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MANIFESTATION OF THE TOPOLOGICAL INDEX FORMULA IN QUANTUM WAVES AND GEOPHYSICAL WAVES

MANIFESTATION DE LA FORMULE DE L'INDICE TOPOLOGIQUE DANS LES ONDES QUANTIQUES ET LES ONDES GÉOPHYSIQUES

ABSTRACT. — Using semi-classical analysis in $\mathbb{R}^n$ we present a quite general model for which the topological index formula of Atiyah–Singer predicts a spectral flow with the transition of a finite number of eigenvalues between clusters (energy bands). This model corresponds to physical phenomena that are well observed for quantum energy levels of small molecules [FZ00, FZ01], also in geophysics for the oceanic or atmospheric equatorial waves [DMV17, Mat66] and expected to be observed in plasma physics [QF22].

RÉSUMÉ. — En utilisant l'analyse semi-classique dans $\mathbb{R}^n$ nous présentons un modèle assez général pour lequel la formule de l'indice topologique d'Atiyah–Singer prédit un flot spectral avec la transition d'un nombre fini de valeurs propres entre des clusters (bandes) d'énergie. Ce modèle correspond à des phénomènes physiques qui sont bien observés pour les niveaux

d'énergie quantiques de petites molécules [FZ00, FZ01], également en géophysique pour les ondes équatoriales océaniques ou atmosphériques [DMV17, Mat66] et que l'on s'attend à observer en physique des plasmas [QF22].

Remark 0.1. — On this pdf file, you can click on the colored words, they contain an hyper-link to wikipedia or other multimedia contents.

1. Introduction

The famous index theorem of Atiyah Singer obtained in the 60' relates two different domains of mathematics: spectral theory of pseudo-differential operators and differential topology [BB85, Fed96]. This theorem has a strong importance in mathematics with many applications (e.g. the Riemann–Roch–Hirzebruch index formula that is used in geometric quantization [Haw00]) but also in physics: in quantum field theory with anomalies [PS95, Chapter 19], in molecular physics with energy spectrum [FZ00, FZ01, FZ02b]. Recently P. Delplace, J.B. Marston and A. Venaille [DMV17] have discovered that a famous model of oceanic equatorial waves established by Matsuno in 1966 [Mat66] has remarkable topological properties, namely that the existence of $\mathcal{N} = +2$ equatorial modes in the Matsuno's model is related to the fact that the dispersion equation of this model defines a vector bundle over[1] S^2 whose topology is characterized by a Chern index with value $\mathcal{C} = +2$. In the similar context of waves but in plasma physics, Hong Qin, Yichen Fu [QF22] have recently predicted a manifestation of the index formula.

In this paper we propose a general mathematical model that contains as particular cases the normal form used for molecular physics in [FZ00, FZ01] and the model of Matsuno [DMV17, Mat66] of equatorial waves. For this general model we have on one side a spectral index $\mathcal{N} \in \mathbb{Z}$ that counts the number of eigenvalues that move upwards as a parameter μ increases and on the other side a topological Chern index $\mathcal{C} \in \mathbb{Z}$ associated to a vector bundle that characterizes the equivalence class of the model. We establish the index formula $\mathcal{N} = \mathcal{C}$.

There are many studies about topological phenomena in condensed matter physics. Closely related to this paper are the works related to bulk-interface correspondence by Guillaume Bal [Bal19], Chris Bourne, Johannes Kellendonk, and Adam Rennie [BKR17, BR18], Alexis Drouot [Dro21], A Elgart, GM Graf, and J.H. Schenker [EGS05], Gian Michele Graf and Marcello Porta [GP13], Yosuke Kubota [Kub17], Emil Prodan and Hermann Schulz-Baldes [PSB16], and Julio Cesar Avila, Hermann Schulz-Baldes, and Carlos Villegas-Blas [ASBVB13]. In particular the work of Alexis Drouot [Dro21] uses microlocal analysis as here. There are also the works by C Dembowski, H.-D. Gräf, H. L. Harney, A. Heine, W. D. Heiss, H. Rehfeld, and A. Richter [DGH+01], Jacob Shapiro and Clément Tauber [ST19], Alex Bols, Jeffrey Schenker, and Jacob Shapiro [BSS21].

The paper is organized as follows. In Section 2 we present the general model and the main result of this paper, Theorem 2.7. In Section 2.5 we give the proof of

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[1] Here S^2 is not related to the surface of the earth but is a surface in $\mathbb{R}^3$ that enclosed a singularity at the origin.

Theorem 2.7. The proof relies on the index theorem on Euclidean space of Fedosov–Hörmander given in [Hör79, Theorem 7.3 p. 422],[BB85, Theorem 1, p. 252] and explained in the appendix.

Sections 3 and 4 are applications of this general model in physics. In Section 3 we present a simple model used in [FZ00, FZ01] to show the manifestation of the index formula in experimental molecular spectra of quantum waves. In Section 4 we present the model of equatorial geophysics waves of Matsuno [Mat66] and the topological interpretation from [DMV17].

The reader may prefer to read first Section 3 and 4 that present the examples with detailed computations before Section 2 that presents the general but more abstract model.

Appendix A gives a short overview of symbols and pseudo-differential operators. Appendix B gives a short overview of vector bundles over spheres.

This article is made from the lecture notes in French [Fau18].

2. A general model on $\mathbb{R}^n$ and index formula

In this Section we propose a general framework that will contains the particular models of molecular physics of Section 3 and of geophysics of Section 4. For this general model we define a spectral index $\mathcal{N}$ that corresponds to the number of eigenvalues that move upwards with respect to an external parameter μ and we define a topological index (Chern index) $\mathcal{C}$ of a vector bundle that characterizes the (stable) isomorphism class of the model. We establish the index formula $\mathcal{N} = \mathcal{C}$.

2.1. Admissible family of symbols $(H_\mu)_\mu$

Let $\mu \in]-2, 2[$ be a parameter. Let $n \in \mathbb{N}\backslash\{0\}$ and $(x, \xi) \in T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ a point on the cotangent space $T^*\mathbb{R}^n$ called "*slow phase space*". Let $d \geqslant 2$ be an integer and $\mathrm{Herm}(\mathbb{C}^d)$ denotes Hermitian operators on $\mathbb{C}^d$. We consider a function $(\mu, x, \xi) \to H_\mu(x, \xi)$ smooth with respect to μ, x, ξ and valued in $\mathrm{Herm}(\mathbb{C}^d)$:

$$(2.1) \qquad H_\mu : \begin{cases} T^*\mathbb{R}^n & \to \mathrm{Herm}\left(\mathbb{C}^d\right) \\ (x, \xi) & \mapsto H_\mu\left(x, \xi\right) \end{cases}$$

called *symbol* (we suppose that $H_\mu \in S_{\rho,\delta}^m(T^*\mathbb{R}^n)$ belongs to the class of Hörmander symbols. This corresponds to suitable hypothesis of regularity at infinity, see Section A).

For fixed values of μ, x, ξ, the eigenvalues of the matrix $H_\mu(x, \xi)$ are real and are denoted

$$(2.2) \qquad \omega_1\left(\mu, x, \xi\right) \leqslant \ldots \leqslant \omega_d\left(\mu, x, \xi\right).$$

We will assume the following hypothesis[2] for the family of symbols $(H_\mu)_\mu$. This hypothesis is illustrated on Figure 2.1.

[2] Here $\|(\mu, x, \xi)\| := \sqrt{\mu^2 + \sum_{j=1}^n (x_j^2 + \xi_j^2)}$ is the Euclidean distance from (μ, x, ξ) to the origin in $\mathbb{R}^{2n+1}$.
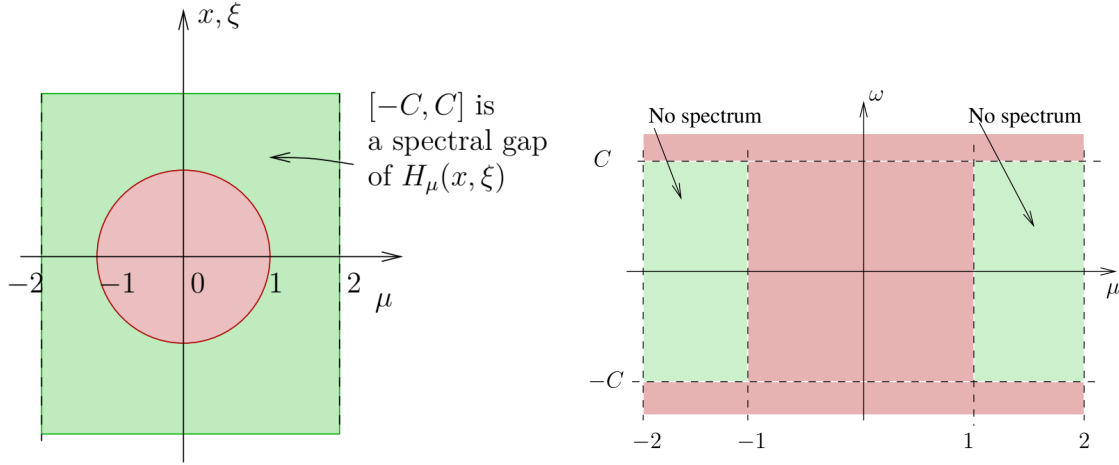
Figure 2.1. Illustration of Assumption 2.1. On Figure (a), for parameters $(\mu, x, \xi) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ in the green domain, we assume that the spectrum of the hermitian matrix $H_\mu(x, \xi)$, has r eigenvalues smaller than $-C$ and that the others are greater than $C > 0$. Equivalently, on Figure (b), the spectrum of $H_\mu(x, \xi)$ for any (x, ξ) is contained in the red domain.

ASSUMPTION 2.1 ("*Spectral gap assumption*"). — *For the family of symbols* $(H_\mu)_\mu$, *(2.1), we suppose that there exists an index* $r \in \{1, \ldots d - 1\}$ *and* $C > 0$ *such that for every* $(\mu, x, \xi) \in \mathbb{R}^3$ *such that* $\|(\mu, x, \xi)\| \geqslant 1$ *and* $|\mu| \leqslant 2$, *we have*

$$\omega_r(\mu, x, \xi) < -C \text{ and } \omega_{r+1}(\mu, x, \xi) > +C.$$

2.2. Spectral index $\mathcal{N}$ for the family of symbols $(H_\mu)_\mu$

The reader may read first the Appendix A that gives an introduction with examples to *pseudo-differential operators* (PDO) and *pseudo-differential calculus*.

Let us introduce a new parameter $\epsilon > 0$ called *adiabatic parameter* or *semi-classical parameter*. We define the *pseudo-differential operator*[3] (PDO)

$$(2.3) \qquad \widehat{H}_{\mu,\epsilon} := \mathrm{Op}_\epsilon(H_\mu) \in \mathrm{Herm}\left(L^2(\mathbb{R}^n) \otimes \mathbb{C}^d\right),$$

obtained by Weyl quantization of the symbol H_μ.

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[3] The operator $\widehat{H}_{\mu,\epsilon}$ belongs to $\mathrm{Herm}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^d) \equiv \mathrm{Herm}(L^2(\mathbb{R}^n; \mathbb{C}^d))$, i.e. is a self-adjoint operator in the space of functions on $\mathbb{R}^n$ with d complex components.
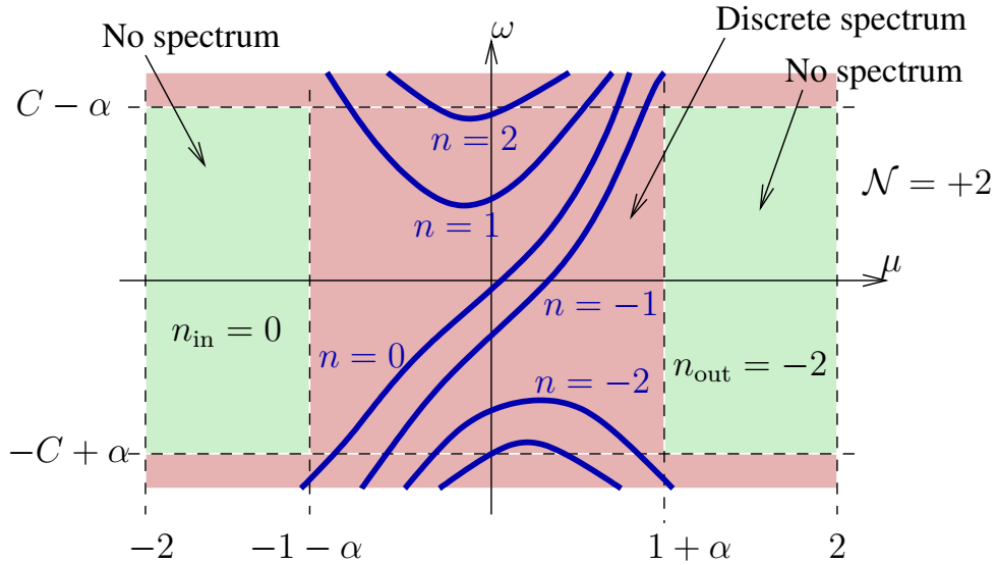
Figure 2.2. For $\epsilon > 0$ fixed, this is a schematic picture of the spectrum of the operator $\widehat{H}_{\mu,\epsilon}$. From Theorem 2.2, in the green domain, there is no spectrum. In the red domain, the spectrum is discrete: discrete eigenvalues are shown in blue and depend continuously on μ, ϵ. Consequently we can label the eigenvalues by some increasing number n and label each spectral gap by the index n of the first eigenvalue below it. We define then the spectral index of the family of symbols $(H_\mu)_\mu$ by $\mathcal{N} = n_{\mathrm{in}} - n_{\mathrm{out}}$. In this example, $\mathcal{N} = n_{\mathrm{in}} - n_{\mathrm{out}} = 0 - (-2) = +2$ corresponding to the fact that $\mathcal{N} = +2$ eigenvalues are moving upward as μ increases.

THEOREM 2.2. — We do the Assumption 2.1. Then for every $\alpha > 0$ there exists $\epsilon_0 > 0$ such that for every $0 < \epsilon < \epsilon_0$,
- for any μ such that $1 + \alpha < |\mu| < 2$, the operator $\widehat{H}_{\mu,\epsilon}$ has no spectrum in the interval $] - C + \alpha, +C - \alpha[$.
- for any μ such that $|\mu| \leqslant 1 + \alpha$, the operator $\widehat{H}_{\mu,\epsilon}$ has discrete spectrum in the interval $] - C + \alpha, C - \alpha[$ that depends continuously on μ, ϵ.

See Figure 2.2.

Proof. — We follow quite standard techniques from micro-local analysis. From Assumption 2.1, if $|\mu| > 1 + \alpha$ then the symbol has no spectrum in the interval $[-C, C]$. Hence for any $z \in I := [-C + \alpha, C - \alpha]$, the operator $(z\mathrm{Id} - \widehat{H}_{\mu,\epsilon})$ is invertible with approximate inverse given by $\mathrm{Op}_\epsilon((z - H_\mu(x, \xi))^{-1})$. This means that there is no spectrum for $\widehat{H}_\mu$ in this interval I.

If $|\mu| < 1 + \alpha$, the symbol $H_\mu(x, \xi)$ may have some spectrum in this spectral range I. However, from Assumption 2.1, the points (μ, x, ξ) for which $\mathrm{Ran}(H_\mu(x, \xi)) \cap I$ is non empty are included in the compact ball $\|(\mu, x, \xi)\| \leqslant 1$ (here $\mathrm{Ran}(H_\mu(x, \xi))$ stands for the numerical range of the matrix $H_\mu(x, \xi)$). Let us take $\chi_\mu : (x, \xi) \to \chi_\mu(x, \xi) \geqslant 0$ that is a smooth regularization of the characteristic function of the

set $\|(\mu, x, \xi)\| \leqslant 1$. We take $A > 0$ large enough such that the perturbed symbol $H'_\mu(x, \xi) = H_\mu(x, \xi) + A\chi_\mu(x, \xi)$ has no spectrum in $[-C, C]$ for every μ such that $|\mu| < 1 + \alpha$. Since χ_μ has compact support, then $\mathrm{Op}_\epsilon(\chi_\mu)$ is trace class hence compact, see (A.4). Then as before, $\mathrm{Op}_\epsilon(H'_\mu)$ has no spectrum in $[-C + \alpha, C - \alpha]$ for μ s.t. $|\mu| < 1 + \alpha$ (i.e. the perturbation $A\chi_\mu$ has pushed the spectrum above). For $z \in [-C + \alpha, C - \alpha]$, we write

$$(z - H_\mu)^{-1} = \left(z - H'_\mu + A\chi_\mu\right)^{-1} = \left(z - H'_\mu\right)^{-1}\left(1 + \left(z - H'_\mu\right)^{-1}A\chi_\mu\right)^{-1}$$

By quantization of this relation, we have that $\mathrm{Op}_\epsilon((z - H'_\mu)^{-1})$ is bounded and analytic in z, $\mathrm{Op}_\epsilon(A\chi_\mu)$ is compact, hence from analytic Fredholm theorem [RS72, p. 201], $\mathrm{Op}_\epsilon(1 + (z - H'_\mu)^{-1}A\chi_\mu)^{-1}$ and therefore $(z - \widehat{H}_{\mu,\epsilon})^{-1}$ are meromorphic in z with residues that are operators of finite rank, i.e. the spectrum is discrete. $\square$

As a consequence of Theorem 2.2 we can define the spectral index $\mathcal{N}$ as follows, as shown on Figure 2.2.

