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BOUNDED WEAK SOLUTIONS TO A CLASS OF DEGENERATE CROSS-DIFFUSION SYSTEMS

SOLUTIONS FAIBLES BORNÉES POUR UNE CLASSE DE SYSTÈMES À DIFFUSION CROISÉE DÉGÉNÉRÉS

Abstract. — Bounded weak solutions are constructed for a degenerate parabolic system with a full diffusion matrix, which is a generalized version of the thin film Muskat system. Boundedness is achieved with the help of a sequence $(\mathcal{E}_n)_{n\geqslant 2}$ of Liapunov functionals such that $\mathcal{E}_n$ is equivalent to the L_n-norm for each $n \geqslant 2$ and $\mathcal{E}_n^{1/n}$ controls the L_∞-norm in the limit $n \to \infty$. Weak solutions are built by a compactness approach, special care being needed in the construction of the approximation in order to preserve the availability of the above-mentioned Liapunov functionals.

Résumé. — Des solutions faibles bornées sont construites pour un système parabolique dégénéré avec une matrice de diffusion pleine, qui est une version généralisée d'une approximation de type « film mince » du système de Muskat. Le caractère borné des solutions est obtenu à l'aide d'une suite $(\mathcal{E}_n)_{n\geqslant 2}$ de fonctionnelles de Liapunov avec les propriétés suivantes : $\mathcal{E}_n$ est équivalente à la norme L_n pour chaque $n \geqslant 2$ et $\mathcal{E}_n^{1/n}$ contrôle la norme L_∞

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dans la limite $n \to \infty$. Les solutions faibles sont construites par une méthode de compacité, la construction des approximations requérant une attention particulière afin d'être compatibles avec les fonctionnelles de Liapunov mentionnées ci-dessus.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geqslant 1$, with smooth boundary $\partial\Omega$ and let R and μ be two positive real numbers. In a recent paper [LM22], we noticed that there is an infinite family $(\mathcal{E}_n)_{n\geqslant 1}$ of Liapunov functionals associated with the thin film Muskat system

$$\begin{aligned} \partial_t f &= \operatorname{div}\left(f\nabla\left[(1+R)f + Rg\right]\right) \quad \text{in} \quad (0,\infty)\times\Omega\,, \\ \partial_t g &= \mu R \operatorname{div}\left(g\nabla\left[f+g\right]\right) \quad \text{in} \quad (0,\infty)\times\Omega\,, \end{aligned}$$

supplemented with homogeneous Neumann boundary conditions and initial conditions, with the following properties: for all $n \geqslant 2$, there are $0 < c_n < C_n$ such that

$$c_n\|f+g\|_n^n \leqslant \mathcal{E}_n(f,g) \leqslant C_n\|f+g\|_n^n\,, \qquad (f,g)\in L_{n,+}\left(\Omega,\mathbb{R}^2\right)\,,$$

and there are $0 < c_\infty < C_\infty$ such that

$$c_\infty\|f+g\|_\infty \leqslant \liminf_{n\to\infty}\mathcal{E}_n(f,g)^{1/n} \leqslant \limsup_{n\to\infty}\mathcal{E}_n(f,g)^{1/n} \leqslant C_\infty\|f+g\|_\infty$$

for $(f,g)\in L_{\infty,+}(\Omega,\mathbb{R}^2)$, where $L_{p,+}(\Omega,\mathbb{R}^m)$ denotes the positive cone of $L_p(\Omega,\mathbb{R}^m)$ for $m \geqslant 1$ and $p\in[1,\infty]$.

