dots, g_n) = \mu \left(h_{(g_2, \dots, g_n)} \right) \quad \text{for every } (g_2, \dots, g_n) \in G^{n-1} .$$

For every $g \in G$ let us denote by $r_g : G \rightarrow G$ the right multiplication by g . Since ω is a cocycle, for every $(g_2, \dots, g_{n+1}) \in G^n$ the function

$$h_{(g_3, \dots, g_{n+1})} \circ r_{g_2} - \left(\sum_{i=2}^{n+1} (-1)^i h_{(g_2, \dots, g_i g_{i+1}, \dots, g_{n+1})} \right) - (-1)^{n+1} h_{(g_2, \dots, g_n)}$$

takes the constant value $\omega(g_2, \dots, g_{n+1})$ on G . Therefore, by using that μ is linear and right-invariant, we have

$$\begin{aligned} & \omega(g_2, \dots, g_{n+1}) \\ &= \mu \left(h_{(g_3, \dots, g_{n+1})} \right) - \left(\sum_{i=2}^{n+1} (-1)^i \mu \left(h_{(g_2, \dots, g_i g_{i+1}, \dots, g_{n+1})} \right) \right) - (-1)^{n+1} \mu \left(h_{(g_2, \dots, g_n)} \right) \\ &= \delta(f)(g_2, \dots, g_{n+1}) . \end{aligned}$$

Thus ω is a coboundary, and $\alpha = 0$, as desired. \square

Putting together Corollary 2.9 and Proposition 5.1 we get the following:

COROLLARY 5.2. — *Every amenable group satisfies Property QITB.*

Remark 5.3. — Together with Proposition 1.11, Proposition 5.1 implies that, if G is amenable, then the map

$$\iota^n: H^n(G, \mathbb{R}) \rightarrow H_{(\infty)}^n(G, \mathbb{R})$$

is injective. This result was first proved in [Ger, Theorem 10.13] under the assumption that G admits an Eilenberg–MacLane model with a finite n -skeleton, and by Wienhard [Wie12, Proposition 5.3] in the general case (see also [Bla14, Section 6.3]).

In fact, Proposition 5.1 could be deduced from Proposition 1.11 and [Wie12, Proposition 5.3]. We preferred to add here our proof of Proposition 5.1 because it is very elementary and self-contained.

Other examples

Let G be a group. We say that a class $\alpha \in H_2(G, \mathbb{R})$ is *amenable* if there exist an amenable group A and a homomorphism $f: A \rightarrow G$ such that α lies in the image of $f_*: H_2(A, \mathbb{R}) \rightarrow H_2(G, \mathbb{R})$. The amenable classes generate a linear subspace of $H_2(G, \mathbb{R})$ that we denote by $H_2^{\text{am}}(G, \mathbb{R})$. It readily follows from the definitions that, if $f: G_1 \rightarrow G_2$ is a homomorphism, then

$$f_*(H_2^{\text{am}}(G_1, \mathbb{R})) \subseteq H_2^{\text{am}}(G_2, \mathbb{R}) .$$

Remark 5.4. — It is well known that any element $\alpha \in H_2(G, \mathbb{R})$ is represented by a surface, i.e. that $\alpha = f_*(\beta)$ for some $\beta \in H_2(\Gamma_g, \mathbb{R})$, where Γ_g is the fundamental group of the closed connected orientable surface of genus g , $g \geq 1$. In fact, one may define the *genus* of a class α as the minimal $g \in \mathbb{N}$ such that $\alpha = f_*(\beta)$ for some $\beta \in H_2(\Gamma_g, \mathbb{R})$. The class α is *toral* if its genus is equal to or smaller than 1.

