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SMOOTH BRANCH OF RAREFACTION PULSES FOR THE NONLINEAR SCHRÖDINGER EQUATION AND THE EULER–KORTEWEG SYSTEM IN 2D

BRANCHE RÉGULIÈRE D'ONDES DE RARÉFACTION POUR L'ÉQUATION DE SCHRÖDINGER NON LINÉAIRE ET LE SYSTÈME D'EULER–KORTEWEG EN 2D

Abstract. — We are interested in the construction of a smooth branch of travelling waves to the Nonlinear Schrödinger Equation and the Euler–Korteweg system for capillary fluids with nonzero condition at infinity. This branch is defined for speeds close to the speed of sound and looks qualitatively, after rescaling, as a rarefaction pulse described by the Kadomtsev–Petviashvili equation. The proof relies on a fixed point theorem based on the nondegeneracy of the lump solitary wave of the Kadomtsev–Petviashvili equation.

Keywords: Travelling waves, Nonlinear Schrödinger Equation, Euler–Korteweg system, Kadomtsev–Petviashvili equation, lump.
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Résumé. — Nous nous intéressons à la construction d'une branche régulière d'ondes progressives pour l'équation de Schrödinger non linéaire et le système d'Euler–Korteweg pour les fluides capillaires avec condition non nulle à l'infini. Cette branche est définie pour des vitesses proches de la vitesse du son et ressemblent qualitativement à des ondes de raréfaction décrites, après remise à l'échelle, par l'équation de Kadomtsev–Petviashvili. La démonstration repose sur un théorème de point fixe et sur la non-dégénérescence de l'onde solitaire de l'équation de Kadomtsev–Petviashvili appelée lump.

1. Introduction

The nonlinear Schrödinger equation (NLS)

$$i\frac{\partial\Psi}{\partial t} + \Delta\Psi = \Psi f\left(|\Psi|^2\right) \tag{NLS}$$

in $\mathbb{R}^d$ with nonzero condition at infinity appears in a variety of physical problems: condensed matter physics (see [Pis99]), Bose–Einstein condensates and superfluidity (cf. [AHM+03, RB01]), as well as nonlinear Optics (see [KLD98]). Depending on the physical problem, several nonlinearities may be of interest (see the examples and references quoted in [Chi12, CS16]). The most popular one is of cubic type and leads to the equation sometimes called the Gross–Pitaevskii equation:

$$i\frac{\partial\Psi}{\partial t} + \Delta\Psi = \Psi\left(|\Psi|^2 - 1\right).$$

At spatial infinity, we impose $|\Psi| \to \rho_0$, where $\rho_0 > 0$ is a constant such that $f(\rho_0) = 0$. Without loss of generality, we may scale so that $\rho_0 = 1$. At least formally, NLS is a hamiltonian flow, associated with the energy

$$\mathcal{E}(v) \stackrel{\text{def}}{=} \frac{1}{2}\int_{\mathbb{R}^d} |\nabla v|^2 + F\left(|v|^2\right)\, dx,$$

where $F(\rho) \stackrel{\text{def}}{=} \int_1^\rho f(\varrho)\, d\varrho$.

If Ψ is a solution of NLS which does not vanish, we may use the Madelung transform

$$\Psi = A\exp(i\phi)$$

and rewrite NLS as an hydrodynamical system with an additional quantum pressure

$$\begin{cases} \partial_t A + 2\nabla\phi\cdot\nabla A + A\Delta\phi = 0 \\ \\ \partial_t\phi + |\nabla\phi|^2 + f(A^2) - \dfrac{\Delta A}{A} = 0, \end{cases} \tag{1.1}$$

or

$$\begin{cases} \partial_t\rho + 2\nabla\cdot(\rho U) = 0 \\ \\ \partial_t U + 2U\cdot\nabla U + \nabla(f(\rho)) - \nabla\left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = 0 \end{cases}$$

with $(\rho, U) \stackrel{\text{def}}{=} (A^2, \nabla\phi)$. When neglecting the quantum pressure and linearizing this Euler type system around the particular trivial solution $\Psi = 1$ (or $(A, U) = (1, 0)$), we obtain the free wave equation

$$\begin{cases} \partial_t \bar{A} + \nabla \cdot \bar{U} = 0 \\ \\ \partial_t \bar{U} + 2f'(1)\nabla \bar{A} = 0 \end{cases}$$

with associated speed of sound

$$\mathfrak{c} \stackrel{\text{def}}{=} \sqrt{2f'(1)} > 0$$

provided $f'(1) > 0$, which we will assume throughout the paper. This means that the Euler system is hyperbolic in the region $\rho \simeq 1$, or that NLS is defocusing at least near $\rho \simeq 1$.

In this paper, we work only in space dimension two. We shall be interested in the construction of travelling waves for NLS, that is particular solutions of the form

$$\Psi(t, x) = u(x_1 - ct, x_2)$$

that should play some important role in the dynamics of NLS. The profile u then solves the elliptic PDE

$$-ic\frac{\partial u}{\partial x_1} + \Delta u = uf\left(|u|^2\right), \tag{TW$_c$}$$

with the condition $|u| \to 1$ for $|x| \to +\infty$. In the papers [JR82, JPR86], a study of the travelling waves for NLS (with cubic nonlinearity) is carried on, through numerical and formal computations.

Since then, several mathematical results have been established through variational methods: [BS99, (2d)], [BOS04, (3d and higher)], [Chi04, (3d and higher)], [BGS09, (2d and 3d)], [Mar13], [CM17, (2d and higher)], [BR23, (2d and 3d)]. On the other hand, the paper [CS16] provides some numerical study of travelling waves (in 2d) for general nonlinearities, exhibiting some cusps, self-intersections of curves in the energy-momentum diagram, etc., and in [CS18], some excited states have been put forward (in 2d, cubic nonlinearity).

1.1. The KP-I limit for the travelling waves

The formal convergence to the (KP-I) solitary wave in dimensions $d = 2$ or $d = 3$ is given in [JR82] (see also [IS78], and [KP00] in the context of nonlinear optics) for the Gross–Pitaevskii equation (i.e. NLS with $f(\varrho) = \varrho - 1$), where the speed of sound is $\mathfrak{c} = \sqrt{2}$. This limit arises for speeds c close to the speed of sound $\mathfrak{c}$. The argument is as follows. For some small parameter ε, we define the speed $c = c(\varepsilon)$ by the relation

$$c(\varepsilon) \stackrel{\text{def}}{=} \sqrt{\mathfrak{c}^2 - \varepsilon^2} \in (0, \mathfrak{c})$$

and then insert the ansatz

$$u(x) = \left(1 + \varepsilon^2 A_\varepsilon(z)\right) \exp\left(i\varepsilon\phi_\varepsilon(z)\right) \qquad z_1 \stackrel{\text{def}}{=} \varepsilon x_1, \quad z_2 \stackrel{\text{def}}{=} \varepsilon^2 x_2 \tag{1.2}$$

in $(\mathrm{TW}_{c(\varepsilon)})$, cancel the phase factor and separate real and imaginary parts to obtain the system

$$\tag{1.3}\begin{cases} -c(\varepsilon)\partial_1 A_\varepsilon + 2\varepsilon^2\partial_1\phi_\varepsilon\partial_1 A_\varepsilon + 2\varepsilon^4\partial_2\phi_\varepsilon\partial_2 A_\varepsilon \\ \qquad + \left(1+\varepsilon^2 A_\varepsilon\right)\left(\partial_1^2\phi_\varepsilon + \varepsilon^2\partial_2^2\phi_\varepsilon\right) = 0 \\ -c(\varepsilon)\partial_1\phi_\varepsilon + \varepsilon^2(\partial_1\phi_\varepsilon)^2 + \varepsilon^4(\partial_2\phi_\varepsilon)^2 + \dfrac{1}{\varepsilon^2} f\left(\left(1+\varepsilon^2 A_\varepsilon\right)^2\right) \\ \qquad -\varepsilon^2\dfrac{\partial_1^2 A_\varepsilon + \varepsilon^2\partial_2^2 A_\varepsilon}{1+\varepsilon^2 A_\varepsilon} = 0. \end{cases}$$

On the formal level, if $A_\varepsilon \to A$ and $\phi_\varepsilon \to \phi$ as $\varepsilon \to 0$ in some reasonable sense, then, to leading order, we obtain $-\mathfrak{c}\partial_{z_1} A + \partial_{z_1}^2\phi = 0$ for the first equation of (1.3), and, using the Taylor expansion

$$f\left(\left(1+\varepsilon^2 A_\varepsilon\right)^2\right) = f(1) + \mathfrak{c}^2\varepsilon^2 A_\varepsilon + \mathcal{O}(\varepsilon^4)$$

with $f(1) = 0$, the second one implies $-\mathfrak{c}\partial_1\phi + \mathfrak{c}^2 A = 0$. In both cases, we obtain the constraint

$$\tag{1.4} \mathfrak{c}A = \partial_{z_1}\phi.$$

We now add $c(\varepsilon)/\mathfrak{c}^2$ times the first equation of (1.3) and $\partial_1/\mathfrak{c}^2$ times the second one. Using the Taylor expansion

$$\tag{1.5} f\left((1+\alpha)^2\right) = \mathfrak{c}^2\alpha + \left(\frac{\mathfrak{c}^2}{2} + 2f''(1)\right)\alpha^2 + f_3(\alpha),$$

with $f_3(\alpha) = \mathcal{O}(\alpha^3)$ as $\alpha \to 0$, this gives

$$\tag{1.6} \begin{aligned} &\frac{\mathfrak{c}^2 - c^2(\varepsilon)}{\varepsilon^2\mathfrak{c}^2}\,\partial_{z_1} A_\varepsilon - \frac{1}{\mathfrak{c}^2}\,\partial_{z_1}\left(\frac{\partial_{z_1}^2 A_\varepsilon + \varepsilon^2\partial_{z_2}^2 A_\varepsilon}{1+\varepsilon^2 A_\varepsilon}\right) + \frac{c(\varepsilon)}{\mathfrak{c}^2}\left(1+\varepsilon^2 A_\varepsilon\right)\Delta_{z_\perp}\phi_\varepsilon \\ &+ \left\{2\frac{c(\varepsilon)}{\mathfrak{c}^2}\partial_{z_1}\phi_\varepsilon\partial_{z_1} A_\varepsilon + \frac{c(\varepsilon)}{\mathfrak{c}^2}\,A_\varepsilon\partial_{z_1}^2\phi_\varepsilon + \frac{1}{\mathfrak{c}^2}\,\partial_{z_1}\left[(\partial_{z_1}\phi_\varepsilon)^2\right] + \left[\frac{1}{2} + 2\frac{f''(1)}{\mathfrak{c}^2}\right]\partial_{z_1}\left(A_\varepsilon^2\right)\right\} \\ &= -2\varepsilon^2\frac{c(\varepsilon)}{\mathfrak{c}^2}\partial_{z_2}\phi_\varepsilon\partial_{z_2} A_\varepsilon - \frac{\varepsilon^2}{\mathfrak{c}^2}\partial_{z_1}\left[(\partial_{z_2}\phi_\varepsilon)^2\right] - \frac{1}{\mathfrak{c}^2\varepsilon^4}\,\partial_{z_1}\left[f_3\left(\varepsilon^2 A_\varepsilon\right)\right]. \end{aligned}$$

It then follows that if $A_\varepsilon \to A$ and $\phi_\varepsilon \to \phi$ as $\varepsilon \to 0$ in a suitable sense, we can infer from (1.4) that $\partial_{z_1}^{-1} A = \phi/\mathfrak{c}$, and since $\mathfrak{c}^2 - c^2(\varepsilon) = \varepsilon^2$, (1.6) gives the solitary waves equation for the KP-I equation

$$\tag{SW} \frac{1}{\mathfrak{c}^2}\,\partial_{z_1} A - \frac{1}{\mathfrak{c}^2}\,\partial_{z_1}^3 A + \Gamma A\partial_{z_1} A + \partial_{z_2}^2\partial_{z_1}^{-1} A = 0.$$

Here, the coefficient Γ depends on f through the formula

$$\Gamma \stackrel{\text{def}}{=} 6 + \frac{4}{\mathfrak{c}^2} f''(1).$$

For mathematical results (variational characterization, decay at infinity) on the solitary waves of KP-I, see [S97] and [S96], as well as [WW96]. Complete justifications of the KP-I solitary wave limit for the travelling waves of GP have been given in [BGS08] for the two-dimensional cubic NLS equation and in [CM14] (for a general nonlinearity and in dimensions two and three). A weak version of the convergence results in space dimension two is the following. It relies on the existence of travelling waves shown in [BGS09] and [CM17] respectively by means of variational methods. By definition, a ground state for KP-I is a nontrivial solution $\mathcal{W}_*$ of SW which minimizes the action among all such nontrivial solution of SW. It has been shown in [S96, Lemma 2.1] that the KP-I equation possesses at least one ground state.

Theorem 1.1 ([BGS08, CM14]). — *Assume that the nonlinearity f is of class $\mathcal{C}^3$ near $\varrho = 1$ and that $\Gamma \neq 0$. Then, there exists a sequence (ε_j) tending to zero, a sequence (U_j) of travelling waves for NLS with speed $c(\varepsilon_j) = \sqrt{\mathfrak{c}^2 - \varepsilon_j^2}$, and a ground state $\mathcal{W}_*$ for KP-I such that*

$$U_j(x) = \left(1 + \varepsilon_j^2 A_j(z)\right) \exp\left(i\varepsilon_j \phi_j(z)\right), \qquad \text{where} \quad z_1 = \varepsilon_j x_1, \quad z_2 = \varepsilon_j^2 x_2,$$

and, as $j \to +\infty$ and for any $1 < p < \infty$,

$$A_j \to \mathcal{W}_*, \qquad \partial_{z_1} A_j \to \partial_{z_1}\mathcal{W}_*, \qquad \partial_{z_1}\phi_j \to \mathfrak{c}\mathcal{W}_* \quad \text{and} \quad \partial_{z_1}^2 \phi_j \to \mathfrak{c}\partial_{z_1}\mathcal{W}_*$$

in $W^{1,p}(\mathbb{R}^2)$.

1.2. A smooth branch of travelling waves associated with the Lump

The KP-I equation is integrable in space dimension 2, and explicit solitary waves solutions are then known. Due to their algebraic decay at infinity, they are called Lumps, and [MZB+77] gives the first explicit Lump, expected to be the ground state:

$$\mathcal{W}_1(z) \stackrel{\text{def}}{=} -24\frac{3 - z_1^2 + z_2^2}{\left(3 + z_1^2 + z_2^2\right)^2} = -24\partial_{z_1}\left(\frac{z_1}{3 + z_1^2 + z_2^2}\right) = \partial_{z_1}\phi_1,$$

which solves the adimensionnalized version of SW

$$\partial_{z_1}\mathcal{W}_1 - \partial_{z_1}^3\mathcal{W}_1 + \mathcal{W}_1\partial_{z_1}\mathcal{W}_1 + \partial_{z_2}^2\partial_{z_1}^{-1}\mathcal{W}_1 = 0.$$

Using the scaling properties of the KP-I equation, we then see that

$$\mathcal{W}_1^{\mathrm{sc}}(z) \stackrel{\text{def}}{=} \frac{1}{\mathfrak{c}^2\Gamma}\mathcal{W}_1\left(z_1, \frac{z_2}{\mathfrak{c}}\right) \tag{1.7}$$

solves SW.

Up to our knowledge, the conjecture that $\mathcal{W}_1$ is the unique ground state of KP-I is not proved yet. The article [LW19] has made a substantial progress in this direction, by showing that $\mathcal{W}_1$ is a non-degenerate solution of Morse index 1.

Theorem 1.2 ([LW19]). — *Suppose w is a smooth solution to the equation*

$$\partial_{z_1}^2 w - \partial_{z_1}^4 w + \partial_{z_1}^2(w\mathcal{W}_1) + \partial_{z_2}^2 w = 0$$

satisfying $w \to 0$ at infinity. Then, there exists ν_1, $\nu_2 \in \mathbb{R}$ such that

$$w = \nu_1\partial_{z_1}\mathcal{W}_1 + \nu_2\partial_{z_2}\mathcal{W}_1.$$

Furthermore, the linearized operator

$$\mathfrak{L} : w \mapsto w - \partial_{z_1}^2 w + w\mathcal{W}_1 + \partial_{z_2}^2 \partial_{z_1}^{-2} w$$

as an operator on the energy space $\mathcal{H}$ has exactly one negative eigenvalue. The energy space $\mathcal{H}$ is defined as the completion (in L^2) of $\partial_{z_1}\mathcal{C}_c^\infty(\mathbb{R}^2)$ for the norm

$$\|w\|_{\mathcal{H}}^2 = \int_{\mathbb{R}^2} w^2 + (\partial_{z_1} w)^2 + \left(\partial_{z_2}\partial_{z_1}^{-1} w\right)^2 \, dz.$$

In the numerical computations in [JR82] (and also [CS16]), when we approach the speed of sound $\mathfrak{c}$, the KP-I solitary wave we observe is always the $\mathcal{W}_1$ Lump, in agreement with the fact that $\mathcal{W}_1$ is presumably the unique ground state for KP-I (up to space translations). In [CS18], it has been numerically computed that the only negative eigenvalue of $\mathfrak{L}$ is ≈ -2.3539.

In the opposite limit $c \to 0$, where we expect travelling waves with vortices, several results of existence have been established *via* a Liapounov–Schmidt type reduction. This approach has originated in [PMK06] for the construction of stationary solutions to the Ginzburg–Landau model in a bounded domain, and relies on the nondegeneracy of the vortex of degree one shown in [dPFK04]. This has been extended in [CP23] and [CP21], where a smooth branch of travelling waves is constructed, in [LW20], where the existence of travelling waves of small speed with several vortices is shown, in [DPMMR21], where travelling waves with several vortex helices are constructed, and finally by J. Wei et al. to other Schrödinger type models.

Let us now state our main result. For $1 < p \leqslant \infty$, the Sobolev space

$$W^{1,p} \stackrel{\text{def}}{=} \left\{u \in L^p(\mathbb{R}^2) \text{ s.t. } \nabla u \in L^p(\mathbb{R}^2)\right\}$$

is naturally endowed with the norm

$$\|u\|_{W^{1,p}} \stackrel{\text{def}}{=} \|u\|_{L^p(\mathbb{R}^2)} + \|\nabla u\|_{L^p(\mathbb{R}^2)}.$$

Theorem 1.3. — *We assume that f is of class $\mathcal{C}^3$ near 1, with $\Gamma \neq 0$, and let $\mathcal{W}_1^{\rm sc}$ be defined by (1.7). Let $1 < p \leqslant \infty$ be given. There exist $\varepsilon_*(p) > 0$ small and a $\mathcal{C}^1$ mapping*

$$]0, \varepsilon_*(p)] \ni \varepsilon \mapsto (A_\varepsilon, \phi_\varepsilon) \in W^{1,p} \times W^{1,p+1}$$

such that:

(i) *for every $0 < \varepsilon \leqslant \varepsilon_*(p)$,*

$$U_{c(\varepsilon)}(x) \stackrel{\text{def}}{=} \left(1 + \varepsilon^2 A_\varepsilon(z)\right) \exp\left(i\varepsilon\phi_\varepsilon(z)\right), \qquad \text{where} \quad z_1 = \varepsilon x_1, \quad z_2 = \varepsilon^2 x_2,$$

is a travelling wave for NLS of speed $c(\varepsilon) = \sqrt{\mathfrak{c}^2 - \varepsilon^2}$ tending to 1 at infinity;

(ii) *when $\varepsilon \to 0$, there holds*

$$\|A_\varepsilon - \mathcal{W}_1^{\rm sc}\|_{W^{1,p}} + \left\|\phi_\varepsilon - \mathfrak{c}\partial_{z_1}^{-1}\mathcal{W}_1^{\rm sc}\right\|_{W^{1,p+1}} \leqslant C(p)\varepsilon^2 |\ln \varepsilon|^2 \to 0;$$

(iii) *for $c \in]c(\varepsilon_*(p)), \mathfrak{c}[$, we have the Hamilton group relation*

$$\frac{d}{dc} E(U_c) = c \frac{d}{dc} P(U_c)$$

with

$$P(U_c) \stackrel{\text{def}}{=} \frac{1}{2}\int_{\mathbb{R}^2} \frac{|U_c|^2-1}{|U_c|^2}\langle iU_c|\partial_{x_1}U_c\rangle\, dx = \frac{1}{2}\int_{\mathbb{R}^2}\langle i(U_c-1)|\partial_{x_1}U_c\rangle\, dx,$$

where $\langle\cdot|\cdot\rangle$ *denotes the real scalar product in* $\mathbb{C}$*;*

(iv) *when* $\varepsilon\to 0$*, there holds*

$$E\left(U_{c(\varepsilon)}\right) \sim \mathfrak{c}P\left(U_{c(\varepsilon)}\right) \sim \frac{\varepsilon}{\mathfrak{c}\Gamma^2}\int_{\mathbb{R}^2}\mathcal{W}_1^2\, dz = \varepsilon\frac{96\pi}{\mathfrak{c}\Gamma^2},$$
$$E\left(U_{c(\varepsilon)}\right) - c(\varepsilon)P\left(U_{c(\varepsilon)}\right) \sim \frac{\varepsilon^3}{3\mathfrak{c}^3\Gamma^2}\int_{\mathbb{R}^2}\mathcal{W}_1^2\, dz = \varepsilon^3\frac{32\pi}{\mathfrak{c}^3\Gamma^2}$$

and

$$E\left(U_{c(\varepsilon)}\right) - \mathfrak{c}P\left(U_{c(\varepsilon)}\right) \sim -\frac{\varepsilon^3}{6\mathfrak{c}^3\Gamma^2}\int_{\mathbb{R}^2}\mathcal{W}_1^2\, dz = -\varepsilon^3\frac{16\pi}{\mathfrak{c}^3\Gamma^2}.$$

Let us make a few remarks on this result.

Remark 1.4. — Actually, we shall not work with L^p estimates but with weighted L^∞ estimates. In particular, we have the stronger statements, for any $\sigma\in]0,1[$,

$$\left\|(1+|z|)^{1+\sigma}\left(A_\varepsilon - \mathcal{W}_1^{\mathrm{sc}}\right)\right\|_{L^\infty(\mathbb{R}^2)} + \left\|(1+|z|)^{\sigma}\left(\phi_\varepsilon - \mathfrak{c}\partial_{z_1}^{-1}\mathcal{W}_1^{\mathrm{sc}}\right)\right\|_{L^\infty(\mathbb{R}^2)} \leqslant C(\sigma)\varepsilon^2|\ln\varepsilon|^2$$

and

$$\left\|(1+|z|)^{2+\sigma}\nabla(A_\varepsilon - \mathcal{W}_1^{\mathrm{sc}})\right\|_{L^\infty(\mathbb{R}^2)} + \left\|(1+|z|)^{1+\sigma}\nabla\left(\phi_\varepsilon - \mathfrak{c}\partial_{z_1}^{-1}\mathcal{W}_1^{\mathrm{sc}}\right)\right\|_{L^\infty(\mathbb{R}^2)} \leqslant C(\sigma)\varepsilon^2|\ln\varepsilon|^2.$$

Remark 1.5. — If $1<p<p^\dagger\leqslant\infty$ are arbitrary exponents, it will follow form our arguments that, as expected, the mappings $]0,\varepsilon_*(p)]\ni\varepsilon\mapsto(A_\varepsilon,\phi_\varepsilon)\in W^{1,p}\times W^{1,p+1}$ and $]0,\varepsilon_*(p^\dagger)]\ni\varepsilon\mapsto(A_\varepsilon,\phi_\varepsilon)\in W^{1,p^\dagger}\times W^{1,p^\dagger+1}$ coincide on $]0,\min(\varepsilon_*(p),\varepsilon_*(p^\dagger))]$.

Remark 1.6. — The quantity $P(U_c)$ is the momentum of the travelling wave U_c. This is also, formally, a conserved quantity by the NLS flow. Its precise definition requires however some care when we deal with general functions on the energy space and not only travelling waves: see [Mar13] in dimension three and higher, [CM17] in dimension two. The first integral defining $P(U_c)$ is well defined since $|U_c(x)| = 1+\varepsilon^2A_\varepsilon(z)$ is uniformly close to 1 as $\varepsilon\to 0$.

Remark 1.7. — The proof of Theorem 1.3 works as well for any solitary wave $\mathcal{W}$ of KP-I, that is any nontrivial smooth solution of SW such that, on the one hand, $\mathcal{W}(z)=\mathcal{O}(1/|z|^2)$ and $\nabla\mathcal{W}(z)=\mathcal{O}(1/|z|^3)$ at infinity and on the other hand that the only smooth solution $w\in\mathcal{H}$ of

$$w - \partial_{z_1}^2 w + w\mathcal{W} + \partial_{z_2}^2\partial_{z_1}^{-2}w = 0$$

which is even in z_1 and in z_2 is trivial. The decay at infinity is immediate if $\mathcal{W}$ belongs to the energy space $\mathcal{H}$, thanks to the results in [Gra08, Theorem 1.11] (with $N=2=p+1$).

1.3. Extension to the Euler–Korteweg model

The Euler–Korteweg model

$$\text{(EK)}\qquad \begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathfrak{u}) = 0 \\ \\ \partial_t \mathfrak{u} + (\mathfrak{u} \cdot \nabla)\mathfrak{u} + \nabla(g(\rho)) = \nabla\Big(\kappa(\rho)\Delta\rho + \dfrac{1}{2}\kappa'(\rho)|\nabla\rho|^2\Big), \end{cases}$$

is a dispersive perturbation of the Euler equations which includes capillary effects through the capillarity coefficient κ, which depends smoothly on the density and is positive. When writing the Madelung transform $\Psi = \mathcal{A}\mathrm{e}^{i\varphi}$, the NLS equation becomes EK with

$$\rho = \mathcal{A}^2, \qquad \mathfrak{u} = 2\nabla\varphi, \qquad g(\rho) = 2f(\rho), \qquad \kappa(\rho) = \frac{1}{\rho}.$$

The EK system then includes NLS as this particular case. We shall consider the condition $\rho \to \rho_0$ at infinity, and we may rescale so that $\rho_0 = 1$. The speed of sound reads now

$$\mathfrak{c}_{\mathrm{EK}} \stackrel{\mathrm{def}}{=} \sqrt{g'(1)},$$

under the hypothesis that $g'(1) > 0$. The energy and the momentum are defined by

$$\mathscr{E}_{\mathrm{EK}}(\rho, \mathfrak{u}) = \frac{1}{2}\int_{\mathbb{R}^d} \rho|\mathfrak{u}|^2 + \kappa(\rho)|\nabla\rho|^2 + 2G(\rho)\,dx,$$

where $G(\rho) \stackrel{\mathrm{def}}{=} \int_1^\rho g(\varrho)\,d\varrho$, and

$$P_{\mathrm{EK}}(\rho, \mathfrak{u}) = \int_{\mathbb{R}^d} (\rho - 1)\mathfrak{u}_1\,dx.$$

The travelling waves for EK are less studied than for NLS. We may quote [BG13] in 1D and [Aud17] in 2d (and in Appendix B there, a result for nonlinear instability in 1d). In [Aud17], the existence of nontrivial travelling waves of small energy to EK has been shown in 2d, in the case where the flow is potential. The approach relies on the minimization of a modified energy under constraint of fixed, small, momentum, hence yielding nontrivial travelling waves of small energy to EK. Indeed, since then ρ is uniformly close to 1, the modified energy is the EK energy. The stability properties (analogous to [CM17]) are then more difficult to recover. The KP-I limit of these travelling waves (that is the analog of [BGS08] for the EK system) has been recently investigated in [Vas22].

In order to keep the same notations as for NLS, we insert in EK the ansatz

$$\rho(t,x) = \Big(1 + \varepsilon^2 A_\varepsilon(z)\Big)^2, \qquad \mathfrak{u}(t,x) = 2\varepsilon\Big(\partial_1\phi_\varepsilon(z), \varepsilon\partial_2\phi_\varepsilon(z)\Big),$$

where

$$z_1 = \varepsilon\Big(x_1 - c(\varepsilon)t\Big),\ z_2 = \varepsilon^2 x_2,$$

and with, as before $c(\varepsilon)^2 + \varepsilon^2 = \mathfrak{c}^2$, which implies

$$(1.8)\qquad \begin{cases} -c(\varepsilon)\partial_1 A_\varepsilon + 2\varepsilon^2\partial_1\phi_\varepsilon\partial_1 A_\varepsilon + 2\varepsilon^4\partial_2\phi_\varepsilon\partial_2 A_\varepsilon \\ \quad + \left(1+\varepsilon^2 A_\varepsilon\right)\left(\partial_1^2\phi_\varepsilon + \varepsilon^2\partial_2^2\phi_\varepsilon\right) = 0 \\ \\ -c(\varepsilon)\partial_1\phi_\varepsilon + \varepsilon^2(\partial_1\phi_\varepsilon)^2 + \varepsilon^4(\partial_2\phi_\varepsilon)^2 + \dfrac{1}{2\varepsilon^2}g\Big(\left(1+\varepsilon^2 A_\varepsilon\right)^2\Big) \\ \quad = \varepsilon^2\left(1+\varepsilon^2 A_\varepsilon\right)\kappa\Big(\left(1+\varepsilon^2 A_\varepsilon\right)^2\Big)\Big(\partial_1^2 A_\varepsilon + \varepsilon^2\partial_2^2 A_\varepsilon\Big) \\ \qquad + \varepsilon^4\widetilde{\kappa}\Big(\left(1+\varepsilon^2 A_\varepsilon\right)^2\Big)\Big((\partial_1 A_\varepsilon)^2 + \varepsilon^2(\partial_2 A_\varepsilon)^2\Big), \end{cases}$$

with

$$\widetilde{\kappa}(\varrho) \stackrel{\text{def}}{=} \kappa(\varrho) + \varrho\kappa'(\varrho) = \frac{d}{d\varrho}\Big(\varrho\kappa(\varrho)\Big),$$

and we see that this quantity vanishes in the NLS case $\kappa(\varrho) = 1/\varrho$. Then, computations similar to those above yield

$$(\mathrm{SW_{EK}})\qquad \frac{1}{\mathfrak{c}_{\mathrm{EK}}^2}\,\partial_{z_1}A - \frac{\kappa(1)}{\mathfrak{c}_{\mathrm{EK}}^2}\,\partial_{z_1}^3 A + \Gamma_{\mathrm{EK}} A\partial_{z_1}A + \partial_{z_2}^2\partial_{z_1}^{-1}A = 0,$$

where the coefficient Γ_{EK} depends on g through the formula

$$\Gamma_{\mathrm{EK}} \stackrel{\text{def}}{=} 6 + \frac{2}{\mathfrak{c}_{\mathrm{EK}}^2}g''(1).$$

The scaling corresponding to (1.7) is then

$$(1.9)\qquad \mathcal{W}_1^{\mathrm{sc},EK}(z) \stackrel{\text{def}}{=} \frac{1}{\mathfrak{c}_{\mathrm{EK}}^2\Gamma_{\mathrm{EK}}}\mathcal{W}_1\left(z_1, \frac{z_2}{\mathfrak{c}_{\mathrm{EK}}}\right).$$

Theorem 1.8. — *We assume that g is of class $\mathcal{C}^3$ near 1, with $\Gamma_{\mathrm{EK}} \neq 0$, and let $\mathcal{W}_1^{\mathrm{sc},EK}$ be defined by (1.9). Let $1 < p \leqslant \infty$ be given. There exist $\varepsilon_*(p) > 0$ small and a $\mathcal{C}^1$ mapping*

$$]0, \varepsilon_*(p)] \ni \varepsilon \mapsto (A_\varepsilon, \phi_\varepsilon) \in W^{1,p} \times W^{1,p+1}$$

such that:

(i) *for every $0 < \varepsilon \leqslant \varepsilon_*(p)$, $(\rho_{c(\varepsilon)}, \mathfrak{u}_{c(\varepsilon)})$, where*

$$\rho_{c(\varepsilon)}(x) \stackrel{\text{def}}{=} \Big(1 + \varepsilon^2 A_\varepsilon(z)\Big)^2, \qquad \mathfrak{u}_{c(\varepsilon)}(x) \stackrel{\text{def}}{=} 2\varepsilon\Big(\partial_1\phi_\varepsilon(z), \varepsilon\partial_2\phi_\varepsilon(z)\Big),$$

with

$$z_1 = \varepsilon x_1, \quad z_2 = \varepsilon^2 x_2,$$

is a travelling wave for EK of speed $c(\varepsilon) = \sqrt{\mathfrak{c}_{\mathrm{EK}}^2 - \varepsilon^2}$ with density $\rho_{c(\varepsilon)}$ tending to 1 at infinity;

(ii) *when $\varepsilon \to 0$, there holds*

$$\Big\|A_\varepsilon - \mathcal{W}_1^{\mathrm{sc},EK}\Big\|_{W^{1,p}} + \Big\|\phi_\varepsilon - \mathfrak{c}_{\mathrm{EK}}\partial_{z_1}^{-1}\mathcal{W}_1^{\mathrm{sc},EK}\Big\|_{W^{1,p+1}} \leqslant C(p)\varepsilon^2|\ln\varepsilon|^2 \to 0;$$

(iii) *for $c \in]c(\varepsilon_*(p)), \mathfrak{c}_{\mathrm{EK}}[$, we have the Hamilton group relation*

$$\frac{d}{dc}E_{\mathrm{EK}}(U_c) = c\frac{d}{dc}P_{\mathrm{EK}}(U_c);$$

(iv) *when $\varepsilon \to 0$, there holds*

$$E_{\mathrm{EK}}\left(\rho_{c(\varepsilon)}, \mathfrak{u}_{c(\varepsilon)}\right) \sim \mathfrak{c}_{\mathrm{EK}} P_{\mathrm{EK}}\left(\rho_{c(\varepsilon)}, \mathfrak{u}_{c(\varepsilon)}\right) \sim \frac{4\varepsilon}{\mathfrak{c}_{\mathrm{EK}}\Gamma_{\mathrm{EK}}^2} \int_{\mathbb{R}^2} \mathcal{W}_1^2 \, dz = \varepsilon \frac{384\pi}{\mathfrak{c}_{\mathrm{EK}}\Gamma_{\mathrm{EK}}^2},$$

$$E_{\mathrm{EK}}\left(\rho_{c(\varepsilon)}, \mathfrak{u}_{c(\varepsilon)}\right) - c(\varepsilon) P_{\mathrm{EK}}\left(\rho_{c(\varepsilon)}, \mathfrak{u}_{c(\varepsilon)}\right) \sim \frac{4\varepsilon^3}{3\mathfrak{c}_{\mathrm{EK}}^3\Gamma_{\mathrm{EK}}^2} \int_{\mathbb{R}^2} \mathcal{W}_1^2 \, dz = \varepsilon^3 \frac{128\pi}{\mathfrak{c}_{\mathrm{EK}}^3\Gamma_{\mathrm{EK}}^2}$$

and

$$E_{\mathrm{EK}}\left(\rho_{c(\varepsilon)}, \mathfrak{u}_{c(\varepsilon)}\right) - \mathfrak{c}_{\mathrm{EK}} P_{\mathrm{EK}}\left(\rho_{c(\varepsilon)}, \mathfrak{u}_{c(\varepsilon)}\right) \sim -\frac{2\varepsilon^3}{3\mathfrak{c}_{\mathrm{EK}}^3\Gamma_{\mathrm{EK}}^2} \int_{\mathbb{R}^2} \mathcal{W}_1^2 \, dz = -\varepsilon^3 \frac{64\pi}{\mathfrak{c}_{\mathrm{EK}}^3\Gamma_{\mathrm{EK}}^2}.$$

Concerning the approximation of NLS by the KP-I equation (for the time dependent problem), we refer to [CR10, Chi14] (with error bounds when comparing to some weakly transverse Boussinesq system), and [BGC18] for the Euler–Korteweg system.

During the completion of our work, we have learned that [LWWY21] had the same kind of result. Their result also relies on suitable estimates on the kernel $\mathcal{K}^\varepsilon$ appearing in the linearization at infinity of the equation (see (2.7) below). Our estimates, given in Proposition 2.3 below, are much sharper than those given in [LWWY21, Lemmas 3.1 and 3.2], and are, as we show, optimal up to some logarithmic factors. This has the advantage of giving simpler spaces and norms, which are much more in the spirit of the works [BGS08, CM17]. Furthermore, our approach allows to derive $\mathcal{O}(\varepsilon^2|\ln\varepsilon|^2)$ bounds on the error $\phi_\varepsilon - \mathfrak{c}\mathcal{W}_1^{\mathrm{sc}}$ and its gradient (resp. $A_\varepsilon - \mathcal{W}_1^{\mathrm{sc}}$ and its gradient) in weighted L^∞ spaces corresponding to a decay like r^{-1+0} and r^{-2+0} for the gradient (resp. r^{-2+0} and r^{-3+0} for the gradient). The result in [LWWY21] gives, for the phase $\phi_\varepsilon - \mathfrak{c}\mathcal{W}_1^{\mathrm{sc}}$, the decay r^{-1+0} but they have the weaker decay $r^{-3/2+0}$ for the gradient (and an additional loss in ε for the ∂_2-derivative); and for the amplitude, the decay seems to be r^{-1+0} and $r^{-3/2+0}$ for the gradient (and an additional loss in ε for the ∂_2-derivative). Our methods include more complicated model, such as the Euler–Korteweg system.

2. The scheme of the proof

2.1. A fix point problem

We look for $(\phi_\varepsilon, A_\varepsilon)$ under the form

$$\phi_\varepsilon = \phi^{(0)} + \phi^{(1)}, \qquad A_\varepsilon = A^{(0)} + A^{(1)},$$

where

$$A^{(0)}(z) \stackrel{\text{def}}{=} \frac{1}{\mathfrak{c}^2\Gamma}\mathcal{W}_1\left(z_1, \frac{z_2}{\mathfrak{c}}\right), \qquad \phi^{(0)} = \mathfrak{c}\partial_{z_1}^{-1} A^{(0)}. \tag{2.1}$$

Then, we shall recast (1.3), written under the form

$$\begin{cases} \dfrac{c(\varepsilon)\partial_1 A_\varepsilon - 2\varepsilon^2\partial_1\phi_\varepsilon\partial_1 A_\varepsilon - 2\varepsilon^4\partial_2\phi_\varepsilon\partial_2 A_\varepsilon}{1+\varepsilon^2 A_\varepsilon} - \left(\partial_1^2 + \varepsilon^2\partial_2^2\right)\phi_\varepsilon = 0 \\ \left(1+\varepsilon^2 A_\varepsilon\right) \times \left(- c(\varepsilon)\partial_1\phi_\varepsilon + \varepsilon^2(\partial_1\phi_\varepsilon)^2 + \varepsilon^4(\partial_2\phi_\varepsilon)^2 + \dfrac{1}{\varepsilon^2} f\Big(\left(1+\varepsilon^2 A_\varepsilon\right)^2\Big)\right) \\ \qquad -\varepsilon^2\left(\partial_1^2+\varepsilon^2\partial_2^2\right) A_\varepsilon = 0. \end{cases}$$

First, we define the consistency type errors for the phase

$$-\mathrm{Err}_{\mathrm{ph}} \stackrel{\mathrm{def}}{=} c(\varepsilon)\partial_1 A^{(0)} - \varepsilon^2 c(\varepsilon) A^{(0)}\partial_1 A^{(0)} - 2\varepsilon^2\partial_1\phi^{(0)}\partial_1 A^{(0)} - \left(\partial_1^2+\varepsilon^2\partial_2^2\right)\phi^{(0)}$$

and the amplitude

$$\begin{aligned} - \mathrm{Err}_{\mathrm{am}} \stackrel{\mathrm{def}}{=} -c(\varepsilon)\partial_1\phi^{(0)} + \varepsilon^2\left(\partial_1\phi^{(0)}\right)^2 - \varepsilon^2 c(\varepsilon) A^{(0)}\partial_1\phi^{(0)} + \mathfrak{c}^2 A^{(0)} \\ + \varepsilon^2\frac{\mathfrak{c}^2}{2}(\Gamma-3)\left[A^{(0)}\right]^2 - \varepsilon^2\partial_1^2 A^{(0)}. \end{aligned}$$

Then, we consider, for the phase, the quantity

$$\begin{aligned} \mathcal{N}_{\mathrm{ph}}(A,\phi) \stackrel{\mathrm{def}}{=} & -c(\varepsilon)\varepsilon^2 A\partial_1 A - c(\varepsilon)\varepsilon^4\frac{\left(A^{(0)}+A\right)^2}{1+\varepsilon^2A^{(0)}+\varepsilon^2A}\partial_1\left(A^{(0)}+A\right) \\ & + 2\varepsilon^2\partial_1\phi\partial_1 A - 2\varepsilon^4\left(A^{(0)}+A\right)\frac{\partial_1\left(\phi^{(0)}+\phi\right)\partial_1(A^{(0)}+A)}{1+\varepsilon^2A^{(0)}+\varepsilon^2A} \\ & + 2\varepsilon^4\partial_2\phi\partial_2 A^{(0)} - 2\varepsilon^4\frac{\partial_2\left(\phi^{(0)}+\phi\right)\partial_2(A^{(0)}+A)}{1+\varepsilon^2A^{(0)}+\varepsilon^2A} \end{aligned} \tag{2.2}$$

gathering the nonlinear terms and the terms that are $\mathcal{O}(\varepsilon^4)$ formally[(1)], and similarly for the amplitude:

$$\begin{aligned} \mathcal{N}_{\mathrm{am}}(A,\phi) = {} & \varepsilon^2 c(\varepsilon) A\partial_1\phi - \varepsilon^4\left(A^{(0)}+A\right)\left(\partial_1\phi^{(0)}+\partial_1\phi\right)^2 - \varepsilon^2(\partial_1\phi)^2 \\ & - \varepsilon^4\left(1+\varepsilon^2A^{(0)}+\varepsilon^2A\right)\left(\partial_2\phi^{(0)}+\partial_2\phi\right)^2 \\ & - \varepsilon^2\frac{\mathfrak{c}^2}{2}(\Gamma-3)A^2 - \varepsilon^4\frac{\mathfrak{c}^2}{2}(\Gamma-5)(A^{(0)}+A)^3 \\ & - \frac{1}{\varepsilon^2}\left(1+\varepsilon^2A^{(0)}+\varepsilon^2A\right)f_3\left(\varepsilon^2A^{(0)}+\varepsilon^2A\right), \end{aligned} \tag{2.3}$$

where (see (1.5))

$$f\Big((1+\alpha)^2\Big) = \mathfrak{c}^2\alpha + \frac{\mathfrak{c}^2}{2}(\Gamma-5)\alpha^2 + f_3(\alpha), \qquad \text{with } f_3(\alpha) = \mathcal{O}(\alpha^3) \text{ as } \alpha\to 0.$$

Finally, we set

$$\Sigma_{\mathrm{ph}}(A,\phi) \stackrel{\mathrm{def}}{=} \mathrm{Err}_{\mathrm{ph}} + \mathcal{N}_{\mathrm{ph}}(A,\phi), \qquad \Sigma_{\mathrm{am}}(A,\phi) \stackrel{\mathrm{def}}{=} \mathrm{Err}_{\mathrm{am}} + \mathcal{N}_{\mathrm{am}}(A,\phi).$$

(1) The term $2\varepsilon^4\partial_2\phi\partial_2 A^{(0)}$ could be in the right-hand side: we have left it in the left-hand side only to preserve some self-adjointness structure, see Subsection 4.4.3.

With these notations, (1.3) becomes

$$
(2.4)\qquad \begin{cases} c(\varepsilon)\partial_1 A^{(1)} - \varepsilon^2 c(\varepsilon)\partial_1\left(A^{(0)}A^{(1)}\right) - \left(\partial_1^2+\varepsilon^2\partial_2^2\right)\phi^{(1)} \\ -2\varepsilon^2\partial_1\phi^{(0)}\partial_1 A^{(1)} - 2\varepsilon^2\partial_1\phi^{(1)}\partial_1 A^{(0)} - 2\varepsilon^4\partial_2\phi^{(1)}\partial_2 A^{(0)} \\ = \Sigma_{\mathrm{ph}}\left(A^{(1)},\phi^{(1)}\right) \\ \\ -c(\varepsilon)\partial_1\phi^{(1)} - \varepsilon^2 c(\varepsilon)A^{(0)}\partial_1\phi^{(1)} - \varepsilon^2 c(\varepsilon)A^{(1)}\partial_1\phi^{(0)} + 2\varepsilon^2\partial_1\phi^{(0)}\partial_1\phi^{(1)} \\ +\mathfrak{c}^2 A^{(1)} + \varepsilon^2\mathfrak{c}^2(\Gamma-3)A^{(0)}A^{(1)} - \varepsilon^2\left(\partial_1^2+\varepsilon^2\partial_2^2\right)A^{(1)} \\ = \Sigma_{\mathrm{am}}(A^{(1)},\phi^{(1)}). \end{cases}
$$

When using the hydrodynamical form of the equation, we obviously assume that $1+\varepsilon^2 A_\varepsilon$ never vanishes. Since, clearly, $\varepsilon^2 A^{(0)} \geqslant -1/4$ for ε small enough, we shall then require

$$
(2.5)\qquad \varepsilon^2 A^{(1)} \geqslant -1/4.
$$

We now define the Banach spaces we shall use. For $\mu\in\mathbb{R}_+$, we set

$$
X_\mu \stackrel{\mathrm{def}}{=} \left\{v\in\mathcal{C}\left(\mathbb{R}^2,\mathbb{R}\right) \text{ s.t. } z\mapsto(1+|z|)^\mu v(z)\in L^\infty(\mathbb{R}^2)\right\}
$$

equipped with the norm

$$
\|v\|_{X_\mu} \stackrel{\mathrm{def}}{=} \|(1+|z|)^\mu v(z)\|_{L^\infty(\mathbb{R}^2)}.
$$

We also set

$$
X_\mu^1 \stackrel{\mathrm{def}}{=} \left\{v\in\mathcal{C}^1\left(\mathbb{R}^2,\mathbb{R}\right) \text{ s.t. } v\in X_\mu \text{ and } \nabla v\in X_{\mu+1}(\mathbb{R}^2)\right\},
$$

equipped with the norm

$$
\|v\|_{X_\mu^1} \stackrel{\mathrm{def}}{=} \|(1+|z|)^\mu v(z)\|_{L^\infty(\mathbb{R}^2)} + \left\|(1+|z|)^{\mu+1}\nabla v(z)\right\|_{L^\infty(\mathbb{R}^2)} = \|v\|_{X_\mu} + \|\nabla v\|_{X_{\mu+1}}.
$$

We shall impose some symmetries on the functions, namely that they are always even in z_2 and odd/even in z_1. We therefore define

$$
\begin{aligned}
X_{\mu,a} &\stackrel{\mathrm{def}}{=} \left\{v\in X_\mu \text{ s.t. } \forall\, z\in\mathbb{R}^2,\quad -v(-z_1,z_2)=v(z_1,z_2)=v(z_1,-z_2)\right\},\\
X_{\mu,s} &\stackrel{\mathrm{def}}{=} \left\{v\in X_\mu \text{ s.t. } \forall\, z\in\mathbb{R}^2,\quad v(-z_1,z_2)=v(z_1,z_2)=v(z_1,-z_2)\right\}
\end{aligned}
$$

and

$$
X_{\mu,a}^1 \stackrel{\mathrm{def}}{=} X_{\mu,a}\cap X_\mu^1, \qquad X_{\mu,s}^1 \stackrel{\mathrm{def}}{=} X_{\mu,s}\cap X_\mu^1
$$

and finally

$$
\begin{aligned}
\dot{X}_{\mu,a}^1 \stackrel{\mathrm{def}}{=} \Big\{\phi\in\mathcal{C}^2(\mathbb{R}^2) \text{ s.t. } \nabla\phi\in X_\mu^1 & \\
\text{and } \forall\, z\in\mathbb{R}^2,\quad & -\phi(-z_1,z_2)=\phi(z_1,z_2)=\phi(z_1,-z_2)\Big\}.
\end{aligned}
$$

We define, for $\mu>1$, the operator $\mathcal{I}$ on $X_{\mu,a}$ by

$$
(2.6)\qquad \mathcal{I}w(z) \stackrel{\mathrm{def}}{=} \int_{-\infty}^{z_1} w(\zeta,z_2)\,d\zeta = -\int_{z_1}^{+\infty} w(\zeta,z_2)\,d\zeta,
$$

the equality coming from the fact that $w\in X_{\mu,a}$, hence $\int_{\mathbb{R}} w(\zeta,z_2)\,d\zeta=0$ for any $z_2\in\mathbb{R}$.

