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SYMPLECTIC COHOMOLOGY OF COMPOUND DU VAL SINGULARITIES

COHOMOLOGIE SYMPLECTIQUE DES SINGULARITÉS CDVS

Abstract. — We compute symplectic cohomology for Milnor fibres of certain compound Du Val singularities that admit small resolution by using homological mirror symmetry. Our computations suggest a new conjecture that the existence of a small resolution has strong implications for the symplectic cohomology and conversely. We also use our computations to give a contact invariant of the link of the singularities and thereby distinguish many contact structures on connected sums of $S^2 \times S^3$.

Résumé. — Nous calculons la cohomologie symplectique des fibres de Milnor de certaines singularités Du Val composites qui admettent une petite résolution en utilisant la symétrie miroir homologique. Nos calculs suggèrent une nouvelle conjecture comme quoi l'existence d'une petite résolution a de fortes implications pour la cohomologie symplectique et inversement. Nous utilisons également nos calculs pour donner un invariant de contact du link des singularités et ainsi distinguer de nombreuses structures de contact sur des sommes connexes de $S^2 \times S^3$.

Keywords: Symplectic cohomology, compound Du Val, terminal, singularities, contact geometry, links, homological mirror symmetry.
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1. Introduction

1.1. Links

Let $X \subset \mathbb{C}^N$ be a normal n-dimensional algebraic variety over $\mathbb{C}$ and let $P \in X$ be a point; we will write $[P \in X]$ for the germ of X at P considered up to local analytic equivalence. Recall that the *link* of $P \in X$, written $\mathrm{Link}(P)$, is the intersection of a small Euclidean sphere centred at P with X. If P is a smooth point or isolated singularity then the link is a smooth, compact $(2n-1)$-dimensional manifold; we will focus on hypersurface singularities, whose link is $(n-2)$-connected. How much information do we retain about $[P \in X]$ if we only remember the manifold $\mathrm{Link}(P)$?

Mumford [Mum61] proves that if $n = 2$ then $\mathrm{Link}(P)$ is a simply-connected 3-manifold if and only if $P \in X$ is a smooth point. By contrast, in higher dimensions, the topology of the link exerts less influence. For example, if Σ is any homotopy 7-sphere, Brieskorn [Bri66] constructs singular complex 4-folds $P_k \in X_k$, $k \in \mathbb{N}$, with $[P_i \in X_i] \neq [P_j \in X_j]$ for $i \neq j$ and $\mathrm{Link}(P_k) \cong \Sigma$. More generally, when $n \geqslant 3$, surgery theory tells us there are not very many $(n-2)$-connected $(2n-1)$-manifolds[(1)], but there are lots of singularities.

The field of complex tangencies ξ forms a contact distribution on $\mathrm{Link}(P)$ [Var80]. McLean [McL16] demonstrates that the contact manifold $(\mathrm{Link}(P), \xi)$ retains much more information about $[P \in X]$. For example, he shows that $(\mathrm{Link}(P), \xi)$ is contactomorphic to the standard contact 5-sphere if and only if $P \in X$ is a smooth point, and that the minimal discrepancy of a canonical $\mathbb{Q}$-Gorenstein singularity $P \in X$ is determined by $(\mathrm{Link}(P), \xi)$.

An interesting corollary of McLean's work relates the purely algebro-geometric notion of *terminal* singularities to the purely contact geometric notion of *dynamical convexity*.

- A singularity is called terminal if its minimal discrepancy is positive. Terminal singularities emerged in the work of Reid [Rei83] as a natural class of singularities that should appear on minimal models of smooth 3-folds. The 3-fold terminal singularities were classified by Mori [Mor85].
- A Reeb flow on a contact manifold is called *dynamically convex* if every closed Reeb orbit γ satisfies $\mu_{CZ}(\gamma) + n - 3 > 0$, where μ_{CZ} is the Conley–Zehnder index. A contact manifold which admits a dynamically convex Reeb flow is called *index positive*.

Theorem 1.1 ([McL16]). — *Suppose that $P \in X$ is an isolated $\mathbb{Q}$-Gorenstein singularity with $H^1(\mathrm{Link}(P); \mathbb{Q}) = 0$ (e.g. a hypersurface singularity of dimension $n \geqslant 3$). The singularity $P \in X$ is terminal if and only if its link $(\mathrm{Link}(P), \xi)$ is index positive.*

Proof. — If $P \in X$ is terminal then the minimal discrepancy is positive, so McLean's theorem implies that the highest minimal SFT index of the link is positive, which is precisely the statement that there is a dynamically convex Reeb flow on $(\mathrm{Link}(P), \xi)$. If there is no dynamically convex Reeb flow then the highest minimal

[(1)] For a classification, see Wall [Wal67].

SFT index is nonpositive so, by McLean's theorem, the minimal discrepancy is also nonpositive; therefore $P \in X$ is not terminal. □

Invariants of contact manifolds (like contact homology or symplectic field theory) are notoriously difficult to define because of bubbling of pseudoholomorphic curves in symplectisations. The condition of index positivity allows us to bypass many of these problems to get useful contact invariants. For example, if Y is a contact manifold and V is a simply-connected strong symplectic filling of Y with $c_1(V) = 0$, then we can define symplectic cohomology

$$\mathrm{SH}^*(V;\mathbb{C})$$

as a $\mathbb{Z}$-graded(2) $\mathbb{C}$-vector space (with various additional algebraic structures on it) which usually depends on V. However, if Y is simply-connected and index positive then the dependence of $\mathrm{SH}^*(V;\mathbb{C})$ on the filling is very mild. The positive symplectic cohomology $\mathrm{SH}^*_+(V;\mathbb{C})$, constructed as the cohomology of a quotient complex of the cochain complex of $\mathrm{SH}^*(V;\mathbb{C})$ by the cochains coming from the interior of the filling, is known to be a contact invariant [CO18, Proposition 9.17]. This has been used successfully by Uebele to distinguish some contact structures on $S^2 \times S^3$ [Ueb16].

We explore a refinement of this in Corollary 4.5. In particular, for $n = 3$, we are able prove by a standard neck-stretching technique that the Lie algebra structure on $\mathrm{SH}^1(V;\mathbb{C})$ and its Lie algebra representation on $\bigoplus_{d<0} \mathrm{SH}^d(V;\mathbb{C})$ is a contact invariant.

Our goal in this paper is to compute symplectic cohomology for some further examples of links of terminal 3-fold hypersurface singularities, observe some patterns which emerge, and use it to distinguish a variety of links.

1.2. Compound Du Val (cDV) singularities

It is a theorem of Reid [Rei83, Theorem 1.1] that the Gorenstein terminal 3-fold singularities are precisely the isolated *compound Du Val (cDV) singularities*. These are hypersurface singularities which (in suitable local analytic coordinates (w,x,y,z)) are cut out by an equation of the form

$$f(x,y,z) + wg(x,y,z,w) = 0$$

where f is one of the following polynomials:

$$\begin{aligned}
A_\ell : &\quad x^2 + y^2 + z^{\ell+1},\\
D_\ell : &\quad x^2 + y\left(z^2 + y^{\ell-2}\right),\\
E_6 : &\quad x^2 + y^3 + z^4,\\
E_7 : &\quad x^2 + y\left(y^2 + z^3\right),\\
E_8 : &\quad x^2 + y^3 + z^5.
\end{aligned}$$

(2) More generally, the possible $\mathbb{Z}$-gradings on $\mathrm{SH}^*(V;\mathbb{C})$ form a torsor over $\mathrm{H}^1(V;\mathbb{Z})$. Note that with our grading conventions an orbit with Conley–Zehnder index μ lives in degree $n-\mu$ where $2n = \dim V$. In particular the unit lives in degree zero and a constant orbit corresponding to a critical point of Morse index k lives in degree k.

The $w = 0$ hyperplane section has an ADE singularity at 0. If Γ is the ADE type of this hyperplane section, we refer to the 3-fold singularity as a *compound* Γ or $c\Gamma$ singularity.

As we have explained in Theorem 1.1, the links of these singularities are index positive and so we can use SH^* of the Milnor fibre for $* < 0$ as a contact invariant.

Remark 1.2. — Observe that if we define $B \subset \mathrm{Link}(0)$ to be the intersection $\{w = 0\} \cap \mathrm{Link}(0)$ then we get a Milnor open book

$$w/|w| \colon \mathrm{Link}(0) \setminus B \to S^1$$

with binding B. The page is a copy of the corresponding 4-dimensional ADE Milnor fibre and the contact structure determined by the open book is contactomorphic to ξ.

Example 1.3. — Consider the family of cA_1 singularities

$$A_\ell := \left\{x^2 + y^2 + z^2 + w^{\ell+1} = 0\right\}, \qquad \ell \geqslant 1.$$

In fact, any cA_1 singularity is equivalent to one of these. The link is either S^5 (if ℓ is even) or $S^2 \times S^3$ (if ℓ is odd). The page of the Milnor open book is the A_1-Milnor fibre T^*S^2, and the monodromy is the $(\ell + 1)$st power of a Dehn twist in the zero-section. The symplectic cohomology of the Milnor fibre V_ℓ behaves differently if ℓ is odd or even. If ℓ is even then, by [LU21, Section 5.2], we have

$$\mathrm{SH}^*(V_\ell; \mathbb{C}) = \begin{cases} \mathbb{C}^\ell & \text{if } * = 3 \\ \mathbb{C} & \text{if } * = -q(\ell+3) - r \text{ for } r \in \{0, \ldots, \ell-1\},\ r = q (\mathrm{mod} 2) \\ \mathbb{C} & \text{if } * = -q(\ell+3) - r + 1 \text{ for } r \in \{0, \ldots, \ell-1\},\ r = q (\mathrm{mod} 2) \\ 0 & \text{otherwise,} \end{cases}$$

for $q \in \mathbb{N}$. In particular, we see that SH^* can be either 0 or $\mathbb{C}$ for $* < 0$.

If ℓ is odd then we will see below that

$$\mathrm{SH}^*(V_\ell; \mathbb{C}) = \begin{cases} \mathbb{C}^\ell & \text{if } * = 3 \\ \mathbb{C} & \text{if } * = 1 \text{ or } * < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Write ξ_ℓ for the contact structure on the link of $0 \in A_\ell$. Since the contact invariant SH^*_+ coincides with SH^* if $* < 0$, this shows that

(a) the links $\{(S^5, \xi_\ell) \ : \ \ell = 2, 4, 6, \ldots\}$ are pairwise nonisomorphic as contact manifolds,
(b) we cannot distinguish the links $\{(S^2 \times S^3, \xi_\ell) \ : \ \ell = 1, 3, 5, \ldots\}$ using SH^*_+ with coefficients in $\mathbb{C}$.

A similar phenomenon was observed by Van Koert [Koe08, Example 3.1.1] for these contact structures on $S^2 \times S^3$: they are not distinguished by their cylindrical contact homology. Interestingly, Uebele [Ueb16] does distinguish them using SH^*_+ with coefficients in $\mathbb{Z}/2$. We will give a second way to distinguish them below.

From an algebro-geometric perspective, the singularities A_ℓ have different behaviour when ℓ is even/odd. For example, these singularities admit *small resolutions*(3) if and only if ℓ is odd; indeed, if ℓ is odd, there is a resolution whose exceptional set is an irreducible rational curve. However, if ℓ is even then there cannot be a small resolution because the link is not diffeomorphic to a nontrivial connected sum of copies of $S^2 \times S^3$.

Inspired by this example, we record an optimistic conjecture, which provides the main motivation for the calculations in this paper. We will establish this conjecture in a range of examples (Theorem 1.8).

Conjecture 1.4. — *Suppose that $P \in X$ is a cDV singularity and let V be the Milnor fibre of the singularity. Then $P \in X$ admits a small resolution such that the exceptional set has ℓ irreducible components if and only if $\mathrm{SH}^*(V;\mathbb{C})$ has rank ℓ in every negative degree.*

Remark 1.5. — In this paper, we have focused on providing evidence for one direction of this conjecture: that the existence of a small resolution constrains the symplectic cohomology. The converse is plausible: we have calculated many examples and found no counterexample. This would give an a priori way of detecting whether a cDV singularity admits a small resolution just by knowing its link.

Remark 1.6. — If $P \in X$ admits a small resolution whose exceptional set has ℓ irreducible components then the link is diffeomorphic to $\sharp_\ell(S^2 \times S^3)$. Small resolutions can be constructed by thinking of the 3-fold as a 1-parameter deformation of an ADE singularity. This gives a classifying map from the disc to the versal deformation space of the ADE singularity such that the 3-fold is the pullback along the classifying map of the versal family. Brieskorn [Bri68], Tjurina [Tju70] and Pinkham [Pin83] constructed branched coverings of the versal ADE deformation space (branched over the discriminant locus) such that the pullback of the versal family to the branched covering admits a simultaneous (partial) resolution. More precisely, the fundamental group of the complement of the discriminant locus is the ADE Artin braid group; Brieskorn and Tjurina constructed the branched covering corresponding to the kernel of the homomorphism to the ADE Weyl group and found a full simultaneous resolution, while Pinkham constructed simultaneous partial resolutions for intermediate covering spaces. For a specific 3-fold, if the classifying map from the disc lifts (in the sense of algebraic topology) to one of these branched covers, then you get a small resolution by pulling back the simultaneous partial resolution of the versal family. In particular, the existence of a small resolution can be read off from the monodromy of the Milnor open book mentioned in Remark 1.2 (which is the element of the fundamental group of the ADE Artin braid group represented by the boundary of the disc under the classifying map).

Remark 1.7. — Remark 1.6 provides a sanity check on Conjecture 1.4. Consider what happens if we deform the germ of the singularity at P. Namely, suppose we have

(3) Recall that a small resolution is a resolution whose exceptional set has codimension at least 2. Note that, by [Rei83, Theorem 1.14], a resolution of an isolated cDV singularity is small if and only if it is crepant.

a family $h_s(w,x,y,z) = f(x,y,z) + g_s(w,x,y,z)$ of cDV singularities parametrised by $s \in \mathbb{C}$. Suppose that there are balls $B \subset \mathbb{C}$ and $B' \subset \mathbb{C}^4$ such that for $s \in B$, the origin is the only singularity of the hypersurface $h_s^{-1}(0) \cap B'$. Gray's stability theorem tells us that the contact geometry of the link of the singularity is independent of $s \in B$. Moreover, if $0 \in h_0^{-1}(0)$ admits a small resolution then so do all the singularities $0 \in h_s^{-1}(0)$ because the monodromy of the Milnor open book is stable under perturbations.

We now summarise our evidence for Conjecture 1.4. These calculations will be explained in Section 3. Throughout, we work over $\mathbb{C}$.

Theorem 1.8. — *The table below summarises our calculations of symplectic cohomology for Milnor fibres of some cDV singularities. The left-most column is a polynomial* $\check{\mathbf{w}}$ *and the singularity is defined by* $0 \in \check{\mathbf{w}}^{-1}(0)$. *The columns* SH^* *give the ranks of the various graded pieces of* $\mathrm{SH}(\check{\mathbf{w}}^{-1}(1))$. *In all cases,* $\mathrm{SH}^d(\check{\mathbf{w}}^{-1}(1)) = 0$ *if* $d = 2$ *or* $d \geqslant 4$. *The final column gives a reference for the calculation. Case 4 is conditional on Conjecture 2.2 or Conjecture 2.3, so we have marked it with an asterisk.*

	Singularity	*ADE type*	SH^3	$\mathrm{SH}^{d\leqslant 1}$	*See Theorem…*
1.	$x_1^2 + x_2^2 + x_3^{\ell+1} + x_4^{k(\ell+1)}$	A_ℓ	$\ell(k(\ell+1)-1)$	ℓ	*3.7 (1), (2)*
2.	$x_1^2 + x_2^2 + x_3x_4(x_3^{\ell-1} + x_4^{k(\ell-1)})$	A_ℓ	$(k\ell+1)(\ell-1)$	ℓ	*3.13*
3.	$x_1^2 + x_2^3 + x_3^3 + x_4^{6k}$	D_4	$24k-4$	4	*3.7 (3)*
4.*	$x_1^3 + x_1x_2^{2k+1} + x_2x_3^2 + x_4^2$	D_4	$6k+5$	1	*3.10*
5.	$x_1^2 + x_2^3 + x_3^4 + x_4^{12k}$	E_6	$72k-6$	6	*3.7 (4)*
6.	$x_1^2 + x_2^3 + x_3^5 + x_4^{30k}$	E_8	$240k-8$	8	*3.7 (5)*

Remark 1.9. — In all cases, these singularities admit small resolutions and the number of exceptional curves in the resolution equals the rank of SH^d for $d \leqslant 1$; this is explained case-by-case in Section 3. In particular, this establishes Conjecture 1.4 for these examples.

Remark 1.10. — The examples in Theorem 1.8 are all *invertible polynomials* (see Section 2), and our strategy for calculating symplectic cohomology uses mirror symmetry for invertible polynomials to relate SH with the Hochschild cohomology of a mirror dg-category of equivariant matrix factorisations. In all cases except case 4, the required mirror symmetry conjecture is proven. Case 4 is only proved conditionally (see Section 2.2). This example is the base of the Laufer flop [Lau81].