Observe that, if $K < \Gamma_g$ and $g \geq 2$, then $H_2^{\text{am}}(K, \mathbb{R}) = 0$. In fact, if the index of K in Γ_g is infinite, then K is free, and $H_2(K, \mathbb{R}) = 0$ (hence, a fortiori, $H_2^{\text{am}}(K, \mathbb{R}) = 0$), while if the index of K in Γ_g is finite, then $K = \Gamma_{g'}$ for some $g' \geq g \geq 2$. Since any amenable subgroup of $\Gamma_{g'}$ is either trivial or infinite cyclic, and the degree-2 homology with real coefficients of infinite cyclic groups vanishes, also in this case we may conclude that $H_2^{\text{am}}(K, \mathbb{R}) = 0$. Building on this remark, one may wonder whether amenable classes defined above should in fact be toral. However, it is shown in [BG88] that, for every $g \in \mathbb{N}$, there exist a nilpotent (hence, amenable) group N and a class $\alpha \in H_2(N, \mathbb{R})$ which does not lie in the subspace generated by classes of $H_2(N, \mathbb{R})$ with genus smaller than g .

Recall that there exists a duality pairing

$$\langle \cdot, \cdot \rangle: H^2(G, \mathbb{R}) \times H_2(G, \mathbb{R}) \rightarrow \mathbb{R} .$$

We denote by $\text{Ann}(H_2^{\text{am}}(G, \mathbb{R})) \subseteq H^2(G, \mathbb{R})$ the annihilator of $H_2^{\text{am}}(G, \mathbb{R})$ in $H^2(G, \mathbb{R})$, i.e. the subspace of coclasses $\varphi \in H^2(G, \mathbb{R})$ for which

$$\langle \varphi, \alpha \rangle = 0 \quad \forall \alpha \in H_2^{\text{am}}(G, \mathbb{R}) .$$

Notice that $\varphi \in \text{Ann}(H_2^{\text{am}}(G, \mathbb{R}))$ if and only if $\langle \varphi, \alpha \rangle = 0$ for all amenable classes α (amenable classes might form a proper subset of $H_2^{\text{am}}(G, \mathbb{R})$). If $f: G_1 \rightarrow G_2$ is a homomorphism, then

$$f^*(\text{Ann}(H_2^{\text{am}}(G_2, \mathbb{R}))) \subseteq \text{Ann}(H_2^{\text{am}}(G_1, \mathbb{R})) .$$

DEFINITION 5.5. — We say that a group G has Property $(*)$ if $H_2^{\text{am}}(G, \mathbb{R}) = H_2(G, \mathbb{R})$.

Of course, any amenable group has Property $(*)$. Interesting non-amenable examples are given by:

Remark 5.6. — If G is a right-angled Artin group, then G admits a classifying space (the Salvetti complex) whose 2-skeleton is obtained by gluing tori. From this, one can deduce that $H_2(G, \mathbb{R})$ is generated by toral classes, so that, in particular, we see that right-angled Artin groups have Property $(*)$.

DEFINITION 5.7. — We say that a group G has Property $(**)$ if every class in $\text{Ann}(H_2^{\text{am}}(G, \mathbb{R}))$ is bounded, i.e.

$$\text{Ann}(H_2^{\text{am}}(G, \mathbb{R})) \subseteq c\left(H_b^2(G, \mathbb{R})\right) ,$$

where $c: H_b^2(G, \mathbb{R}) \rightarrow H^2(G, \mathbb{R})$ is the comparison map.

Of course, if a group G has Property $(*)$ then it also has Property $(**)$. Property $(**)$ is significant in our context due to the following:

PROPOSITION 5.8. — Let $\alpha \in H^2(G, \mathbb{R})$ be weakly bounded. Then $\alpha \in \text{Ann}(H_2^{\text{am}}(G, \mathbb{R}))$.

Proof. — Let $\alpha \in H^2(G, \mathbb{R})$ be a weakly bounded class, and let β be an amenable class. Then there exist an amenable group A and a homomorphism $f: A \rightarrow G$ such that $\beta = f_*(\beta_A)$ for some $\beta_A \in H_2(A, \mathbb{R})$. Being the pull-back of a weakly bounded class, the element $f^*(\alpha) \in H^2(A, \mathbb{R})$ is weakly bounded itself. Since A is amenable, Proposition 5.1 ensures that $f^*(\alpha) = 0$, hence

$$\langle \alpha, \beta \rangle = \langle \alpha, f_*(\beta_A) \rangle = \langle f^*(\alpha), \beta_A \rangle = 0 .$$

We have thus shown that α belongs to $\text{Ann}(H_2^{\text{am}}(G, \mathbb{R}))$. \square

Proposition 5.8 and Corollary 2.9 readily imply the following:

COROLLARY 5.9. — Suppose G has Property $(**)$. Then G satisfies QITB.