Proposition 2.1. — *Let $\mu > 1$. Then, $\mathcal{I}$ maps continuously $X_{\mu,a}$ into $X_{\mu-1}$: there exists $C(\mu)$ such that, for any $w \in X_{\mu,a}$,*

$$\|\mathcal{I}w\|_{X_{\mu-1}} \leqslant C(\mu)\|w\|_{X_\mu}.$$

Proof. — It suffices to write, for $z_1 \geqslant 0$,

$$\begin{aligned}\left|\int_{z_1}^{+\infty} w(\zeta, z_2)\, d\zeta\right| &\leqslant \int_{z_1}^{+\infty} \frac{\|w\|_{X_\mu}}{(1+\zeta+|z_2|)^\mu}\, d\zeta \\ &= \frac{\|w\|_{X_\mu}}{(\mu-1)(1+z_1+|z_2|)^{\mu-1}} \\ &\leqslant \frac{\|w\|_{X_\mu}}{(\mu-1)(1+|z|)^{\mu-1}}.\end{aligned}$$

For $z_1 \leqslant 0$, we use the first formula in (2.6) and argue similarly. □

As a first step, we provide a consistency estimate.

Proposition 2.2. — *There exists $C > 0$, depending on the lump $\mathcal{W}_1$, such that, for $0 < \varepsilon < 1$, we have*

$$\|\mathrm{Err}_{\mathrm{am}}\|_{X^1_2} \leqslant C\varepsilon^2 \qquad \text{and} \qquad \|\mathrm{Err}_{\mathrm{ph}}\|_{X^1_3} \leqslant C\varepsilon^2.$$

Moreover,

$$\|\mathfrak{c}\mathcal{I}\mathrm{Err}_{\mathrm{ph}} - \mathrm{Err}_{\mathrm{am}}\|_{X^1_2} \leqslant C\varepsilon^4.$$

In order to solve the fix point problem, one natural approach would be to use a standard implicit function theorem near $\varepsilon = 0$. However, this shall not work. Indeed, the KP-I limit for the solitary wave (as well as for the time dependent problem) is formally shown in [JR82] (see also [BR04]) and in [KP00] by expansion in powers of ε ($A_\varepsilon = A_0 + \varepsilon A_1 + \dots$, $\phi_\varepsilon = \phi_0 + \varepsilon\phi_1 + \dots$), whereas it has been put forward in [Chi14, Section 1.3] that this is not formally correct since some terms (such as $\partial^2_{z_2}\partial^{-2}_{z_1} A_0$ or $\partial^4_{z_2}\partial^{-3}_{z_1} A_0$) become meaningless. Actually, in [Chi14], the issue was the derivation of the time dependent KP-I, but the argument remains valid for the SW equation. Similar issues for the consistency estimate (for the time-dependent problem) of KP-I with some fluid systems have been studied in [Lan03]. This is related to logarithmic divergences estimates for the kernel $\mathcal{K}^\varepsilon$ (see Proposition 2.3 below). Therefore, we need to use the well-preparedness assumption (1.4), and this forces us to employ ε-dependent norms.

2.2. Properties of the kernel

In our analysis, the kernel associated with the linearized problem at infinity will play a crucial role. The linearization of the system (1.3) at infinity is

$$\begin{cases} -\left(\partial_1^2 + \varepsilon^2\partial_2^2\right)\phi + c(\varepsilon)\partial_1 A = 0 \\[2ex] -c(\varepsilon)\partial_1\phi + \mathfrak{c}^2 A - \varepsilon^2\left(\partial_1^2 + \varepsilon^2\partial_2^2\right) A = 0. \end{cases}$$

The determinant of the underlying operator matrix is, since $\mathfrak{c}^2 = c(\varepsilon)^2 + \varepsilon^2$,

$$c(\varepsilon)^2\partial_1^2 - \left(\partial_1^2 + \varepsilon^2\partial_2^2\right)\left(\mathfrak{c}^2 - \varepsilon^2\left(\partial_1^2 + \varepsilon^2\partial_2^2\right)\right) = \varepsilon^2\left(-\partial_1^2 - \mathfrak{c}^2\partial_2^2 + \left(\partial_1^2 + \varepsilon^2\partial_2^2\right)^2\right),$$

which yields us to define in the sense of tempered distributions the kernel (FP stands for the Finite Part)

$$\mathcal{K}^\varepsilon \stackrel{\text{def}}{=} \mathscr{F}^{-1}\left(\mathrm{FP}\frac{1}{\xi_1^2 + \mathfrak{c}^2\xi_2^2 + (\xi_1^2 + \varepsilon^2\xi_2^2)^2}\right) \in \mathscr{S}'(\mathbb{R}^2). \tag{2.7}$$

Here, $\mathscr{F}$ stands for the Fourier transform with the following normalizations: for $u \in L^1(\mathbb{R}^2)$, then $\mathscr{F}(u)(\xi) = \int_{\mathbb{R}^2} u(x)\mathrm{e}^{-ix\cdot\xi}\,dx$ ($\xi \in \mathbb{R}^2$). When $\varepsilon = 0$, $\partial_1^2\mathcal{K}^\varepsilon$ is the kernel K_0 studied in [Gra08].

Proposition 2.3. — *The kernel $\mathcal{K}^\varepsilon$ is a real valued function, even in z_1 and in z_2.*

(i) *There exists $C > 0$ such that, for $\varepsilon \in]0, \min(\mathfrak{c}, \mathfrak{c}^{-1})/2]$ and $z \in \mathbb{R}^2$ with $|z| \geqslant 1$, we have*

$$|z| \times |\nabla\mathcal{K}^\varepsilon(z)| + |z|^2\left|\nabla^2\mathcal{K}^\varepsilon(z)\right| + |z|^3\left|\nabla^3\mathcal{K}^\varepsilon(z)\right| + \frac{|\mathcal{K}^\varepsilon(z)|}{1 + \ln|z|} \leqslant C.$$

(ii) *There exists $C > 0$ such that, for $\varepsilon \in]0, 1/2[$, we have*

$$\int_{D(0,1)} |\nabla\mathcal{K}^\varepsilon(z)| + \left|\partial_1^2\mathcal{K}^\varepsilon(z)\right| + |\partial_1\partial_2\mathcal{K}^\varepsilon(z)| + \frac{1}{|\ln\varepsilon|^2}\left|\partial_2^2\mathcal{K}^\varepsilon(z)\right|\,dz \leqslant C$$

and

$$\int_{D(0,1)} \left|\partial_1^3\mathcal{K}^\varepsilon(z)\right| + \frac{1}{|\ln\varepsilon|^2}\left|\partial_1^2\partial_2\mathcal{K}^\varepsilon(z)\right| + \frac{\varepsilon}{|\ln\varepsilon|}\left|\partial_1\partial_2^2\mathcal{K}^\varepsilon(z)\right| + \frac{\varepsilon^2}{|\ln\varepsilon|}\left|\partial_2^3\mathcal{K}^\varepsilon(z)\right|\,dz \leqslant C.$$

Remark 2.4. — Since, for any $\alpha \in \mathbb{N}^2$ with $\alpha_1 + \alpha_2 = 4$, the function

$$\xi^\alpha\mathscr{F}\mathcal{K}^\varepsilon = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\mathscr{F}\mathcal{K}^\varepsilon = \frac{\xi_1^{\alpha_1}\xi_2^{\alpha_2}}{\xi_1^2 + \mathfrak{c}^2\xi_2^2 + (\xi_1^2 + \varepsilon^2\xi_2^2)^2}$$

does not tend to 0 at infinity, we can not have $\partial_\xi^\alpha\mathcal{K}^\varepsilon \in L^1(\mathbb{R}^2)$ (by Riemann–Lebesgue's lemma). In the same way, since

$$\xi_2^2\mathscr{F}\mathcal{K}^0 = \frac{\xi_2^2}{\xi_1^2 + \mathfrak{c}^2\xi_2^2 + \xi_1^4}, \quad \xi_1^2\xi_2\mathscr{F}\mathcal{K}^0 = \frac{\xi_1^2\xi_2}{\xi_1^2 + \mathfrak{c}^2\xi_2^2 + \xi_1^4},$$
$$\xi_1\xi_2^2\mathscr{F}\mathcal{K}^0 = \frac{\xi_1\xi_2^2}{\xi_1^2 + \mathfrak{c}^2\xi_2^2 + \xi_1^4}, \quad \xi_2^3\mathscr{F}\mathcal{K}^0 = \frac{\xi_2^3}{\xi_1^2 + \mathfrak{c}^2\xi_2^2 + \xi_1^4}$$

do not tend to 0 at infinity (take, respectively, ($\xi_1 = 1$ and $\xi_2 \to \infty$), $\xi_2 = \xi_1^2 \to +\infty$, $\xi_2 = \xi_1^2 \to +\infty$ and ($\xi_1 = 1$ and $\xi_2 \to +\infty$)) we must have, as $\varepsilon \to 0$,

$$\left\|\partial_2^2\mathcal{K}^\varepsilon\right\|_{L^1(\mathbb{R}^2)} \to +\infty, \quad \left\|\partial_1^2\partial_2\mathcal{K}^\varepsilon\right\|_{L^1(\mathbb{R}^2)} \to +\infty,$$
$$\left\|\partial_1\partial_2^2\mathcal{K}^\varepsilon\right\|_{L^1(\mathbb{R}^2)} \to +\infty, \quad \left\|\partial_2^3\mathcal{K}^\varepsilon\right\|_{L^1(\mathbb{R}^2)} \to +\infty.$$

As a matter of fact,

$$\begin{aligned}\left\|\partial_1\partial_2^2\mathcal{K}^\varepsilon\right\|_{L^1(\mathbb{R}^2)} &\geqslant \left\|\mathscr{F}\left(\partial_1\partial_2^2\mathcal{K}^\varepsilon\right)\right\|_{L^\infty(\mathbb{R}^2)}\\ &\geqslant \frac{\xi_1\xi_2^2}{\xi_1^2+\mathfrak{c}^2\xi_2^2+\left(\xi_1^2+\varepsilon^2\xi_2^2\right)^2}_{|\xi=(\varepsilon^{-1},\varepsilon^{-2})} \sim \frac{\varepsilon^{-1}}{\mathfrak{c}^2+4}\end{aligned}$$

and

$$\left\|\partial_2^3\mathcal{K}^\varepsilon\right\|_{L^1(\mathbb{R}^2)} \geqslant \left\|\mathscr{F}\left(\partial_2^3\mathcal{K}^\varepsilon\right)\right\|_{L^\infty(\mathbb{R}^2)} \geqslant \frac{\xi_2^3}{\xi_1^2+\mathfrak{c}^2\xi_2^2+\left(\xi_1^2+\varepsilon^2\xi_2^2\right)^2}_{|\xi=(1,\varepsilon^{-2})} \sim \frac{\varepsilon^{-2}}{\mathfrak{c}^2+1}.$$

It then follows from (i) that the estimates for $\partial_1\partial_2^2\mathcal{K}^\varepsilon$ and $\partial_2^3\mathcal{K}^\varepsilon$ in $L^1(D(0,1))$ in (ii) are optimal up to logarithmic factors.

Remark 2.5. — The above estimates are of course in agreement with the asymptotic behaviors obtained in [Gra08] for the kernel $\partial_1^2\mathcal{K}^0$ (denoted K_0 there).

2.3. Mapping properties of the convolution kernel $\mathcal{K}^\varepsilon$

The next Proposition gives some estimates between X_μ spaces for the convolution with the kernels $\partial^\alpha\mathcal{K}^\varepsilon$. Some of them depend whether the second factor of the convolution has vanishing integral over $\mathbb{R}^2$ or not (note that $X_\mu \subset L^1$ when $\mu > 2$).

Proposition 2.6. — *The following estimates hold.*

(i) $\forall\, 1<\mu<2$, $\forall\, v\in X_{\mu+1}$ *such that* $\int_{\mathbb{R}^2} v\,dz = 0$,

$$\|\partial_1\mathcal{K}^\varepsilon\star v\|_{X_\mu} + \|\partial_2\mathcal{K}^\varepsilon\star v\|_{X_\mu} \leqslant C(\mu)\|v\|_{X_{\mu+1}};$$

(ii) $\forall\, 0<\mu'<\mu<2$, $\forall\, v\in X_\mu$,

$$\left\|\partial_1^2\mathcal{K}^\varepsilon\star v\right\|_{X_{\mu'}} + \|\partial_1\partial_2\mathcal{K}^\varepsilon\star v\|_{X_{\mu'}} + |\ln\varepsilon|^{-2}\left\|\partial_2^2\mathcal{K}^\varepsilon\star v\right\|_{X_{\mu'}} \leqslant C(\mu,\mu')\|v\|_{X_\mu};$$

(ii') $\forall\, 2<\mu'<\mu<3$, $\forall\, v\in X_\mu$ *such that* $\int_{\mathbb{R}^2} v\,dz = 0$,

$$\left\|\partial_1^2\mathcal{K}^\varepsilon\star v\right\|_{X_{\mu'}} + \|\partial_1\partial_2\mathcal{K}^\varepsilon\star v\|_{X_{\mu'}} + |\ln\varepsilon|^{-2}\left\|\partial_2^2\mathcal{K}^\varepsilon\star v\right\|_{X_{\mu'}} \leqslant C(\mu,\mu')\|v\|_{X_\mu};$$

(iii) $\forall\, 0<\mu<3$, $\forall\, v\in X_\mu$,

$$\begin{aligned}&\left\|\partial_1^3\mathcal{K}^\varepsilon\star v\right\|_{X_\mu} + |\ln\varepsilon|^{-2}\left\|\partial_1^2\partial_2\mathcal{K}^\varepsilon\star v\right\|_{X_\mu}\\ &\qquad + \varepsilon|\ln\varepsilon|^{-1}\left\|\partial_1\partial_2^2\mathcal{K}^\varepsilon\star v\right\|_{X_\mu} + \varepsilon^2|\ln\varepsilon|^{-1}\left\|\partial_2^3\mathcal{K}^\varepsilon\star v\right\|_{X_\mu} \leqslant C(\mu)\|v\|_{X_\mu}.\end{aligned}$$

Remark 2.7. — The estimate in (i) is not expected to hold if $\int_{\mathbb{R}^2} v(z)\,dz \neq 0$. Indeed, in this case, we should have $\nabla\mathcal{K}^\varepsilon\star v(z) \approx \frac{z}{|z|^2}\int_{\mathbb{R}^2} v$ at infinity, which does not belong to X_μ when $\mu > 1$.

Let us now consider the linear system with source term

$$\left\{\begin{aligned}&c(\varepsilon)\partial_1 A - \left(\partial_1^2+\varepsilon^2\partial_2^2\right)\phi = S_{\mathrm{ph}}\\ &-c(\varepsilon)\partial_1\phi + \left(\mathfrak{c}^2-\varepsilon^2\left(\partial_1^2+\varepsilon^2\partial_2^2\right)\right)A = S_{\mathrm{am}}\end{aligned}\right. \tag{2.8}$$

which is associated with (2.4) at spatial infinity. When we impose the symmetries, its solution is given, as we shall see, by

$$\tag{2.9}\begin{cases} A = -\dfrac{1}{\varepsilon^2}\Big(\left(\partial_1^2 + \varepsilon^2\partial_2^2\right) \mathcal{K}^\varepsilon \star S_{\mathrm{am}} - c(\varepsilon)\partial_1\mathcal{K}^\varepsilon \star S_{\mathrm{ph}}\Big) \\ \\ \phi = -\dfrac{1}{\varepsilon^2}\left(c(\varepsilon)\partial_1\mathcal{K}^\varepsilon \star S_{\mathrm{am}} - \left(\mathfrak{c}^2 - \varepsilon^2\left(\partial_1^2 + \varepsilon^2\partial_2^2\right)\right)\mathcal{K}^\varepsilon \star S_{\mathrm{ph}}\right). \end{cases}$$

Assuming $S_{\mathrm{ph}} \in X_{2+\sigma,a}$ and $S_{\mathrm{am}} \in X_{1+\sigma',s}$ for some $0 < \sigma' < 1$, the crude estimate coming from Proposition 2.6 on $(A, \nabla\phi)$ is, for $0 < \sigma < \sigma'$,

$$\varepsilon^2\|A\|_{X_{1+\sigma}} + \varepsilon^2\|\nabla\phi\|_{X_{1+\sigma}} \leqslant C(\sigma,\sigma')\left(\|S_{\mathrm{ph}}\|_{X_{2+\sigma}} + \|S_{\mathrm{am}}\|_{X_{1+\sigma'}}\right).$$

Notice that Proposition 2.6 (i) does not apply to $\partial_1\mathcal{K}^\varepsilon \star S_{\mathrm{am}}$, since S_{am} is not odd, thus there is no decay estimate claimed on ϕ. The estimate below shall then rely on suitable cancellation properties imposed on the source terms S_{am} and S_{ph}.

Proposition 2.8. — *Let $0 < \sigma < \sigma' < 1$ be given. Then, there exists a positive constant $C(\sigma,\sigma') > 0$ such that, if $(\mathcal{S}_{\mathrm{ph}}, \mathcal{S}_{\mathrm{am}}) \in X^1_{2+\sigma',a} \times X^1_{1+\sigma',s}$ and $0 < \varepsilon < 1/2$, then (2.8) has a unique weak solution (ϕ, A) in $\dot{X}^1_{1+\sigma,a} \times X^1_{1+\sigma,s}$, and it is given by (2.9). In addition, $A \in X^2_\sigma$ and*

$$\|A\|_{X^1_{1+\sigma}} + \|\nabla\phi\|_{X^1_{1+\sigma}} + \left\|\partial_1^2 A\right\|_{X_{2+\sigma}} + \varepsilon\|\partial_1\partial_2 A\|_{X_{2+\sigma}} + \varepsilon^2\left\|\partial_2^2 A\right\|_{X_{2+\sigma}}$$
$$\leqslant C(\sigma,\sigma')|\ln\varepsilon|^2\left(\|\mathcal{S}_{\mathrm{ph}}\|_{X^1_{2+\sigma'}} + \|\mathcal{S}_{\mathrm{am}}\|_{X^1_{1+\sigma'}} + \frac{1}{\varepsilon^2}\|\mathfrak{c}\mathcal{I}\mathcal{S}_{\mathrm{ph}} - \mathcal{S}_{\mathrm{am}}\|_{X^1_{1+\sigma'}}\right).$$

Moreover, if $\mathcal{S}_{\mathrm{am}} = 0$, then we have

$$\|A\|_{X^1_{1+\sigma}} + \|\nabla\phi\|_{X^1_{1+\sigma}} + \left\|\partial_1^2 A\right\|_{X_{2+\sigma}} + \varepsilon\|\partial_1\partial_2 A\|_{X_{2+\sigma}} + \varepsilon^2\left\|\partial_2^2 A\right\|_{X_{2+\sigma}}$$
$$\leqslant C(\sigma,\sigma')|\ln\varepsilon|^2\left(\|\mathcal{S}_{\mathrm{ph}}\|_{X^1_{2+\sigma'}} + \frac{1}{\varepsilon^2}\|\mathcal{S}_{\mathrm{ph}}\|_{X_{2+\sigma'}} + \frac{1}{\varepsilon^2}\|\partial_1\mathcal{S}_{\mathrm{ph}}\|_{X_{2+\sigma'}}\right).$$

Remark 2.9. — The second estimate is adapted to source terms with ∂_2-derivatives, since we do not want to loose too much through a bound involving $\partial_2\mathcal{S}_{\mathrm{ph}}$. This will be the case of the term $\partial_2\phi^{(0)}\partial_2 A^{(1)}$.

2.4. Invertibility for the linear problem

Proposition 2.8 allows to solve the linear problem at infinity. We shall now state our invertibility result for the linear problem on the plane, with non constant coefficient.

Proposition 2.10. — *Let $0 < \sigma < \sigma' < 1$ be given. Then, there exists positive constants $\epsilon(\sigma,\sigma') \in]0,1[$ and $C(\sigma,\sigma') > 0$ such that, if $0 < \varepsilon < \epsilon(\sigma,\sigma')$, for any $(\mathcal{S}_{\mathrm{ph}}, \mathcal{S}_{\mathrm{am}}) \in X^1_{2+\sigma',a} \times X^1_{1+\sigma',s}$, then the linear system*

$$\tag{2.10}\begin{cases} c(\varepsilon)\partial_1 A - \left(\partial_1^2 + \varepsilon^2\partial_2^2\right)\phi - 2\varepsilon^2\partial_1\phi\partial_1 A^{(0)} - 2\varepsilon^4\partial_2\phi\partial_2 A^{(0)} \\ \quad -\varepsilon^2 c(\varepsilon)\partial_1\left(A^{(0)}A\right) - 2\varepsilon^2\partial_1\phi^{(0)}\partial_1 A = \mathcal{S}_{\mathrm{ph}} \\ \\ -c(\varepsilon)\partial_1\phi - \varepsilon^2 c(\varepsilon)A^{(0)}\partial_1\phi - \varepsilon^2 c(\varepsilon)A\partial_1\phi^{(0)} + 2\varepsilon^2\partial_1\phi^{(0)}\partial_1\phi \\ \quad +\mathfrak{c}^2 A + \varepsilon^2\mathfrak{c}^2(\Gamma-3)A^{(0)}A - \varepsilon^2\left(\partial_1^2 A + \varepsilon^2\partial_2^2 A\right) = \mathcal{S}_{\mathrm{am}}, \end{cases}$$

has a unique weak solution $(\phi, A) \in \dot{X}^1_{1+\sigma,a} \times (X^1_{1+\sigma,s} \cap X^2_\sigma)$, *and*

$$\|A\|_{X^1_{1+\sigma}} + \|\nabla\phi\|_{X^1_{1+\sigma}} + \left\|\partial_1^2 A\right\|_{X_{2+\sigma}} + \varepsilon\|\partial_1\partial_2 A\|_{X_{2+\sigma}} + \varepsilon^2\left\|\partial_2^2 A\right\|_{X_{2+\sigma}} \leqslant C(\sigma,\sigma')|\ln\varepsilon|^2\left(\|\mathcal{S}_{\mathrm{ph}}\|_{X^1_{2+\sigma'}} + \|\mathcal{S}_{\mathrm{am}}\|_{X^1_{1+\sigma'}} + \frac{1}{\varepsilon^2}\|\mathfrak{c}\mathcal{I}\mathcal{S}_{\mathrm{ph}} - \mathcal{S}_{\mathrm{am}}\|_{X^1_{1+\sigma'}}\right).$$

Moreover, if $\mathcal{S}_{\mathrm{am}} = 0$, *then we have*

$$\|A\|_{X^1_{1+\sigma}} + \|\nabla\phi\|_{X^1_{1+\sigma}} + \left\|\partial_1^2 A\right\|_{X_{2+\sigma}} + \varepsilon\|\partial_1\partial_2 A\|_{X_{2+\sigma}} + \varepsilon^2\left\|\partial_2^2 A\right\|_{X_{2+\sigma}} \leqslant C(\sigma,\sigma')|\ln\varepsilon|^2\left(\|\mathcal{S}_{\mathrm{ph}}\|_{X^1_{2+\sigma'}} + \frac{1}{\varepsilon^2}\|\mathcal{S}_{\mathrm{ph}}\|_{X_{2+\sigma'}} + \frac{1}{\varepsilon^2}\|\partial_1\mathcal{S}_{\mathrm{ph}}\|_{X_{2+\sigma'}}\right).$$

This result contains an a priori estimate as well as an existence result. The a priori estimate shall follow from the nondegeneracy of the lump $\mathcal{W}_1$ (see [LW19, Theorem 1.2]), and assuming the symmetries will guarantee that there is no kernel. The existence part shall rely on solving the same problem on a large disk in a standard H^1_0 Hilbert space and showing that the adjoint problem is injective.

2.5. Solving the fix point problem

We now fix $0 < \sigma < \sigma' < 1$ arbitrarily and define the spaces

$$\mathcal{Y}^\varepsilon_{\sigma'} \stackrel{\text{def}}{=} X^1_{2+\sigma',a} \times X^1_{1+\sigma',s} \qquad \text{and} \qquad \mathcal{X}^\varepsilon_\sigma \stackrel{\text{def}}{=} \dot{X}^1_{1+\sigma,a} \times \left(X^1_{1+\sigma,s} \cap X^2_{\sigma,s}\right)$$

endowed with the norms

$$\|(\mathcal{S}_{\mathrm{ph}}, \mathcal{S}_{\mathrm{am}})\|_{\mathcal{Y}^\varepsilon_{\sigma'}} \stackrel{\text{def}}{=} \|\mathcal{S}_{\mathrm{ph}}\|_{X^1_{2+\sigma'}} + \|\mathcal{S}_{\mathrm{am}}\|_{X^1_{1+\sigma'}} + \frac{1}{\varepsilon^2}\|\mathfrak{c}\mathcal{I}\mathcal{S}_{\mathrm{ph}} - \mathcal{S}_{\mathrm{am}}\|_{X^1_{1+\sigma'}}$$

and

$$\|(\phi, A)\|_{\mathcal{X}^\varepsilon_\sigma} \stackrel{\text{def}}{=} \|A\|_{X^1_{1+\sigma}} + \|\nabla\phi\|_{X^1_{1+\sigma}} + \left\|\left(\partial_1^2 A, \varepsilon\partial_1\partial_2 A, \varepsilon^2\partial_2^2 A\right)\right\|_{X_{2+\sigma}}$$

respectively. The parameter ε does not appear in the definition of the vector spaces, but only in the norms. The result of Proposition 2.10 then allows us to define the operator

$$\mathbb{M} : \mathcal{Y}^\varepsilon_{\sigma'} \to \mathcal{X}^\varepsilon_\sigma$$

by

$$\mathbb{M}\begin{pmatrix}\mathcal{S}_{\mathrm{ph}}\\ \mathcal{S}_{\mathrm{am}}\end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix}A\\ \phi\end{pmatrix},$$

where $(\phi, A) \in \mathcal{X}^\varepsilon_\sigma$ is the solution to (2.10); in addition,

$$\|\mathbb{M}\|_{\mathcal{Y}^\varepsilon_{\sigma'} \to \mathcal{X}^\varepsilon_\sigma} \leqslant C(\sigma,\sigma')|\ln\varepsilon|^2$$

with $C(\sigma,\sigma')$ independent of ε.

Then, the system (2.4) may be recast into

$$\begin{pmatrix}\phi^{(1)}\\ A^{(1)}\end{pmatrix} = \mathbb{M}\begin{pmatrix}\mathrm{Err}_{\mathrm{ph}}\\ \mathrm{Err}_{\mathrm{am}}\end{pmatrix} + \mathbb{M}\begin{pmatrix}\mathcal{N}_{\mathrm{ph}}\left(A^{(1)}, \phi^{(1)}\right)\\ \mathcal{N}_{\mathrm{am}}\left(A^{(1)}, \phi^{(1)}\right)\end{pmatrix}. \tag{2.11}$$

We then fix $\sigma\in]0,1[$, pick some $\sigma'\in]\sigma,1[$ such that $\sigma'<2\sigma$, and define the mapping

$$\begin{aligned}\Upsilon:\bar{B}_{\mathcal{X}^\varepsilon_\sigma}(0,1)&\to\mathcal{X}^\varepsilon_\sigma\\ (A,\phi)&\mapsto\mathbb{M}\begin{pmatrix}\mathrm{Err}_{\mathrm{ph}}\\ \mathrm{Err}_{\mathrm{am}}\end{pmatrix}+\mathbb{M}\begin{pmatrix}\mathcal{N}_{\mathrm{ph}}(A,\phi)\\ \mathcal{N}_{\mathrm{am}}(A,\phi)\end{pmatrix}.\end{aligned}$$

It is well-defined for ε sufficiently small so that $\varepsilon^2A\geqslant-\varepsilon^2\|A\|_{L^\infty(\mathbb{R}^2)}\geqslant-1/4$ (see (2.5)).

Proposition 2.11. — *Let $0<\sigma<1$ be given. Then, there exists $\epsilon(\sigma)\in]0,1[$ and $C_3(\sigma)\geqslant1$ such that, if $0<\varepsilon<\epsilon(\sigma)$, then*

(i) *$\Upsilon(\bar{B}_{\mathcal{X}^\varepsilon_\sigma}(0,C_3(\sigma)\varepsilon^2|\ln\varepsilon|^2))\subset\bar{B}_{\mathcal{X}^\varepsilon_\sigma}(0,C_3(\sigma)\varepsilon^2|\ln\varepsilon|^2)$;*

(ii) *the mapping Υ is 1/2-Lipschitz continuous.*

As a consequence, Υ has a unique fix point $(\phi^{(1)}_\varepsilon,A^{(1)}_\varepsilon)$ in $\bar{B}_{\mathcal{X}^\varepsilon_\sigma}(0,C_3(\sigma)\varepsilon^2|\ln\varepsilon|^2)$. It follows that we have $\|(A^{(1)}_\varepsilon,A^{(1)}_\varepsilon)\|_{\mathcal{X}^\varepsilon_\sigma}\leqslant C_3(\sigma)\varepsilon^2|\ln\varepsilon|^2$.

Remark 2.12. — As already seen in [Chi14], we do not expect the estimate $\|(A^{(1)}_\varepsilon,\phi^{(1)}_\varepsilon)\|_{\mathcal{X}^\varepsilon_\sigma}\leqslant C_3(\sigma)\varepsilon^2$ to hold.

3. Estimates on the kernel

3.1. Properties and partial integration of the kernels

The kernel $\mathcal{K}^\varepsilon$ is the tempered distribution defined by

$$\mathcal{K}^\varepsilon=\mathscr{F}^{-1}\left(\mathrm{FP}\frac{1}{\mathfrak{c}^2\xi_1^2+\xi_2^2+(\xi_1^2+\varepsilon^2\xi_2^2)^2}\right)\in\mathscr{S}'(\mathbb{R}^2).$$

It is easily checked that for $\varepsilon>0$ and $j=1$ or 2, then $\xi_j\mathscr{F}\mathcal{K}^\varepsilon\in L^1(\mathbb{R}^2)$.

At fixed $\xi_1\in\mathbb{R}$, we shall use the Cauchy residue theorem in order to compute the integrals in ξ_2. We set

$$\Xi^\pm_\varepsilon(\xi_1)\stackrel{\mathrm{def}}{=}\frac{1}{\varepsilon^2\sqrt{2}}\sqrt{1+2\varepsilon^2\xi_1^2\pm\sqrt{(1+2\varepsilon^2\xi_1^2)^2-4\varepsilon^4\left(\mathfrak{c}^2\xi_1^2+\xi_1^4\right)}}.$$

Notice that

$$\left(1+2\varepsilon^2\xi_1^2\right)^2\geqslant\left(1+2\varepsilon^2\xi_1^2\right)^2-4\varepsilon^4\left(\mathfrak{c}^2\xi_1^2+\xi_1^4\right)=1+4\varepsilon^2\xi_1^2\left(1-\varepsilon^2\mathfrak{c}^2\right)\geqslant1$$

since $\varepsilon\mathfrak{c}<1$, hence all the roots are roots of nonnegative real numbers. Moreover, we have $\Xi^+_\varepsilon(\xi_1)>\Xi^-_\varepsilon(\xi_1)\geqslant0$, and $\Xi^-_\varepsilon(\xi_1)=0$ if and only if $\xi_1=0$. Then, we may factorize the polynomial in ξ_2 on the denominator:

$$\begin{aligned}&\mathfrak{c}^2\xi_1^2+\xi_2^2+\left(\xi_1^2+\varepsilon^2\xi_2^2\right)^2\\ &\quad=\varepsilon^4\left(\xi_2-i\Xi^+_\varepsilon(\xi_1)\right)\times\left(\xi_2-i\Xi^-_\varepsilon(\xi_1)\right)\times\left(\xi_2+i\Xi^+_\varepsilon(\xi_1)\right)\times\left(\xi_2+i\Xi^-_\varepsilon(\xi_1)\right).\end{aligned}$$

Except if $\xi_1=0$, these roots are simple and nonzero. Notice that Ξ^+_ε is everywhere positive and smooth, whereas Ξ^-_ε vanishes at the origin and is smooth except at $\xi_1=0$.

Proposition 3.1. — *The function $\mathcal{K}^\varepsilon$ is of class $\mathcal{C}^1$ in $\mathbb{R}^2$. Moreover, we have*

$$4\pi\partial_1\mathcal{K}^\varepsilon(z) = i\int_{\mathbb{R}}\sum_{s\,\in\,\{\pm\}}\frac{s\xi_1\mathsf{e}^{-|z_2|\Xi^s_\varepsilon(\xi_1)}\mathsf{e}^{iz_1\xi_1}}{\Xi^s_\varepsilon(\xi_1)\sqrt{1+4\varepsilon^2\xi_1^2\left(1-\mathfrak{c}^2\varepsilon^2\right)}}\,d\xi_1$$

and

$$4\pi\partial_2\mathcal{K}^\varepsilon(z) = \int_{\mathbb{R}}\sum_{s\,\in\,\{\pm\}}\frac{s\mathsf{e}^{-|z_2|\Xi^s_\varepsilon(\xi_1)}\mathsf{e}^{iz_1\xi_1}}{\sqrt{1+4\varepsilon^2\xi_1^2\left(1-\mathfrak{c}^2\varepsilon^2\right)}}\,d\xi_1.$$

When $z_2\neq 0$, each function $\xi_1\mapsto\frac{s\xi_1\mathsf{e}^{-|z_2|\Xi^s_\varepsilon(\xi_1)}\mathsf{e}^{iz_1\xi_1}}{\Xi^s_\varepsilon(\xi_1)\sqrt{1+4\varepsilon^2\xi_1^2(1-\mathfrak{c}^2\varepsilon^2)}}$, $s=\pm$, is integrable over $\mathbb{R}$. For $z_2=0$, none of these functions is integrable over $\mathbb{R}$, but they have an improper convergent integral; moreover, the sum in $s\in\{\pm\}$ is integrable over $\mathbb{R}$.

Proof. — For fixed $\varepsilon>0$, we have

$$\frac{|\xi|}{\mathfrak{c}^2\xi_1^2+\xi_2^2+(\xi_1^2+\varepsilon^2\xi_2^2)^2}\in L^1(\mathbb{R}^2),$$

thus $\partial_1\mathcal{K}^\varepsilon$ and $\partial_2\mathcal{K}^\varepsilon$ are (distributions associated with) continuous functions tending to zero at infinity (by Riemann–Lebesgue lemma). This implies that $\mathcal{K}^\varepsilon\in\mathcal{C}^1(\mathbb{R}^2)$. Furthermore, by Fubini's theorem, we then have, for $z\in\mathbb{R}^2$,

$$(2\pi)^2\partial_1\mathcal{K}^\varepsilon(z) = \int_{\mathbb{R}}\int_{\mathbb{R}}\frac{i\xi_1\mathsf{e}^{iz_1\xi_1+iz_2\xi_2}}{\mathfrak{c}^2\xi_1^2+\xi_2^2+\left(\xi_1^2+\varepsilon^2\xi_2^2\right)^2}\,d\xi_2d\xi_1.$$

At fixed ξ_1, we may use the Cauchy residue theorem (see [Lan99, Chap. VI, § 2, Theorem 2.2 p. 194] or [How03, Chap. 9, Theorem 9.1 p. 154]) for rational functions (in ξ_2) and get

$$(2\pi)^2\partial_1\mathcal{K}^\varepsilon(z) = i\pi\int_{\mathbb{R}}\sum_{s\,\in\,\{\pm\}}\frac{s\xi_1\mathsf{e}^{-|z_2|\Xi^s_\varepsilon(\xi_1)}\mathsf{e}^{iz_1\xi_1}}{\Xi^s_\varepsilon(\xi_1)\sqrt{1+4\varepsilon^2\xi_1^2\left(1-\mathfrak{c}^2\varepsilon^2\right)}}\,d\xi_1$$

and

$$(2\pi)^2\partial_2\mathcal{K}^\varepsilon(z) = \pi\int_{\mathbb{R}}\sum_{s\,\in\,\{\pm\}}\frac{s\mathsf{e}^{-|z_2|\Xi^s_\varepsilon(\xi_1)}\mathsf{e}^{iz_1\xi_1}}{\left(1+2\varepsilon^2\left(\xi_1^2-\varepsilon^2\Xi^s_\varepsilon(\xi_1)^2\right)\right)}\,d\xi_1.\qquad\square$$

In a similar way, we may wish to integrate at fixed $\xi_2\in\mathbb{R}$. Therefore, we let, for $s=\pm$,

$$T^s\overset{\text{def}}{=}\begin{cases}\dfrac{1}{2}\left(-\mathfrak{c}^2-2\varepsilon^2\xi_2^2-s\sqrt{\mathfrak{c}^4-4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)}\right) & \text{if } 4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)\leqslant\mathfrak{c}^4\\[2ex] \dfrac{1}{2}\left(-\mathfrak{c}^2-2\varepsilon^2\xi_2^2-is\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}\right) & \text{if } 4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)\geqslant\mathfrak{c}^4\end{cases}$$

be the two roots of the polynomial

$$\mathfrak{c}^2T+\xi_2^2+\left(T+\varepsilon^2\xi_2^2\right)^2 = T^2+T\left(\mathfrak{c}^2+2\varepsilon^2\xi_2^2\right)+\xi_2^2+\varepsilon^4\xi_2^4.$$

For $4\xi_2^2(1-\varepsilon^2\mathfrak{c}^2) < \mathfrak{c}^4$, the roots are real and we have $T^+ < T^- \leqslant 0$ (and $T^- = 0$ only for $\xi_2 = 0$). For $4\xi_2^2(1-\varepsilon^2\mathfrak{c}^2) > \mathfrak{c}^4$, the roots are complex conjugate with negative real part. We further define

$$\widetilde{\Xi}^s_\varepsilon(\xi_2) \stackrel{\text{def}}{=} \sqrt{-T^s} \in \{\mathrm{Re} > 0\},$$

where $\sqrt{\cdot}$ is the principal square root of complex numbers. Clearly, denoting

$$\xi_2^\varepsilon \stackrel{\text{def}}{=} \frac{\mathfrak{c}^2}{2}\sqrt{1-\varepsilon^2\mathfrak{c}^2},$$

we see that $\widetilde{\Xi}^-_\varepsilon$ is not smooth at $\xi_2 = \pm\xi_2^\varepsilon$, but is continuous and piecewise smooth. Then, we have the factorization

$$\begin{aligned}\mathfrak{c}^2\xi_1^2 + \xi_2^2 + \left(\xi_1^2+\varepsilon^2\xi_2^2\right)^2 \\ = \left(\xi_1 - i\widetilde{\Xi}^+_\varepsilon(\xi_2)\right) \times \left(\xi_1 - i\widetilde{\Xi}^-_\varepsilon(\xi_2)\right) \times \left(\xi_1 + i\widetilde{\Xi}^+_\varepsilon(\xi_2)\right) \times \left(\xi_1 + i\widetilde{\Xi}^-_\varepsilon(\xi_2)\right),\end{aligned}$$

where $\mathrm{Im}(i\widetilde{\Xi}^\pm_\varepsilon(\xi_2)) > 0$.

Proposition 3.2. — *The function $\mathcal{K}^\varepsilon$ is of class $\mathcal{C}^1$ in $\mathbb{R}^2$. Moreover, we have*

$$\begin{aligned}4\pi\partial_1\mathcal{K}^\varepsilon(z) = \int_{\{|\xi_2|\leqslant\xi_2^\varepsilon\}}\sum_{s\in\{\pm\}}\frac{s\mathrm{e}^{-|z_1|\widetilde{\Xi}^s_\varepsilon(\xi_2)+iz_2\xi_2}}{\sqrt{\mathfrak{c}^4-4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)}}\,d\xi_2 \\ - \int_{\{|\xi_2|\geqslant\xi_2^\varepsilon\}}\sum_{s\in\{\pm\}}\frac{is\mathrm{e}^{-|z_1|\widetilde{\Xi}^s_\varepsilon(\xi_2)+iz_2\xi_2}}{\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\,d\xi_2\end{aligned}$$

and

$$\begin{aligned}4\pi\partial_2\mathcal{K}^\varepsilon(z) = -i\int_{\{|\xi_2|\leqslant\xi_2^\varepsilon\}}\frac{s\xi_2\mathrm{e}^{-|z_1|\widetilde{\Xi}^s_\varepsilon(\xi_2)+iz_2\xi_2}}{\widetilde{\Xi}^s_\varepsilon(\xi_2)\sqrt{\mathfrak{c}^4-4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)}}\,d\xi_2 \\ - \int_{\{|\xi_2|\geqslant\xi_2^\varepsilon\}}\sum_{s\in\{\pm\}}\frac{s\xi_2\mathrm{e}^{-|z_1|\widetilde{\Xi}^s_\varepsilon(\xi_2)+iz_2\xi_2}}{\widetilde{\Xi}^s_\varepsilon(\xi_2)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\,d\xi_2.\end{aligned}$$

When $z_2 \neq 0$, each function $\xi_1 \mapsto \frac{s\xi_1\mathrm{e}^{-|z_2|\widetilde{\Xi}^s_\varepsilon(\xi_1)}\mathrm{e}^{iz_1\xi_1}}{\widetilde{\Xi}^s_\varepsilon(\xi_1)\sqrt{1+4\varepsilon^2\xi_1^2(1-\mathfrak{c}^2\varepsilon^2)}}$, $s = \pm$ is integrable over $\mathbb{R}$. For $z_2 = 0$, none of these functions is integrable over $\mathbb{R}$, but they have an improper convergent integral; moreover, the sum in $s \in \{\pm\}$ is integrable over $\mathbb{R}$.

Proof. — The proof is very similar to the one of Proposition 3.1 and therefore is omitted. ☐

In order to prove our estimates on the kernel, we shall need the following Propositions on the behaviors of $\Xi^\pm_\varepsilon$ and of $\widetilde{\Xi}^\pm_\varepsilon$.

Proposition 3.3. — *There exist C and C_0 such that we have the following inequalities.*

(i) *If $\varepsilon|\xi_1| \geqslant C_0$, then*

$$\frac{|\xi_1|}{C\varepsilon} \leqslant \Xi^\pm_\varepsilon(\xi_1) \leqslant C\frac{|\xi_1|}{\varepsilon} \qquad \text{and} \qquad 0 \leqslant \Xi^+_\varepsilon(\xi_1) - \Xi^-_\varepsilon(\xi_1) \leqslant \frac{C}{\varepsilon^2}.$$

(ii) *If* $0 < \varepsilon|\xi_1| \leqslant C_0$, *then*

$$\frac{1}{C\varepsilon^2} \leqslant |\Xi_\varepsilon^+(\xi_1)| \leqslant \frac{C}{\varepsilon^2}, \qquad \frac{1}{C}\left(|\xi_1| + \xi_1^2\right) \leqslant |\Xi_\varepsilon^-(\xi_1)| \leqslant C\left(|\xi_1| + \xi_1^2\right).$$

Proof. —

(i) We write, for $\xi_1 \neq 0$,

$$\begin{aligned}\Xi_\varepsilon^s(\xi_1) &= \frac{1}{\varepsilon^2\sqrt{2}}\sqrt{1 + 2\varepsilon^2\xi_1^2 + s\sqrt{1 + 4\varepsilon^2\xi_1^2\left(1 - \varepsilon^2\mathfrak{c}^2\right)}} \\ &= \frac{|\xi_1|}{\varepsilon}\sqrt{1 + \frac{1}{2\varepsilon^2\xi_1^2} + \frac{s}{\varepsilon|\xi_1|}\sqrt{1 - \varepsilon^2\mathfrak{c}^2 + \frac{1}{4\varepsilon^2\xi_1^2}}}\end{aligned} \tag{3.1}$$

and the result follows by choosing a suitably large constant C_0.

We also have

$$0 \leqslant \Xi_\varepsilon^+(\xi_1) - \Xi_\varepsilon^-(\xi_1) = \frac{\Xi_\varepsilon^+(\xi_1)^2 - \Xi_\varepsilon^-(\xi_1)^2}{\Xi_\varepsilon^+(\xi_1) + \Xi_\varepsilon^-(\xi_1)} = \frac{\sqrt{1 + 4\varepsilon^2\xi_1^2\left(1 - \varepsilon^2\mathfrak{c}^2\right)}}{\varepsilon^4\left(\Xi_\varepsilon^+(\xi_1) + \Xi_\varepsilon^-(\xi_1)\right)} \leqslant \frac{C}{\varepsilon^2}.$$

(ii) Recalling

$$\Xi_\varepsilon^+(\xi_1) = \frac{1}{\varepsilon^2\sqrt{2}}\sqrt{1 + 2\varepsilon^2\xi_1^2 + \sqrt{1 + 4\varepsilon^2\xi_1^2\left(1 - \varepsilon^2\mathfrak{c}^2\right)}},$$

the inequalities for Ξ_ε^+ and $\varepsilon|\xi_1| \leqslant C_0$ are immediate.