Remark 1.11. — Theorem 1.8 seems to indicate that symplectic cohomology (over $\mathbb{C}$) of the Milnor fibre is not a useful invariant for distinguishing contact structures on links. We are nonetheless able to distinguish all these examples by studying a certain bigrading on symplectic cohomology, as we discuss in Section 1.3. Note that Uebele's work (discussed in Example 1.3 above) shows that Conjecture 1.4

breaks down if we work over a field of characteristic $\neq 0$, which gives an alternative way to distinguish contact structures on links.

1.3. Families of inequivalent contact structures

We introduce the following notation for the contact structures on the links of our singularities:

Table 1.1. Contact structures on links of our cDV singularities.

	Singularity	Link	Contact structure
1.	$x_1^2+x_2^2+x_3^{\ell+1}+x_4^{k(\ell+1)}$	$\sharp_\ell(S^2\times S^3)$	$\alpha_{\ell,k}$
2.	$x_1^2+x_2^2+x_3x_4(x_3^{\ell-1}+x_4^{k(\ell-1)})$ $(\ell\geqslant 2)$	$\sharp_\ell(S^2\times S^3)$	$\beta_{\ell,k}$
3.	$x_1^2+x_2^3+x_3^3+x_4^{6k}$	$\sharp_4(S^2\times S^3)$	$\delta_{4,k}$
4.	$x_1^3+x_1x_2^{2k+1}+x_2x_3^2+x_4^2$	$S^2\times S^3$	$\lambda_{1,k}$
5.	$x_1^2+x_2^3+x_3^4+x_4^{12k}$	$\sharp_6(S^2\times S^3)$	$\epsilon_{6,k}$
6.	$x_1^2+x_2^3+x_3^5+x_4^{30k}$	$\sharp_8(S^2\times S^3)$	$\epsilon_{8,k}$

Remark 1.12. — Note that $\alpha_{\ell,1}\cong\beta_{\ell,1}$: the two singularities are related by a change of variables.

Let Ξ_ℓ denote the list of all contact structures on $\sharp_\ell(S^2\times S^3)$ from this table. For example,

$$\begin{aligned}\Xi_1 &= (\alpha_{1,1},\alpha_{1,2},\ \ldots,\ \lambda_{1,1},\lambda_{1,2},\ \ldots)\\ \Xi_4 &= (\alpha_{4,1},\alpha_{4,2},\ \ldots,\ \beta_{4,1},\beta_{4,2},\ \ldots,\delta_{4,1},\delta_{4,2},\ \ldots).\end{aligned}$$

Theorem 1.13. — *For each ℓ, the contact structures in the list Ξ_ℓ are pairwise nonisomorphic except for $\alpha_{\ell,1}\cong\beta_{\ell,1}$.*

Remark 1.14. — We remind the reader that all results about $\lambda_{1,k}$ are conditional on a mirror symmetry statement.

Remark 1.15. — What makes this an interesting theorem is that all of these links have the same positive symplectic cohomology over $\mathbb{C}$. We equip SH^* with a contact-invariant bigrading to distinguish these contact manifolds. This bigrading will be the weight decomposition of $\bigoplus_{d<0}\mathrm{SH}^d$ under the action of the Lie algebra SH^1.

Remark 1.16. — As explained in Example 1.3, the fact that $\alpha_{1,i}\ncong\alpha_{1,j}$ if $i\neq j$ was proved by Uebele [Ueb16] using positive symplectic cohomology with coefficients in $\mathbb{Z}/2$ (rather than a bigrading).

Remark 1.17. — If one focuses on cDV singularities which do not admit a small resolution, one finds very many more contact structures which can be distinguished by SH^*_+ already without using the bigrading. This is not so surprising: it is much easier for 5-manifolds to be diffeomorphic than contactomorphic.

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2. Symplectic cohomology for invertible polynomials

2.1. Symplectic cohomology

Let V be a Liouville manifold with $c_1(V) = 0$. Associated to V we can define an invariant $\mathrm{SH}^*(V)$ called the symplectic cohomology of V. Symplectic cohomology was introduced by Cieliebak, Floer, Hofer [Hof90, FH94, CFH95] and Viterbo [Vit99]. An excellent exposition can be found in [Sei08]. More recent results can be learned from [CO18]. See also [LP16, Sec 2.1] for a fast review of our sign and grading conventions. In particular, our conventions are cohomological and the unit lives in degree zero!

Briefly, $\mathrm{SH}^*(V)$ is an algebra over the homology operad of framed little discs over an arbitrary commutative ring $\mathbf{k}$ (in this paper $\mathbf{k} = \mathbb{C}$). In particular, it has a (graded) commutative product, a Gerstenhaber bracket $[\,,\,]$ (i.e a Lie bracket of degree -1), and a Batalin–Vilkovisky operator Δ (i.e. a degree -1 operator whose Hochschild coboundary is the bracket).

In general, symplectic cohomology is rather difficult to compute explicitly. A fruitful approach to do such computations goes via the open string A-model. Namely, we have an isomorphism

$$\mathrm{SH}^*(V) \simeq \mathrm{HH}^*(\mathcal{W}(V))$$

where $\mathcal{W}(V)$ is the wrapped Fukaya category of V. An early version of this result based on Legendrian surgery is due to Bourgeois–Ekholm–Eliashberg ([BEE12], elaborated in [EL17]) which concerned Hochschild homology; a definitive version based on duality appeared in [Gan13, Theorem 1.1] (see also the more recent [CDGG17]).

On the other hand, even if one achieved a good understanding of $\mathcal{W}(V)$, in general, it is still a difficult algebraic problem to compute Hochschild cohomology of A_∞ categories.

In [LU18, LU21], a method to compute symplectic cohomology for certain Milnor fibres was given based on the homological mirror symmetry conjecture for *invertible polynomials.*

2.2. Invertible polynomials and mirror symmetry

Definition 2.1. — *To an $(n+1)$-by-$(n+1)$ integer matrix $A = (a_{ij})$ with nonzero determinant, we associate the polynomial*

$$\mathbf{w}(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} \prod_{j=1}^{n+1} x_j^{a_{ij}}.$$

We write $\check{\mathbf{w}}$ *for the polynomial associated to* A^T *(the* Berglund–Hübsch mirror *to* $\mathbf{w}$, *see* [BH92]*).*

An invertible polynomial is weighted homogeneous, that is there is a uniquely determined weight system $(d_1, d_2, \ldots, d_{n+1}; h)$ satisfying $\gcd(d_1, d_2, \ldots, d_{n+1}, h) = 1$ for which

$$\mathbf{w}\left(\lambda^{d_1} x_1, \ldots, \lambda^{d_n} x_{n+1}\right) = \lambda^h \mathbf{w}(x_1, \ldots, x_{n+1})$$

for all $\lambda \in \mathbb{G}_m$. In this paper, we are primarily concerned with the log Fano case, i.e. when

$$h - \sum_{i=1}^{n+1} d_i =: d_0 < 0$$

In fact, there is a finite extension $\Gamma_{\mathbf{w}}$ of $\mathbb{G}_m$ acting on $\mathbb{A}^{n+1}$ which preserves $\mathbf{w}$, namely

$$\Gamma_{\mathbf{w}} := \left\{ (t_0, t_1, \ldots, t_{n+1}) \in \mathbb{G}_m^{n+2} \;:\; \prod_{j=1}^{n+1} t_j^{a_{ij}} = t_0 t_1 \cdots t_{n+1},\ i = 1, \ldots, n+1 \right\},$$

acting on $\mathbb{A}^{n+1}$ via $(x_1, \ldots, x_{n+1}) \mapsto (t_1 x_1, \ldots, t_{n+1} x_{n+1})$. This group also acts on $\mathbb{A}^{n+2}$ via $(x_0, x_1, \ldots, x_{n+1}) \mapsto (t_0 x_0, t_1 x_1, \ldots, t_{n+1} x_{n+1})$, and this $\Gamma_{\mathbf{w}}$-action preserves the polynomial

$$\mathbf{w}(x_1, \ldots, x_{n+1}) + x_0 \cdots x_{n+1}.$$

With this setup, we can formulate the following mirror symmetry conjectures. A version of Conjecture 2.2 appeared in [FU09] (see also [LU18, Conjecture 1.2] and references therein), and Conjecture 2.3 appeared in [LU18].

Conjecture 2.2. — *There is a quasi-equivalence of idempotent complete A_∞-categories*

$$\mathcal{F}(\check{\mathbf{w}}) \simeq \mathrm{mf}\left(\mathbb{A}^{n+1}, \Gamma_{\mathbf{w}}, \mathbf{w}\right)$$

between the Fukaya–Seidel category of a Morsification of $\check{\mathbf{w}}$ *and the dg-category of* $\Gamma_{\mathbf{w}}$*-equivariant matrix factorisations of* $\mathbf{w}$. *Moreover, there exists a full exceptional collection* $\Delta_1, \ldots, \Delta_K$ *of vanishing thimbles for the Morsification of* $\check{\mathbf{w}}$ *such that the A_∞-algebra* $\mathcal{A} := \mathrm{end}_{\mathcal{F}(\check{\mathbf{w}})}(\bigoplus_i \Delta_i)$ *has its cohomology* $A := H(\mathcal{A})$ *supported in degree zero. In particular, this entails that (a)* $\mathcal{A}$ *is quasi-isomorphic to* A *and (b) both* $\mathcal{F}(\check{\mathbf{w}})$ *and* $\mathrm{mf}(\mathbb{A}^{n+1}, \Gamma_{\mathbf{w}}, \mathbf{w})$ *are quasi-equivalent to* $\mathrm{perf}(A)$.

Conjecture 2.3 ([LU18, Conjecture 1.4]). — *There is a quasi-equivalence of idempotent complete A_∞ categories*

$$\mathcal{W}\left(\check{\mathbf{w}}^{-1}(1)\right) \simeq \mathrm{mf}\left(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w} + x_0 x_1 \cdots x_{n+1}\right)$$

between the wrapped Fukaya category of the Milnor fibre $\check{\mathbf{w}}^{-1}(1)$ *and the dg-category of* $\Gamma_{\mathbf{w}}$*-equivariant matrix factorisations of* $\mathbf{w} + x_0 \cdots x_{n+1}$.

These conjectures are established in the following situations:

- If the matrix A is diagonal (so $\mathbf{w}$ defines a Brieskorn–Pham singularity) then Conjecture 2.2 was proved by Futaki and Ueda [FU11]. More generally, if the matrix A is block diagonal and its blocks are either 1-by-1 or 2-by-2 equal to $\left(\begin{smallmatrix} 2 & 1 \\ 0 & k \end{smallmatrix}\right)$ (so that $\mathbf{w}$ is a Sebastiani–Thom sum of ADE polynomials of type

A or D), Conjecture 2.2 was proved by Futaki and Ueda [FU13]. Polishchuk and Varolgunes [PV21] make significant progress towards establishing Conjecture 2.2 in the chain case which includes the Laufer flop (Case 4 in the Table of Theorem 1.8).

- If $n = 1$, Conjecture 2.2 was proved by Habermann and Smith [HS20]. In fact, this means Conjecture 2.2 holds for any invertible polynomial $\mathbf{w}$ of the form $\mathbf{w}(x_1, \ldots, x_{n+1}) = x_1^2 + \cdots + x_{n-1}^2 + f(x_n, x_{n+1})$. This is because stabilising $\mathbf{w}$ and $\check{\mathbf{w}}$ by adding quadratic terms in extra variables changes neither the Fukaya–Seidel nor the matrix factorisation category.
- In [LU18], various cases of Conjecture 2.3 were verified. The sequel paper [LU21] focused on the log Fano case and established Conjecture 2.3 for the Milnor fibres of simple singularities. The $n = 1$ case of Conjecture 2.3 was proved by Habermann [Hab21]. Conjecture 2.3 was proved in full generality by Gammage [Gam20] in the $\mathbb{Z}/2$-graded case using a microlocal sheaf category version of wrapped Fukaya categories. For our purposes, we will need to work with $\mathbb{Z}$-graded categories; a careful chase of $\mathbb{Z}$-gradings in [Gam20] might allow us to assume Conjecture 2.3 in all cases.

Remark 2.4. — The main theorem statements from Futaki–Ueda and Habermann–Smith do not mention the formality of $\mathcal{A}$, but in either case the authors construct a full exceptional collection whose cohomology is supported in degree zero, hence formality follows for degree reasons

Remark 2.5. — The examples $\check{\mathbf{w}}$ from Theorem 1.8 all fall into one of these cases except for $\check{\mathbf{w}} = x_1^3 + x_1x_2^{2k+1} + x_2x_3^2 + x_4^2$. In this case, our results are conditional on one of the two Conjecture 2.2 or 2.3 holding.

We now explain how knowing one or other of these conjectures can help one to calculate symplectic cohomology.

2.3. Using mirror symmetry to compute symplectic cohomology

Pick a Morsification of $\check{\mathbf{w}}$. Let $\mathcal{F}(\check{\mathbf{w}})$ denote the Fukaya–Seidel category of the Morsification, let $V := \check{\mathbf{w}}^{-1}(1)$ denote the Milnor fibre, and let $\mathcal{W}(V)$ (respectively $\mathcal{F}(V)$) denote the wrapped (respectively compact) Fukaya category of V. Choose a collection of vanishing paths for the Morsification and let $\Delta_1, \ldots, \Delta_K$ (respectively $S_1, \ldots, S_K$) be the corresponding vanishing thimbles (respectively vanishing cycles). Let $\mathcal{A} = \mathrm{end}_{\mathcal{F}(\check{\mathbf{w}})}(\bigoplus_i \Delta_i)$ and $\mathcal{B} = \mathrm{end}_{\mathcal{F}(V)}(\bigoplus_i S_i)$. Let $A = H(\mathcal{A})$ and $B = H(\mathcal{B})$ denote the cohomology algebras of $\mathcal{A}$ and $\mathcal{B}$ (considered as A_∞-algebras with zero higher products).

Theorem 2.6. — *Assume that* $\mathrm{HH}^2(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w})) = 0$, *that* $d_0 \neq 0$, *and either Conjecture 2.2 or 2.3 holds. Then*

$$\mathrm{SH}^*(V) \cong \mathrm{HH}^*\left(\mathrm{mf}\left(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}\right)\right) \tag{2.1}$$

as Gerstenhaber algebras.

In the next section, we give a formula to compute $\mathrm{HH}^*(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}))$.

Proof that Conjecture 2.2 implies Equation (2.1). — If $d_0 \neq 0$ then [LU18, Theorem 6.2] implies that the inclusion of categories $\mathcal{B} \to \mathcal{W}(V)$ induces an isomorphism on Hochschild cohomology. Since this map comes from a functor, it is a morphism of Gerstenhaber algebras. Ganatra [Gan13, Theorem 1.1] shows that $\mathrm{SH}^*(V) \cong \mathrm{HH}^*(\mathcal{W}(V))$ as Gerstenhaber algebras. Therefore, we need to show

$$\mathrm{HH}^*(\mathcal{B}) \cong \mathrm{HH}^*\left(\mathrm{mf}\left(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}\right)\right). \tag{2.2}$$

As a first step, we calculate $\mathrm{HH}^*(B)$, where $B = H(\mathcal{B})$ is the cohomology algebra of $\mathcal{B}$.

Lemma 2.7. — *We have* $\mathrm{HH}^*(B) \cong \mathrm{HH}^*(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}))$ *as Gerstenhaber algebras.*

Proof. — We continue to write $\mathcal{A}$ for the endomorphism A_∞-algebra of the vanishing thimbles and A for its cohomology. Recall that the *trivial extension algebra* $T_n(A)$ is defined to be $A \oplus A^\vee[-n]$ with the product $(a, b)(a', b') = (aa', ab' + a'b)$. For any Lefschetz fibration with $(n+1)$-(complex-)dimensional total space with $n > 0$, the Floer cohomology algebra $B = H(\mathcal{B})$ of the vanishing cycles is an extension of $A^\vee[-n]$ by A, where A is the directed Fukaya–Seidel Floer cohomology algebra for the vanishing thimbles [Sei10, Equation 4.1 and Proposition 5.1]. If A is supported in degree zero (as asserted by Conjecture 2.2) then this is the trivial extension $T_n(A)$: the products in B which are not determined by the A-module structure of $A^\vee[-n]$ vanish for degree reasons. To prove the lemma, it therefore suffices to show that $\mathrm{HH}^*(T_n(A)) \simeq \mathrm{HH}^*(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}))$.