THEOREM 5.10. — Let G be relatively hyperbolic w.r.t. the finite collection of subgroups $\mathcal{H} = \{H_1, \dots, H_k\}$, and suppose that every H_i has Property $(*)$. Then G has Property $(**)$.

Proof. — Let us consider the commutative diagram

$$\begin{array}{ccccc} H_b^2(G, \mathcal{H}, \mathbb{R}) & \xrightarrow{i_b} & H_b^2(G, \mathbb{R}) & & \\ \downarrow c_1 & & \downarrow c_2 & & \\ H^2(G, \mathcal{H}, \mathbb{R}) & \xrightarrow{i} & H^2(G, \mathbb{R}) & \xrightarrow{j} & \bigoplus_{i=1}^k H^2(H_i, \mathbb{R}) , \end{array}$$

where the bottom row is exact (see e.g. [BE78]). Let $\alpha \in \text{Ann}(H_2^{\text{am}}(G, \mathbb{R}))$. Since group homomorphisms preserve the annihilators of amenable classes, $j(\alpha) \in \bigoplus_{i=1}^k \text{Ann}(H_2^{\text{am}}(H_i, \mathbb{R}))$. Since each H_i has Property $(*)$, this means that $j(\alpha) = 0$, hence $\alpha = i(\beta)$ for some $\beta \in H_b^2(G, \mathcal{H}, \mathbb{R})$. Now the comparison map in relative cohomology for relative hyperbolic pairs is surjective [Fra18], hence $\beta = c_1(\eta)$ for some $\eta \in H_b^2(G, \mathcal{H}, \mathbb{R})$, and $\alpha = c_2(i_b(\eta))$ is a bounded class in $H^2(G, \mathbb{R})$, as desired. \square

COROLLARY 5.11. — *Let G be relatively hyperbolic w.r.t. the finite collection of subgroups $\mathcal{H} = \{H_1, \dots, H_k\}$, and suppose that every H_i is amenable. Then G satisfies Property QITB.*

As already mentioned in the statement of Theorem 1.4, the previous corollary implies that every hyperbolic group satisfies Property QITB. In order to prove this fact, however, there is no need to introduce the machinery above: indeed, Neumann and Reeves [NR97] and Mineyev [Min01] proved that, if G is hyperbolic, then every class in $H^2(G, \mathbb{Z})$ is bounded, hence G trivially satisfies Property QITB.

Virtually trivial central extensions

DEFINITION 5.12. — *We say that a central extension*

$$1 \longrightarrow Z \longrightarrow E \xrightarrow{\pi} G \longrightarrow 1$$

is virtually trivial if there exists a finite-index subgroup G' of G such that the induced extension

$$1 \longrightarrow Z \longrightarrow \pi^{-1}(G') \xrightarrow{\pi} G' \longrightarrow 1$$

is trivial.

LEMMA 5.13. — *A central extension by a finitely generated torsion-free abelian group is virtually trivial if and only if its Euler class has finite order.*

Proof. — Let $1 \longrightarrow Z \longrightarrow E \xrightarrow{\pi} G \longrightarrow 1$ be a central extension with Euler class $\alpha \in H^2(G, Z)$.

Suppose first that α has order $n \in \mathbb{N}$, $n > 0$, and let ω be a representative of α . Then $n\omega = \delta f$ for some 1-cochain $f \in C^1(G, Z)$. Thus f defines a homomorphism $\hat{f}: G \rightarrow Z/nZ$. Since the target group is finite, the kernel is a finite index subgroup $G' < G$. Take $g \in G'$. Then $f(g) \in nZ$. Since Z is torsion-free, there exists a unique element $h(g) = f(g)/n \in Z$, and we have $\omega = \delta h$ on G' . This implies that the restriction of ω to G' is a coboundary, which in turn shows that the induced extension of G' is trivial.