On the one hand, the thin film Muskat system being of cross-diffusion type (i.e., featuring a diffusion matrix with no zero entry), the availability of such a family of Liapunov functionals is rather seldom within this class of systems and paves the way towards the construction of bounded weak solutions, a result that we were only able to show in one space dimension $N = 1$ in [LM22]. On the other hand, it is tempting to figure out whether this property is peculiar to the thin film Muskat system or extends to the generalization thereof

$$\partial_t f = \operatorname{div}\left(f\nabla\left[af+bg\right]\right) \quad \text{in} \quad (0,\infty)\times\Omega\,, \tag{1.1a}$$

$$\partial_t g = \operatorname{div}\left(g\nabla\left[cf+dg\right]\right) \quad \text{in} \quad (0,\infty)\times\Omega\,, \tag{1.1b}$$

with $(a,\,b,\,c,\,d)\in(0,\infty)^4$, supplemented with homogeneous Neumann boundary conditions

$$\nabla f\cdot\mathbf{n} = \nabla g\cdot\mathbf{n} = 0 \quad \text{on} \quad (0,\infty)\times\partial\Omega\,, \tag{1.1c}$$

and non-negative initial conditions

$$(f,g)(0) = \left(f^{in},g^{in}\right) \quad \text{in} \quad \Omega\,, \tag{1.1d}$$

which is proposed in [BGHP85, Section 4] to describe the dispersal of two interacting population species and is also a particular case of a model of interacting particles derived in [GS14]. Obviously, the thin film Muskat system is a particular case of (1.1a)-(1.1b), corresponding to the choice

$$(a,\,b,\,c,\,d) = (1+R,\,R,\,\mu R,\,\mu R)\,.$$

It is worth mentioning at this point that the existence of global weak solutions to several cross-diffusion systems relies on the availability of a Liapunov functional or an entropy and we refer to [Jün16, Chapter 4] and the references therein for results in that direction. In the most favourable cases, an a priori L_∞-bound can even be retrieved from the structure of the entropy functional, see [BDFPS10] for instance. In contrast, the cornerstone of our approach is the construction of countably infinitely many Liapunov functionals, leading to L_∞-bounds after performing a suitable limiting process. Our contribution is somewhat closer in spirit to [JM06], where an algorithmic method for the construction of Liapunov functionals is developed. Let us also mention [Mie23], where a system of two coupled degenerate parabolic equations (without cross-diffusion) is studied which also features an infinite family of Liapunov functionals. This family of functionals provides L_n-estimates for all $n \geqslant 1$, but no L_∞-bound as in [LM22] and herein.

Coming back to (1.1), the main result of this paper is to show that, for any quadruple (a, b, c, d) satisfying

$$(a, b, c, d) \in (0, \infty)^4 \qquad \text{and} \qquad ad > bc\,, \tag{1.2}$$

we can associate a countably infinite family of Liapunov functionals with (1.1) and prove the global existence of bounded non-negative weak solutions to (1.1), whatever the dimension $N \geqslant 1$. More precisely, given a quadruple (a, b, c, d) satisfying (1.2), we define a sequence $(\Phi_n)_{n\geqslant 1}$ of functions as follows. Setting $L(r) := r\ln r - r + 1 \geqslant 0$ for $r \geqslant 0$, we first define the function Φ_1 by the relation

$$\Phi_1(X) := L(X_1) + \frac{b^2}{ad} L(X_2)\,, \qquad X = (X_1, X_2) \in [0,\infty)^2\,. \tag{1.3}$$

Next, for each integer $n \geqslant 2$, let Φ_n be the homogeneous polynomial of degree n defined by

$$\Phi_n(X) := \sum_{j=0}^{n} a_{j,n} X_1^j X_2^{n-j}\,, \qquad X = (X_1, X_2) \in \mathbb{R}^2\,, \tag{1.4}$$

with $a_{0,n} := 1$ and

$$a_{j,n} := \binom{n}{j} \prod_{k=0}^{j-1} \frac{ak + c(n-k-1)}{bk + d(n-k-1)} > 0\,, \qquad 1 \leqslant j \leqslant n\,. \tag{1.5}$$

We then define, for $n \geqslant 1$, the functional

$$\mathcal{E}_n(u) := \int_\Omega \Phi_n(u(x))\,dx, \qquad u = (f,g) \in L_{\max\{2,n\},+}\left(\Omega, \mathbb{R}^2\right)\,. \tag{1.6}$$