Let $\mathbf{k}$ be the semisimple ring $\bigoplus_{i=1}^K \mathbb{C}e_i$ where $e_i \in A$ is the identity element of $HF(\Delta_i, \Delta_i)$. The projection $T_n(A) \to A \to \mathbf{k}$ makes $\mathbf{k}$ into an $T_n(A)$-module (augmentation). Keller [Kel11, Section 4.1] defines a Koszul-dual algebra called the *n-Calabi–Yau completion* $\Pi_n(A) \cong \mathrm{RHom}_{T_n(A)}(\mathbf{k}, \mathbf{k})$. This is Koszul-dual in the sense that $\mathbf{k}$ is a $(T_n(A), \Pi_n(A))$-bimodule and $T_n(A) \cong \mathrm{RHom}_{\Pi_n(A)}(\mathbf{k}, \mathbf{k})$. Koszul duality ensures that we can apply [Kel03, Theorem in Section 3.2] to deduce that the Hochschild cohomologies $\mathrm{HH}^*(T_n(A))$ and $\mathrm{HH}^*(\Pi_n(A))$ are isomorphic as Gerstenhaber algebras. For any algebra C (more generally A_∞-algebra), $\mathrm{HH}^*(C) \cong \mathrm{HH}^*(\mathrm{perf}(C))$, so

$$\mathrm{HH}^*(B) \cong \mathrm{HH}^*(T_n(A)) \cong \mathrm{HH}^*(\Pi_n(A)) \cong \mathrm{HH}^*\left(\mathrm{perf}(\Pi_n(A))\right),$$

and it suffices to prove that $\mathrm{HH}^*(\mathrm{perf}(\Pi_n(A))) = \mathrm{HH}^*(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}))$. In fact, we will show a stronger result: that

$$\mathrm{perf}(\Pi_n(A)) \simeq \mathrm{mf}\left(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}\right).$$

To see this stronger result, recall that Keller's construction of $\Pi_n(\cdot)$ works more generally when the input is a dg-algebra or category, and satisfies [LU21, Eq. (2.2)]

$$\mathrm{perf}(\Pi_n(A)) \simeq \Pi_n(\mathrm{perf}(A)).$$

It was shown in [LU21] (Eq. (1.7) for the statement and Section 4 for the proof) that

$$\mathrm{mf}\left(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}\right) \simeq \Pi_n\left(\mathrm{mf}\left(\mathbb{A}^{n+1}, \Gamma_{\mathbf{w}}, \mathbf{w}\right)\right) \tag{2.3}$$

and Conjecture 2.2 is the assumption that

$$\mathrm{perf}(A) \simeq \mathrm{mf}\left(\mathbb{A}^{n+1}, \Gamma_{\mathbf{w}}, \mathbf{w}\right)$$

so

$$\mathrm{perf}(\Pi_n(A)) \simeq \Pi_n(\mathrm{perf}(A)) \simeq \Pi_n\left(\mathrm{mf}\left(\mathcal{A}^{n+1}, \Gamma_{\mathbf{w}}, \mathbf{w}\right)\right) \simeq \mathrm{mf}\left(\mathcal{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}\right),$$

as required. □

Lemma 2.8. — *The A_∞-algebra $\mathcal{B}$ is quasi-isomorphic to its cohomology algebra B.*

Proof. — By Lemma 2.7, $\mathrm{HH}^2(B) \cong \mathrm{HH}^2(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}))$, which vanishes by assumption, so $\mathcal{B}$ is intrinsically formal, and hence quasi-isomorphic to $B = H(\mathcal{B})$. □

Together, these two lemmas show that $\mathrm{HH}^*(\mathcal{B}) \cong \mathrm{HH}^*(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}))$ as Gerstenhaber algebras, establishing Equation (2.2), so Equation (2.1) follows. □

Proof that Conjecture 2.3 implies Equation (2.1). — We will show in Theorem 2.15 below that if $\mathrm{HH}^2(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w})) = 0$ then we can make a $\Gamma_{\mathbf{w}}$-equivariant formal change of coordinates along the critical locus of $\mathbf{w} + x_0 \cdots x_{n+1}$ such that the pullback of $\mathbf{w} + x_0 \cdots x_{n+1}$ in these new coordinates equals $\mathbf{w}$. If we can make such a formal change of coordinates, it follows from [Orl11, Theorem 2.10] that $\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w} + x_0 \cdots x_{n+1})$ is quasi-equivalent to $\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w})$, so Conjecture 2.3 implies

$$\mathrm{HH}^*(\mathcal{W}(V)) \cong \mathrm{HH}^*\left(\mathrm{mf}\left(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}\right)\right)$$

as Gerstenhaber algebras. By [Gan13, Theorem 1.1], $\mathrm{HH}^*(\mathcal{W}(V)) \cong \mathrm{SH}^*(V)$ as Gerstenhaber algebras, so Equation (2.1) follows. □

2.4. Calculating $\mathrm{HH}^*(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}))$

There is a formula for $\mathrm{HH}^*(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}))$ which expresses it as a sum of $\Gamma_{\mathbf{w}}$-invariant pieces of twisted Koszul cohomologies; this formula appeared in [BFK14, Theorem 1.2], where its context and history are discussed. It is also explained and used in [LU18, Theorem 3.1] and [LU21, Section 5.1]. We now briefly describe how to perform calculations in practice with this formula; Theorem 2.14 below summarises the answer and its proof explains how our notation fits with the notation from [LU21]. We will use the notation from this section in our calculations in Section 3.

Definition 2.9. — *Define the character*

$$\chi\colon \Gamma_{\mathbf{w}} \to \mathbb{G}_m, \quad \chi(t_0, \ldots, t_{n+1}) = t_0 \cdots t_{n+1}.$$

Its kernel $\ker\chi$ *is the finite group*

$$\ker\chi = \left\{(t_0, \ldots, t_{n+1}) \in \mathbb{G}_m^{n+2} : \prod_{j=1}^{n+1} t_j^{a_{ij}} = 1,\ t_0 = t_1^{-1} \cdots t_{n+1}^{-1}\right\}.$$

Definition 2.10. — *Given an element* $\gamma \in \ker \chi$, *let*

$$\{1, \ldots, n+1\} = \{i_1, \ldots, i_k\} \cup \{j_1, \ldots, j_{n+1-k}\}$$

be the partition for which each x_{i_m} *is fixed under the action of* γ *and each* x_{j_m} *is not fixed under the action of* γ. *Let* J_γ *be a monomial basis for the Jacobian ring of* $\mathbf{w}|_{x_{j_1}=\cdots=x_{j_{n+1-k}}=0}$.

Definition 2.11. — *The set* M_γ *of* γ-monomials *is the union* $M_\gamma = A_\gamma \cup B_\gamma \cup C_\gamma$ *where*

$$A_\gamma = \begin{cases} \left\{x_0^\beta p x_{j_1}^\vee \cdots x_{j_{n+1-k}}^\vee \ : \ p \in J_\gamma,\ \beta = 0,1,2,\ldots\right\} & \text{if } x_0 \text{ is fixed by } \gamma \\ \emptyset & \text{otherwise.} \end{cases}$$

$$B_\gamma = \begin{cases} \left\{x_0^\beta p x_0^\vee x_{j_1}^\vee \cdots x_{j_{n+1-k}}^\vee \ : \ p \in J_\gamma,\ \beta = 0,1,2,\ldots\right\} & \text{if } x_0 \text{ is fixed by } \gamma \\ \emptyset & \text{otherwise.} \end{cases}$$

$$C_\gamma = \begin{cases} \emptyset & \text{if } x_0 \text{ is fixed by } \gamma \\ \left\{p x_0^\vee x_{j_1}^\vee \cdots x_{j_{n+1-k}}^\vee \ : \ p \in J_\gamma\right\} & \text{otherwise.} \end{cases}$$

Definition 2.12. — *Let* $\zeta\colon \Gamma_{\mathbf{w}} \to \mathbb{G}_m$ *be a character of* $\Gamma_{\mathbf{w}}$. *We say that a polynomial or formal power series* $p(x_0, \ldots, x_{n+1})$ *is* ζ-isotypical *if* $p(g\boldsymbol{x}) = \zeta(g)p(\boldsymbol{x})$ *for all* $g \in \Gamma_{\mathbf{w}}$. *Note that every monomial* m *determines a character* $\xi(m)$ *such that* m *is* $\xi(m)$*-isotypical. The space of formal power series* $K := \mathbb{C}[\![x_0, \ldots, x_{n+1}]\!]$ *is therefore the completed direct sum of its* ζ*-isotypical summands*

$$K = \widehat{\bigoplus}_{\zeta \in \hat{\Gamma}_{\mathbf{w}}} K_\zeta, \quad K_\zeta = \left\{p \in K \ : \ p(g\boldsymbol{x}) = \zeta(g)p(\boldsymbol{x}) \ \ \forall\, g \in \Gamma_{\mathbf{w}}\right\}.$$

Definition 2.13. — *Given a* γ*-monomial* m, *we write* b_k *for the total exponent of* x_k *in* m, *where* $x_k^\vee$ *contributes* -1 *to* b_k. *The character* $\xi(m)$ *is determined by these exponents:*

$$\xi(m)(t_0, \ldots, t_{n+1}) = t_0^{b_0} \cdots t_{n+1}^{b_{n+1}}.$$

We now assume the following. For each $\gamma \in \Gamma_{\mathbf{w}}$ let $\mathbf{w}_\gamma$ (respectively $\mathbf{w}'_\gamma$) denote the restriction of the polynomial $\mathbf{w}$ to the subspace where the unfixed variables $x_{j_1}, \cdots, x_{j_{n-k+1}}$ (respectively $x_0, x_{j_1}, \cdots, x_{j_{n-k+1}}$) vanish. We assume that $\mathbf{w}'_\gamma$ has an isolated singularity at the origin for all $\gamma \in \Gamma$, which is the case for all our examples.

Theorem 2.14. — *Under this assumption, the Hochschild cohomology*

$$\mathrm{HH}^*\left(\mathrm{mf}\left(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}\right)\right)$$

is a direct sum of 1-dimensional contributions, one from each pair (γ, m) *with* $m \in M_\gamma$ *such that* $\xi(m) = \chi^{\otimes u}$ *for some* $u \in \mathbb{Z}$. *In this case,* (γ, m) *contributes to*

$$\mathrm{HH}^{2u+n-k+1} \text{ if } m \in A_\gamma,$$
$$\mathrm{HH}^{2u+n-k+2} \text{ if } m \in B_\gamma,$$
$$\mathrm{HH}^{2u+n-k+2} \text{ if } m \in C_\gamma,$$

where k *is the number of variables amongst* $\{x_1, \ldots, x_{n+1}\}$ *fixed by* γ.

Proof. — This is just a repackaging of [BFK14, Theorem 1.2], based on the exposition in [LU21, Section 5.1]. We briefly explain how to translate between our notation and the notation of [LU21]. For each γ there are three kinds of contribution to Hochschild cohomology, enumerated by [LU21, Equations 5.5-5.7]:

- If x_0 is not fixed by γ then the Hochschild cohomology picks up a contribution given by [LU21, Equation 5.5]:
$$\left(\mathrm{Jac}_{\mathbf{w}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee\right)_{\chi^{\otimes u}}$$
where N_γ is the vector space spanned by the non-fixed variables $x_0, x_{j_1}, \ldots, x_{j_{n-k+1}}$, Jac denotes the Jacobian ring, and $\chi^{\otimes u}$ means taking the isotypical part. Our γ-monomials from C_γ form an explicit basis of this space: J_γ is a basis for the Jacobian $\mathrm{Jac}_{\mathbf{w}_\gamma}$ and $x_0^\vee \otimes x_{j_1}^\vee \otimes \cdots \otimes x_{j_{n-k+1}}^\vee$ is a generator of $\Lambda^{\dim N_\gamma} N_\gamma^\vee$; the $\chi^{\otimes u}$ subscript is precisely telling us to restrict attention to γ-monomials with $\xi(m) = \chi^{\otimes u}$. This contributes to $\mathrm{HH}^{2u+\dim N_\gamma} = \mathrm{HH}^{2u+n-k+2}$.
- If x_0 is fixed by γ then there are contributions [LU21, Equations 5.6 and 5.7]:
$$\left(\mathrm{Jac}_{\mathbf{w}'_\gamma} \otimes \mathbb{C}[x_0] \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee\right)_{\chi^{\otimes u}}, \qquad \left(\mathbb{C}x_0^\vee \otimes \mathrm{Jac}_{\mathbf{w}'_\gamma} \otimes \mathbb{C}[x_0] \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee\right)_{\chi^{\otimes u}}$$
to $\mathrm{HH}^{2u+\dim N_\gamma} = \mathrm{HH}^{2u+n-k+1}$ and $\mathrm{HH}^{2u+\dim N_\gamma+1} = \mathrm{HH}^{2u+n-k+2}$ respectively. Our γ-monomials of type A_γ and B_γ give bases for these vector spaces. □

2.5. Formal change of coordinates

In this section we prove the last remaining ingredient (Theorem 2.15 below) that was used in Section 2.3 above (in the proof that Conjecture 2.3 implies Equation (2.1)).

Recall that there exist weights $d_0, d_1, \ldots, d_{n+1}$ such that if we give x_i weight d_i then both $\mathbf{w}(x_1, \ldots, x_{n+1})$ and $x_0 \cdots x_{n+1}$ are quasihomogeneous of degree $h = \sum_{i=0}^{n+1} d_i$ and χ-isotypical. Let $|\cdot|_0$ be the x_0-valuation on the space $K = \mathbb{C}[\![x_0, \ldots, x_{n+1}]\!]$ of formal power series, i.e. $|p|_0 = k$ if x_0^k divides p but x_0^{k+1} does not.

Theorem 2.15. — *Suppose that $p_0(x_0, \ldots, x_{n+1})$ is a χ-isotypical formal power series which is quasihomogeneous of degree h and $|p_0|_0 > 0$. If $\mathrm{HH}^2(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w})) = 0$ then there is a formal change of variables $\boldsymbol{z} = (x_0, z_1(\boldsymbol{x}), \ldots, z_{n+1}(\boldsymbol{x}))$ such that $\mathbf{w}(\boldsymbol{z}) = \mathbf{w}(\boldsymbol{x}) + p_0(\boldsymbol{x})$.*

Remark 2.16. — In particular, the theorem applies when $p_0(\boldsymbol{x}) = x_0 \cdots x_{n+1}$. To prove Theorem 2.15, we first establish a sequence of lemmas.

Lemma 2.17. — *Suppose that $\mathrm{HH}^2(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w})) = 0$. Then the image of K_χ in the Jacobian ring is trivial.*

Proof. — If m is a χ-isotypical monomial which is nontrivial in the Jacobian ring then we can use it as part of our monomial basis J_γ for $\gamma = id$. It will then contribute as a type A id-monomial to $\mathrm{HH}^2(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}))$. Thus if $\mathrm{HH}^2(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w})) = 0$, we deduce that any monomial $m \in K_\chi$ is trivial in the Jacobian ring, and hence the image of K_χ in the Jacobian ring is zero. □

Lemma 2.18. — *If $p \in K_\chi$ is trivial in the Jacobian ring then $p = \sum_{i=1}^{n+1} v_i \frac{\partial \mathbf{w}}{\partial x_i}$ for some $v_1, \ldots, v_{n+1} \in K$ where $v_i \in K_{t_i}$. Here, t_i denotes the character of $\Gamma_{\mathbf{w}}$ which projects to t_i.*

Proof. — Consider the map $\partial\colon K^{n+1} \to K$ defined by $\partial(v_1, \ldots, v_{n+1}) = \sum_{i=1}^{n+1} v_i \frac{\partial \mathbf{w}}{\partial x_i}$. The cokernel of ∂ is the Jacobian ring. Because $\mathbf{w} \in K_\chi$, we have $\partial \mathbf{w}/\partial x_i \in K_{\chi \otimes t_i^{-1}}$ for all i, so $\partial(v_1, \ldots, v_{n+1}) \in K_\chi$ if and only if $v_i \in K_{t_i}$ for all $i = 1, \ldots, n+1$. ☐

If $\boldsymbol{v} \in K_{t_1} \oplus \cdots \oplus K_{t_{n+1}}$ then we call $\boldsymbol{v}$ a $\Gamma_{\mathbf{w}}$-equivariant vector field because the components v_i of $\boldsymbol{v}$ transform under $\Gamma_{\mathbf{w}}$ like the coordinates x_i. We have now seen that, under the hypotheses of Theorem 2.15, $p_0 = \partial \boldsymbol{v}$ for a $\Gamma_{\mathbf{w}}$-equivariant vector field $\boldsymbol{v}$.

Lemma 2.19. — *In the setting of Theorem 2.15, there exists a formal change of variables $\boldsymbol{y}$ such that*

$$p_1(\boldsymbol{y}) := \mathbf{w}(\boldsymbol{x}) + p_0(\boldsymbol{x}) - \mathbf{w}(\boldsymbol{y})$$

is χ-isotypical and satisfies $|p_1|_0 > |p_0|_0$.

Proof. — This is a small modification of [AGV12, Section 12.6]. We know that $p_0 = \partial \boldsymbol{v}$ for a $\Gamma_{\mathbf{w}}$-equivariant vector field $\boldsymbol{v}$. We define $\boldsymbol{y}$ implicitly by $y_0 = x_0$, $x_i = y_i - v_i(\boldsymbol{y})$. Since $\boldsymbol{v}$ is $\Gamma_{\mathbf{w}}$-equivariant, this formal change of coordinates is $\Gamma_{\mathbf{w}}$-equivariant. As in the proof[(4)] of cite[Section 12.6]Arnold, we find that $p_1(\boldsymbol{y}) := \mathbf{w}(\boldsymbol{x}) + p_0(\boldsymbol{x}) - \mathbf{w}(\boldsymbol{y})$ has $|p_1|_0 > |p_0|_0$. Moreover, since $\boldsymbol{v}$ is $\Gamma_{\mathbf{w}}$-equivariant, p_1 is χ-isotypical. ☐

Proof of Theorem 2.15. — We can apply Lemma 2.19 iteratively and compose the formal diffeomorphisms we get at each stage. Composition makes sense because $|\boldsymbol{v}|_0 > 0$, so the x_0^k-term in the composition of formal diffeomorphisms only involves summing finitely many terms. In this way, we construct a sequence of perturbation terms $p_1, p_2, \ldots$ with $|p_1|_0 < |p_2|_0 < \cdots$. In the limit, we obtain a formal change of coordinates $\boldsymbol{z}$ with perturbation term $p_\infty(\boldsymbol{z}) := \mathbf{w}(\boldsymbol{x}) + p_0(\boldsymbol{x}) - \mathbf{w}(\boldsymbol{z})$ satisfying $|p_\infty|_0 = \infty$. Therefore $p_\infty = 0$ and we have proved the theorem. ☐

3. Compendium of examples

In this section, we calculate $\mathrm{HH}^* := \mathrm{HH}^*(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}))$ for the invertible polynomials $\mathbf{w}$ which are mirror-dual to the polynomials in Theorem 1.8. We now summarise how this leads to a proof of that theorem.