Then, the expression

$$\begin{aligned}\Xi_\varepsilon^-(\xi_1) &= \frac{1}{\varepsilon^2\sqrt{2}}\sqrt{1 + 2\varepsilon^2\xi_1^2 - \sqrt{1 + 4\varepsilon^2\xi_1^2\left(1 - \varepsilon^2\mathfrak{c}^2\right)}} \\ &= \frac{|\xi_1|\sqrt{2\mathfrak{c}^2 + 2\xi_1^2}}{\sqrt{1 + 2\varepsilon^2\xi_1^2 + \sqrt{1 + 4\varepsilon^2\xi_1^2\left(1 - \varepsilon^2\mathfrak{c}^2\right)}}}\end{aligned}$$

yields easily the conclusion for $0 \leqslant \varepsilon\xi_1 \leqslant C_0$. $\square$

Proposition 3.4. — *There exist* C *and* C_1 *such that we have the following inequalities.*

(i) *If* $\varepsilon^2|\xi_2| \geqslant C_1$, *then*

$$\frac{\varepsilon|\xi_2|}{C} \leqslant \mathrm{Re}\left(\widetilde{\Xi}_\varepsilon^\pm(\xi_2)\right) \leqslant \left|\widetilde{\Xi}_\varepsilon^\pm(\xi_2)\right| \leqslant C\varepsilon|\xi_2| \qquad \textit{and} \qquad \left|\mathrm{Im}\widetilde{\Xi}_\varepsilon^\pm(\xi_2)\right| \leqslant 1/\varepsilon,$$

$$\frac{\varepsilon}{C} \leqslant \mathrm{Re}\left(\partial_2\widetilde{\Xi}_\varepsilon^\pm(\xi_2)\right) \leqslant \left|\partial_2\widetilde{\Xi}_\varepsilon^\pm(\xi_2)\right| \leqslant C\varepsilon,$$

$$\frac{1}{C\varepsilon^3|\xi_2|^3} \leqslant \mathrm{Re}\left(\partial_2^2\widetilde{\Xi}_\varepsilon^\pm(\xi_2)\right) \leqslant \left|\partial_2^2\widetilde{\Xi}_\varepsilon^\pm(\xi_2)\right| \leqslant \frac{C}{\varepsilon^3|\xi_2|^3}$$

and

$$\left|\partial_2\widetilde{\Xi}_\varepsilon^s(\xi_2) - \varepsilon\right| \leqslant \frac{C}{\varepsilon^3\xi_2^2},$$

$$\left|\partial_2\left(\xi_2/\widetilde{\Xi}_\varepsilon^\pm(\xi_2)\right)\right| \leqslant \frac{C}{\varepsilon^3|\xi_2|^2}, \qquad \left|\partial_2\left(\xi_2^2/\widetilde{\Xi}_\varepsilon^\pm(\xi_2)\right) - \varepsilon^{-1}\right| \leqslant \frac{C}{\varepsilon^5|\xi_2|^2}.$$

(ii) *If* $\mathfrak{c}^2 < |\xi_2|^2 \leqslant C_1/\varepsilon^2$, *then*

$$\frac{\xi_2^{1/2}}{C} \leqslant \mathrm{Re}\left(\widetilde{\Xi}_\varepsilon^+(\xi_2)\right) \leqslant \left|\widetilde{\Xi}_\varepsilon^+(\xi_2)\right| \leqslant C\xi_2^{1/2},$$

$$\frac{1}{C\xi_2^{1/2}} \leqslant \mathrm{Re}\left(\partial_2\widetilde{\Xi}_\varepsilon^+(\xi_2)\right) \leqslant \left|\partial_2\widetilde{\Xi}_\varepsilon^+(\xi_2)\right| \leqslant \frac{C}{\xi_2^{1/2}},$$

and

$$\frac{1}{C\xi_2^{3/2}} \leqslant \mathrm{Re}\left(\partial_2^2\widetilde{\Xi}_\varepsilon^+(\xi_2)\right) \leqslant \left|\partial_2^2\widetilde{\Xi}_\varepsilon^+(\xi_2)\right| \leqslant \frac{C}{\xi_2^{3/2}}.$$

(iii) *If* $|\xi_2|^2 \leqslant \mathfrak{c}^2$, *then*

$$\frac{1}{C} \leqslant \frac{|\xi_2|}{\left|\widetilde{\Xi}_\varepsilon^-(\xi_2)\right|} \leqslant C.$$

Proof. — We work for $\xi_2 > 0$.
(i) There holds

$$\begin{aligned}-2T^s &= \mathfrak{c}^2 + 2\varepsilon^2\xi_2^2 + is\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right) - \mathfrak{c}^4}\\ &= 2\varepsilon^2\xi_2^2\left(1 + \frac{\mathfrak{c}^2}{2\varepsilon^2\xi_2^2} + \frac{is}{2\varepsilon^2\xi_2}\sqrt{4\left(1-\varepsilon^2\mathfrak{c}^2\right) - \mathfrak{c}^4/\xi_2^2}\right),\end{aligned}$$

thus

$$\begin{aligned}\mathrm{Re}\left(\widetilde{\Xi}_\varepsilon^s(\xi_2)\right) &= \mathrm{Re}\left(\sqrt{-T^s}\right)\\ &= \varepsilon\xi_2\mathrm{Re}\sqrt{1 + \frac{\mathfrak{c}^2}{2\varepsilon^2\xi_2^2} + \frac{is}{2\varepsilon^2\xi_2}\sqrt{4\left(1-\varepsilon^2\mathfrak{c}^2\right) - \mathfrak{c}^4/\xi_2^2}} \geqslant \frac{\varepsilon\xi_2}{2},\end{aligned}$$

and

$$\mathrm{Im}\left(\widetilde{\Xi}_\varepsilon^s(\xi_2)\right) \leqslant \frac{C}{\varepsilon}$$

provided $\varepsilon^2|\xi_2| \geqslant C_1$, with C_1 sufficiently large (but absolute). Moreover, from its definition, we have

$$\left|\widetilde{\Xi}_\varepsilon^s(\xi_2)\right| = \sqrt{|T^s|} \leqslant C\left(1 + \varepsilon^2\xi_2^2 + |\xi_2|\right)^{1/2} \leqslant C\varepsilon\xi_2.$$

We now differentiate:

$$\begin{aligned}\partial_2\widetilde{\Xi}_\varepsilon^s(\xi_2) = \varepsilon&\sqrt{1 + \frac{\mathfrak{c}^2}{2\varepsilon^2\xi_2^2} + \frac{is}{2\varepsilon^2\xi_2}\sqrt{4\left(1-\varepsilon^2\mathfrak{c}^2\right) - \mathfrak{c}^4/\xi_2^2}}\\ &+ \varepsilon\xi_2\frac{-\frac{\mathfrak{c}^2}{\varepsilon^2\xi_2^3} - \frac{is}{2\varepsilon^2\xi_2^2}\sqrt{4\left(1-\varepsilon^2\mathfrak{c}^2\right) - \mathfrak{c}^4/\xi_2^2} + \frac{is\mathfrak{c}^4}{\varepsilon^2\xi_2^4}\left(4\left(1-\varepsilon^2\mathfrak{c}^2\right) - \mathfrak{c}^4/\xi_2^2\right)^{-1/2}}{2\sqrt{1 + \frac{\mathfrak{c}^2}{2\varepsilon^2\xi_2^2} + \frac{is}{2\varepsilon^2\xi_2}\sqrt{4\left(1-\varepsilon^2\mathfrak{c}^2\right) - \mathfrak{c}^4/\xi_2^2}}}\end{aligned}$$

and deduce

$$\left|\partial_2\widetilde{\Xi}_\varepsilon^s(\xi_2)\right| \leqslant C\varepsilon, \qquad \left|\partial_2\widetilde{\Xi}_\varepsilon^s(\xi_2) - \varepsilon\right| \leqslant \frac{C}{\varepsilon^3\xi_2^2}$$

and, for ε small enough (depending on C_1),

$$\mathrm{Re}\left(\partial_2\widetilde{\Xi}_\varepsilon^s(\xi_2)\right) \geqslant \frac{\varepsilon}{4}.$$

Differentiating once again and noticing the cancellation of the terms

$$\frac{-\frac{is}{\varepsilon\xi_2^2}\sqrt{4\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4/\xi_2^2}}{\sqrt{1+\frac{\mathfrak{c}^2}{2\varepsilon^2\xi_2^2}+\frac{is}{2\varepsilon^2\xi_2}\sqrt{4\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4/\xi_2^2}}},$$

we obtain

$$\left|\partial_2^2\widetilde{\Xi}_\varepsilon^s(\xi_2)\right|\leqslant\frac{C}{\varepsilon^3\xi_2^3}\qquad\text{and}\qquad\mathrm{Re}\left(\partial_2^2\widetilde{\Xi}_\varepsilon^s(\xi_2)\right)\geqslant\frac{1}{4\varepsilon^3\xi_2^3}.$$

In addition,

$$\left|\partial_2\left(\xi_2/\widetilde{\Xi}_\varepsilon^\pm(\xi_2)\right)\right|=\frac{1}{\varepsilon}\left|\partial_2\left(1+\frac{\mathfrak{c}^2}{2\varepsilon^2\xi_2^2}+\frac{is}{2\varepsilon^2\xi_2}\sqrt{4\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4/\xi_2^2}\right)^{-1/2}\right|\leqslant\frac{C}{\varepsilon^3\xi_2^2}.$$

(ii) The estimates in the region $\mathfrak{c}^2\leqslant\xi_2\leqslant C_1/\varepsilon^2$ are analogous.
(iii) For $0\leqslant\xi_2\leqslant\mathfrak{c}^2$ and $s=-$, we use the expression, when $4\xi_2^2(1-\varepsilon^2\mathfrak{c}^2)\leqslant\mathfrak{c}^4$,

$$-2T^-=\mathfrak{c}^2+2\varepsilon^2\xi_2^2-\sqrt{\mathfrak{c}^4-4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)}=\frac{4\xi_2^2\left(1+\varepsilon^2\xi_2^2\right)}{\mathfrak{c}^2+2\varepsilon^2\xi_2^2+\sqrt{\mathfrak{c}^4-4\xi_2^2(1-\varepsilon^2\mathfrak{c}^2)}},$$

which yields, for $|\xi_2|\leqslant\mathfrak{c}^2$,

$$\frac{1}{C}\leqslant\frac{|\xi_2|}{\left|\widetilde{\Xi}_\varepsilon^-(\xi_2)\right|}\leqslant C,$$

and a similar inequality holds if $4\xi_2^2(1-\varepsilon^2\mathfrak{c}^2)\geqslant\mathfrak{c}^4$. □

3.2. Proof of Proposition 2.3 (i) (behaviors at infinity)

3.2.1. Estimates in the region $\{|z_2|\geqslant\max(1,|z_1|)\}$

In this region, $1\leqslant|z_2|\leqslant|z|\leqslant2|z_2|$. By parity, we may assume $z_2>0$. Thanks to the exponential decay at infinity in ξ_1 coming from Ξ_ε^s, we may use rough estimates. We shall use that, for $k\geqslant0$, $|z_2|\geqslant1\geqslant\varepsilon^2$ and $\alpha\in\mathbb{R}$, we have

$$\int_{C_0/\varepsilon}^{+\infty}\xi_1^\alpha\mathsf{e}^{-|z_2|\xi_1/(C\varepsilon)}\,d\xi_1=\left(\frac{C\varepsilon}{|z_2|}\right)^{\alpha+1}\int_{\frac{C_0|z_2|}{C\varepsilon^2}}^{+\infty}t^\alpha\mathsf{e}^{-t}\,dt\leqslant\frac{C(\alpha)}{|z_2|\varepsilon^{\alpha-1}}\mathsf{e}^{-C_0|z_2|/(C\varepsilon^2)}\leqslant\frac{C_k\varepsilon^k}{z_2^k},\tag{3.2}$$

$$\int_0^1\xi_1^k\mathsf{e}^{-z_2\xi_1/C}\,d\xi_1\leqslant\int_0^{+\infty}\xi_1^k\mathsf{e}^{-z_2\xi_1/C}\,d\xi_1\leqslant\frac{C_k}{z_2^{k+1}},\tag{3.3}$$

and

$$\int_1^{+\infty}\xi_1^\alpha\mathsf{e}^{-|z_2|\xi_1^2/C}\,d\xi_1=\frac{1}{2|z_2|^{(\alpha+1)/2}}\int_{|z_2|}^{+\infty}t^{(\alpha-1)/2}\mathsf{e}^{-t/C}\,dt\leqslant\frac{C(\alpha)}{|z_2|}\mathsf{e}^{-|z_2|/C}\leqslant\frac{C_k}{z_2^k}.\tag{3.4}$$

Let us fix $\beta \in \mathbb{N}^2$. Then, using the formula in Proposition 3.1 and applying Proposition 3.3, we obtain, by separating the contributions ($\xi_1 \geqslant C_0/\varepsilon$ and $s = \pm$), then ($\xi_1 \leqslant C_0/\varepsilon$ and $s = +$), then ($\xi_1 \leqslant 1$ and $s = -$) and finally ($1 \leqslant \xi_1 \leqslant C_0/\varepsilon$ and $s = -$),

$$\begin{aligned}\left|\partial_1 \partial_z^\beta \mathcal{K}^\varepsilon(z)\right| &\leqslant C \sum_{s \in \{\pm\}} \int_0^{+\infty} \frac{\xi_1^{1+\beta_1} \Xi_\varepsilon^s(\xi_1)^{\beta_2} \mathsf{e}^{-|z_2| \Xi_\varepsilon^s(\xi_1)}}{\Xi_\varepsilon^s(\xi_1)\sqrt{1+4\varepsilon^2\xi_1^2\,(1-\mathfrak{c}^2\varepsilon^2)}}\, d\xi_1 \\ &\leqslant C \int_{C_0/\varepsilon}^{+\infty} \frac{\xi_1^{\beta_1+\beta_2-1}}{\varepsilon^{\beta_2}} \mathsf{e}^{-z_2\xi_1/(C\varepsilon)}\, d\xi_1 + C\varepsilon^{2-2\beta_2} \int_0^{C_0/\varepsilon} \xi_1^{1+\beta_1} \mathsf{e}^{-z_2/(C\varepsilon^2)}\, d\xi_1 \\ &\quad + C \int_0^1 \xi_1^{\beta_1+\beta_2} \mathsf{e}^{-z_2\xi_1/C}\, d\xi_1 + C \int_1^{C_0/\varepsilon} \xi_1^{2\beta_2+\beta_1-1} \mathsf{e}^{-z_2\xi_1^2/C}\, d\xi_1 \\ &\leqslant \frac{C}{|z|^{\beta_1+\beta_2+1}},\end{aligned}$$

by (3.3), (3.2) and (3.4). The estimates for $\partial_2 \partial_z^\beta$ are similar:

$$\begin{aligned}\left|\partial_2 \partial_z^\beta \mathcal{K}^\varepsilon(z)\right| &\leqslant C \sum_{s \in \{\pm\}} \int_0^{+\infty} \frac{\xi_1^{\beta_1} \Xi_\varepsilon^s(\xi_1)^{\beta_2} \mathsf{e}^{-|z_2| \Xi_\varepsilon^s(\xi_1)}}{\sqrt{1+4\varepsilon^2\xi_1^2\,(1-\mathfrak{c}^2\varepsilon^2)}}\, d\xi_1 \\ &\leqslant C \int_{C_0/\varepsilon}^{+\infty} \frac{\xi_1^{\beta_1+\beta_2}}{\varepsilon^{\beta_2}} \mathsf{e}^{-z_2\xi_1/(C\varepsilon)}\, d\xi_1 + C\varepsilon^{-2\beta_2} \int_0^{C_0/\varepsilon} \xi_1^{\beta_1} \mathsf{e}^{-z_2/(C\varepsilon^2)}\, d\xi_1 \\ &\quad + C \int_0^1 \xi_1^{\beta_1+\beta_2} \mathsf{e}^{-z_2\xi_1/C}\, d\xi_1 + C \int_1^{C_0/\varepsilon} \xi_1^{\beta_1+2\beta_2} \mathsf{e}^{-z_2\xi_1^2/C}\, d\xi_1 \\ &\leqslant \frac{C}{|z|^{\beta_1+\beta_2+1}}.\end{aligned}$$

3.2.2. Estimates in the region $\{|z_1| \geqslant \max(1, |z_2|)\}$

This time, we shall take the expressions given in Proposition 3.2. As in the previous case, where $|z_2| \geqslant \max(1, |z_1|)$, we see that the contributions associated with the intervals $[C_1/\varepsilon^2, +\infty[$ are exponentially small. This comes from the inequalities, for $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$,

$$\begin{aligned}&\int_{C_1/\varepsilon^2}^{+\infty} \xi_2^\alpha \mathsf{e}^{-|z_1|\varepsilon\xi_2/C}\, d\xi_2 \leqslant C_{\alpha,k} \frac{\varepsilon^k}{z_1^k}, \\ &\int_{\mathfrak{c}^2}^{C_1/\varepsilon^2} \xi_2^\alpha \mathsf{e}^{-|z_1|\xi_2^{1/2}/C}\, d\xi_2 \leqslant \frac{C}{|z_1|^{2+2\alpha}} \int_{|z_1|\mathfrak{c}/C}^{+\infty} t^{2\alpha+1} \mathsf{e}^{-t}\, dt \leqslant C_{\alpha,k} \frac{\varepsilon^k}{z_1^k}\end{aligned}$$

and

$$\int_0^{\mathfrak{c}^2} \xi_2^\alpha \mathsf{e}^{-|z_1|/C}\, d\xi_2 \leqslant C_\alpha \mathsf{e}^{-|z_1|/C} \leqslant \frac{C_{\alpha,k}}{z_1^k}.$$

Therefore,

$$\begin{aligned}\left|\partial_1\partial_z^\beta\mathcal{K}^\varepsilon(z)\right| &\leqslant C\int_0^{+\infty}\sum_{s\in\{\pm\}}\frac{\left|\widetilde{\Xi}_\varepsilon^s(\xi_2)\right|^{\beta_1}\xi_2^{\beta_2}\mathsf{e}^{-|z_1|\mathrm{Re}\widetilde{\Xi}_\varepsilon^s(\xi_2)}}{\sqrt{|4\xi_2^2\,(1-\varepsilon^2\mathfrak{c}^2)-\mathfrak{c}^4|}}\,d\xi_2\\ &\leqslant C\varepsilon^{\beta_1}\int_{C_1/\varepsilon^2}^{+\infty}\xi_2^{\beta_1+\beta_2-1}\mathsf{e}^{-|z_1|\varepsilon\xi_2/C}\,d\xi_2+C\int_{\mathfrak{c}^2}^{C_1/\varepsilon^2}\xi_2^{\beta_1/2+\beta_2-1}\mathsf{e}^{-|z_1|\xi_2^{1/2}/C}\,d\xi_2\\ &\quad+C\int_0^{\mathfrak{c}^2}\frac{\xi_2^{\beta_2}\mathsf{e}^{-|z_1|/C}}{\sqrt{|4\xi_2^2\,(1-\varepsilon^2\mathfrak{c}^2)-\mathfrak{c}^4|}}\,d\xi_2+C\int_0^{\mathfrak{c}^2}\frac{\xi_2^{\beta_1+\beta_2}\mathsf{e}^{-|z_1|\xi_2/C}}{\sqrt{|4\xi_2^2\,(1-\varepsilon^2\mathfrak{c}^2)-\mathfrak{c}^4|}}\,d\xi_2\\ &\leqslant\frac{C}{|z|^{\beta_1+\beta_2+1}}\end{aligned}$$

and, similarly,

$$\begin{aligned}\left|\partial_2\partial_z^\beta\mathcal{K}^\varepsilon(z)\right| &\leqslant C\int_0^{+\infty}\sum_{s\in\{\pm\}}\frac{\left|\widetilde{\Xi}_\varepsilon^s(\xi_2)\right|^{\beta_1-1}\xi_2^{\beta_2+1}\mathsf{e}^{-|z_1|\mathrm{Re}\widetilde{\Xi}_\varepsilon^s(\xi_2)}}{\sqrt{|4\xi_2^2\,(1-\varepsilon^2\mathfrak{c}^2)-\mathfrak{c}^4|}}\,d\xi_2\\ &\leqslant C\varepsilon^{\beta_1-1}\int_{C_1/\varepsilon^2}^{+\infty}\xi_2^{\beta_1+\beta_2-1}\mathsf{e}^{-|z_1|\varepsilon\xi_2/C}\,d\xi_2\\ &\quad+C\int_{\mathfrak{c}^2}^{C_1/\varepsilon^2}\xi_2^{\beta_1/2+\beta_2-1/2}\mathsf{e}^{-|z_1|\xi_2^{1/2}/C}\,d\xi_2\\ &\quad+C\int_0^{\mathfrak{c}^2}\frac{\xi_2^{\beta_2+1}\mathsf{e}^{-|z_1|/C}}{\sqrt{|4\xi_2^2\,(1-\varepsilon^2\mathfrak{c}^2)-\mathfrak{c}^4|}}\,d\xi_2+C\int_0^{\mathfrak{c}^2}\frac{\xi_2^{\beta_1+\beta_2}\mathsf{e}^{-|z_1|\xi_2/C}}{\sqrt{|4\xi_2^2\,(1-\varepsilon^2\mathfrak{c}^2)-\mathfrak{c}^4|}}\,d\xi_2\\ &\leqslant\frac{C}{|z|^{\beta_1+\beta_2+1}}.\end{aligned}$$

3.3. Proof of Proposition 2.3 (ii) (behaviors at the origin)

In this region, the estimates are more delicate and we shall use either compensations between the signs $s=\pm$, either integration by parts, either both.

3.3.1. Estimates in the region $\{|z_1|\leqslant|z_2|\leqslant 1\}$

We have to replace (3.2) by the following inequality, for $|z_2|\leqslant 1$ and $\alpha>-1$,

$$\int_{C_0/\varepsilon}^{+\infty}\xi_1^\alpha\mathsf{e}^{-|z_2|\xi_1/(C\varepsilon)}\,d\xi_1=\left(\frac{C\varepsilon}{|z_2|}\right)^{\alpha+1}\int_{\frac{C_0|z_2|}{C\varepsilon^2}}^{+\infty}t^\alpha\mathsf{e}^{-t}\,dt\leqslant\frac{C(\alpha)}{|z_2|\varepsilon^{\alpha-1}}\mathsf{e}^{-C_0|z_2|/(C\varepsilon^2)}$$

as before, when $|z_2|\geqslant\varepsilon^2$, and if $|z_2|\leqslant\varepsilon^2$, then the integral in t is $\leqslant C(\alpha)$ since $\alpha>-1$. Therefore,

$$\begin{aligned}(3.5)\quad&\int_{C_0/\varepsilon}^{+\infty}\xi_1^\alpha\mathsf{e}^{-|z_2|\xi_1/(C\varepsilon)}\,d\xi_1\\ &\qquad\leqslant\frac{C(\alpha)}{|z_2|\varepsilon^{\alpha-1}}\mathsf{e}^{-C_0|z_2|/(C\varepsilon^2)}\mathbf{1}_{1\geqslant|z_2|\geqslant\varepsilon^2}+C(\varepsilon/|z_2|)^{\alpha+1}\mathbf{1}_{|z_2|\leqslant\varepsilon^2}.\end{aligned}$$

For $\alpha > -1$, we have

$$\begin{aligned}\int_0^{C_0/\varepsilon} \xi_1^\alpha \mathsf{e}^{-|z_2|\xi_1^2/C}\, d\xi_1 &= \frac{1}{|z_2|^{(1+\alpha)/2}} \int_0^{C_0\sqrt{|z_2|/\varepsilon^2}} t^\alpha \mathsf{e}^{-t^2/C}\, dt \\ &\leqslant \frac{C}{|z_2|^{(1+\alpha)/2}} \mathbf{1}_{|z_2|\geqslant\varepsilon^2} + \frac{C(\alpha)}{|z_2|^{(1+\alpha)/2}} (\sqrt{|z_2|}/\varepsilon)^{\alpha+1} \mathbf{1}_{|z_2|\leqslant\varepsilon^2} \\ &\leqslant \frac{C(\alpha)}{(|z_2|+\varepsilon^2)^{(1+\alpha)/2}}.\end{aligned} \tag{3.6}$$

Furthermore, we shall use the exponential integral function $E_1 : \mathbb{R}_+^* \to \mathbb{R}$ given by

$$E_1(y) = \int_y^{+\infty} \frac{\mathsf{e}^{-z}}{z}\, dz.$$

It is known (see for instance [AS64]) that

$$E_1(y) \sim_{y\to+\infty} \frac{\mathsf{e}^{-y}}{y}, \qquad E_1(y) \sim_{y\to 0} -\ln y. \tag{3.7}$$

• *Estimate for* $\partial_1\mathcal{K}^\varepsilon$. From proposition 3.1, we have

$$-4i\pi\partial_1\mathcal{K}^\varepsilon(z) = \int_{\mathbb{R}} \sum_{s\,\in\,\{\pm\}} \frac{s\xi_1 \mathsf{e}^{-|z_2|\Xi_\varepsilon^s(\xi_1)} \mathsf{e}^{iz_1\xi_1}}{\Xi_\varepsilon^s(\xi_1)\sqrt{1+4\varepsilon^2\xi_1^2\,(1-\mathfrak{c}^2\varepsilon^2)}}\, d\xi_1.$$

We first estimate the contributions for $|\xi_1| \geqslant C_0/\varepsilon$ and $s = \pm$:

$$\begin{aligned}\left|\int_{\{|\xi_1|\,\geqslant\, C_0/\varepsilon\}} \frac{s\xi_1 \mathsf{e}^{-|z_2|\Xi_\varepsilon^s(\xi_1)} \mathsf{e}^{iz_1\xi_1}}{\Xi_\varepsilon^s(\xi_1)\sqrt{1+4\varepsilon^2\xi_1^2\,(1-\mathfrak{c}^2\varepsilon^2)}}\, d\xi_1\right| &\leqslant C\int_{C_0/\varepsilon}^{+\infty} \frac{\mathsf{e}^{-|z_2|\xi_1/(C\varepsilon)}}{\xi_1}\, d\xi_1 \\ &\leqslant C\int_{|z_2|/(C\varepsilon^2)}^{+\infty} \frac{\mathsf{e}^{-t}}{t}\, dt \\ &= CE_1\left(|z_2|/(C\varepsilon^2)\right).\end{aligned}$$

We now consider the case $s = -$. By using Proposition 3.3, we obtain

$$\begin{aligned}&\left|\int_{\{|\xi_1|\,\leqslant\, C_0/\varepsilon\}} \frac{s\xi_1 \mathsf{e}^{-|z_2|\Xi_\varepsilon^-(\xi_1)} \mathsf{e}^{iz_1\xi_1}}{\Xi_\varepsilon^-(\xi_1)\sqrt{1+4\varepsilon^2\xi_1^2\,(1-\mathfrak{c}^2\varepsilon^2)}}\, d\xi_1\right| \\ &\qquad\leqslant C\int_0^1 \mathsf{e}^{-|z_2|\xi_1/C}\, d\xi_1 + C\int_1^{C_0/\varepsilon} \frac{\mathsf{e}^{-|z_2|\xi_1^2/C}}{\xi_1}\, d\xi_1 \\ &\qquad\leqslant C + C\int_{|z_2|/C}^{|z_2|/(C'\varepsilon^2)} \frac{\mathsf{e}^{-t}}{t}\, dt \\ &\qquad\leqslant C\left(1+\left|\ln\left(|z_2|+\varepsilon^2\right)\right|\right),\end{aligned}$$

since, for $|z_2| \leqslant \varepsilon^2$, $\frac{\mathsf{e}^{-t}}{t} \leqslant \frac{1}{t}$, and for $|z_2| \geqslant \varepsilon^2$, the integral is $\leqslant E_1(|z_2|/C) \leqslant C(1-\ln|z_2|)$.

For the case $s=+$, we have similarly

$$\left|\int_{\{|\xi_1|\leqslant C_0/\varepsilon\}} \frac{s\xi_1 \mathrm{e}^{-|z_2|\Xi_\varepsilon^+(\xi_1)}\mathrm{e}^{iz_1\xi_1}}{\Xi_\varepsilon^+(\xi_1)\sqrt{1+4\varepsilon^2\xi_1^2\left(1-\mathfrak{c}^2\varepsilon^2\right)}}\,d\xi_1\right| \leqslant C\int_0^{C_0/\varepsilon} \varepsilon^2\xi_1\mathrm{e}^{-|z_2|/(C\varepsilon^2)}\,d\xi_1 \leqslant C\mathrm{e}^{-|z_2|/(C\varepsilon^2)}.$$

As a consequence,

$$\begin{aligned}|\partial_1\mathcal{K}^\varepsilon(z)| &\leqslant CE_1\left(|z_2|/(C\varepsilon^2)\right)+C\left(1+\left|\ln\left(|z_2|+\varepsilon^2\right)\right|\right)+C\mathrm{e}^{-|z_2|/(C\varepsilon^2)}\\ &\leqslant C(1+|\ln z_2|).\end{aligned}$$

• *Estimate for* $\partial_1^2\mathcal{K}^\varepsilon$. The differentiation under the integral sign is here again easily justified and yields

$$\begin{aligned}|\partial_1^2\mathcal{K}^\varepsilon(z)| &\leqslant C\int_{C_0/\varepsilon}^{+\infty}\mathrm{e}^{-z_2\xi_1/(C\varepsilon)}\,d\xi_1+C\varepsilon^2\int_0^{C_0/\varepsilon}\xi_1^2\mathrm{e}^{-z_2/(C\varepsilon^2)}\,d\xi_1\\ &\qquad +C\int_0^1\xi_1\mathrm{e}^{-z_2\xi_1/C}\,d\xi_1+C\int_1^{C_0/\varepsilon}\mathrm{e}^{-z_2\xi_1^2/C}\,d\xi_1\\ &\leqslant \frac{C\varepsilon}{z_2}\mathrm{e}^{-z_2/(C\varepsilon^2)}+\frac{C}{\varepsilon}\mathrm{e}^{-z_2/(C\varepsilon^2)}+C+\frac{C}{(z_2+\varepsilon^2)^{1/2}}\\ &\leqslant \frac{C\varepsilon\mathbf{1}_{z_2\leqslant\varepsilon^2}}{z_2}+\frac{C\mathbf{1}_{z_2\geqslant\varepsilon^2}}{\sqrt{z_2}},\end{aligned}\tag{3.8}$$

by (3.6) with $\alpha=0>-1$.

• *Estimate for* $\partial_1\partial_2\mathcal{K}^\varepsilon$. Similarly, by (3.6) with $\alpha=1>-1$,

$$\begin{aligned}|\partial_1\partial_2\mathcal{K}^\varepsilon(z)| &\leqslant C\int_{C_0/\varepsilon}^{+\infty}\frac{\mathrm{e}^{-z_2\xi_1/(C\varepsilon)}}{\varepsilon}\,d\xi_1+C\int_0^{C_0/\varepsilon}\xi_1\mathrm{e}^{-z_2/(C\varepsilon^2)}\,d\xi_1\\ &\qquad +C\int_0^1\xi_1\mathrm{e}^{-z_2\xi_1/C}\,d\xi_1+C\int_1^{C_0/\varepsilon}\xi_1\mathrm{e}^{-z_2\xi_1^2/C}\,d\xi_1\\ &\leqslant \frac{C}{z_2}\mathrm{e}^{-z_2/(C\varepsilon^2)}+\frac{C}{\varepsilon^2}\mathrm{e}^{-z_2/(C\varepsilon^2)}+C+\frac{C}{z_2+\varepsilon^2}\\ &\leqslant \frac{C}{z_2}.\end{aligned}$$

• *Estimate for* $\partial_2\mathcal{K}^\varepsilon$. We use the formula in Proposition 3.1. By Proposition 3.3, we have

$$\begin{aligned}|\partial_2\mathcal{K}^\varepsilon(z)| &\leqslant C\int_{C_0/\varepsilon}^{+\infty}\frac{\mathrm{e}^{-z_2\xi_1/(C\varepsilon)}}{\varepsilon\xi_1}\,d\xi_1+C\int_0^{C_0/\varepsilon}\mathrm{e}^{-z_2/(C\varepsilon^2)}\,d\xi_1\\ &\qquad +C\int_0^1\mathrm{e}^{-z_2\xi_1/C}\,d\xi_1+C\int_1^{C_0/\varepsilon}\mathrm{e}^{-z_2\xi_1^2/C}\,d\xi_1\\ &\leqslant \frac{C}{\varepsilon}E_1\left(z_2/(C\varepsilon^2)\right)+\frac{C}{\varepsilon}\mathrm{e}^{-z_2/(C\varepsilon^2)}+C+\frac{C}{\sqrt{z_2+\varepsilon^2}}\\ &\leqslant \frac{C\mathbf{1}_{\varepsilon^2\leqslant z_2}}{\sqrt{z_2}}+\frac{C}{\varepsilon}\mathbf{1}_{\varepsilon^2\geqslant z_2}\left(1+\left|\ln\left(z_2/\varepsilon^2\right)\right|\right),\end{aligned}$$

with (3.7) and $\alpha = 0 > -1$ for (3.6).

• *Estimate for $\partial_2^2\mathcal{K}^\varepsilon$.* By similar computations, we obtain, by (3.6) with $\alpha = 2 > -1$,

$$\begin{aligned}
|\partial_2^2\mathcal{K}^\varepsilon(z)| &\leqslant C\int_{C_0/\varepsilon}^{+\infty}\frac{1}{\varepsilon^2}\mathsf{e}^{-z_2\xi_1/(C\varepsilon)}\,d\xi_1 + \frac{C}{\varepsilon^2}\int_0^{C_0/\varepsilon}\mathsf{e}^{-z_2/(C\varepsilon^2)}\,d\xi_1\\
&\quad + C\int_0^1 \xi_1\mathsf{e}^{-z_2\xi_1/C}\,d\xi_1 + C\int_1^{C_0/\varepsilon}\xi_1^2\mathsf{e}^{-z_2\xi_1^2/C}\,d\xi_1\\
&\leqslant \frac{C}{z_2\varepsilon}\mathsf{e}^{-z_2/(C\varepsilon^2)} + \frac{C}{\varepsilon^3}\mathsf{e}^{-z_2/(C\varepsilon^2)} + C + \frac{C}{(z_2+\varepsilon^2)^{3/2}}\\
&\leqslant \frac{C\mathbf{1}_{\varepsilon^2\leqslant z_2}}{z_2^{3/2}} + \frac{C\mathbf{1}_{z_2\leqslant\varepsilon^2}}{\varepsilon z_2}.
\end{aligned}$$

From all these estimates, we infer

$$\begin{aligned}
&\int_{\{|z_1|\leqslant|z_2|\leqslant 1\}}|\partial_1\mathcal{K}^\varepsilon(z)|\,dz\\
&\qquad\leqslant C\int_0^1\int_0^{z_2}(1+|\ln z_2|)\,dz_1dz_2 = C\int_0^1 z_2(1+|\ln z_2|)\,dz_2 \leqslant C,
\end{aligned}$$

$$\begin{aligned}
&\int_{\{|z_1|\leqslant|z_2|\leqslant 1\}}|\partial_2\mathcal{K}^\varepsilon(z)|\,dz\\
&\qquad\leqslant \frac{C}{\varepsilon}\int_0^{\varepsilon^2} z_2\times\left(1+\left|\ln\left(z_2/\varepsilon^2\right)\right|\right)\,dz_2 + C\int_{\varepsilon^2}^1 z_2\times\frac{1}{\sqrt{z_2}}\,dz_2 \leqslant C
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\{|z_1|\leqslant|z_2|\leqslant 1\}}\left|\partial_1^2\mathcal{K}^\varepsilon(z)\right| + \left|\partial_2^2\mathcal{K}^\varepsilon(z)\right| + |\partial_1\partial_2\mathcal{K}^\varepsilon(z)|\,dz\\
&\quad\leqslant C\int_0^{\varepsilon^2} z_2\left(\frac{\varepsilon}{z_2}+\frac{1}{\varepsilon z_2}+\frac{1}{z_2}\right)dz_2 + C\int_{\varepsilon^2}^1 z_2\left(\frac{1}{z_2^{1/2}}+\frac{1}{z_2^{3/2}}+\frac{1}{z_2}\right)dz_2 \leqslant C.
\end{aligned}$$

We now turn to third order derivatives. Then, we have to be a little more careful for the contributions with $\varepsilon|\xi_1| \geqslant C_0$, since applying (3.5) with $\alpha \geqslant 1$ yields an upper bound which shall not be integrable (for the measure z_2dz_2).

• *Estimate for $\partial_1\partial_2^2\mathcal{K}^\varepsilon$.* The estimates for $\varepsilon|\xi_1| \leqslant C_0$ shall be similar to those previously used. We then consider, by parity, the case $\xi_1 \geqslant C_0/\varepsilon$ only. First, we have

$$\begin{aligned}
\sum_{s=\pm} s\frac{\xi_1\Xi_\varepsilon^s(\xi_1)}{\sqrt{1+4\varepsilon^2\xi_1^2\left(1-\varepsilon^2\mathfrak{c}^2\right)}}\exp\left(iz_1\xi_1 - |z_2|\Xi_\varepsilon^s(\xi_1)\right) = \frac{\xi_1\mathsf{e}^{iz_1\xi_1}}{\sqrt{1+4\varepsilon^2\xi_1^2\left(1-\varepsilon^2\mathfrak{c}^2\right)}}\\
\times\left(\left(\Xi_\varepsilon^+(\xi_1)-\Xi_\varepsilon^-(\xi_1)\right)\mathsf{e}^{-|z_2|\Xi_\varepsilon^+(\xi_1)} + \Xi_\varepsilon^-(\xi_1)\left(\mathsf{e}^{-|z_2|\Xi_\varepsilon^+(\xi_1)} - \mathsf{e}^{-|z_2|\Xi_\varepsilon^-(\xi_1)}\right)\right).
\end{aligned}$$

From Proposition 3.3, we have

$$0 \leqslant \Xi_\varepsilon^+(\xi_1) - \Xi_\varepsilon^-(\xi_1) \leqslant \frac{C}{\varepsilon^2}, \tag{3.9}$$

thus

$$\left|\mathsf{e}^{-|z_2|\Xi_\varepsilon^+(\xi_1)} - \mathsf{e}^{-|z_2|\Xi_\varepsilon^-(\xi_1)}\right| = \\ \mathsf{e}^{-|z_2|\Xi_\varepsilon^-(\xi_1)}\left|\mathsf{e}^{-|z_2|(\Xi_\varepsilon^+(\xi_1)-\Xi_\varepsilon^-(\xi_1))} - 1\right| \leqslant C\mathsf{e}^{-|z_2|\xi_1/(C\varepsilon)}\left(1-\mathsf{e}^{-C|z_2|/\varepsilon^2}\right). \tag{3.10}$$

As a consequence, from (3.5) with $\alpha = 1 > -1$,

$$\begin{aligned}\left|\int_{C_0/\varepsilon}^{+\infty}\sum_{s=\pm} s\frac{\xi_1\Xi_\varepsilon^s(\xi_1)}{\sqrt{1+4\varepsilon^2\xi_1^2\left(1-\varepsilon^2\mathfrak{c}^2\right)}}\exp\left(iz_1\xi_1 - z_2\Xi_\varepsilon^s(\xi_1)\right)\,d\xi_1\right|& \\ \leqslant C\int_{C_0/\varepsilon}^{+\infty}\frac{1}{\varepsilon^3}\mathsf{e}^{-z_2\xi_1/(C\varepsilon)} + \frac{\xi_1}{\varepsilon^2}\mathsf{e}^{-z_2\xi_1/(C\varepsilon)}\left(1-\mathsf{e}^{-Cz_2/\varepsilon^2}\right)\,d\xi_1& \\ \leqslant \frac{C}{\varepsilon^2 z_2}\mathsf{e}^{-z_2/(C\varepsilon^2)} + C\left(1-\mathsf{e}^{-C|z_2|/\varepsilon^2}\right)\left(\mathbf{1}_{z_2\geqslant\varepsilon^2}\frac{\mathsf{e}^{-z_2/(C'\varepsilon^2)}}{|z_2|} + \mathbf{1}_{z_2\leqslant\varepsilon^2}\frac{\varepsilon^2}{z_2^2}\right)& \\ \leqslant \frac{C}{\varepsilon^2 z_2}\mathsf{e}^{-z_2/(C\varepsilon^2)} + \frac{C\mathbf{1}_{z_2\leqslant\varepsilon^2}}{z_2},&\end{aligned}$$

by the basic inequality $1-\mathsf{e}^{-C|z_2|/\varepsilon^2} \leqslant C|z_2|/\varepsilon^2$. We then infer

$$\left|\int_{C_0/\varepsilon}^{+\infty}\sum_{s=\pm} s\frac{\xi_1\Xi_\varepsilon^s(\xi_1)}{\sqrt{1+4\varepsilon^2\xi_1^2\left(1-\varepsilon^2\mathfrak{c}^2\right)}}\exp\left(iz_1\xi_1 - |z_2|\Xi_\varepsilon^s(\xi_1)\right)\,d\xi_1\right| \leqslant \frac{C}{\varepsilon^2|z_2|}\mathsf{e}^{-|z_2|/(C\varepsilon^2)}.$$

We then obtain, by (3.6) with $\alpha = 3 > -1$,

$$\begin{aligned}\left|\partial_1\partial_2^2\mathcal{K}^\varepsilon(z)\right| &\leqslant \frac{C}{\varepsilon^2|z_2|}\mathsf{e}^{-|z_2|/(C\varepsilon^2)} + \frac{C}{\varepsilon^2}\int_0^{C_0/\varepsilon}\xi_1\mathsf{e}^{-z_2/(C\varepsilon^2)}\,d\xi_1 \\ &\quad + C\int_0^1\xi_1^2\mathsf{e}^{-z_2\xi_1/C}\,d\xi_1 + C\int_1^{C_0/\varepsilon}\xi_1^3\mathsf{e}^{-z_2\xi_1^2/C}\,d\xi_1 \\ &\leqslant \frac{C}{\varepsilon^2|z_2|}\mathsf{e}^{-|z_2|/(C\varepsilon^2)} + \frac{C}{\varepsilon^4}\mathsf{e}^{-z_2/(C\varepsilon^2)} + C + \frac{C}{\left(|z_2|+\varepsilon^2\right)^2} \\ &\leqslant \frac{C}{|z_2|\left(\varepsilon^2+|z_2|\right)}.\end{aligned}$$

• *Estimate for* $\partial_1^3\mathcal{K}^\varepsilon$. Here again, we have to pay attention to the contributions with $\varepsilon|\xi_1| \geqslant C_0$. We have

$$\begin{aligned}\sum_{s=\pm} s\frac{\exp(-|z_2|\Xi_\varepsilon^s(\xi_1))}{\Xi_\varepsilon^s(\xi_1)} &= \frac{\Xi_\varepsilon^-(\xi_1)-\Xi_\varepsilon^+(\xi_1)}{\Xi_\varepsilon^-(\xi_1)\Xi_\varepsilon^+(\xi_1)}\exp(-|z_2|\Xi_\varepsilon^+(\xi_1)) \\ &\quad + \frac{1}{\Xi_\varepsilon^-(\xi_1)}\Big(\exp(-|z_2|\Xi_\varepsilon^+(\xi_1)) - \exp(-|z_2|\Xi_\varepsilon^-(\xi_1))\Big),\end{aligned}$$

hence, by (3.9) and (3.10),

$$\left|\sum_{s=\pm} s\frac{\exp(-z_2\Xi_\varepsilon^s(\xi_1))}{\Xi_\varepsilon^s(\xi_1)}\right| \leqslant \frac{C}{\xi_1^2}\mathsf{e}^{-z_2\xi_1/(C\varepsilon)} + \frac{C\varepsilon}{\xi_1}\mathsf{e}^{-z_2\xi_1/(C\varepsilon)}\left(1-\mathsf{e}^{-Cz_2/\varepsilon^2}\right).$$

Arguing as for $\partial_1\partial_2^2\mathcal{K}^\varepsilon$, we then infer

$$\begin{aligned}
&\left|\int_{C_0/\varepsilon}^{+\infty}\sum_{s=\pm} s\frac{\xi_1^3}{\Xi_\varepsilon^s(\xi_1)\sqrt{1+4\varepsilon^2\xi_1^2\left(1-\varepsilon^2\mathfrak{c}^2\right)}}\exp\left(iz_1\xi_1 - z_2\Xi_\varepsilon^s(\xi_1)\right)\,d\xi_1\right|\\
&\leqslant C\int_{C_0/\varepsilon}^{+\infty}\frac{1}{\varepsilon}\mathsf{e}^{-z_2\xi_1/(C\varepsilon)} + \xi_1\mathsf{e}^{-z_2\xi_1/(C\varepsilon)}(1-\mathsf{e}^{-z_2/(C\varepsilon^2)})\,d\xi_1\\
&\leqslant \frac{C}{z_2}\mathsf{e}^{-z_2/(C\varepsilon^2)} + \mathbf{1}_{z_2\geqslant\varepsilon^2}\frac{\mathsf{e}^{-z_2/(C'\varepsilon^2)}}{z_2} + \frac{C}{z_2}\mathbf{1}_{z_2\leqslant\varepsilon^2}\\
&\leqslant \frac{C}{z_2}\mathsf{e}^{-z_2/(C\varepsilon^2)},
\end{aligned}$$

by (3.5) with $\alpha = 1 > -1$.