Conversely, suppose that there exists a subgroup G' of G of index $n \in \mathbb{N}$, $n > 0$, such that the induced extension of G' is trivial. Let $\text{res}: H^2(G, Z) \rightarrow H^2(G', Z)$ and $\text{trans}: H^2(G', Z) \rightarrow H^2(G, Z)$ be the restriction and the transfer map, respectively, and recall that $\text{trans} \circ \text{res}: H^2(G, Z) \rightarrow H^2(G, Z)$ is the multiplication by n (see e.g. [Bro82, Proposition 9.5]). We then have $\text{res}(\alpha) = 0$, whence $n\alpha = \text{trans}(\text{res}(\alpha)) = 0$, i.e. α has finite order in $H^2(G, Z)$. \square

LEMMA 5.14. — *Let G be a finitely generated group, let Z be a finitely generated abelian group, and let $j: H^2(G, Z) \rightarrow H^2(G, Z \otimes \mathbb{R})$ be the change of coefficients map induced by the natural map $Z \rightarrow Z \otimes \mathbb{R}$. Then $\ker j$ is equal to the torsion subgroup of $H^2(G, Z)$.*

Proof. — Since $H^2(G, Z \otimes \mathbb{R})$ is torsion-free, every element of $H^2(G, Z)$ of finite order lies in $\ker j$.

Let now $\alpha \in \ker j$. By looking at the commutative diagram

$$\begin{array}{ccc} H^2(G, Z) & \xrightarrow{\varphi} & \operatorname{Hom}_{\mathbb{Z}}(H_2(G, \mathbb{Z}), Z) \\ \downarrow j & & \downarrow j' \\ H^2(G, Z \otimes \mathbb{R}) & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}(H_2(G, \mathbb{Z}), Z \otimes \mathbb{R}) \end{array}$$

we obtain that $\varphi(\alpha)$ lies in $\ker j'$, and this readily implies that $\varphi(\alpha)$ has finite order, i.e. there exists $n \in \mathbb{Z} \setminus \{0\}$ such that $\varphi(n\alpha) = 0$. Indeed, for every $c \in H_2(G, \mathbb{Z})$ we have that $\varphi(\alpha)(c)$ is an element of the kernel of the map $Z \mapsto Z \otimes \mathbb{R}$, and there exists some (uniform) n such that for any such element z we have $nz = 0$.

By the Universal Coefficient Theorem for cohomology we now have the exact sequence

$$0 \rightarrow \operatorname{Ext}^1(H_1(G, \mathbb{Z}), Z) \rightarrow H^2(G, Z) \rightarrow \operatorname{Hom}_{\mathbb{Z}}(H_2(G, \mathbb{Z}), Z) \rightarrow 0.$$

Since G and Z are finitely generated, $\operatorname{Ext}^1(H_1(G, \mathbb{Z}), Z)$ is a torsion group. Thus $n\alpha$, which belongs to (the image of) $\operatorname{Ext}^1(H_1(G, \mathbb{Z}), Z)$, has finite order, and hence α has finite order as well. \square

Since amenable groups and right-angled Artin groups satisfy Property (*), the following result implies Theorem 1.5 from the introduction.

THEOREM 5.15. — *Suppose that the group G satisfies Property (*). Then a central extension of G by the finitely generated abelian group Z is quasi-isometrically trivial if and only if its Euler class has finite order. Moreover, if Z is torsion-free then this happens if and only if the extension is virtually trivial.*

Proof. — Let Z be any finitely generated abelian group. We first prove that a class in $H^2(G, Z)$ is bounded if and only if it has finite order.

By Proposition 5.8, any bounded class $\alpha \in H^2(G, \mathbb{R})$ vanishes on $H_2^{\text{am}}(G, \mathbb{R}) = H_2(G, \mathbb{R})$. By the Universal Coefficient Theorem, this implies that $\alpha = 0$. Therefore, the comparison map $c^2: H_b^2(G, \mathbb{R}) \rightarrow H^2(G, \mathbb{R})$ is null. Observe now that $Z \cong \mathbb{Z}^k \oplus F$, where F is finite. Then $Z \otimes \mathbb{R} \cong \mathbb{R}^k$, thus the comparison map $H_b^2(G, Z \otimes \mathbb{R}) \rightarrow H^2(G, Z \otimes \mathbb{R})$ is null. By looking at the commutative diagram

$$\begin{array}{ccc} H_b^2(G, Z) & \longrightarrow & H^2(G, Z) \\ \downarrow & & \downarrow j \\ H_b^2(G, Z \otimes \mathbb{R}) & \longrightarrow & H^2(G, Z \otimes \mathbb{R}) \end{array}$$

we can then deduce that every bounded class in $H^2(G, Z)$ is contained in $\ker j$. By Lemma 5.14, we conclude that bounded classes have finite order in $H^2(G, Z)$.