Proof of Theorem 1.8. — These examples are log Fano, so $d_0 < 0$, and, in all cases, we will see that $\mathrm{HH}^2(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w})) = 0$. In Cases 1–3 and 5–6 of Theorem 1.8, Conjecture 2.2 holds, so that Theorem 2.6 applies. As a consequence, we can conclude Theorem 1.8 unconditionally in these cases. Case 4 holds conditionally on Conjecture 2.2 or 2.3. ☐

(4) In [AGV12], they have no variable x_0 and filter by the weighted degree of the perturbation rather than the x_0-valuation. Since the perturbation terms are quasihomogeneous of degree h, the weighted degree of the perturbation term with respect to $x_1, \ldots, x_{n+1}$ is proportional to the x_0-valuation, so our strategy is equivalent.

3.1. Brieskorn–Pham

A Brieskorn–Pham singularity is an isolated hypersurface singularity given by the vanishing of the polynomial

$$\mathbf{w}(x_1, \ldots, x_{n+1}) = x_1^{a_1} + \cdots + x_{n+1}^{a_{n+1}}$$

for a collection of integers $a_i \geqslant 2$. This is an invertible polynomial with $\check{\mathbf{w}} = \mathbf{w}$.

Let $\boldsymbol{\mu}_k$ denote the cyclic group of kth roots of unity, and let

$$l = \operatorname{lcm}(a_1, \ldots, a_{n+1}), \qquad \nu = 1 - \sum_{i=1}^{n+1} \frac{1}{a_i}.$$

We have a surjective l-to-1 homomorphism

$$\begin{aligned} &T \colon \boldsymbol{\mu}_{a_1} \times \cdots \times \boldsymbol{\mu}_{a_{n+1}} \times \mathbb{G}_m \to \Gamma_{\mathbf{w}}, \\ &T(\mu_1, \ldots, \mu_{n+1}, \tau) = \left(\tau^{l\nu} \mu_1^{-1} \cdots \mu_{n+1}^{-1}, \tau^{l/a_1}\mu_1, \ldots, \tau^{l/a_{n+1}}\mu_{n+1}\right) \end{aligned}$$

Remark 3.1. — Under T, the subgroup $\boldsymbol{\mu}_{a_1} \times \cdots \times \boldsymbol{\mu}_{a_{n+1}}$ maps isomorphically onto $\ker\chi$; we will use this identification to write elements of $\ker\chi$ as $(n+1)$-tuples of roots of unity.

Fix an element $\gamma \in \ker\chi$. Restricting $\mathbf{w}$ to the fixed variables $x_{i_1}, \ldots, x_{i_k}$ we get $\sum_{m=1}^{k} x_{i_m}^{a_{i_m}}$, and we pick the monomial basis

$$J_\gamma = \left\{ x_{i_1}^{b_{i_1}} \cdots x_{i_k}^{b_{i_k}} \;:\; 0 \leqslant b_{i_m} \leqslant a_{i_m} - 2 \text{ for } m = 1, 2, \ldots k \right\}$$

for its Jacobian ring.

Lemma 3.2. — *Let m be a γ-monomial with total exponents $b_0, \ldots, b_{n+1}$ and suppose that (γ, m) contributes to Hochschild cohomology. Then $b_i = b_0 \bmod a_i$ for $i = 1, \ldots, n+1$ and $\xi(m) = \chi^{\otimes(b_0 - \sum_{i=1}^{n+1} m_i)}$, where the integers m_i are determined by $b_0 = b_i + m_i a_i$.*

Proof. — The γ-monomial m with total exponents $b_0, \ldots, b_{n+1}$ has character[5]

$$\xi(m)(\mu_1, \ldots, \mu_{n+1}, \tau) = \tau^{b_0 l\nu + \sum_{i=1}^{n+1} b_i l/a_i} \mu_1^{b_1 - b_0} \cdots \mu_{n+1}^{b_{n+1} - b_0}$$

This coincides with a power of χ if and only if $b_0 = b_i \bmod a_i$ for $i = 1, \ldots, n+1$. More precisely, if $b_0 = b_i + m_i a_i$ for integers $m_1, \ldots, m_n$ then $\xi(m) = \chi^{\otimes(b_0 - \sum_{i=1}^{n+1} m_i)}$. $\square$

Remark 3.3. — In fact, if $b_0 \geqslant 0$, then b_0 uniquely determines monomials $m_A(b_0)$ and $m_B(b_0)$ of types A and B respectively which have total exponents $b_i = b_0 \bmod a_i$. Namely, we multiply together factors of $x_i^{b_i}$, $i = 1, \ldots, n+1$, where x_i^{-1} means $x_i^\vee$. To obtain $m_A(b_0)$ we include a factor of $x_0^{b_0}$; to obtain $m_B(b_0)$ we include a factor of $x_0^{b_0+1} x_0^\vee$. Similarly, $b_0 = -1$ determines unique monomials $m_B(-1) = x_0^\vee \cdots x_{n+1}^\vee$ of type B and $m_C(-1) = x_0^\vee \cdots x_{n+1}^\vee$ of type C.

[5] The characters of $\Gamma_{\mathbf{w}}$ induce characters of $\boldsymbol{\mu}_{a_1} \times \cdots \times \boldsymbol{\mu}_{a_{n+1}} \times \mathbb{G}_m$ by precomposing with T and we will often write characters of $\Gamma_{\mathbf{w}}$ by giving a character of the bigger group which factors through T.

Remark 3.4. — By the Sun Zi remainder theorem, given any collection of total exponents $0 \leqslant b_1 < a_1, \ldots, 0 \leqslant b_{n+1} < a_{n+1}$, we can solve this system of congruences for b_0 uniquely modulo l if and only if $b_i = b_j \bmod \gcd(a_i, a_j)$ for all $i, j \in \{1, \ldots, n+1\}$.

Our approach to calculating HH^* will therefore be to consider each possible value b_0 and find the number of elements $\gamma \in \ker \chi$ such that $(\gamma, m_A(b_0))$, $(\gamma, m_B(b_0))$ or $(\gamma, m_C(b_0))$ is a contributing γ-monomial.

The contributions from $b_0 = -1$ are easy to calculate.

Lemma 3.5. — *The contributions from monomials with total exponent $b_0 = -1$ come from*

$$\left(\gamma, x_0^\vee \cdots x_{n+1}^\vee\right) \in \mathrm{HH}^n$$

for all $\gamma \in (\boldsymbol{\mu}_{a_1} \setminus \{1\}) \times \cdots \times (\boldsymbol{\mu}_{a_{n+1}} \setminus \{1\})$.

Proof. — We have $m_B(-1) = m_C(-1) = x_0^\vee \cdots x_{n+1}^\vee$. This contributes as a γ-monomial if and only if either:

- γ leaves all variables $x_0, \ldots, x_{n+1}$ unfixed. In this case we get a type C contribution from $(\gamma, m_C(-1))$.
- γ fixes x_0 and does not fix any other variable. In this case we get a type B contribution from $(\gamma, m_B(-1))$. □

Remark 3.6. — In fact, in our examples (but not in general), these will be the only contributions to HH^n, which gives $\dim \mathrm{HH}^n = (a_1 - 1) \cdots (a_{n+1} - 1)$. Note that this equals the Milnor number of the singularity.

We now proceed to the specific examples of interest to compute the contributions explicitly. These examples are:

$$\begin{aligned}
cA_\ell:&\quad x_1^2 + x_2^2 + x_3^{\ell+1} + x_4^{k(\ell+1)}, && k, \ell \geqslant 1\\
cD_4:&\quad x_1^2 + x_2^3 + x_3^3 + x_4^{6k}, && k \geqslant 1\\
cE_6:&\quad x_1^2 + x_2^3 + x_3^4 + x_4^{12k}, && k \geqslant 1\\
cE_8:&\quad x_1^2 + x_2^3 + x_3^5 + x_4^{30k}, && k \geqslant 1
\end{aligned}$$

In all cases, the $x_4 = 0$ slice has an ADE singularity at the origin, having the type indicated. The 3-folds admit small resolutions which fully resolve the singularity of the slice; this follows from [Bri68, Satz 0.2] because the exponent of x_4 is a multiple of the Coxeter number of the ADE singularity.

Theorem 3.7. — *For each $\mathbf{w}$ below, we will compute $\mathrm{HH}^* = \mathrm{HH}^*(\mathrm{mf}(\mathbb{A}^5, \Gamma_{\mathbf{w}}, \mathbf{w}))$. In all cases, HH^d vanishes when $d = 2$ or $d \geqslant 4$.*

(1) **Type cA_ℓ:** *Let $\mathbf{w} = x_1^2 + x_2^2 + x_3^{\ell+1} + x_4^{k(\ell+1)}$. Then*

$$\dim \mathrm{HH}^3 = \ell(k(\ell+1) - 1), \qquad \dim \mathrm{HH}^d = \ell \text{ for } d \leqslant 1.$$

(2) **Type cD_4:** *Let $\mathbf{w} = x_1^2 + x_2^3 + x_3^3 + x_4^{6k}$. Then*

$$\dim \mathrm{HH}^3 = 24k - 4, \qquad \dim \mathrm{HH}^d = 4 \text{ for } d \leqslant 1.$$

(3) **Type** cE_6**:** *Let* $\mathbf{w} = x_1^2 + x_2^3 + x_3^4 + x_4^{12k}$. *Then*

$$\dim \mathrm{HH}^3 = 72k - 6, \qquad \dim \mathrm{HH}^d = 6 \text{ for } d \leqslant 1.$$

(4) **Type** cE_8**:** *Let* $\mathbf{w} = x_1^2 + x_2^3 + x_3^5 + x_4^{30k}$. *Then*

$$\dim \mathrm{HH}^3 = 240k - 8, \qquad \dim \mathrm{HH}^d = 8 \text{ for } d \leqslant 1.$$

Proof. — In each case, the HH^3 contributions come from Lemma 3.5. We will consider the contributions from $m_A(b_0)$ with $b_0 \geqslant 0$.

In the various cases we will use Euclid's algorithm to write:

Type	$b_0 =$	$q \in$	$r \in$
cA_ℓ	$k(\ell+1)p + (\ell+1)q + r$	$\{0, 1, \ldots, k-1\}$	$r \in \{0, 1, \ldots, \ell\}$
cD_4	$6kp + 6q + r$	$\{0, 1, \ldots, k-1\}$	$r \in \{0, 1, \ldots, 5\}$
cE_6	$12kp + 12q + r$	$\{0, 1, \ldots, k-1\}$	$r \in \{0, 1, \ldots, 11\}$
cE_8	$30kp + 30q + r$	$\{0, 1, \ldots, k-1\}$	$r \in \{0, 1, \ldots, 29\}$

In the following tables, we indicate: the type A monomials $m_A(b_0)$; the γ for which $(\gamma, m_A(b_0))$ contribute to HH^*; the number of such γ; and the degree of HH^* to which they contribute. We omit monomials m for which there are no γ such that (γ, m) contributes.

In every case, we will see that HH^* has the rank stated in the theorem in every even degree $d \leqslant 0$. The type B contributions, other than those appearing in Lemma 3.5, will differ only in replacing $m_A(b_0)$ with $x_0 x_0^\vee m_A(b_0)$ and yield the same ranks in every odd degree $d \leqslant 1$.

Table 3.1. Table for cA_ℓ. Note that $b_0 = k(\ell+1)p + (\ell+1)q + r$ with $p \geqslant 0$, $0 \leqslant q \leqslant k-1$, $0 \leqslant r \leqslant \ell$. The top two row give us rank ℓ in every degree $d \neq -2k \bmod 2(k+1)$. The bottom row gives us rank ℓ in degrees $d = -2k \bmod 2(k+1)$(ω is a chosen primitive $(\ell+1)^{\text{th}}$ root of unity and $a \in \{1, 2, \ldots, \ell\}$).

q	r	$m_A(b_0)$	γ	$\#\,\gamma$	HH^* degree
any	$< \ell$	$x_0^{b_0} x_3^r x_4^{(\ell+1)q+r} \begin{cases} 1 & \text{if } b_0 \text{ even} \\ x_1^\vee x_2^\vee & \text{if } b_0 \text{ odd} \end{cases}$	$\begin{cases} (1,1,1,1) \\ (-1,-1,1,1) \end{cases}$	1	$-2(k+1)p - 2q$
$k-1$	ℓ	$x_0^{b_0} x_3^\vee x_4^\vee \begin{cases} 1 & \text{if } b_0 \text{ even} \\ x_1^\vee x_2^\vee & \text{if } b_0 \text{ odd} \end{cases}$	$\begin{cases} (-1,-1,\omega^a,\omega^{-a}) \\ (1,1,\omega^a,\omega^{-a}) \end{cases}$	ℓ	$-2(k+1)p - 2k$

Table 3.2. Table for cD_4. The top three rows give us rank 4 in every degree $d \neq -2k \bmod 2(k{+}1)$. The bottom row gives us rank 4 in degrees $d = -2k \bmod 2(k{+}1)$ (on the second and fourth rows, ω is a chosen primitive 3rd root of unity and $a, b \in \{1, 2\}$).

q	r	$m_A(b_0)$	γ	$\#\,\gamma$	HH* degree
any	0	$x_0^{6kp+6q}x_4^{6q}$	$(1,1,1,1)$	1	$-2(k+1)p-2q$
any	2	$x_0^{6kp+6q+2}x_4^{6q+2}x_2^\vee x_3^\vee$	$(1,\omega^a,\omega^{-a},1),$	2	$-2(k+1)p-2q$
any	4	$x_0^{6kp+6q+4}x_2x_3x_4^{6q+4}$	$(1,1,1,1)$	1	$-2(k+1)p-2q$
$k-1$	5	$x_0^{6kp+6k-1}x_1^\vee x_2^\vee x_3^\vee x_4^\vee$	$(-1,\omega^a,\omega^b,-\omega^{-a-b})$	4	$-2(k+1)p-2k$

Table 3.3. Table for cE_6. The top six rows give us rank 6 in every degree $d \neq -2k \bmod 2(k{+}1)$. The bottom row gives us rank 6 in degrees $d = -2k \bmod 2(k{+}1)$ (ω and i are chosen primitive 3rd and 4th roots of unity; and $a \in \{1, 2\}$ and $b \in \{1, 2, 3\}$).

q	r	$m_A(b_0)$	γ	$\#\,\gamma$	HH* degree
any	0	$x_0^{12kp+12q}x_4^{12q}$	$(1,1,1,1)$	1	$-2(k+1)p-2q$
any	3	$x_0^{12kp+12q+3}x_4^{12q+3}x_1^\vee x_3^\vee$	$(-1,1,-1,1)$	1	$-2(k+1)p-2q$
any	4	$x_0^{12kp+12q+4}x_2x_4^{12q+4}$	$(1,1,1,1)$	1	$-2(k+1)p-2q$
any	6	$x_0^{12kp+12q+6}x_3^2x_4^{12q+6}$	$(1,1,1,1)$	1	$-2(k+1)p-2q$
any	7	$x_0^{12kp+12q+7}x_2x_4^{12q+7}x_1^\vee x_3^\vee$	$(-1,1,-1,1)$	1	$-2(k+1)p-2q$
any	10	$x_0^{12kp+12q+10}x_2x_3^2x_4^{12q+10}$	$(1,1,1,1)$	1	$-2(k+1)p-2q$
$k-1$	11	$x_0^{12kp+12k-1}x_1^\vee x_2^\vee x_3^\vee x_4^\vee$	$(-1,\omega^a,i^b,-\omega^{-a}i^{-b})$	6	$-2(k+1)p-2k$

Table 3.4. Table for cE_8. The top eight rows give us rank 8 in every degree $d \neq -2k \bmod 2(k{+}1)$. The bottom row gives us rank 8 in degrees $d = -2k \bmod 2(k{+}1)$ (ω and ζ are chosen primitive 3rd and 5th roots of unity, $a \in \{1, 2\}$, and $b \in \{1, 2, 3, 4\}$).

q	r	$m_A(b_0)$	γ	$\#\,\gamma$	HH* degree
any	0	$x_0^{30kp+30q}x_4^{30q}$	$(1,1,1,1)$	1	$-2(k+1)p-2q$
any	6	$x_0^{30kp+30q+6}x_3x_4^{30q+6}$	$(1,1,1,1)$	1	$-2(k+1)p-2q$
any	10	$x_0^{30kp+30q+10}x_2x_4^{30q+10}$	$(1,1,1,1)$	1	$-2(k+1)p-2q$
any	12	$x_0^{30kp+30q+12}x_3^2x_4^{30q+12}$	$(1,1,1,1)$	1	$-2(k+1)p-2q$
any	16	$x_0^{30kp+30q+16}x_2x_3x_4^{30q+16}$	$(1,1,1,1)$	1	$-2(k+1)p-2q$
any	18	$x_0^{30kp+30q+18}x_3^3x_4^{30q+18}$	$(1,1,1,1)$	1	$-2(k+1)p-2q$
any	22	$x_0^{30kp+30q+22}x_2x_3^2x_4^{30q+22}$	$(1,1,1,1)$	1	$-2(k+1)p-2q$
any	28	$x_0^{30kp+30q+28}x_2x_3^3x_4^{30q+28}$	$(1,1,1,1)$	1	$-2(k+1)p-2q$
$k-1$	29	$x_0^{30kp+30k-1}x_1^\vee x_2^\vee x_3^\vee x_4^\vee$	$(-1,\omega^a,\xi^b,-\omega^{-a}\xi^{-b})$	8	$-2(k+1)p-2k$

□

3.2. Laufer's examples

Let

$$\check{\mathbf{w}} = x_1^3 + x_1 x_2^{2k+1} + x_2 x_3^2 + x_4^2$$

This polynomial defines a cD_4 singularity: the $x_1 = x_2$ slice has an isolated D_4 singularity at the origin. Laufer [Lau81] showed that this admits a small resolution with a single exceptional curve; the small resolution yields a partial resolution of the D_4 slice (the map from the minimal resolution to the partial resolution collapses the three peripheral curves in the D_4 configuration).