By computations similar to those for $\partial_1\partial_2^2\mathcal{K}^\varepsilon$ and by (3.6) with $\alpha = 1$, we then obtain

$$\begin{aligned}
\left|\partial_1^3\mathcal{K}^\varepsilon(z)\right| &\leqslant \frac{C}{z_2}\mathsf{e}^{-z_2/(C\varepsilon^2)} + C\varepsilon^2\int_0^{C_0/\varepsilon}\xi_1^3\mathsf{e}^{-z_2/(C\varepsilon^2)}\,d\xi_1\\
&\quad + C\int_0^1\xi_1^2\mathsf{e}^{-z_2\xi_1/C}\,d\xi_1 + C\int_1^{C_0/\varepsilon}\xi_1\mathsf{e}^{-z_2\xi_1^2/C}\,d\xi_1\\
&\leqslant \frac{C}{z_2}\mathsf{e}^{-z_2/(C\varepsilon^2)} + \frac{C}{\varepsilon^2}\mathsf{e}^{-z_2/(C\varepsilon^2)} + C + \frac{C}{z_2+\varepsilon^2}\\
&\leqslant \frac{C}{z_2}.
\end{aligned}$$

• *Estimate for* $\partial_2^3\mathcal{K}^\varepsilon$. First, we write

$$\begin{aligned}
\sum_{s=\pm} s\frac{\left[\Xi_\varepsilon^s(\xi_1)\right]^2}{\sqrt{1+4\varepsilon^2\xi_1^2\left(1-\varepsilon^2\mathfrak{c}^2\right)}}\exp\left(iz_1\xi_1 - z_2\Xi_\varepsilon^s(\xi_1)\right) &= \frac{\mathsf{e}^{iz_1\xi_1}}{\sqrt{1+4\varepsilon^2\xi_1^2\left(1-\varepsilon^2\mathfrak{c}^2\right)}}\\
\times\left(\left(\left[\Xi_\varepsilon^+(\xi_1)\right]^2 - \left[\Xi_\varepsilon^-(\xi_1)\right]^2\right)\mathsf{e}^{-z_2\Xi_\varepsilon^+(\xi_1)} + \left[\Xi_\varepsilon^-(\xi_1)\right]^2\left(\mathsf{e}^{-z_2\Xi_\varepsilon^+(\xi_1)} - \mathsf{e}^{-z_2\Xi_\varepsilon^-(\xi_1)}\right)\right)&.
\end{aligned}$$

By using (3.9), we have

$$\left|\left[\Xi_\varepsilon^+(\xi_1)\right]^2 - \left[\Xi_\varepsilon^-(\xi_1)\right]^2\right| = \left|\Xi_\varepsilon^+(\xi_1) + \Xi_\varepsilon^-(\xi_1)\right| \times \left|\Xi_\varepsilon^+(\xi_1) - \Xi_\varepsilon^-(\xi_1)\right| \leqslant \frac{C\xi_1}{\varepsilon^3}$$

and we deduce, by (3.10),

$$\begin{aligned}
&\left|\sum_{s=\pm} s\frac{\left[\Xi_\varepsilon^s(\xi_1)\right]^2}{\sqrt{1+4\varepsilon^2\xi_1^2\left(1-\varepsilon^2\mathfrak{c}^2\right)}}\exp\left(iz_1\xi_1 - z_2\Xi_\varepsilon^s(\xi_1)\right)\right|\\
&\qquad\leqslant \frac{C}{\varepsilon\xi_1}\mathsf{e}^{-z_2\xi_1/(C\varepsilon)}\left(\frac{\xi_1}{\varepsilon^3} + \frac{\xi_1^2}{\varepsilon^2}\left(1-\mathsf{e}^{-z_2/(C\varepsilon^2)}\right)\right)\\
&\qquad= C\mathsf{e}^{-z_2\xi_1/(C\varepsilon)}\left(\frac{1}{\varepsilon^4} + \frac{\xi_1}{\varepsilon^3}\left(1-\mathsf{e}^{-z_2/(C\varepsilon^2)}\right)\right).
\end{aligned}$$

As a consequence, by (3.5) with $\alpha = 1 > -1$,

$$\begin{aligned}\left|\int_{C_0/\varepsilon}^{+\infty} \sum_{s=\pm} s \frac{\left[\Xi_\varepsilon^s(\xi_1)\right]^2}{\sqrt{1+4\varepsilon^2\xi_1^2\left(1-\varepsilon^2\mathfrak{c}^2\right)}} \exp\left(iz_1\xi_1 - z_2\Xi_\varepsilon^s(\xi_1)\right)\, d\xi_1\right| \\ \leqslant C\int_{C_0/\varepsilon}^{+\infty} \frac{1}{\varepsilon^4}\mathsf{e}^{-z_2\xi_1/(C\varepsilon)} + \frac{\xi_1}{\varepsilon^3}\mathsf{e}^{-z_2\xi_1/(C\varepsilon)}\left(1-\mathsf{e}^{-z_2/(C\varepsilon^2)}\right)\, d\xi_1 \\ \leqslant \frac{C}{\varepsilon^3 z_2}\mathsf{e}^{-z_2/(C\varepsilon^2)} + \frac{C}{\varepsilon^3 z_2}\mathsf{e}^{-z_2/(C\varepsilon^2)}\mathbf{1}_{z_2\geqslant\varepsilon^2} + \frac{C}{\varepsilon^3 z_2}\mathbf{1}_{z_2\leqslant\varepsilon^2} \\ \leqslant \frac{C}{\varepsilon^3 z_2}\mathsf{e}^{-z_2/(C\varepsilon^2)}.\end{aligned}$$

By similar computations, we then obtain

$$\begin{aligned}\left|\partial_2^3\mathcal{K}^\varepsilon(z)\right| &\leqslant \frac{C}{\varepsilon^3 z_2}\mathsf{e}^{-z_2/(C\varepsilon^2)} + \frac{C}{\varepsilon^4}\int_0^{C_0/\varepsilon} \mathsf{e}^{-z_2/(C\varepsilon^2)}\, d\xi_1 \\ &\quad + C\int_0^1 \xi_1^2\mathsf{e}^{-z_2\xi_1/C}\, d\xi_1 + C\int_1^{C_0/\varepsilon} \xi_1^4\mathsf{e}^{-z_2\xi_1^2/C}\, d\xi_1 \\ &\leqslant \frac{C}{\varepsilon^3 z_2}\mathsf{e}^{-z_2/(C\varepsilon^2)} + \frac{C}{\varepsilon^5}\mathsf{e}^{-z_2/(C\varepsilon^2)} + C + \frac{C}{\left(\varepsilon^2+z_2\right)^{5/2}} \\ &\leqslant \frac{C}{z_2(\varepsilon^2+z_2)^{3/2}}.\end{aligned}$$

• *Estimate for* $\partial_1^2\partial_2\mathcal{K}^\varepsilon$. In a similar way, we obtain, by (3.9) and (3.10),

$$\begin{aligned}\int_{C_0/\varepsilon}^{+\infty} \left|\sum_{s=\pm} s \frac{\xi_1^2\exp\left(-iz_1\xi_1 - z_2\Xi_\varepsilon^s(\xi_1)\right)}{\sqrt{1+4\varepsilon^2\xi_1^2\left(1-\varepsilon^2\mathfrak{c}^2\right)}}\right| d\xi_1 \\ \leqslant \int_{C_0/\varepsilon}^{+\infty} \frac{C\xi_1}{\varepsilon}\mathsf{e}^{-z_2\xi_1/(C\varepsilon)}\left(1-\mathsf{e}^{-z_2/(C\varepsilon^2)}\right)\, d\xi_1 \\ \leqslant \frac{C\mathbf{1}_{z_2\geqslant\varepsilon^2}}{\varepsilon z_2}\mathsf{e}^{-z_2/(C\varepsilon^2)} + \frac{C\mathbf{1}_{z_2\leqslant\varepsilon^2}}{\varepsilon z_2} \\ \leqslant \frac{C}{\varepsilon z_2}\mathsf{e}^{-z_2/(C\varepsilon^2)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\left|\partial_1^2\partial_2\mathcal{K}^\varepsilon(z)\right| &\leqslant \frac{C}{\varepsilon(z_2+\varepsilon^2)}\mathsf{e}^{-z_2/(C\varepsilon^2)} + C\int_0^{C_0/\varepsilon} \xi_1^2\mathsf{e}^{-z_2/(C\varepsilon^2)}\, d\xi_1 \\ &\quad + C\int_0^1 \xi_1^2\mathsf{e}^{-z_2\xi_1/C}\, d\xi_1 + C\int_1^{C_0/\varepsilon} \xi_1^2\mathsf{e}^{-z_2\xi_1^2/C}\, d\xi_1 \\ &\leqslant \frac{C}{\varepsilon(z_2+\varepsilon^2)}\mathsf{e}^{-z_2/(C\varepsilon^2)} + \frac{C}{\varepsilon^3}\mathsf{e}^{-z_2/(C\varepsilon^2)} + C + \frac{C}{(z_2+\varepsilon^2)^{3/2}} \\ &\leqslant \frac{C}{(z_2+\varepsilon^2)^{3/2}}.\end{aligned}$$

From all these estimates, we infer

$$\int_{\{|z_1|\leqslant|z_2|\leqslant 1\}}\left|\partial_1\partial_2^2\mathcal{K}^\varepsilon(z)\right|\,dz\leqslant C\int_0^1 z_2\frac{dz_2}{(z_2+\varepsilon^2)^{3/2}}\leqslant C,$$

$$\int_{\{|z_1|\leqslant|z_2|\leqslant 1\}}\left|\partial_1^3\mathcal{K}^\varepsilon(z)\right|\,dz\leqslant C\int_0^1 \frac{z_2dz_2}{z_2}\leqslant C,$$

$$\int_{\{|z_1|\leqslant|z_2|\leqslant 1\}}\left|\partial_2^3\mathcal{K}^\varepsilon(z)\right|\,dz\leqslant C\int_0^1 \frac{z_2dz_2}{z_2(\varepsilon^2+z_2)^{3/2}}\leqslant \frac{C}{\varepsilon}$$

and finally

$$\int_{\{|z_1|\leqslant|z_2|\leqslant 1\}}\left|\partial_1^2\partial_2\mathcal{K}^\varepsilon(z)\right|\,dz\leqslant C\int_0^1 \frac{z_2\mathrm{e}^{-z_2/(C\varepsilon^2)}\,dz_2}{\varepsilon z_2}\leqslant C\varepsilon.$$

3.3.2. Estimates in the region $\{|z_2|\leqslant|z_1|\leqslant 1\}$

This time, we shall take the expressions given in Proposition 3.2.

First, we may use rough estimates for the low frequency part: for $\beta\in\mathbb{N}^2$ with $\beta_1+\beta_2\leqslant 2$ and $z\in D(0,1)$, we have

$$\left|\partial_z^\beta\int_{\{|\xi_2|\leqslant\xi_2^\varepsilon\}}\sum_{s\in\{\pm\}}\frac{s\mathrm{e}^{-|z_1|\tilde{\Xi}_\varepsilon^s(\xi_2)+iz_2\xi_2}}{\sqrt{\mathfrak{c}^4-4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)}}\,d\xi_2\right| \\ +\left|\partial_z^\beta\int_{\{\xi_2^\varepsilon\leqslant|\xi_2|\leqslant\mathfrak{c}^2\}}\sum_{s\in\{\pm\}}\frac{is\mathrm{e}^{-|z_1|\tilde{\Xi}_\varepsilon^s(\xi_2)+iz_2\xi_2}}{\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\,d\xi_2\right|\leqslant C.$$

Similarly, for the z_2 derivative of $\mathcal{K}^\varepsilon$, using the inequality (see Proposition 3.4)

$$\frac{|\xi_2|}{\left|\tilde{\Xi}_\varepsilon^-(\xi_2)\right|}\leqslant C,$$

we have, for $\beta\in\mathbb{N}^2$ with $\beta_1+\beta_2\leqslant 2$ and $z\in D(0,1)$,

$$\left|\partial_z^\beta\int_{\{|\xi_2|\leqslant\xi_2^\varepsilon\}}\sum_{s\in\{\pm\}}\frac{s\xi_2\mathrm{e}^{-|z_1|\tilde{\Xi}_\varepsilon^s(\xi_2)+iz_2\xi_2}}{\tilde{\Xi}_\varepsilon^s(\xi_2)\sqrt{\mathfrak{c}^4-4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)}}\,d\xi_2\right| \\ +\left|\partial_z^\beta\int_{\{\xi_2^\varepsilon\leqslant|\xi_2|\leqslant\mathfrak{c}^2\}}\sum_{s\in\{\pm\}}\frac{is\xi_2\mathrm{e}^{-|z_1|\tilde{\Xi}_\varepsilon^s(\xi_2)+iz_2\xi_2}}{\tilde{\Xi}_\varepsilon^s(\xi_2)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\,d\xi_2\right|\leqslant C.$$

• *Estimate for $\partial_1\mathcal{K}^\varepsilon$.* We first easily estimate the contributions for $\xi_2\geqslant\varepsilon^{-2}$. For $s=\pm$, we have

$$\left|\int_{\{|\xi_2|\geqslant C_1/\varepsilon^2\}}\frac{s\mathrm{e}^{-|z_1|\tilde{\Xi}_\varepsilon^s(\xi_2)+iz_2\xi_2}}{\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\,d\xi_2\right|\leqslant C\int_{C_1/\varepsilon^2}^{+\infty}\frac{\mathrm{e}^{-\varepsilon z_1\xi_2/C}}{\xi_2}\,d\xi_2=CE_1(z_1/(C\varepsilon)).$$

Moreover,

$$\begin{aligned}\left|\int_{\{\mathfrak{c}^2\leqslant|\xi_2|\leqslant C_1/\varepsilon^2\}}\frac{s\mathrm{e}^{-|z_1|\tilde{\Xi}^s_\varepsilon(\xi_2)+iz_2\xi_2}}{\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\,d\xi_2\right|&\\ \leqslant C\int_{\mathfrak{c}^2}^{C_1/\varepsilon^2}\frac{\mathrm{e}^{-z_1\sqrt{\xi_2}/C}}{\xi_2}\,d\xi_2&\\ \leqslant C\int_{\mathfrak{c}z_1/C}^{z_1\sqrt{C_1}/(C\varepsilon)}\frac{\mathrm{e}^{-t}}{t}\,dt&\\ \leqslant C(1+|\ln\varepsilon|)\mathbf{1}_{z_1\leqslant C\varepsilon}+C(1+|\ln z_1|)\mathbf{1}_{z_1\geqslant C\varepsilon}.&\end{aligned}$$

- *Estimate for $\partial_2\mathcal{K}^\varepsilon$.* Similarly, for $s=\pm$, we have

$$\begin{aligned}\left|\int_{\{|\xi_2|\geqslant\varepsilon^{-2}\}}\frac{s\xi_2\mathrm{e}^{-|z_1|\tilde{\Xi}^s_\varepsilon(\xi_2)+iz_2\xi_2}}{\widetilde{\Xi}^s_\varepsilon(\xi_2)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\,d\xi_2\right|\leqslant C\int_{C_1/\varepsilon^2}^{+\infty}\frac{\mathrm{e}^{-\varepsilon z_1\xi_2/C}}{\varepsilon\xi_2}\,d\xi_2&\\ =\frac{C}{\varepsilon}E_1(z_1/(C\varepsilon))&\end{aligned}$$

and

$$\begin{aligned}\left|\int_{\{\mathfrak{c}^2\leqslant|\xi_2|\leqslant\varepsilon^{-2}\}}\frac{s\xi_2\mathrm{e}^{-|z_1|\tilde{\Xi}^s_\varepsilon(\xi_2)+iz_2\xi_2}}{\widetilde{\Xi}^s_\varepsilon(\xi_2)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\,d\xi_2\right|\leqslant C\int_{\mathfrak{c}^2}^{C_1/\varepsilon^2}\frac{\mathrm{e}^{-z_1\sqrt{\xi_2}/C}}{\sqrt{\xi_2}}\,d\xi_2&\\ \leqslant C\frac{\mathrm{e}^{-\mathfrak{c}z_1/C}-\mathrm{e}^{-z_1/(C\varepsilon)}}{z_1}\leqslant C\frac{\mathrm{e}^{-\mathfrak{c}z_1/C}}{z_1}.&\end{aligned}$$

This yields

$$\begin{aligned}\int_{D(0,1)}|\nabla\mathcal{K}^\varepsilon(z)|\,dz\leqslant C+C\int_0^1 z_1\frac{E_1(z_1/(C\varepsilon))}{\varepsilon}+\mathrm{e}^{-\mathfrak{c}z_1/C}\,dz_1&\\ +C\int_0^{C\varepsilon}z_1(1+|\ln\varepsilon|)\,dz_1+C\int_{C\varepsilon}^1 z_1(1+|\ln z_1|)\,dz_1\leqslant C,&\end{aligned}$$

since $y\mapsto yE_1(y)\in L^1(]0,+\infty[)$ (by (3.7)).

- *Estimate for $\partial_1^2\mathcal{K}^\varepsilon$.* We have, for $s=\pm$,

$$\left|\int_{\{|\xi_2|\geqslant C_1/\varepsilon^2\}}\frac{s\widetilde{\Xi}^s_\varepsilon(\xi_2)\mathrm{e}^{-|z_1|\tilde{\Xi}^s_\varepsilon(\xi_2)+iz_2\xi_2}}{\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\,d\xi_2\right|\leqslant C\int_{C_1/\varepsilon^2}^{+\infty}\varepsilon\mathrm{e}^{-\varepsilon z_1\xi_2/C}\,d\xi_2=\frac{C}{z_1}\mathrm{e}^{-z_1/(C\varepsilon)},$$

and

$$\left|\int_{\{\mathfrak{c}^2\leqslant|\xi_2|\leqslant C_1/\varepsilon^2\}}\frac{s\widetilde{\Xi}^s_\varepsilon(\xi_2)\mathrm{e}^{-|z_1|\tilde{\Xi}^s_\varepsilon(\xi_2)+iz_2\xi_2}}{\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\,d\xi_2\right|\leqslant C\int_{\mathfrak{c}^2}^{C_1/\varepsilon^2}\frac{\mathrm{e}^{-z_1\sqrt{\xi_2}/C}}{\sqrt{\xi_2}}\,d\xi_2\leqslant C\frac{\mathrm{e}^{-\mathfrak{c}z_1/C}}{z_1}.$$

This yields

$$\int_{D(0,1)}\left|\partial_1^2\mathcal{K}^\varepsilon(z)\right|\,dz\leqslant C+C\int_0^1\mathrm{e}^{-z_1/(C\varepsilon)}+\mathrm{e}^{-\mathfrak{c}z_1/C}\,dz_1\leqslant C.$$

• *Estimate for* $\partial_1\partial_2\mathcal{K}^\varepsilon$. We have, for $s=\pm$,

$$\left|\int_{\{|\xi_2|\geqslant C_1/\varepsilon^2\}} \frac{\xi_2 \mathsf{e}^{-|z_1|\tilde{\Xi}^s_\varepsilon(\xi_2)+iz_2\xi_2}}{\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\,d\xi_2\right| \leqslant C\int_{C_1/\varepsilon^2}^{+\infty} \mathsf{e}^{-\varepsilon z_1\xi_2/C}\,d\xi_2 = \frac{C}{\varepsilon z_1}\mathsf{e}^{-z_1/(C\varepsilon)}.$$

For the contribution $\mathfrak{c}^2\leqslant|\xi_2|\leqslant C_1/\varepsilon^2$, we first integrate by parts. For $s=\pm$, we have

$$\begin{aligned}
&\int_{\mathfrak{c}^2}^{C_1/\varepsilon^2} \frac{\xi_2 \mathsf{e}^{-|z_1|\tilde{\Xi}^s_\varepsilon(\xi_2)+iz_2\xi_2}}{\sqrt{4\xi_2^2(1-\varepsilon^2\mathfrak{c}^2)-\mathfrak{c}^4}}\,d\xi_2\\
&= \int_{\mathfrak{c}^2}^{C_1/\varepsilon^2} \mathsf{e}^{-z_1\tilde{\Xi}^s_\varepsilon(\xi_2)+iz_2\xi_2}\partial_{\xi_2}\left(\frac{\xi_2}{\left(z_1\partial_2\tilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\right)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\right)d\xi_2\\
&\quad+\frac{\mathfrak{c}^2\mathsf{e}^{-z_1\tilde{\Xi}^s_\varepsilon(\mathfrak{c}^2)+iz_2\mathfrak{c}^2}}{\left(z_1\partial_2\tilde{\Xi}^s_\varepsilon(\mathfrak{c}^2)-iz_2\right)\sqrt{4\mathfrak{c}^4\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\\
&\quad-\frac{C_1\mathsf{e}^{-z_1\tilde{\Xi}^s_\varepsilon\left(C_1/\varepsilon^2\right)+iz_2C_1/\varepsilon^2}}{\varepsilon^2\left(z_1\partial_2\tilde{\Xi}^s_\varepsilon\left(C_1/\varepsilon^2\right)-iz_2\right)\sqrt{4C_1^2\varepsilon^{-4}\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}.
\end{aligned}\tag{3.11}$$

In addition, by Proposition 3.4,

$$\begin{aligned}
&\left|\frac{\mathfrak{c}^2\mathsf{e}^{-z_1\tilde{\Xi}^s_\varepsilon(\mathfrak{c}^2)+iz_2\mathfrak{c}^2}}{\left(z_1\partial_2\tilde{\Xi}^s_\varepsilon(\mathfrak{c}^2)-iz_2\right)\sqrt{4\mathfrak{c}^4\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\right.\\
&\qquad\left.-\frac{C_1\mathsf{e}^{-z_1\tilde{\Xi}^s_\varepsilon\left(C_1/\varepsilon^2\right)+iz_2C_1/\varepsilon^2}}{\varepsilon^2\left(z_1\partial_2\tilde{\Xi}^s_\varepsilon\left(C_1/\varepsilon^2\right)-iz_2\right)\sqrt{4C_1^2\varepsilon^{-4}\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\right| \leqslant \frac{C}{z_1+z_2}\leqslant\frac{C}{z_1}
\end{aligned}$$

and

$$\begin{aligned}
&\partial_{\xi_2}\left(\frac{\xi_2}{\left(z_1\partial_2\tilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\right)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\right)\\
&\qquad= -\frac{z_1\partial_2^2\tilde{\Xi}^s_\varepsilon(\xi_2)}{\left(z_1\partial_2\tilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\right)^2\sqrt{4\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4/\xi_2^2}}\\
&\qquad\qquad+\frac{2\mathfrak{c}^4}{\xi_2^3\left(z_1\partial_2\tilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\right)\left[4\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4/\xi_2^2\right]^{3/2}},
\end{aligned}\tag{3.12}$$

from which we deduce

$$\begin{aligned}
&\left|\partial_{\xi_2}\left(\frac{\xi_2}{\left(z_1\partial_2\tilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\right)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\right)\right|\\
&\qquad\leqslant\frac{Cz_1}{\xi_2^{3/2}\left(z_1/\sqrt{\xi_2}+z_2\right)^2}+\frac{C}{\xi_2^3\left(z_1/\sqrt{\xi_2}+z_2\right)},
\end{aligned}$$

by using Proposition 3.4 and the fact that

$$\left|z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\right|^2\geqslant z_1^2\mathrm{Re}\left(\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2)\right)+\left(z_1\mathrm{Im}\left(\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2)\right)-z_2\right)^2\geqslant\frac{1}{C}\left(z_1^2/\xi_2+z_2^2\right)$$

both when $z_2\leqslant Cz_1/\sqrt{\xi_2}$ and when $z_2\geqslant Cz_1/\sqrt{\xi_2}$, by Proposition 3.4. This yields

$$\int_{D(0,1)}|\partial_1\partial_2\mathcal{K}^\varepsilon(z)|\,dz\leqslant C+C\int_0^1 z_1\left(\frac{\mathsf{e}^{-z_1/(C\varepsilon)}}{\varepsilon z_1}+\frac{1}{z_1}\right)dz_1+Q_{12}\leqslant C+Q_{12},$$

where Q_{12} contains the terms coming from (3.12):

$$Q_{12}\overset{\text{def}}{=}C\int_0^1\int_0^{z_1}\int_{\mathfrak{c}^2}^{\varepsilon^{-2}}\frac{z_1\mathsf{e}^{-z_1\sqrt{\xi_2}}}{\xi_2^{3/2}\left(z_1/\sqrt{\xi_2}+z_2\right)^2}+\frac{\mathsf{e}^{-z_1\sqrt{\xi_2}}}{\xi_2^3\left(z_1/\sqrt{\xi_2}+z_2\right)}\,d\xi_2dz_2dz_1.$$

Since, by explicit integration,

$$\int_0^{z_1}\frac{1}{\left(z_1/\sqrt{\xi_2}+z_2\right)^2}\,dz_2=\frac{\sqrt{\xi_2}}{z_1\left(1/\sqrt{\xi_2}+1\right)}\leqslant C\frac{\sqrt{\xi_2}}{z_1}$$

and

$$\int_0^{z_1}\frac{1}{z_1/\sqrt{\xi_2}+z_2}\,dz_2=\ln\left(1+\sqrt{\xi_2}\right)\leqslant C\ln\xi_2,$$

we infer

$$\begin{aligned}Q_{12}&\leqslant C\int_0^1\int_{\mathfrak{c}^2}^{\varepsilon^{-2}}\frac{\mathsf{e}^{-z_1\sqrt{\xi_2}}}{\xi_2}\,d\xi_2dz_1+C\int_0^1\int_{\mathfrak{c}^2}^{\varepsilon^{-2}}\mathsf{e}^{-z_1\sqrt{\xi_2}}\frac{\ln\xi_2}{\xi_2^3}\,d\xi_2dz_1\\&\leqslant C\int_{\mathfrak{c}^2}^{\varepsilon^{-2}}\frac{d\xi_2}{\xi_2^{1.5}}+C\int_{\mathfrak{c}^2}^{\varepsilon^{-2}}\frac{\ln\xi_2}{\xi_2^{3.5}}\,d\xi_2\leqslant C.\end{aligned}$$

Therefore, as wished,

$$\int_{D(0,1)}|\partial_1\partial_2\mathcal{K}^\varepsilon(z)|\,dz\leqslant C.$$

• *Estimate for $\partial_2^2\mathcal{K}^\varepsilon$.* For $s=\pm$, we integrate by parts

$$\begin{aligned}(3.13)\quad&\int_{\mathfrak{c}^2}^{+\infty}\frac{\xi_2^2\mathsf{e}^{-|z_1|\tilde{\Xi}^s_\varepsilon(\xi_2)+iz_2\xi_2}}{\widetilde{\Xi}^s_\varepsilon(\xi_2)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\,d\xi_2\\&=\int_{\mathfrak{c}^2}^{+\infty}\mathsf{e}^{-z_1\tilde{\Xi}^s_\varepsilon(\xi_2)+iz_2\xi_2}\partial_{\xi_2}\left(\frac{\xi_2^2}{\widetilde{\Xi}^s_\varepsilon(\xi_2)\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\right)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\right)d\xi_2\\&\qquad+\frac{\mathfrak{c}^2\mathsf{e}^{-z_1\tilde{\Xi}^s_\varepsilon(\mathfrak{c}^2)+iz_2\mathfrak{c}^2}}{\widetilde{\Xi}^s_\varepsilon(\mathfrak{c}^2)\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\mathfrak{c}^2)-iz_2\mathfrak{c}^2\right)\sqrt{4\mathfrak{c}^4\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}.\end{aligned}$$

In addition,

$$\left|\frac{\mathfrak{c}^2\mathsf{e}^{-z_1\tilde{\Xi}^s_\varepsilon(\mathfrak{c}^2)+iz_2\mathfrak{c}^2}}{\widetilde{\Xi}^s_\varepsilon(\mathfrak{c}^2)\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\mathfrak{c}^2)-iz_2\mathfrak{c}^2\right)\sqrt{4\mathfrak{c}^4\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\right|\leqslant\frac{C}{z_1}.$$

We have

$$\partial_{\xi_2}\left(\frac{\xi_2^2}{\widetilde{\Xi}^s_\varepsilon(\xi_2)\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\right)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\right)$$
$$=-\frac{z_1\xi_2\partial_2^2\widetilde{\Xi}^s_\varepsilon(\xi_2)}{\widetilde{\Xi}^s_\varepsilon(\xi_2)\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\right)^2\sqrt{4\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4/\xi_2^2}}$$
$$-\frac{\mathfrak{c}^4}{\xi_2^2\widetilde{\Xi}^s_\varepsilon(\xi_2)\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\right)\left(4\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4/\xi_2^2\right)^{3/2}}$$
$$+\frac{\partial_2\left(\xi_2/\widetilde{\Xi}^s_\varepsilon(\xi_2)\right)}{\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\right)\left(4\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4/\xi_2^2\right)^{1/2}}$$

thus, in the interval $[C_1/\varepsilon^2,+\infty[$,

$$\left|\partial_{\xi_2}\left(\frac{\xi_2^2}{\widetilde{\Xi}^s_\varepsilon(\xi_2)\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\right)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\right)\right|$$
$$\leqslant\frac{Cz_1}{\varepsilon^4\xi_2^3\left(z_1^2\varepsilon^2+z_2^2\right)}+\frac{C}{\varepsilon^3\xi_2^2(\varepsilon|z_1|+|z_2|)}\leqslant\frac{C}{\varepsilon^3\xi_2^2(\varepsilon|z_1|+|z_2|)}$$

and in the interval $[\mathfrak{c}^2, C_1/\varepsilon^2]$,

$$\left|\partial_{\xi_2}\left(\frac{\xi_2^2}{\widetilde{\Xi}^s_\varepsilon(\xi_2)\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\xi_2\right)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\right)\right|$$
$$\leqslant\frac{z_1}{\xi_2(z_1\xi_2^{-1/2}-iz_2)^2}+\frac{C}{\xi_2^{1/2}(|z_1|\xi_2^{-1/2}+|z_2|)}\leqslant\frac{C}{\xi_2^{1/2}\left(|z_1|\xi_2^{-1/2}+|z_2|\right)}.$$

As a consequence,

$$\int_{D(0,1)}\left|\int_{\{|\xi_2|\geqslant\mathfrak{c}^2\}}\frac{\xi_2^2\mathsf{e}^{-|z_1|\widetilde{\Xi}^s_\varepsilon(\xi_2)+iz_2\xi_2}}{\widetilde{\Xi}^s_\varepsilon(\xi_2)\sqrt{4\xi_2^2(1-\varepsilon^2\mathfrak{c}^2)-\mathfrak{c}^4}}\,d\xi_2\right|dz$$
$$\leqslant C\int_0^1 z_1\frac{C}{z_1}\,dz_1+C\int_{C_1/\varepsilon^2}^{+\infty}\int_0^1\int_0^{z_1}\frac{\mathsf{e}^{-\varepsilon z_1\xi_2/C}}{\varepsilon^3\xi_2^2(\varepsilon z_1+z_2)}\,dz_2dz_1d\xi_2$$
$$+C\int_{\mathfrak{c}^2}^{C_1/\varepsilon^2}\int_0^1\int_0^{z_1}\frac{\mathsf{e}^{-z_1\sqrt{\xi_2}/C}}{\xi_2^{1/2}\left(z_1/\sqrt{\xi_2}+z_2\right)}\,dz_2dz_1d\xi_2$$
$$\leqslant C+C|\ln\varepsilon|\int_{C_1/\varepsilon^2}^{+\infty}\int_0^1\frac{\mathsf{e}^{-\varepsilon z_1\xi_2/C}}{\varepsilon^3\xi_2^2}\,dz_1d\xi_2+C\int_{\mathfrak{c}^2}^{\varepsilon^{-2}}\int_0^1\ln\xi_2\frac{\mathsf{e}^{-z_1\sqrt{\xi_2}/C}}{\xi_2^{1/2}}\,dz_1d\xi_2$$
$$=C+C|\ln\varepsilon|\int_{C_1/\varepsilon^2}^{+\infty}\frac{1}{\varepsilon^4\xi_2^3}\,d\xi_2+C\int_{\mathfrak{c}^2}^{\varepsilon^{-2}}\frac{\ln\xi_2}{\xi_2}\,d\xi_2\leqslant C|\ln\varepsilon|^2$$

and it follows that

$$\int_{D(0,1)} \left|\partial_2^2 \mathcal{K}^\varepsilon(z)\right| \, dz \leqslant C|\ln\varepsilon|^2.$$

- *Estimate for $\partial_1^3 \mathcal{K}^\varepsilon$.* There holds, for $\xi_2 \geqslant \mathfrak{c}^2$ (hence $\widetilde{\Xi}_\varepsilon^- = \overline{\widetilde{\Xi}_\varepsilon^+}$)

$$\begin{aligned}&\left[\widetilde{\Xi}_\varepsilon^+(\xi_2)\right]^2 \mathsf{e}^{-|z_1|\widetilde{\Xi}_\varepsilon^+(\xi_2)} - \left[\widetilde{\Xi}_\varepsilon^-(\xi_2)\right]^2 \mathsf{e}^{-|z_1|\widetilde{\Xi}_\varepsilon^-(\xi_2)} \\ &\qquad = i\mathsf{e}^{-z_1 \mathrm{Re}\widetilde{\Xi}_\varepsilon^+(\xi_2)} \mathrm{Im}\left[\mathsf{e}^{-iz_1\mathrm{Im}\widetilde{\Xi}_\varepsilon^+(\xi_2)} \left(-\mathfrak{c}^2 - 2\varepsilon^2\xi_2^2 - i\sqrt{4\xi_2^2(1-\varepsilon^2\mathfrak{c}^2) - \mathfrak{c}^4}\right)\right].\end{aligned}$$

It then follows that, for $\xi_2 \geqslant C_1/\varepsilon^2$,

$$\begin{aligned}&\left|\left[\widetilde{\Xi}_\varepsilon^+(\xi_2)\right]^2 \mathsf{e}^{-|z_1|\widetilde{\Xi}_\varepsilon^+(\xi_2)} - \left[\widetilde{\Xi}_\varepsilon^-(\xi_2)\right]^2 \mathsf{e}^{-|z_1|\widetilde{\Xi}_\varepsilon^-(\xi_2)}\right| \\ &\qquad \leqslant \mathsf{e}^{-z_1 \mathrm{Re}\widetilde{\Xi}_\varepsilon^+(\xi_2)} \left|\mathrm{Im}\left[\mathsf{e}^{-iz_1\mathrm{Im}\widetilde{\Xi}_\varepsilon^+(\xi_2)} \left(-\mathfrak{c}^2 - 2\varepsilon^2\xi_2^2 - i\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right) - \mathfrak{c}^4}\right)\right]\right| \\ &\qquad \leqslant C\mathsf{e}^{-z_1\varepsilon\xi_2/C}\left[\varepsilon^2\xi_2^2\left|\sin\left(z_1\mathrm{Im}\widetilde{\Xi}_\varepsilon^+(\xi_2)\right)\right| + \xi_2\right] \\ &\qquad \leqslant C\mathsf{e}^{-z_1\varepsilon\xi_2/C}\left[\varepsilon^2\xi_2^2\min(1, Cz_1/\varepsilon) + \xi_2\right].\end{aligned}$$

Consequently,

$$\begin{aligned}&\left|\int_{C_1/\varepsilon^2}^{+\infty} \sum_{s=\pm} \frac{s\widetilde{\Xi}_\varepsilon^s(\xi_2)^2 \mathsf{e}^{-|z_1|\widetilde{\Xi}_\varepsilon^s(\xi_2)+iz_2\xi_2}}{\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right) - \mathfrak{c}^4}} \, d\xi_2\right| \\ &\qquad\qquad \leqslant C\int_{C_1/\varepsilon^2}^{+\infty} \mathsf{e}^{-z_1\varepsilon\xi_2/C}\left[\varepsilon^2\xi_2\min(1, Cz_1/\varepsilon) + 1\right] d\xi_2 \\ &\qquad\qquad \leqslant C\mathbf{1}_{Cz_1\leqslant\varepsilon}\frac{1}{\varepsilon z_1} + C\mathbf{1}_{Cz_1\geqslant\varepsilon}\frac{\mathsf{e}^{-z_1/(C\varepsilon)}}{\varepsilon z_1} + C\frac{\mathsf{e}^{-z_1/(C\varepsilon)}}{\varepsilon z_1} \leqslant C\frac{\mathsf{e}^{-z_1/(C\varepsilon)}}{\varepsilon z_1}.\end{aligned}$$

For the contributions for $\mathfrak{c}^2 \leqslant \xi_2 \leqslant C_1/\varepsilon^2$, we integrate by parts:

$$\begin{aligned}&\int_{\mathfrak{c}^2}^{C_1/\varepsilon^2} \frac{\widetilde{\Xi}_\varepsilon^s(\xi_2)^2\mathsf{e}^{-|z_1|\widetilde{\Xi}_\varepsilon^s(\xi_2)+iz_2\xi_2}}{\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\, d\xi_2 \\ &\quad = \int_{\mathfrak{c}^2}^{C_1/\varepsilon^2} \mathsf{e}^{-z_1\widetilde{\Xi}_\varepsilon^s(\xi_2)+iz_2\xi_2}\partial_{\xi_2}\left(\frac{\widetilde{\Xi}_\varepsilon^s(\xi_2)^2}{\left(z_1\partial_2\widetilde{\Xi}_\varepsilon^s(\xi_2) - iz_2\right)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\right) d\xi_2 \\ &\qquad + \frac{\widetilde{\Xi}_\varepsilon^s(\mathfrak{c}^2)^2\mathsf{e}^{-z_1\widetilde{\Xi}_\varepsilon^s(\mathfrak{c}^2)+iz_2\mathfrak{c}^2}}{\left(z_1\partial_2\widetilde{\Xi}_\varepsilon^s(\mathfrak{c}^2) - iz_2\right)\sqrt{4\mathfrak{c}^4\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}} \\ &\qquad - \frac{\widetilde{\Xi}_\varepsilon^s\left(C_1/\varepsilon^2\right)^2\mathsf{e}^{-z_1\widetilde{\Xi}_\varepsilon^s\left(C_1/\varepsilon^2\right)+iz_2C_1/\varepsilon^2}}{\left(z_1\partial_2\widetilde{\Xi}_\varepsilon^s\left(C_1/\varepsilon^2\right) - iz_2\right)\sqrt{4C_1^2\varepsilon^{-4}\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}.\end{aligned} \tag{3.14}$$

In addition,

$$\left| \frac{\widetilde{\Xi}^s_\varepsilon(\mathfrak{c}^2)^2 \mathsf{e}^{-z_1 \widetilde{\Xi}^s_\varepsilon(\mathfrak{c}^2) + iz_2\mathfrak{c}^2}}{\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\mathfrak{c}^2) - iz_2\right)\sqrt{4\mathfrak{c}^4\left(1-\varepsilon^2\mathfrak{c}^2\right) - \mathfrak{c}^4}} \right. \\ \left. - \frac{\widetilde{\Xi}^s_\varepsilon\left(C_1/\varepsilon^2\right)^2 \mathsf{e}^{-z_1\widetilde{\Xi}^s_\varepsilon\left(C_1/\varepsilon^2\right) + iz_2\varepsilon^{-2}}}{\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon\left(C_1/\varepsilon^2\right) - iz_2\right)\sqrt{4C_1^2\varepsilon^{-4}\left(1-\varepsilon^2\mathfrak{c}^2\right) - \mathfrak{c}^4}} \right| \leqslant \frac{C}{z_1}$$

and

$$\partial_{\xi_2}\left(\frac{\widetilde{\Xi}^s_\varepsilon(\xi_2)^2}{\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2) - iz_2\right)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right) - \mathfrak{c}^4}} \right) \\ = -\frac{z_1\partial_2^2\widetilde{\Xi}^s_\varepsilon(\xi_2) \times \left(\widetilde{\Xi}^s_\varepsilon(\xi_2)^2/\xi_2\right)}{\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2) - iz_2\right)^2\sqrt{4\left(1-\varepsilon^2\mathfrak{c}^2\right) - \mathfrak{c}^4/\xi_2^2}} \\ + \frac{\partial_2\left(\widetilde{\Xi}^s_\varepsilon(\xi_2)^2/\xi_2\right)}{\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2) - iz_2\right)\left[4\left(1-\varepsilon^2\mathfrak{c}^2\right) - \mathfrak{c}^4/\xi_2^2\right]^{1/2}} \\ + \frac{2\mathfrak{c}^4\widetilde{\Xi}^s_\varepsilon(\xi_2)^2/\xi_2}{\xi_2^3\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2) - iz_2\right)\left[4\left(1-\varepsilon^2\mathfrak{c}^2\right) - \mathfrak{c}^4/\xi_2^2\right]^{3/2}}, \tag{3.15}$$

from which we deduce, by Proposition 3.4,

$$\left|\partial_{\xi_2}\left(\frac{\widetilde{\Xi}^s_\varepsilon(\xi_2)^2}{\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2) - iz_2\right)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right) - \mathfrak{c}^4}} \right)\right| \\ \leqslant \frac{Cz_1}{\xi_2^{3/2}\left(z_1/\sqrt{\xi_2} + z_2\right)^2} + \frac{C}{\xi_2(z_1/\sqrt{\xi_2} + z_2)} + \frac{C}{\xi_2^3\left(z_1/\sqrt{\xi_2} + z_2\right)} \\ \leqslant \frac{C}{\xi_2\left(z_1/\sqrt{\xi_2} + z_2\right)}.$$

Then, arguing as for $\partial_1\partial_2\mathcal{K}^\varepsilon$, we obtain

$$\int_{D(0,1)}\left|\partial_1^3\mathcal{K}^\varepsilon(z)\right| dz \leqslant C + C\int_{\mathfrak{c}^2}^{C_1/\varepsilon^2}\int_0^1 \mathsf{e}^{-z_1\sqrt{\xi_2}/C}\frac{\ln\xi_2}{\xi_2}\,dz_1 d\xi_2 \\ \leqslant C + C\int_{\mathfrak{c}^2}^{\varepsilon^{-2}}\frac{\ln\xi_2}{\xi_2^{1.5}}\,d\xi_2 \leqslant C.$$

• *Estimate for* $\partial_1^2\partial_2\mathcal{K}^\varepsilon$. For $s=\pm$, we integrate by parts

$$\begin{aligned}(3.16)\quad &\int_{\mathfrak{c}^2}^{+\infty}\frac{\xi_2\widetilde{\Xi}_\varepsilon^s(\xi_2)\mathsf{e}^{-|z_1|\widetilde{\Xi}_\varepsilon^s(\xi_2)+iz_2\xi_2}}{\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\,d\xi_2\\ &=\int_{\mathfrak{c}^2}^{+\infty}\mathsf{e}^{-z_1\widetilde{\Xi}_\varepsilon^s(\xi_2)+iz_2\xi_2}\partial_{\xi_2}\left(\frac{\xi_2\widetilde{\Xi}_\varepsilon^s(\xi_2)}{\left(z_1\partial_2\widetilde{\Xi}_\varepsilon^s(\xi_2)-iz_2\right)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\right)d\xi_2\\ &\qquad+\frac{\mathfrak{c}^2\widetilde{\Xi}_\varepsilon^s(\mathfrak{c}^2)\mathsf{e}^{-z_1\widetilde{\Xi}_\varepsilon^s(\mathfrak{c}^2)+iz_2\mathfrak{c}^2}}{\left(z_1\partial_2\widetilde{\Xi}_\varepsilon^s(\mathfrak{c}^2)-iz_2\mathfrak{c}^2\right)\sqrt{4\mathfrak{c}^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}.\end{aligned}$$

In addition,

$$\left|\frac{\mathfrak{c}^2\widetilde{\Xi}_\varepsilon^s(\mathfrak{c}^2)\mathsf{e}^{-z_1\widetilde{\Xi}_\varepsilon^s(\mathfrak{c}^2)+iz_2\mathfrak{c}^2}}{\left(z_1\partial_2\widetilde{\Xi}_\varepsilon^s(\mathfrak{c}^2)-iz_2\mathfrak{c}^2\right)\sqrt{4\mathfrak{c}^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\right|\leqslant\frac{C}{z_1}.$$

As for $\partial_2^2\mathcal{K}^\varepsilon$, we obtain, on the one hand, in the interval $[C_1/\varepsilon^2,+\infty[$,

$$\begin{aligned}&\left|\partial_{\xi_2}\left(\frac{\xi_2\widetilde{\Xi}_\varepsilon^s(\xi_2)}{\left(z_1\partial_2\widetilde{\Xi}_\varepsilon^s(\xi_2)-iz_2\right)\sqrt{4\xi_2^2(1-\varepsilon^2\mathfrak{c}^2)-\mathfrak{c}^4}}\right)-\frac{\partial_2\widetilde{\Xi}_\varepsilon^s(\xi_2)}{2\sqrt{1-\varepsilon^2\mathfrak{c}^2}\left(z_1\partial_2\widetilde{\Xi}_\varepsilon^s(\xi_2)-iz_2\right)}\right|\\ &\qquad\leqslant\frac{z_1}{\varepsilon^2\xi_2^2(z_1\varepsilon+z_2)^2}+\frac{C\varepsilon}{\xi_2(z_1\varepsilon+z_2)}\leqslant\frac{C\varepsilon}{\xi_2(z_1\varepsilon+z_2)},\end{aligned}$$

which yields, using Proposition 3.4,

$$\begin{aligned}(3.17)\quad &\left|\partial_{\xi_2}\left(\frac{\xi_2\widetilde{\Xi}_\varepsilon^s(\xi_2)}{\left(z_1\partial_2\widetilde{\Xi}_\varepsilon^s(\xi_2)-iz_2\right)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\right)\right.\\ &\qquad\left.-\frac{\varepsilon}{2\sqrt{1-\varepsilon^2\mathfrak{c}^2}\left(z_1\partial_2\widetilde{\Xi}_\varepsilon^s(\xi_2)-iz_2\right)}\right|\leqslant\frac{C}{\varepsilon\xi_2(z_1\varepsilon+z_2)},\end{aligned}$$

and on the other hand, in the interval $[\mathfrak{c}^2,C_1/\varepsilon^2]$,

$$\begin{aligned}&\left|\partial_{\xi_2}\left(\frac{\xi_2\widetilde{\Xi}_\varepsilon^s(\xi_2)}{\left(z_1\partial_2\widetilde{\Xi}_\varepsilon^s(\xi_2)-iz_2\xi_2\right)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\right)\right|\\ &\qquad\leqslant\frac{Cz_1}{\xi_2\left(z_1\xi_2^{-1/2}+z_2\right)^2}+\frac{C}{\xi_2^{2.5}\left(z_1\xi_2^{-1/2}+z_2\right)}+\frac{C}{\xi_2^{1/2}\left(z_1\xi_2^{-1/2}+z_2\right)}\\ &\qquad\leqslant\frac{C}{\xi_2^{1/2}\left(z_1\xi_2^{-1/2}+z_2\right)}.\end{aligned}$$

The estimate for the integral over $[\mathfrak{c}^2, C_1/\varepsilon^{-2}]$ is as for $\partial_2^2\mathcal{K}^\varepsilon$. We have to use a compensation in the remaining terms for the integral over $[C_1/\varepsilon^2, +\infty[$:

$$\varepsilon \sum_{s=\pm} \frac{s\mathsf{e}^{-|z_1|\tilde{\Xi}_\varepsilon^s(\xi_2)+iz_2\xi_2}}{\left(z_1\partial_2\tilde{\Xi}_\varepsilon^s(\xi_2) - iz_2\right)} = \frac{\varepsilon\mathsf{e}^{-|z_1|\mathrm{Re}\tilde{\Xi}_\varepsilon^s(\xi_2)+iz_2\xi_2}}{\left(z_1\partial_2\tilde{\Xi}_\varepsilon^+(\xi_2) - iz_2\right)\left(z_1\partial_2\overline{\tilde{\Xi}}_\varepsilon^+(\xi_2) - iz_2\right)}$$
$$\times \left(\mathsf{e}^{-i|z_1|\mathrm{Im}\tilde{\Xi}_\varepsilon^+(\xi_2)}(z_1\varepsilon - iz_2) - \mathsf{e}^{i|z_1|\mathrm{Im}\tilde{\Xi}_\varepsilon^s(\xi_2)}(z_1\varepsilon - iz_2) + \mathcal{T}\right),$$

where

$$\mathcal{T} \stackrel{\text{def}}{=} z_1\mathsf{e}^{-i|z_1|\mathrm{Im}\tilde{\Xi}_\varepsilon^+(\xi_2)}\left(\partial_2\tilde{\Xi}_\varepsilon^+(\xi_2) - \varepsilon\right) - z_1\mathsf{e}^{i|z_1|\mathrm{Im}\tilde{\Xi}_\varepsilon^+(\xi_2)}\left(\partial_2\overline{\tilde{\Xi}}_\varepsilon^+(\xi_2) - \varepsilon\right),$$

so that, by using Proposition 3.4,

$$\left|\frac{\varepsilon\mathcal{T}}{\left(z_1\partial_2\tilde{\Xi}_\varepsilon^+(\xi_2) - iz_2\right)\left(z_1\partial_2\overline{\tilde{\Xi}}_\varepsilon^+(\xi_2) - iz_2\right)}\right| \leqslant \frac{C\varepsilon z_1}{\varepsilon^3\xi_2^2(z_1\varepsilon + z_2)^2} \leqslant \frac{C}{\varepsilon\xi_2(z_1\varepsilon + z_2)}$$

which is as the right-hand side of (3.17). Observing that, from Proposition 3.4,

$$\left|\mathsf{e}^{-i|z_1|\mathrm{Im}\tilde{\Xi}_\varepsilon^s(\xi_2)} - \mathsf{e}^{i|z_1|\mathrm{Im}\tilde{\Xi}_\varepsilon^s(\xi_2)}\right| = 2\left|\sin\left(|z_1|\mathrm{Im}\tilde{\Xi}_\varepsilon^s(\xi_2)\right)\right| \leqslant 2\min(1, Cz_1/\varepsilon),$$

we deduce

$$\begin{aligned}\int_{D(0,1)}|\partial_1^2\partial_2\mathcal{K}^\varepsilon(z)|\,dz &\leqslant C|\ln\varepsilon|^2 + C\int_{C_1/\varepsilon^2}^{+\infty}\int_0^1\int_0^{z_1}\frac{\mathsf{e}^{-z_1\varepsilon\xi_2/C}}{\varepsilon\xi_2(\varepsilon z_1 + z_2)}\,dz_2dz_1d\xi_2\\
&\quad + C\varepsilon\int_0^1\int_{C_1/\varepsilon^2}^{+\infty}\int_0^{z_1}\frac{\mathsf{e}^{-z_1\varepsilon\xi_2/C}}{(\varepsilon z_1 + z_2)}\min(1, Cz_1/\varepsilon)\,dz_2d\xi_2dz_1\\
&\leqslant C|\ln\varepsilon|^2 + \int_{C_1/\varepsilon^2}^{+\infty}\frac{|\ln\varepsilon|}{\varepsilon^2\xi_2^2}\,d\xi_2\\
&\quad + C\varepsilon|\ln\varepsilon|\int_0^1\int_{C_1/\varepsilon^2}^{+\infty}\mathsf{e}^{-z_1\varepsilon\xi_2/C}\min(1, Cz_1/\varepsilon)\,d\xi_2dz_1\\
&\leqslant C|\ln\varepsilon|^2 + C|\ln\varepsilon|\int_0^1\frac{\mathsf{e}^{-z_1/(C\varepsilon)}}{z_1}\min(1, Cz_1/\varepsilon)\,dz_1 \leqslant C|\ln\varepsilon|^2.\end{aligned}$$