> DEFINITION 2.3 (*"Spectral index $\mathcal{N}$ of the family of symbols $(H_\mu)_\mu$"*). — *With Assumption 2.1 and from Theorem 2.2, for fixed ϵ, each spectral gap can be labeled as follows. Let $(\omega_n(\mu, \epsilon))_{n \in \mathbb{Z}}$ be the eigenvalues of the operator $\widehat{H}_{\mu,\epsilon}$ that belongs to the interval $I_\alpha :=]-C + \alpha, +C - \alpha[$, labeled by $n \in \mathbb{Z}$ and sorted by increasing values (this is well defined up to a constant). For a given n, the eigenvalue $\omega_n(\mu, \epsilon) \in \mathbb{R}$ is continuous w.r.t. μ, ϵ. For a point $(\mu, \omega) \in (-1 - \alpha, 1 + \alpha) \times I_\alpha$ different from an eigenvalue, we associate the index $n(\mu, \omega) \in \mathbb{Z}$ of the eigenvalue just below it, i.e. such that $\omega_n(\mu, \epsilon) < \omega < \omega_{n+1}(\mu, \epsilon)$. We denote $n_{\mathrm{in}} := n(-1, 0)$ the index of the first gap and $n_{\mathrm{out}} := n(1, 0)$ the index of the last gap. This defines an integer*
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> (2.4) $$\mathcal{N} := n_{\mathrm{in}} - n_{\mathrm{out}} \in \mathbb{Z}$$
>
> *called the spectral index of the family of symbols $(H_\mu)_\mu$. This integer $\mathcal{N}$ counts the number of eigenvalues that go upwards as μ increases. $\mathcal{N}$ is independent on ϵ and more generally invariant under any continuous variation of the symbol $(H_\mu)_\mu$ family satisfying the assumption 2.1. Hence, $\mathcal{N}$ is a topological index.*

Remark 2.4. — The last remark that $\mathcal{N}$ is invariant under continuous variation of the symbol comes from the fact that the map $H_\mu \to \mathcal{N} \in \mathbb{Z}$ is continuous hence locally constant

Remark 2.5. — We have used Weyl quantization in (2.3) to define the operator $\widehat{H}_{\mu,\epsilon}$. We could have choose any other quantization procedure. The index $\mathcal{N}$ does not depend on the choice of quantization.

QUESTION 2.6. — *How to compute the spectral index $\mathcal{N} \in \mathbb{Z}$ directly from the symbol $(H_\mu)_\mu$?*

Answer: in the next section, with Theorem 2.7, we will see that $\mathcal{N}$ is simply related to the degree of a certain map $f : S^{2n-1} \to S^{2n-1}$ that is obtained from the symbol $(H_\mu)_\mu$.

2.3. Chern topological index $\mathcal{C}$ and index formula

The reader may read first the Appendix B.3 that gives an introduction and general information about topology of vector bundles over spheres.

Let

$$S^{2n} := \left\{ (\mu, x, \xi) \in \mathbb{R}^{1+2n}, \quad \|(\mu, x, \xi)\| = 1 \right\},$$

be the unit sphere in the space of parameters. From assumption 2.1, for every parameter $(\mu, x, \xi) \in S^{2n}$, we have a spectral gap between eigenvalues $\omega_r(\mu, x, \xi)$ and $\omega_{r+1}(\mu, x, \xi)$. Then we can define the spectral projector associated to the first r eigenvalues by Cauchy formula

$$\Pi_{1;r}(\mu, x, \xi) := \frac{i}{2\pi} \oint_\gamma \left(z - H_\mu(x, \xi) \right)^{-1} dz$$

where the integration path $\gamma \subset \mathbb{C}$ enclosed the segment $[\omega_1(\mu, x, \xi), \omega_r(\mu, x, \xi)]$ and crosses the spectral gaps. The spectral space associated to the first r eigenvalues $\omega_1 \dots \omega_r$ is then the image of this projector

$$(2.5) \qquad\qquad F(\mu, x, \xi) := \mathrm{Ran}\Pi_{1;r}(\mu, x, \xi).$$

The linear space $F(\mu, x, \xi) \subset \mathbb{C}^d$ has complex dimension r and defines a smooth complex vector bundle of rank r over the sphere S^{2n}, that we denote $F \to S^{2n}$. From Remark 2.8 below, we can suppose that $r \geqslant n$.

From Bott's theorem B.20, the topology of $F \to S^{2n}$ is characterized by an integer $\mathcal{C} \in \mathbb{Z}$ called *Chern index* defined in (B.18) from the degree $\deg(f)$ of a map $f : S^{2n-1} \to S^{2n-1}$ in (B.17), by $\mathcal{C} = \frac{\deg(f)}{(n-1)!}$, and f is directly obtained from the clutching function

$$(2.6) \qquad\qquad g : S^{2n-1} \to U(r)$$

of the bundle $F \to S^{2n}$ on the equator S^{2n-1} with respect to some local trivialization. In dimension $n = 1$ this is more simple because $\mathcal{C}$ is just the winding number of the clutching function $g : S^1 \to U(1) \equiv S^1$ on the equator S^1. The physical applications considered later in this paper correspond to dimension $n = 1$.

> THEOREM 2.7 (*"Index formula"*). — *Let $(H_\mu)_\mu$ be a family of symbols that satisfies Assumption 2.1. Let $\mathcal{N} \in \mathbb{Z}$ be the spectral index defined in (2.4) and let $\mathcal{C} \in \mathbb{Z}$ be the Chern topological index defined from the vector bundle $F \to S^{2n}$ by (B.18). We have*
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> $$(2.7) \qquad\qquad \mathcal{N} = \mathcal{C}.$$

The proof of Theorem 2.7 is given in Section 2.5. It is based on the index theorem on Euclidean space of Fedosov–Hörmander given in [Hör79, Theorem 7.3, p. 422], [BB85, Theorem 1, page 252].

Remark 2.8. — If one replaces the symbol $H_\mu(x,\xi) \in \text{Herm}(\mathbb{C}^d)$ in (2.1) by the symbol $\widetilde{H}_\mu(x,\xi) \in \text{Herm}(\mathbb{C}^{d+m})$ obtained by adding a constant diagonal term

$$\widetilde{H}_\mu(x,\xi) = \begin{pmatrix} H_\mu(x,\xi) & 0 \\ 0 & \omega_0 \text{Id}_{\mathbb{C}^m} \end{pmatrix},$$

with $\omega_0 < -C$ then one observes that
- $\widetilde{H}_\mu$ satisfies Assumption 2.1.
- The spectral index of $\widetilde{H}_\mu$ and H_μ are equal, i.e. $\widetilde{\mathcal{N}} = \mathcal{N}$. This is because $\text{Op}_\epsilon(\widetilde{H}_\mu) \equiv \text{Op}_\epsilon(H_\mu) \oplus \text{Op}_\epsilon(\omega_0 \text{Id}_{\mathbb{C}^m})$ and the spectrum of $\text{Op}_\epsilon(\omega_0 \text{Id}_{\mathbb{C}^m}) = \omega_0 \text{Op}_\epsilon(\text{Id}_{\mathbb{C}^m})$ is on the constant horizontal line $\omega = \omega_0 < -C$, so does not give moving eigenvalues.
- The associated vector bundle $\widetilde{F} \to S^{2n}$ is $\widetilde{F} = F \oplus T_m$ where $T_m = S^{2n} \times \mathbb{C}^m$ is the trivial bundle and $\text{rank}(\widetilde{F}) = \text{rank}(F) + m$.

This remark shows that the spectral index does not change if one adds a trivial bundle T_m to the bundle F. It means that $\mathcal{N}$ depends only on the equivalence class of F (or H) in the K-theory group $\widetilde{K}(S^{2n})$, cf [Hat98].

2.4. Special case of matrix symbols that are linear in (μ, x, ξ)

In this section, we give a simple but important remark to understand why the model of Matsuno presented in Section 4 does not depend on a small parameter ϵ but nevertheless belongs to the general model presented here. This is the same for the normal form model presented in Section 3.

Suppose that

$$\widetilde{H} : \left(\widetilde{\mu}, \widetilde{x}, \widetilde{\xi}\right) \in \mathbb{R}^{1+2n} \to H\left(\widetilde{\mu}, \widetilde{x}, \widetilde{\xi}\right) \in \text{Herm}\left(\mathbb{C}^d\right)$$

is a *linear map* with respect to $(\widetilde{\mu}, \widetilde{x}, \widetilde{\xi})$ and consider the quantization rule $\text{Op}_1(\widetilde{\xi}) = -i\partial_{\widetilde{x}}$ (i.e. with $\epsilon = 1$). For example, see the normal form symbol (3.1) or the Matsuno's symbol (4.3).

For any $\epsilon > 0$, we do the change of variables

$$\mu = \sqrt{\epsilon}\widetilde{\mu}, \quad x = \sqrt{\epsilon}\widetilde{x},$$

that gives

$$\text{Op}_\epsilon(\xi) = -i\epsilon\partial_x = -i\sqrt{\epsilon}\partial_{\widetilde{x}} = \sqrt{\epsilon}\text{Op}_1\left(\widetilde{\xi}\right).$$

Hence the symbol $H(\mu, x, \xi) = \sqrt{\epsilon}\widetilde{H}(\mu, x, \xi)$ satisfies

$$\text{Op}_\epsilon(H_\mu) = \sqrt{\epsilon}\text{Op}_1\left(\widetilde{H}_{\widetilde{\mu}}\right).$$

In other words all these models with different ϵ are equivalent up to a scaling of the parameters and the operator (and spectrum). The benefit to consider an additional semi-classical (or adiabatic) parameter $\epsilon \ll 1$ is that *one can perturb the linear symbol to a non linear symbol* and still get the index formula $\mathcal{N} = \mathcal{C}$ from Theorem 2.7.

2.5. Proof of the index formula (2.7)

In this section we give a proof of Formula (2.7). This proof relies on the index Theorem on Euclidean space of Fedosov–Hörmander given in [Hör79, Theorem 7.3 p. 422], [BB85, Theorem 1, p. 252].

For a given family of symbols $H = (H_\mu)_{\mu \in (-2,2)}$ with Assumption 2.1, we have defined two topological indices $\mathcal{N}_H \in \mathbb{Z}$ and $\mathcal{C}_H \in \mathbb{Z}$. These indices are topological, i.e. they depend only on the class of equivalence of the symbols and we want to show that they are equal, i.e. $\mathcal{N}_H = \mathcal{C}_H$.

Let us denote $F \to S^{2n}$ the smooth vector bundle of rank r defined from H in (2.5). We will construct a new symbol in the same equivalence class, so having the same indices $\mathcal{N}_H, \mathcal{C}_H$, but that will be easier to handle to show that $\mathcal{N}_H = \mathcal{C}_H$. Let $g : S^{2n-1} \to U(r)$ be the clutching function on the equator of the bundle F, as defined in (2.6) or Appendix B.3.2. We extend g outside of $S^{2n-1} \subset \mathbb{R}^{2n}_{x,\xi}$ giving a 1-homogeneous function $\widetilde{g} : \mathbb{R}^{2n}_{x,\xi} \to \mathrm{Mat}(\mathbb{C}^r)$ by

$$(2.8) \qquad \widetilde{g} : (x,\xi) \in \mathbb{R}^{2n} \to \widetilde{g}(x,\xi) := \|(x,\xi)\| \, g\left(\frac{(x,\xi)}{\|(x,\xi)\|}\right) \in \mathrm{Mat}_r(\mathbb{C}).$$

Then we define the (new) symbol H_μ as follows. For $\mu \in \mathbb{R}, (x,\xi) \in \mathbb{R}^{2n}$, let

$$(2.9) \qquad H_\mu(x,\xi) := \begin{pmatrix} -\mu \mathrm{Id}_r & -\widetilde{g}(x,\xi) \\ -\widetilde{g}^\dagger(x,\xi) & \mu \mathrm{Id}_r \end{pmatrix} \in \mathrm{Herm}\left(\mathbb{C}^{2r}\right).$$

> LEMMA 2.9. — *There are two eigenvalues of $H_\mu(x,\xi)$ defined in (2.9), given by $\omega_\pm(\mu,x,\xi) = \pm\|(\mu,x,\xi)\|$, each with multiplicity r. For $(\mu,x,\xi) \in S^{2n}$, the eigenspace $F_-(\mu,x,\xi)$ associated to $\omega_-(\mu,x,\xi) = -1$ defines a vector bundle $F_- \to S^{2n}$ of rank r isomorphic to the initial given vector bundle $F \to S^{2n}$.*

Remark 2.10. — Eq. (2.9) car be related to a more general construction of a projector from a given vector bundle, see [Fed96, p. 14].

Proof. — For $(\mu,x,\xi) \in \mathbb{R}^3$, we denote $R = \|(\mu,x,\xi)\|$. Since g is unitary on S^{2n-1}, we get that $\widetilde{g}^\dagger \widetilde{g} = \widetilde{g}\widetilde{g}^\dagger = \|(x,\xi)\|^2 = R^2 - \mu^2$ and easily check that eigenvalues $\omega^\pm$ and eigenvectors $U_j^\pm$ defined by $H_\mu(x,\xi)U_j^\pm = \omega^\pm U_j^\pm$ are given for $j = 1, \ldots r$ by

$$\omega_\pm(\mu,x,\xi) = \pm R, \qquad U_j^\pm(\mu,x,\xi) = \begin{pmatrix} (-\mu \pm R)\delta_j \\ -\widetilde{g}^\dagger \delta_j \end{pmatrix},$$

where

$$\delta_j := \Big(0, \ldots, \underbrace{1}_{j}, 0\ldots\Big) \in \mathbb{C}^r$$

denotes the canonical basis vector of $\mathbb{C}^r$. So there are two eigenvalues $\omega_\pm(\mu,x,\xi)$ each with multiplicity r. We denote $F_\pm(\mu,x,\xi) := \mathrm{Vect}(U_j^\pm, j \in \{1\ldots r\}) \subset \mathbb{C}^2$ the associated eigenspaces. We compute that

$$\left\|U_j^\pm\right\|^2 = (-\mu \pm R)^2 + \sum_{j'}\left|\left\langle \delta_{j'} \,\middle|\, \widetilde{g}^\dagger \delta_j\right\rangle\right|^2 = (-\mu \pm R)^2 + R^2 - \mu^2 = 2R(R \mp \mu).$$

Since the vectors $(U_j^\pm)_j$ are orthogonal, the spectral projector π_- on F_- is given by

$$(2.10) \qquad \pi^- = \sum_{j=1}^{r} \frac{1}{\left\| U_j^- \right\|^2} U_j^- \left\langle U_j^- | . \right\rangle : \mathbb{C}^{2r} \to F_- \left(\mu, x, \xi \right).$$

Consider $S^{2n} = \{ (\mu, x, \xi) \in \mathbb{R}^{2n+1}, R = 1 \}$ the unit sphere in the parameter space, the northern hemisphere $H_1 := \{ (\mu, x, \xi) \in S^{2n}, \mu \geqslant 0 \}$ and southern hemisphere $H_2 := \{ (\mu, x, \xi) \in S^{2n}, \mu \leqslant 0 \}$. For a given $j \in \{ 1, \ldots, r \}$, the orthogonal projection of the fixed vector $\binom{\delta_j}{0} \in \mathbb{C}^{2r}$ onto $F_-(\mu, x, \xi)$ gives the global section:

$$(2.11) \qquad s_1^{(j)} \left(\mu, x, \xi \right) := \pi_- \begin{pmatrix} \delta_j \\ 0 \end{pmatrix} \underset{(2.10)}{=} -\frac{1}{2} U_j^-$$

We compute $\| s_1^{(j)} \|^2 = \frac{1}{2}(1 + \mu)$ hence $\| s_1^{(j)} \|^2 \neq 0$ does not vanish on H_1. Hence $(s_1^{(j)})_{j \in \{1, \ldots r\}}$ is a trivialization of $F_- \to H_1$. We consider also the following trivialization of $F_- \to H_2$:

$$(2.12) \qquad s_2^{(j)} \left(\mu, x, \xi \right) := \pi_- \begin{pmatrix} 0 \\ \delta_j \end{pmatrix} = \frac{-1}{2 \left(1 - \mu \right)} \sum_{j'=1}^{r} U_{j'}^- \left\langle \widetilde{g}^\dagger \delta_{j'} \, \middle| \, \delta_j \right\rangle,$$

We have $\| s_2^{(j)} \|^2 = \frac{1}{2}(1 - \mu)$ hence $\| s_2^{(j)} \|^2 \neq 0$ on H_2 and $(s_2^{(j)})_{j \in \{1, \ldots r\}}$ is a trivialization of $F_- \to H_2$. We observe that

$$s_2^{(j)} = \frac{-1}{2 \left(1 - \mu \right)} \sum_{j'=1}^{r} U_{j'}^- \widetilde{g}_{j',j} \underset{2.11}{=} \frac{1}{\left(1 - \mu \right)} \sum_{j'=1}^{r} \widetilde{g}_{j',j} s_1^{(j)}$$

Hence on the equator $S^{2n-1} = \{ \mu = 0, (x, \xi) \in S^{2n-1} \}$ the clutching function $f_{21} : S^{2n-1} \to U(r)$ of F_- defined by $s_2^{(j)}(0, x, \xi) = \sum_{k=1}^{r} f_{21}^{(j,k)}(x, \xi) s_1^{(k)}(0, x, \xi)$ is given by $f_{21}(x, \xi) = \widetilde{g}(x, \xi) = g(x, \xi)$, that is the clutching function of F. Hence F_- and F are isomorphic. $\square$

From Lemma 2.9, we see that the symbol H_μ in (2.9) satisfies the Assumption 2.1. As in (2.3) we define the operator

$$(2.13) \quad \widehat{H}_{\mu,\epsilon} \underset{(2.3)}{=} \mathrm{Op}_\epsilon \left(H_\mu \right) \underset{(2.9)}{=} \begin{pmatrix} -\mu \mathrm{Id} & -\mathrm{Op}_\epsilon \left(\widetilde{g} \right) \\ -\mathrm{Op}_\epsilon \left(\widetilde{g} \right)^\dagger & \mu \mathrm{Id} \end{pmatrix} \quad \in \mathrm{Herm} \left(L^2 \left(\mathbb{R}^n \right) \otimes \mathbb{C}^{2r} \right).$$

and from Theorem 2.2 we can define the spectral index $\mathcal{N}_H$ in (2.4).