The Berglund–Hübsch transpose is

$$\mathbf{w} = x_1^3 x_2 + x_2^{2k+1} x_3 + x_3^2 + x_4^2$$

which has

$$\Gamma_{\mathbf{w}} = \left\{(t_0, t_1, t_2, t_3, t_4) : t_1^3 t_2 = t_2^{2k+1} t_3 = t_3^2 = t_4^2 = t_0 t_1 t_2 t_3 t_4\right\}.$$

Lemma 3.8. — *There is a* 3*-to-*1 *surjective homomorphism*

$$T \colon \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_3 \times \mathbb{C}^\times \to \Gamma_{\mathbf{w}},$$
$$T(s, \mu, \tau) = \left(s\mu^{-1}\tau^{-(4k+4)}, \mu\tau^{4k+1}, \tau^3, \tau^{6k+3}, s\tau^{6k+3}\right).$$

The composition $\chi \circ T$ *is given by* $(s, \mu, \tau) \mapsto \tau^{12k+6}$.

Proof. — We first show that the stated homomorphism is surjective. Since $t_2^{2k+1} t_3 = t_3^2$ we get $t_3 = t_2^{2k+1}$. Since $t_4^2 = t_3^2$, we have $t_4 = \pm t_2^{2k+1}$. Since $t_1^3 t_2 = t_3^2 = t_2^{4k+2}$, we get $t_1^3 = t_2^{4k+1}$. If $t_2 = \tau^3$ for some $\tau \in \mathbb{C}^\times$ then $t_1 = \mu\tau^{4k+1}$ for some cube root μ of unity. Finally, t_0 is determined by $t_0 \cdots t_4 = t_3^2$, which gives $t_0 = \pm\mu^{-1}\tau^{-4k-4}$.

To see that the homomorphism is 3-to-1, observe that its kernel consists of triples (s, μ, τ) such that

$$\mu\tau^{4k+1} = \tau^3 = \tau^{6k+3} = s\tau^{6k+3} = 1.$$

In particular, this means $s = 1$ and $\tau^3 = 1$. The condition $1 = \mu\tau^{4k+1} = \mu\tau^{k+1}$ means that $\mu = \tau^{-k-1}$, so the kernel is $\{(1, \tau^{-k-1}, \tau) \ : \ \tau^3 = 1\}$, which has size 3. ☐

The kernel $\ker(\chi \circ T)$ is then $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_3 \times \boldsymbol{\mu}_{12k+6}$; recall that T is 3-to-1, so this is three times the size of $\ker\chi$. We now identify which combinations of fixed and unfixed variables are possible for $\gamma \in \ker\chi$.

Fixed variables	Number of $\gamma = T(s,\mu,\tau) \in \ker\chi$	s	μ	τ
$\{0,1,2,3,4\}$	1	1	$\tau^{-4(k+1)}$	$\tau^3 = 1$
$\{0\}$	1	1	$\tau^{-4(k+1)}$	$\tau^3 = -1$
$\{2,3\}$	2	-1	$\mu \neq \tau^{-k-1}$	$\tau^3 = 1$
$\emptyset$	$6k+2$	See below		
Five further cases which do not contribute to HH*: $\{3\}$, $\{4\}$, $\{3,4\}$, $\{1,2,3\}$ and $\{2,3,4\}$				

Lemma 3.9. — *Let $\gamma = T(s,\mu,\tau) \in \ker\chi$. The possible combinations of fixed and unfixed variables for γ are given by the table before. We state the conditions on $(s,\mu,\tau) \in \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_3 \times \boldsymbol{\mu}_{12k+6}$ such that $\gamma = T(s,\mu,\tau)$ fixes this combination of variables, and also the number of suchγ (remembering that T is 3-to-1).*

Proof. — Let $\gamma = T(s,\mu,\tau)$ with $s \in \{\pm 1\}$, $\mu \in \boldsymbol{\mu}_3$, $\tau \in \boldsymbol{\mu}_{12k+6}$.

If x_0 is fixed then $\tau^{-4(k+1)} = s\mu$. This means that $\tau^{24(k+1)} = 1$, but $\tau^{12k+6} = 1$, so $\tau^6 = 1$. Therefore $\tau^{-4(k+1)}$ is a cube root of unity, which means that $s = 1$. This means that the other variables transform as $\mu\tau^{4k+1} = \tau^{-3}$, τ^3, τ^{6k+3} and τ^{6k+3}. There are two possibilities: $\tau^3 = 1$ (which fixes all variables) or $\tau^3 = -1$ (which fixes none).

If x_1 is fixed then $\mu = \tau^{-4k-1}$, so $\tau^{-12k-3} = 1$, but $\tau^{12k+6} = 1$, so $\tau^3 = 1$. This means that x_2 and x_3 are also fixed. The variable x_0 transforms as $s\mu^{-1}\tau^{-4k-4} = s\tau^{-3} = s$, so either x_0 is fixed (as in the previous case) or $s = -1$, in which case both x_0 and x_4 are unfixed.

If x_2 is fixed then $\tau^3 = 1$ so $\tau^{6k+3} = 1$ and x_3 is also fixed. If x_0 or x_1 is fixed then we are in a previous case; assume they are not. Then $\mu \neq \tau^{-k-1}$ and s can take on either value because both μ and τ^{-k-1} are in $\boldsymbol{\mu}_3$, so $\mu = -\tau^{-k-1}$ is impossible. If $s = 1$ then x_4 is fixed (yielding fixed variables $\{2,3,4\}$); otherwise we get fixed variables $\{2,3\}$.

Finally, if none of x_0, x_1, x_2 are fixed then the remaining variables can be fixed in any combination. We will see in Theorem 3.10 that the only such γ which contribute γ-monomials to HH^* are those which fix no variables. There are $6k+2$ of these. To see this, we argue as follows. If x_3 is not fixed then $\tau^{6k+3} \neq 1$, so $\tau^{6k+3} = -1$. If x_4 is not fixed then $s\tau^{6k+3} = -1$ means that $s = 1$. The remaining conditions become

$$\mu \neq \tau^{-4(k+1)}, \qquad \mu \neq \tau^{-4k-1}.$$

The second condition always holds because $\tau^{6k+3} = -1$, so τ^{-4k-1} is not a cube root of unity ($(\tau^{-4k-1})^3 = \tau^{-12k-3} = \tau^3 \neq 1$). The first condition implies $1 = (\tau^{-4(k+1)})^3 = \tau^{12k+12} = \tau^6$, which can hold only if $\tau^3 = -1$. Therefore there are $6k$ roots of $\tau^{6k+3} = -1$ for which μ can take on any value and 3 roots of $\tau^3 = -1$ for which μ can be two out of the three roots of unity. This gives a total of $3(6k+2)$ combinations $(1,\mu,\tau)$, and this triple-counts the available γs because T is 3-to-1. $\square$

We now pick the following monomial bases J_γ for the relevant Jacobian rings:

$$\begin{aligned}
\mathrm{Jac}\left(\mathbf{w}_{|x_1,x_2,x_3,x_4}\right) &= \mathrm{Jac}\left(\mathbf{w}_{|x_1,x_2,x_3}\right)\\
&= \mathbb{C}\left[x_1,x_2,x_3\right]/\left(3x_1^2x_2, x_1^3 + (2k+1)x_2^{2k}x_3, x_2^{2k+1} + 2x_3\right)\\
&= \mathbb{C}\left\{1, x_2, x_2^2, \ldots, x_2^{4k+1}, x_1, x_1x_2, x_1x_2^2, \ldots, x_1x_2^{4k+1}, x_1^2\right\}\\
\mathrm{Jac}\left(\mathbf{w}_{|x_2,x_3,x_4}\right) &= \mathrm{Jac}\left(\mathbf{w}_{|x_2,x_3}\right)\\
&= \mathbb{C}\left[x_2,x_3\right]/\left((2k+1)x_2^{2k}x_3, x_2^{2k+1} + 2x_3\right)\\
&= \mathbb{C}\cdot\left\{1, x_2, x_2^2, \ldots, x_2^{4k}\right\}\\
\mathrm{Jac}\left(\mathbf{w}_{|x_3,x_4}\right) &= \mathbb{C}\left[x_3,x_4\right]/(2x_3, 2x_4) = \mathbb{C}\cdot 1\\
\mathrm{Jac}(\mathbf{w}_{|x_3}) &= \mathrm{Jac}(\mathbf{w}_{|x_4}) = \mathbb{C}\cdot 1
\end{aligned}$$

Theorem 3.10. — *If* $\mathbf{w} = x_1^3 x_2 + x_2^{2k+1} x_3 + x_3^2 + x_4^2$ *then* $\mathrm{HH}^*(\mathbb{A}^5, \Gamma_{\mathbf{w}}, \mathbf{w})$ *satisfies*

$$\dim \mathrm{HH}^3 = 6k+5, \qquad \dim \mathrm{HH}^d = 1 \text{ for } d \leqslant 1$$

and $\dim \mathrm{HH}^d = 0$ *for* $d = 2$ *and* $d \geqslant 4$.

The HH^* *contributions for these singularities are as follows:*

Monomial	*Type*	*Degree in* HH^*	*Number of contributions*
$x_0^\vee x_1^\vee x_2^\vee x_3^\vee x_4^\vee$	C	3	$6k+2$
$x_0^\vee x_1^\vee x_2^\vee x_3^\vee x_4^\vee$	B	3	1
$x_0^\vee x_1^\vee x_2^{2k} x_4^\vee$	C	3	2
$x_0^{(6k+3)p-1} x_1^\vee x_2^\vee x_3^\vee x_4^\vee$	A	$-4(k+1)p+2$	$p \geqslant 1$ *even*
$x_0^{(6k+3)p+3q} x_2^{2q}$	A	$-4(k+1)p-2q$	$p \geqslant 0$, $0 \leqslant q \leqslant 2k$, $p = q \bmod 2$
$x_0^{(6k+3)p+3q+4} x_1 x_2^{2q+3}$	A	$-4(k+1)p-2q-2$	$p \geqslant 0$, $0 \leqslant q \leqslant 2k-1$, $p = q \bmod 2$
$x_0^{(6k+3)p+6k+4} x_1 x_2$	A	$-4(k+1)(p+1)$	$p \geqslant 0$ *even*
$x_0^{(6k+3)p+6k+2} x_1^2$	A	$-4(k+1)p-4k-2$	$p \geqslant 0$ *even*
Type B contributions in HH^{d+1} *for each type A monomial contributing to* HH^d.			

Proof. — Assuming the stated monomials are correct, the patient reader can check that every degree less than or equal to 1 picks up precisely one contribution as p and q vary (it suffices to check this over the degree range from 0 to $-8(k+1)$). We will therefore focus on establishing the list of contributing γ-monomials.

We work one set of fixed variables at a time and figure out which γ-monomials can contribute.

$\{0,1,2,3,4\}$: The possible A-type monomials are as follows.

$x_0^{b_0} x_2^{b_2}$ with $0 \leqslant b_2 \leqslant 4k+1$. For this to contribute, we need the existence of a $u \in \mathbb{Z}$ such that $s^{b_0}\mu^{-b_0}\tau^{3b_2-4(k+1)b_0} = \tau^{(12k+6)u}$ for all $(s,\mu,\tau) \in \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_3 \times \boldsymbol{\mu}_{12k+6}$. By taking $(s,\mu,\tau) = (-1, e^{2\pi i/3}, 1)$ we see that $b_0 = 0 \bmod 6$, which leaves $\tau^{3b_2-4(k+1)b_0} = \tau^{(12k+6)u}$, so $3b_2 - 4(k+1)b_0 = (12k+6)u$. If we write $b_0 = 6\beta_0$ and $2\beta_0 = (2k+1)p+q$ with $p \geqslant 0$, $q \in \{0,1\ldots,2k\}$, $p = q \bmod 2$, then we get

$$b_2 = (4k+2)(2(k+1)p+u) + 4(k+1)q,$$

so if we reduce modulo $4k+2$ we get $b_2 = 2q \bmod 4k+2$. Since $0 \leqslant b_2 \leqslant 4k+1$ and $q \leqslant 2k$, this determines b_2. The result is a contribution $x_0^{(6k+3)p+3q} x_2^{2q} \in \mathrm{HH}^{-4(k+1)p-2q}$ for all $p \geqslant 0$, $q \in \{0,1,\ldots,2k\}$ with $p = q \bmod 2$.

$x_0^{b_0} x_1 x_2^{b_2}$ with $0 \leqslant b_2 \leqslant 4k+1$. For this to contribute, we need $b_0 = 1 \bmod 3$, $b_0 = 0 \bmod 2$ (so $b_0 = 4 \bmod 6$) and $3b_2 + 4k + 1 - 4(k+1)b_0 = (12k+6)u$ for some $u \in \mathbb{Z}$. As in the previous case, we write $b_0 = 6\beta_0 + 4$ and $2\beta_0 = (2k+1)p+q$ with $p \geqslant 0$, $q \in \{0,1,\ldots,2k\}$, $p = q \bmod 2$. Arguing as before, we deduce that $b_2 = 2q+3 \bmod 4k+2$. This means $b_2 = 2q+3$ except in the case $q = 2k$, when $b_2 = 1$. We get contributions $x_0^{(6k+3)p+3q+4} x_1 x_2^{2q+3} \in \mathrm{HH}^{-4p(k+1)-2q-2}$ for $p \geqslant 0$, $q \in \{0,1,\ldots,2k-1\}$ with $p = q \bmod 2$ and a contribution $x_0^{(6k+3)p+6k+4} x_1 x_2 \in \mathrm{HH}^{-4(p+1)(k+1)}$.

$x_0^{b_0}x_1^2$. For this to contribute, we need $b_0 = 0 \bmod 2$, $b_0 = 2 \bmod 3$ (so $b_0 = 2 \bmod 6$) and $8k+2-4(k+1)b_0 = (12k+6)u$ for some $u \in \mathbb{Z}$. If we write $b_0 = 6\beta_0 + 2$ and $2\beta_0 = (2k+1)p + q$ with $p \geqslant 0$, $q \in \{0,1,\ldots,2k\}$, $p = q \bmod 2$ then we get $q = 2k \bmod 2k+1$ and so $x_0^{(6k+3)p+6k+2}x_1^2 \in \mathrm{HH}^{-4(k+1)p-4k-2}$ (with p even).

We also get corresponding B-type monomials by replacing $x_0^{b_0}$ with $x_0^{b_0+1}x_0^\vee$.

$\{0\}$: Any type A contribution is $x_0^{b_0}x_1^\vee\cdots x_4^\vee$. This transforms as

$$\begin{aligned}\xi\left(x_0^{b_0}x_1^\vee\cdots x_4^\vee\right)(s,\mu,\tau) &= \left(s\mu^{-1}\tau^{-4(k+1)}\right)^{b_0}\left(\mu\tau^{4k+1}\right)^{-1}\tau^{-3}\tau^{-(6k+3)}s^{-1}\tau^{-(6k+3)}\\ &= s^{b_0-1}\mu^{-b_0-1}\tau^{-2(6k+3)-3-(4k+1)-4(k+1)b_0}.\end{aligned}$$

For this to coincide with $\tau^{(12k+6)u}$ for all $(s,\mu,\tau) \in \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_3 \times \boldsymbol{\mu}_{12k+6}$ we need

$$b_0 = 1 \bmod 2, \quad b_0 = -1 \bmod 3 \Rightarrow b_0 = 5 \bmod 6$$

and $-2(6k+3)-3-(4k+1)-4(k+1)b_0 = (12k+6)u$ for some $u \in \mathbb{Z}$. Write $b_0 = 6\beta_0 - 1$. Then we get $-4(k+1)\beta_0 = (2k+1)(u+1)$. Since $\gcd(4(k+1), 2k+1) = 1$, we deduce that $\beta_0 = (2k+1)p$ and $u+1 = -4(k+1)p$ for some p. In other words, we get $x_0^{(12k+6)p-1}x_1^\vee\cdots x_4^\vee \in \mathrm{HH}^{2-4(k+1)p}$ ($p \geqslant 1$). There is a corresponding B-type monomial $x_0^{(12k+6)p}x_0^\vee x_1^\vee\cdots x_4^\vee \in \mathrm{HH}^{3-4(k+1)p}$ ($p \geqslant 0$).