- *Estimate for $\partial_1\partial_2^2\mathcal{K}^\varepsilon$.* For $s = \pm$, we integrate by parts

$$(3.18)\quad \int_{\mathfrak{c}^2}^{+\infty}\frac{\xi_2^2\mathsf{e}^{-|z_1|\tilde{\Xi}_\varepsilon^s(\xi_2)+iz_2\xi_2}}{\sqrt{4\xi_2^2(1-\varepsilon^2\mathfrak{c}^2) - \mathfrak{c}^4}}\,d\xi_2$$
$$= \int_{\mathfrak{c}^2}^{+\infty}\mathsf{e}^{-z_1\tilde{\Xi}_\varepsilon^s(\xi_2)+iz_2\xi_2}\partial_{\xi_2}\left(\frac{\xi_2^2}{\left(z_1\partial_2\tilde{\Xi}_\varepsilon^s(\xi_2) - iz_2\right)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right) - \mathfrak{c}^4}}\right)d\xi_2$$
$$+ \frac{\mathfrak{c}^4\mathsf{e}^{-z_1\tilde{\Xi}_\varepsilon^s(\mathfrak{c}^2)+iz_2\mathfrak{c}^2}}{\left(z_1\partial_2\tilde{\Xi}_\varepsilon^s(\mathfrak{c}^2) - iz_2\mathfrak{c}^2\right)\sqrt{4\mathfrak{c}^2\left(1-\varepsilon^2\mathfrak{c}^2\right) - \mathfrak{c}^4}}.$$

In addition,

$$\left|\frac{\mathfrak{c}^4 \mathsf{e}^{-z_1\tilde{\Xi}^s_\varepsilon(\mathfrak{c}^2)+iz_2\mathfrak{c}^2}}{\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\mathfrak{c}^2)-iz_2\mathfrak{c}^2\right)\sqrt{4\mathfrak{c}^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\right| \leqslant \frac{C}{z_1}.$$

We have

$$\begin{aligned}\partial_{\xi_2}&\left(\frac{\xi_2^2}{\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\right)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\right)\\ &= -\frac{z_1\xi_2\partial_2^2\widetilde{\Xi}^s_\varepsilon(\xi_2)}{\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\right)^2\sqrt{4\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4/\xi_2^2}}\\ &\quad-\frac{\mathfrak{c}^4}{\xi_2^2\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\right)\left(4\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4/\xi_2^2\right)^{3/2}}\\ &\quad+\frac{1}{\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\right)\left(4\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4/\xi_2^2\right)^{1/2}}\end{aligned}$$

thus, in the interval $[C_1/\varepsilon^2,+\infty[$,

$$\begin{aligned}&\left|\partial_{\xi_2}\left(\frac{\xi_2^2}{\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\right)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\right)\right.\\ &\qquad\left.-\frac{1}{\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\right)\left(4\left(1-\varepsilon^2\mathfrak{c}^2\right)\right)^{1/2}}\right|\\ &\qquad\qquad\leqslant \frac{Cz_1}{\varepsilon^3\xi_2^2(z_1\varepsilon+z_2)^2}+\frac{C}{\xi_2^2(z_1\varepsilon+z_2)}\leqslant\frac{C}{\varepsilon^4\xi_2^2(z_1\varepsilon+z_2)}\end{aligned}$$

and in the interval $[\mathfrak{c}^2, C_1/\varepsilon^2]$,

$$\begin{aligned}&\left|\partial_{\xi_2}\left(\frac{\xi_2^2}{\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\xi_2\right)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\right)\right|\\ &\leqslant \frac{Cz_1}{\xi_2^{1/2}\left(z_1\xi_2^{-1/2}+z_2\right)^2}+\frac{C}{\xi_2^2\left(z_1\xi_2^{-1/2}+z_2\right)}+\frac{C}{\left(z_1\xi_2^{-1/2}+z_2\right)}\leqslant\frac{C}{\left(z_1\xi_2^{-1/2}+z_2\right)}.\end{aligned} \tag{3.19}$$

Therefore,

$$\int_{D(0,1)}\left|\sum_{s=\pm}\int_{\{|\xi_2|\geqslant\mathfrak{c}^2\}}\frac{s\xi_2^2\mathsf{e}^{-|z_1|\tilde{\Xi}^s_\varepsilon(\xi_2)+iz_2\xi_2}}{\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\,d\xi_2\right|\,dz$$
$$\leqslant C\int_{C_1/\varepsilon^2}^{+\infty}\int_0^1\int_0^{z_1}\frac{\mathsf{e}^{-\varepsilon z_1\xi_2/C}}{\varepsilon^4\xi_2^2(z_1\varepsilon+z_2)}\,dz_2dz_1d\xi_2$$
$$+C\int_{\mathfrak{c}^2}^{C_1/\varepsilon^2}\int_0^1\int_0^{z_1}\frac{\mathsf{e}^{-z_1\sqrt{\xi_2}/C}}{\left(z_1/\sqrt{\xi_2}+z_2\right)}\,dz_2dz_1d\xi_2+Q_{122}$$
$$\leqslant C\int_{C_1/\varepsilon^2}^{+\infty}\int_0^1|\ln\varepsilon|\frac{\mathsf{e}^{-\varepsilon z_1\xi_2/C}}{\varepsilon^4\xi_2^2}\,dz_1d\xi_2+C\int_{\mathfrak{c}^2}^{C_1/\varepsilon^2}\int_0^1\ln\xi_2\mathsf{e}^{-z_1\sqrt{\xi_2}/C}\,dz_1d\xi_2+Q_{122}$$
$$\leqslant C\int_{C_1/\varepsilon^2}^{+\infty}\frac{|\ln\varepsilon|}{\varepsilon^5\xi_2^3}\,d\xi_2+C\int_{\mathfrak{c}^2}^{C_1/\varepsilon^2}\frac{\ln\xi_2}{\xi_2^{1/2}}\,d\xi_2+Q_{122}\leqslant C\frac{|\ln\varepsilon|}{\varepsilon}+Q_{122}.$$

We then estimate the terms in Q_{122} containing the contribution of $\sum_{s=\pm}\frac{s}{z_1\partial_2\tilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\xi_2}$ in (3.19) in the following way. First, as for $\partial_1^2\partial_2\mathcal{K}^\varepsilon$,

$$\left|\sum_{s=\pm}\frac{s\mathsf{e}^{-iz_1\mathrm{Im}\tilde{\Xi}^s_\varepsilon(\xi_2)}}{(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2)-iz_2)}\right|\leqslant C\frac{\min(1,Cz_1/\varepsilon)}{(z_1\varepsilon+z_2)}+\frac{C}{\varepsilon^4\xi_2^2(z_1\varepsilon+z_2)},$$

hence

$$Q_{122}\leqslant C\int_{C_1/\varepsilon^2}^{+\infty}\int_0^1\int_0^{z_1}\frac{\mathsf{e}^{-\varepsilon z_1\xi_2/C}}{\varepsilon^4\xi_2^2(z_1\varepsilon+z_2)}\,dz_2dz_1d\xi_2$$
$$+C\int_0^1\int_{C_1/\varepsilon^2}^{+\infty}\int_0^{z_1}\mathsf{e}^{-\varepsilon z_1\xi_2/C}\frac{\min(1,Cz_1/\varepsilon)}{(z_1\varepsilon+z_2)}\,dz_2d\xi_2dz_1$$
$$\leqslant C|\ln\varepsilon|\int_{C_1/\varepsilon^2}^{+\infty}\frac{d\xi_2}{\varepsilon^5\xi_2^3}+C|\ln\varepsilon|\int_0^1\mathsf{e}^{-z_1/(C\varepsilon)}\frac{\min(1,Cz_1/\varepsilon)}{\varepsilon z_1}\,dz_1$$
$$\leqslant\frac{C|\ln\varepsilon|}{\varepsilon}.$$

This gives

$$\int_{D(0,1)}\left|\partial_1\partial_2^2\mathcal{K}^\varepsilon(z)\right|\,dz\leqslant C\frac{|\ln\varepsilon|}{\varepsilon}.$$

• *Estimate for* $\partial_2^3\mathcal{K}^\varepsilon$. For $s=\pm$, we integrate by parts

$$\int_{\mathfrak{c}^2}^{+\infty}\frac{\xi_2^3\mathsf{e}^{-|z_1|\tilde{\Xi}^s_\varepsilon(\xi_2)+iz_2\xi_2}}{\widetilde{\Xi}^s_\varepsilon(\xi_2)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\,d\xi_2 \tag{3.20}$$
$$=\int_{\mathfrak{c}^2}^{+\infty}\mathsf{e}^{-z_1\tilde{\Xi}^s_\varepsilon(\xi_2)+iz_2}\partial_{\xi_2}\left(\frac{\xi_2^3}{\widetilde{\Xi}^s_\varepsilon(\xi_2)\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\right)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\right)\,d\xi_2$$
$$+\frac{\mathfrak{c}^6\mathsf{e}^{-z_1\tilde{\Xi}^s_\varepsilon(\mathfrak{c}^2)+iz_2}}{\widetilde{\Xi}^s_\varepsilon(\mathfrak{c}^2)\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\mathfrak{c}^2)-iz_2\mathfrak{c}^2\right)\sqrt{4\mathfrak{c}^4\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}.$$

In addition,

$$\left|\frac{\mathfrak{c}^6 \mathrm{e}^{-z_1\widetilde{\Xi}^s_\varepsilon(\mathfrak{c}^2)+iz_2\mathfrak{c}^2}}{\widetilde{\Xi}^s_\varepsilon(\mathfrak{c}^2)\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\mathfrak{c}^2)-iz_2\mathfrak{c}^2\right)\sqrt{4\mathfrak{c}^4\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\right| \leqslant \frac{C}{z_1}.$$

We have

$$\begin{aligned}
\partial_{\xi_2}&\left(\frac{\xi_2^3}{\widetilde{\Xi}^s_\varepsilon(\xi_2)\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\right)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\right)\\
&= -\frac{z_1\xi_2^2\partial_2^2\widetilde{\Xi}^s_\varepsilon(\xi_2)}{\widetilde{\Xi}^s_\varepsilon(\xi_2)\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\right)^2\sqrt{4\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4/\xi_2^2}}\\
&\quad-\frac{\mathfrak{c}^4}{\xi_2\widetilde{\Xi}^s_\varepsilon(\xi_2)\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\right)\left(4\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4/\xi_2^2\right)^{3/2}}\\
&\quad+\frac{\partial_2\left(\xi_2^2/\widetilde{\Xi}^s_\varepsilon(\xi_2)\right)}{\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\right)\left(4\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4/\xi_2^2\right)^{1/2}}
\end{aligned}$$

thus, in the interval $[C_1/\varepsilon^2,+\infty[$,

$$\begin{aligned}
&\left|\partial_{\xi_2}\left(\frac{\xi_2^3}{\widetilde{\Xi}^s_\varepsilon(\xi_2)\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\right)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\right)\right.\\
&\qquad\left.-\frac{\partial_2\left(\xi_2^2/\widetilde{\Xi}^s_\varepsilon(\xi_2)\right)}{2\sqrt{1-\varepsilon^2\mathfrak{c}^2}\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\right)}\right|\\
&\qquad\leqslant\frac{z_1}{\varepsilon^4\xi_2^2(z_1\varepsilon+z_2)^2}+\frac{C}{\varepsilon\xi_2^2(z_1\varepsilon+z_2)}\leqslant\frac{C}{\varepsilon\xi_2^2(z_1\varepsilon+z_2)}
\end{aligned}\tag{3.21}$$

and in the interval $[\mathfrak{c}^2, C_1/\varepsilon^2]$,

$$\begin{aligned}
&\left|\partial_{\xi_2}\left(\frac{\xi_2^3}{\widetilde{\Xi}^s_\varepsilon(\xi_2)\left(z_1\partial_2\widetilde{\Xi}^s_\varepsilon(\xi_2)-iz_2\xi_2\right)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\right)\right|\\
&\quad\leqslant\frac{z_1}{\left(z_1\xi_2^{-1/2}+z_2\right)^2}+\frac{C}{\xi_2^{1.5}\left(z_1\xi_2^{-1/2}+z_2\right)}+\frac{C\xi_2^{1/2}}{\left(z_1\xi_2^{-1/2}+z_2\right)}\leqslant\frac{C\xi_2^{1/2}}{\left(z_1\xi_2^{-1/2}+z_2\right)}.
\end{aligned}$$

As a consequence,

$$\begin{aligned}
\int_{D(0,1)} &\left| \sum_{s=\pm} \int_{\{|\xi_2| \geqslant \mathfrak{c}^2\}} \frac{s\xi_2^3 \mathsf{e}^{-|z_1|\tilde{\Xi}_\varepsilon^s(\xi_2)+iz_2\xi_2}}{\widetilde{\Xi}_\varepsilon^s(\xi_2)\sqrt{4\xi_2^2\left(1-\varepsilon^2\mathfrak{c}^2\right)-\mathfrak{c}^4}}\, d\xi_2 \right| dz \\
&\leqslant C \int_0^1 \int_{C_1/\varepsilon^2}^{+\infty} \int_0^{z_1} \frac{\mathsf{e}^{-\varepsilon z_1 \xi_2/C}}{\varepsilon\xi_2^2(z_1\varepsilon+z_2)}\, dz_2 dz_1 d\xi_2 + Q_{222} \\
&\quad + C \int_0^1 \int_{\mathfrak{c}^2}^{C_1/\varepsilon^2} \int_0^{z_1} \frac{\xi_2^{1/2}\mathsf{e}^{-z_1\sqrt{\xi_2}}}{\left(z_1/\sqrt{\xi_2}+z_2\right)}\, dz_2 d\xi_2 dz_1 \\
&\leqslant C \int_0^1 \int_{C_1/\varepsilon^2}^{+\infty} |\ln\varepsilon| \frac{\mathsf{e}^{-\varepsilon z_1\xi_2/C}}{\varepsilon\xi_2^2}\, d\xi_2 dz_1 + Q_{222} \\
&\quad + C \int_0^1 \int_{\mathfrak{c}^2}^{C_1/\varepsilon^2} \ln\xi_2 \xi_2^{1/2} \mathsf{e}^{-z_1\sqrt{\xi_2}/C}\, d\xi_2 dz_1 \\
&= C \int_{C_1/\varepsilon^2}^{+\infty} \frac{|\ln\varepsilon|}{\varepsilon^2\xi_2^3}\, d\xi_2 + C \int_{\mathfrak{c}^2}^{C_1/\varepsilon^2} \ln\xi_2\, d\xi_2 + Q_{222} \leqslant C\frac{|\ln\varepsilon|}{\varepsilon^2} + Q_{222},
\end{aligned}$$

and Q_{222} contains the terms put aside in (3.21) for which there is a cancellation: by Proposition 3.4,

$$\begin{aligned}
\int_{D(0,1)} &\left| \sum_{s=\pm} \int_{C_1/\varepsilon^2} \frac{s\partial_2\left(\xi_2^2/\widetilde{\Xi}_\varepsilon^s(\xi_2)\right)\mathsf{e}^{-|z_1|\tilde{\Xi}_\varepsilon^s(\xi_2)+iz_2\xi_2}}{2\sqrt{1-\varepsilon^2\mathfrak{c}^2}\left(z_1\partial_2\widetilde{\Xi}_\varepsilon^s(\xi_2)-iz_2\right)}\, d\xi_2 \right| dz_2 dz_1 \\
&\leqslant C \int_{C_1/\varepsilon^2}^{+\infty} \int_0^1 \int_0^{z_1} \frac{\mathsf{e}^{-\varepsilon z_1\xi_2/C}}{\varepsilon^5\xi_2^2(z_1\varepsilon+z_2)}\, dz_2 dz_1 d\xi_2 \\
&\quad + \frac{C}{\varepsilon} \int_{C_1/\varepsilon^2}^{+\infty} \int_0^1 \int_0^{z_1} \frac{\mathsf{e}^{-\varepsilon z_1\xi_2/C}}{(z_1\varepsilon+z_2)} \min(1, Cz_1/\varepsilon)\, dz_2 dz_1 d\xi_2 \leqslant C\frac{|\ln\varepsilon|}{\varepsilon^2},
\end{aligned}$$

by the same kind of computations. This concludes the proof for the estimates on the kernel $\mathcal{K}^\varepsilon$.

4. Proofs

4.1. Proof of Proposition 2.2

We recall that

$$\begin{aligned}
\mathrm{Err}_{\mathrm{ph}} = -c(\varepsilon)\partial_1 A^{(0)} + \varepsilon^2 c(\varepsilon) A^{(0)}\partial_1 A^{(0)} + 2\varepsilon^2\partial_1\phi^{(0)}\partial_1 A^{(0)} \\
+ 2\varepsilon^4\partial_2\phi^{(0)}\partial_2 A^{(0)} + \left(\partial_1^2+\varepsilon^2\partial_2^2\right)\phi^{(0)}.
\end{aligned}$$

Since $A^{(0)} \in X_2^2$, $\nabla\phi^{(0)} \in X_2^2$, we infer

$$\left\|\partial_2^2\phi^{(0)}\right\|_{X_3^1} + \left\|A^{(0)}\partial_1 A^{(0)}\right\|_{X_5^1} + \left\|\partial_2\phi^{(0)}\partial_2 A^{(0)}\right\|_{X_5^1} + \left\|\partial_1\phi^{(0)}\partial_1 A^{(0)}\right\|_{X_5^1} \leqslant C,$$

thus, by the choice (2.1),

$$\begin{aligned}\|\mathrm{Err}_{\mathrm{ph}}\|_{X^1_3} &\leqslant \left\|\partial_1^2\phi^{(0)} - c(\varepsilon)\partial_1 A^{(0)}\right\|_{X^1_3} + C\varepsilon^2\\ &= (\mathfrak{c} - c(\varepsilon))\left\|\partial_1 A^{(0)}\right\|_{X^1_3} + C\varepsilon^2 \leqslant C\varepsilon^2.\end{aligned}$$

Analogously,

$$\left\|\left(\partial_1\phi^{(0)}\right)^2\right\|_{X^1_4} + \left\|A^{(0)}\partial_1\phi^{(0)}\right\|_{X^1_4} + \left\|\left[A^{(0)}\right]^2\right\|_{X^1_4} + \left\|\partial_1^2 A^{(0)}\right\|_{X^1_4} \leqslant C,$$

hence

$$\begin{aligned}\|\mathrm{Err}_{\mathrm{am}}\|_{X^1_2} &\leqslant \left\|c(\varepsilon)\partial_1\phi^{(0)} - \mathfrak{c}^2 A^{(0)}\right\|_{X^1_2} + C\varepsilon^2\\ &= \mathfrak{c}\Big(\mathfrak{c} - c(\varepsilon)\Big)\left\|A^{(0)}\right\|_{X^1_2} + C\varepsilon^2 \leqslant C\varepsilon^2.\end{aligned}$$

Furthermore, we may put forward some cancellation:

$$\|\mathfrak{c}\mathcal{I}\mathrm{Err}_{\mathrm{ph}} - \mathrm{Err}_{\mathrm{am}}\|_{X^1_2} \leqslant C\varepsilon^4.$$

In order to justify this estimate, we use (2.1), i.e. $\mathfrak{c}A^{(0)} = \partial_1\phi^{(0)}$, and the relation $\mathfrak{c}^2 = c(\varepsilon)^2 + \varepsilon^2$ (hence $c(\varepsilon) = \mathfrak{c} + \mathcal{O}(\varepsilon^2)$), and write

$$\begin{aligned}&\|c(\varepsilon)\mathcal{I}\mathrm{Err}_{\mathrm{ph}} - \mathrm{Err}_{\mathrm{am}}\|_{X^1_2}\\ &\quad= \Big\|c(\varepsilon)\mathcal{I}\Big(c(\varepsilon)\partial_1 A^{(0)} - \varepsilon^2 c(\varepsilon)A^{(0)}\partial_1 A^{(0)} - 2\varepsilon^2\partial_1\phi^{(0)}\partial_1 A^{(0)} - \left(\partial_1^2 + \varepsilon^2\partial_2^2\right)\phi^{(0)}\Big)\\ &\qquad - \Big(- c(\varepsilon)\partial_1\phi^{(0)} + \varepsilon^2\left(\partial_1\phi^{(0)}\right)^2 - \varepsilon^2 c(\varepsilon)A^{(0)}\partial_1\phi^{(0)}\\ &\qquad\quad + \mathfrak{c}^2 A^{(0)} + \varepsilon^2\frac{\mathfrak{c}^2}{2}(\Gamma - 3)\left[A^{(0)}\right]^2 - \varepsilon^2\partial_1^2 A^{(0)}\Big)\Big\|_{X^1_2}\\ &\quad\leqslant \varepsilon^2\Big\|\mathfrak{c}\mathcal{I}\Big(- \mathfrak{c}A^{(0)}\partial_1 A^{(0)} - 2\mathfrak{c}A^{(0)}\partial_1 A^{(0)} - \partial_2^2\phi^{(0)}\Big)\\ &\qquad - \Big(- A^{(0)} + \frac{\mathfrak{c}^2}{2}(\Gamma - 3)\left[A^{(0)}\right]^2 - \partial_1^2 A^{(0)}\Big)\Big\|_{X^1_2} + C\varepsilon^4,\end{aligned}$$

since $\mathcal{I}\partial_1^2\phi^{(0)} = \partial_1\phi^{(0)}$, $\mathcal{I}\partial_1 A^{(0)} = A^{(0)}$ (the right-hand sides go to zero at infinity). Using that $2\mathcal{I}(A^{(0)}\partial_1 A^{(0)}) = \mathcal{I}(\partial_1[A^{(0)}]^2) = [A^{(0)}]^2$ and that $\partial_2^2\phi^{(0)} = \mathfrak{c}\partial_1^{-1}\partial_2^2 A^{(0)}$, we infer that the last norm is actually that of

$$\mathfrak{c}^2\mathcal{I}\left(-\partial_1^{-1}\partial_2^2 A^{(0)} + \frac{1}{\mathfrak{c}^2}\partial_1 A^{(0)} - \Gamma A^{(0)}\partial_1 A^{(0)} + \frac{1}{\mathfrak{c}^2}\partial_1^3 A^{(0)}\right) = 0,$$

since $A^{(0)}$ solves SW. This is the required result.

4.2. Proof of Proposition 2.6

We start with some easy estimates of convolution in X_β spaces in $\mathbb{R}^2$.

Lemma 4.1. —

(i) *Let be given* μ *and* ν *such that* $0 < \mu < \nu$ *and* $\nu \geqslant 2$ *and let us fix* μ' *such that* ($\nu > 2$ *and* $\mu' = \mu$) *or* ($\nu = 2$ *and* $\mu' < \mu$). *Assume that* $u \in X_\nu$ *and* $v \in X_\mu$. *Then,* $u \star v \in X_{\mu'}$ *and*

$$\|u \star v\|_{X_{\mu'}} \leqslant C(\nu, \mu, \mu')\|u\|_{X_\nu}\|v\|_{X_\mu}.$$

(ii) *Let be given* μ *and* ν *such that* $1 < \mu < 2$ *and* $0 \leqslant \nu < 2$. *Assume that* $u \in X^1_\nu$ *and* $v \in X_{\mu+1}$ *is such that* $\int_{\mathbb{R}^2} v\,dz = 0$. *Then,* $u \star v \in X_{\mu+\nu-1}$ *and*

$$\|u \star v\|_{X_{\mu+\nu-1}} \leqslant C(\nu, \mu, \mu')\|u\|_{X^1_\nu}\|v\|_{X_{\mu+1,0}}.$$

(iii) *Assume that* $u \in L^1$ *is compactly supported in* $\bar{D}(0,2)$. *Let* $\mu > 0$ *be given and* $v \in X_\mu$. *Then,* $u \star v \in X_\mu$ *and*

$$\|u \star v\|_{X_\mu} \leqslant C(\mu)\|u\|_{L^1}\|v\|_{X_\mu}.$$

Proof. — We easily check that, in all the above cases, $u \star v$ is well-defined and continuous on $\mathbb{R}^2$ (since $u(z-y)v(y) = \mathcal{O}(1/|y|^{2+\sigma})$ at infinity for some $\sigma = \sigma(\mu,\nu) > 0$ for (i) and (ii)). Therefore, only the decay at infinity has to be shown.
(i) For $z \in \mathbb{R}^2$ with $|z| \geqslant 2$, we have

$$\begin{aligned}
|u \star v(z)| &\leqslant \int_{\mathbb{R}^2} |u(z-y)| \times |v(y)|\,dy \\
&\leqslant \int_{\mathbb{R}^2} |u(z-y)| \times |v(y)|\mathbf{1}_{|y|\geqslant 2|z|}\,dy \\
&\quad + \int_{\mathbb{R}^2} |u(z-y)| \times |v(y)|\mathbf{1}_{|z-y|\leqslant |z|/2}\,dy \\
&\quad + \int_{\mathbb{R}^2} |u(z-y)| \times |v(y)|\mathbf{1}_{|y|\leqslant 2|z| \text{ and } |z-y|\geqslant |z|/2}\,dy
\end{aligned}$$

(notice that if $|z-y| \leqslant |z|/2$, then $|y| \leqslant 3|z|/2 \leqslant 2|z|$) and estimate all the contributions separately.

For the first integral, we have $|z-y| \geqslant |y| - |z| \geqslant |y|/2$, thus, since $\nu + \mu > 2$,

$$\begin{aligned}
\int_{\mathbb{R}^2} |u(z-y)| \times |v(y)|\mathbf{1}_{|y|\geqslant 2|z|}\,dy &\leqslant C(\nu,\mu)\|u\|_{X_\nu}\|v\|_{X_\mu} \int_{\{|y|\geqslant 2|z|\}} \frac{dy}{|y|^{\nu+\mu}} \\
&\leqslant \frac{C(\nu,\mu)}{|z|^{\nu+\mu-2}}\|u\|_{X_\nu}\|v\|_{X_\mu}.
\end{aligned}$$

For the second integral, we have $|y| \geqslant |z| - |y-z| \geqslant |z|/2$, and this implies

$$\begin{aligned}
&\int_{\mathbb{R}^2} |u(z-y)| \times |v(y)|\mathbf{1}_{|z-y|\leqslant |z|/2}\,dy \\
&\qquad\qquad \leqslant \frac{C(\nu,\mu)\|u\|_{X_\nu}\|v\|_{X_\mu}}{|z|^\mu} \int_{\{|z-y|\leqslant |z|/2\}} \frac{dy}{(1+|y-z|)^\nu}.
\end{aligned}$$

The integral in the right hand side is $\leqslant C\ln|z|$ if $\nu = 2$ (and then $\mu' < \mu$) and $\leqslant C$ if $\nu > 2$ (and then $\mu' = \mu$). In both cases, we then have

$$\int_{\mathbb{R}^2} |u(z-y)| \times |v(y)|\mathbf{1}_{|z-y|\leqslant |z|/2}\,dy \leqslant \|u\|_{X_\nu}\|v\|_{X_\mu}\frac{C(\nu,\mu,\mu')}{|z|^{\mu'}}.$$

Finally, for the third integral, $|z-y| \geqslant |z|/2$ and thus

$$\int_{\mathbb{R}^2} |u(z-y)| \times |v(y)| \mathbf{1}_{|y| \leqslant 2|z| \text{ and } |z-y| \geqslant |z|/2} \, dy \leqslant \frac{C(\nu,\mu)}{|z|^\nu} \|u\|_{X_\nu} \|v\|_{X_\mu} \int_{\mathbb{R}^2} \frac{\mathbf{1}_{|y| \leqslant 2|z|}}{(1+|y|)^\mu} \, dy.$$

If $\mu < 2$, the integral in the right-hand side is $\leqslant C(\mu)|z|^{2-\mu}$ and we are done since $\mu' \leqslant \mu + \nu - 2$. If $\mu > 2$ (resp. $\mu = 2$), then the integral in the right-hand side is $\leqslant C(\mu)|z|^{2-\mu}$ (resp. $\leqslant C \ln |z|$), and we are done since $\mu' \leqslant \mu < \nu$.

(ii) Since v has zero integral, for $z \in \mathbb{R}^2$ with $|z| \geqslant 2$, we have

$$\begin{aligned}
|u \star v(z)| &= \left| \int_{\mathbb{R}^2} u(z-y) v(y) \, dy \right| \\
&= \left| \int_{\mathbb{R}^2} \big(u(z-y) - u(z)\big) v(y) \, dy \right| \\
&\leqslant \int_{\mathbb{R}^2} |u(z-y) - u(z)| \times |v(y)| \, dy \\
&\leqslant \int_{\mathbb{R}^2} |u(z-y) - u(z)| \times |v(y)| \mathbf{1}_{|y| \geqslant 2|z|} \, dy \\
&\quad + \int_{\mathbb{R}^2} |u(z-y) - u(z)| \times |v(y)| \mathbf{1}_{|z-y| \leqslant |z|/2} \, dy \\
&\quad + \int_{\mathbb{R}^2} |u(z-y) - u(z)| \times |v(y)| \mathbf{1}_{|y| \leqslant 2|z| \text{ and } |z-y| \geqslant |z|/2} \, dy
\end{aligned}$$

and estimate all the contributions separately.

For the first integral, we have $|z-y| \geqslant |y| - |z| \geqslant |z|$, thus,

$$|u(z-y) - u(z)| \leqslant |u(z-y)| + |u(z)| \leqslant \frac{C\|u\|_{X_\nu}}{|z|^\nu},$$

and this implies, since $\mu > 1$,

$$\begin{aligned}
\int_{\mathbb{R}^2} |u(z-y) - u(z)| \times |v(y)| \mathbf{1}_{|y| \geqslant 2|z|} \, dy &\leqslant \frac{C\|u\|_{X_\nu}}{|z|^\nu} \|v\|_{X_{\mu+1}} \int_{\{|y| \geqslant 2|z|\}} \frac{dy}{|y|^{\mu+1}} \\
&\leqslant \frac{C(\mu,\nu)}{|z|^{\mu+\nu-1}} \|u\|_{X_\nu} \|v\|_{X_{\mu+1}}.
\end{aligned}$$

For the second integral, we have ($\nu \geqslant 0$)

$$|u(z-y) - u(z)| \leqslant |u(z-y)| + |u(z)| \leqslant \frac{\|u\|_{X_\nu}}{(1+|z-y|)^\nu} + \frac{\|u\|_{X_\nu}}{|z|^\nu} \leqslant \frac{2\|u\|_{X_\nu}}{(1+|z-y|)^\nu},$$

and, combining this with $|y| \geqslant |z| - |y-z| \geqslant |z|/2$, this implies, since $\nu < 2$,

$$\begin{aligned}
&\int_{\mathbb{R}^2} |u(z-y) - u(z)| \times |v(y)| \mathbf{1}_{1 \leqslant |z-y| \leqslant |z|/2} \, dy \\
&\qquad \leqslant \frac{C(\mu)}{|z|^{\mu+1}} \|u\|_{X_\nu} \|v\|_{X_{\mu+1}} \int_{\{|z-y| \leqslant |z|/2\}} \frac{dy}{(1+|y-z|)^\nu} \\
&\qquad \leqslant \frac{C(\mu)}{|z|^{\mu+\nu-1}} \|u\|_{X_\nu} \|v\|_{X_{\mu+1}}.
\end{aligned}$$

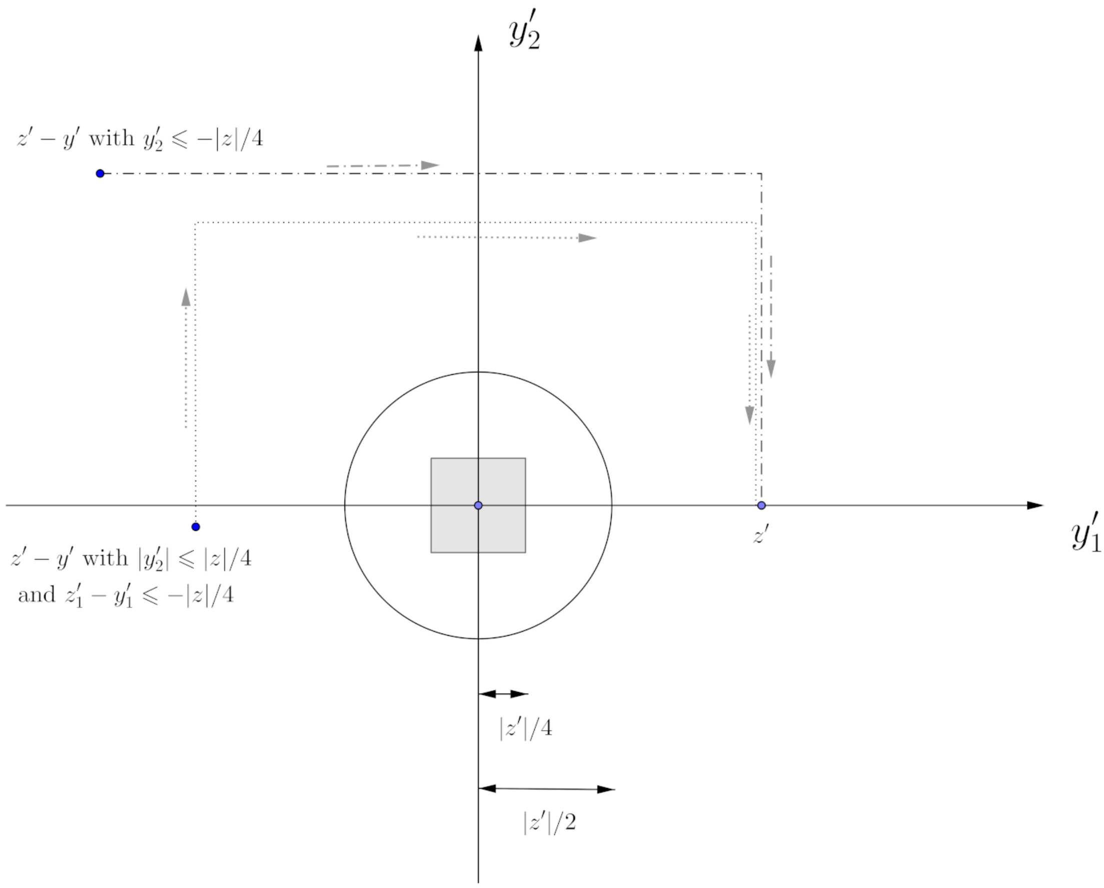

Figure 4.1. The path $\gamma_{y,z}$

For the third integral, we shall use the Mean Value Theorem along an appropriate path $\gamma_{y,z}$ defined in the following way. Let us denote (y'_1, y'_2) the coordinates of y in the orthonormal basis $(z/|z|, z^\perp/|z|)$ of $\mathbb{R}^2$, so that z has now coordinates $(z'_1 = |z|, z'_2 = 0)$. Keep in mind that $|z-y| = |z'-y'| \geqslant |z|/2 = |z'|/2$, thus $|y'_2| \geqslant |z|/4$ or $|z'_1 - y'_1| \geqslant |z|/4$. Then, if $y'_2 \leqslant -|z|/4$, the path $\gamma_{y,z}$ (in rotated coordinates) goes from $z'-y'$ to $(z'_1, z'_2 - y'_2)$ along $[0, y'_1] \ni t \mapsto (z'_1 - y'_1 + t, z'_2 - y'_2)$ and then from $(z'_1, z'_2 - y'_2)$ to (z'_1, z'_2) along $[0, y'_2] \ni t \mapsto (z'_1, z'_2 - y'_2 + t)$ (see the dashdot line in Figure 4.1). The length of this path is $\leqslant |y'_1| + |y'_2| \leqslant 2|y|$ and, along this path, we have

$$|\nabla u| \leqslant \frac{4\|u\|_{X^1_\nu}}{|z|^{\nu+1}}$$

(for the first part, $|\gamma_{y,z}| \geqslant |(\gamma_{y,z})'_2| = |-y'_2| \geqslant |z|/4$, and, on the second part, $|\gamma_{y,z}| \geqslant |(\gamma_{y,z})'_1| = |z|$), hence

$$|u(z-y) - u(z)| \leqslant \frac{C\|u\|_{X^1_\nu}|y|}{|z|^{\nu+1}}. \tag{4.1}$$

If $y_2' \geqslant |z|/4$, we use the symmetric path with respect to the z_1' axis and also get (4.1). If $|y_2'| \leqslant |z|/4$ and $z_1' - y_1' \geqslant |z|/4$, the estimate (4.1) remains true with the first path, simply noticing that, for the first part of the path, we also have $|\gamma_{y,z}| \geqslant |(\gamma_{y,z})_1'| = |z_1' - y_1' + t| \geqslant |z|/4$. Finally, if $|y_2'| \leqslant |z|/4$ and $z_1' - y_1' \leqslant -|z|/4$, we follow (see the dotted line in Figure 4.1) the path $(z_1' - y_1', z_2' - y_2') \to (z_1' - y_1', |z|)$ parallel to the z_2'-axis, then $(z_1' - y_1', |z|) \to (z_1' = |z|, |z|)$ parallel to the z_1'-axis, and finally $(z_1' = |z|, |z|) \to (z_1' = |z|, 0)$ parallel to the z_2'-axis. The length of this path is $\leqslant C|z|$, and, by hypothesis, $z_1' - y_1' \leqslant -|z|/4 \leqslant 0$, thus $|y| \geqslant y_1' \geqslant z_1' = |z|$, and then $|\nabla u| \leqslant C\|u\|_{X^1_\nu}/|z|^{\nu+1}$. Therefore,

$$|u(z-y) - u(z)| \leqslant \frac{C\|u\|_{X^1_\nu}|z|}{|z|^{\nu+1}} \leqslant \frac{C\|u\|_{X^1_\nu}|y|}{|z|^{\nu+1}},$$

which proves (4.1) in this last case. We then infer, for the third integral,

$$\begin{aligned}
&\int_{\mathbb{R}^2} |u(z-y) - u(z)| \times |v(y)| \mathbf{1}_{|y| \leqslant 2|z| \text{ and } |z-y| \geqslant |z|/2}\, dy \\
&\qquad \leqslant \frac{C\|u\|_{X^1_\nu}}{|z|^{\nu+1}} \|v\|_{X_{\mu+1}} \int_{\mathbb{R}^2} \frac{|y|}{(1+|y|)^{\mu+1}} \mathbf{1}_{|y| \leqslant 2|z| \text{ and } |z-y| \geqslant |z|/2}\, dy \\
&\qquad \leqslant \frac{C}{|z|^{\nu+1}} \|u\|_{X^1_\nu} \|v\|_{X_{\mu+1}} \int_0^{2|z|} \frac{r^2\, dr}{(1+r)^{\mu+1}} \\
&\qquad \leqslant \frac{C(\mu)}{|z|^{\mu+\nu-1}} \|u\|_{X^1_\nu} \|v\|_{X_{\mu+1}},
\end{aligned}$$

since $\mu < 2$. Gathering all these estimates, we have shown (i).

(iii) For $z \in \mathbb{R}^2$ with $|z| \geqslant 4$, we have

$$|u \star v(z)| \leqslant \int_{D(z,2)} |u(z-y)| \times |v(y)|\, dy.$$

Moreover, using that $|y| \geqslant |z| - |z-y| \geqslant |z| - 2 \geqslant |z|/2$, we deduce

$$\int_{D(z,2)} |u(z-y)| \times |v(y)|\, dy \leqslant C(\mu) \frac{\|v\|_{X_\mu}}{|z|^\mu} \int_{D(z,2)} |u(z-y)|\, dy = C(\mu) \|u\|_{L^1} \frac{\|v\|_{X_\mu}}{|z|^\mu},$$

as wished. □

We now turn to the proof of Proposition 2.6. We fix $\chi_0 \in \mathcal{C}(\mathbb{R}^2)$ such that $\chi_0 \equiv 1$ in $\bar{D}(0,1)$ and $\chi_0 \equiv 0$ outside $\bar{D}(0,2)$. We shall then split the convolution kernel $\partial^\alpha \mathcal{K}^\varepsilon$ as

$$\partial^\alpha \mathcal{K}^\varepsilon = \chi_0 \partial^\alpha \mathcal{K}^\varepsilon + (1 - \chi_0) \partial^\alpha \mathcal{K}^\varepsilon.$$

Proof of Proposition 2.6 (iii). — For the term $\chi_0 \partial^\alpha \mathcal{K}^\varepsilon \in L^1(\mathbb{R}^2)$ compactly supported in $\bar{D}(0,2)$, we apply Lemma 4.1 (iii) and Proposition 2.3 (ii) to infer, for $\mu > 0$ and $v \in X_\mu$,

$$\begin{aligned}
&\left\| \left(\chi_0 \partial_1^3 \mathcal{K}^\varepsilon\right) \star v \right\|_{X_\mu} + |\ln \varepsilon|^{-1} \left\| \left(\chi_0 \partial_1^2 \partial_2 \mathcal{K}^\varepsilon\right) \star v \right\|_{X_\mu} \\
&\qquad + \varepsilon |\ln \varepsilon|^{-1} \left\| \left(\chi_0 \partial_1 \partial_2^2 \mathcal{K}^\varepsilon\right) \star v \right\|_{X_\mu} + \varepsilon^2 |\ln \varepsilon|^{-1} \left\| \left(\chi_0 \partial_2^3 \mathcal{K}^\varepsilon\right) \star v \right\|_{X_\mu} \leqslant C(\mu) \|v\|_{X_\mu}.
\end{aligned}$$

For any $\alpha \in \mathbb{N}^2$ with $\alpha_1 + \alpha_2 = 3$, the term $(1 - \chi_0)\partial^\alpha \mathcal{K}^\varepsilon$ belongs to X_3 by Proposition 2.3 (i). It then follows from Lemma 4.1 (i) (with $\nu = 3 > 2$) that, for $0 < \mu < 3$,

$$\|[(1-\chi_0)\partial^\alpha \mathcal{K}^\varepsilon] \star v\|_{X_\mu} \leqslant C(\mu)\|v\|_{X_\mu}. \qquad \square$$

Proof of Proposition 2.6 (ii). — The estimates for the local term $\chi_0 \partial^\alpha \mathcal{K}^\varepsilon$ (with $\alpha_1 + \alpha_2 = 2$) and for the term $(1-\chi_0)\partial^\alpha \mathcal{K}^\varepsilon \in X_2$ (with $\nu = 2$) follow as for (iv). $\square$

Proof of Proposition 2.6 (i). — The estimate for the local term $\chi_0 \nabla \mathcal{K}^\varepsilon$ follows as for (iv). For the term $(1-\chi_0)\nabla \mathcal{K}^\varepsilon \in X_1^1$, we apply Lemma 4.1 (ii) (with $\nu = 1 < 2$) and deduce from Proposition 2.3 (ii) that, for $1 < \mu < 2$ and $v \in X_{\mu+1,a}$,

$$\|[(1-\chi_0)\nabla \mathcal{K}^\varepsilon] \star v\|_{X_\mu} \leqslant C(\mu)\,\|(1-\chi_0)\nabla \mathcal{K}^\varepsilon\|_{X_1^1}\,\|v\|_{X_{\mu+1}} \leqslant C(\mu)\|v\|_{X_{\mu+1}}. \qquad \square$$

Proof of Proposition 2.6 (ii'). — The estimate for the local terms $\chi_0 \nabla^2 \mathcal{K}^\varepsilon$ follows as for (iv). For the terms $(1-\chi_0)\nabla^2 \mathcal{K}^\varepsilon \in X_2^1$, we apply Proposition 2.3 (i) and Lemma 4.1 (ii) (with "μ" $= \mu - 1 \in]1, 2[$ and "ν" $= 2 + \mu' - \mu \in [0, 2[$, so that "$\mu$" + "$\nu$" $- 1 = \mu'$) and deduce, for $v \in X_{\mu,a}$,

$$\left\|\left[(1-\chi_0)\nabla^2 \mathcal{K}^\varepsilon\right] \star v\right\|_{X_{\mu'}} \leqslant C(\mu,\mu')\left\|(1-\chi_0)\nabla^2 \mathcal{K}^\varepsilon\right\|_{X^1_{2+\mu'-\mu}} \|v\|_{X_\mu} \leqslant C(\mu,\mu')\|v\|_{X_\mu}.$$

$\square$

4.3. Proof of Proposition 2.8

Let us first prove the claimed estimates on (A, ϕ) given by (2.9). It is easy to check that all the terms are well-defined thanks to the decays imposed on $\mathcal{S}_{\rm am}$ and $\mathcal{S}_{\rm ph}$.