> LEMMA 2.11. — *The operator* $\mathrm{Op}_\epsilon(\widetilde{g}) \in \mathrm{Herm}(L^2(\mathbb{R}^n) \otimes \mathbb{C}^r)$ *is Fredholm with index*
>
> $$(2.14) \qquad\qquad \mathrm{Ind} \left(\mathrm{Op}_\epsilon \left(\widetilde{g} \right) \right) = \mathcal{N}_H.$$

Proof. — For simplicity of notation, we denote the operator $A := \mathrm{Op}_\epsilon(\widetilde{g})$. Since $\widetilde{g}^\dagger \widetilde{g} = \widetilde{g} \widetilde{g}^\dagger = \| (x, \xi) \|^2$ we see that A is elliptic hence Fredholm [BB85, Theorem 3, p. 185], with index [BB85, Theorem 2, p. 16]

$$(2.15) \qquad\qquad \mathrm{Ind} A = \dim \mathrm{Ker} A - \dim \mathrm{Ker} A^\dagger.$$

Since $\langle u | A^\dagger A u \rangle = \| A u \|^2 \geqslant 0$, we have that $A^\dagger A$ has discrete and positive spectrum denoted $A^\dagger A = \sum_{k \in \mathbb{N}^*} \lambda_k \pi_k$, with positive eigenvalues $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots$ and π_k

being the spectral projector associated to λ_k. We denote π_0 the projector on $\mathrm{Ker}A$. Similarly we denote $AA^\dagger = \sum_{k \in \mathbb{N}^*} \lambda'_k \pi'_k$ and π'_0 the projector on $\mathrm{Ker}A^\dagger$. In fact for a given $k > 0$, we have $\lambda'_k = \lambda_k > 0$ and $\pi'_k = \frac{1}{\lambda_k} A\pi_k A^\dagger$, $\pi_k = \frac{1}{\lambda_k} A^\dagger \pi'_k A$, because $\mathrm{Tr}(A\pi_k A^\dagger) = \mathrm{Tr}(A^\dagger A\pi_k) = \lambda_k > 0$. For $k > 0$, we have the isomorphism $A : \mathrm{Im}\pi_k \to \mathrm{Im}\pi'_k$ and $A^\dagger : \mathrm{Im}\pi'_k \to \mathrm{Im}\pi_k$. If $(e_l)_{l=1\ldots \dim \mathrm{Im}\pi_k}$ is an orthonormal basis of $\mathrm{Im}\pi_k$ then $\frac{1}{\sqrt{\lambda_k}}(Ae_l)_l, (e_l)_l$ is an orthonormal basis of $\mathrm{Im}\pi'_k \oplus \mathrm{Im}\pi_k$. In this basis, the operator $\widehat{H}_{\mu,\epsilon}$ is represented by the matrix

$$\widehat{H}_{\mu,\epsilon} \equiv \begin{pmatrix} -\mu & -\sqrt{\lambda_k} \\ -\sqrt{\lambda_k} & \mu \end{pmatrix}$$

The eigenvalues of this matrix are $\omega_k^\pm = \pm(\mu^2 + \lambda_k)^{1/2}$ and never vanish for any $\mu \in \mathbb{R}$, since $\lambda_k > 0$. Additionally, for $k = 0$, we have $\lambda_0 = 0$, hence $\widehat{H}_{\mu,\epsilon} \equiv \left(\begin{smallmatrix} -\mu & 0 \\ 0 & \mu \end{smallmatrix} \right)$ has eigenvalue $-\mu$ with multiplicity $\mathrm{rank}\pi'_0$, and eigenvalue μ with multiplicity $\mathrm{rank}\pi_0$. As a function of $\mu \in \mathbb{R}$, these eigenvalues vanish transversely for $\mu = 0$, as on Figure 3.2 and we get the index $\mathcal{N}_H = \mathrm{rank}\pi_0 - \mathrm{rank}\pi'_0$. Consequently

$$\mathcal{N}_H = \mathrm{rank}\pi_0 - \mathrm{rank}\pi'_0 = \dim \mathrm{Ker}A - \dim \mathrm{Ker}A^\dagger \underset{(2.15)}{=} \mathrm{Ind}A. \qquad \square$$

The index Theorem on Euclidean space of Fedosov–Hörmander given in [Hör79, Theorem 7.3, p. 422], [BB85, Theorem 1, p. 252] or Eq. (B.27), gives

$$(2.16) \qquad\qquad\qquad \mathrm{Ind}\,(\mathrm{Op}_\epsilon\,(\tilde{g})) = \mathcal{C}_H.$$

So we conclude that $\mathcal{N}_H \underset{(2.14),(2.16)}{=} \mathcal{C}_H$.

2.6. Some models with topological contact without exchange of states

In Section 2, we have seen a model constructed from a symbol $H_\mu(x,\xi)$ on a phase space $(x,\xi) \in \mathbb{R}^{2n}$ (i.e. n degrees of freedom) and parameter $\mu \in (-2,2)$, with a spectral gap for $\mu < -1$ and $\mu > 1$ and with a *spectral index* $\mathcal{N} \in \mathbb{Z}$ that counts the exchange of discrete energy eigenvalues (or states) between *two energy bands,* as the parameter μ increases (energy bands are the spectrum below the gap and the spectrum above the gap). We have seen that $\mathcal{N}$ is equal to the Chern index $\mathcal{C}$ of a vector bundle $F \to S^{2n}$ of rank r that is defined from the symbol.

- If the *vector bundle F is trivial*, it means that the two bands are not "topologically coupled" and we can perturb continuously the symbol $(H_\mu)_\mu$ so that the gap may exist for every values of $\mu \in (-2,2)$, i.e. we can "*open the gap*".
- If the *vector bundle F is non trivial,* it means that the two bands are "topologically coupled" with a "*topological contact*" and we *can not "open the gap"*, or remove the contact between the two bands.

If $\mathcal{N} = \mathcal{C} \neq 0$ then the bundle F is not trivial and we can not open the gap, since some energy levels pass through it, and this situation cannot be changed by continuous perturbations. From *Bott's theorem* B.20, if $r = \mathrm{rank}\,(F) \geqslant n$ then $\mathcal{C} \in \mathbb{Z}$ characterizes the topology of F. In other words, if $r \geqslant n$ then $\mathcal{C} = \mathcal{N} = 0 \Leftrightarrow F$ is trivial.

However for vector bundles F of smaller ranks, $r < n$ this is not always true (we only have the obvious fact F is trivial $\Rightarrow \mathcal{C} = \mathcal{N} = 0$ but not the converse). There exist some non trivial bundles $F \to S^{2n}$ with Chern index $\mathcal{C}(F) = 0$. From Table B.2, the simplest example is for $F \to S^6$, i.e. $n = 3$ degrees of freedom, with rank $r = 2$, because $\mathrm{Vect}^2(S^6) = \mathbb{Z}_2 = \{0, 1\}$. Suppose for example that $F \to S^6$ is non trivial and with topological class $[F] = 1 \in \mathrm{Vect}^2(S^6) = \mathbb{Z}_2$. It means that the two bands have a "topological contact", i.e. that we can not open the gap. Nevertheless $\mathcal{N} = \mathcal{C} = 0$, i.e. there is no exchange of states between the two bands at the contact (since the spectrum is discrete, there is some small gap that goes to zero as $\epsilon \to 0$). See figure below.
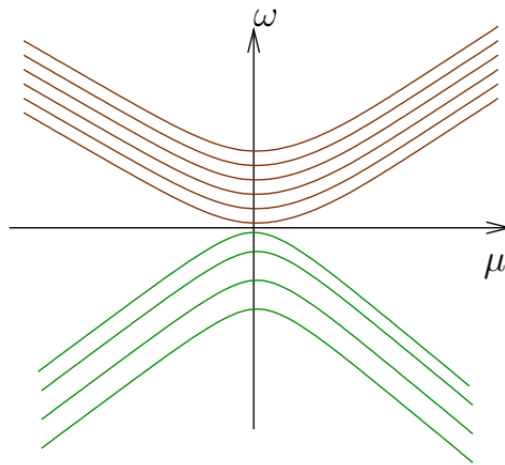
If one adds a second similar contact (at some other value of μ), then since $1 + 1 = 0$ in $\mathbb{Z}_2$, the result is that the two contact annihilate themselves and one can finally "open the gap". See figure below.
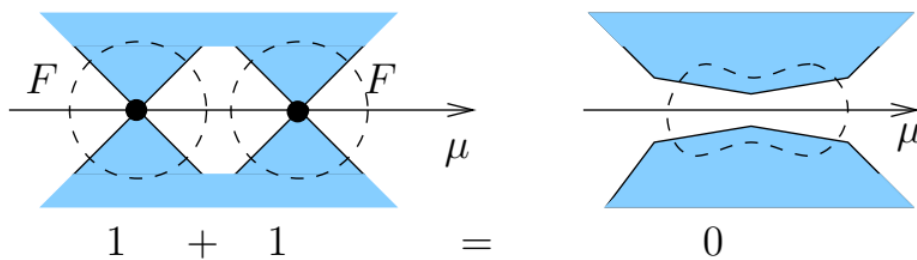
These kind of phenomena may occur with vector bundles $F \to S^{2n}$ that are in the "*non stable range*", where the homotopy groups are very complicated, see the Appendix B.3.

For a different example of the role of topology in spectral phenomena, in the paper [FZ02b] there is a simple model used molecular physics, for which the energy bands are topological coupled and associated to a rank 2 vector bundle that can not be split into two rank 1 vector bundles. This involves Chern numbers C_1, C_2 and shows the manifestation of algebraic topology in quantum mechanics of molecules or more generally quantum interacting systems.

3. Spectral flow and index formula for quantum waves in molecules

References for this Section are [Fau06, FZ00, FZ01, FZ02a].

3.1. Introduction

A small molecule is a set of atoms (electrons and nuclei) and can be considered as an isolated but complex quantum system since many degrees of freedom interact strongly on different time scales: the electrons that are light evolve on very short scales of time $\tau_e \in [10^{-16}s, 10^{-15}s]$, which are small compared to the time scales of the vibration motion of the atoms $\tau_{\mathrm{vib}} \in [10^{-15}s, 10^{-14}s]$, themselves small compared to the slower rotation of the molecule $\tau_{\mathrm{rot.}} \in [10^{-12}s, 10^{-10}s]$. In quantum mechanics the state of the molecule is described by a multivariate "quantum wave function" and a stationary state of the molecule corresponds to an eigenfunction of the Hamiltonian operator. The corresponding eigenvalue is the energy of this state. If the molecule is sufficiently isolated from its environment, one can experimentally measure its quantum energy levels (discrete spectrum) by spectroscopy. These quantum energy levels correspond to stationary collective states of all the internal interactions between all these different degrees of freedom. It seems to be (and it is) a very complicated problem, but these different time scales allows to approximate the dynamics by some "fiber bundle description". This is called the adiabatic theory. In simple words the fast motion phase space is a fiber bundle over the slow motion phase space. In quantum mechanics (or more generally in wave mechanics, like optics, acoustics...) one has to quantize this fiber bundle description. Although this adiabatic approach does not solve completely the problem it gives a geometric description and some rough (and robust under perturbations) first description of the spectrum can be obtained from topological properties of these fiber bundles. This is the subject of this Section. See Figure 3.1.

3.2. Simple model (normal form)

References for this section: [FZ00, FZ01]. The following model not only is relevant in molecular physics to illustrate the spectral behavior of rotational / vibrational (slow / fast) energy levels of nuclei, but also plays an important role in the general theory because it is an "elementary topological normal form".

Let $\mu \in \mathbb{R}$ be a parameter that is fixed. Let $(x, \xi) \in T^*\mathbb{R} \equiv \mathbb{R} \times \mathbb{R}$ "slow variables" on phase space $\mathbb{R}^2$. We introduce the "symbol"

$$(3.1) \qquad H_\mu(x, \xi) := \begin{pmatrix} -\mu & x + i\xi \\ x - i\xi & +\mu \end{pmatrix} \in \mathrm{Herm}\left(\mathbb{C}^2\right).$$

We will call $\mathcal{H} = \mathbb{C}^2$ the fast Hilbert space. The space of "slow Hilbert" is $L^2(\mathbb{R})$ and corresponds to the quantification of the phase space $T^*\mathbb{R}$ of "slow variables" x, ξ and replace them by quantum operators. Let $\epsilon > 0$, the "adiabatic parameter" and set
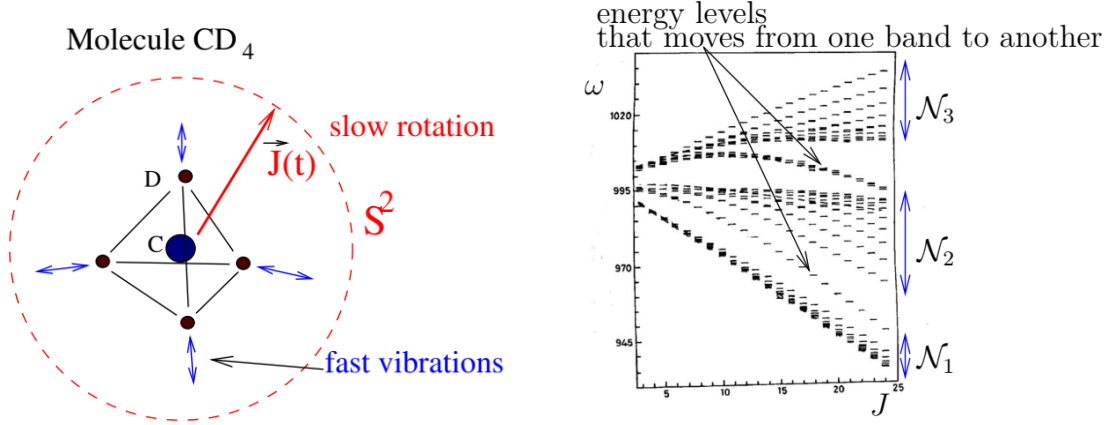
Figure 3.1. *Energy levels (in cm^{-1}) of the molecule CD_4 (carbon with 4 deuterium atoms) as a function of the total angular momentum $J \in \mathbb{N}$ (rotation energy and which is a preserved quantity). The fine structure of the spectrum corresponds to the slow rotation motion and the broad structure to the faster vibration motion. There are groups of levels and levels that pass between these groups. The index formula gives the exact values of number of levels $\mathcal{N}_j$ in each group* [Fau06, FZ00, FZ01, FZ02a].

$$(3.2) \qquad \widehat{H}_\mu := \mathrm{Op}_\epsilon\left(H_\mu\right) := \begin{pmatrix} -\mu\,\mathrm{Id} & \widehat{x} + i\widehat{\xi} \\ \widehat{x} - i\widehat{\xi} & \mu\,\mathrm{Id} \end{pmatrix} \in \mathrm{Herm}\left(L^2\left(\mathbb{R}_x\right) \otimes \mathbb{C}^2\right)$$

where $\mathrm{Id}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $\widehat{\xi} := \mathrm{Op}_\epsilon(\xi) := -i\epsilon\frac{d.}{dx} \in \mathrm{Herm}(L^2(\mathbb{R}))$, and $\widehat{x}$ is the multiplication operator x in $L^2(\mathbb{R}_x)$, see Section A for more details.

Remark 3.1. — In [FZ00, FZ01] it is shown how this normal form gives a microlocal description of the interaction between the fast vibration motion and the slow rotational motion of the molecule of Figure 3.1. In few words, (x, ξ) are local coordinates on the sphere S^2 of rotation in a vicinity of a point where two spectral bands have a contact, and the $\mathbb{C}^2$ space describes the quantum dynamics of the fast vibrations by restricting to an effective two level problem.