$\{1,2,3\}$: The possible γ-monomials are of type C. They have the form $x_0^\vee x_1^{b_1}x_2^{b_2}x_4^\vee$ with $b_1 = 0,1$, $b_2 = 0,1,\ldots,4k+1$ or $b_1 = 2$, $b_2 = 0$. If this contributes then we have $b_1 = b_0 = -1 \bmod 3$, which leaves the only possibility as $x_0^\vee x_1^2 x_4^\vee$. This transforms under the action of $T(1,1,\tau)$ as τ^{6k+3}, which is not an integer power of τ^{12k+6}, so this monomial does not contribute.

$\{2,3,4\}$: The possible γ-monomials are $x_0^\vee x_1^\vee x_2^{b_2} \in C_\gamma$, which transform nontrivially under the action of $T(-1,1,1)$ and hence do not contribute to HH^*.

$\{2,3\}$: There are two γ fixing precisely x_2, x_3. The only γ-monomials are $x_0^\vee x_1^\vee x_2^{b_2}x_4^\vee$ with $b_2 = 0,1,\ldots,4k$. These transform according to the character τ^{3b_2-6k}, which is an integer power of τ^{12k+6} if and only if $b_2 = 2k$. This yield two contributions $(\gamma, x_0^\vee x_1^\vee x_2^{2k}x_4^\vee) \in \mathrm{HH}^3$.

$\{3,4\}$: The only γ-monomial is $x_0^\vee x_1^\vee x_2^\vee \in C_\gamma$, which transforms nontrivially under the action of $T(-1,1,1)$ and hence does not contribute to HH^*.

$\{3\}$: The only γ-monomial $x_0^\vee x_1^\vee x_2^\vee x_4^\vee \in C_\gamma$ which transforms as τ^{6k+3} under the action of $T(1,1,\tau)$ and hence does not contribute to HH^*.

$\{4\}$: The only γ-monomial $x_0^\vee x_1^\vee x_2^\vee x_3^\vee \in C_\gamma$ which transforms as τ^{6k+3} under the action of $T(1,1,\tau)$ and hence does not contribute to HH^*.

$\emptyset$: The C-type monomial $x_0^\vee\cdots x_4^\vee \in \mathrm{HH}^3$ contributes whenever γ has no fixed variables; there are precisely $6k+2$ such elements γ. ☐

3.3. More cA_ℓ examples

By [Kat91, Theorem 1.1], any cA_ℓ singularity with a small resolution is given by an equation $x_1^2 + x_2^2 + f(x_3,x_4) = 0$ where germ of the plane curve $f = 0$ at the origin has $\ell+1$ distinct smooth branches, and conversely, any such singularity admits a small resolution (the converse was also proved in [Fri86, p. 676]).

Let

$$\mathbf{w} = \check{\mathbf{w}} = x_1^2 + x_2^2 + x_3x_4\left(x_3^{\ell-1} + x_4^{k(\ell-1)}\right).$$

The singularity $\check{\mathbf{w}} = 0$ is of type cA_ℓ: the $x_3 = x_4$ slice has an A_ℓ singularity at the origin. The curve $x_3x_4(x_3^{\ell-1} + x_4^{k(\ell-1)}) = 0$ has multiplicity $\ell + 1$ and $\ell + 1$ distinct branches at the origin:

$$x_3 = 0, \quad x_4 = 0, \quad \text{and} \quad x_3 + \mu x_4^k = 0 \text{ for } \mu^{\ell-1} = -1.$$

Therefore, this singularity admits a small resolution.

Lemma 3.11. — *There exists a surjective* 2*-to-*1 *homomorphism* $T\colon \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_{2(\ell-1)} \times \mathbb{C}^\times \to \Gamma_{\mathbf{w}}$ *which we will construct in the proof. The composition* $\chi \circ T$ *is given by* $(\pm 1, \sigma, \tau) \mapsto \sigma^2\tau^{2k\ell+2}$.

Proof. — The group $\Gamma_{\mathbf{w}}$ is defined by the equations

$$t_0t_1t_2t_3t_4 = t_1^2 = t_2^2 = t_3^\ell t_4 = t_3t_4^{k(\ell-1)+1},$$

which imply $t_3^{\ell-1} = t_4^{k(\ell-1)}$, so $t_3 = \xi t_4^k$ for some ξ with $\xi^{\ell-1} = 1$. Substituting back, we get

$$t_1^2 = t_2^2 = \xi t_4^{k\ell+1}.$$

Pick a square root σ for ξ and a square root τ for t_4 such that $t_1 = \sigma\tau^{k\ell+1}$; then $t_2 = \pm\sigma\tau^{k\ell+1}$, $t_3 = \sigma^2\tau^{2k}$, $t_4 = \tau^2$, $t_0 = \pm\sigma^{-2}\tau^{-2(k+1)}$. This shows that the homomorphism

$$T(\pm 1, \sigma, \tau) = \left(\pm\sigma^{-2}\tau^{-2(k+1)}, \sigma\tau^{k\ell+1}, \pm\sigma\tau^{k\ell+1}, \sigma^2\tau^{2k}, \tau^2\right)$$

is surjective. To see that it is 2-to-1, note that its kernel consists of triples $(1, \sigma, \tau)$ for which $\tau^2 = 1$ (so $\tau = \pm 1$) and $\sigma = \tau^{-k\ell-1}$. This has size 2. □

The kernel $\ker(\chi \circ T)$ is the subgroup

$$\left\{(\pm 1, \sigma, \tau) \in \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_{2(\ell-1)} \times \boldsymbol{\mu}_{2(k\ell+1)(\ell-1)} \ : \ \tau^{2(k\ell+1)} = \sigma^{-2}\right\}.$$

The projection to $\tau \in \boldsymbol{\mu}_{2(k\ell+1)(\ell-1)}$ is surjective and split by the map $\tau \mapsto (1, \tau^{-k\ell-1}, \tau)$; its kernel consists of triples $(\pm 1, \pm 1, 1)$, so there is an isomorphism

$$\begin{aligned} \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_{2(k\ell+1)(\ell-1)} &\to \ker(\chi \circ T) \\ (s_1, s_2, \tau) &\mapsto \left(s_1, s_2\tau^{-(k\ell+1)}, \tau\right). \end{aligned}$$

We will work with elements of this group; since T is 2-to-1, this will mean that we overcount contributions to HH^* by a factor of 2. We now identify which combinations of fixed and unfixed variables are possible for $\gamma \in \ker\chi$.

Lemma 3.12. — *The possible combinations of fixed variables are given in the table below, along with the number of elements* $\gamma \in \ker\chi$ *which give rise to these fixed variables.*

Proof. — Let $\gamma = T(s_1, s_2\tau^{-(k\ell+1)}, \tau)$ with $s_1, s_2 \in \{1, -1\}$ and $\tau \in \boldsymbol{\mu}_{2(k\ell+1)(\ell-1)}$.

If x_0 is fixed then $s_1\tau^{2(k\ell+1)-2k-2} = 1$, so $\tau^{2(\ell-1)k} = s_1$. Since $\tau^{2(\ell-1)(k\ell+1)} = 1$ this implies $\tau^{2(\ell-1)} = s_1^\ell$, and therefore $s_1^{\ell(k\ell+1)} = 1$. If $s_1 = 1$ then this always holds. If $s_1 = -1$ then this holds if and only if $\ell(k\ell + 1)$ is even. Therefore the element which fix x_0 are those of the form $T(1, s_2\tau^{-(k\ell+1)}, \tau)$ with $\tau^{2(\ell-1)} = 1$ and (if $\ell(k\ell + 1)$ is even) $T(-1, s_2\tau^{-(k\ell+1)}, \tau)$ with $\tau^{2(\ell-1)} = (-1)^\ell$.

Fixed variables	$\#\gamma$
$\emptyset$	$k\ell(\ell-1)$
$\{0\}$	$\ell-2$
$\{0,3,4\}$	1
$\{0,1,2\}$	$\ell-2$
$\{0,1,2,3,4\}$	1

The following cases occur, but do not contribute to HH^**:*
$\{1\}$, $\{2\}$, $\{1,2\}$, $\{0,1\}$, $\{0,2\}$, $\{0,1,3,4\}$, $\{0,2,3,4\}$

- x_1 is fixed if and only if $\sigma\tau^{k\ell+1} = s_2 = 1$.
- x_2 is fixed if and only if $s_1\sigma\tau^{k\ell+1} = s_1 s_2 = 1$, that is $s_1 = s_2$.
- x_4 is fixed if and only if $\tau^2 = 1$. That is $\tau = \pm 1$.
- x_3 is fixed if and only if $\sigma^2\tau^{2k} = \tau^{-2k\ell-2+2k} = \tau^{-2(k(\ell-1)+1)} = 1$. Note that

  $$\gcd(k\ell+1, k(\ell-1)+1) = 1 \quad \text{and} \quad \gcd(\ell-1, k(\ell-1)+1) = 1,$$

  so the only way we can simultaneously solve $\tau^{2(k\ell+1)(\ell-1)} = 1$ and $\tau^{-2(k(\ell-1)+1)} = 1$ is if $\tau^2 = 1$. This means that x_3 is fixed if and only if $\tau = \pm 1$ (if and only if x_4 is also fixed).

s_1	s_2	τ^2	$\tau^{2(\ell-1)}$	fixed variables	$\#\gamma$
1	1	1	1	0 1 2 3 4	1
		$\neq 1$	1	0 1 2	$\ell-2$
			$\neq 1$	1 2	
1	-1	1	1	0 3 4	1
		$\neq 1$	1	0	$\ell-2$
			$\neq 1$		$k\ell(\ell-1)$
-1	1	1	1	0 1 3 4	
		$\neq 1$	1	0 1	
			$(-1)^\ell$	0 1	
			else	1	
-1	-1	1	1	0 2 3 4	
		$\neq 1$	1	0 2	
			$(-1)^\ell$	0 2	
			else	2	

The table before enumerates the possibilities for combinations of fixed variables and the counts[(6)] of $\gamma \in \ker\chi$ which fix this combination of variables (we omit the $\#\gamma$ data for any combinations which turn out not to contribute to HH^*; in particular this allows us to ignore the distinction between $\ell(k\ell+1)$ even/odd). $\square$

[(6)] Recall that if we count elements of $\ker(\chi \circ T)$ then we overcount by a factor of 2. We have removed this factor of 2 in the table.

We pick the monomial basis $x_3^a x_4^b$, $0 \leqslant a \leqslant \ell - 1$, $0 \leqslant b \leqslant k(\ell-1)-1$ for the Jacobian ring of $\mathbf{w}|_{x_{j_1}} = \cdots = x_{j_{\ell-k}} = 0$ when x_3 and x_4 are fixed and the monomial basis 1 when they are not.

Theorem 3.13. — *If* $\mathbf{w} = x_1^2 + x_2^2 + x_3x_4(x_3^{\ell-1} + x_4^{k(\ell-1)})$ *then* $\mathrm{HH}^*(\mathbb{A}^5, \Gamma_{\mathbf{w}}, \mathbf{w})$ *satisfies*

$$\dim \mathrm{HH}^3 = (k\ell+1)(\ell-1), \qquad \dim \mathrm{HH}^d = \ell \text{ for } d \leqslant 1$$

and $\dim \mathrm{HH}^d = 0$ *for* $d = 2$ *and* $d \geqslant 4$.

The HH^* *contributions for these singularities are given by the following table.*

Monomial	*Type*	*Degree in* HH^*	*Number of contributions*
$x_0^\vee x_1^\vee x_2^\vee x_3^\vee x_4^\vee$	C	3	$k\ell(\ell-1)$
$x_0^\vee x_1^\vee x_2^\vee x_3^\vee x_4^\vee$	B	3	$\ell-1$
$x_0^{(k\ell+1)p+q\ell+r} v x_3^r x_4^{q(\ell-1)+r}$	A	$-2(k+1)p-2q$	$\left\{\begin{array}{l} 0 \leqslant q \leqslant k-1 \\ 0 \leqslant r \leqslant \ell-1 \\ p \geqslant 0 \end{array}\right.$
$x_0^{(k\ell+1)p+k\ell} v x_3^{\ell-1}$	A	$-2(k+1)p-2k$	$p \geqslant 0$
$x_0^{(k\ell+1)p+k\ell} v x_4^{k(\ell-1)}$	A	$-2(k+1)p-2k$	$p \geqslant 0$
$x_0^{(k\ell+1)p+k\ell} v x_3^\vee x_4^\vee$	A	$-2(k+1)p-2k$	$\ell-2$, $p \geqslant 0$
Type B contributions in HH^{d+1} *for each type A monomial contributing to* HH^d			

In this table, we have written

$$v = \begin{cases} 1 & \text{if } b_0 = 0 \bmod 2, \\ x_1^\vee x_2^\vee & \text{if } b_0 = 1 \bmod 2. \end{cases}$$

Proof. — For each γ-monomial m, let $b_0, \ldots, b_4$ be the total exponents of $x_0, \ldots, x_4$ in m. This monomial transforms under $T(s,\sigma,\tau)$ as

$$s^{b_0+b_2}\sigma^{-2b_0+b_1+b_2+2b_3}\tau^{-2b_0(k+1)+(k\ell+1)(b_1+b_2)+2kb_3+2b_4},$$

which agrees with $(\chi \circ T)^{\otimes u}(s,\sigma,\tau)$ for all (s,σ,τ) if and only if

$$b_0 = b_2 \bmod 2, \tag{3.1}$$

$$b_1 + b_2 + 2b_3 = 2b_0 + 2u \bmod 2(\ell-1), \tag{3.2}$$

$$(k\ell+1)(b_1+b_2) + 2kb_3 + 2b_4 = 2(k\ell+1)u + 2b_0(k+1). \tag{3.3}$$

Reducing Equation (3.2) modulo 2 tells us that $b_1 = b_2 \bmod 2$. For $i = 1, 2$, the only possibilities for b_i are 0 (if x_i is fixed) or -1 (if x_i is not fixed). Thus, if (γ, m) contributes to HH^* then either x_1 and x_2 are both fixed or neither is fixed. This immediately rules out the contributions from γ with fixed variables $\{1\}$, $\{2\}$, $\{0,1\}$, $\{0,2\}$ $\{0,1,3,4\}$, $\{0,2,3,4\}$. Moreover, if x_2 is fixed then $b_2 = 0$ so $b_0 = 0$ so x_0 must also be fixed (or else we would have $b_0 = -1$). This rules out contributions with fixed variables $\{1,2\}$.

We now dispose of the type C contributions. These come from $(\gamma, x_0^\vee x_1^\vee x_2^\vee x_3^\vee x_4^\vee) \in \mathrm{HH}^3$ where γ fixes no variables; there are $k\ell(\ell-1)$ of these.

Since $b_1 = b_2 \bmod 2$ and $b_1, b_2 \in \{0, -1\}$, Equations (3.2) and (3.3) become

$$b_3 = b_0 + u - b_1 \bmod \ell - 1 \tag{3.4}$$

$$kb_3 + b_4 = k(\ell-1)(u-b_1) + (k+1)(b_0+u-b_1). \tag{3.5}$$

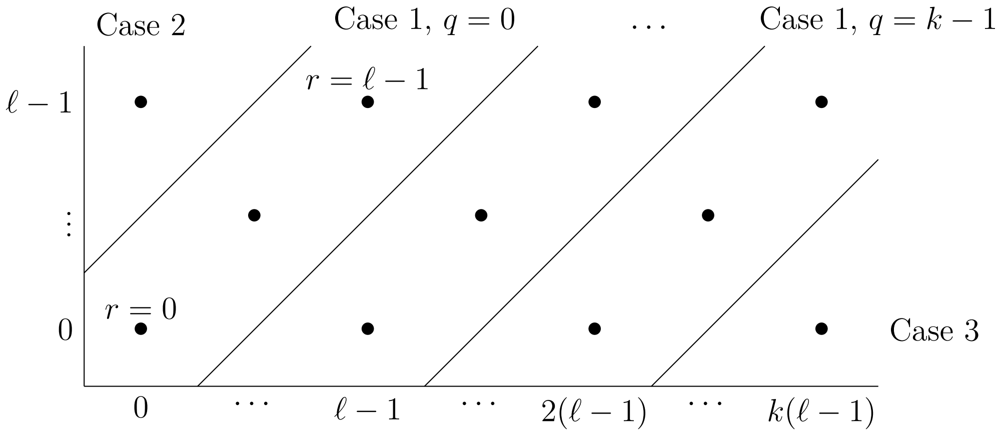
Reducing (3.5) modulo $\ell - 1$ yields

$$b_4 = u - b_1 + b_0 = b_3 \bmod \ell - 1.$$

We distinguish the following cases:

(**Case 1**) $b_3 = r$, $b_4 = q(\ell-1) + r$ with $q = 0, 1, \ldots, k-1$ and $r = 0, 1, \ldots, \ell - 1$.
(**Case 2**) $b_3 = \ell - 1$, $b_4 = 0$
(**Case 3**) $b_3 = 0$, $b_4 = k(\ell-1)$
(**Case 4**) $b_3 = b_4 = -1$.