Estimate for A. By Proposition 2.6 (ii), we have

$$\left\|\partial_2^2 \mathcal{K}^\varepsilon \star \mathcal{S}_{\rm am}\right\|_{X_{1+\sigma}} \leqslant C(\sigma,\sigma')|\ln \varepsilon|^2 \|\mathcal{S}_{\rm am}\|_{X_{1+\sigma'}}$$

and, by Proposition 2.6 (i), since $\mathcal{S}_{\rm ph}$ is odd in z_1, thus has vanishing integral,

$$\|\partial_1 \mathcal{K}^\varepsilon \star \mathcal{S}_{\rm ph}\|_{X_{1+\sigma}} \leqslant C(\sigma,\sigma')\|\mathcal{S}_{\rm ph}\|_{X_{2+\sigma'}}.$$

Therefore, using $c(\varepsilon) = \mathfrak{c} + \mathcal{O}(\varepsilon^2)$,

$$\|A\|_{X_{1+\sigma}} \leqslant \|\mathcal{T}\|_{X_{1+\sigma}} + C(\sigma,\sigma')|\ln \varepsilon|^2 \|\mathcal{S}_{\rm am}\|_{X_{1+\sigma'}} + C(\sigma,\sigma')\|\mathcal{S}_{\rm ph}\|_{X_{2+\sigma'}},$$

with

$$\mathcal{T} = \varepsilon^{-2}\left(\partial_1^2 \mathcal{K}^\varepsilon \star \mathcal{S}_{\rm am} - \mathfrak{c}\partial_1 \mathcal{K}^\varepsilon \star \mathcal{S}_{\rm ph}\right) = \varepsilon^{-2}\partial_1^2 \mathcal{K}^\varepsilon \star (\mathcal{S}_{\rm am} - \mathfrak{c}\mathcal{I}\mathcal{S}_{\rm ph}),$$

hence, by Proposition 2.6 (i),

$$\varepsilon^2 \|\mathcal{T}\|_{X_{1+\sigma}} \leqslant C(\sigma,\sigma')\|\mathcal{S}_{\rm am} - \mathfrak{c}\mathcal{I}\mathcal{S}_{\rm ph}\|_{X_{1+\sigma'}}.$$

If $\mathcal{S}_{\rm am} = 0$, then we may write, by Proposition 2.6 (i) ($\mathcal{S}_{\rm ph}$ is odd in z_1, thus has vanishing integral),

$$\varepsilon^2 \|\mathcal{T}\|_{X_{1+\sigma}} = \mathfrak{c}\,\|\partial_1 \mathcal{K}^\varepsilon \star \mathcal{S}_{\rm ph}\|_{X_{1+\sigma}} \leqslant C(\sigma,\sigma')\|\mathcal{S}_{\rm ph}\|_{X_{2+\sigma'}}.$$

Estimate for $\nabla\phi$. By Proposition 2.6 (ii) and (iii), we have

$$\|\nabla \partial_1 \mathcal{K}^\varepsilon \star \mathcal{S}_{\rm am}\|_{X_{1+\sigma}} \leqslant C(\sigma,\sigma')\|\mathcal{S}_{\rm am}\|_{X_{1+\sigma'}}$$

and

$$\left\|\nabla\partial_1^2\mathcal{K}^\varepsilon\star\mathcal{S}_{\mathrm{ph}}\right\|_{X_{1+\sigma}}+\varepsilon^2\left\|\nabla\partial_2^2\mathcal{K}^\varepsilon\star\mathcal{S}_{\mathrm{ph}}\right\|_{X_{1+\sigma}}\leqslant C(\sigma,\sigma')|\ln\varepsilon|^2\left\|\mathcal{S}_{\mathrm{ph}}\right\|_{X_{1+\sigma'}}.$$

Therefore, using $c(\varepsilon)=\mathfrak{c}+\mathcal{O}(\varepsilon^2)$,

$$\|\nabla\phi\|_{X_{1+\sigma}}\leqslant\|\mathcal{T}\|_{X_{1+\sigma}}+C(\sigma,\sigma')\|\mathcal{S}_{\mathrm{am}}\|_{X_{1+\sigma'}}+C(\sigma,\sigma')|\ln\varepsilon|^2\|\mathcal{S}_{\mathrm{ph}}\|_{X_{1+\sigma'}},$$

with

$$\mathcal{T}=\frac{\mathfrak{c}}{\varepsilon^2}\left(\nabla\partial_1\mathcal{K}^\varepsilon\star\mathcal{S}_{\mathrm{am}}-\mathfrak{c}\nabla\mathcal{K}^\varepsilon\star\mathcal{S}_{\mathrm{ph}}\right)=\frac{\mathfrak{c}}{\varepsilon^2}\nabla\partial_1\mathcal{K}^\varepsilon\star\left(\mathcal{S}_{\mathrm{am}}-\mathfrak{c}\mathcal{I}\mathcal{S}_{\mathrm{ph}}\right),$$

hence, by Proposition 2.6 (ii),

$$\varepsilon^2\|\mathcal{T}\|_{X_{1+\sigma}}\leqslant C(\sigma,\sigma')\left\|\mathcal{S}_{\mathrm{am}}-\mathfrak{c}\mathcal{I}\mathcal{S}_{\mathrm{ph}}\right\|_{X_{1+\sigma'}}.$$

If $\mathcal{S}_{\mathrm{am}}=0$, then we may write, by Proposition 2.6 (i) ($\mathcal{S}_{\mathrm{ph}}$ is odd in z_1),

$$\varepsilon^2\|\mathcal{T}\|_{X_{1+\sigma}}=\mathfrak{c}\|\nabla\mathcal{K}^\varepsilon\star\mathcal{S}_{\mathrm{ph}}\|_{X_{1+\sigma}}\leqslant C(\sigma,\sigma')\|\mathcal{S}_{\mathrm{ph}}\|_{X_{2+\sigma'}}.$$

Combining these two estimates, we infer

$$\begin{aligned}\|A\|_{X_{1+\sigma}}+\|\nabla\phi\|_{X_{1+\sigma}}\leqslant C(\sigma,\sigma')|\ln\varepsilon|^2&\Big(\|\mathcal{S}_{\mathrm{ph}}\|_{X_{2+\sigma'}}+\|\mathcal{S}_{\mathrm{am}}\|_{X_{1+\sigma'}}\Big)\\&+\frac{C(\sigma,\sigma')}{\varepsilon^2}\|\mathcal{S}_{\mathrm{am}}-\mathfrak{c}\mathcal{I}\mathcal{S}_{\mathrm{ph}}\|_{X_{1+\sigma'}},\end{aligned}$$

and, when $\mathcal{S}_{\mathrm{am}}=0$, we have

$$\begin{aligned}&\|A\|_{X_{1+\sigma}}+\|\nabla\phi\|_{X_{1+\sigma}}\\&\quad\leqslant C(\sigma,\sigma')\|\mathcal{S}_{\mathrm{am}}\|_{X_{1+\sigma'}}+C(\sigma,\sigma')|\ln\varepsilon|^2\|\mathcal{S}_{\mathrm{ph}}\|_{X_{2+\sigma'}}+\frac{C(\sigma,\sigma')}{\varepsilon^2}\|\mathcal{S}_{\mathrm{ph}}\|_{X_{1+\sigma'}}.\end{aligned}$$

Estimate for ∇A. By Proposition 2.6 (ii'), we have, since $\int\nabla\mathcal{S}_{\mathrm{am}}\,dz=0$,

$$\left\|\partial_2^2\mathcal{K}^\varepsilon\star\nabla\mathcal{S}_{\mathrm{am}}\right\|_{X_{2+\sigma}}\leqslant C(\sigma,\sigma')|\ln\varepsilon|^2\left\|\nabla\mathcal{S}_{\mathrm{am}}\right\|_{X_{2+\sigma'}}$$

and, since $\mathcal{S}_{\mathrm{ph}}\in X_{2+\sigma,a}$ has also vanishing integral,

$$\left\|\nabla\partial_1\mathcal{K}^\varepsilon\star\mathcal{S}_{\mathrm{ph}}\right\|_{X_{2+\sigma}}\leqslant C(\sigma,\sigma')\left\|\mathcal{S}_{\mathrm{ph}}\right\|_{X_{2+\sigma'}}.$$

Therefore, using $c(\varepsilon)=\mathfrak{c}+\mathcal{O}(\varepsilon^2)$,

$$\|\nabla A\|_{X_{2+\sigma}}\leqslant\|\mathcal{T}\|_{X_{2+\sigma}}+C(\sigma,\sigma')|\ln\varepsilon|^2\|\nabla\mathcal{S}_{\mathrm{am}}\|_{X_{2+\sigma'}}+C(\sigma,\sigma')\|\mathcal{S}_{\mathrm{ph}}\|_{X_{2+\sigma'}},$$

with

$$\mathcal{T}=\varepsilon^{-2}\nabla\left(\partial_1^2\mathcal{K}^\varepsilon\star\mathcal{S}_{\mathrm{am}}-\mathfrak{c}\partial_1\mathcal{K}^\varepsilon\star\mathcal{S}_{\mathrm{ph}}\right)=\varepsilon^{-2}\partial_1^2\mathcal{K}^\varepsilon\star\nabla(\mathcal{S}_{\mathrm{am}}-\mathfrak{c}\mathcal{I}\mathcal{S}_{\mathrm{ph}}),$$

hence, by Proposition 2.6 (ii') (since $\int\nabla(\mathcal{S}_{\mathrm{am}}-\mathfrak{c}\mathcal{I}\mathcal{S}_{\mathrm{ph}})\,dz=0$),

$$\varepsilon^2\|\mathcal{T}\|_{X_{2+\sigma}}\leqslant C(\sigma,\sigma')\left\|\nabla\left(\mathcal{S}_{\mathrm{am}}-\mathfrak{c}\mathcal{I}\mathcal{S}_{\mathrm{ph}}\right)\right\|_{X_{2+\sigma'}}.$$

If $\mathcal{S}_{\mathrm{am}}=0$, then, by Proposition 2.6 (ii'), ($\mathcal{S}_{\mathrm{ph}}$ is odd in z_1), we may write

$$\varepsilon^2\|\mathcal{T}\|_{X_{2+\sigma}}=\mathfrak{c}\left\|\nabla\partial_1\mathcal{K}^\varepsilon\star\mathcal{S}_{\mathrm{ph}}\right\|_{X_{2+\sigma}}\leqslant C(\sigma,\sigma')\|\mathcal{S}_{\mathrm{ph}}\|_{X_{2+\sigma'}}.$$

Estimate for $\nabla^2\phi$. Let $j,\,k\in\{1,2\}$. By Proposition 2.6 (ii') ($\int\partial_k\mathcal{S}_{\rm am}\,dz=0$) and (iii), we have

$$\|\partial_j\partial_k\left(\partial_1\mathcal{K}^\varepsilon\star\mathcal{S}_{\rm am}\right)\|_{X_{2+\sigma}}=\|\partial_j\partial_1\mathcal{K}^\varepsilon\star\partial_k\mathcal{S}_{\rm am}\|_{X_{2+\sigma}}\leqslant C(\sigma,\sigma')\,\|\nabla\mathcal{S}_{\rm am}\|_{X_{2+\sigma'}}$$

and

$$\left\|\partial_j\partial_1^2\mathcal{K}^\varepsilon\star\partial_k\mathcal{S}_{\rm ph}\right\|_{X_{2+\sigma}}+\varepsilon^2\left\|\partial_j\partial_2^2\mathcal{K}^\varepsilon\star\partial_k\mathcal{S}_{\rm ph}\right\|_{X_{2+\sigma}}\leqslant C(\sigma,\sigma')|\ln\varepsilon|^2\,\|\nabla\mathcal{S}_{\rm ph}\|_{X_{2+\sigma'}}\,.$$

Therefore, using $c(\varepsilon)=\mathfrak{c}+\mathcal{O}(\varepsilon^2)$,

$$\|\partial_j\partial_k\phi\|_{X_{2+\sigma}}\leqslant\|\mathcal{T}\|_{X_{2+\sigma}}+C(\sigma,\sigma')\|\nabla\mathcal{S}_{\rm am}\|_{X_{2+\sigma'}}+C(\sigma,\sigma')|\ln\varepsilon|^2\|\nabla\mathcal{S}_{\rm ph}\|_{X_{2+\sigma'}},$$

with

$$\mathcal{T}=\frac{\mathfrak{c}}{\varepsilon^2}\partial_j\partial_k\left(\partial_1\mathcal{K}^\varepsilon\star\mathcal{S}_{\rm am}-\mathfrak{c}\mathcal{K}^\varepsilon\star\mathcal{S}_{\rm ph}\right)=\frac{\mathfrak{c}}{\varepsilon^2}\partial_j\partial_1\mathcal{K}^\varepsilon\star\partial_k\left(\mathcal{S}_{\rm am}-\mathfrak{c}\mathcal{I}\mathcal{S}_{\rm ph}\right),$$

hence, by Proposition 2.6 (ii') ($\partial_k(\mathcal{S}_{\rm am}-\mathfrak{c}\mathcal{I}\mathcal{S}_{\rm ph})$ has zero integral),

$$\varepsilon^2\|\mathcal{T}\|_{X_{2+\sigma}}\leqslant C(\sigma,\sigma')\,\|\nabla\left(\mathcal{S}_{\rm am}-\mathfrak{c}\mathcal{I}\mathcal{S}_{\rm ph}\right)\|_{X_{2+\sigma'}}\,.$$

If $\mathcal{S}_{\rm am}=0$, then we may write, by Proposition 2.3 (ii')($\mathcal{S}_{\rm ph}$ is odd in z_1, thus has vanishing integral),

$$\varepsilon^2\|\mathcal{T}\|_{X_{2+\sigma}}=\mathfrak{c}\,\|\partial_j\partial_k\mathcal{K}^\varepsilon\star\mathcal{S}_{\rm ph}\|_{X_{2+\sigma}}\leqslant C(\sigma,\sigma')|\ln\varepsilon|^2\|\mathcal{S}_{\rm ph}\|_{X_{2+\sigma'}}.$$

Combining these two estimates, we infer

$$\begin{aligned}\|\nabla A\|_{X_{2+\sigma}}+\left\|\nabla^2\phi\right\|_{X_{2+\sigma}}\leqslant{}& C(\sigma,\sigma')|\ln\varepsilon|^2\Big(\|\mathcal{S}_{\rm am}\|_{X^1_{1+\sigma'}}+\|\mathcal{S}_{\rm ph}\|_{X^1_{2+\sigma'}}\Big)\\&+\frac{C(\sigma,\sigma')}{\varepsilon^2}\,\|\mathcal{S}_{\rm am}-\mathfrak{c}\mathcal{I}\mathcal{S}_{\rm ph}\|_{X^1_{1+\sigma'}}\,,\end{aligned}$$

and when $\mathcal{S}_{\rm am}=0$, then we have

$$\|\nabla A\|_{X_{2+\sigma}}+\left\|\nabla^2\phi\right\|_{X_{2+\sigma}}\leqslant C(\sigma,\sigma')|\ln\varepsilon|^2\|\mathcal{S}_{\rm ph}\|_{X^1_{2+\sigma'}}+\frac{C(\sigma,\sigma')}{\varepsilon^2}|\ln\varepsilon|\|\mathcal{S}_{\rm ph}\|_{X_{1+\sigma'}}.$$

Estimate for ∇^2A. We start with estimating ∂_1^2A. By Proposition 2.6 (iii), we have,

$$\left\|\partial_1^2\partial_2^2\left(\mathcal{K}^\varepsilon\star\mathcal{S}_{\rm am}\right)\right\|_{X_{2+\sigma}}=\left\|\partial_1^2\partial_2\mathcal{K}^\varepsilon\star\partial_2\mathcal{S}_{\rm am}\right\|_{X_{2+\sigma}}\leqslant C(\sigma,\sigma')|\ln\varepsilon|^2\,\|\nabla\mathcal{S}_{\rm am}\|_{X_{2+\sigma'}}$$

and

$$\left\|\partial_1^2\partial_1\mathcal{K}^\varepsilon\star\mathcal{S}_{\rm ph}\right\|_{X_{2+\sigma}}\leqslant C(\sigma,\sigma')\|\mathcal{S}_{\rm ph}\|_{X_{2+\sigma'}}.$$

Therefore, using $c(\varepsilon)=\mathfrak{c}+\mathcal{O}(\varepsilon^2)$,

$$\left\|\partial_1^2A\right\|_{X_{2+\sigma}}\leqslant\|\mathcal{T}\|_{X_{2+\sigma}}+C(\sigma,\sigma')|\ln\varepsilon|^2\,\|\nabla\mathcal{S}_{\rm am}\|_{X_{2+\sigma'}}+C(\sigma,\sigma')\,\|\mathcal{S}_{\rm ph}\|_{X_{2+\sigma'}}\,,$$

with

$$\mathcal{T}=\varepsilon^{-2}\partial_1^2\left(\partial_1^2\mathcal{K}^\varepsilon\star\mathcal{S}_{\rm am}-\mathfrak{c}\partial_1\mathcal{K}^\varepsilon\star\mathcal{S}_{\rm ph}\right)=\varepsilon^{-2}\partial_1^3\mathcal{K}^\varepsilon\star\partial_1\left(\mathcal{S}_{\rm am}-\mathfrak{c}\mathcal{I}\mathcal{S}_{\rm ph}\right),$$

hence, by Proposition 2.6 (iii),

$$\varepsilon^2\|\mathcal{T}\|_{X_{2+\sigma}}\leqslant C(\sigma,\sigma')\|\nabla(\mathcal{S}_{\rm am}-\mathfrak{c}\mathcal{I}\mathcal{S}_{\rm ph})\|_{X_{2+\sigma'}}.$$

If $\mathcal{S}_{\mathrm{am}}=0$, then we may write
$$\varepsilon^2\|\mathcal{T}\|_{X_{2+\sigma}}=\mathfrak{c}\left\|\partial_1^3\mathcal{K}^\varepsilon\star\mathcal{S}_{\mathrm{ph}}\right\|_{X_{2+\sigma}}\leqslant C(\sigma,\sigma')\|\mathcal{S}_{\mathrm{ph}}\|_{X_{2+\sigma'}}.$$
This gives, as wished, the estimate for ∂_1^2A in $X_{2+\sigma}$.

We then consider $\partial_1\partial_2A$. By Proposition 2.6 (iii), we have, since $\nabla\mathcal{S}_{\mathrm{am}}\in X_{2+\sigma'}$ has zero integral,
$$\left\|\partial_1\partial_2^3\left(\mathcal{K}^\varepsilon\star\mathcal{S}_{\mathrm{am}}\right)\right\|_{X_{2+\sigma}}=\left\|\partial_1\partial_2^2\mathcal{K}^\varepsilon\star\partial_2\mathcal{S}_{\mathrm{am}}\right\|_{X_{2+\sigma}}\leqslant C(\sigma,\sigma')\frac{|\ln\varepsilon|}{\varepsilon}\left\|\nabla\mathcal{S}_{\mathrm{am}}\right\|_{X_{2+\sigma'}}$$
and, since $\partial_2\mathcal{S}_{\mathrm{ph}}\in X_{2+\sigma,0}$,
$$\left\|\partial_2\partial_1^2\mathcal{K}^\varepsilon\star\mathcal{S}_{\mathrm{ph}}\right\|_{X_{2+\sigma}}=\left\|\partial_1^2\mathcal{K}^\varepsilon\star\partial_2\mathcal{S}_{\mathrm{ph}}\right\|_{X_{2+\sigma}}\leqslant C(\sigma,\sigma')\left\|\nabla\mathcal{S}_{\mathrm{ph}}\right\|_{X_{2+\sigma'}}.$$
Therefore, using $c(\varepsilon)=\mathfrak{c}+\mathcal{O}(\varepsilon^2)$,
$$\|\partial_1\partial_2A\|_{X_{2+\sigma}}\leqslant\|\mathcal{T}\|_{X_{2+\sigma}}+C(\sigma,\sigma')\frac{|\ln\varepsilon|}{\varepsilon}\|\nabla\mathcal{S}_{\mathrm{am}}\|_{X_{2+\sigma'}}+C(\sigma,\sigma')\left\|\nabla\mathcal{S}_{\mathrm{ph}}\right\|_{X_{2+\sigma'}},$$
with
$$\mathcal{T}=\varepsilon^{-2}\partial_1\partial_2\left(\partial_1^2\mathcal{K}^\varepsilon\star\mathcal{S}_{\mathrm{am}}-\mathfrak{c}\partial_1\mathcal{K}^\varepsilon\star\mathcal{S}_{\mathrm{ph}}\right)=\varepsilon^{-2}\partial_1^3\mathcal{K}^\varepsilon\star\partial_2\left(\mathcal{S}_{\mathrm{am}}-\mathfrak{c}\mathcal{I}\mathcal{S}_{\mathrm{ph}}\right),$$
hence, by Proposition 2.6 (iii),
$$\varepsilon^2\|\mathcal{T}\|_{X_{2+\sigma}}\leqslant C(\sigma,\sigma')\left\|\nabla\left(\mathcal{S}_{\mathrm{am}}-\mathfrak{c}\mathcal{I}\mathcal{S}_{\mathrm{ph}}\right)\right\|_{X_{2+\sigma'}}.$$
If $\mathcal{S}_{\mathrm{am}}=0$, then we may write ($\int\partial_1\mathcal{S}_{\mathrm{ph}}\,dz=0$)
$$\varepsilon^2\|\mathcal{T}\|_{X_{2+\sigma}}\leqslant C(\sigma,\sigma')\left\|\partial_1\partial_2\mathcal{K}^\varepsilon\star\partial_1\mathcal{S}_{\mathrm{ph}}\right\|_{X_{2+\sigma}}\leqslant C(\sigma,\sigma')\left\|\partial_1\mathcal{S}_{\mathrm{ph}}\right\|_{X_{2+\sigma'}}.$$
This gives the desired estimate for $\partial_1\partial_2A$ in $X_{2+\sigma}$.

We finally consider ∂_2^2A. By Proposition 2.6 (iii), we have
$$\left\|\partial_2^4\left(\mathcal{K}^\varepsilon\star\mathcal{S}_{\mathrm{am}}\right)\right\|_{X_{2+\sigma}}=\left\|\partial_2^3\mathcal{K}^\varepsilon\star\partial_2\mathcal{S}_{\mathrm{am}}\right\|_{X_{2+\sigma}}\leqslant C(\sigma,\sigma')\frac{|\ln\varepsilon|}{\varepsilon^2}\|\nabla\mathcal{S}_{\mathrm{am}}\|_{X_{2+\sigma'}}$$
and, by Proposition 2.6 (ii') ($\int\partial_2\mathcal{S}_{\mathrm{ph}}\,dz=0$),
$$\left\|\partial_2^2\partial_1\mathcal{K}^\varepsilon\star\mathcal{S}_{\mathrm{ph}}\right\|_{X_{2+\sigma}}=\|\partial_1\partial_2\mathcal{K}^\varepsilon\star\partial_2\mathcal{S}_{\mathrm{ph}}\|_{X_{2+\sigma}}\leqslant C(\sigma,\sigma')\left\|\nabla\mathcal{S}_{\mathrm{ph}}\right\|_{X_{2+\sigma'}}.$$
Therefore, using $c(\varepsilon)=\mathfrak{c}+\mathcal{O}(\varepsilon^2)$,
$$\left\|\partial_2^2A\right\|_{X_{2+\sigma}}\leqslant\|\mathcal{T}\|_{X_{2+\sigma}}+C(\sigma,\sigma')\frac{|\ln\varepsilon|}{\varepsilon^2}\left\|\nabla\mathcal{S}_{\mathrm{am}}\right\|_{X_{2+\sigma'}}+C(\sigma,\sigma')\left\|\nabla\mathcal{S}_{\mathrm{ph}}\right\|_{X_{2+\sigma'}},$$
with
$$\mathcal{T}=\varepsilon^{-2}\partial_2^2\left(\partial_1^2\mathcal{K}^\varepsilon\star\mathcal{S}_{\mathrm{am}}-\mathfrak{c}\partial_1\mathcal{K}^\varepsilon\star\mathcal{S}_{\mathrm{ph}}\right)=\varepsilon^{-2}\partial_2\partial_1^2\mathcal{K}^\varepsilon\star\partial_2\left(\mathcal{S}_{\mathrm{am}}-\mathfrak{c}\mathcal{I}\mathcal{S}_{\mathrm{ph}}\right),$$
hence, by Proposition 2.6 (iii),
$$\varepsilon^2\|\mathcal{T}\|_{X_{2+\sigma}}\leqslant C(\sigma,\sigma')|\ln\varepsilon|^2\left\|\nabla\left(\mathcal{S}_{\mathrm{am}}-\mathfrak{c}\mathcal{I}\mathcal{S}_{\mathrm{ph}}\right)\right\|_{X_{2+\sigma'}}.$$
If $\mathcal{S}_{\mathrm{am}}=0$, then we may write, by Proposition 2.6 (ii')($\int\partial_2\mathcal{S}_{\mathrm{ph}}\,dz=0$),
$$\varepsilon^2\|\mathcal{T}\|_{X_{2+\sigma}}=\mathfrak{c}\left\|\partial_1\partial_2\mathcal{K}^\varepsilon\star\partial_2\mathcal{S}_{\mathrm{ph}}\right\|_{X_{2+\sigma}}\leqslant C(\sigma,\sigma')\left\|\partial_1\mathcal{S}_{\mathrm{ph}}\right\|_{X_{2+\sigma'}}.$$
This gives the required estimate for ∂_2^2A in $X_{2+\sigma}$.

Remark 4.2. — It may be noticed that the proof does not require $\nabla\mathcal{S}_{\mathrm{ph}} \in X_{3+\sigma'}$, but only $\nabla\mathcal{S}_{\mathrm{ph}} \in X_{2+\sigma'}$. We shall not need this information during the proof.

Let us now turn to the uniqueness part. By using Fourier transform in the sense of tempered distributions, we see that (2.9) is indeed a solution of the system (2.8). We now consider (A, ϕ) solution of the homogeneous problem

$$\begin{cases} c(\varepsilon)\partial_1 A - \left(\partial_1^2 + \varepsilon^2\partial_2^2\right)\phi = 0 \\ \\ -c(\varepsilon)\partial_1\phi + \left(\mathfrak{c}^2 - \varepsilon^2\left(\partial_1^2 + \varepsilon^2\partial_2^2\right)\right)A = 0. \end{cases}$$

By taking Fourier transform and combining the two equations, we infer

$$\left(\xi_1^2 + \mathfrak{c}^2\xi_2^2 + \left(\xi_1^2 + \varepsilon^2\xi_2^2\right)^2\right)\mathscr{F}(A) = \left(\xi_1^2 + \mathfrak{c}^2\xi_2^2 + \left(\xi_1^2 + \varepsilon^2\xi_2^2\right)^2\right)\mathscr{F}(\phi) = 0.$$

Since the symbol $\xi_1^2 + \mathfrak{c}^2\xi_2^2 + (\xi_1^2 + \varepsilon^2\xi_2^2)^2$ vanishes only at the origin, a classical result due to L. Schwartz asserts that $\mathscr{F}(A)$ and $\mathscr{F}(\phi)$ are linear combination of Dirac masses at the origin and their derivatives, which means that A and ϕ are polynomial functions. Since $A \in X_{1+\sigma}$, the only possibility is $A \equiv 0$. Concerning ϕ, we only have $\nabla\phi \in X_{1+\sigma}$, which imposes that ϕ is constant in $\mathbb{R}^2$, and since it is assumed odd in z_1, we deduce $\phi \equiv 0$, as wished.

4.4. Proof of Proposition 2.10

4.4.1. A priori estimate in the plane

We first establish an a priori estimate on the solutions to (2.10) (assuming existence).

Lemma 4.3. — *Let $0 < \sigma < \sigma' < 1$ be given. Then, there exists $\epsilon(\sigma, \sigma') \in]0, 1[$ and $C(\sigma, \sigma') > 0$ such that, for any $0 < \varepsilon < \epsilon(\sigma, \sigma')$, any $(\mathcal{S}_{\mathrm{ph}}, \mathcal{S}_{\mathrm{am}}) \in X^1_{2+\sigma',a} \times X^1_{1+\sigma',s}$ and any $(A, \nabla\phi) \in X^1_{1+\sigma'} \times X^1_{1+\sigma'}$, satisfying the linear system* (2.10)*, namely*

$$\begin{cases} c(\varepsilon)\partial_1 A - (\partial_1^2 + \varepsilon^2\partial_2^2)\,\phi - 2\varepsilon^2\partial_1\phi\partial_1 A^{(0)} - 2\varepsilon^4\partial_2\phi\partial_2 A^{(0)} \\ \qquad -\varepsilon^2 c(\varepsilon)\partial_1(A^{(0)}A) + 2\varepsilon^2\partial_1\phi^{(0)}\partial_1 A = \mathcal{S}_{\mathrm{ph}} \\ \\ -c(\varepsilon)\partial_1\phi - \varepsilon^2 c(\varepsilon)A^{(0)}\partial_1\phi - \varepsilon^2 c(\varepsilon)A\partial_1\phi^{(0)} + 2\varepsilon^2\partial_1\phi^{(0)}\partial_1\phi + \mathfrak{c}^2 A \\ \qquad -\varepsilon^2\mathfrak{c}^2(\Gamma - 3)A^{(0)}A - \varepsilon^2\left(\partial_1^2 A + \varepsilon^2\partial_2^2 A\right) = \mathcal{S}_{\mathrm{am}}, \end{cases}$$

then we have

$$\|A\|_{X^1_{1+\sigma}} + \|\nabla\phi\|_{X^1_{1+\sigma}} \leqslant C(\sigma, \sigma')|\ln\varepsilon|^2\left(\|\mathcal{S}_{\mathrm{ph}}\|_{X^1_{2+\sigma}} + \|\mathcal{S}_{\mathrm{am}}\|_{X^1_{1+\sigma}} + \frac{1}{\varepsilon^2}\left\|\mathfrak{c}\mathcal{I}\mathcal{S}_{\mathrm{ph}} - \mathcal{S}_{\mathrm{am}}\right\|_{X^1_{1+\sigma'}}\right).$$

Proof. — We argue by contradiction and assume that there exist sequences $(\varepsilon_n)_{n\in\mathbb{N}}$ tending to 0, $(\mathcal{S}_{\mathrm{ph},n}, \mathcal{S}_{\mathrm{am},n}) \in X^1_{2+\sigma} \times X^1_{1+\sigma'}$ and $(\phi_n, A_n) \in \dot{X}^1_{1+\sigma} \times X^1_{1+\sigma}$, satisfying (2.10), that is

$$\left\{\begin{aligned} & c(\varepsilon_n)\partial_1 A_n - (\partial_1^2 + \varepsilon_n^2\partial_2^2)\,\phi_n - 2\varepsilon_n^2 \partial_1\phi_n\partial_1 A^{(0)} - 2\varepsilon_n^4 \partial_2\phi_n\partial_2 A^{(0)} \\ & \quad -\varepsilon_n^2 c(\varepsilon_n)\partial_1\left(A^{(0)}A_n\right) + 2\varepsilon_n^2\partial_1\phi^{(0)}\partial_1 A_n = \mathcal{S}_{\mathrm{ph},n} \\ & -c(\varepsilon_n)\partial_1\phi_n - \varepsilon_n^2 c(\varepsilon_n)A^{(0)}\partial_1\phi_n - \varepsilon_n^2 c(\varepsilon_n)A_n\partial_1\phi^{(0)} + 2\varepsilon_n^2\partial_1\phi^{(0)}\partial_1\phi_n \\ & \quad + \mathfrak{c}^2 A_n + \varepsilon_n^2\mathfrak{c}^2(\Gamma-3)A^{(0)}A_n - \varepsilon_n^2\left(\partial_1^2 A_n + \varepsilon_n^2\partial_2^2 A_n\right) = \mathcal{S}_{\mathrm{am},n}, \end{aligned}\right. \tag{4.2}$$

such that

$$\|A_n\|_{X^1_{1+\sigma}} + \|\nabla\phi_n\|_{X^1_{1+\sigma}} + \left\|\left(\partial_1^2 A_n, \varepsilon_n\partial_1\partial_2 A_n, \varepsilon_n^2\partial_2^2 A_n\right)\right\|_{X_{2+\sigma}} = 1 \tag{4.3}$$

and

$$|\ln\varepsilon_n|^2\left(\|\mathcal{S}_{\mathrm{ph},n}\|_{X^1_{2+\sigma',a}} + \|\mathcal{S}_{\mathrm{am},n}\|_{X^1_{1+\sigma'}} + \frac{1}{\varepsilon_n^2}\left\|c\mathcal{I}\mathcal{S}_{\mathrm{ph},n} - \mathcal{S}_{\mathrm{am},n}\right\|_{X^1_{1+\sigma'}}\right) \to 0. \tag{4.4}$$

Step 1 (Local convergence). — First, we notice that since ϕ_n is odd in z_1, we have $\phi(0, z_2) = 0$ for any $z_2 \in \mathbb{R}$, and this implies, for $z_1 \in \mathbb{R}_+$, $z_2 \in \mathbb{R}$,

$$\begin{aligned} |\phi_n(z_1,z_2)| \leqslant \int_0^{z_1} |\partial_1\phi_n(\zeta,z_2)|\,d\zeta \leqslant \|\nabla\phi_n\|_{X_{1+\sigma}} \int_0^{z_1} \frac{d\zeta}{(1+|(\zeta,z_2)|)^{1+\sigma}} \\ \leqslant C(\sigma)\|\nabla\phi_n\|_{X_{1+\sigma}}. \end{aligned}$$

By Ascoli's Theorem, we may assume, up to a subsequence, that there exists $(A_\infty, \phi_\infty) \in X^1_{1+\sigma}$ such that

$$A_n \to A_\infty \qquad \phi_n \to \phi_\infty \qquad \nabla\phi_n \to \nabla\phi_\infty \qquad \text{strongly in } \mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^2).$$

In view of the bounds $\|A_n\|_{X_{1+\sigma}} + \|\nabla\phi_n\|_{X_{1+\sigma}} \leqslant 1$, we actually have

$$A_n \to A_\infty \qquad \nabla\phi_n \to \nabla\phi_\infty \qquad \text{strongly in } \mathcal{C}_b(\mathbb{R}^2).$$

Furthermore, we may assume that $(\nabla A_n)_{n\in\mathbb{N}}$ is weakly convergent in L^2. The weak limit can only be ∇A_∞ since $A_n \to A_\infty$ in $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^2)$. Arguing in a similar way, we may then assume

$$\nabla A_n \rightharpoonup \nabla A_\infty, \qquad \partial_1^2 A_n \rightharpoonup \partial_1^2 A_\infty \qquad \nabla^2\phi_n \rightharpoonup \nabla^2\phi_\infty \qquad \text{weakly in } L^2(\mathbb{R}^2).$$

In view of the bounds (4.3) and (4.4), we may easily pass to the (weak, or distributional) limit in (4.2) and get

$$\left\{\begin{aligned} & -\mathfrak{c}\partial_1 A_\infty + \partial_1^2\phi_\infty = 0 \\ & \mathfrak{c}\partial_1\phi_\infty - \mathfrak{c}^2 A_\infty = 0, \end{aligned}\right.$$

that is the preparedness relation

$$\mathfrak{c}A_\infty = \partial_1\phi_\infty.$$

As for the derivation of the KP-I solitary wave equation, we now subtract $c(\varepsilon_n)/\mathfrak{c}^2$ times the first equation of (2.10) and $\partial_1/\mathfrak{c}^2$ times the second one. This gives

$$\begin{aligned}(4.5)\quad &\frac{\mathfrak{c}^2 - c^2(\varepsilon_n)}{\varepsilon_n^2\mathfrak{c}^2}\partial_1 A_n - \frac{1}{\mathfrak{c}^2}\partial_1\left(\partial_1^2 A_n + \varepsilon_n^2\partial_2^2 A_n\right) + \frac{c(\varepsilon_n)}{\mathfrak{c}^2}\partial_2^2\phi_n \\ &\quad + \left\{2\frac{c(\varepsilon_n)}{\mathfrak{c}^2}\partial_1\phi^{(0)}\partial_1 A_n + 2\frac{c(\varepsilon_n)}{\mathfrak{c}^2}\partial_1\phi_n\partial_1 A^{(0)} + \frac{c(\varepsilon_n)^2}{\mathfrak{c}^2}\partial_1(A^{(0)}A_n)\right. \\ &\quad - \frac{c(\varepsilon_n)}{\mathfrak{c}^2}\partial_1\left(A^{(0)}\partial_1\phi_n\right) - \frac{c(\varepsilon_n)}{\mathfrak{c}^2}\partial_1\left(A_n\partial_1\phi^{(0)}\right) \\ &\quad \left. + \frac{2}{\mathfrak{c}^2}\partial_1\left[\partial_1\phi_n\partial_1\phi^{(0)}\right] + (\Gamma - 3)\partial_1\left(A_n A^{(0)}\right)\right\} \\ &= -\frac{1}{\varepsilon_n^2\mathfrak{c}^2}\partial_1\left(\mathfrak{c}\mathcal{I}\mathcal{S}_{\mathrm{ph},n} - \mathcal{S}_{\mathrm{am},n}\right) + \frac{\mathfrak{c} - c(\varepsilon_n)}{\varepsilon_n^2\mathfrak{c}^2}\mathcal{S}_{\mathrm{ph},n} - 2\varepsilon_n^2\partial_2\phi_n\partial_2 A^{(0)}\,.\end{aligned}$$

Since $\mathfrak{c}^2 - c^2(\varepsilon_n) = \varepsilon_n^2$, it then follows that $A_\infty = \partial_1\phi_\infty/\mathfrak{c}$ solves, in the distributional sense, the linearized KP-I solitary waves equation

$$(4.6)\qquad \frac{1}{\mathfrak{c}^2}\partial_1 A_\infty - \frac{1}{\mathfrak{c}^2}\partial_1^3 A_\infty + \Gamma\partial_1\left(A_\infty A^{(0)}\right) + \partial_2^2\partial_1^{-1}A_\infty = 0.$$

By the result in [LW19, Theorem 1.2], we deduce that

$$A_\infty \in \mathrm{Span}\left(\partial_1 A^{(0)}, \partial_2 A^{(0)}\right).$$

Furthermore, we have imposed the symmetries A_n even in z_1 and in z_2, whereas $\partial_1 A^{(0)}$ is odd in z_1 and even in z_2, and $\partial_2 A^{(0)}$ is even in z_1 and odd in z_2. It then follows that

$$A_\infty \equiv 0, \qquad \phi_\infty \equiv 0,$$

since ϕ_∞ is odd in z_1.

Step 2 (Smallness in L^2). — We multiply the first equation of (4.2) by ϕ_n and integrate over $\mathbb{R}^2$ by parts:

$$\begin{aligned}(4.7)\quad &\int |\nabla^{\varepsilon_n}\phi_n|^2 - c(\varepsilon_n)A_n\partial_1\phi_n\,dz \\ &= \int\Big(\mathcal{S}_{\mathrm{ph},n}\phi_n + 2\varepsilon_n^2\partial_1\phi^{(0)}\partial_1 A_n\phi_n + 2\varepsilon_n^2\partial_1\phi_n\partial_1 A^{(0)}\phi_n \\ &\qquad - 2\varepsilon_n^4\partial_2\phi_n\partial_2 A^{(0)}\phi_n - \varepsilon_n^2 c(\varepsilon_n)A^{(0)}A_n\partial_1\phi_n\Big)\,dz.\end{aligned}$$

Here, we have set $\nabla^\varepsilon = (\partial_1, \varepsilon\partial_2)^T$. We notice that

$$\left|\partial_1\phi^{(0)}\partial_1 A_n\phi_n\right| \leqslant \left\|\partial_1\phi^{(0)}\right\|_{X_2}\frac{|\phi_n|\|\partial_1 A_n\|_{X_{2+\sigma}}}{(1+|z|)^{4+\sigma}} \leqslant \frac{C|\phi_n|}{(1+|z|)^{4+\sigma}}$$

and that, similarly,

$$\left|\partial_1\phi_n\partial_1 A^{(0)}\phi_n\right| \leqslant \frac{C|\phi_n|}{(1+|z|)^3}, \quad \left|c(\varepsilon_n)A^{(0)}A_n\partial_1\phi_n\right| \leqslant \frac{C|A_n|}{(1+|z|)^{3+\sigma}}$$

and

$$\left|-2\varepsilon_n^4\partial_2\phi_n\partial_2 A^{(0)}\phi_n\right| \leqslant \frac{C\varepsilon_n^4}{(1+|z|)^3}.$$

Thus, as $n \to +\infty$ and by Dominated Convergence ($\|\phi_n\|_{L^\infty}$ is bounded and $\phi_n \to 0$ pointwise and similarly for A_n),

$$\int 2\partial_1\phi^{(0)}\partial_1 A_n\phi_n + 2\partial_1\phi_n\partial_1 A^{(0)}\phi_n - 2\varepsilon_n^2\partial_2\phi_n\partial_2 A^{(0)}\phi_n - c(\varepsilon_n)A^{(0)}A_n\partial_1\phi_n\, dz \to 0.$$

Inserting this into (4.7) then yields

$$\int |\nabla^{\varepsilon_n}\phi_n|^2 - c(\varepsilon_n)A_n\partial_1\phi_n\, dz = \int \mathcal{S}_{\mathrm{ph},n}\phi_n\, dz + o(\varepsilon_n^2).$$

Multiplying the second equation of (4.2) by A_n, integrating by parts over $\mathbb{R}^2$ and arguing similarly, we obtain

$$\int \varepsilon_n^2|\nabla^{\varepsilon_n}A_n|^2 + \mathfrak{c}^2A_n^2 - c(\varepsilon_n)\partial_1\phi_n A_n\, dz = \int \mathcal{S}_{\mathrm{am},n}A_n\, dz + o(\varepsilon_n^2).$$

Adding the two equations and using $\mathfrak{c}^2 = c(\varepsilon_n)^2 + \varepsilon_n^2$ and the fact that $\mathcal{S}_{\mathrm{ph},n} = \partial_1\mathcal{I}\mathcal{S}_{\mathrm{ph},n}$ yields

$$\begin{aligned}\int (\partial_1\phi_n)^2 &+ \varepsilon_n^2(\partial_2\phi_n)^2 - 2c(\varepsilon_n)A_n\partial_1\phi_n + c(\varepsilon_n)^2A_n^2 + \varepsilon_n^2A_n^2 + \varepsilon_n^2(\partial_1 A_n)^2\, dz\\ &= \frac{1}{\mathfrak{c}}\int \left(-\mathfrak{c}\mathcal{I}\mathcal{S}_{\mathrm{ph},n} + \mathcal{S}_{\mathrm{am},n}\right)\partial_1\phi_n\, dz - \frac{1}{\mathfrak{c}}\int \mathcal{S}_{\mathrm{am},n}(\partial_1\phi_n - \mathfrak{c}A_n)\, dz + o(\varepsilon_n^2).\end{aligned}$$

Observing the expanded square $(\partial_1\phi_n - c(\varepsilon_n)A_n)^2$, and using (4.3), (4.4) and Young's inequality, we obtain

$$\begin{aligned}\int \frac{1}{\varepsilon_n^2}\left(\partial_1\phi_n - c(\varepsilon_n)A_n\right)^2 + (\partial_2\phi_n)^2 + A_n^2 + (\partial_1 A_n)^2\, dz&\\ \leqslant \frac{1}{10\varepsilon_n^2}\int (\partial_1\phi_n - \mathfrak{c}A_n)^2\, dz + o(1).&\end{aligned}$$

This shows that

$$\int \frac{1}{\varepsilon_n^2}\left(\partial_1\phi_n - c(\varepsilon_n)A_n\right)^2 + \Big(\partial_2\phi_n\Big)^2 + A_n^2 + \Big(\partial_1 A_n\Big)^2\, dz = o(1),$$

and hence that

$$\int \frac{1}{\varepsilon_n^2}\left(\partial_1\phi_n - c(\varepsilon_n)A_n\right)^2 + |\nabla\phi_n|^2 + A_n^2 + (\partial_1 A_n)^2\, dz = o(1). \tag{4.8}$$

Step 3 (Smallness in H^1). — Let $j \in \{1, 2\}$. We multiply the first equation of (4.2) by $\partial_j^2\phi_n$ and integrate over $\mathbb{R}^2$ by parts:

$$\begin{aligned}\int |\nabla^{\varepsilon_n}\partial_j\phi_n|^2 &- c(\varepsilon_n)\partial_j A_n\partial_1\partial_j\phi_n\, dz\\ &= \int \Big(\partial_j\mathcal{S}_{\mathrm{ph},n}\partial_j\phi_n + 2\varepsilon_n^2\partial_1\phi^{(0)}\partial_1 A_n\partial_j^2\phi_n\\ &+2\varepsilon_n^2\partial_1\phi_n\partial_1 A^{(0)}\partial_j^2\phi_n - 2\varepsilon_n^4\partial_2\phi_n\partial_2 A^{(0)}\partial_j^2\phi_n + \varepsilon_n^2c(\varepsilon_n)\partial_1\left(A^{(0)}A_n\right)\partial_j^2\phi_n\Big)\, dz.\end{aligned}$$

We notice that, by (4.8),

$$\left|\int \partial_1\phi^{(0)}\partial_1 A_n\partial_j^2\phi_n\, dz\right| \leqslant \left\|\partial_1\phi^{(0)}\right\|_{L^2}\left\|\partial_j^2\phi_n\right\|_{L^\infty}\|\partial_1 A_n\|_{L^2} = o(1)$$

and that, similarly,

$$\left|\int \partial_1\phi_n\partial_1 A^{(0)}\partial_j^2\phi_n\,dz\right| \leqslant \|\partial_1\phi_n\|_{L^2}\left\|\partial_1 A^{(0)}\right\|_{L^2}\left\|\partial_j^2\phi_n\right\|_{L^\infty} = o(1),$$

$$\left|\int \partial_1(A^{(0)}A_n)\partial_j^2\phi_n\,dz\right| \leqslant \left\|\partial_j^2\phi_n\right\|_{L^\infty}\left(\left\|A^{(0)}\right\|_{L^2}\|\partial_1 A_n\|_{L^2} + \left\|\partial_1 A^{(0)}\right\|_{L^2}\|A_n\|_{L^2}\right) \to 0$$

and

$$\left|\int -2\varepsilon_n^2\partial_2\phi_n\partial_2 A^{(0)}\partial_j^2\phi_n\,dz\right| \leqslant 2\varepsilon_n^2\|\partial_2\phi_n\|_{L^\infty}\left\|\partial_2 A^{(0)}\right\|_{L^1}\left\|\partial_j^2\phi_n\right\|_{L^\infty} \to 0.$$

Thus, as $n\to+\infty$,

$$\int |\nabla^{\varepsilon_n}\partial_j\phi_n|^2 - c(\varepsilon_n)\partial_j A_n\partial_1\partial_j\phi_n\,dz = \int \partial_j\mathcal{S}_{\mathrm{ph},n}\partial_j\phi_n\,dz + o(\varepsilon_n^2).$$

Analogously, we obtain

$$\int \varepsilon_n^2|\nabla^{\varepsilon_n}\partial_j A_n|^2 + \mathfrak{c}^2(\partial_j A_n)^2 - c(\varepsilon_n)\partial_1\partial_j\phi_n\partial_j A_n\,dz = \int \partial_j\mathcal{S}_{\mathrm{am},n}\partial_j A_n + o\left(\varepsilon_n^2\right)$$

and this allows to derive as before

$$\int \frac{1}{\varepsilon_n^2}|\nabla(\partial_1\phi_n - c(\varepsilon_n)A_n)|^2 + \left|\nabla^2\phi_n\right|^2 + |\nabla A_n|^2 + |\partial_1\nabla A_n|^2\,dz = o(1). \tag{4.9}$$

We then write (4.2) under the form

$$\begin{cases} c(\varepsilon_n)\partial_1 A_n - (\partial_1^2+\varepsilon_n^2\partial_2^2)\,\phi_n \\ \quad = \mathcal{S}_{\mathrm{ph},n} + 2\varepsilon_n^2\partial_1\phi_n\partial_1 A^{(0)} + 2\varepsilon_n^4\partial_2\phi_n\partial_2 A^{(0)} \\ \qquad + \varepsilon_n^2 c(\varepsilon_n)\partial_1\left(A^{(0)}A_n\right) - 2\varepsilon_n^2\partial_1\phi^{(0)}\partial_1 A_n \\ \quad = \mathcal{S}_{\mathrm{ph},n} + \varepsilon_n^2\mathcal{S}'_{\mathrm{ph},n} \\ \\ -c(\varepsilon_n)\partial_1\phi_n + \mathfrak{c}^2 A_n - \varepsilon_n^2\left(\partial_1^2 A_n + \varepsilon_n^2\partial_2^2 A_n\right) \\ \quad = \mathcal{S}_{\mathrm{am},n} + \varepsilon_n^2 c(\varepsilon_n)A^{(0)}\partial_1\phi_n + \varepsilon_n^2 c(\varepsilon_n)A_n\partial_1\phi^{(0)} \\ \qquad - 2\varepsilon_n^2\partial_1\phi^{(0)}\partial_1\phi_n - \varepsilon_n^2\mathfrak{c}^2(\Gamma-3)A^{(0)}A_n \\ \quad = \mathcal{S}_{\mathrm{am},n} + \varepsilon_n^2\mathcal{S}'_{\mathrm{am},n} \end{cases}$$

(with obvious notations) which is the linear problem studied in Proposition 2.8 with modified source terms. This yields the expression

$$\begin{cases} A_n &= -\dfrac{1}{\varepsilon_n^2}\left(\left(\partial_1^2+\varepsilon_n^2\partial_2^2\right)\mathcal{K}^{\varepsilon_n}\star\mathcal{S}_{\phi,n} + c(\varepsilon_n)\partial_1\mathcal{K}^{\varepsilon_n}\star\mathcal{S}_{\mathrm{ph},n}\right) \\ &\quad - \left((\partial_1^2+\varepsilon_n^2\partial_2^2)\,\mathcal{K}^{\varepsilon_n}\star\mathcal{S}'_{\phi,n} + c(\varepsilon_n)\partial_1\mathcal{K}^{\varepsilon_n}\star\mathcal{S}'_{\mathrm{ph},n}\right) \\ &= A_n^\dagger + A_n' \\ \\ \phi_n &= -\dfrac{1}{\varepsilon_n^2}\left(c(\varepsilon_n)\partial_1\mathcal{K}^{\varepsilon_n}\star\mathcal{S}_{\phi,n} + \left(\mathfrak{c}^2 - \varepsilon_n^2\left(\partial_1^2+\varepsilon_n^2\partial_2^2\right)\right)\mathcal{K}^{\varepsilon_n}\star\mathcal{S}_{A,n}\right) \\ &\quad - \left(c(\varepsilon_n)\partial_1\mathcal{K}^{\varepsilon_n}\star\mathcal{S}'_{\phi,n} + (\mathfrak{c}^2 - \varepsilon_n^2(\partial_1^2+\varepsilon_n^2\partial_2^2))\mathcal{K}^{\varepsilon_n}\star\mathcal{S}'_{A,n}\right) \\ &= \phi_n^\dagger + \phi_n'. \end{cases}$$

Applying Proposition 2.8, we deduce

$$\left\|A_n^\dagger\right\|_{X^1_{1+\sigma}} + \left\|\nabla\phi_n^\dagger\right\|_{X^1_{1+\sigma}} \to 0 \qquad \text{as } n\to+\infty. \tag{4.10}$$

We shall then need some $L^q - L^q$ convolution estimates involving the kernel $\mathcal{K}^\varepsilon$ and its derivatives, as used in [BGS08, Lemma 5.2]. These estimates rely on Fourier multiplier properties (see [Liz67, MPTT]).