On the other hand, if $\alpha \in H^2(G, Z)$ has finite order, then $j(\alpha) = 0$ in $H^2(G, Z \otimes \mathbb{R})$. The very same argument as in the proof of Lemma 2.8 now shows that α is bounded.

We have thus shown that a class in $H^2(G, Z)$ is bounded if and only if it has finite order. By Corollary 5.9, this implies that a central extension of G by Z is quasi-isometrically trivial if and only if its Euler class has finite order. We conclude applying Lemma 5.13. \square

Remark 5.16. — Lemma 5.13 (and Theorem 5.15) cannot hold in general for extensions by finitely generated abelian groups with torsion. For example, let us consider Thompson's group T . It is shown in [GS87] that $H^1(T, \mathbb{Z}) = 0$, $H^2(T, \mathbb{Z}) = \mathbb{Z}^2$. If $n > 1$ is any integer, we then deduce from the Universal Coefficient Theorem that $H^2(T, \mathbb{Z}_n) = \mathbb{Z}_n^2$. In particular, there exists a non-trivial class $\alpha \in H^2(T, \mathbb{Z}_n)$. This class has obviously finite order. Nevertheless, the unique finite-index subgroup of T is T itself, thus α does not vanish on any finite-index subgroup of T . The central extension of T by \mathbb{Z}_n with Euler class α is quasi-isometrically trivial (since its Euler class is obviously bounded), but not virtually trivial.

(Amalgamated) products of groups with Property QITB

We now prove Propositions 1.6 and 1.7 from the introduction.

PROPOSITION 5.17 (Proposition 1.6.). — *Let G_1, G_2 be groups satisfying Property QITB. Then the direct product $G_1 \times G_2$ satisfies Property QITB.*

Proof. — As usual, we prove that every weakly bounded class in $H^2(G, \mathbb{Z})$ is bounded, under the assumption that the same condition holds in $H^2(G_i, \mathbb{Z})$, $i = 1, 2$.

Let $\alpha \in H^2(G, \mathbb{Z})$ be weakly bounded, and denote by $p_i: G \rightarrow G_i$ the projection, and by $j_i: G_i \rightarrow G_1 \times G_2$ the inclusion. By the Künneth formula, we have

$$H_2(G_1 \times G_2, \mathbb{Z}) \cong \left(H_1(G_1, \mathbb{Z}) \otimes H_1(G_2, \mathbb{Z}) \right) \oplus H_2(G_1, \mathbb{Z}) \oplus H_2(G_2, \mathbb{Z}) .$$

More precisely, any element $\beta \in H_2(G_1 \times G_2, \mathbb{Z})$ may be decomposed as

$$\beta = \bar{\beta} + (j_1)_*(\beta_1) + (j_2)_*(\beta_2) ,$$

where $\bar{\beta}$ corresponds to a class in $H_1(G_1, \mathbb{Z}) \otimes H_1(G_2, \mathbb{Z})$ under the above identification, and $\beta_i \in H_2(G_i, \mathbb{Z})$ for $i = 1, 2$. Let now $\alpha_{\mathbb{R}} \in H^2(G_1 \times G_2, \mathbb{R})$, $\bar{\beta}_{\mathbb{R}} \in H_2(G_1 \times G_2, \mathbb{R})$ be the classes corresponding to $\alpha, \bar{\beta}$ under the change of coefficients homomorphism (thus, $\alpha_{\mathbb{R}}$ is also weakly bounded).