We illustrate Cases 1–3 in the diagram below for $\ell = 2$, $k = 3$:



In what follows, we let $d = \gcd(\ell-1, k+1) = \gcd(k+1, k\ell+1) = \gcd(\ell-1, k\ell+1)$ and define x, y, z by

$$k + 1 = dx, \quad k\ell + 1 = dy, \quad \ell - 1 = dz.$$

We will focus on type A contributions (there will be corresponding type B contributions obtained by multiplying with $x_0 x_0^\vee$).

In **Case 1**, Equation (3.5) becomes

$$(\ell-1)q + (k+1)r = k(\ell-1)(u-b_1) + (k+1)(b_0+u-b_1),$$

so $k(u-b_1) = q - sx$, $b_0 + u - b_1 = r + sz$ for some integer s. Equation (3.4) tells us that $r + sz = r \bmod \ell - 1$, so $s = dP$ for some integer P. If we write $P = kp + q$ for some p, we get

$$k(u-b_1) = q - sx = q - dPx = q - dxkp - dxq = -k(q + (k+1)p),$$

giving $u - b_1 = -(q + (k+1)p)$, and

$$\begin{aligned} b_0 &= r + sz - (u - b_1) \\ &= r + dz(kp + q) + q + (k+1)p \\ &= r + kp(\ell - 1) + q(\ell - 1) + q + (k+1)p. \\ &= q\ell + r + (k\ell + 1)p. \end{aligned}$$

This yields a contribution of

$$x_0^{(k\ell+1)p+q\ell+r} v x_3^r x_4^{q(\ell-1)+r} \in \mathrm{HH}^{-2((k+1)p+q)} \text{ where } v = \begin{cases} 1 & \text{if } b_0 = 0 \bmod 2, \\ x_1^\vee x_2^\vee & \text{if } b_0 = 1 \bmod 2. \end{cases}$$

In each case, there is precisely one γ contributing this monomial (according to whether the fixed variables are $\{0,1,2,3,4\}$ or $\{0,3,4\}$). There are ℓ contributions in each degree (as r varies) and we get every degree congruent to $-2q \bmod 2(k+1)$ for $q = 0, 1, \ldots, k-1$, that is, HH^d has rank ℓ for every even $d \neq -2k \bmod 2(k+1)$, $d \leqslant 0$. The corresponding type B contributions give $\dim \mathrm{HH}^d = \ell$ for every odd $d \neq 1 - 2k \bmod 2(k+1)$, $d \leqslant 1$.

In **Case 2** and **Case 3**, $kb_3 + b_4 = k(\ell - 1)$, so

$$k(\ell - 1) = k(\ell - 1)(u - b_1) + (k+1)(b_0 + u - b_1),$$

which implies

$$k(u - b_1) = k - sx, \quad b_0 + u - b_1 = sz$$

for some s. As before, Equation (3.4) implies $s = dP$, so $k(u - b_1) = k - P(k+1)$. This means $P = kp$ for some p, so $u - b_1 = 1 - (k+1)p$ and $b_0 = (k\ell + 1)p - 1$. Thus we get contributions

$$x_0^{(k\ell+1)p-1} vw \in \mathrm{HH}^{-2(k+1)p+2} \text{ where } w \in \left\{x_3^{\ell-1}, x_4^{k(\ell-1)}\right\}$$
$$\text{and } v = \begin{cases} 1 & \text{if } b_0 = 0 \bmod 2, \\ x_1^\vee x_2^\vee & \text{if } b_0 = 1 \bmod 2. \end{cases}$$

In both cases there is precisely one γ contributing this monomial (according to whether the fixed variables are $\{0,1,2,3,4\}$ or $\{0,3,4\}$). This gives two contributions in every even degree $d = 2 \bmod 2(k+1)$, $d \leqslant -2k$.

Finally, in **Case 4** we have $kb_3 + b_4 = -(k+1)$, which yields

$$u - b_1 = -(k+1)p, \quad b_0 = (k\ell + 1)p - 1,$$

and we get a contribution

$$x_0^{(k\ell+1)p-1} v x_3^\vee x_4^\vee \in \mathrm{HH}^{-2(k+1)p+2} \text{ where } v = \begin{cases} 1 & \text{if } b_0 = 0 \bmod 2, \\ x_1^\vee x_2^\vee & \text{if } b_0 = 1 \bmod 2. \end{cases}$$

In both cases, there are $\ell - 2$ elements γ contributing these monomials (according to whether the fixed variables are $\{0,1,2\}$ or $\{0\}$). Together with the contributions from Cases 2 and 3, this yields $\dim \mathrm{HH}^d = \ell$ for every even $d = 2 \bmod 2(k+1)$, $d \leqslant -2k$. The corresponding type B contributions give $\dim \mathrm{HH}^d = \ell$ in every odd degree $d = 3 \bmod 2(k+1)$, $d \leqslant 3$.

Altogether, we get $\dim \mathrm{HH}^d = \ell$ if $d \leqslant 0$ and $\dim \mathrm{HH}^3 = (k\ell + 1)(\ell - 1)$. $\square$

4. Bigrading

4.1. Scale-equivalence of bigradings

In this section, we need to work over $\mathbb{C}$ (or at least an algebraically closed field of characteristic zero).

Definition 4.1. — *A $\mathbb{Z} \times \mathbb{C}$-grading on a vector space V (or* bigrading *for short) is a decomposition*

$$V = \bigoplus_{(p,q) \in \mathbb{Z}\times\mathbb{C}} V^{p,q}.$$

Two $\mathbb{Z} \times \mathbb{C}$-graded vector spaces $V = \bigoplus V^{p,q}$ and $W = \bigoplus W^{p,q}$ are scale-equivalent *if there is a nonzero $c \in \mathbb{C}$ such that $\dim(V^{p,q}) = \dim(W^{p,cq})$ for all p, q.*

Our contact invariant will be a scale-equivalence class of $\mathbb{Z} \times \mathbb{C}$-graded vector spaces (in fact, we will be able to find a representative which takes values in $\mathbb{Z} \times \mathbb{Z}$). We now explain how to construct a $\mathbb{Z} \times \mathbb{C}$-graded vector spaces out of a certain class of Gerstenhaber algebras.

4.2. Bigradings from Gerstenhaber algebras

Let $\mathfrak{g}^*$ be a Gerstenhaber algebra over $\mathbb{C}$; in particular, there is a Gerstenhaber bracket $[\cdot,\cdot]$ on $\mathfrak{g}^*$ satisfying:

$$[x, y] = (-1)^{|x||y|}[y, x], \quad (-1)^{|x||z|}[[x, y], z]+(-1)^{|y||x|}[[y, z], x]+(-1)^{|z||y|}[[z, x], y] = 0$$

The subset $\mathfrak{g}^1 \subset \mathfrak{g}^*$ is a complex Lie algebra and the bracket gives a representation $\rho^d \colon \mathfrak{g}^1 \to \mathfrak{gl}(\mathfrak{g}^d)$ for each d. We will assume that each graded piece of $\mathfrak{g}^*$ is finite-dimensional.

Let $\mathfrak{h} \subset \mathfrak{g}^1$ be a Cartan subalgebra, that is a nilpotent, self-normalising subalgebra. A Cartan subalgebra exists and is unique up to automorphisms of $\mathfrak{g}^1$; for example, you can construct one by taking the generalised 0-eigenspace of a regular element (an element $\zeta \in \mathfrak{g}^1$ is regular if the generalised 0-eigenspace of ad_ζ has the least possible dimension). If $\rho \colon \mathfrak{g}^1 \to \mathfrak{gl}(V)$ is a finite-dimensional complex representation then we get a weight-space decomposition $V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha$ where

$$V^\alpha := \left\{v \in V \;:\; (\rho(H) - \alpha(H))^N v = 0 \text{ for some } N\right\}.$$

In other words, V^α is a simultaneous generalised eigenspace for $\{\rho(H) \;:\; H \in \mathfrak{h}\}$, with eigenvalues $\alpha(H)$. The weight-space decomposition $\mathfrak{g}^1 = \bigoplus_\alpha \mathfrak{g}^{1,\alpha}$ of the adjoint representation has $\mathfrak{h} = \mathfrak{g}^{1,0}$.

If $\mathfrak{h}$ has rank 1 then we have $\mathfrak{h}^* \cong \mathbb{C}$. If we pick such an identification then the weight-space decomposition gives us a $\mathbb{Z} \times \mathbb{C}$-bigrading $\mathfrak{g}^* = \bigoplus_{p,q} \mathfrak{g}^{p,q}$. Changing our identification $\mathfrak{h}^* \cong \mathbb{C}$ yields a scale-equivalent $\mathbb{Z} \times \mathbb{C}$-grading.

Example 4.2. — Let A^* be a $\mathbb{Z}$-graded associative algebra and suppose that its Hochschild cohomology $\mathrm{HH}^*(A,A)$ has finite dimension in each degree. The Hochschild cochains can be given an additional $\mathbb{Z}$-grading so that a graded multilinear map $A^{\otimes p} \to A[-q]$ contributes to $\mathrm{HH}^{p,q}(A,A) \subset \mathrm{HH}^{p+q}(A,A)$. This $\mathbb{Z}\times\mathbb{Z}$-bigrading fits into our setting above. We write $\mathrm{HH}^*(A,A) \cong \bigoplus_{p,q} \mathrm{HH}^{p,q}(A,A)$. There is an element $\mathfrak{eu} \in \mathrm{CC}^{1,0}(A,A)$ defined on the graded piece A^q by $\mathfrak{eu}(a) = qa$. This is a Hochschild cocycle and defines a class (which we also write as $\mathfrak{eu}$) in $\mathrm{HH}^{1,0}(A,A)$. This satisfies $[\mathfrak{eu}, c] = qc$ for $c \in \mathrm{CC}^{*,q}(A,A)$. In particular, the generalised[7] 0-eigenspace of $\mathrm{ad}_{\mathfrak{eu}}$ is $\mathrm{HH}^{1,0}(A,A)$. If $\mathfrak{eu}$ is a regular element of the Lie algebra $\mathrm{HH}^1(A,A)$ then $\mathrm{HH}^{1,0}(A,A)$ is a Cartan subalgebra. In particular, if $\dim(\mathrm{HH}^{1,0}(A,A)) = 1$ then $\mathfrak{eu}$ is necessarily regular and we can take $\mathfrak{h} = \mathrm{HH}^{1,0}(A,A)$. In this case, if we identify $\mathfrak{h}^*$ with $\mathbb{C}$ by sending $\mathfrak{eu}^*$ to 1 then the weight decomposition gives us the usual bigrading $\mathrm{HH}^*(A,A) \cong \bigoplus_{p,q} \mathrm{HH}^{p,q}(A,A)$.

4.3. Bigradings on symplectic cohomology

If V is a Liouville domain with $c_1(V) = 0$, the symplectic cohomology $\mathrm{SH}^*(V)$ is a Gerstenhaber algebra. We will sketch how the bracket is defined; for more detail, see [Sei14, Section 4] or [Abo15, Section 2.5.1]. The bracket $[x,y]$ is defined by

$$[x,y] = \bigoplus_z \left(\sharp\mathcal{M}(z;x,y,H,J)\right) z,$$

where $\mathcal{M}(z;x,y,J)$ is the moduli space of solutions $u\colon \Sigma \to \widehat{V}$ to Floer's equation

$$(du + X_H \otimes \beta)^{0,1} = 0$$

where:

- $\widehat{V}$ is the symplectic completion of V;
- Σ is a pair-of-pants $\mathbb{CP}^1 \setminus \{0,1,\infty\}$, where we consider $0,1$ to be positive punctures and ∞ as a negative puncture;
- we equip Σ with a 1-parameter family of positive/negative cylindrical ends, specified by asymptotic markers which rotate once for each puncture. As the parameter varies from 0 to 2π, the markers at the positive punctures rotate once clockwise and the marker at the negative puncture rotates once anticlockwise;
- β is a subclosed 1-form on Σ compatible with the cylindrical ends;
- u has asymptotes x, y, z respectively at the punctures 0, 1, ∞.

The bracket has degree -1, that is

$$|x| + |y| = |z| + 1,$$

where the degree is related to the Conley–Zehnder index by $|x| = n - \mu_{CZ}(x)$. Equivalently,

$$n = \mu_{CZ}(x) + \mu_{CZ}(y) - \mu_{CZ}(z) + 1.$$

(7) Since $\mathrm{ad}_{\mathfrak{eu}}$ is semisimple on the level of cochains, it remains semisimple in its action on cohomology, so generalised eigenspaces are actual eigenspaces.

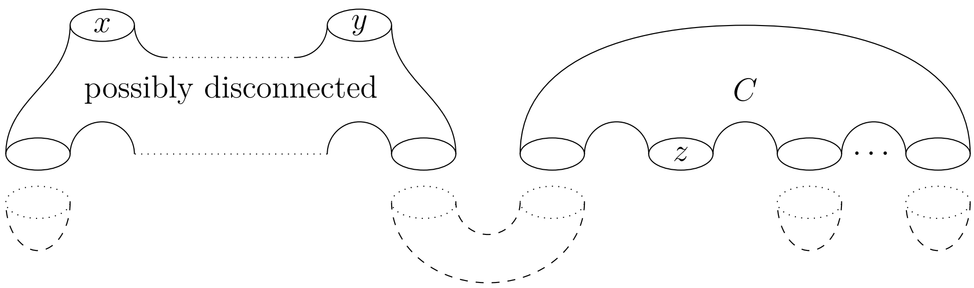
Lemma 4.3. — *Let V be a $2n$-dimensional Liouville domain with simply-connected boundary and suppose that there is a contact form on $Y = \partial V$ such that every Reeb orbit γ on Y satisfies the inequality*

$$\mu_{CZ}(\gamma) \geqslant \max(5-n, n-1).$$

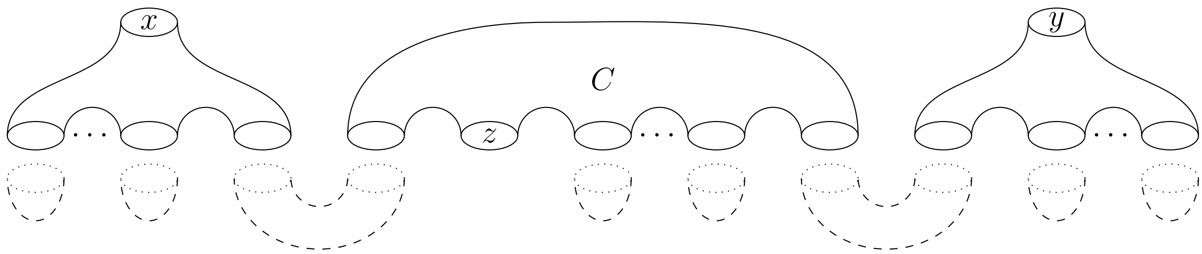
If x, y, z are Reeb orbits then there exists a J such that any $u \in \mathcal{M}(z; x, y, H, J)$ avoids the interior of V, that is, every u stays in the cylindrical end $\widehat{V} \setminus V$.

Proof. — Suppose this is not true. Pick a neck-stretching sequence of almost complex structures J_k around Y and assume our Hamiltonian is constant on the neck as in [CO18, Figure 8] so that our solutions to Floer's equation are genuinely holomorphic in that region and the standard SFT analysis of neck-stretching applies. Suppose we have a sequence of curves $u_k \in \mathcal{M}(z; x, y, H, J_k)$ which enter the interior of V. By the SFT compactness theorem, we can find a convergent subsequence which breaks into levels. There are several cases we need to consider.

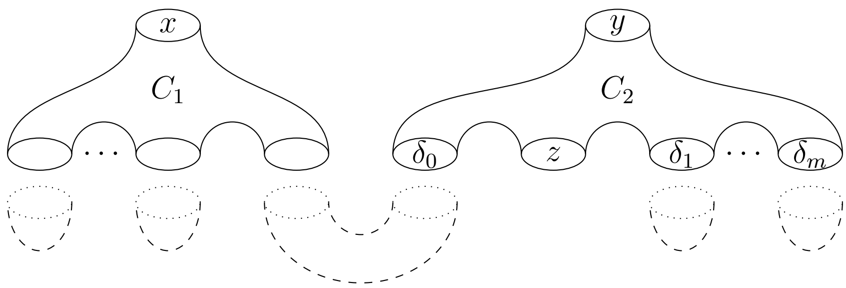
Case 1. — A break occurs along a separating curve parallel to z (and possibly other curves).