Proposition 4.4. — *Let $1 < q < \infty$ be given. Then, there exists $C(q)$, depending only on q, such that, for any $h \in L^q(\mathbb{R}^2)$ and any $\varepsilon \in]0,1[$, we have*

$$\begin{aligned}\left\|\partial_1^2\mathcal{K}^\varepsilon \star h\right\|_{L^q} + \left\|\partial_2^2\mathcal{K}^\varepsilon \star h\right\|_{L^q} + \|\partial_1\partial_2\mathcal{K}^\varepsilon \star h\|_{L^q} + \left\|\partial_1^3\mathcal{K}^\varepsilon \star h\right\|_{L^q} + \left\|\partial_1^2\partial_2\mathcal{K}^\varepsilon \star h\right\|_{L^q}\\ + \left\|\partial_1^4\mathcal{K}^\varepsilon \star h\right\|_{L^q} + \varepsilon\left\|\partial_1^3\partial_2\mathcal{K}^\varepsilon \star h\right\|_{L^q} + \varepsilon\left\|\partial_1\partial_2^2\mathcal{K}^\varepsilon \star h\right\|_{L^q} + \varepsilon^2\left\|\partial_1^2\partial_2^2\mathcal{K}^\varepsilon \star h\right\|_{L^q}\\ + \varepsilon^2\left\|\partial_2^3\mathcal{K}^\varepsilon \star h\right\|_{L^q} + \varepsilon^3\left\|\partial_1\partial_2^3\mathcal{K}^\varepsilon \star h\right\|_{L^q} + \varepsilon^4\left\|\partial_2^4\mathcal{K}^\varepsilon \star h\right\|_{L^q} \leqslant C(q)\|h\|_{L^q}.\end{aligned}$$

Step 4 (Smallness in $W^{1,q}$). — Let $1 < q < \infty$ be given. We infer from (4.9), Sobolev imbedding $H^1 \hookrightarrow L^{2q}$ (in $2d$) and Rogers-Hölder inequality ($L^{2q} \times L^{2q} \subset L^q$) that

$$\left\|\mathcal{S}'_{\mathrm{am},n}\right\|_{L^q} \leqslant C\left(\left\|A^{(0)}\right\|_{L^{2q}} + \left\|\partial_1\phi^{(0)}\right\|_{L^{2q}}\right) \times \left(\|A_n\|_{L^{2q}} + \|\partial_1\phi_n\|_{L^{2q}}\right) = o_{n\to+\infty}(1)$$

and, in a similar way,

$$\begin{aligned}\left\|\mathcal{S}'_{\mathrm{ph},n}\right\|_{L^q} \leqslant{}& C\|\partial_1\phi_n\|_{L^{2q}}\left\|\partial_1 A^{(0)}\right\|_{L^{2q}}\\ &+ C\|A_n\|_{L^{2q}}\left\|\partial_1 A^{(0)}\right\|_{L^{2q}} + C\left\|A^{(0)}\right\|_{L^{2q}}\|\partial_1 A_n\|_{L^{2q}}\\ &+ C\left\|\partial_1\phi^{(0)}\right\|_{L^{2q}}\|\partial_1 A_n\|_{L^{2q}} + C\varepsilon_n^2\|\partial_2\phi_n\|_{L^\infty}\left\|\partial_2 A^{(0)}\right\|_{L^q} = o_{n\to+\infty}(1).\end{aligned}$$

Therefore, applying Proposition 4.4, we deduce, on the one hand,

$$\begin{aligned}\|\nabla\partial_1\phi'_n\|_{L^q} &\leqslant C\left(\left\|\nabla\partial_1^2\mathcal{K}^{\varepsilon_n} \star \mathcal{S}'_{\mathrm{am},n}\right\|_{L^q} + \left\|\nabla\partial_1\mathcal{K}^{\varepsilon_n} \star \mathcal{S}'_{\mathrm{ph},n}\right\|_{L^q}\right.\\ &\qquad\left.+\varepsilon_n^2\left\|\nabla\partial_1\left(\partial_1^2 + \varepsilon_n^2\partial_2^2\right)\mathcal{K}^{\varepsilon_n} \star \mathcal{S}'_{\mathrm{ph},n}\right\|_{L^q}\right)\\ &\leqslant C(q)\left(\left\|\mathcal{S}'_{\mathrm{am},n}\right\|_{L^q} + \left\|\mathcal{S}'_{\mathrm{ph},n}\right\|_{L^q}\right) = o_{n\to+\infty}(1)\end{aligned}\tag{4.11}$$

and on the other hand,

$$\begin{aligned}\left\|\partial_1^2 A'_n\right\|_{L^q} &\leqslant C\left(\left\|\partial_1^2\mathcal{K}^{\varepsilon_n} \star \mathcal{S}'_{\mathrm{ph},n}\right\|_{L^q} + \left\|\partial_1^2\left(\partial_1^2 + \varepsilon_n^2\partial_2^2\right)\mathcal{K}^{\varepsilon_n} \star \mathcal{S}'_{\mathrm{am},n}\right\|_{L^q}\right)\\ &\leqslant C(q)\left(\left\|\mathcal{S}'_{\mathrm{am},n}\right\|_{L^q} + \left\|\mathcal{S}'_{\mathrm{ph},n}\right\|_{L^q}\right) = o_{n\to+\infty}(1).\end{aligned}\tag{4.12}$$

The inequalities (4.10), (4.11) and (4.12) allow to improve the estimate on $\mathcal{S}'_{\mathrm{am},n}$ and $\mathcal{S}'_{A,n}$:

$$\left\|\partial_1\mathcal{S}'_{\mathrm{ph},n}\right\|_{L^q} + \left\|\partial_1\mathcal{S}'_{\mathrm{am},n}\right\|_{L^q} = o_{n\to+\infty}(1).\tag{4.13}$$

Applying Proposition 4.4 and using (4.13), we then infer, on the one hand,

$$\begin{aligned}\left\|\partial_2^2\phi'_n\right\|_{L^q} &\leqslant C\left(\left\|\partial_2^2\mathcal{K}^{\varepsilon_n} \star \partial_1\mathcal{S}'_{\mathrm{am},n}\right\|_{L^q} + \left\|\partial_2^2\mathcal{K}^{\varepsilon_n} \star \mathcal{S}'_{\mathrm{ph},n}\right\|_{L^q}\right.\\ &\qquad\left.+\varepsilon_n^2\left\|\partial_2^2\left(\partial_1^2 + \varepsilon_n^2\partial_2^2\right)\mathcal{K}^{\varepsilon_n} \star \mathcal{S}'_{\mathrm{ph},n}\right\|_{L^q}\right)\\ &\leqslant C(q)\left(\left\|\partial_1\mathcal{S}'_{\mathrm{am},n}\right\|_{L^q} + \left\|\mathcal{S}'_{\mathrm{ph},n}\right\|_{L^q}\right) = o_{n\to+\infty}(1)\end{aligned}\tag{4.14}$$

and on the other hand,

$$\begin{aligned}\|\partial_1\partial_2 A'_n\|_{L^q} &\leqslant C\left(\left\|\partial_2^2\mathcal{K}^{\varepsilon_n}\star\partial_1\mathcal{S}'_{\mathrm{ph},n}\right\|_{L^q}+\left\|\partial_2\left(\partial_1^2+\varepsilon_n^2\partial_2^2\right)\mathcal{K}^{\varepsilon_n}\star\partial_1\mathcal{S}'_{\mathrm{am},n}\right\|_{L^q}\right)\\ &\leqslant C(q)\left(\left\|\partial_1\mathcal{S}'_{\mathrm{am},n}\right\|_{L^q}+\left\|\partial_1\mathcal{S}'_{\mathrm{ph},n}\right\|_{L^q}\right)=o_{n\to+\infty}(1).\end{aligned}\tag{4.15}$$

The above estimates combined with (4.10) then yield

$$\|\partial_2\mathcal{S}_{\mathrm{ph},n}\|_{L^q}+\|\partial_2\mathcal{S}_{\mathrm{am},n}\|_{L^q}=o_{n\to+\infty}(1).\tag{4.16}$$

Another application of Proposition 4.4 then gives

$$\begin{aligned}\left\|\partial_2^2 A'_n\right\|_{L^q} &\leqslant C\left(\left\|\partial_2\partial_1\mathcal{K}^{\varepsilon_n}\star\partial_2\mathcal{S}'_{\mathrm{ph},n}\right\|_{L^q}+\left\|\partial_2\left(\partial_1^2+\varepsilon_n^2\partial_2^2\right)\mathcal{K}^{\varepsilon_n}\star\partial_2\mathcal{S}'_{\mathrm{am},n}\right\|_{L^q}\right)\\ &\leqslant C(q)\left(\left\|\partial_2\mathcal{S}'_{\mathrm{am},n}\right\|_{L^q}+\left\|\partial_2\mathcal{S}'_{\mathrm{ph},n}\right\|_{L^q}\right)=o_{n\to+\infty}(1).\end{aligned}\tag{4.17}$$

We have then shown (in view of (4.10)) that, for any $1<q<\infty$, we have, as $n\to+\infty$,

$$\|\nabla A_n\|_{W^{1,q}}+\|\nabla\phi_n\|_{W^{1,q}}\to 0.$$

In particular, by Sobolev imbedding (picking some $q>2$),

$$\|\nabla A_n\|_{L^\infty}+\|\nabla\phi_n\|_{L^\infty}\to 0\tag{4.18}$$

and

$$\left\|\nabla\mathcal{S}'_{\mathrm{am},n}\right\|_{W^{1,q}}+\|\nabla\mathcal{S}'_{\mathrm{ph},n}\|_{W^{1,q}}=o_{n\to+\infty}(1).$$

Step 5 (Contradiction). — Repeating the argument once again, we obtain

$$\left\|\nabla^2 A'_n\right\|_{W^{1,q}}+\left\|\nabla^2\phi'_n\right\|_{W^{1,q}}\to 0$$

hence, by Sobolev imbedding (picking some $q>2$) and (4.10),

$$\left\|\nabla^2 A_n\right\|_{L^\infty}+\left\|\nabla^2\phi_n\right\|_{L^\infty}\to 0.\tag{4.19}$$

Since $\mathfrak{c}A^{(0)}=\partial_1\phi^{(0)}\in X_2^1$, it follows that

$$\left\|\mathcal{S}'_{\mathrm{am},n}\right\|_{X_2^1}+\left\|\mathcal{S}'_{A,n}\right\|_{X_2^1}\to 0.$$

By Proposition 2.8 once again, we finally obtain

$$\|A_n\|_{X^1_{1+\sigma}}+\|\nabla\phi_n\|_{X^1_{1+\sigma}}+\left\|\left(\partial_1^2 A_n,\varepsilon_n\partial_1\partial_2 A_n,\varepsilon_n^2\partial_2^2 A_n\right)\right\|_{X_{2+\sigma}}\to 0,$$

in contradiction with our hypothesis (4.3). This concludes the proof of Lemma 4.3, and shows in particular that (2.10) has at most one solution (satisfying the symmetries).

The case where $\mathcal{S}_{\mathrm{am}}\equiv 0$ is similar.

We now turn to the existence part (at fixed $\varepsilon\in]0,\epsilon(\sigma,\sigma')[$). The next lemma shows that it is sufficient to look for a solution (ϕ,A) such that $A\in H^1(\mathbb{R}^2)$, $\phi\in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ and $\nabla\phi\in L^2(\mathbb{R}^2)$.

Lemma 4.5. — *Let $0 < \sigma < \sigma' < 1$ be given and $0 < \varepsilon < \epsilon(\sigma, \sigma')$ (the latter is given in Lemma 4.3). Let $(\mathcal{S}_{\rm ph}, \mathcal{S}_{\rm am}) \in X^1_{2+\sigma',a} \times X^1_{1+\sigma',s}$ and $(\phi, A) \in H^1_{\rm loc} \times H^1$ with $\nabla\phi \in L^2$, satisfying the linear system* (2.10)*, namely*

$$\begin{cases} c(\varepsilon)\partial_1 A - \left(\partial_1^2 + \varepsilon^2\partial_2^2\right)\phi - 2\varepsilon^2\partial_1\phi\partial_1 A^{(0)} - 2\varepsilon^4\partial_2\phi\partial_2 A^{(0)} \\ \quad -\varepsilon^2 c(\varepsilon)\partial_1\left(A^{(0)}A\right) + 2\varepsilon^2\partial_1\phi^{(0)}\partial_1 A = \mathcal{S}_{\rm ph} \\ \\ -c(\varepsilon)\partial_1\phi - \varepsilon^2 c(\varepsilon)A^{(0)}\partial_1\phi - \varepsilon^2 c(\varepsilon)A\partial_1\phi^{(0)} + 2\varepsilon^2\partial_1\phi^{(0)}\partial_1\phi + \mathfrak{c}^2 A \\ \quad -\varepsilon^2\mathfrak{c}^2(\Gamma-3)A^{(0)}A - \varepsilon^2\left(\partial_1^2 A + \varepsilon^2\partial_2^2 A\right) = \mathcal{S}_{\rm am}, \end{cases}$$

then we have $(A, \nabla\phi) \in X^1_{1+\sigma'} \times X^1_{1+\sigma'}$ and

$$\begin{aligned} &\|A\|_{X^1_{1+\sigma}} + \|\nabla\phi\|_{X^1_{1+\sigma}} \\ &\qquad\qquad \leqslant C(\sigma,\sigma')\left(\|\mathcal{S}_{\rm ph}\|_{X^1_{2+\sigma,a}} + \|\mathcal{S}_{\rm am}\|_{X^1_{1+\sigma}} + \frac{1}{\varepsilon^2}\|\mathfrak{c}\mathcal{I}\mathcal{S}_{\rm ph} - \mathcal{S}_{\rm am}\|_{X^1_{1+\sigma'}}\right). \end{aligned}$$

Proof. — From the facts that, on the one hand, $(A, \nabla\phi) \in H^1 \times L^2$ and on the other hand, $(A^{(0)}, \phi^{(0)}) \in X^1_2 \times X^1_1$, we deduce

$$\left(\partial_1^2 + \varepsilon^2\partial_2^2\right)A \in L^2 \qquad \text{and} \qquad \left(\partial_1^2 + \varepsilon^2\partial_2^2\right)\phi \in L^2.$$

It then follows from standard L^2 elliptic regularity that $A \in H^2$ and $\nabla\phi \in H^1$. By two dimensional Sobolev imbedding, we deduce ∇A, $\nabla\phi \in L^q$ for any $2 < q < \infty$, then

$$\left(\partial_1^2 + \varepsilon^2\partial_2^2\right)A \in L^q \qquad \text{and} \qquad \left(\partial_1^2 + \varepsilon^2\partial_2^2\right)\phi \in L^q,$$

hence $A \in W^{2,q}$ and $\nabla\phi \in W^{1,q}$ by L^q elliptic regularity (see, for instance, [GT01, Theorem 9.9]). By Sobolev imbedding ($W^{1,q} \hookrightarrow \mathcal{C}^{0,1-2/q}$) and Schauder estimates (since $\mathcal{S}_{\rm ph}$, $\mathcal{S}_{\rm am} \in X^1_{1+\sigma} \subset \mathcal{C}^{0,1-2/q}$), we obtain ∇A, $\nabla\phi \in \mathcal{C}^{1,1-2/q}$. Therefore, we have $A \in \mathcal{C}^2 \cap W^{2,\infty}$ and $\nabla\phi \in \mathcal{C}^1 \cap W^{1,\infty}$.

We finally write the system under the form

$$\begin{cases} c(\varepsilon)\partial_1 A - \left(\partial_1^2 + \varepsilon^2\partial_2^2\right)\phi \\ = S_{\rm ph} + 2\varepsilon^2\partial_1\phi^{(0)}\partial_1 A + 2\varepsilon^2\partial_1\phi\partial_1 A^{(0)} + 2\varepsilon^4\partial_2\phi\partial_2 A^{(0)} + \varepsilon^2 c(\varepsilon)\partial_1\left(A^{(0)}A\right) \\ \\ -c(\varepsilon)\partial_1\phi + \left(\mathfrak{c}^2 - \varepsilon^2\left(\partial_1^2 + \varepsilon^2\partial_2^2\right)\right)A \\ = S_{\rm am} + \varepsilon^2 c(\varepsilon)A^{(0)}\partial_1\phi + \varepsilon^2 c(\varepsilon)A\partial_1\phi^{(0)} - 2\varepsilon^2\partial_1\phi^{(0)}\partial_1\phi - \mathfrak{c}^2 A + \varepsilon^2\mathfrak{c}^2(\Gamma-3)A^{(0)}A. \end{cases}$$

Since ∇A, $\nabla\phi \in \mathcal{C}^1 \cap W^{1,\infty}$, the source terms, denoted $S'_{\rm ph}$ and $S'_{\rm am}$, belong to $X_2 \cap X^1_1$ (but not to $X^1_{2+\sigma'}$ at this stage due to the terms $\phi^{(0)}\partial_1 A$ and $\varepsilon^2 c(\varepsilon)A^{(0)}\partial_1 A$) and $X^1_{2+\sigma'}$ respectively. We then can not apply directly Proposition 2.8, but we may follow the proof of this result. Indeed, we know that the A part of the solution to this system is given by

$$A = -\frac{1}{\varepsilon^2}\Big(\left(\partial_1^2 + \varepsilon^2\partial_2^2\right)\mathcal{K}^\varepsilon \star S'_{\rm am} - c(\varepsilon)\partial_1\mathcal{K}^\varepsilon \star S'_{\rm ph}\Big).$$

By using Proposition 2.6 (ii), we have (with ε-dependent constants)

$$\left\|\left(\partial_1^2 + \varepsilon^2\partial_2^2\right)\mathcal{K}^\varepsilon \star S'_{\rm am}\right\|_{X_{1+\sigma}} \leqslant C(\varepsilon)\,\|S'_{\rm am}\|_{X_{1+\sigma'}}$$

and, similarly,

$$\left\|\nabla\left(\partial_1^2+\varepsilon^2\partial_2^2\right)\mathcal{K}^\varepsilon\star S'_{\mathrm{am}}\right\|_{X_{1+\sigma}}\leqslant C(\varepsilon)\left\|\nabla S'_{\mathrm{am}}\right\|_{X_{1+\sigma'}}.$$

Since the $S'_{\mathrm{ph}}\in X^1_{2+\sigma'}$ contribution has already been estimated, we infer $A\in X_{1+\sigma}\cap X^1_\sigma$. This is sufficient to get $S'_{\mathrm{ph}}\in X^1_{2+\sigma'}$, and thus we can apply the a priori estimate of Proposition 2.8 to conclude. □

4.4.2. A priori ε-dependent estimate in a large disk

For $R>1$, we denote

$$X^1_{2+\sigma',a}\Big(D(0,R)\Big)\overset{\mathrm{def}}{=}\Big\{S_{\mathrm{ph}}\in\mathcal{C}^1(D(0,R))\text{ s.t. }S_{\mathrm{ph}}\text{ is even in }z_2\text{ and odd in }z_1\Big\}$$

and

$$X^1_{1+\sigma',s}\Big(D(0,R)\Big)\overset{\mathrm{def}}{=}\Big\{S_{\mathrm{am}}\in\mathcal{C}^1(D(0,R))\text{ s.t. }S_{\mathrm{am}}\text{ is even in }z_2\text{ and even in }z_1\Big\},$$

that are endowed with the norms

$$\|S_{\mathrm{ph}}\|_{X^1_{2+\sigma',a}(D(0,R))}\overset{\mathrm{def}}{=}\left\|(1+|z|)^{2+\sigma'}S_{\mathrm{ph}}(z)\right\|_{L^\infty(D(0,R))}+\left\|(1+|z|)^{3+\sigma'}\nabla S_{\mathrm{ph}}(z)\right\|_{L^\infty(D(0,R))}$$

and

$$\|S_{\mathrm{am}}\|_{X^1_{1+\sigma',s}(D(0,R))}\overset{\mathrm{def}}{=}\left\|(1+|z|)^{1+\sigma'}S_{\mathrm{am}}(z)\right\|_{L^\infty(D(0,R))}+\left\|(1+|z|)^{2+\sigma'}\nabla S_{\mathrm{am}}(z)\right\|_{L^\infty(D(0,R))}.$$

We further set

$$H^1_a(D(0,R))\overset{\mathrm{def}}{=}\Big\{u\in H^1_0(D(0,R))\text{ s.t. }u\text{ is even in }z_2\text{ and odd in }z_1\Big\}$$

and

$$H^1_s(D(0,R))\overset{\mathrm{def}}{=}\Big\{u\in H^1_0(D(0,R))\text{ s.t. }u\text{ is even in }z_2\text{ and even in }z_1\Big\}.$$

Lemma 4.6. — *Let $0<\sigma<\sigma'<1$ be given and $0<\varepsilon<\epsilon(\sigma,\sigma')$ (the latter is given in Lemma 4.3). Then, there exists $R_*=R_*(\varepsilon,\sigma,\sigma')>1$ such that for any $R\geqslant R_*$, any $(\mathcal{S}_{\mathrm{ph}},\mathcal{S}_{\mathrm{am}})\in X^1_{2+\sigma',a}(D(0,R))\times X^1_{1+\sigma',s}(D(0,R))$ and any $(\phi,A)\in H^1_a(D(0,R))\times H^1_s(D(0,R))$ satisfying (2.10), namely*

$$\begin{cases} c(\varepsilon)\partial_1A-\left(\partial_1^2+\varepsilon^2\partial_2^2\right)\phi-2\varepsilon^2\partial_1\phi\partial_1A^{(0)}-2\varepsilon^4\partial_2\phi\partial_2A^{(0)}\\ \quad-\varepsilon^2c(\varepsilon)\partial_1(A^{(0)}A)-2\varepsilon^2\partial_1\phi^{(0)}\partial_1A-2\varepsilon^4\partial_2\phi^{(0)}\partial_2A=\mathcal{S}_{\mathrm{ph}}\\ \\ -c(\varepsilon)\partial_1\phi-\varepsilon^2c(\varepsilon)A^{(0)}\partial_1\phi-\varepsilon^2c(\varepsilon)A\partial_1\phi^{(0)}+2\varepsilon^2\partial_1\phi^{(0)}\partial_1\phi+\mathfrak{c}^2A\\ \quad+\varepsilon^2\mathfrak{c}^2(\Gamma-3)A^{(0)}A-\varepsilon^2\left(\partial_1^2A+\varepsilon^2\partial_2^2A\right)=\mathcal{S}_{\mathrm{am}}, \end{cases}$$

then we have

$$\|A\|_{H^1(D(0,R))}+\|\nabla\phi\|_{L^2(D(0,R))}\leqslant C(\varepsilon,\sigma,\sigma')\Big(\|\mathcal{S}_{\mathrm{ph}}\|_{X^1_{2+\sigma',a}(D(0,R))}+\|\mathcal{S}_{\mathrm{am}}\|_{X^1_{1+\sigma'}(D(0,R))}\Big),$$

where the constant $C(\varepsilon,\sigma,\sigma')$ does not depend on $R\geqslant R_*$.

Proof. — We emphasize that we look for estimates independent of the radius R but (strongly) depending on ε. The proof follows the ideas used for the proof of Lemma 4.3, Step 2, at the difference that we do not have a priori L^∞ estimates for ϕ. Instead, since ϕ vanishes when $z_1 = 0$ (and on $\partial D(0,R)$), we have by Wirtinger's inequality, for any $r > 0$,

$$\int_{\partial D(0,r)} |\phi|^2 \, d\mathcal{H}^1 \leqslant Cr^2 \int_{\partial D(0,r)} |\nabla \phi|^2 \, d\mathcal{H}^1 \leqslant C_\varepsilon r^2 \int_{\partial D(0,r)} |\nabla^\varepsilon \phi|^2 \, d\mathcal{H}^1. \tag{4.20}$$

The proof is then here again by contradiction. We then assume that (for $0 < \sigma < \sigma' < 1$ fixed and $0 < \varepsilon < \epsilon(\sigma, \sigma')$) there exist sequences $(R_n)_{n\in\mathbb{N}}$ tending to $+\infty$, $(\mathcal{S}_{\mathrm{ph},n}, \mathcal{S}_{\mathrm{am},n}) \in X^1_{2+\sigma',a}(D(0,R_n)) \times X^1_{1+\sigma',s}(D(0,R_n))$ and $(\phi_n, A_n) \in H^1_a(D(0,R_n)) \times H^1_s(D(0,R_n))$, satisfying (2.10) in $D(0,R_n)$, that is

$$\begin{cases} c(\varepsilon)\partial_1 A_n - (\partial_1^2 + \varepsilon^2\partial_2^2)\,\phi_n - 2\varepsilon^2 \partial_1\phi_n \partial_1 A^{(0)} - 2\varepsilon^4 \partial_2 \phi_n \partial_2 A^{(0)} \\ \quad -\varepsilon^2 c(\varepsilon)\partial_1 \left(A^{(0)} A_n\right) - 2\varepsilon^2 \partial_1 \phi^{(0)} \partial_1 A_n = \mathcal{S}_{\mathrm{ph},n} \\ \\ -c(\varepsilon)\partial_1\phi_n - \varepsilon^2 c(\varepsilon) A^{(0)} \partial_1 \phi_n - \varepsilon^2 c(\varepsilon) A_n \partial_1 \phi^{(0)} + 2\varepsilon^2 \partial_1 \phi^{(0)} \partial_1 \phi_n + \mathfrak{c}^2 A_n \\ \quad + \varepsilon^2 \mathfrak{c}^2 (\Gamma - 3) A^{(0)} A_n - \varepsilon^2 \left(\partial_1^2 A_n + \varepsilon^2 \partial_2^2 A_n\right) = \mathcal{S}_{\mathrm{am},n} \end{cases} \tag{4.21}$$

in $D(0,R_n)$ with homogeneous Dirichlet boundary condition, and such that

$$\|A_n\|_{H^1(D(0,R_n))} + \|\nabla \phi_n\|_{L^2(D(0,R_n))} = 1 \tag{4.22}$$

and

$$\|\mathcal{S}_{\mathrm{ph},n}\|_{X^1_{2+\sigma',a}(D(0,R_n))} + \|\mathcal{S}_{\mathrm{am},n}\|_{X^1_{1+\sigma',s}(D(0,R_n))} \to 0. \tag{4.23}$$

From the Wirtinger inequality (4.20), for any $r_0 > 0$, we have

$$\|\phi_n\|_{L^2(D(0,r_0))} \leqslant C(\varepsilon, r_0).$$

Step 1 (local convergence). — In view of the H^1 bound on A_n and of the local L^2 bound on ϕ_n, we may assume, up to a subsequence, that there exists $(A_\infty, \phi_\infty) \in H^1 \times H^1_{\mathrm{loc}}$ such that

$$A_n \rightharpoonup A_\infty \qquad \phi_n \rightharpoonup \phi_\infty \qquad \text{weakly in } H^1_{\mathrm{loc}}(\mathbb{R}^2),$$

with, by lower semicontinuity,

$$\|A_\infty\|_{H^1} + \|\nabla \phi_\infty\|_{L^2} \leqslant 1.$$

Passing to the limit then yields

$$\begin{cases} c(\varepsilon)\partial_1 A_\infty - (\partial_1^2 + \varepsilon^2\partial_2^2)\,\phi_n - 2\varepsilon^2 \partial_1\phi_\infty \partial_1 A^{(0)} - 2\varepsilon^4 \partial_2 \phi_\infty \partial_2 A^{(0)} \\ \quad -\varepsilon^2 c(\varepsilon)\partial_1 \left(A^{(0)} A_\infty\right) - 2\varepsilon^2 \partial_1 \phi^{(0)} \partial_1 A_\infty = 0 \\ \\ -c(\varepsilon)\partial_1\phi_\infty - \varepsilon^2 c(\varepsilon) A^{(0)} \partial_1 \phi_\infty - \varepsilon^2 c(\varepsilon) A_\infty \partial_1 \phi^{(0)} + 2\varepsilon^2 \partial_1 \phi^{(0)} \partial_1 \phi_\infty \\ \quad + \mathfrak{c}^2 A_\infty + \varepsilon^2 \mathfrak{c}^2 (\Gamma - 3) A^{(0)} A_\infty - \varepsilon^2 \left(\partial_1^2 A_\infty + \varepsilon^2 \partial_2^2 A_\infty\right) = 0 \end{cases} \tag{4.24}$$

in $\mathbb{R}^2$. By Lemma 4.5, we infer (ϕ_∞ is odd in z_1)

$$A_\infty = \phi_\infty \equiv 0.$$

By compact Sobolev imbedding (locally in space) and L^2 elliptic regularity, we actually have

$$A_n \to 0 \qquad \phi_n \to 0 \qquad \text{strongly in } H^1_{\mathrm{loc}}(\mathbb{R}^2). \tag{4.25}$$

Step 2 (Smallness in $H^1(D(0,R_n))$). — Following Step 2 in the proof of Lemma 4.3, we multiply the first equation of (4.21) by ϕ_n and integrate over $D(0,R_n)$ by parts (ϕ_n vanishes on $\partial D(0,R_n)$):

$$\begin{aligned}\int_{D(0,R_n)} |\nabla^\varepsilon \phi_n|^2 - c(\varepsilon) A_n \partial_1 \phi_n \, dz \\ = \int_{D(0,R_n)} \Big(\mathcal{S}_{\mathrm{ph},n}\phi_n + 2\varepsilon^2 \partial_1 \phi^{(0)} \partial_1 A_n \phi_n + 2\varepsilon^2 \partial_1 \phi_n \partial_1 A^{(0)} \phi_n \\ + 2\varepsilon^4 \partial_2 \phi_n \partial_2 A^{(0)} \phi_n - \varepsilon^2 c(\varepsilon) A^{(0)} A_n \partial_1 \phi_n \Big) \, dz.\end{aligned} \tag{4.26}$$

We claim that, as $n \to +\infty$,

$$\int_{D(0,R_n)} \partial_1 \phi^{(0)} \partial_1 A_n \phi_n \, dz \to 0.$$

Indeed, for $\rho > 0$, by the Wirtinger's inequality and the fact that $\partial_1 \phi^{(0)} \in X_2$, we have

$$\begin{aligned}\left| \int_{D(0,R_n)\setminus D(0,\rho)} \partial_1 \phi^{(0)} \partial_1 A_n \phi_n \, dz \right| \\ \leqslant C \int_\rho^{R_n} \frac{1}{r^2} \|\partial_1 A_n\|_{L^2(\partial D(0,r))} \|\phi_n\|_{L^2(\partial D(0,r))} \, dr \\ \leqslant C_\varepsilon \int_\rho^{R_n} \frac{1}{r} \Big(\|\partial_1 A_n\|^2_{L^2(\partial D(0,r))} + \|\nabla \phi_n\|^2_{L^2(\partial D(0,r))} \Big) \, dr \\ \leqslant \frac{C_\varepsilon}{\rho} \Big(\|\partial_1 A_n\|^2_{L^2(D(0,R_n))} + \|\nabla \phi_n\|^2_{L^2(D(0,R_n))} \Big) \leqslant \frac{C_\varepsilon}{\rho}.\end{aligned}$$

In view of this uniform decay and the local convergences (4.25), the claim follows. Similar arguments give

$$\begin{aligned}\int_{D(0,R_n)} \partial_1 \phi_n \partial_1 A^{(0)} \phi_n \, dz \to 0, \quad \int_{D(0,R_n)} \partial_1 \phi_n \partial_1 A^{(0)} \phi_n \, dz \to 0, \\ \int_{D(0,R_n)} \partial_2 \phi_n \partial_2 A^{(0)} \phi_n \, dz \to 0, \quad \int_{D(0,R_n)} A^{(0)} A_n \partial_1 \phi_n \, dz \to 0\end{aligned}$$

and

$$\int_{D(0,R_n)} \mathcal{S}_{\mathrm{ph},n} \phi_n \, dz \to 0,$$

since $\mathcal{S}_{\mathrm{ph},n}$ is bounded in $X^1_{2+\sigma'}$, thus, by Cauchy–Schwarz inequality,

$$\begin{aligned}\int_{D(0,R_n)\setminus D(0,1)} &|\mathcal{S}_{\mathrm{ph},n}\phi_n|\, dz\\ &\leqslant \|\mathcal{S}_{\mathrm{ph},n}\|_{X^1_{2+\sigma'}}\int_1^{R_n}\frac{\|\phi_n\|_{L^2(\partial D(0,r))}}{r^{3/2+\sigma'}}\,dr\\ &\leqslant C_\varepsilon\,\|\mathcal{S}_{\mathrm{ph},n}\|_{X^1_{2+\sigma'}}\int_1^{R_n}\frac{\|\nabla\phi_n\|_{L^2(\partial D(0,r))}}{r^{1/2+\sigma'}}\,dr\\ &\leqslant C_\varepsilon\,\|\mathcal{S}_{\mathrm{ph},n}\|_{X^1_{2+\sigma'}}\left(\int_1^{R_n}\|\nabla\phi_n\|^2_{L^2(\partial D(0,r))}\,dr\right)^{1/2}\left(\int_1^{+\infty}\frac{dr}{r^{1+2\sigma'}}\right)^{1/2}\\ &\leqslant C_{\varepsilon,\sigma'}\|\mathcal{S}_{\mathrm{ph},n}\|_{X^1_{2+\sigma'}}\to 0.\end{aligned}$$

As a consequence,

$$\int_{D(0,R_n)}|\nabla^\varepsilon\phi_n|^2 - c(\varepsilon)A_n\partial_1\phi_n\,dz = o_{n\to+\infty}(1). \tag{4.27}$$

Then, we multiply the second equation of (4.21) by A_n, integrate by parts over $D(0,R_n)$ ($A_n=0$ on $\partial D(0,R_n)$) and obtain

$$\begin{aligned}\int_{D(0,R_n)} &\varepsilon^2|\nabla^\varepsilon A_n|^2+\mathfrak{c}^2A_n^2-c(\varepsilon)\partial_1\phi_nA_n\,dz\\ &=\int_{D(0,R_n)}\Big(\mathcal{S}_{\mathrm{am},n}A_n+\varepsilon^2c(\varepsilon)A^{(0)}A_n\partial_1\phi_n+\varepsilon^2c(\varepsilon)A_n^2\partial_1\phi^{(0)}\\ &\qquad\qquad -2\varepsilon^2A_n\partial_1\phi^{(0)}\partial_1\phi_n-\varepsilon^2\mathfrak{c}^2(\Gamma-3)A^{(0)}A_n^2\Big)\,dz.\end{aligned}$$

We control the right-hand side in the following way. Since $A^{(0)}$ and $\partial_1\phi^{(0)}$ tend to 0 at infinity, there exists $R_\varepsilon>1$ such that,

$$\begin{aligned}\int_{D(0,R_n)\setminus D(0,R_\varepsilon)}&\Big(\varepsilon^2c(\varepsilon)A^{(0)}A_n\partial_1\phi_n+\varepsilon^2c(\varepsilon)A_n^2\partial_1\phi^{(0)}\\ &\qquad -2\varepsilon^2A_n\partial_1\phi^{(0)}\partial_1\phi_n-\varepsilon^2\mathfrak{c}^2(\Gamma-3)A^{(0)}A_n^2\Big)\,dz\\ &\leqslant\frac{\varepsilon^2}{10C}\int_{D(0,R_n)\setminus D(0,R_\varepsilon)}|\nabla^\varepsilon A_n|^2+A_n^2+(\partial_1\phi_n)^2\,dz\\ &\leqslant\frac{\varepsilon^2}{10}\int_{D(0,R_n)\setminus D(0,R_\varepsilon)}|\nabla^\varepsilon A_n|^2+A_n^2+\varepsilon^{-2}(\partial_1\phi_n-c(\varepsilon)A_n)^2\,dz.\end{aligned}$$

Furthermore,

$$\int_{D(0,R_n)}\mathcal{S}_{\mathrm{am},n}A_n\,dz\leqslant\|\mathcal{S}_{\mathrm{am},n}\|_{L^2(D(0,R_n))}\|A_n\|_{L^2(D(0,R_n))}\to 0,$$

thus

$$\begin{aligned}\int_{D(0,R_n)}&\varepsilon^2|\nabla^\varepsilon A_n|^2+\mathfrak{c}^2A_n^2-c(\varepsilon)\partial_1\phi_nA_n\,dz\\ &\leqslant o_{n\to+\infty}(1)+\frac{\varepsilon^2}{10}\int_{D(0,R_n)\setminus D(0,R_\varepsilon)}|\nabla^\varepsilon A_n|^2+A_n^2+\varepsilon^{-2}(\partial_1\phi_n-c(\varepsilon)A_n)^2\,dz.\end{aligned} \tag{4.28}$$

Adding (4.28) and (4.27), we infer

$$\int_{D(0,R_n)} \varepsilon^2 |\nabla^\varepsilon A_n|^2 + \varepsilon^2 A_n^2 + \varepsilon^2 (\partial_2 \phi_n)^2 + (\partial_1 \phi_n - c(\varepsilon) A_n)^2 \, dz$$
$$\leqslant o_{n\to+\infty}(1) + \frac{\varepsilon^2}{10} \int_{D(0,R_n)} |\nabla^\varepsilon A_n|^2 + A_n^2 + (\partial_2 \phi_n)^2 + \varepsilon^{-2} (\partial_1 \phi_n - c(\varepsilon) A_n)^2 \, dz,$$

which implies

$$\int_{D(0,R_n)} A_n^2 + |\nabla A_n|^2 + |\nabla \phi_n|^2 \, dz = o_{n\to+\infty}(1). \tag{4.29}$$

This is in contradiction with (4.22). □

Remark 4.7. — It is natural to think that the constant $C(\varepsilon, \sigma, \sigma')$ could actually be chosen independent of ε. We have not tried to show this (this could slightly simplify the proof of the existence below).

4.4.3. Injectivity for the adjoint problem in the plane

The linear system (2.10) may be written under the form

$$\Lambda^\varepsilon \begin{pmatrix} \phi \\ A \end{pmatrix} = \begin{pmatrix} \mathcal{S}_{\mathrm{ph}} \\ \mathcal{S}_{\mathrm{am}} \end{pmatrix},$$

where (Δ^ε stands for $\nabla^\varepsilon \cdot \nabla^\varepsilon = \partial_1^2 + \varepsilon^2 \partial_2^2$)

$$\Lambda^\varepsilon \stackrel{\mathrm{def}}{=} \begin{pmatrix} \Lambda^\varepsilon_{1,1} & \Lambda^\varepsilon_{1,2} \\ \Lambda^\varepsilon_{2,1} & \Lambda^\varepsilon_{2,2} \end{pmatrix}$$

with

$$\begin{cases} \Lambda^\varepsilon_{1,1} \stackrel{\mathrm{def}}{=} -\Delta^\varepsilon - 2\varepsilon^2 \nabla^\varepsilon A^{(0)} \cdot \nabla^\varepsilon \\ \Lambda^\varepsilon_{1,2} \stackrel{\mathrm{def}}{=} c(\varepsilon)\partial_1 - 2\varepsilon^2 \partial_1 \phi^{(0)} \partial_1 - \varepsilon^2 c(\varepsilon) \partial_1 \left(A^{(0)} \cdot \right) \\ \Lambda^\varepsilon_{2,1} \stackrel{\mathrm{def}}{=} -c(\varepsilon)\partial_1 - \varepsilon^2 c(\varepsilon) A^{(0)} \partial_1 + 2\varepsilon^2 \partial_1 \phi^{(0)} \partial_1 \\ \Lambda^\varepsilon_{2,2} \stackrel{\mathrm{def}}{=} -\varepsilon^2 \left(\partial_1^2 + \varepsilon^2 \partial_2^2 \right) + \mathfrak{c}^2 + \varepsilon^2 \mathfrak{c}^2 (\Gamma - 3) A^{(0)} - \varepsilon^2 c(\varepsilon) \partial_1 \phi^{(0)} \end{cases} \tag{4.30}$$

is an unbounded operator on $L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ with domain $H^2(\mathbb{R}^2) \times H^2(\mathbb{R}^2)$. We wish to apply Fredholm alternative, thus we investigate the injectivity of the adjoint of Λ^ε. In order to keep track of the variational structure of the initial problem TW_c, we shall not use the L^2 standard inner product for the computation of the adjoint of Λ^ε but instead we choose the weighted one

$$\left\langle \begin{pmatrix} \phi \\ A \end{pmatrix} \middle| \begin{pmatrix} \phi' \\ A' \end{pmatrix} \right\rangle_{A^{(0)}} \stackrel{\mathrm{def}}{=} \int \mathsf{e}^{2\varepsilon^2 A^{(0)}} \phi \phi' + A A' \, dz,$$

which induces an equivalent norm (with constants independent of ε). This has to be related to the fact that the operator $-\Delta^\varepsilon - 2\varepsilon^2 \nabla^\varepsilon A^{(0)} \cdot \nabla^\varepsilon$ may be written under the div $-\nabla$ form

$$-\Delta^\varepsilon - 2\varepsilon^2 \nabla^\varepsilon A^{(0)} \cdot \nabla^\varepsilon = -\mathsf{e}^{-2\varepsilon^2 A^{(0)}} \nabla^\varepsilon \cdot \left[\mathsf{e}^{2\varepsilon^2 A^{(0)}} \nabla^\varepsilon \cdot \right].$$

For this scalar product, we have

$$[\Lambda^\varepsilon]^* = \begin{pmatrix} [\Lambda^\varepsilon]^*_{11} & [\Lambda^\varepsilon]^*_{12} \\ [\Lambda^\varepsilon]^*_{21} & [\Lambda^\varepsilon]^*_{22} \end{pmatrix}$$

with, by using (2.1),

$$[\Lambda^\varepsilon]^*_{11} = \Lambda^\varepsilon_{11} = -\Delta^\varepsilon - 2\varepsilon^2 \nabla^\varepsilon A^{(0)} \cdot \nabla^\varepsilon,$$

$$\begin{aligned}[\Lambda^\varepsilon]^*_{21}\phi &= -c(\varepsilon)\partial_1\left(\mathsf{e}^{2\varepsilon^2 A^{(0)}}\phi\right) + 2\varepsilon^2\partial_1\left[\mathsf{e}^{2\varepsilon^2 A^{(0)}}\partial_1\phi^{(0)}\phi\right] + \varepsilon^2 c(\varepsilon)A^{(0)}\partial_1\left[\mathsf{e}^{2\varepsilon^2 A^{(0)}}\phi\right] \\ &= -c(\varepsilon)\partial_1\phi + \varepsilon^2 c(\varepsilon)A^{(0)}\partial_1\phi + \mathcal{O}_{X^1_2}(\varepsilon^4)\partial_1\phi + \mathcal{O}_{X^1_3}(\varepsilon^4)\phi,\end{aligned}$$

$$\begin{aligned}[\Lambda^\varepsilon]^*_{12}A &= \mathsf{e}^{-2\varepsilon^2 A^{(0)}}\left(c(\varepsilon)\partial_1 A - \varepsilon^2\partial_1\Big(\left(2\partial_1\phi^{(0)} - c(\varepsilon)A^{(0)}\right)A\Big)\right) \\ &= c(\varepsilon)\partial_1 A - \varepsilon^2\mathfrak{c}\partial_1\left(A^{(0)}A\right) - 2\varepsilon^2 c(\varepsilon)A^{(0)}\partial_1 A + \mathcal{O}_{X^1_2}(\varepsilon^4)\partial_1 A + \mathcal{O}_{X^1_6}(\varepsilon^4)A\end{aligned}$$

and

$$[\Lambda^\varepsilon]^*_{22} = [\Lambda^\varepsilon]_{22} = -\varepsilon^2\left(\partial_1^2 + \varepsilon^2\partial_2^2\right) + \mathfrak{c}^2 + \varepsilon^2\mathfrak{c}^2(\Gamma - 3)A^{(0)} - \varepsilon^2 c(\varepsilon)\partial_1\phi^{(0)}.$$

Lemma 4.8. — *Let $0 < \sigma < \sigma' < 1$ be given. Then, there exists $0 < \epsilon_\sharp(\sigma,\sigma') < \epsilon(\sigma,\sigma')$ such that, if $0 < \varepsilon < \epsilon_\sharp(\sigma,\sigma')$, $(A,\phi) \in X^1_{1+\sigma',s} \times \dot{X}^1_{1+\sigma',a}$ and*

$$[\Lambda^\varepsilon]^*\begin{pmatrix}\phi \\ A\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix},$$

then

$$A \equiv 0 \qquad \textit{and} \qquad \phi \equiv 0.$$

Proof. — From the expression of $[\Lambda^\varepsilon]^*$ above, we see that Λ^ε is self-adjoint up to $\mathcal{O}(\varepsilon^4)$ terms, which is the interest for choosing the weighted inner product $\langle\cdot|\cdot\rangle_{A^{(0)}}$. More precisely, for $(A,\phi) \in X^1_{1+\sigma} \times \dot{X}^1_{1+\sigma}$, we have

$$\left([\Lambda^\varepsilon]^* - \Lambda^\varepsilon\right)\begin{pmatrix}\phi \\ A\end{pmatrix} = \begin{pmatrix}\mathcal{O}_{X^1_2}(\varepsilon^4)\partial_1 A + \mathcal{O}_{X^1_6}(\varepsilon^4)A \\ \mathcal{O}_{X^1_2}(\varepsilon^4)\partial_1\phi + \mathcal{O}_{X^1_3}(\varepsilon^4)\phi\end{pmatrix}. \tag{4.31}$$

Therefore, if $[\Lambda^\varepsilon]^*(\phi, A)^T = 0$, then

$$\Lambda^\varepsilon\begin{pmatrix}\phi \\ A\end{pmatrix} = \begin{pmatrix}\mathcal{O}_{X^1_2}(\varepsilon^4)\partial_1 A + \mathcal{O}_{X^1_6}(\varepsilon^4)A \\ \mathcal{O}_{X^1_2}(\varepsilon^4)\partial_1\phi + \mathcal{O}_{X^1_3}(\varepsilon^4)\phi\end{pmatrix},$$

and, applying Lemma 4.3, we obtain

$$\begin{aligned}&\|A\|_{X^1_{1+\sigma}} + \|\nabla\phi\|_{X^1_{1+\sigma}} + \left\|(\nabla^\varepsilon)^2 A\right\|_{X_{2+\sigma}} \\ &\leqslant C(\sigma,\sigma')|\ln\varepsilon|^2\Big(\left\|\mathcal{O}_{X^1_2}(\varepsilon^4)\partial_1 A + \mathcal{O}_{X^1_6}(\varepsilon^4)A\right\|_{X^1_{1+\sigma'}} + \left\|\mathcal{O}_{X^1_2}(\varepsilon^4)\partial_1\phi + \mathcal{O}_{X^1_3}(\varepsilon^4)\phi\right\|_{X^1_{1+\sigma'}} \\ &\qquad + \frac{1}{\varepsilon^2}\left\|\mathfrak{c}\mathcal{I}(\mathcal{O}_{X^1_2}(\varepsilon^4)\partial_1 A + \mathcal{O}_{X^1_6}(\varepsilon^4)A) - \mathcal{O}_{X^1_2}(\varepsilon^4)\partial_1\phi + \mathcal{O}_{X^1_3}(\varepsilon^4)\phi\right\|_{X^1_{1+\sigma'}}\Big).\end{aligned}$$

Elementary estimates, including $\|\phi\|_{L^\infty} \leqslant C(\sigma)\|\nabla\phi\|_{X^1_{1+\sigma}}$ and $\varepsilon\|\partial_1\partial_2 A\|_{X_{1+\sigma}} \leqslant C(\sigma)\|(\nabla^\varepsilon)^2 A\|_{X_{2+\sigma}}$, then yield

$$\|A\|_{X^1_{1+\sigma}} + \|\nabla\phi\|_{X^1_{1+\sigma}} + \|(\nabla^\varepsilon)^2 A\|_{X_{2+\sigma}} \leqslant C(\sigma,\sigma')\varepsilon|\ln\varepsilon|^2\Big(\|A\|_{X^1_{1+\sigma}} + \|\nabla\phi\|_{X^1_{1+\sigma}} + \|(\nabla^\varepsilon)^2 A\|_{X_{2+\sigma}}\Big),$$

hence $A \equiv 0$ and $\phi \equiv 0$ for $\varepsilon < \epsilon_\sharp(\sigma,\sigma')$ sufficiently small. □

4.4.4. Existence in a large disk

We now prove an existence result by using Fredholm alternative and the injectivity for the adjoint problem on the plane.