3.2.1. Spectral index $\mathcal{N}$

In the following Theorem, $(\varphi_n)_{n \in \mathbb{N}}$ is the orthonormal basis of Hermite functions of $L^2(\mathbb{R})$ defined by the Gaussian function

$$(3.3) \qquad \varphi_0\left(x\right) = \frac{1}{(\pi\epsilon)^{1/4}} e^{-\frac{1}{2}\frac{x^2}{\epsilon}},$$

and

$$(3.4) \qquad \varphi_{n+1} = \frac{1}{\sqrt{n+1}} a^\dagger \varphi_n, \qquad a\varphi_n = \sqrt{n}\varphi_{n-1},$$

with the operators (so called annihilation and creation operators from quantum optics)

$$(3.5) \qquad a := \frac{1}{\sqrt{2\epsilon}}\left(\widehat{x} + i\widehat{\xi}\right), \quad a^\dagger := \frac{1}{\sqrt{2\epsilon}}\left(\widehat{x} - i\widehat{\xi}\right).$$

PROPOSITION 3.2 (*"Spectrum of $\widehat{H}_\mu$".*). — *For each parameter $\mu \in \mathbb{R}$, the operator $\widehat{H}_\mu$, (3.2), has discrete spectrum in $L^2(\mathbb{R}_x) \otimes \mathbb{C}^2$ given by*

$$(3.6) \qquad \widehat{H}_\mu \phi_n^\pm = \omega_n^\pm \phi_n^\pm, \quad n \geqslant 1,$$

with for any $n \in \mathbb{N}\backslash\{0\}$,

$$(3.7) \qquad \frac{\omega_n^\pm}{\sqrt{\epsilon}} = \pm\sqrt{\left(\frac{\mu}{\sqrt{\epsilon}}\right)^2 + 2n}$$

$$\phi_n^\pm = \begin{cases} \frac{\sqrt{2n\epsilon}}{\mu + \omega_n^\pm}\varphi_{n-1} \\ \varphi_n \end{cases}$$

and for $n = 0$,

$$\widehat{H}_\mu \phi_0 = \omega_0 \phi_0,$$

with

$$\omega_0 = \mu$$

$$\phi_0 = \begin{pmatrix} 0 \\ \varphi_0 \end{pmatrix}$$

Observe that there is

$$(3.8) \qquad \mathcal{N} = +1$$

eigenvalue transiting upwards, for μ increasing. See Figure 3.2.

Remark 3.3. — It appears in (3.7) that $\sqrt{\epsilon}$ is a natural parameter of "scaling". See Section 2.4 for a discussion.

For the moment we can not say that (3.8) is a result of topology. For $\mathcal{N}$ to be recognized as a "topological index", it would be necessary for this model to belong to a set of models and to show that this number $\mathcal{N} = +1$ is model independent (robust by continuous perturbation within this set). This is done in Section 2.

A detailed proof of Proposition 3.2 with different methods, can be found in the arxiv version [Fau19].

3.2.2. Topological Chern Index $\mathcal{C}$

We can first consult the Section B which introduces in simple terms the notion of topology of a complex vector bundle of rank 1 on the sphere S^2.
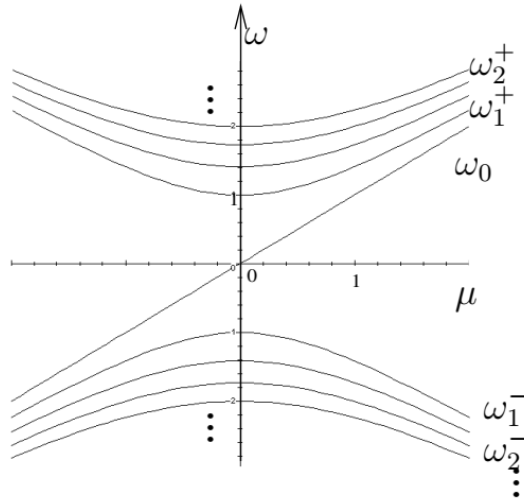
Figure 3.2. Spectrum of (3.6).

PROPOSITION 3.4 (*"Topological aspects of the symbol"*). — *The eigenvalues of the matrix $H_\mu(x,\xi) \in \mathrm{Herm}(\mathbb{C}^2)$, Eq.* (3.1), *are*

$$(3.9) \qquad\qquad \omega_\pm(\mu, x, \xi) = \pm\sqrt{\mu^2 + x^2 + \xi^2}$$

There is therefore a degeneracy $\omega_+ = \omega_-$ for $(\mu, x, \xi) = (0,0,0)$. For $(\mu, x, \xi) \in S^2 = \{(\mu, x, \xi) \in \mathbb{R}^3, |(\mu, x, \xi)| = 1\}$, i.e. on the unit sphere in the parameter space, the eigenspace $F_-(\mu, x, \xi) \subset \mathbb{C}^2$ associated with the eigenvalue ω_- defines a complex vector bundle of rank 1, denoted F_-. Its isomorphism class is characterized by the topological Chern index

$$\mathcal{C}(F_-) = +1.$$

Similarly for eigenvalue ω_+,
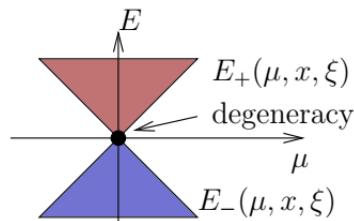
$$\mathcal{C}(F_+) = -1.$$

Figure 3.3. We have $\omega^-(\mu, x, \xi) \leqslant -|\mu|$, $\omega^+(\mu, x, \xi) \geqslant |\mu|$. The red domain represents the possible values of $\omega_-(\mu, x, \xi)$ with μ fixed and $(x, \xi) \in \mathbb{R}^2$. Similarly, the blue domain represent $\omega_-(\mu, x, \xi)$. The degeneracy is at $(\mu, x, \xi) = (0,0,0)$.

A detailed proof of Proposition 3.2.2 with different methods, can be found in the arxiv version [Fau19].

3.2.3. Conclusion on the model (3.2)

In the model defined by (3.2), we observe from the symbol, a vector bundle F_- whose index of Chern is $\mathcal{C}(F_-) = +1$ and we observe that there is $\mathcal{N} = +1$ level transiting (upwards) in the spectrum of the operator. We see in Section 2, Theorem 2.7, that this equality

$$\mathcal{N} = \mathcal{C}$$

is a special case of a more general result, called the *index formula*, valid for a continuous family of symbols and for spaces and bundles of larger dimensions.

Another equivalent formulation given in [FZ00, FZ02a, FZ01] in a more general context: for $|\mu| \gg 1$, there are two groups of levels $j = -, +$ in the spectrum of $\widehat{H}_\mu$. When changing $\mu = -\infty \to +\infty$ each group has a variation $\Delta \mathcal{N}_j \in \mathbb{Z}$ of the number of levels. We have the formula

$$\Delta \mathcal{N}_j = -\mathcal{C}_j$$

where $\mathcal{C}_j$ is the Chern index of the bundle $F_j \to S^2$.

4. Spectral flow and index formula for oceanic equatorial waves

In this Section we present the model of Matsuno (1966) [Mat66] for equatorial waves and the topological interpretation given by P. Delplace, J. B. Marston, and A. Venaille in [DMV17].

4.1. Matsuno's model

We first present the physical meaning of the Matsuno's model [Mat66]. See also this document, [Val17].

The shallow water model

See also Shallow_water_equations on wikipedia. Let $x = (x_1, x_2) \in \mathbb{R}^2$ be local coordinates on the horizontal plane near the equator. x_1 is the longitude and x_2 the latitude. The function $(h(x, t) + H) \in \mathbb{R}$ with $H > 0$ represents the depth of water (or of a layer of hot water) at position x and time $t \in \mathbb{R}$. The vector $u(x, t) = (u_1(x, t), u_2(x, t)) \in \mathbb{R}^2$ represents the (horizontal) velocity of this water. Water is submitted to gravity ($g = 9.81 \, m/s^2$ is the g-force) and since the earth is rotating with frequency Ω, there is also an effective Coriolis force. The Navier–Stokes equations with shallow water assumptions give

$$\partial_t h + \operatorname{div} ((h + H) u) = 0$$
$$\partial_t u + u \cdot \operatorname{grad} (u) = -g \operatorname{grad} (h) - f n \wedge u$$

with $f(x) = 2\Omega \cdot n(x) \in \mathbb{R}$ and (x) being the unit normal vector at position x. See Figure 4.2.
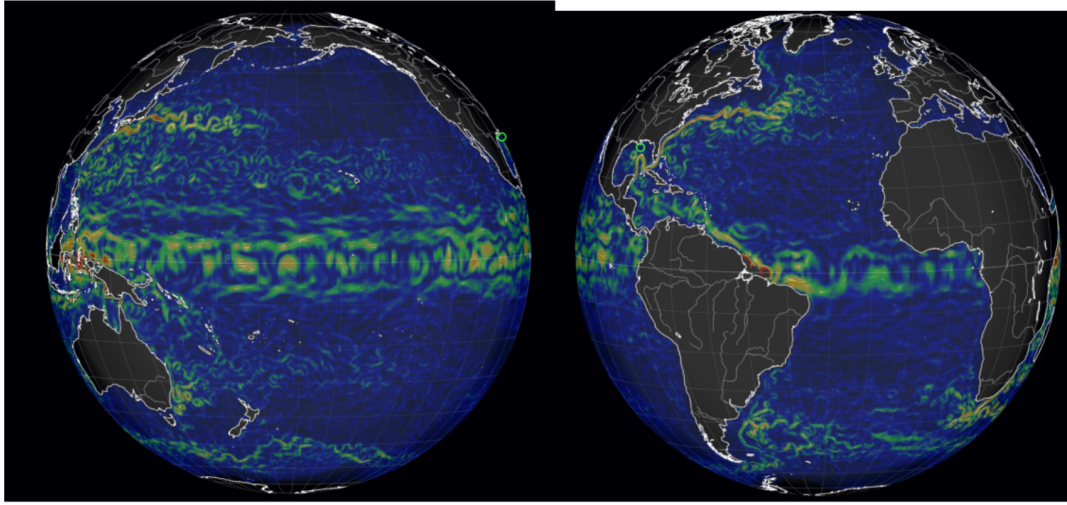
Figure 4.1. Marine currents May 27, 2018, according to the website nullschool. We observe the equatorial waves and the accumulation of energy on the east coasts of continents.
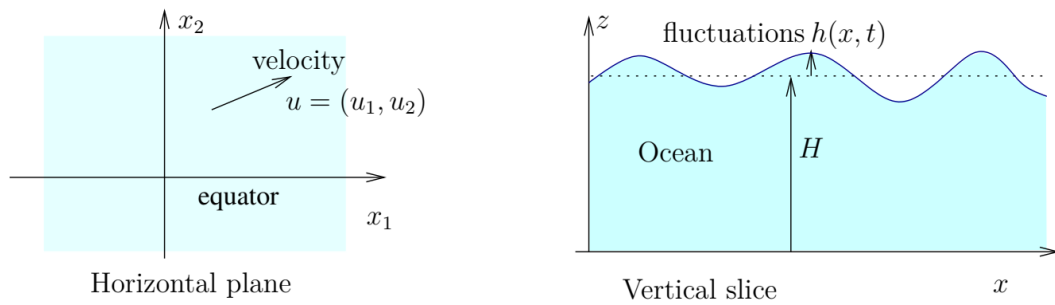

Figure 4.2. Illustration of quantities for the shallow water model (4.1).

Linearization

The idea of Matsuno is to linearize the equations (4.1) in the vicinity of $x_2 = 0$ (the equator), $u = 0$ (small velocities), $h = 0$ (small fluctuations). We assume

$$f(x) = \beta x_2, \quad \beta > 0.$$

Then (4.1) at first order give the following linear equations

(4.1)
$$\partial_t h = -H \text{div}(u)$$

$$\partial_t u = -g\text{grad}(h) - \beta x_2 \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}$$

With $c = \sqrt{gH}$ and the change of variables

$$t' = \sqrt{c\beta}t, \quad x' = \sqrt{\frac{\beta}{c}}x, \quad h' = \sqrt{\frac{\beta}{c}}h, \quad u' = \frac{1}{c}u,$$

we obtain the dimensionless equations, written without $'$ (equivalently we put $H = 1$, $g = 1$, $\beta = 1$):

$$\partial_t h = -\partial_{x_1} u_1 - \partial_{x_2} u_2$$
$$\partial_t u_1 = -\partial_{x_1} h + x_2 u_2$$
$$\partial_t u_2 = -\partial_{x_2} h - x_2 u_1$$

We will write

$$\Psi = \begin{pmatrix} h \\ u_1 \\ u_2 \end{pmatrix} \in L^2\left(\mathbb{R}^3_{x_1,x_2,t}\right) \otimes \mathbb{C}^3.$$

Then

$$i\partial_t \Psi = \begin{pmatrix} 0 & -i\partial_{x_1} & -i\partial_{x_2} \\ -i\partial_{x_1} & 0 & ix_2 \\ -i\partial_{x_2} & -ix_2 & 0 \end{pmatrix} \Psi.$$

Since the coefficients do not depend on x_1 one can assume the Fourier mode in x_1:

$$\Psi\left(x_1, x_2, t\right) = e^{i\mu x_1} \psi\left(x_2, t\right)$$

with Fourier variable $\mu \in \mathbb{R}$ and $\psi \in L^2(\mathbb{R}^2_{x_2,t}) \otimes \mathbb{C}^3$. In other words, μ is the spatial frequency in x_1 (and $\lambda_1 = \frac{2\pi}{\mu}$ is the wave length).

For simplicity we replace (x_2, ξ_2) by (x, ξ). This gives the Matsuno model:

DEFINITION 4.1. — *The "Matsuno model" is the system of equations for* $\psi : (t, x) \in \mathbb{R}^2 \to \psi(t, x) \in \mathbb{C}^3$ *given by*

$$i\partial_t \psi = \widehat{H}_\mu \psi$$

with the operator

(4.2) $\qquad \widehat{H}_\mu = \begin{pmatrix} 0 & \mu & \widehat{\xi} \\ \mu & 0 & i\widehat{x} \\ \widehat{\xi} & -i\widehat{x} & 0 \end{pmatrix} = \mathrm{Op}\left(H_\mu\right), \quad \in \mathrm{Herm}\left(L^2\left(\mathbb{R}_x\right) \otimes \mathbb{C}^3\right)$

and its symbol

(4.3) $\qquad\qquad H_\mu\left(x, \xi_2\right) = \begin{pmatrix} 0 & \mu & \xi \\ \mu & 0 & ix \\ \xi & -ix & 0 \end{pmatrix} \in \mathrm{Herm}\left(\mathbb{C}^3\right)$

and $\widehat{\xi} = \mathrm{Op}_1(\xi) := -i\partial_x$, $\widehat{x} = \mathrm{Op}_1(x) := x$.

4.2. Spectral index $\mathcal{N}$

The following proposition describes the spectrum of the operator $\widehat{H}_\mu$ with respect to the μ parameter.
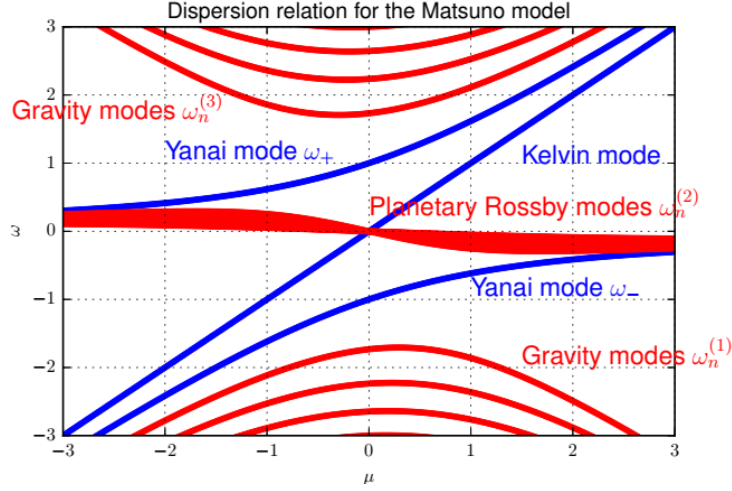
Figure 4.3. Representations of eigenvalues $\omega_n^{(j)}(\mu)$, Eq. (4.4). We observe a spectral index of $\mathcal{N} = +2$ levels, the Kelvin and Yanaï modes.

PROPOSITION 4.2 (*"Spectrum of $\widehat{H}_\mu$"* [Mat66]). — *For each $\mu \in \mathbb{R}$, the operator $\widehat{H}_\mu$, (4.2), has a discrete spectrum in $L^2(\mathbb{R}_x) \otimes \mathbb{C}^3$ given by*

$$(4.4) \qquad \widehat{H}_\mu \phi_n^{(j)} = \omega_n^{(j)} \phi_n^{(j)}, \quad j = 1, 2, 3, \quad n \geqslant 1,$$

with $\omega_n^{(j)}, j = 1, 2, 3$ solutions of the equation of degree 3 in ω:

$$(4.5) \qquad \omega^3 - \left(\mu^2 + 2n + 1\right)\omega - \mu = 0,$$

called gravity waves *for $j = 1, 3$ and* Rossby planetary waves *for $j = 2$. In addition there are the solutions*

$$\widehat{H}_\mu \phi_K = \mu \phi_K \ : \ Kelvin \ mode$$

$$\widehat{H}_\mu \phi_Y^\pm = \omega_\pm \phi_Y^\pm \ : \ Yanaï \ mode$$

with $\omega_\pm = \frac{1}{2}(\mu \pm \sqrt{\mu^2 + 4})$ solutions of $(\omega^2 - \mu\omega - 1) = 0$. We observe in Figure 4.3 that when μ increases, there is

$$\mathcal{N} = +2$$

eigenvalues that are going upward.

See the arxiv version [Fau19] for more physical remarks about this model and a proof of Proposition 4.2.