Since any class in $H_1(G_i, \mathbb{Z})$ is the pushforward of a class in $H_1(\mathbb{Z}, \mathbb{Z})$ via some homomorphism $f_i: \mathbb{Z} \rightarrow G_i$, the class $\bar{\beta}$ is the push-forward of an element of $H_2(\mathbb{Z} \times \mathbb{Z}, \mathbb{Z})$ via a homomorphism $\mathbb{Z} \times \mathbb{Z} \rightarrow G_1 \times G_2$ of the form $f_1 \times f_2$. As a consequence, $\bar{\beta}_{\mathbb{R}}$ is amenable, and $(p_i)_*(\bar{\beta}) = 0$ for $i = 1, 2$. Since $\alpha_{\mathbb{R}}$ is weakly bounded, by Proposition 5.8 we thus get $\langle \alpha, \bar{\beta} \rangle = \langle \alpha_{\mathbb{R}}, \bar{\beta}_{\mathbb{R}} \rangle = 0$. Moreover, for $i = 1, 2$ we have $\langle p_i^*(j_i^*(\alpha)), \bar{\beta} \rangle = \langle j_i^*(\alpha), (p_i)_*(\bar{\beta}) \rangle = 0$.

Let us now set

$$\alpha' = \alpha - p_1^*(j_1^*(\alpha)) - p_2^*(j_2^*(\alpha)) .$$

We will prove that α' vanishes on β , by showing that it vanishes on its summands $\bar{\beta}$, $(j_1)_*(\beta_1)$ and $(j_2)_*(\beta_2)$.

The above computations show that

$$\langle \alpha', \bar{\beta} \rangle = \langle \alpha, \bar{\beta} \rangle - \langle p_1^*(j_1^*(\alpha)), \bar{\beta} \rangle - \langle p_2^*(j_2^*(\alpha)), \bar{\beta} \rangle = 0 .$$

Moreover, we have

$$\begin{aligned} \langle \alpha', (j_1)_*(\beta_1) \rangle &= \langle \alpha, (j_1)_*(\beta_1) \rangle - \langle p_1^*(j_1^*(\alpha)), (j_1)_*(\beta_1) \rangle - \langle p_2^*(j_2^*(\alpha)), (j_1)_*(\beta_1) \rangle \\ &= \langle j_1^*(\alpha), \beta_1 \rangle - \langle j_1^*(\alpha), (p_1 \circ j_1)_*(\beta_1) \rangle - \langle j_2^*(\alpha), (p_2 \circ j_1)_*(\beta_1) \rangle \\ &= \langle j_1^*(\alpha), \beta_1 \rangle - \langle j_1^*(\alpha), \beta_1 \rangle - \langle j_2^*(\alpha), 0 \rangle = 0, \end{aligned}$$

and a similar computation implies that $\langle \alpha', (j_2)_*(\beta_2) \rangle = 0$ too.

We have thus shown that $\alpha = p_1^*(j_1^*(\alpha)) + p_2^*(j_2^*(\alpha)) + \alpha'$, where $\alpha' \in H^2(G, \mathbb{Z})$ is such that $\langle \alpha', \beta \rangle = 0$ for every $\beta \in H_2(G, \mathbb{Z})$. Since α is weakly bounded, $j_i^*(\alpha) \in H^2(G_i, \mathbb{Z})$ is also weakly bounded, for $i = 1, 2$. But G_i satisfies Property QITB, hence $j_i^*(\alpha)$ is bounded for $i = 1, 2$, and also $p_1^*(j_1^*(\alpha)) + p_2^*(j_2^*(\alpha))$ is bounded.

In order to conclude, it is thus sufficient to show that α' is also bounded. However, since $\langle \alpha', \beta \rangle = 0$ for every $\beta \in H_2(G, \mathbb{Z})$, the image of α' in $H^2(G, \mathbb{R})$ is trivial, hence α' is bounded by Lemma 2.8. \square

DEFINITION 5.18. — *An amalgamated product $G = G_1 *_H G_2$ is transverse if, denoting $i_1: H \rightarrow G_1$ and $i_2: H \rightarrow G_2$ the inclusions defining the amalgamated product, the map*

$$(i_1)_* \oplus (i_2)_*: H_1(H, \mathbb{R}) \rightarrow H_1(G_1, \mathbb{R}) \oplus H_1(G_2, \mathbb{R})$$

is injective.