Lemma 4.9. — *Let $0<\sigma<\sigma'<1$ be given and $\epsilon_\sharp(\sigma,\sigma') \in]0,1[$ as in Lemma 4.8. Let $0<\varepsilon<\epsilon_\sharp(\sigma,\sigma')$ be given. Then, there exists $R_\sharp(\varepsilon,\sigma,\sigma') > R_\varepsilon(\varepsilon,\sigma,\sigma')$ such that, for any $R \geqslant R_\sharp(\varepsilon,\sigma,\sigma')$ and any $(\mathcal{S}_{\rm ph},\mathcal{S}_{\rm am}) \in X^1_{2+\sigma',a}(D(0,R)) \times X^1_{1+\sigma',s}(D(0,R))$, there exists $(\phi,A) \in H^1_a(D(0,R)) \times H^1_s(D(0,R))$, satisfying the linear system* (2.10)*, namely*

$$\begin{cases} c(\varepsilon)\partial_1 A - (\partial_1^2 + \varepsilon^2\partial_2^2)\,\phi - 2\varepsilon^2\partial_1\phi\partial_1 A^{(0)} - 2\varepsilon^4\partial_2\phi\partial_2 A^{(0)} \\ \quad -\varepsilon^2 c(\varepsilon)\partial_1\left(A^{(0)}A\right) - 2\varepsilon^2\partial_1\phi^{(0)}\partial_1 A = \mathcal{S}_{\rm ph} \\ \\ -c(\varepsilon)\partial_1\phi - \varepsilon^2 c(\varepsilon)A^{(0)}\partial_1\phi - \varepsilon^2 c(\varepsilon)A\partial_1\phi^{(0)} + 2\varepsilon^2\partial_1\phi^{(0)}\partial_1\phi + \mathfrak{c}^2 A \\ \quad +\varepsilon^2\mathfrak{c}^2(\Gamma-3)A^{(0)}A - \varepsilon^2\left(\partial_1^2 A + \varepsilon^2\partial_2^2 A\right) = \mathcal{S}_{\rm am}. \end{cases}$$

Moreover, there exists $C(\varepsilon,s,\sigma')$, independent of $R \geqslant R_\sharp(\varepsilon,\sigma,\sigma')$ and $(\mathcal{S}_{\rm ph},\mathcal{S}_{\rm am})$ such that

$$\|A\|_{H^1(D(0,R))} + \|\nabla\phi\|_{L^2(D(0,R))} \leqslant C(\varepsilon,\sigma,\sigma')\left(\|\mathcal{S}_{\rm ph}\|_{X^1_{2+\sigma}} + \|\mathcal{S}_{\rm am}\|_{X^1_{1+\sigma}} + \frac{1}{\varepsilon^2}\|\mathfrak{c}\mathcal{I}\mathcal{S}_{\rm ph} - \mathcal{S}_{\rm am}\|_{X^1_{1+\sigma'}}\right).$$

Proof. — The linear system (2.10) may be written under the form

$$\Lambda^\varepsilon_R\begin{pmatrix}\phi\\A\end{pmatrix} = \begin{pmatrix}\mathcal{S}_{\rm ph}\\\mathcal{S}_{\rm am}\end{pmatrix},$$

where Λ^ε_R is the differential operator Λ^ε on $H^1_a(D(0,R)) \times H^1_s(D(0,R))$. It is classical to see that Λ^ε_R has compact resolvent (by applying Lax-Milgram theorem to $\Lambda^\varepsilon_R + \kappa$ for some large constant κ depending actually only on ε). Therefore, we may apply Fredholm alternative: in order to show that Λ^ε_R is onto (from $(H^1_a \cap H^2) \times (H^1_s \cap H^2)$ to $L^2 \times L^2$), it suffices to prove that its adjoint is injective. We proceed with a last proof by contradiction. We then assume that (for $0<\sigma<\sigma'<1$ fixed and $0<\varepsilon<\epsilon_\sharp(\sigma,\sigma')$) there exist sequences $(R_n)_{n\in\mathbb{N}}$ tending to $+\infty$ and $(\phi_n,A_n) \in H^1_a(D(0,R_n)) \times H^1_s(D(0,R_n))$, satisfying (2.10) in $D(0,R_n)$, that is

$$(4.32)\qquad \begin{cases} c(\varepsilon)\partial_1 A_n - (\partial_1^2 + \varepsilon^2\partial_2^2)\,\phi_n - 2\varepsilon^2\partial_1\phi_n\partial_1 A^{(0)} - 2\varepsilon^4\partial_2\phi_n\partial_2 A^{(0)} \\ \quad -\varepsilon^2 c(\varepsilon)\partial_1(A^{(0)}A_n) - 2\varepsilon^2\partial_1\phi^{(0)}\partial_1 A_n = \mathcal{S}_{\mathrm{ph},n} \\ \\ -c(\varepsilon)\partial_1\phi_n - \varepsilon^2 c(\varepsilon)A^{(0)}\partial_1\phi_n - \varepsilon^2 c(\varepsilon)A_n\partial_1\phi^{(0)} + 2\varepsilon^2\partial_1\phi^{(0)}\partial_1\phi_n + \mathfrak{c}^2 A_n \\ \quad +\varepsilon^2\mathfrak{c}^2(\Gamma-3)A^{(0)}A_n - \varepsilon^2\,(\partial_1^2 A_n + \varepsilon^2\partial_2^2 A_n) = \mathcal{S}_{\mathrm{am},n} \end{cases}$$

in $D(0,R_n)$ with homogeneous Dirichlet boundary condition, and such that

$$(4.33)\qquad \|A_n\|_{H^1(D(0,R_n))} + \|\nabla\phi_n\|_{L^2(D(0,R_n))} = 1.$$

Arguing as for the proof of Lemma 4.6, we may assume that

$$A_n \rightharpoonup A_\infty \equiv 0 \qquad \phi_n \rightharpoonup \phi_\infty \equiv 0 \qquad \text{weakly in } H^1_{\mathrm{loc}}(\mathbb{R}^2),$$

since $[\Lambda^\varepsilon]^*$ is injective (see Lemma 4.8). The remaining of the proof is quite similar to that of Lemma 4.6. We multiply the first equation of (4.32) by ϕ_n and integrate by parts over $D(0,R_n)$ ($\phi_n = 0$ on $\partial D(0,R_n)$), respectively the second equation of (4.32) by A_n and integrate by parts over $D(0,R_n)$ ($\phi_n = 0$ on $\partial D(0,R_n)$). We claim that (4.27) and (4.28) remain true, which shall imply as before a contradiction. To see this, is suffices to check that the additional terms in (4.31) lead to small contributions for $n \to +\infty$, which can be done as we before: for instance, by using, for $\rho < R_n$,

$$\begin{aligned} \left|\int_{D(0,R_n)\setminus D(0,\rho)} \mathcal{O}_{X^1_3}(1) A_n\phi_n\,dz\right| &\leqslant C\int_\rho^{R_n} \frac{1}{r^3}\|A_n\|_{L^2(\partial D(0,r))}\|\phi_n\|_{L^2(\partial D(0,r))}\,dr \\ &\leqslant C_\varepsilon \int_\rho^{R_n} \frac{1}{r^2}\Big(\|A_n\|^2_{L^2(\partial D(0,r))} + \|\nabla\phi_n\|^2_{L^2(\partial D(0,r))}\Big)\,dr \\ &\leqslant \frac{C_\varepsilon}{\rho^2}\Big(\|A_n\|^2_{L^2(D(0,R_n))} + \|\nabla\phi_n\|^2_{L^2(D(0,R_n))}\Big) \leqslant \frac{C_\varepsilon}{\rho^2}. \end{aligned}$$

As a consequence, we have existence of solutions $(\phi, A) \in (H^1_a,\cap H^2) \times (H^1_a,\cap H^2)$ for source terms in $L^2\times L^2$. The a priori estimate for source terms in $X^1_{2+\sigma,a}(D(0,R))\times X^1_{1+\sigma}(D(0,R))$ then follows from Lemma 4.6. ☐

4.4.5. Existence in the plane

Let us fix $(\mathcal{S}_{\mathrm{ph}}, \mathcal{S}_{\mathrm{am}}) \in X^1_{2+\sigma',a} \times X^1_{2+\sigma,s}$, $0 < \sigma < \sigma' < 1$ and $0 < \varepsilon < \epsilon_\sharp(\sigma,\sigma')$ (given in Lemma 4.9). For $R \geqslant R_\sharp(\varepsilon,\sigma,\sigma')$, we know by Lemma 4.6 that (2.10) has a solution $(A_R,\phi_R) \in H^1_0(D(0,R)) \times H^1_0(D(0,R))$, and that, by Lemma 4.9, it enjoys the estimate

$$\begin{aligned} &\|A_R\|_{H^1(D(0,R))} + \|\nabla\phi_R\|_{L^2(D(0,R))} \\ &\qquad \leqslant C(\varepsilon,\sigma,\sigma')\Big(\|\mathcal{S}_{\mathrm{ph}}\|_{X^1_{2+\sigma,a}(D(0,R))} + \|\mathcal{S}_{\mathrm{am}}\|_{X^1_{1+\sigma}(D(0,R))}\Big) \\ &\qquad \leqslant C(\varepsilon,\sigma,\sigma')\Big(\|\mathcal{S}_{\mathrm{ph}}\|_{X^1_{2+\sigma,a}} + \|\mathcal{S}_{\mathrm{am}}\|_{X^1_{1+\sigma}}\Big). \end{aligned}$$

Thanks to this bound uniform in $R \geqslant R_\sharp(\varepsilon,\sigma,\sigma')$ and the fact that $\phi_R = 0$ on $\{0\}\times[-R,+R]$, we may extract a sequence $R_n \to +\infty$ such that there exists $A \in H^1(\mathbb{R}^2)$, $\phi \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ satisfying

$$A_{R_n} \rightharpoonup A \qquad \text{in } H^1(\mathbb{R}^2) \qquad \text{and} \qquad \phi_{R_n} \rightharpoonup \phi \qquad \text{in } H^1_{\mathrm{loc}}(\mathbb{R}^2),$$

with

$$\|A\|_{H^1(\mathbb{R}^2)} + \|\nabla\phi\|_{L^2(\mathbb{R}^2)} \leqslant C(\varepsilon,\sigma,\sigma')\Big(\|\mathcal{S}_{\mathrm{ph}}\|_{X^1_{2+\sigma,a}} + \|\mathcal{S}_{\mathrm{am}}\|_{X^1_{1+\sigma}}\Big).$$

Passing to the limit in (2.10) in $D(0,R_n)$, we see that (A,ϕ) solves (2.10) in $\mathbb{R}^2$. We finally apply Lemma 4.5 to deduce $A \in X^1_{1+\sigma,s}$, $\nabla\phi \in X^1_{1+\sigma,s}$ as well as the estimate

$$\begin{aligned} &\|A\|_{X^1_{1+\sigma}} + \|\nabla\phi\|_{X^1_{1+\sigma}} \\ &\qquad \leqslant C(\sigma,\sigma')\left(\|\mathcal{S}_{\mathrm{ph}}\|_{X^1_{2+\sigma,a}} + \|\mathcal{S}_{\mathrm{am}}\|_{X^1_{1+\sigma}} + \frac{1}{\varepsilon^2}\|\mathfrak{c}\mathcal{I}\mathcal{S}_{\mathrm{ph}} - \mathcal{S}_{\mathrm{am}}\|_{X^1_{1+\sigma'}}\right). \end{aligned}$$

This concludes the proof of Proposition 2.10. □

4.5. Proof of Proposition 2.11

In this section, we fix $0<\sigma<1$ and pick some σ' such that

$$0 < \sigma < \sigma' < 1 \qquad \text{and} \qquad \sigma' < 2\sigma.$$

The second assumption ensures that the quadratic terms shall have a sufficiently strong decay. For instance, $\sigma' = \sigma^{1/2}$ will work.

Step 1 (Estimate for the source term). — We have, by Proposition 2.8,

$$\begin{aligned} \left\|\mathbb{M}\begin{pmatrix}\mathrm{Err}_{\mathrm{ph}}\\ \mathrm{Err}_{\mathrm{am}}\end{pmatrix}\right\|_{\mathcal{X}^\varepsilon_\sigma} &\leqslant C(\sigma)|\ln\varepsilon| \left\|\begin{pmatrix}\mathrm{Err}_{\mathrm{ph}}\\ \mathrm{Err}_{\mathrm{am}}\end{pmatrix}\right\|_{\mathcal{Y}^\varepsilon_{\sigma'}} \\ &= C(\sigma)|\ln\varepsilon|^2\left(\|\mathrm{Err}_{\mathrm{ph}}\|_{X^1_{2+\sigma'}} + \|\mathrm{Err}_{\mathrm{am}}\|_{X^1_{1+\sigma'}} + \frac{1}{\varepsilon^2}\|\mathfrak{c}\mathcal{I}\mathrm{Err}_{\mathrm{ph}} - \mathrm{Err}_{\mathrm{am}}\|_{X^1_{1+\sigma'}}\right) \\ &\leqslant C(\sigma)\varepsilon^2|\ln\varepsilon|^2, \end{aligned}$$

by using Proposition 2.2.

Step 2 (Estimate for the terms $\mathcal{N}_{\mathrm{am}}$). — We claim that $\mathcal{N}_{\mathrm{am}} : \bar{B}_{\mathcal{X}^\varepsilon_\sigma}(0,1) \to \mathcal{Y}^\varepsilon_{\sigma'}$ is of class $\mathcal{C}^1$ and that, for any $(\phi_*,A_*) \in \bar{B}_{\mathcal{X}^\varepsilon_\sigma}(0,1)$, we have

$$\|d\mathcal{N}_{\mathrm{am}}(\phi_*,A_*)\|_{\mathcal{X}^\varepsilon_\sigma \to \mathcal{Y}^\varepsilon_{\sigma'}} \leqslant C(\sigma,\sigma')\Big(\varepsilon^2 + \|(\phi_*,A_*)\|_{\mathcal{X}^\varepsilon_\sigma}\Big).$$

We recall that

$$\begin{aligned} \mathcal{N}_{\mathrm{am}}(A,\phi) = {}& \varepsilon^2 c(\varepsilon) A\partial_1\phi - \varepsilon^4\left(A^{(0)}+A\right)\left(\partial_1\phi^{(0)}+\partial_1\phi\right)^2 - \varepsilon^2(\partial_1\phi)^2 \\ & - \varepsilon^4\left(1+\varepsilon^2A^{(0)}+\varepsilon^2A\right)\left(\partial_2\phi^{(0)}+\partial_2\phi\right)^2 \\ & - \varepsilon^2\frac{\mathfrak{c}^2}{2}(\Gamma-3)A^2 - \varepsilon^4\frac{\mathfrak{c}^2}{2}(\Gamma-5)\left(A^{(0)}+A\right)^3 \\ & - \frac{1}{\varepsilon^2}\left(1+\varepsilon^2A^{(0)}+\varepsilon^2A\right) f_3\left(\varepsilon^2A^{(0)}+\varepsilon^2A\right), \end{aligned}$$

where (see (1.5))

$$f_3(\alpha) = f\Big((1+\alpha)^2\Big) - \mathfrak{c}^2\alpha - \frac{\mathfrak{c}^2}{2}(\Gamma-5)\alpha^2 = \mathcal{O}(\alpha^3) \text{ as } \alpha \to 0.$$

Let us consider, for instance, the term $A\partial_1\phi$: there holds

$$\begin{aligned}\left\| \frac{\partial [A\partial_1\phi]}{\partial A}(\phi_*, A_*)\left[\widetilde{\phi}, \widetilde{A}\right] \right\|_{X^1_{1+\sigma'}} & \\ &= \left\| \widetilde{A}\partial_1\phi_* \right\|_{X^1_{1+\sigma'}} \\ &= \left\| \widetilde{A}\partial_1\phi_* \right\|_{X_{1+\sigma'}} + \left\| \partial_1\left(\widetilde{A}\partial_1\phi_*\right)\right\|_{X_{2+\sigma'}} + \left\| \partial_2\left(\widetilde{A}\partial_1\phi_*\right)\right\|_{X_{2+\sigma'}} \\ &\leqslant C\|(\phi_*, A_*)\|_{\mathcal{X}^\varepsilon_\sigma} \left\|\left(\widetilde{\phi}, \widetilde{A}\right)\right\|_{\mathcal{X}^\varepsilon_\sigma},\end{aligned}$$

hence

$$\left\| \begin{pmatrix} 0 \\ \partial[\varepsilon^2 c(\varepsilon) A\partial_1\phi]/\partial A(\phi_*, A_*)\end{pmatrix}\right\|_{\mathcal{Y}^\varepsilon_{\sigma'}} \leqslant C\|(\phi_*, A_*)\|_{\mathcal{X}^\varepsilon_\sigma}.$$

For the term $\varepsilon^4(A^{(0)} + A)(\partial_1\phi^{(0)} + \partial_1\phi)^2$, we write

$$\begin{aligned}\left\| \frac{\partial\left[\left(A^{(0)}+A\right)\left(\partial_1\phi^{(0)}+\partial_1\phi\right)^2\right]}{\partial A}(\phi_*, A_*)\left[\widetilde{\phi}, \widetilde{A}\right]\right\|_{X^1_{1+\sigma'}} &= \left\| \widetilde{A}\left(\partial_1\phi^{(0)} + \partial_1\phi_*\right)^2\right\|_{X^1_{1+\sigma'}} \\ &\leqslant \left(C + \|(\phi_*, A_*)\|^2_{\mathcal{X}^\varepsilon_\sigma}\right)\left\|\left(\widetilde{\phi}, \widetilde{A}\right)\right\|_{\mathcal{X}^\varepsilon_\sigma} \\ &\leqslant C\left\|\left(\widetilde{\phi}, \widetilde{A}\right)\right\|_{\mathcal{X}^\varepsilon_\sigma}.\end{aligned}$$

The other terms are estimated in the same way (using Taylor formula with integral remainder for the term involving f_3).

For the source term $\mathcal{N}_{\rm ph}$, we shall need to single out the term $2\varepsilon^4\partial_2\phi^{(0)}\partial_2 A$ because we need to treat it carefully: the rough bound, in $X^1_{1+\sigma'}$, $\leqslant C\varepsilon^2|\ln\varepsilon|^2\|(\phi, A)\|_{\mathcal{X}^\varepsilon_\sigma}$ is indeed not sufficient since it would yield a bound $\leqslant C|\ln\varepsilon|^2\|(\phi, A)\|_{\mathcal{X}^\varepsilon_\sigma}$. We therefore set

$$\mathcal{N}^\flat_{\rm ph}(\phi, A) \stackrel{\rm def}{=} \mathcal{N}_{\rm ph}(\phi, A) + 2\varepsilon^4\partial_2\phi^{(0)}\partial_2 A.$$

Step 3 (Estimate for the terms $\mathcal{N}^\flat_{\rm ph}$). — We claim that $\mathcal{N}^\flat_{\rm ph} : \bar{B}_{\mathcal{X}^\varepsilon_\sigma}(0,1) \to \mathcal{Y}^\varepsilon_{\sigma'}$ is of class $\mathcal{C}^1$ and that, for any $(\phi_*, A_*) \in \bar{B}_{\mathcal{X}^\varepsilon_\sigma}(0,1)$, we have

$$\left\| d\mathcal{N}^\flat_{\rm ph}(\phi_*, A_*)\right\|_{\mathcal{X}^\varepsilon_\sigma \to \mathcal{Y}^\varepsilon_{\sigma'}} \leqslant C(\sigma, \sigma')\left(\varepsilon^2 + \varepsilon^{-1}\|(\phi_*, A_*)\|_{\mathcal{X}^\varepsilon_\sigma}\right).$$

Let us recall that

$$\begin{aligned}\mathcal{N}^{\flat}_{\mathrm{ph}}(\phi, A) = -c(\varepsilon)\varepsilon^2 A\partial_1 A - c(\varepsilon)\varepsilon^4 \frac{\left(A^{(0)}+A\right)^2}{1+\varepsilon^2 A^{(0)}+\varepsilon^2 A}\partial_1(A^{(0)}+A) \\ + 2\varepsilon^2\partial_1\phi\partial_1 A - 2\varepsilon^4\left(A^{(0)}+A\right)\frac{\partial_1\left(\phi^{(0)}+\phi\right)\partial_1\left(A^{(0)}+A\right)}{1+\varepsilon^2 A^{(0)}+\varepsilon^2 A} \\ + 2\varepsilon^4\partial_2\phi^{(0)}\partial_2 A - 2\varepsilon^4\frac{\partial_2\left(\phi^{(0)}+\phi\right)\partial_2\left(A^{(0)}+A\right)}{1+\varepsilon^2 A^{(0)}+\varepsilon^2 A},\end{aligned}$$

thus we have to treat carefully the terms with ∇A. For the term $A\partial_1 A$, there holds

$$\begin{aligned}\left\|\frac{\partial[A\partial_1 A]}{\partial A}(\phi_*, A_*)\left[\widetilde{\phi}, \widetilde{A}\right]\right\|_{X^1_{2+\sigma'}} \\ = \left\|\widetilde{A}\partial_1 A_* + A_*\partial_1\widetilde{A}\right\|_{X^1_{2+\sigma'}} \\ \leqslant \left\|\widetilde{A}\partial_1 A_*\right\|_{X_{2+\sigma'}} + \left\|\partial_1\left(\widetilde{A}\partial_1 A_*\right)\right\|_{X_{3+\sigma'}} + \left\|\partial_2\left(\widetilde{A}\partial_1 A_*\right)\right\|_{X_{3+\sigma'}} \\ + \left\|A_*\partial_1\widetilde{A}\right\|_{X_{2+\sigma'}} + \left\|\partial_1\left(A_*\partial_1\widetilde{A}\right)\right\|_{X_{3+\sigma'}} + \left\|\partial_2\left(A_*\partial_1\widetilde{A}\right)\right\|_{X_{3+\sigma'}} \\ \leqslant \frac{C}{\varepsilon}\left\|(\phi_*, A_*)\right\|_{\mathcal{X}^\varepsilon_\sigma}\left\|\left(\widetilde{\phi}, \widetilde{A}\right)\right\|_{\mathcal{X}^\varepsilon_\sigma},\end{aligned}$$

since, for instance, $\|\widetilde{A}\partial_j\partial_1 A_*\|_{X_{3+\sigma'}} \leqslant \|\widetilde{A}\|_{X_{1+\sigma}}\|\partial_j\partial_1 A_*\|_{X_{2+\sigma}}$, in view of the assumption $\sigma' < 2\sigma$, and when $j = 2$, we loose ε.

For the term $c(\varepsilon)\varepsilon^4\frac{(A^{(0)}+A)^2}{1+\varepsilon^2 A^{(0)}+\varepsilon^2 A}\partial_1(A^{(0)}+A)$, we have

$$\begin{aligned}\frac{\partial}{\partial A}\left[\frac{\left(A^{(0)}+A\right)^2}{1+\varepsilon^2 A^{(0)}+\varepsilon^2 A}\partial_1\left(A^{(0)}+A\right)\right](A_*)(\widetilde{A}) = \frac{2\widetilde{A}\left(A^{(0)}+A\right)}{1+\varepsilon^2 A^{(0)}+\varepsilon^2 A_*}\partial_1\left(A^{(0)}+A_*\right) \\ + \frac{\left(A^{(0)}+A_*\right)^2}{1+\varepsilon^2 A^{(0)}+\varepsilon^2 A_*}\partial_1\widetilde{A} - \frac{\varepsilon^2\left(A^{(0)}+A_*\right)^2\widetilde{A}}{\left(1+\varepsilon^2 A^{(0)}+\varepsilon^2 A_*\right)^2}\partial_1\left(A^{(0)}+A_*\right)\end{aligned}$$

thus

$$\begin{aligned}\left\|c(\varepsilon)\varepsilon^4\frac{\partial}{\partial A}\left[\frac{\left(A^{(0)}+A\right)^2}{1+\varepsilon^2 A^{(0)}+\varepsilon^2 A}\partial_1\left(A^{(0)}+A\right)\right](A_*)(\widetilde{A})\right\|_{X^1_{2+\sigma'}} \\ \leqslant C\left(\varepsilon^4 + \varepsilon^3\|(\phi_*, A_*)\|_{\mathcal{X}^\varepsilon_\sigma}\right).\end{aligned}$$

Finally, for the term

$$\begin{aligned}\mathcal{T} &\overset{\text{def}}{=} 2\varepsilon^4\partial_2\phi\partial_2A^{(0)} + 2\varepsilon^4\partial_2\phi^{(0)}\partial_2A - 2\varepsilon^4\frac{\partial_2\left(\phi^{(0)}+\phi\right)\partial_2(A^{(0)}+A)}{1+\varepsilon^2A^{(0)}+\varepsilon^2A}\\ &= 2\varepsilon^4\partial_2\phi\partial_2A^{(0)} - \frac{2\varepsilon^4\partial_2\left(\phi^{(0)}+\phi\right)\partial_2A^{(0)}}{1+\varepsilon^2A^{(0)}+\varepsilon^2A} - \frac{2\varepsilon^4\partial_2\phi\partial_2A}{1+\varepsilon^2A^{(0)}+\varepsilon^2A}\\ &\quad + 2\varepsilon^6\left(A^{(0)}+A\right)\frac{\partial_2\phi^{(0)}\partial_2A}{1+\varepsilon^2A^{(0)}+\varepsilon^2A},\end{aligned}$$

we have similarly (we loose ε^2 for $\partial_2(\partial_2\phi\partial_2A)$)

$$\|d\mathcal{T}(\phi_*,A_*)\|_{\mathcal{X}_\sigma^\varepsilon\to\mathcal{Y}_{\sigma'}^\varepsilon} \leqslant C\varepsilon^2\left(\varepsilon^2+\|(\phi_*,A_*)\|_{\mathcal{X}_\sigma^\varepsilon}\right).$$

Step 4 (Estimate for the solution associated with the source term $(2\varepsilon^4\partial_2\phi^{(0)}\partial_2A,0)$). We have

$$\left\|\mathbb{M}\begin{pmatrix}2\varepsilon^4\partial_2\phi^{(0)}\partial_2A\\0\end{pmatrix}\right\|_{\mathcal{X}_\sigma^\varepsilon} \leqslant C(\sigma)\varepsilon|\ln\varepsilon|^2\|(A,\phi)\|_{\mathcal{X}_\sigma^\varepsilon}.$$

We use the second estimate of Proposition 2.8 (the case where $\mathcal{S}_{\text{am}}=0$) and get

$$\begin{aligned}&\left\|\mathbb{M}\begin{pmatrix}2\varepsilon^4\partial_2\phi^{(0)}\partial_2A\\0\end{pmatrix}\right\|_{\mathcal{X}_\sigma^\varepsilon}\\ &\qquad\leqslant C(\sigma)|\ln\varepsilon|^2\Big(\left\|\varepsilon^4\partial_2\phi^{(0)}\partial_2A\right\|_{X^1_{2+\sigma'}} + \varepsilon^{-2}\left\|\varepsilon^4\partial_2\phi^{(0)}\partial_2A\right\|_{X_{2+\sigma'}}\\ &\qquad\qquad + \varepsilon^{-2}\left\|\varepsilon^4\partial_1\left(\partial_2\phi^{(0)}\partial_2A\right)\right\|_{X_{2+\sigma'}}\Big)\\ &\qquad\leqslant C(\sigma)\varepsilon|\ln\varepsilon|^2\|(A,\phi)\|_{\mathcal{X}_\sigma^\varepsilon},\end{aligned}$$

as claimed.

Step 5 (The closed ball $\bar{B}_{\mathcal{X}_\sigma^\varepsilon}(0,C_3\varepsilon^2|\ln\varepsilon|^2)$ is stable by Υ). — From Steps 1-4, we infer

$$\begin{aligned}\left\|\mathbb{M}\begin{pmatrix}\mathcal{N}^\flat_{\text{ph}}(A,\phi)\\\mathcal{N}_{\text{am}}(A,\phi)\end{pmatrix}\right\|_{\mathcal{X}_\sigma^\varepsilon} &\leqslant C(\sigma,\sigma')|\ln\varepsilon|^2\left\|\begin{pmatrix}\mathcal{N}^\flat_{\text{ph}}(A,\phi)\\\mathcal{N}_{\text{am}}(A,\phi)\end{pmatrix}-\begin{pmatrix}\mathcal{N}^\flat_{\text{ph}}(0,0)\\\mathcal{N}_{\text{am}}(0,0)\end{pmatrix}\right\|_{\mathcal{Y}_{\sigma'}^\varepsilon}\\ &\leqslant C(\sigma,\sigma')|\ln\varepsilon|\left(\varepsilon^2+\varepsilon^{-1}\|(A,\phi)\|_{\mathcal{X}_\sigma^\varepsilon}\right)\|(A,\phi)\|_{\mathcal{X}_\sigma^\varepsilon}\end{aligned}$$

by the mean-value theorem, hence

$$\begin{aligned}\left\|\Upsilon\begin{pmatrix}A\\\phi\end{pmatrix}\right\|_{\mathcal{X}_\sigma^\varepsilon} &\leqslant \left\|\mathbb{M}\begin{pmatrix}\text{Err}_{\text{ph}}\\\text{Err}_{\text{am}}\end{pmatrix}\right\|_{\mathcal{X}_\sigma^\varepsilon} + \left\|\mathbb{M}\begin{pmatrix}\mathcal{N}^\flat_{\text{ph}}(A,\phi)\\\mathcal{N}_{\text{am}}(A,\phi)\end{pmatrix}\right\|_{\mathcal{X}_\sigma^\varepsilon} + \left\|\mathbb{M}\begin{pmatrix}2\varepsilon^4\partial_2\phi^{(0)}\partial_2A\\0\end{pmatrix}\right\|_{\mathcal{X}_\sigma^\varepsilon}\\ &\leqslant C(\sigma)|\ln\varepsilon|^2\left(\varepsilon^2+\varepsilon\|(A,\phi)\|_{\mathcal{X}_\sigma^\varepsilon}+\varepsilon^{-1}\|(A,\phi)\|^2_{\mathcal{X}_\sigma^\varepsilon}\right).\end{aligned}$$

In particular, choosing $C_3 > C(\sigma)$, if $(A,\phi)\in\bar{B}_{\mathcal{X}_\sigma^\varepsilon}(0,C_3\varepsilon^2|\ln\varepsilon|^2)$, then the above inequality yields

$$\|\Upsilon(A,\phi)\|_{\mathcal{X}_\sigma^\varepsilon} \leqslant C(\sigma)|\ln\varepsilon|\left(\varepsilon^2+C_3\varepsilon^3|\ln\varepsilon|^2+C_3^2\varepsilon^3|\ln\varepsilon|^4\right) \leqslant C_3\varepsilon^2|\ln\varepsilon|^2, \tag{4.34}$$

provided $\varepsilon < \epsilon(\sigma,C_3)$ is small enough.

Step 6 (Lipschitz estimate for the terms $\mathcal{N}$). — We claim that Υ is 1/2-Lipschitz continuous on $\bar{B}_{\mathcal{X}_\sigma^\varepsilon}(0, C_3\varepsilon^2|\ln\varepsilon|^2)$, provided ε is small enough (depending on σ).

We use the mean value theorem (using Step 2 and Step 3) for the terms $\mathcal{N}_{\mathrm{am}}$ and $\mathcal{N}_{\mathrm{ph}}^\flat$, as well as Step 4 for the linear term $\varepsilon^4\partial_2\phi^{(0)}\partial_2 A$, which yields, for $(\phi_*, A_*) \in \bar{B}_{\mathcal{X}_\sigma^\varepsilon}(0, C_3\varepsilon^2|\ln\varepsilon|)$,

$$\|d\Upsilon(\phi_*, A_*)\|_{\mathcal{X}_\sigma^\varepsilon \to \mathcal{X}_\sigma^\varepsilon} \leqslant C(\sigma)|\ln\varepsilon|^2 \left(\varepsilon + \varepsilon^{-1}\|(\phi_*, A_*)\|_{\mathcal{X}_\sigma^\varepsilon}\right) \leqslant C(\sigma)|\ln\varepsilon|^2 \left(\varepsilon + C_3\varepsilon|\ln\varepsilon|^2\right),$$

and the conclusion follows for $\varepsilon \leqslant \epsilon(\sigma)$ sufficiently small (depending on C_3).

4.6. Proof of Theorem 1.3 completed

At this stage, fixing $0 < \sigma < 1$ (and C_3), we have constructed, for $\varepsilon < \epsilon(\sigma, C_3)$ a solution $(\phi_\varepsilon^{(1)}, A^{(1)}) \in \bar{B}_{\mathcal{X}_\sigma^\varepsilon(0, C_3\varepsilon^2|\ln\varepsilon|^2)}$ to (2.11), which means that

$$(\phi_\varepsilon, A_\varepsilon) = \left(\phi^{(0)} + \phi^{(1)}, A^{(0)} + A^{(1)}\right)$$

solves, as desired, the PDE (1.3). If $0 < \sigma < \sigma^\dagger < 1$, using the uniqueness for σ yields that the functions $(\phi_\varepsilon^{(1)}, A^{(1)})$ obtained with σ or with $\sigma^\dagger$ are the same, at least on $]0, \min(\epsilon(\sigma, C_3), \epsilon(\sigma^\dagger, C_3^\dagger))]$.

We easily check that U_c has finite energy. It then follow from [Gra04] that there exists $\omega \in \mathbb{C}$ with $|\omega| = 1$ such that $U_c(x) \to \omega$ when $|x| \to +\infty$. Since ϕ_ε is odd in z_1 by construction, we may take $x = (0, x_2)$ with $x_2 \to +\infty$ to infer $\mathbb{R}_+ \ni U_c(0, x_2) \to \omega$, which imposes $\omega = 1$, that is $\phi(x) \to 0$ for $|x| \to +\infty$. Since $\nabla\phi^{(1)} \in X_{1+\sigma}$, we deduce from Proposition 2.1 that $\phi^{(1)} \in X_\sigma$ and $\|\phi^{(1)}\|_{X_\sigma} \leqslant C(\sigma)\varepsilon^2|\ln\varepsilon|^2 \to 0$.

4.6.1. $\mathcal{C}^1$ regularity of the branch

Let us fix $\sigma \in]0,1[$ and some constant $C_3' \in]C(\sigma), C_3[$, where $C_3 > C(\sigma)$. Going back to (4.34), we see that we actually have $\|(\phi^{(1)}, A^{(1)})\|_{\mathcal{X}_\sigma^\varepsilon} \leqslant C_3'\varepsilon^2|\ln\varepsilon|^2$ for ε small enough, say $0 < \varepsilon < \varepsilon_*(\sigma, C_3')$. Let us then fix $\varepsilon_0 \in]0, \varepsilon_*(\sigma, C_3')[$.

Once we have constructed an exact solution $(\phi^{(1)}, A^{(1)})$ to the fix point problem, from the computations in subsection 4.5, we know that the mapping Υ is of class $\mathcal{C}^1$ and has small differential, therefore, by the classical implicit function theorem, we may construct a $\mathcal{C}^1$ branch $[\underline{\varepsilon}, \overline{\varepsilon}] \ni \varepsilon \to (\phi_\varepsilon^\dagger, A_\varepsilon^\dagger) \in \mathcal{X}_\sigma^\varepsilon$ of fix points to (2.11) near $\varepsilon = \varepsilon_0$. We may further assume (since $C_3' < C_3$), changing $\underline{\varepsilon}$ and $\overline{\varepsilon}$ if necessary, that $(\phi_\varepsilon^\dagger, A_\varepsilon^\dagger) \in \bar{B}_{\mathcal{X}_\sigma^\varepsilon}(0, C_3\varepsilon^2|\ln\varepsilon|^2)$ for $\varepsilon \in [\underline{\varepsilon}, \overline{\varepsilon}]$. It then follows from the uniqueness of the fix point in $\bar{B}_{\mathcal{X}_\sigma^\varepsilon}(0, C_3\varepsilon^2|\ln\varepsilon|^2)$ that $(\phi_\varepsilon^\dagger, A_\varepsilon^\dagger) = (\phi^{(1)}, A^{(1)})$ in $[\underline{\varepsilon}, \overline{\varepsilon}]$. This shows the smoothness of the branch $]0, \varepsilon_*(\sigma, C_3')[\to \mathcal{X}_\sigma^\varepsilon$.

In view of the X_μ^1 decays on $\phi_\varepsilon^{(1)}$ and $A^{(1)}$), the $W^{1,p}$ smoothness and the $W^{1,p}$ convergences in Theorem 1.3 (ii) are immediate once we have fixed $1 < p \leqslant \infty$ and chosen $2/p - 1 < \sigma < 1$ so that $X_{1+\sigma} \subset L^p$.

4.6.2. Another expression of the momentum

From [CM17, Definition 2.2 and (2.7)] and [Mar13], the momentum is well-defined in the energy space by the formula

$$P(U) = \frac{1}{2}\int_{\mathbb{R}^2} \frac{|U|^2-1}{|U|^2}\langle iU|\partial_1 U\rangle \, dx$$

when U does not vanish. Let us prove that this equals the second expression, namely

$$P(U) = \frac{1}{2}\int_{\mathbb{R}^2} \langle i(U-1)|\partial_1 U\rangle \, dx. \tag{4.35}$$

We may write $U = A\mathsf{e}^{i\phi}$ since U does not vanish. Then, we have, on the one hand,

$$\frac{|U|^2-1}{|U|^2}\langle iU|\partial_1 U\rangle = \left(A^2-1\right)\partial_1\phi,$$

and on the other hand

$$\langle i(U-1)|\partial_{x_1} U\rangle = A^2\partial_1\phi - \partial_1 \mathrm{Im} U = \left(A^2-1\right)\partial_1\phi + \partial_1(\phi - \mathrm{Im} U).$$

Furthermore,

$$\phi - \mathrm{Im} U = \phi - \mathrm{Im}(A\mathsf{e}^{i\phi}) = \phi - \mathrm{Im}\Big((A-1)\mathsf{e}^{i\phi}\Big) + \phi - \sin\phi.$$

For the cubic nonlinearity (Gross–Pitaevskii), by the decay results in [Gra04, Theorem 11], we have $A-1 = \mathcal{O}(|x|^{-2})$, $\nabla A = \mathcal{O}(|x|^{-3})$, $\phi = \mathcal{O}(|x|^{-1})$ and $\nabla\phi = \mathcal{O}(|x|^{-2})$, hence the integrable function $\partial_1(\phi - \mathrm{Im} U)$ has vanishing integral over $\mathbb{R}^2$. This concludes the proof of (4.35) in the case of arbitrary finite energy traveling wave (with cubic nonlinearity).

For the travelling wave coming from our construction, we shall first see that ϕ_ε has a limit at infinity by showing that it satisfies the Bolzano–Cauchy criterion. Let x, $x' \in \mathbb{R}^2$ with $|x'| \geqslant |x| \geqslant 1$. Then, using the path going from x to $|x|x'/|x'|$ along an arc of circle, and then from $|x|x'/|x'|$ to x' along the joining segment, we have

$$|\phi_\varepsilon(x') - \phi_\varepsilon(x)| \leqslant \pi|x|\frac{\|\nabla\phi_\varepsilon\|_{X_{1+\sigma}}}{|x|^{1+\sigma}} + \int_{|x|}^{|x'|}\frac{\|\nabla\phi_\varepsilon\|_{X_{1+\sigma}}}{r^{1+\sigma}}\, dr \leqslant \frac{C(\sigma)}{|x|^\sigma}.$$

This shows that $\lim_{|x|\to+\infty}\phi_\varepsilon(x)$ exists, and taking as before $x = (0, x_2)$ with x_2 tending to $+\infty$, we infer

$$\lim_{|x|\to+\infty}\phi_\varepsilon(x) = 0.$$

It is then sufficient to use the $X^1_{1+\sigma}$ decays to get the result (4.35).

4.6.3. Hamilton group relation

As in [JR82], the Hamilton group relation (iii) is formally shown by multiplying the travelling wave equation TW_c by $\frac{dU_c}{dc}$ and then integrating by parts and differentiating under the integral sign. We need then some decay estimates on $\frac{dU_c}{dc}$, or, equivalently, on $\partial_\varepsilon A^{(1)}$ and $\partial_\varepsilon\phi^{(1)}$. The standard implicit function theorem yields that $\varepsilon \mapsto (\phi^{(1)}, A^{(1)}) \in \mathcal{C}^1(]0,\varepsilon_*], \mathcal{X}^\varepsilon_\sigma)$, hence $\frac{dU_c}{dc} \in \mathcal{C}^0(]0,\varepsilon_*], X_\sigma)$ (the worst term is the phase). It is then easily checked that we have strong enough decays to justify integration by parts. For instance, we may justify that $\int \Delta U_c \partial_c U_c \, dx = -\int \nabla U_c \cdot \nabla \partial_c U_c \, dx$ by the fact that $\nabla U_c \partial_c U_c = \mathcal{O}(|x|^{-2-\sigma})$ for $|x|$ large.

4.6.4. Asymptotics for the energy and the momentum

First, elementary integration through polar coordinates gives

$$\int_{\mathbb{R}^2} \mathcal{W}_1^2 \, dz = 24^2 \times \frac{\pi}{6} = 96\pi.$$

Let us then consider the momentum. We have, by (ii),

$$\begin{aligned} \mathfrak{c} P(U_{c(\varepsilon)}) &= \frac{\mathfrak{c}}{2} \int_{\mathbb{R}^2} \frac{|U_c|^2 - 1}{|U_c|^2} \langle i U_c | \partial_{x_1} U_c \rangle \, dx \\ &= \frac{\varepsilon \mathfrak{c}}{2} \int_{\mathbb{R}^2} \left(2A_\varepsilon + \varepsilon^2 A_\varepsilon^2 \right) \partial_1 \phi_\varepsilon \, dz \sim \varepsilon \int_{\mathbb{R}^2} [\mathcal{W}_1^{\mathrm{sc}}]^2 \, dz = \frac{\varepsilon}{\mathfrak{c}\Gamma^2} \int_{\mathbb{R}^2} \mathcal{W}_1^2 \, dz \end{aligned} \tag{4.36}$$

by (1.7). We obtain similarly

$$E(U_{c(\varepsilon)}) \sim \frac{\varepsilon}{\mathfrak{c}\Gamma^2} \int_{\mathbb{R}^2} \mathcal{W}_1^2 \, dz.$$

We now consider

$$\frac{d}{dc} \Big(E(U_c) - cP(U_c) \Big) = -P(U_c), \tag{4.37}$$

by the Hamilton group relation (iii). The asymptotic (4.36) reads, for c close to $\mathfrak{c}$,

$$P(U_c) \sim \frac{\varepsilon}{\mathfrak{c}^2\Gamma^2} \|\mathcal{W}_1\|_{L^2}^2 \sim \frac{\sqrt{2\mathfrak{c}(\mathfrak{c}-c)}}{\mathfrak{c}^2\Gamma^2} \|\mathcal{W}_1\|_{L^2}^2.$$

Integrating (4.37) then yields

$$E(U_c) - cP(U_c) \sim \frac{[2\mathfrak{c}(\mathfrak{c}-c)]^{3/2}}{3\mathfrak{c}^3\Gamma^2} \|\mathcal{W}_1\|_{L^2}^2 \sim \frac{\varepsilon^3}{3\mathfrak{c}^3\Gamma^2} \|\mathcal{W}_1\|_{L^2}^2.$$

Finally, using $c(\varepsilon) = \sqrt{\mathfrak{c}^2 - \varepsilon^2} = \mathfrak{c} - \varepsilon^2/(2\mathfrak{c}) + \mathcal{O}(\varepsilon^4)$ and (4.36), we infer

$$E(U_{c(\varepsilon)}) - \mathfrak{c} P(U_{c(\varepsilon)}) \sim -\frac{\varepsilon^3}{6\mathfrak{c}^3\Gamma^2} \|\mathcal{W}_1\|_{L^2}^2.$$

5. Modifications for the Euler–Korteweg model

When considering the Euler–Korteweg model, we only have to include the extra term $\varepsilon^4 \widetilde{\kappa}((1+\varepsilon^2 A_\varepsilon)^2)((\partial_1 A_\varepsilon)^2 + \varepsilon^2(\partial_2 A_\varepsilon)^2)$ in the source term $\mathcal{N}_{\mathrm{am}}$. This term is easily estimated: for $(\phi, A) \in \bar{B}_{\mathcal{X}_\sigma^\varepsilon}(0,1)$,

$$\begin{aligned} &\left\| \frac{\varepsilon^4 \widetilde{\kappa}\Big(\left(1 + \varepsilon^2 A^{(0)} + \varepsilon^2 A\right)^2 \Big) \Big(\left(\partial_1 A^{(0)} + \partial_1 A\right)^2 + \varepsilon^2 \left(\partial_2 A^{(0)} + \partial_2 A\right)^2 \Big)}{(1 + \varepsilon^2 A^{(0)} + \varepsilon^2 A)^2 \, \kappa\Big((1 + \varepsilon^2 A^{(0)} + \varepsilon^2 A)^2 \Big)} \right\|_{X^1_{2+\sigma'}} \\ &\qquad \leqslant C(\sigma, \sigma') \varepsilon^2 \Big(\varepsilon^2 + \varepsilon \|(\phi, A)\|_{\mathcal{X}_\sigma^\varepsilon} \Big) \end{aligned}$$

and then this does not change the computations of Subsections 4.5 and 4.6.

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