We finally observe that (1.2) guarantees that

$$\Theta_1 := \frac{b(ad+bc)}{2ad} > 0 \qquad \text{and} \qquad \Theta_2 := \frac{b^2(ad-bc)(3ad+bc)}{4a^2d^2} > 0\,. \tag{1.7}$$

With this notation, the main result of this paper is the following:

Theorem 1.1. — *Assume* (1.2) *and let* $u^{in} := (f^{in}, g^{in}) \in L_{\infty,+}(\Omega, \mathbb{R}^2)$ *be given. Then, there is a bounded weak solution* $u = (f,g)$ *to* (1.1) *such that:*

(i) *for each* $T > 0$,

$$(f,g) \in L_{\infty,+}\left((0,T)\times\Omega,\mathbb{R}^2\right) \cap L_2\left((0,T),H^1\left(\Omega,\mathbb{R}^2\right)\right) \cap W_2^1\left((0,T),H^1\left(\Omega,\mathbb{R}^2\right)'\right)\,; \tag{1.8}$$

(ii) *for all* $\varphi \in H^1(\Omega)$ *and* $t \geqslant 0$,

$$\int_\Omega \left(f(t,x)-f^{in}(x)\right)\varphi(x)\,\mathrm{d}x + \int_0^t \int_\Omega f(s,x)\nabla[af+bg](s,x)\cdot\nabla\varphi(x)\,\mathrm{d}x\mathrm{d}s = 0 \tag{1.9a}$$

and

$$\int_\Omega \left(g(t,x)-g^{in}(x)\right)\varphi(x)\,\mathrm{d}x + \int_0^t \int_\Omega g(s,x)\nabla[cf+dg](s,x)\cdot\nabla\varphi(x)\,\mathrm{d}x\mathrm{d}s = 0\,; \tag{1.9b}$$

(iii) *for all* $t \geqslant 0$,

$$\mathcal{E}_1(u(t)) + \frac{1}{a}\int_0^t\int_\Omega \left[|\nabla(af+\Theta_1 g)|^2 + \Theta_2|\nabla g|^2\right](s,x)\,\mathrm{d}x\mathrm{d}s \leqslant \mathcal{E}_1(u^{in})\,, \tag{1.10}$$

where the positive constants Θ_1 *and* Θ_2 *are defined in* (1.7)*;*

(iv) *for all* $n \geqslant 2$ *and all* $t \geqslant 0$,

$$\mathcal{E}_n(u(t)) \leqslant \mathcal{E}_n(u^{in})\,; \tag{1.11}$$

(v) *for* $t \geqslant 0$,

$$\|f(t)+g(t)\|_\infty \leqslant \frac{d}{b}\frac{\max\{a,\,b\}}{\min\{c,\,d\}}\left\|f^{in}+g^{in}\right\|_\infty\,. \tag{1.12}$$

Let us first mention that Theorem 1.1 improves [LM22] in two directions: on the one hand, it shows that the structural properties (1.10), (1.11), and (1.12), uncovered there for the thin film Muskat system, are also available for the whole class (1.1). On the other hand, it provides the existence of non-negative bounded weak solutions to (1.1) in all space dimensions, a result which was only established in one space dimension in [LM22]. Theorem 1.1 may also be viewed as a partial extension of [GS14], where global weak solutions to (1.1) are constructed in space dimensions $N \in \{1,\,2,\,3\}$ when the coefficients a, b, c, d are non-negative bounded functions which satisfy a more restrictive condition than (1.2), namely the inequality $4ad-(b+c)^2 > \lambda$ for some positive constant λ. Whether the analysis performed below could be adapted to non-constant coefficients is yet unclear. Let us also mention that global weak solutions to the thin film Muskat system are also constructed in [AIJM18, BGB19, ELM11, LM13, LM17, ACCL19], but their boundedness is an open question, to which an affirmative answer is only provided in [BGB19]. The latter however requires some smallness condition on the initial data, in contrast to Theorem 1.1. Finally, the local well-posedness of the thin film Muskat system in the classical sense is investigated in [EMM12].