PROPOSITION 5.19 (Proposition 1.7). — *Let $G = G_1 *_H G_2$ be a transverse amalgamated product, where H is amenable. If G_1, G_2 satisfy Property QITB, then G satisfies Property QITB.*

Proof. — Let $\alpha \in H^2(G, \mathbb{Z})$ be weakly bounded, and denote by $\alpha_{\mathbb{R}} \in H^2(G, \mathbb{R})$ the image of α via the change of coefficients map. Then $\alpha_{\mathbb{R}}$ is weakly bounded, and by Lemma 2.8 we are left to show that $\alpha_{\mathbb{R}}$ is bounded.

By definition of transverse amalgamated product, the map $H_1(H, \mathbb{R}) \rightarrow H_1(G_1, \mathbb{R}) \oplus H_1(G_2, \mathbb{R})$ is injective. By looking at the Mayer-Vietoris sequence for the triple G_1, G_2, H , one can then deduce that the map $H_2(G_1, \mathbb{R}) \oplus H_2(G_2, \mathbb{R}) \rightarrow H_2(G, \mathbb{R})$ is surjective, which implies in turn that the restriction map $r: H^2(G, \mathbb{R}) \rightarrow H^2(G_1, \mathbb{R}) \oplus H^2(G_2, \mathbb{R})$ is injective.

Let us now consider the commutative diagram

$$\begin{array}{ccc} H_b^2(G, \mathbb{R}) & \xrightarrow{r_b} & H_b^2(G_1, \mathbb{R}) \oplus H_b^2(G_2, \mathbb{R}) \\ \downarrow c & & \downarrow c' \\ H^2(G, \mathbb{R}) & \xrightarrow{r} & H^2(G_1, \mathbb{R}) \oplus H^2(G_2, \mathbb{R}) \end{array}$$

Since α is weakly bounded, its restrictions in $H^2(G_1, \mathbb{Z})$ and in $H^2(G_2, \mathbb{Z})$ are also weakly bounded, hence bounded, since G_1 and G_2 satisfy QITB. Together with the fact that the change of coefficients map takes bounded cochains to bounded cochains (and commutes with restrictions), this implies that $r(\alpha_{\mathbb{R}})$ lies in the image of the comparison map c' . Since H is amenable, the map r_b is surjective (see [BBF⁺14]), hence there exists $\beta \in H_b^2(G, \mathbb{R})$ such that $c'(r_b(\beta)) = r(\alpha_{\mathbb{R}})$. Using that r is injective we then get $c(\beta) = \alpha_{\mathbb{R}}$, i.e. $\alpha_{\mathbb{R}}$ lies in the image of the comparison map. This concludes the proof. \square

Remark 5.20. — Let G_1, G_2, H and G be as in the statements of the previous propositions. The proofs above may be easily adapted to show the following:

- (1) If both G_1 and G_2 have Property $(*)$ (resp. $(**)$), then $G_1 \times G_2$ has Property $(*)$ (resp. $(**)$).
- (2) If both G_1 and G_2 have Property $(*)$ (resp. $(**)$), then $G_1 *_H G_2$ has Property $(*)$ (resp. $(**)$).

3-manifold groups

In order to prove that 3-manifold groups also satisfy Property QITB we first recall that $C_2(G, \mathbb{R})$ is endowed with an ℓ^1 -norm such that $\|c\|_1 = \sum |a_{(g_1, g_2)}|$ for every chain $c = \sum a_{(g_1, g_2)}(g_1, g_2)$. We then endow $H_2(G, \mathbb{R})$ with the induced quotient ℓ^1 seminorm (which is sometimes called the *Gromov seminorm*) such that, if $\beta \in H_2(G, \mathbb{R})$, then $\|\beta\|_1$ is the infimum of the ℓ^1 -norms of the representatives of β in $C_2(G, \mathbb{R})$. Let us denote by $N_2(G, \mathbb{R})$ the subspace of $H_2(G, \mathbb{R})$ given by classes with vanishing ℓ^1 -seminorm. It is well known that, if A is an amenable group, then the ℓ^1 -seminorm vanishes on $H_2(A, \mathbb{R})$. Since group homomorphisms induce seminorm non-increasing maps on homology, this readily implies that $H_2^{\text{am}}(G, \mathbb{R}) \subseteq N_2(G, \mathbb{R})$.