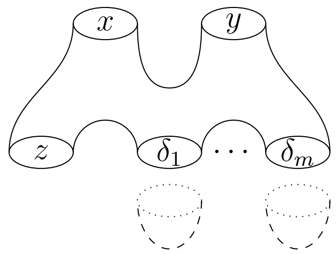
Case 2. — Not case 1, but a break occurs along separating curves parallel to x and to y.



Case 3. — Not cases 1–2, but a break occurs along a separating curve parallel to x (Case 3_y similar).



Case 4. — Not cases 1–3, but a break occurs along a contractible loop.



In Cases 1–2, we are left with a component C which violates the maximum principle (see also the argument from Bourgeois–Oancea [BO09, Proof of Proposition 5, Step 1] or an alternative argument based on action from Cieliebak–Oancea [CO18, Proof of Proposition 9.17]).

The argument for Case 3_x is inspired by [CO18, Appendix A] and [Ueb19, Lemma 3.13]. In this case, there are at least two components C_1 and C_2 in the SFT limit, where C_1 has x as a positive asymptote and C_2 has y as a positive asymptote. The component C_2 has a negative asymptote at z, a negative asymptote δ_0 which connects through lower levels to the component C_1, and possibly further negative asymptotes $\delta_1, \ldots, \delta_m$, which are capped off by planes in other levels. The index of C_2 is (we justify this in Remark 4.4 below):

$$\mu_{CZ}(y) - \mu_{CZ}(z) + 1 - \sum_{i=0}^{m}(\mu_{CZ}(\delta_i) + n - 3). \tag{4.1}$$

We have

$$\mu_{CZ}(y) - \mu_{CZ}(z) + 1 = n - \mu_{CZ}(x) \leqslant 1$$

because $\mu_{CZ}(x) \geqslant n-1$ by assumption. Moreover $\mu_{CZ}(\delta_i)+n-3 \geqslant 2$ by assumption, so $\mu_{CZ}(y)-\mu_{CZ}(z)+1-\sum_{i=0}^{m}\mu_{CZ}(\delta_i) \leqslant 1-2 = -1$, which contradicts the regularity of C_2.

The argument for Case 3_y is the same as for Case 3_x with the roles of x and y interchanged.

Case 4 yields a regular component C in the SFT limit which has punctures asymptotic to x, y and z as well as further negative punctures with asymptotes $\delta_1, \ldots, \delta_m$. The index of C is equal to the index of the original moduli space minus $\sum_{i=1}^{m}(\mu_{CZ}(\delta_i) + n - 3) \geqslant 2$, so becomes negative. This is a contradiction. □

Remark 4.4. — We now explain the index formula (4.1) from the proof. If we fix the positions of the punctures and all the asymptotic markers, the virtual dimension of this moduli space is (see Schwarz's thesis [Sch95, Theorem 3.3.11]):

$$\mu_{CZ}(y) - \mu_{CZ}(z) - \sum_{i=0}^{m}\mu_{CZ}(\delta_i) - n(m+1)$$

since $-m-1$ is the Euler characteristic of the domain. However, the bubbling/breaking which gives rise to the punctures at δ_i can happen anywhere, with any asymptotic marker, and the asymptotic markers on y and z can move in a 1-parameter family, so we get an additional $3(m+1)+1$, which gives Equation (4.1). Note that this is intermediate between the formula in Schwarz's thesis and the formula [Bou02, Corollary 5.4] from Bourgeois's thesis, where all punctures and markers are allowed to move.

Corollary 4.5. — *Suppose that (Y, ξ) is a $(2n-1)$-dimensional contact manifold which admits a contact form α for which every closed Reeb orbit γ satisfies*

$$\mu_{CZ}(\gamma) \geqslant \max(5-n, n-1).$$

Let V_1, V_2 be Liouville domains with $c_1(V_i) = 0$ and $\partial V_i = Y$. Suppose that V_i admits a Morse function with no critical points of index 1. Then (a) there is an isomorphism of Lie algebras $f^1\colon \mathrm{SH}^1(V_1) \to \mathrm{SH}^1(V_2)$, and (b) for each $d < 0$ there is an isomorphism $f^d\colon \mathrm{SH}^d(V_1) \to \mathrm{SH}^d(V_2)$ which intertwines the representations

$$\mathrm{ad}\colon \mathrm{SH}^1(V_i) \to \mathfrak{gl}(\mathrm{SH}^d(V_i)).$$

That is, for each $d < 0$, we have a commutative diagram:

$$\begin{tikzcd} \mathrm{SH}^1(V_1) \arrow[r, "\mathrm{ad}"] \arrow[d, "f^1"'] & \mathfrak{gl}\left(\mathrm{SH}^d(V_1)\right) \arrow[d, "f^d"] \\ \mathrm{SH}^1(V_2) \arrow[r, "\mathrm{ad}"'] & \mathfrak{gl}\left(\mathrm{SH}^d(V_2)\right) \end{tikzcd}$$

Proof. — Under the assumptions of the corollary, every element of $\mathrm{SH}^1(V_i)$ or of $\mathrm{SH}^d(V_i)$ with $d < 0$ can be represented using Reeb orbits for the contact form α (rather than critical points of a Morse function on the filling). These Reeb orbits lie in the cylindrical end of the symplectic completion $\widehat{V}_i$ (rather than in the filling), and these cylindrical ends are both symplectomorphic to the half-symplectisation $[0, \infty) \times Y$, so in a suitable cochain model of symplectic cohomology, we get identifications $f^d\colon SC^d(V_1) \to SC^d(V_2)$ when $d = 1$ or $d < 0$. Since the contact boundary is index-positive, we know that the differential on these cochain groups is independent of the filling [CO18, Prop. 9.17], which tells us that f^1 and f^d induce isomorphisms on cohomology.

By Lemma 4.3, we know there exist almost complex structures for which the Gerstenhaber bracket between these orbits does not involve any contributions from curves entering the filling. This implies that f^1 is an isomorphism of Lie algebras and that f^d intertwines the adjoint action of SH^1. □

4.4. Bigrading on $\mathrm{HH}^*(\mathrm{mf})$

In all our examples, we calculated $\mathrm{HH}^*(\mathrm{mf}(\mathbb{A}^{n+1}, \Gamma_{\mathbf{w}}, \mathbf{w}))$ and saw that $\mathrm{HH}^2 = 0$. Moreover, we saw in Lemma 2.8 that there is an intrinsically formal algebra B such that

$$\mathrm{HH}^*\left(\mathrm{mf}(\mathbb{A}^{n+1}, \Gamma_{\mathbf{w}}, \mathbf{w})\right) = \mathrm{HH}^*(B).$$

We now compute the usual algebra bigrading on $\mathrm{HH}^*(B)$ in terms of the γ-monomial contributions from Theorem 2.14.

Lemma 4.6. — *A γ-monomial m contributing to $\mathrm{HH}^d(B)$ contributes to the bigraded piece $\mathrm{HH}^{d-nb_0, nb_0}(B)$, where b_0 is the total exponent of x_0 in m.*

Remark 4.7. — Recall that $\mathrm{HH}^d = \bigoplus_q \mathrm{HH}^{d-q,q}$, so this is really just saying that the bigrading of m is nb_0.

Proof. — Consider the $\mathbb{G}_m$-action $t \cdot (x_0, \ldots, x_{n+1}) = (tx_0, \ldots, x_{n+1})$. Since this action leaves $\mathbf{w}$ invariant, its weights give a second grading on $\mathrm{HH}^*(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}))$. Theorem 2.14 comes from an isomorphism between $\mathrm{HH}^*(\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}))$ and a suitable twisted Koszul cohomology group (whose generators are γ-monomials) [BFK14]. This isomorphism respects the $\mathbb{G}_m$-action, hence this additional grading is given by the total exponent of x_0 in the γ-monomials contributing to HH.

As in Section 2.3, let $\mathcal{S} = \bigoplus_i S_i$ be the generator of $\mathcal{F}(V)$ given by a direct sum of vanishing cycles. In [LU18, Theorem 4.2] it is shown that, we have a generator $\mathcal{S}$ of $\mathrm{mf}(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}\mathbf{w})$ mirror to $\mathcal{S}$, where $\mathcal{S}$ is the pushforward of a generator $\mathcal{E}$ of $\mathrm{mf}(\mathbb{A}^{n+1}, \Gamma_{\mathbf{w}}, \mathbf{w})$ under the inclusion $(x_1, \ldots, x_{n+1}) \mapsto (0, x_1, \ldots, x_{n+1})$. In particular, $\mathcal{S}$ is $\mathbb{G}_m$-invariant. Using this, [LU18, Theorem 4.2] shows that the endomorphism A_∞-algebra $\mathcal{B} = \mathrm{end}(\mathcal{S})$ is a formal algebra and the grading on $B = \mathrm{H}(\mathcal{B})$ is n times the weight of the $\mathbb{G}_m$-action. Therefore, the $\mathbb{G}_m$-weight on B can be understood in terms of the grading of the algebra B. Indeed, we see it as the weight decomposition for ad_b associated to the derivation $b = n \cdot \mathfrak{eu}$, where $\mathfrak{eu} \in \mathrm{HH}^1$ is the Euler derivation from Example 4.2. □

In fact, in all of our examples we have $\dim \mathrm{HH}^{1,0} = 1$, which means, as in Example 4.2, that the weight decomposition for the representation $\mathrm{ad}\colon \mathrm{HH}^1 \to \bigoplus_d \mathfrak{gl}(\mathrm{HH}^d)$ gives a $\mathbb{Z} \times \mathbb{C}$-bigrading which is scale-equivalent to the algebra bigrading, and hence to the bigrading by the total exponent of x_0 by Lemma 4.6.

4.5. Proof of Theorem 1.13

Let X be a cDV singularity and let V be its Milnor fibre. Let μ be the Milnor number of X. By [Mil68, Theorem 6.6], the Milnor fibre admits a Morse function with precisely one minimum and μ critical points of index 3; in particular, none of index 1. Since X is terminal, McLean's theorem [McL16, Theorem 1.1] tells us that there exists a contact form for which every closed Reeb orbit γ satisfies $\mu_{CZ}(\gamma) \geqslant 2\,\mathrm{md}(X) = 2$, where $\mathrm{md}(X)$ is the minimal discrepancy of X, which equals 1 by a theorem of Markushevich [Mar96]. Since $n = 3$, we have $\max(5-n, n-1) = 2$, so that all the assumptions of Corollary 4.5 are satisfied.

Consider the Lie algebra $\mathrm{SH}^1(V)$ and its representation $\bigoplus_{d<0} \mathrm{SH}^d(V)$ (where $\mathrm{SH}^1(V)$ acts by the Gerstenhaber bracket). By Corollary 4.5, this Lie algebra representation is a contact invariant of the link of X.

We know that Conjecture 2.2 holds for all our Brieskorn–Pham and cA_n examples, and we are going to assume that it holds for the Laufer examples too. By Theorem 2.6, this tells us that, if V is the Milnor fibre of $\check{\mathbf{w}}$, then

$$\mathrm{SH}^*(V) \cong \mathrm{HH}^{*}\left(\mathrm{mf}\left(\mathbb{A}^{n+2}, \Gamma_{\mathbf{w}}, \mathbf{w}\right)\right)$$

as Gerstenhaber algebras. Therefore, the contact invariant Lie algebra representation is equivalent to the representation $\mathrm{ad}\colon \mathrm{HH}^1(B) \to \bigoplus_{d<0} \mathfrak{gl}(\mathrm{HH}^d(B))$ discussed in

Section 4.4. In particular, this gives a $\mathbb{Z}\times\mathbb{C}$-grading on $\bigoplus_{d<x0}\mathrm{SH}^d(V)$ which we can compute in terms of the x_0-powers of the contributing γ-monomials by Lemma 4.6.

We now show that, for all of our examples, these scale-equivalence classes of $\mathbb{Z}\times\mathbb{C}$-gradings distinguish the contact structures.

4.5.1. $\ell = 1$

In this case we need to distinguish the contact structures $\{\alpha_{1,k} \;:\; k=1,2,3,\ldots\}$ and $\{\lambda_{1,k} \;:\; k=1,2,3,\ldots\}$ on $S^2\times S^3$.

The unique contribution to HH^{-2} is:

$$\begin{cases} x_0x_1^\vee x_2^\vee x_3^\vee x_4^\vee & \text{for } \alpha_{1,1}, \\ x_0^2x_4^2 & \text{for } \alpha_{1,k},\ (k\geqslant 2), \end{cases} \qquad x_0^4x_1x_2^3 \text{ for } \lambda_{1,k}.$$

To compare the $\mathbb{Z}\times\mathbb{C}$-gradings, we rescale to ensure $\mathrm{SH}^{-2,4}\neq 0$ in all cases. The $\mathbb{C}$-bigrading of a monomial $x_0^{b_0}\cdots x_4^{b_4}\in\mathrm{HH}^d$ is therefore given by:

$$\begin{cases} 4b_0 & \text{for } \alpha_{1,1}, \\ 2b_0 & \text{for } \alpha_{1,k}, (k\geqslant 2), \end{cases} \qquad b_0 \text{ for } \lambda_{1,k}.$$

The unique contribution to HH^{-4} is:

$$\begin{cases} x_0^4\in\mathrm{SH}^{-4,16} & \text{for } \alpha_{1,1}, \\ x_0^3x_1^\vee x_2^\vee x_3^\vee x_4^\vee\in\mathrm{SH}^{-4,6} & \text{for } \alpha_{1,2}, \\ x_0^4x_4^4\in\mathrm{SH}^{-4,8} & \text{for } \alpha_{1,k}, (k\geqslant 3), \end{cases} \qquad x_0^6x_2^4\in\mathrm{SH}^{-4,6} \text{ for } \lambda_{1,k}.$$

This already distinguishes $\alpha_{1,1}$ from everything, $\alpha_{1,2}$ from the other αs, and the λs from the $\alpha_{1,k}$, $k\neq 2$.

To distinguish $\lambda_{1,k}$ from $\lambda_{1,K}$ with $k<K$, observe that the unique contribution to SH^{-4k-2} is $x_0^{6k+2}x_1^2\in\mathrm{SH}^{-4k-2,6k+2}$ respectively $x_0^{6k+4}x_1x_2^{4k+3}\in\mathrm{SH}^{-4k-2,6k+4}$.

To distinguish $\alpha_{1,k}$ from $\alpha_{1,K}$ with $2\leqslant k<K$, observe that the unique contribution to SH^{-2k} is $x_0^{2k-1}x_1^\vee x_2^\vee x_3^\vee x_4^\vee\in\mathrm{SH}^{-2k,4k-2}$ respectively $x_0^{2k}x_4^{2k}\in\mathrm{SH}^{-2k,4k}$.

To distinguish $\alpha_{1,2}$ from $\lambda_{1,k}$, $k\geqslant 2$, observe that the unique contribution to SH^{-6} is $x_0^4\in\mathrm{SH}^{-6,8}$ respectively $x_0^{10}x_1x_2^2\in\mathrm{SH}^{-6,10}$.

To distinguish $\alpha_{1,2}$ from $\lambda_{1,1}$, observe that the unique contribution to SH^{-8} is $x_0^6x_2^2\in\mathrm{SH}^{-8,12}$ respectively $x_0^{10}\in\mathrm{SH}^{-8,10}$.

4.6. $\ell\geqslant 2$

The contact structures $\xi_{\ell,k}$ in Theorem 1.13 live on the manifold $\sharp_\ell(S^2\times S^3)$. We can see from the tables in Theorems 3.7, 3.10 and 3.13 that for any of $\alpha_{\ell,k}$, $\beta_{\ell,k}$, $\delta_{4,k}$, $\lambda_{\ell,k}$, $\epsilon_{6,k}$, $\epsilon_{8,k}$, the symplectic cohomology SH^d, $d<0$, is supported in a single $\mathbb{C}$-bigrading if and only if $d=-2k$ or $1-2k \bmod 2(k+1)$. Therefore, the only possibility for two contact structures $\xi_{\ell,k}$, $\ell\geqslant 2$, to agree is for the indices ℓ and k to agree.

We also see that, if we bigrade by the total exponent of x_0, SH^1 is supported in bidegrees $0, 1, 2, \ldots, \ell - 1$. This is enough to fix our $\mathbb{Z} \times \mathbb{C}$-grading completely up to scale so that, in all cases, the $\mathbb{C}$-bigrading coincides with the total exponent of x_0.

To distinguish $\alpha_{\ell,k}$ from $\beta_{\ell,k}$ when $k \neq 1$ (the singularities are locally analytically equivalent when $\ell = 1$), observe that the contributions to SH^{-2k} have total x_0 exponent $k\ell$ respectively $k(\ell+1) - 1$. These are different if $k \neq 1$.

To distinguish $\alpha_{4,k}$ and $\beta_{4,k}$ from $\delta_{4,k}$, note that the contributions to SH^{-2k} have total x_0 exponents respectively equal to $5k - 1$, $4k$ and $6k - 1$.

To distinguish $\alpha_{6,k}$ and $\beta_{6,k}$ from $\epsilon_{6,k}$, note that the contributions to SH^{-2k} have total x_0 exponents respectively equal to $7k - 1$, $6k$ and $12k - 1$.

To distinguish $\alpha_{8,k}$ and $\beta_{8,k}$ from $\epsilon_{8,k}$, note that the contributions to SH^{-2k} have total x_0 exponents respectively equal to $9k - 1$, $8k$ and $30k - 1$.

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