4.3. Topological Chern index $\mathcal{C}$

We can first consult the section B which introduces the notion of topology of a complex vector bundle of rank 1 on the sphere S^2.
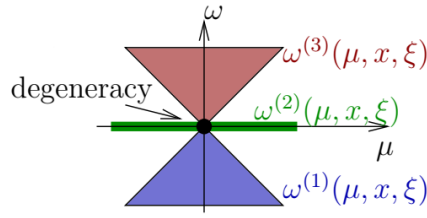
Figure 4.4. Domains that represent the eigenvalues $\omega^{(j)}(\mu, x, \xi)$ for μ fixed and all possible values of $(x, \xi) \in \mathbb{R}^2$, $j = 1, 2, 3$. We have $\omega^{(1)} \leqslant -|\mu|$, $\omega^{(2)} = 0$, $\omega^{(3)} \geqslant |\mu|$.

PROPOSITION 4.3 ("*Topological aspects of the $H_\mu(x, \xi)$ (3.1)*" [DMV17]). — *The eigenvalues of the matrix $H_\mu(x, \xi) \in \mathrm{Herm}(\mathbb{C}^3)$ are*

$$
\begin{aligned}
\omega^{(1)}(\mu, x, \xi) &= -\sqrt{\mu^2 + x^2 + \xi^2} \\
\omega^{(2)}(\mu, x, \xi) &= 0 \\
\omega^{(3)}(\mu, x, \xi) &= +\sqrt{\mu^2 + x^2 + \xi^2}
\end{aligned}
$$

(4.6)

There is therefore a degeneracy at $(\mu, x, \xi) = (0, 0, 0)$. For $(\mu, x, \xi) \in S^2 \subset \mathbb{R}^3$, and $j = 1, 2, 3$, the eigenspace $F^{(j)}(\mu, x, \xi) \subset \mathbb{C}^2$ associated with the eigenvalue $\omega^{(j)}(\mu, x, \xi)$ defines a complex vector bundle of rank 1 above S^2, whose topological indices of Chern $\mathcal{C}_j$ are respectively

$$
\mathcal{C}_1 = +2, \quad \mathcal{C}_2 = 0, \quad \mathcal{C}_3 = -2.
$$

See the arxiv version [Fau19] for a detailed proof of Proposition 4.3.

4.4. Conclusion on the model (3.2)

Formulation given in [FZ00, FZ02a, FZ01] in a more general context: for $|\mu| \gg 1$, there are three groups of levels $j = 1, 2, 3$ in the spectrum of $\widehat{H}_\mu$. When changing $\mu = -\infty \to +\infty$ each group has a variation $\Delta \mathcal{N}_j \in \mathbb{Z}$ of the number of levels. We have the formula

$$
\Delta \mathcal{N}_j = -\mathcal{C}_j
$$

where $\mathcal{C}_j$ is the Chern index of the bundle $F_j \to S^2$.

Another possible formulation: In the model defined by (3.2), one observes from the symbol, a vector bundle F_1 (or $F_1 \oplus F_2$) whose index of Chern is $\mathcal{C} = +2$ and we observe that there is $\mathcal{N} = +2$ levels that transits (upwards) in the spectrum of the operator. We see in Section 2, Theorem 2.7, that this equality

$$
\mathcal{N} = \mathcal{C}
$$

is a special case of a more general result, called the *index formula*, valid for a continuous family of symbols and for spaces and bundles of larger dimensions.

Appendix A. Quantization, pseudo-differential-operators, semi-classical analysis on $\mathbb{R}^{2d}$

A.1. Quantization and pseudo-differential-operators (PDO)

References for this Section are [Mar02, NR11, Zwo12].

We denote $x \in \mathbb{R}^n$ the "*position*" and $\xi \in \mathbb{R}^n$ its dual variable, called "*momentum*". Let $\epsilon > 0$ be a small parameter called *semi-classical parameter*.

DEFINITION A.1. — *If* $a(x,\xi) \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{C})$ *is a function on* phase space $T^*\mathbb{R}^n = \mathbb{R}^{2n}$ *called* symbol, *we associate a* pseudo-differential operator (PDO) *denoted* $\widehat{a} = \mathrm{Op}_\epsilon(a)$ *defined on a function* $\psi \in \mathcal{S}(\mathbb{R}^n)$ *by*

(A.1)
$$(\widehat{a}\psi)(x) = (\mathrm{Op}_\epsilon(a)\,\psi)(x)$$
$$= \frac{1}{(2\pi\epsilon)^n} \int a\left(\frac{x+y}{2}, \xi\right) e^{i\xi \cdot (x-y)/\epsilon} \psi(y)\, dy\, d\xi$$

The operation

$$\mathrm{Op}_\epsilon: \quad a \to \widehat{a} = \mathrm{Op}_\epsilon(a)$$

that gives an operator $\widehat{a}$ *from a symbol* a *is called* Weyl quantization.

Remark A.2. — For example,

- For a function $V(x)$ (function of x only) we get that $\mathrm{Op}_\epsilon(V(x)) = V(x)$, is the multiplication operator by V. For example $\widehat{x}_j = \mathrm{Op}_\epsilon(x_j)$ is called the position operator.
- We have $\widehat{\xi}_j = \mathrm{Op}_\epsilon(\xi_j) = -i\epsilon\frac{\partial.}{\partial x^j}$ called the *momentum operator* and for a function $W: \mathbb{R}^n \to \mathbb{R}$ we have $\mathrm{Op}_\epsilon(W(\xi)) = W((\mathrm{Op}_\epsilon(\xi_j))_j)$, hence $\mathrm{Op}_\epsilon(|\xi|^2) = \sum_j(\mathrm{Op}_\epsilon(\xi_j))^2 = -\epsilon^2\Delta$. The *Schrödinger or Hamiltonian operator* $\widehat{H}$ in quantum mechanics is obtained from the Hamilton function $H(x,\xi)$ by Weyl quantization:

$$H(x,\xi) = \frac{|\xi^2|}{2m} + V(x) \;\to\; \widehat{H} = \mathrm{Op}(H) = -\frac{\epsilon^2}{2m}\Delta + V(x)$$

- We have $\mathrm{Op}_\epsilon(\overline{a}) = (\mathrm{Op}_\epsilon(a))^\dagger$ (the L^2-adjoint).

A.2. Algebra of operators PDO

The following proposition shows that the product of two PDO is a PDO

PROPOSITION A.3. —

- [Zwo12, Section 4.3] "Composition of PDO and star product of symbols": For any $a, b \in \mathcal{S}(\mathbb{R}^{2n})$ we have for $\epsilon \ll 1$

(A.2)
$$\mathrm{Op}_\epsilon(a) \circ \mathrm{Op}_\epsilon(b) = \mathrm{Op}_\epsilon(a \star b)$$

  with $a \star b \in \mathcal{S}(\mathbb{R}^{2n})$ given by

$$a \star b = \left(e^{i\epsilon \hat{A}} \left(a(x, \xi) b(y, \eta) \right) \right)_{y=x, \eta=\xi}$$

$$= ab + \epsilon \frac{1}{2i} \{a, b\} + \epsilon^2 \dots$$

  and $\hat{A} = \frac{1}{2}(\partial_x \partial_\eta - \partial_\xi \partial_y)$.

- "Commutator of PDO and Poisson brackets of symbols":

(A.3)
$$\left[\left(-\frac{i}{\epsilon} \right) \mathrm{Op}_\epsilon(a), \left(-\frac{i}{\epsilon} \right) \mathrm{Op}_\epsilon(b) \right] = \left(-\frac{i}{\epsilon} \right) \mathrm{Op}_\epsilon(\{a, b\}) (1 + O(\epsilon))$$

  i.e.:

$$[a, b]_\star := a \star b - b \star a = i\epsilon \{a, b\} + O(\epsilon^3)$$

- Trace of PDO:

(A.4)
$$\mathrm{Tr}(\mathrm{Op}_\epsilon(a)) = \frac{1}{(2\pi\epsilon)^n} \int_{\mathbb{R}^{2n}} a(x, \xi) \, dx d\xi$$

- Theorem of boundedness: see [NR11, Section 1.4].

Example A.4. — In dimension $n = 1$, we compute directly that $x(-i\epsilon \frac{d}{dx})\psi - (-i\epsilon \frac{d}{dx})(x\psi) = i\epsilon\psi$ and $\{x, \xi\} = 1$. This gives $[\hat{x}, \hat{\xi}] = i\epsilon \mathrm{Id}$ or

$$\left[\left(-\frac{i}{\epsilon} \right) \mathrm{Op}_\epsilon(x), \left(-\frac{i}{\epsilon} \right) \mathrm{Op}_\epsilon(\xi) \right] = \left(-\frac{i}{\epsilon} \right) \mathrm{Op}_\epsilon(\{x, \xi\})$$

in accordance with (A.3).

A.3. Classes of symbols

The relations of Proposition A.3 are a little bit formal. In order to make them useful, one has to control the remainders in terms of operator norm. For this we need to make some assumption on the symbols that express their "slow variation at the Plank scale $dxd\xi \sim \epsilon$" (i.e. uncertainty principle). We call class of symbol the set of symbols that forms an algebra for the operator of composition $\star$. For example, the following classes of symbols have been introduced by Hörmander [Hör83]. Let M be a smooth compact manifold. For $x \in \mathbb{R}^n$, we denote $\langle x \rangle := (1 + |x|^2)^{1/2} \in \mathbb{R}^+$, called the Japanese bracket.

DEFINITION A.5. — Let $m \in \mathbb{R}$ called the order. Let $0 \leqslant \delta < \frac{1}{2} < \rho \leqslant 1$. The class of symbols $S_{\rho,\delta}^m$ contains smooth functions $a \in C^\infty(T^*M)$ such that on any charts of $U \subset M$ with coordinates $x = (x_1, \dots x_n)$ and associated

dual coordinates $\xi = (\xi_1, \ldots \xi_n)$ on $T_x^* U$, any multi-index $\alpha, \beta \in \mathbb{N}^n$, there is a constant $C_{\alpha,\beta}$ such that

$$(A.5) \qquad\qquad \left| \partial_\xi^\alpha \partial_x^\beta a\,(x,\xi) \right| \leqslant C_{\alpha,\beta} \, \langle \xi \rangle^{m - \rho|\alpha| + \delta|\beta|}$$

The case $\rho = 1, \delta = 0$ is very common. We denote $S^m := S_{1,0}^m$.

For example on a chart, $p(x,\xi) = \langle \xi \rangle^m$ is a symbol $p \in S^m$. If $m \leqslant m'$ then $S^m \subset S^{m'}$. We have $S^{-\infty} := \bigcap_{m \in \mathbb{R}} S^m = \mathcal{S}(T^*M)$.

Remark A.6. — The geometric meaning of Definition A.5 may be not very clear a priori. Hörmander improved the geometrical meaning in [Hör79, Hör83] by introducing an associated metric on phase space T^*M. See also [FT17, NR11].

Appendix B. Vector bundles and topology

Some references for this appendix are Fedosov [Fed96, p. 11], Hatcher [Hat98, p. 14].

We will give precise definitions in Section B.3. We begin in Section B.1 and B.2 by a description of vector bundles based on examples and sufficient to understand the case of dimension $n = 1$ used in this paper.

A complex (or real) vector bundle $F \to B$ of rank r is a collection of complex (or real) vector spaces F_x of dimension r, called fiber, and continuously parametrized by points x on a manifold B, called "base space". Locally over $U \subset B$, F is isomorphic to a direct product $U \times \mathbb{C}^r$.

B.1. Topology of a real vector bundle of rank 1 on S^1

B.1.1. Construction of a real vector bundle of rank 1 on S^1

The simplest example is the case where the base space is the circle $B = S^1$ and the rank is $r = 1$, i.e. each fiber is isomorphic (as a vector space) to the real line $\mathbb{R}$.

One can easily imagine two examples of real fiber space of rank 1 on S^1:

- The *trivial bundle* $S^1 \times \mathbb{R}$ that we obtain from the trivial bundle $[0,1] \times \mathbb{R}$ on the segment $x \in [0,1]$ (i.e. direct product) and identifying the points $(0,t) \sim (1,t)$, for all $t \in \mathbb{R}$.
- The *Moebius bundle*, which is obtained from the bundle $[0,1] \times \mathbb{R}$ on the segment $x \in [0,1]$, identifying $(0,t) \sim (1,-t)$, $t \in \mathbb{R}$.

The Moebius bundle is not isomorphic to the trivial bundle. One way to justify this is that in the case of the trivial bundle, the complement of the null section $(s(x) = 0, \forall\, x)$ has two connected components, whereas for the bundle of Moebius, the complement has only one component. (Make a paper construction that is cut with scissors according to $s(x) = 0$ to observe this).

Fibers

Base space S^1

(a) trivial bundle $S^1 \times \mathbb{R}$
SW=0

(b) Moebius bundle
SW=1
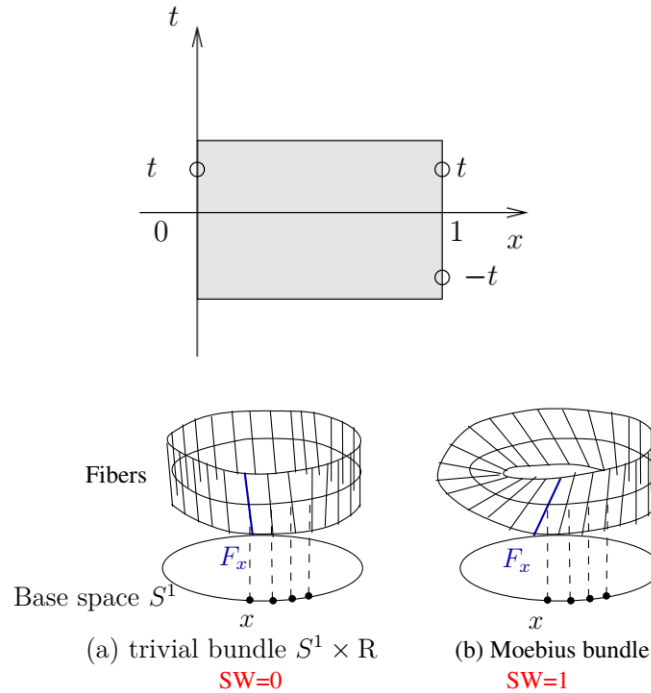
THEOREM B.1. — *Any real vector bundle $F \to S^1$ of rank 1 is isomorphic to the trivial bundle or to the Möebius's bundle. In other words, there are only two classes of equivalences:*

$$\mathrm{Vect}_{\mathbb{R}}^1\left(S^1\right) = \{0, 1\}$$

*associated with the Stiefel-Whitney index $SW = 0$: trivial bundle, $SW = 1$: bundle of Moebius.*

Proof. — Starting from any bundle $F \to S^1$ of rank 1, we cut the base space S^1 at a point, and we are left with the bundle $[0, 1] \times \mathbb{R}$ over $x \in [0, 1]$. To reconstruct the initial bundle F, there are two possibilities: for all $t \in \mathbb{R}$, identify $(0, t) \sim (1, t)$, or $(0, t) \sim (1, -t)$, which gives the trivial or Moebius bundle respectively. $\square$

Remark B.2. —
- the Stiefel–Whitney index $SW = 0, 1$ gives the number of half turns that the fibers make above the base space S^1. The case $SW = 2$ (one full turn) is isomorphic to the trivial bundle. We therefore agree that the index $SW \in \mathbb{Z}/(2\mathbb{Z})$, i.e. SW is an integer modulo 2. It is interesting to have the additive structure on the SW indices ($1 + 1 = 0$ for example).
- Note that in the space $\mathbb{R}^3$, a ribbon making a turn, i.e. $SW = 2$, can not be deformed continuously towards the trivial bundle. [4]. This restriction is due to the embedding in the space $\mathbb{R}^3$ (in $\mathbb{R}^4$, this would be possible), and is not an intrinsic property of the bundle that is nevertheless trivial.

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[4] Because if we cut this ribbon on the section $s = 0$, we obtain two ribbons interlaced, whereas the same cut for a trivial ribbon gives two separate ribbons.
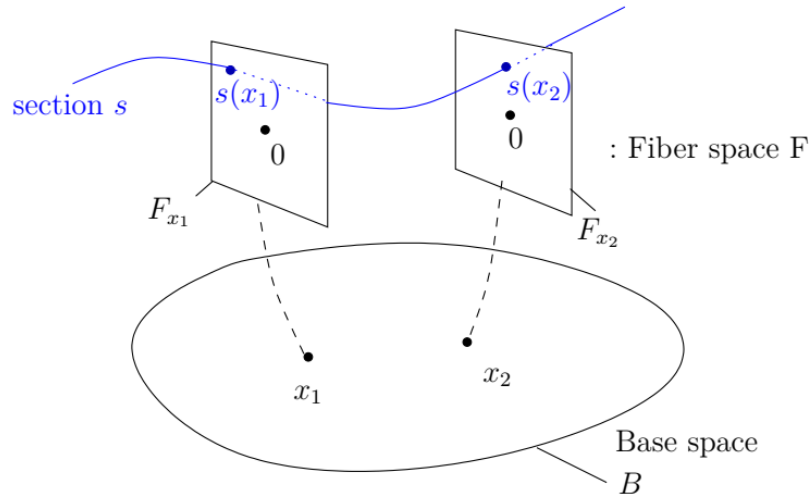
B.1.2. Topology of a real vector bundle of rank 1 over S^1 from the zeros of a section

DEFINITION B.3. — *If $F \to B$ is a vector space, a global section of the bundle is an application (continuous or C^∞) $s : B \to F$ such that each base point $x \in B$ is mapped to a point in the fiber $s(x) \in F_x$. We note*

(B.1)
$$C^\infty(B, F)$$

the space of the smooth sections of the bundle F.