PROPOSITION 5.21. — *Let G be a group such that $H_2(G, \mathbb{R})$ is finite dimensional (this is the case, e.g., if G is finitely presented). Then G satisfies $(**)$ if and only if $N_2(G, \mathbb{R}) = H_2^{\text{am}}(G, \mathbb{R})$.*

Proof. — Suppose first that $N_2(G, \mathbb{R}) = H_2^{\text{am}}(G, \mathbb{R})$, and take $\alpha \in \text{Ann}(H_2^{\text{am}}(G, \mathbb{R})) = \text{Ann}(N_2(G, \mathbb{R}))$. Then α defines a linear map $H_2(G, \mathbb{R})/N_2(G, \mathbb{R}) \rightarrow \mathbb{R}$. Since $H_2(G, \mathbb{R})$ is finite dimensional, this map is continuous with respect to the quotient ℓ^1 -norm on $H_2(G, \mathbb{R})/N_2(G, \mathbb{R})$. By [BG88, Proposition 1.1], this implies that α may be represented by a bounded cocycle. Thus G satisfies $(**)$.

Suppose now that $N_2(G, \mathbb{R}) \neq H_2^{\text{am}}(G, \mathbb{R})$, and take an element $\beta \in N_2(G, \mathbb{R}) \setminus H_2^{\text{am}}(G, \mathbb{R})$. By the Universal Coefficient Theorem, we may construct an element $\alpha \in \text{Ann}(H_2^{\text{am}}(G, \mathbb{R}))$ such that $\langle \alpha, \beta \rangle = 1$. Since $\|\beta\|_1 = 0$, the class α cannot be represented by any bounded cocycle. Thus G does not satisfy $(**)$. \square

THEOREM 5.22. — *Let G be the fundamental group of a compact orientable 3-manifold. Then G satisfies Property QITB.*

Proof. — Let M be a compact orientable 3-manifold. The decomposition of M into prime summands M_1, \dots, M_k decomposes G as the free product of the groups $G_i = \pi_1(M_i)$, $i = 1, \dots, k$. By Corollary 1.8, we may thus assume that M is prime. Moreover, if $M \cong S^2 \times S^1$, then $\pi_1(M) = \mathbb{Z}$ obviously satisfies Property QITB, hence we are reduced to study the case when M is irreducible. We will show that, under this assumption, we have that $G = \pi_1(M)$ satisfies $(**)$, hence Property QITB.

Being the fundamental group of a compact manifold, G is finitely presented, hence by Proposition 5.21 it suffices to show that $N_2(G, \mathbb{R}) = H_2^{\text{am}}(G, \mathbb{R})$. Recall that the inclusion $H_2^{\text{am}}(G, \mathbb{R}) \subseteq N_2(G, \mathbb{R})$ always holds, and take an element $\beta \in N_2(G, \mathbb{R})$. If G is finite, then of course $\beta \in H_2^{\text{am}}(G, \mathbb{R})$. Since irreducible 3-manifolds with infinite fundamental groups are aspherical, we may thus identify $H_2(G, \mathbb{R})$ with $H_2(M, \mathbb{R})$.

The module $H_2(M, \mathbb{R})$ is itself endowed with an ℓ^1 -seminorm (see e.g. [Gro82]), and the identification $H_2(G, \mathbb{R}) \cong H_2(M, \mathbb{R})$ is isometric, hence we may consider β as an element of $H_2(M, \mathbb{R})$ with vanishing seminorm. A result of Gabai [Gab83, Corollary 6.18] now ensures that, since $\|\beta\|_1 = 0$, also the Thurston norm of β vanishes. Therefore, as an element of $H_2(M, \mathbb{R})$, the class β is represented by a finite union of spheres and tori (in fact, since M is aspherical, by a finite union of tori) [Thu86]. This immediately implies that $\beta \in H_2^{\text{am}}(G, \mathbb{R})$, whence the conclusion. \square

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