We call zeros of the section s the points $x \in B$ such that $s(x) = 0$. Let us first consider the very simple and instructive case of a real bundle of rank 1 on S^1. A section is locally like a real value numerical function, so generically, it vanishes transversely at isolated points. Note that "generic" means "except for exceptional case". The following figure shows that we have the following result:

THEOREM B.4. — *If $F \to S^1$ is a real bundle of rank 1 on S^1, and s is a "generic" section, then the topological index $SW(F)$ is given by*
$$SW(F) = \sum_{x \ t.q. \ s(x)=0} \sigma_s(x)$$
where $\sigma_s(x) = 1$ for a generic zero of section s. The sum is obtained modulo 2, and so $SW(F) \in \mathbb{Z}_2 = \{0, 1\}$. The result is independent of the chosen section s.

B.2. Topology of a complex rank 1 vector bundle over S^2

We proceed similarly to the previous Section B.1.
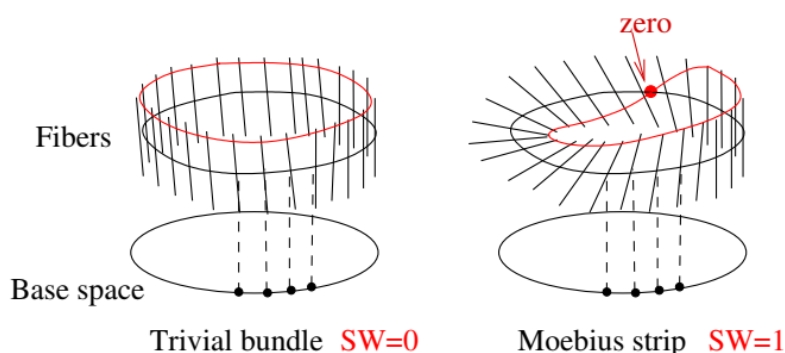
B.2.1. Construction of a complex vector bundle of rank 1 on S^2

Let's first see how to build a complex fiber space of rank 1 over S^2. We cut the sphere S^2 along the equator S^1, obtaining two hemispheres H_1 and H_2. We get two trivial bundles $F_1 = H_1 \times \mathbb{C}$ and $F_2 = H_2 \times \mathbb{C}$ on each hemisphere. To construct a bundle on S^2, it is enough to decide how to "connect" or "identify" the fibers of F_1 above the equator with those of F_2. Note $\theta \in S^1$ the angle[5] (longitude) that characterizes a point on the equator. Note $\varphi(\theta) \in S^1$ the angle which means that the fiber $F_2(\theta)$ is identified to the fiber $F_1(\theta)$ after a rotation of angle $\varphi(\theta)$: a $v \in F_1(\theta) \equiv \mathbb{C}$ is identified with the $e^{i\varphi(\theta)}v \in F_2(\theta)$. After gluing that way the two hemispheres and the fibers above the equator, we obtain a complex vector bundle $F \to S^2$ of rank 1. Thus the bundle F that we have just built is defined by its clutching function on the equator

$$\varphi : \theta \in S^1 \to \varphi(\theta) \in S^1$$

It is a continuous and periodic function so: $\varphi(2\pi) \equiv \varphi(0)[2\pi]$, or

(B.2) $$\varphi(2\pi) = \varphi(0) + 2\pi\mathcal{C}, \qquad \mathcal{C} \in \mathbb{Z},$$

with the integer $\mathcal{C} \in \mathbb{Z}$ that represents the number of revolutions that φ makes when θ goes around. We call $\mathcal{C}$ the degree of the application $\varphi : S^1 \to S^1$. It is clear that two functions φ, φ' are homotopic if and only if they have the same degree $\mathcal{C} = \mathcal{C}'$, and therefore the bundles F and F' are isomorphic if and only if $\mathcal{C} = \mathcal{C}'$.

> THEOREM B.5. — *Any complex fiber bundle $F \to S^2$ of rank 1 is isomorphic to a bundle constructed as above with a clutching function φ on the equator. Its topology is characterized by an integer $\mathcal{C} \in \mathbb{Z}$ called (1st) Chern index given by $\mathcal{C} = \deg(\varphi)$. In other words the equivalence class of rank 1 complex vector bundle on S^2 is*
> $$\mathrm{Vect}^1_{\mathbb{C}}\left(S^2\right) = \mathbb{Z}$$

Proof. — We must show that every bundle F is isomorphic to a bundle constructed as above. Starting from a given bundle F, we cut the base space S^2 along the equator denoted S^1 to obtain two bundles $F_1 \to H_1$ and $F_2 \to H_2$. Each of these bundles is

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[5] Here, we note S^1 the circle. $\theta \in S^1$ is therefore marked with an angle $\theta \in [0, 2\pi]$.
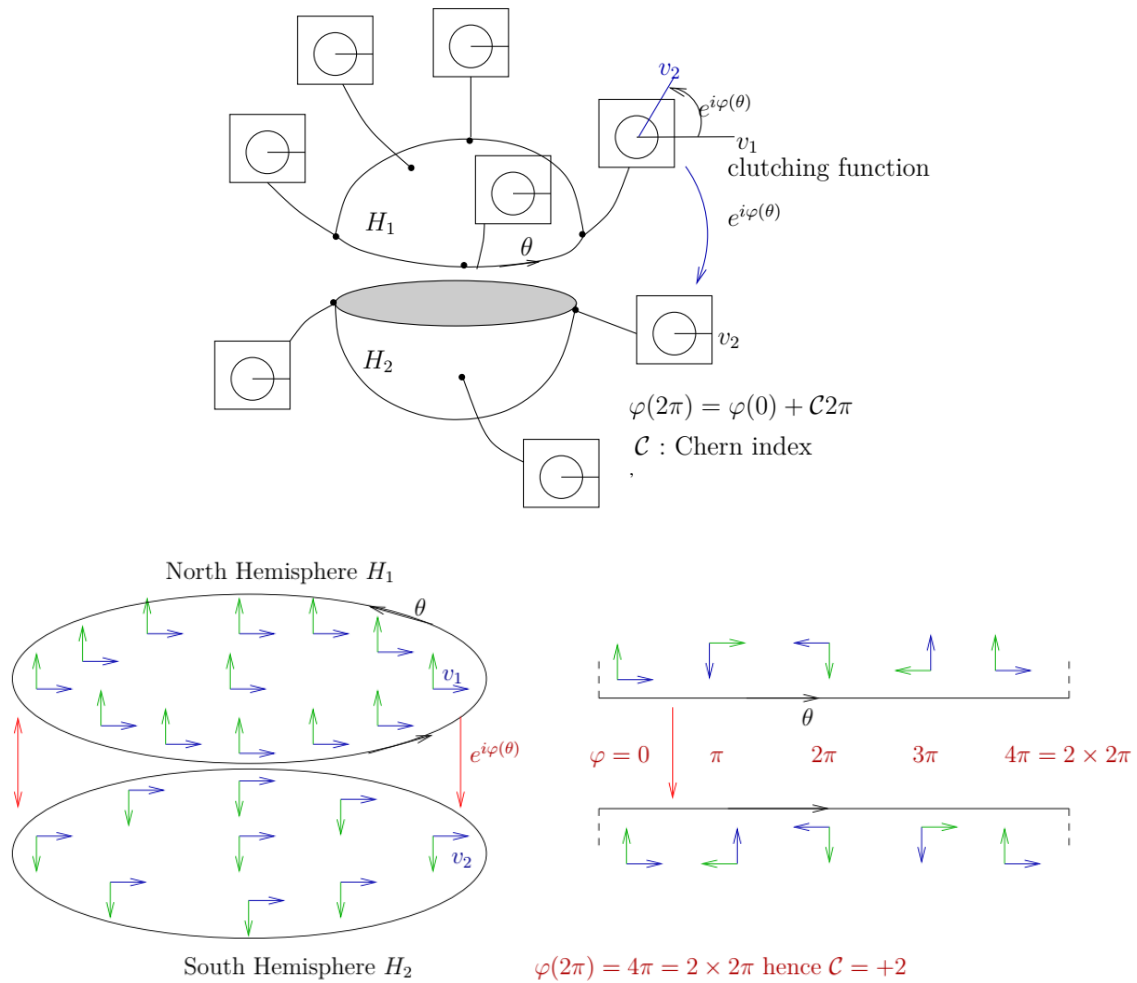
trivial because[Hat98, Corrollaire 1.8, p. 21] the base spaces are disks (contractile spaces). The bundle F is thus defined by its clutching function above the equator S^1, $\varphi : S^1 \to S^1$. □

Consider the example of the tangent bundle TS^2 of the sphere. TS^2 can be identified with a complex bundle of rank 1 because S^2 is orientable.

> THEOREM B.6. — *The tangent bundle TS^2 has Chern index*
>
> (B.3) $$\mathcal{C}(TS^2) = +2$$
>
> *and is therefore non trivial.*

Proof. — We will calculate the degree C of its recollection function defined by Eq. (B.15). We proceed as in the proof above. We trivialize the bundle above H_1, and H_2, and we deduce the degree C of the gluing function. See figure that represents the two hemispheres seen from above and below with a vector field on each. We find $\mathcal{C} = +2$. □

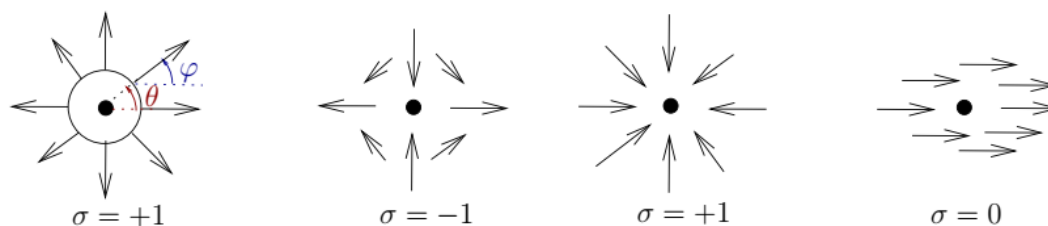*Remark B.7.* — The trivial bundle $S^2 \times \mathbb{C}$ has the Chern index $\mathcal{C} = 0$.

Figure B.1. Index $\sigma \in \mathbb{Z}$ of a zero of a vector field computed from the degree of the map $\theta \in S^1 \to \varphi(\theta) \in S^1$.

B.2.2. Topology of the rank 1 vector bundle on S^2 from the zeros of a section

There is a result analogous to Theorem B.4 for a complex bundle $F \to S^2$ of rank 1 on S^2. Before establishing it, let us notice that a section s of such a bundle is locally like a function with two variables and with values in $\mathbb{C}$, so generically, it vanishes transversely at isolated points. If $\theta \in S^1$ parameterizes a small circle of points x_θ around a zero $x \in S^2$ of s, then by hypothesis, the value of the section $s(x_\theta) \in F_{x_\theta} \equiv \mathbb{C}$ is non-zero for all x_θ, and we write $\varphi \in S^1$ his argument. For each zero x of the section s is therefore associated an application $\varphi : \theta \to \varphi(\theta)$ whose degree, also called index of the zero (defined by Eq. (B.15)), will be noted $\sigma_s(x) \in \mathbb{Z}$. Generically, $\sigma_s(x) = \pm 1$. (Note that the sign of $\sigma_s(x)$ depends on the chosen orientation of the base space and the fiber. In the case of the tangent bundle on S^2, these two orientations are not independent, and the result $\sigma_s(x)$ becomes independent of the choice of orientation). See Figure B.1.

> THEOREM B.8. — *If $F \to S^2$ is a complex bundle of rank 1 on S^2, and s is a "generic" section, then the topological index of Chern $\mathcal{C}(F)$ is given by*
>
> (B.4) $$\mathcal{C}(F) = \sum_{x \ t.q. \ s(x)=0} \sigma_s(x) \qquad \in \mathbb{Z}$$
>
> *where $\sigma_s(x) = \pm 1$ characterizes the degree of zero. The result is independent of the chosen section s.*

Proof. — In the proof of the Theorem B.5, we have constructed sections v_1, v_2 for the respectively bundles $F \to H_1$, $F \to H_2$, that never vanish. If we modify these sections v_1, v_2 to make them coincide on the equator for the purpose of constructing a global section s of the bundle $F \to S^2$, we can get do this except in points isolated, which will be the zeros of s, and one realizes that the sum of the indices will be equal to the degree of the clutching function φ therefore equal to $\mathcal{C}(F)$. $\qquad \square$
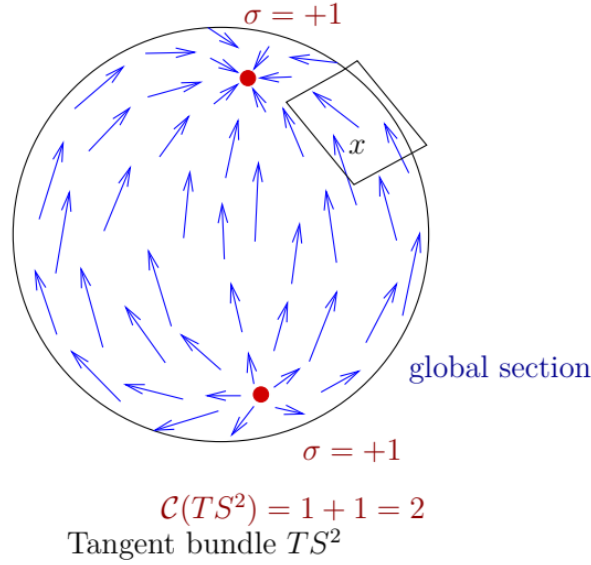
Example of the bundle TS^2. The following figure shows a vector field on the S^2 sphere. It is a global section of the tangent bundle. This vector field has two zeros with indices $+1$ each. Thus we find $\mathcal{C}(TS^2) = +2$, i.e. Eq. (B.3).

Remark B.9. — If we want to give an explicit computation we need an explicit global section (or vector field on TS^2). We can take the fixed vector in $\mathbb{R}^3$: $V = (0, 0, 1)$ oriented along the z axis. Then for a given point $x \in S^2$ we choose:

$$s(x) = P_x V \in T_x S^2 \tag{B.5}$$

where $P_x : \mathbb{R}^3 \to T_x S^2$ is the orthogonal projector given by $P_x = \mathrm{Id} - |x\rangle\langle x|.\rangle$. We get

$$s(x) = V - x\langle x|V\rangle = \left(-x_3 x_1, -x_3 x_2, 1 - x_3^2\right). \tag{B.6}$$

$\sigma = +1$

x

global section

$\sigma = +1$

$\mathcal{C}(TS^2) = 1 + 1 = 2$

Tangent bundle TS^2

The vector field $s(x)$ vanishes at the north and south pole. At distance ϵ of north pole $(0, 0, 1)$, we use local oriented coordinates $(x_1, x_2) \equiv \epsilon e^{i\theta}$ and get $s(x) = (-x_1, -x_2, 0) + O(\epsilon^2) = -e^{i\theta} + O(\epsilon^2)$. The map $e^{i\theta} \in S^1 \to -e^{i\theta} \in S^1$ has degree 1 hence the zero has index $\sigma = +1$. At distance ϵ of south pole $(0, 0, -1)$, we use local oriented coordinates $(x_2, x_1) \equiv \epsilon e^{i\theta}$ and get $s(x) = (x_1, x_2, 0) + O(\epsilon^2) = e^{i\theta} + O(\epsilon^2)$. The map $e^{i\theta} \in S^1 \to e^{i\theta} \in S^1$ has degree 1 hence the zero has again index $\sigma = +1$. Formula (B.4) gives

$$\mathcal{C}\left(TS^2\right) = +1 + 1 = +2.$$

B.2.3. Topology of the rank 1 vector bundle on S^2 from a curvature integral in differential geometry

Let $F \to S^2$ be a complex vector bundle of rank 1 over S^2. Let us assume[6] that there exists a fixed vector space $\mathbb{C}^d$ such that for every $x \in S^2$, the fiber $F_x \subset \mathbb{C}^d$ is a linear subspace of $\mathbb{C}^d$ for some $d \geqslant 1$. For every point $x \in S^2$, let us denote $P_x : \mathbb{C}^d \to \mathbb{C}^d$ the orthogonal projector onto F_x. Then if $s \in C^\infty(S^2; F)$ is a smooth section we can consider $s \in C^\infty(S^2; \mathbb{C}^d)$ as a d multi-components function on S^2.

[6] This is the case in the model of Section 2 and every vector bundle can be realized like this, see [Fed96].

If $V \in T_x S^2$ is a tangent vector at point $x \in S^2$, the derivative $V(s) \in \mathbb{C}^d$ can be projected onto F_x. We get

$$(D_V s)(x) := P_x V(s) \in F_x$$

called the *covariant derivative* of s along V at point x. It measures the variations of s within the fibers F. Since $V(s) = ds(V)$ where ds means the differential[7], we usually write

$$Ds := Pds$$

for the *covariant derivative* or *Levi–Civita connection* (in differential geometry, $Pds \in C^\infty(S^2; \Lambda^1 \otimes F)$ is a one form valued in F).

Suppose that $U \subset S^2$ and for every point $x \in U$ one has $v(x) \in F_x$ a unitary vector that depends smoothly on $x \in U$. This is called a local *unitary trivialization* of $F \to U$ (as in the proof of Theorem B.6). Since the fiber F_x is dimension 1, the vector $v(x)$ is a unitary basis of F_x and if $V \in T_x S^2$, the covariant derivative $(D_V v)(x) = P_x V(v) \in F_x$ can expressed in this basis with one complex component:

$$(D_V v)(x) = \underbrace{(A(x))(V)}_{\in \mathbb{R}} v(x)$$

where $A(x) = i\mathcal{A}(x)$ is $i\mathbb{R}$ valued[8] linear form on $T_x S^2$ (a cotangent vector) called[9] *connection one form.* In short,

(B.7)
$$Dv = Av.$$

Let

(B.8)
$$\Omega := dA$$

---

[7] In local coordinates $x = (x_1, x_2) \in \mathbb{R}^2$ on S^2, if $f(x_1, x_2)$ is a function, then its *differential* is written

$$df = \sum_k \left(\frac{\partial f}{\partial x_k} \right) dx_k$$

and a *tangent vector* is written $V = \sum_k V_k \frac{\partial}{\partial x_k}$. Then since $df(V) = V(f)$ gives in particular for the function x_k that $dx_k(\frac{\partial}{\partial x_l}) = \frac{\partial x_k}{\partial x_l} = \delta_{k=l}$, we get that $df(V) = \sum_k (\frac{\partial f}{\partial x_k}) V_k$.

[8] A is imaginary valued from the fact that $\langle v|v \rangle = 1$ hence

$$0 = d\langle v|v \rangle = \langle Dv|v \rangle + \langle v|Dv \rangle = 2\text{Re}(\langle v|Av \rangle) = 2\text{Re}(A).$$

[9] If $s \in C(S^2; F)$ is an arbitrary section, then locally one can write $s(x) = \phi(x)v(x)$ with some complex component $\phi(x) \in \mathbb{C}$. Then

$$Ds = D(\phi v) = (d\phi)v + \phi Dv = (d\phi)v + \phi Av$$

$$= (d\phi + \phi A)v = \sum_k \left(\frac{\partial \phi}{\partial x_k} + A_k \phi \right) (dx_k)v,$$

with $A = \sum_k A_k dx_k$. Writing $A = i\mathcal{A}$, it shows that the components of the covariant derivative Ds with respect to the unitary trivialization $v(x)$ and local coordinates $(x_k)_k$ on U are $(\frac{\partial \phi}{\partial x_k} + i\mathcal{A}_k \phi)_k$. In quantum physics books it is common to see the expression $(\frac{\partial \phi}{\partial x_k} + i\mathcal{A}_k \phi)_k$ for a definition of the "covariant derivative" or "minimal coupling", e.g. [BIZ80, p. 31].

be the two form[10] called the *curvature* of the connection.

> LEMMA B.10. — *Let $F \to S^2$ be a rank 1 complex vector bundle with $F_x \subset \mathbb{C}^d$. Let $v(x) \in F_x$ a given local unitary trivialization and $Dv = Av$ with A the connection one form and Ω the curvature two form. Then*
>
> $$A = \langle v \,|\, dv \rangle_{\mathbb{C}^d} = \sum_k \left\langle v \,\middle|\, \frac{\partial v}{\partial x_k} \right\rangle dx_k$$
>
> (B.9) $$\Omega = \langle dv \,|\, \wedge dv \rangle = \sum_{k,l} \left\langle \frac{\partial v}{\partial x_k} \,\middle|\, \frac{\partial v}{\partial x_l} \right\rangle dx_k \wedge dx_l.$$
>
> *and Ω does not depend on the trivialization, hence is globally defined on S^2. Finally the topological Chern index $\mathcal{C}$ defined in (B.2) is given by the* curvature integral
>
> (B.10) $$\mathcal{C} = \frac{1}{2\pi} \iint_{S^2} i\Omega.$$

Proof. — The orthogonal projector is given by

$$P_x = |v_x\rangle\langle v_x|.\rangle$$

hence the covariant derivative is given by $Ds = Pds = P_x = |v_x\rangle\langle v_x|ds\rangle$ and since by definition $Dv = Av$ we get $A = \langle v|dv\rangle$ and

$$\Omega = dA = d\left(\sum_k \left\langle v \,\middle|\, \frac{\partial v}{\partial x_k} \right\rangle dx_k \right)$$

$$= \sum_{k,l} \left\langle \frac{\partial v}{\partial x_l} \,\middle|\, \frac{\partial v}{\partial x_k} \right\rangle dx_l \wedge dx_k + \underbrace{\sum_{k,l} \left\langle v \,\middle|\, \frac{\partial^2 v}{\partial x_l \partial x_k} \right\rangle dx_l \wedge dx_k}_{=0}.$$

The second term vanishes since $\left(\frac{\partial^2 v}{\partial x_l \partial x_k}\right)_{k,l}$ is a symmetric array and $(dx_l \wedge dx_k)_{k,l}$ is antisymmetric. If we replace v by another trivialization $v'(x) = e^{i\alpha(x)}v(x)$ (this is called a *Gauge transformation*) then

$$A' = \langle v' \,|\, dv' \rangle = e^{-i\alpha} \sum_k \left\langle v \,\middle|\, \frac{\partial e^{i\alpha}v}{\partial x_k} \right\rangle dx_k = \sum_k \left(i\frac{\partial\alpha}{\partial x_k} \right) dx_k + \left\langle v \,\middle|\, \frac{\partial v}{\partial x_k} \right\rangle dx_k$$

$$= id\alpha + A$$

is changed but

$$\Omega' = dA' = idd\alpha + dA = \Omega$$

is unchanged because $dd\alpha = \sum_{k,l}\left(\frac{\partial^2\alpha}{\partial x_l \partial x_k}\right)dx_l \wedge dx_k = 0$.

As in Section B.2.1, let H_1, H_2 be the north and south hemispheres of S^2 and suppose that for every point $x \in H_1$, $v_1(x) \in F_x$ is a unitary vector that depends smoothly on x, i.e. v_1 is a trivialization of $F \to H_1$. Suppose that v_2 is a trivialization

---

[10] In local coordinates if $A = \sum_k A_k dx_k$ is a one form with components $A_k(x)$ then $dA = \sum_{k,l} \frac{\partial A_k}{\partial x_l} dx_l \wedge dx_k$.

of $F \to H_2$ (as in the proof of Theorem B.6). Let $x \equiv (\theta, \varphi)$ denotes the spherical coordinates on S^2. For a given $0 \leqslant \theta \leqslant \frac{\pi}{2}$ on Hemisphere H_1, let $\gamma_\theta : \varphi \in [0, 2\pi] \to \gamma_\theta(\varphi) \in S^2$ be the closed path. Let $\psi_\theta^{(1)}(0) \in F_{\theta,0}$ and

$$\psi_\theta^{(1)}(\varphi) = e^{i\alpha_\theta^{(1)}(\varphi)} v_1(\theta, \varphi) \in F_{\theta,\varphi}$$

obtained for $0 \leqslant \varphi \leqslant 2\pi$ by *parallel transport*, i.e. under the condition of zero covariant derivative

$$\frac{D\psi_\theta^{(1)}}{d\varphi} = 0 \Leftrightarrow \frac{D\left(e^{i\alpha_\theta^{(1)}} v_1\right)}{d\varphi} = 0 \underset{(B.7)}{\Leftrightarrow} i\frac{d\alpha_\theta^{(1)}}{d\varphi} v_1 + A v_1 = 0$$

giving that

(B.11) $$\alpha_\theta^{(1)}(2\pi) - \alpha_\theta^{(1)}(0) = \int_{\gamma_\theta} iA \underset{(B.8),\, \text{Stokes}}{=} \iint_{H_\theta} i\Omega$$

where $H_\theta = \{(\theta', \varphi'), \theta' \geqslant \theta, \varphi' \in [0, 2\pi]\} \subset S^2$ is a surface with boundary γ_θ. The angle $\alpha_\theta^{(1)}(2\pi)$ is called the *holonomy* of the connection on the closed path γ_θ and also called *Berry's phase* after the paper of M. Berry [Ber84] that shows its natural manifestation in quantum mechanics, see also [Fau18]. We can do the same on the south hemisphere H_2 with v_2 and angles $\alpha_\theta^{(2)}$, giving at $\theta = 0$,

(B.12) $$\alpha_0^{(2)}(2\pi) - \alpha_0^{(2)}(0) = -\iint_{H_2} i\Omega$$

with opposite sign because the orientation of γ_0 is reversed. In particular, on the equator $\theta = 0$ that belongs to both Hemisphere, we have for every φ that

$$v_2(0, \varphi) = e^{i\beta(\varphi)} v_1(0, \varphi)$$

and by definition of Chern index $\mathcal{C}$,

$$\beta(2\pi) \underset{(B.2)}{=} \beta(0) + 2\pi\mathcal{C}$$

Also

$$\psi_0^{(1)}(\varphi) = e^{i\alpha_0^{(1)}(\varphi)} v_1(0, \varphi), \quad \psi_0^{(2)}(\varphi) = e^{i\alpha_0^{(2)}(\varphi)} v_2(0, \varphi),$$

and since the parallel transport preserves the angles, $\psi_0^{(2)}(\varphi) = e^{ic}\psi_0^{(1)}(\varphi)$ with a constant c (independent on φ). Finally we get

$$v_2(0, \varphi) = e^{i\beta(\varphi)} v_1(0, \varphi) = e^{i\left(\beta(\varphi) - \alpha_0^{(1)}(\varphi)\right)} \psi_0^{(1)}(\varphi) = e^{i\left(\beta(\varphi) - \alpha_0^{(1)}(\varphi) - c\right)} \psi_0^{(2)}(\varphi)$$

$$= e^{i\left(\beta(\varphi) - \alpha_0^{(1)}(\varphi) - c + \alpha_0^{(2)}(\varphi)\right)} v_2(0, \varphi)$$

hence

$$\beta(\varphi) = \alpha_0^{(1)}(\varphi) + c - \alpha_0^{(2)}(\varphi)$$

and

$$\mathcal{C} = \frac{1}{2\pi}\left(\beta\left(2\pi\right) - \beta\left(0\right)\right) = \frac{1}{2\pi}\left(\left(\alpha_0^{(1)}\left(2\pi\right) - \alpha_0^{(1)}\left(0\right)\right) - \left(\alpha_0^{(2)}\left(2\pi\right) - \alpha_0^{(2)}\left(0\right)\right)\right)$$

$$\underset{\text{(B.11), (B.12)}}{=} \frac{1}{2\pi}\left(\iint_{H_1} i\Omega + \iint_{H_2} i\Omega\right) = \frac{1}{2\pi}\iint_{S^2} i\Omega \qquad \square$$

Remark B.11. — Formula (B.10) is a special case of a more general Chern–Weil formula, Formula (B.19) given below for a general vector bundle $F \to S^{2n}$ of rank r.

Example B.12. — For the special case of the tangent bundle TS^2, with fiber $T_x S^2 \subset \mathbb{R}^3$, if $i\Omega$ is the (2 form) Gauss curvature of the sphere (that is, the curvature of the tangent bundle TS^2, which is the solid angle), the Gauss–Bonnet formula gives:

$$(\text{B.13}) \qquad \mathcal{C} = \frac{1}{2\pi}\int_{S^2} i\Omega = \frac{4\pi}{2\pi} = 2$$

as in (B.3).

B.3. General vector bundles over sphere S^k

B.3.1. Definitions

DEFINITION B.13. — *We say that (F, π, B) is a complex vector bundle of rank r if F, B are manifolds, $\pi : F \to B$ a map such that there exists a covering $(U_i)_i$ of B and diffeomorphisms $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$ such that*
  (1) *$\pi : \pi^{-1}(U_i) \to U_i$ is the composition of φ_i with projection onto U_i*
  (2) *if $U_i \cap U_j \neq \emptyset$ then $\varphi_i \varphi_j^{-1} : (U_i \cap U_j) \times \mathbb{C}^r \to (U_i \cap U_j) \times \mathbb{C}^r$ is given by $(x, u) \to (x, f_{ij}(x)u)$ with $f_{ij}(x) \in GL(r, \mathbb{C})$.*

We say that φ_i are *trivialization functions*, and f_{ij} are *transition functions*.

PROPOSITION B.14. — *The transition functions satisfy the cocycle conditions:*

$$f_{ji} = f_{ij}^{-1}, \forall\, x \in U_i \cap U_j \qquad f_{ij}f_{jk}f_{ki} = 1, \forall\, x \in U_i \cap U_j \cap U_k$$

Conversely functions f_{ij} with cocycle conditions, define a unique vector bundle.

Proof. — $f_{ji}^{-1} = (\varphi_j\varphi_i^{-1})^{-1} = \varphi_i\varphi_j^{-1} = f_{ij}$. And $f_{ij}f_{jk}f_{ki} = (\varphi_i\varphi_j^{-1})(\varphi_j\varphi_k^{-1})(\varphi_k\varphi_i^{-1}) = 1$. $\qquad \square$

DEFINITION B.15. — *Two vector bundles (F, π, B) and (F', π', B) (with same base B) are* isomorphic *if there exists $h : F \to F'$ which preserves the fibers and such that $h : F_x \to F'_x$ is an isomorphism of linear spaces.*

We write $\mathrm{Vect}^r_{\mathbb{C}}(B)$ for the isomorphism class of complex vector bundles of rank r over B.

> PROPOSITION B.16. — *Two vector bundles F and F' are isomorphic if and only if there exists functions $h_i : U_i \to GL(n, \mathbb{C})$ such that*
> $$f'_{ij} = h_i f_{ij} h_j^{-1}$$
> *where f_{ij}, f'_{ij} are the transition functions.*

Proof. — If h is an isomorphism, define $h_i = \varphi'_i h \varphi_j^{-1}$. Conversely, define $h = (\varphi'_i)^{-1} h_i \varphi_i$ on U_i which does not depend on i. $\qquad\square$

B.3.2.   Complex Vector bundles over spheres S^k

Reference: Hatcher [Hat98, p. 22]. We treat the case where the base space is a sphere
$$B = S^k := \left\{ (x_1, \ldots x_{k+1}) \in \mathbb{R}^{k+1}, \quad \sum_j x_j^2 = 1 \right\}.$$
The sphere $S^k = D_1^k \cup D_2^k$ can be decomposed in two disks (or hemispheres), the north hemisphere D_1^k where $x_{k+1} \geqslant 0$ and the south hemisphere D_2^k where $x_{k+1} \leqslant 0$. The common set is the equator $S^{k-1} = D_1^k \cap D_2^k = \{x \in \mathbb{R}^{k+1}, x_{k+1} = 0\}$ which is also a sphere S^{k-1}. So a vector bundle is described by the transition function at the equator: $f_{21} : S^{k-1} \to GL(r, \mathbb{C})$, which is called the *clutching function*. Let us denote $[f_{21}]$ the homotopy class of the map f_{21}. The set of homotopy classes is $[S^{k-1}, GL(r, \mathbb{C})] \equiv [S^{k-1}, U(r)] =: \pi_{k-1}(U(r))$ is called homotopy group of $U(r)$.

> PROPOSITION B.17. — *Two vector bundles $F \to S^k$, $F' \to S^k$ are isomorphic if and only if their clutching functions are homotopic $[f_{21}] = [f'_{21}]$. In other words the group of equivalence classes of vector bundles coincide with the homotopy groups:*
> $$\mathrm{Vect}^r_{\mathbb{C}}\left(S^k \right) \equiv \pi_{k-1}\left(U\left(r \right) \right).$$

Homotopy groups of spheres. The groups $\mathrm{Vect}^r(S^k) = \pi_{k-1}(U(r))$ can be obtained from homotopy groups of the spheres $\pi_m(S^n)$ from the fact that

(B.14) $$U\left(r \right) / U\left(r - 1 \right) \equiv S^{2r-1}.$$

This is obtained by observing that the unit sphere in $\mathbb{C}^r$ is S^{2r-1} and thus, for $f \in U(r)$ and $e_r = (0, \ldots, 0, 1) \in \mathbb{C}^r$ we have $f(e_r) \in S^{2r-1} \subset \mathbb{C}^r$ that characterizes f up to $U(r-1)$, i.e. its action on $\mathbb{C}^{r-1}$. See Table B.1. See Hatcher's book.

We have
$$\pi_n(S^n) = \mathbb{Z}$$
which is the degree and is computed as follows.

Table B.1. *Homotopy groups of the spheres* $\pi_m(S^n)$.

$\pi_n(S^m)$	π_1	π_2	π_3	π_4	π_5	π_6	π_7
S^1	$\boxed{\mathbb{Z}}$	0	0	0	0	0	0
S^2	0	$\boxed{\mathbb{Z}}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$
S^3	0	0	$\boxed{\mathbb{Z}}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_6$
S^4	0	0	0	$\boxed{\mathbb{Z}}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$
S^5	0	0	0	0	$\boxed{\mathbb{Z}}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
S^6	0	0	0	0	0	$\boxed{\mathbb{Z}}$	$\mathbb{Z}_2$
S^7	0	0	0	0	0	0	$\boxed{\mathbb{Z}}$

DEFINITION B.18. — *The degree of a map* $f : S^m \to S^m$ *is*

$$(\text{B.15}) \qquad \deg(f) := \sum_{x \in f^{-1}(y)} \operatorname{sign}(\det(D_x f)) \in \mathbb{Z},$$

which is independent of the choice of the generic point $y \in S^m$. *In the case* $f : S^1 \to S^1$, *the degree* $\deg(f)$ *is also called "winding number of* f*".*

For $m < n$ we have

$$\pi_m(S^n) = 0,$$

because the image of $f : S^m \to S^n$ is not onto and therefore gives $f : \mathbb{R}^m \to S^n$ which can be retracted to a point because $\mathbb{R}^m$ is contractible. For $m > n$, the homotopy groups of the spheres $\pi_n(S^m)$ are quite complicated and are not all known.

Homotopy groups of $U(r)$. From the fibration (B.14) and Table B.1 we deduce Table B.2. See [Hat98, Hat02].

Table B.2. *Equivalence groups of complex vector bundles of rank* r *over sphere* S^k. $\operatorname{Vect}^r(S^k) = \pi_{k-1}(U(r))$.

$\pi_k(U(r))$	π_1	π_2	π_3	π_4	π_5	π_6
$U(1)$	$\boxed{\mathbb{Z}}$	0	0	0	0	0
$U(2)$	$\mathbb{Z}$	$\boxed{0}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$
$U(3)$	$\mathbb{Z}$	0	$\mathbb{Z}$	$\boxed{0}$	$\mathbb{Z}$	$\mathbb{Z}_6$
$U(4)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\boxed{0}$
$U(5)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\widetilde{K}(S^k)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

$\operatorname{Vect}^r(S^k)$	S^2	S^3	S^4	S^5	S^6	S^7
Vect^1	$\boxed{\mathbb{Z}}$	0	0	0	0	0
Vect^2	$\mathbb{Z}$	$\boxed{0}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$
Vect^3	$\mathbb{Z}$	0	$\mathbb{Z}$	$\boxed{0}$	$\mathbb{Z}$	$\mathbb{Z}_6$
Vect^4	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\boxed{0}$
Vect^5	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\widetilde{K}(S^k)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

Observations on Table B.2.
- $\operatorname{Vect}^2(S^5) \equiv \mathbb{Z}_2 = \{0,1\}$: means that there is only one class of non trivial bundles of rank 2 over S^5.

- $\text{Vect}^r(S^3) \equiv 0, \forall\, r \geqslant 1$, means that complex vector bundles over S^3 are all trivial.
- $\text{Vect}^1(S^{k \geqslant 2}) \equiv 0$ means that all vector bundles of rank 1 over $S^{k \geqslant 2}$ are all trivial.

A remarkable observation is the following theorem: (K-theory[11])

> THEOREM B.19 (*"Bott periodicity Theorem 1959".*). — *If* $2r \geqslant k$ *then* $Vect^r(S^k)$ *is independent on* r. *We denote* $\widetilde{K}(S^k) := \text{Vect}^r(S^k)$ *called group of K-theory. Moreover there is the periodicity property:*
> $$\widetilde{K}\left(S^{k+2}\right) = \widetilde{K}\left(S^k\right) = \mathbb{Z} \text{ if } k \text{ is even}$$
> $$= 0 \text{ if } k \text{ is odd}$$

For the proof, see [Hat98].

B.3.3. Topological Chern index $\mathcal{C}$ of a complex vector bundle $F \to S^{2n}$ of rank $r \geqslant n$

From the Table B.2, if $F \to S^{2n}$ is a complex vector bundle of rank r, with $r \geqslant n$, then its isomorphism class is characterized by an integer $\mathcal{C} \in \mathbb{Z}$ called *topological Chern index.*

Here is an explicit expression for $\mathcal{C}$. The equivalence class of the bundle F is characterized by the homotopy class of the clutching function at the equator $g = f_{21}$,

(B.16) $$g : S^{2n-1} \to U\left(r\right).$$

which is the transition function from north hemisphere to south hemisphere.

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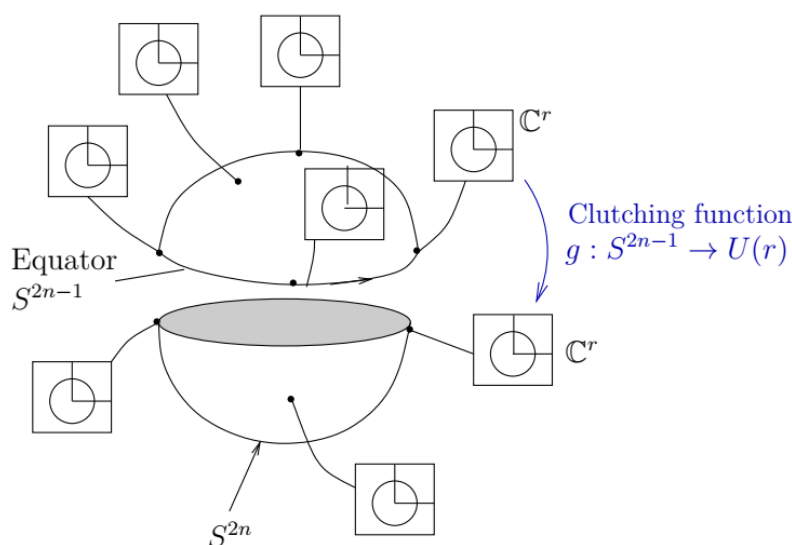
[11] The symbol $K(X)$ comes from "Klassen" in german, by A. Grothendieck 1957, see Lectures of Karoubi. The symbol $C(X)$ was already used.

If $r > n$, we can continuously deform g so that $\forall\ x \in S^{2n-1}, g_x(e_r) = e_r$, where $(e_1, \ldots e_r)$ is the canonical basis of $\mathbb{C}^r$. Cf. [BB85, Section III.1.B, p. 271]. Then g restricted to $\mathbb{C}^{r-1} \subset \mathbb{C}^r$ gives a function $g : S^{2n-1} \to U(r-1)$. By iteration we get the case $r = n$ with a clutching function $g : S^{2n-1} \to U(n)$. Then using g we define the function

(B.17)
$$f : \begin{cases} S^{2n-1} & \to S^{2n-1} \subset \mathbb{C}^n \\ x & \to g_x\left(e_1\right) \end{cases}$$

The degree $\deg(f)$ has been defined in Definition B.18.

THEOREM B.20 ((Bott 1958) [BB85, Section III.1.B., p. 271]). — *Let $F \to S^{2n}$ be a complex vector bundle of rank $r \geqslant n$. The topological index*

(B.18)
$$\mathcal{C} := \frac{\deg\left(f\right)}{(n-1)!}.$$

is an integer $\mathcal{C} \in \mathbb{Z}$ (not only a rational number!) and characterizes the topology of F. Namely, if $F \to S^{2n}$ and $F' \to S^{2n}$ are fiber bundles of same rank $r \geqslant n$ with the same index $\mathcal{C}$ then F and F' are isomorphic.

Remark B.21. — If the vector bundle $F \to S^{2n}$ has a (arbitrary) connection, the Chern-Weil theory permits to express the topological index $\mathcal{C}$ from the curvature Ω of the connection, considered as a imaginary valued 2-form on S^{2n} as follows. We first define $\mathrm{Ch}(F)$ called the *Chern Character* which is a differential form on S^{2n}:

$$\mathrm{Ch}\left(F\right) := \mathrm{Tr}\left(\exp\left(\frac{i\Omega}{2\pi}\right)\right)$$
$$= \mathrm{Tr}\left(1 + \frac{i\Omega}{2\pi} + \frac{1}{2!}\left(\frac{i\Omega}{2\pi}\right) \wedge \left(\frac{i\Omega}{2\pi}\right) + \ldots\right)$$
$$= \mathrm{Ch}_0\left(F\right) + \mathrm{Ch}_2\left(F\right) + \ldots$$

We denote $\mathrm{Ch}_{2n}(F)$ its component of exterior degree $2n$ which is a volume form on S^{2n}:

$$\mathrm{Ch}_{2n}\left(F\right) = \frac{1}{n!}\left(\frac{i\Omega}{2\pi}\right)^{\wedge n}$$

Then

(B.19)
$$\mathcal{C} = \int_{S^{2n}} \mathrm{Ch}_{2n}\left(F\right).$$

Formula (B.19) is a generalization of Gauss-Bonnet formula (B.13). For example, for a rank 1 complex vector bundle $F \to S^2$, i.e. $n = 1$, we have $\mathrm{Ch}_2(F) = \frac{i\Omega}{2\pi}$ and (B.19) gives (B.10).

B.3.4. A normal form bundle $F_n \to S^{2n-1}$ in each K-isomorphism class

We have seen in Theorem B.19 that for $r \geqslant n$ then the isomorphism class of complex vector bundles of rank r over S^{2n-1} is $\mathrm{Vect}^r(S^k) \equiv \mathbb{Z}$. In this section we

provide and *explicit model for the generator* in this class, i.e. giving the topological index $\mathcal{C} = +1 \in \text{Vect}^r(S^k) \equiv \mathbb{Z}$.

These models can be considered as canonical forms (or normal forms). We will consider $S^{2n-1} := \{(z_1, z_2, \ldots z_n) \in \mathbb{C}^n \text{ s.t. } \sum_{j=1}^{n} |z_j| = 1\}$ as the unit sphere.

DEFINITION B.22 ("*Normal form bundles*" [PR03, Section 1.2]). — *For $n \in \mathbb{N}^*$, we define a normal (canonical) vector bundle $F_n \to S^{2n}$ of rank $r = 2^{n-1}$ from the normal (canonical) form clutching function*

$$g_n : S^{2n-1} \to U\left(2^{n-1}\right)$$

by

(B.20)
$$g_1 : \begin{cases} S^1 \subset \mathbb{C} & \to U(1) \subset \mathbb{C} \\ z & \to z \end{cases}$$

and iteration

(B.21)
$$g_{n+1}\left(z_1, \underbrace{z_2, \ldots z_{n+1}}_{z}\right) = \begin{pmatrix} z_1 \text{Id}_{2^{n-1}} & -(g_n(z))^\dagger \\ g_n(z) & \overline{z_1}\text{Id}_{2^{n-1}} \end{pmatrix}$$

where $\text{Id}_{2^{n-1}}$ denotes the $2^{n-1} \times 2^{n-1}$ identity matrix.

Remark B.23. — The map $\mathcal{B} : g_n \to g_{n+1}$ in (B.21) is called the *Bott map*, see [PR03, Section 1.1] and references therein. Here are the first few expressions of g_n:

$$g_1(z_1) = z_1, \quad g_2(z_1, z_2) = \begin{pmatrix} z_1 & -\overline{z_2} \\ z_2 & \overline{z_1} \end{pmatrix}, \quad g_3(z_1, z_2, z_3)$$

(B.22)
$$= \begin{pmatrix} z_1 & 0 & -\overline{z_2} & -\overline{z_3} \\ 0 & z_1 & z_3 & -z_2 \\ z_2 & -\overline{z_3} & \overline{z_1} & 0 \\ z_3 & \overline{z_2} & 0 & \overline{z_1} \end{pmatrix}, \ldots$$

Remark B.24. — These normal forms g_n correspond to Hurwitz–Radon matrices [Eck94] and are related to gamma matrices, of generalized gamma matrices.

PROPOSITION B.25. — *The normal form bundle $F_n \to S^{2n}$ of rank $r = 2^{n-1}$ in Definition B.22 is a generator of the K-theory group $\widetilde{K}(S^{2n}) \equiv \text{Vect}^r(S^{2n}) \equiv \mathbb{Z}$, hence has topological index*

(B.23)
$$\mathcal{C} = +1.$$

For the proof, see [PR03, Section 1.1] and references therein. Here let us observe that taking the first column in (B.22) and removing zero elements we get the vector

$$\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in S^{2n-1} \subset \mathbb{C}^n.$$

Equivalently, with $\delta_1 = (1, 0, \ldots 0) \in \mathbb{C}^n$, the map $g_n \delta_1 : S^{2n-1} \to \mathbb{C}^{2^{2n-1}}$ retracts to the identity map $\mathrm{Id} : \mathbb{C}^n \to \mathbb{C}^n$.

B.3.5. Quantization of the normal form bundle

In the next Proposition, we consider the normal clutching function given in (B.21) as a function

$$g_n : (z_1, z_2, \ldots z_n) \in \mathbb{C}^n \to U\left(2^{n-1}\right)$$

see examples (B.22). By writing $z_j = x_j + i\xi_j$, with $j = 1 \ldots n$, we get a function $g_n(x_1, \xi_1, x_2, \xi_2, \ldots x_n, \xi_n) \in U(2^{n-1})$ considered as a symbol on $\mathbb{R}^{2n}$ valued in unitary matrices. Following definition (A.1) we quantize this symbol, giving an operator

$$\widehat{g}_n := \mathrm{Op}_1(g_n) : \mathcal{S}\left(\mathbb{R}^{2n}; \mathbb{C}^{2^{n-1}}\right) \to \mathcal{S}\left(\mathbb{R}^{2n}; \mathbb{C}^{2^{n-1}}\right).$$

In fact this operation is quite simple since the symbol is linear: to get the operator $\widehat{g}_n$ from the symbol g_n, we only have to replace each complex variable $z_j = x_j + i\xi_j$ by $\mathrm{Op}_1(z_j) = \mathrm{Op}_1(x_j) + i\mathrm{Op}_1(\xi_j) =: \sqrt{2}a_j$ where a_j is called the annihilation operator. For example from (B.22), we get

$$\widehat{g}_1 = \sqrt{2}a_1, \quad \widehat{g}_2 = 2\begin{pmatrix} a_1 & -a_2^\dagger \\ a_2 & a_1^\dagger \end{pmatrix}, \quad \widehat{g}_3 = 2^{3/2}\begin{pmatrix} a_1 & 0 & -a_2^\dagger & -a_3^\dagger \\ 0 & a_1 & a_3 & -a_2 \\ a_2 & -a_3^\dagger & a_1^\dagger & 0 \\ a_3 & a_2^\dagger & 0 & a_1^\dagger \end{pmatrix}, \ldots \ldots$$

> PROPOSITION B.26 (*"Normal form quantum operator"*). — *The operator* $\widehat{g}_n := \mathrm{Op}_1(g_n)$ *is Fredholm with index*
>
> (B.24) $\mathrm{Ind}\left(\widehat{g}_n\right) = +1.$

Proof. — For the symbols we compute $(g_1^\dagger g_1)(z) \underset{(\mathrm{B.20})}{=} |z|^2$ and

$$g_{n+1}^\dagger g_{n+1} \underset{(\mathrm{B.21})}{=} \begin{pmatrix} |z_1|^2 + g_n^\dagger g_n & 0 \\ 0 & |z_1|^2 + g_n g_n^\dagger \end{pmatrix}.$$

Recursively we deduce that for any $n \in \mathbb{N}^*$, $g_{n+1}^\dagger g_{n+1}$ vanishes only at $z = 0$, the operators $\widehat{g}_n^\dagger \widehat{g}_n$ and $\widehat{g}_n$ are elliptic hence Fredholm [BB85, Theorem 3, p. 185]. From (B.21) we compute recursively that

$$\mathrm{Ker}\left(\widehat{g}_n\right) = \mathrm{Span}\left\{\begin{pmatrix} \varphi_0 \\ 0 \\ \vdots \end{pmatrix}\right\}, \quad \mathrm{Ker}\left(\widehat{g}_n^\dagger\right) = \{0\},$$

where φ_0 is the Gaussian function (3.3) spanning the kernel of $a_1 := \frac{1}{\sqrt{2}}(\mathrm{Op}_1(x_j) + i\mathrm{Op}_1(\xi_j))$. We deduce that the index is [BB85, Theorem 2, p. 16]

(B.25) $\mathrm{Ind}\left(\widehat{g}_n\right) = \dim \mathrm{Ker}\left(\widehat{g}_n\right) - \dim \mathrm{Ker}\left(\widehat{g}_n^\dagger\right) = 1 - 0 = 1.$ $\qquad\square$

B.3.6. The index formula on Euclidean space of Fedosov–Hörmander

For the previous canonical vector bundle $F_n \to S^{2n}$ with topological index $\mathcal{C}$ and clutching function g_n we have observed that

$$(B.26) \qquad \text{Ind}\left(\text{Op}_1\left(g_n\right)\right) \underset{(B.24),\,(B.23)}{=} \mathcal{C},$$

and that $\mathcal{C} = +1$, meaning that this vector bundle F_n is the generator of its equivalence class in K-theory. Since both indices $\text{Ind}(\text{Op}_1(g_n))$ and $\mathcal{C}$ are additive under direct sum of vector bundles in K-theory, we deduce the next Theorem showing that (B.26) is generally true.

We consider $F \to S^{2n}$, a general complex vector bundle of rank r with topological index $\mathcal{C} \in \mathbb{Z}$ as defined in (B.18) and clutching function $g : S^{2n-1} \to U(r)$ on the equator S^{2n-1} as defined in (B.16). We extend g from S^{2n-1} to 1-homogeneous function on $\mathbb{R}^{2n}\setminus\{0\}$ by $g(z) := |z|g\left(\frac{z}{|z|}\right)$ and consider this extension as a symbol $g : \mathbb{R}^{2n}\setminus\{0\} \to \text{GL}(r)$. Quantization (A.1) gives an operator $\text{Op}_1(g)$.

> **Theorem B.27** ([Hör79, Theorem 7.3, p. 422], [BB85, Theorem 1, p. 252] *"The index formula on Euclidean space of Fedosov-Hörmander"*). — *Let $F \to S^{2n}$ be a complex vector bundle of rank r with topological index $\mathcal{C} \in \mathbb{Z}$ and clutching function $g : S^{2n-1} \to U(r)$ on the equator S^{2n-1}. We have*
>
> $$(B.27) \qquad \text{Ind}\left(\text{Op}_1\left(g\right)\right) = \mathcal{C}.$$

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