We next outline the main steps of the proof of Theorem 1.1. As in [LM22], the starting point is to notice that, introducing the mobility matrix

$$M(X) = (m_{jk}(X))_{1\leqslant j,k\leqslant 2} := \begin{pmatrix} aX_1 & bX_1 \\ cX_2 & dX_2 \end{pmatrix}, \qquad X=(X_1,X_2)\in\mathbb{R}^2, \tag{1.13}$$

and $u := (f,g)$, an alternative formulation of the system (1.1a)-(1.1b) is

$$\partial_t u = \sum_{i=1}^N \partial_i(M(u)\partial_i u) \ \text{ in } \ (0,\infty)\times\Omega\,. \tag{1.14}$$

Then, given $\Phi\in C^2(\mathbb{R}^2,\mathbb{R})$, it readily follows from (1.14), the homogeneous Neumann boundary conditions (1.1c), and the symmetry of the Hessian matrix $D^2\Phi$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Phi(u)\ \mathrm{d}x + \sum_{i=1}^N\int_\Omega \left\langle D^2\Phi(u)M(u)\partial_i u, \partial_i u\right\rangle\ \mathrm{d}x = 0\,, \tag{1.15}$$

where $\langle\cdot,\cdot\rangle$ stands for the scalar product on $\mathbb{R}^2$. As a straightforward consequence of (1.15) we note that $\int_\Omega \Phi(u)\ \mathrm{d}x$ is a Liapunov functional for (1.14) when the matrix $D^2\Phi(u)M(u)$ is positive semidefinite. We shall then show in Appendix A that, for all $n\geqslant 2$, it is possible to construct an homogeneous polynomial $\Phi_n\in\mathbb{R}[X_1,X_2]$ of degree n which is convex on $[0,\infty)^2$ and such that the matrix $D^2\Phi_n(X)M(X)$ is positive semidefinite for all $X\in[0,\infty)^2$. A closed form formula is actually available for the polynomial Φ_n, see (1.4) and (1.5).

We next construct weak solutions to (1.14) by a compactness method. It is here of utmost importance to construct approximations which do not alter the inequalities (1.15) for $\Phi=\Phi_n$ and $n\geqslant 1$. As a first step, it is well-known that implicit time discrete schemes are well-suited in that direction. Thus, given $\tau>0$, we shall first prove the existence of a sequence $(u_l^\tau)_{l\geqslant 0}$ which satisfies $u_0^\tau = u^{in} := (f^{in},g^{in})$ and, for $l\geqslant 0$,

$$u_{l+1}^\tau - \tau\sum_{i=1}^N \partial_i\Big(M(u_{l+1}^\tau)\partial_i u_{l+1}^\tau\Big) = u_l^\tau \quad \text{in} \quad \Omega\,, \tag{1.16}$$

supplemented with homogeneous Neumann boundary conditions. Furthermore, the sequence $(u_l^\tau)_{l\geqslant 0}$ has the property that, for $n\geqslant 1$ and $l\geqslant 0$,

$$\mathcal{E}_n\left(u_{l+1}^\tau\right) + \tau\sum_{i=1}^N\int_\Omega \left\langle D^2\Phi_n\left(u_{l+1}^\tau\right)M\left(u_{l+1}^\tau\right)\partial_i u_{l+1}^\tau, \partial_i u_{l+1}^\tau\right\rangle\ \mathrm{d}x \leqslant \mathcal{E}_n(u_l^\tau)\,, \tag{1.17}$$

so that the structural property (1.15) is indeed preserved by the time discrete scheme. The existence of a solution to (1.16) is achieved by a compactness method relying on an approximation of the matrix $M(\cdot)$ by bounded ones. This step is actually the more delicate one, as we have to construct matrices approximating $M(\cdot)$ which do not alter (1.17). To this end, a two-parameter approximation procedure is required and it is detailed in Section 2.2. The existence of a weak solution to (1.16) satisfying (1.17) is shown in Section 2.4, building upon preliminary and intermediate results established in Section 2.1 and Section 2.3.

Remark 1.2. — A common feature of system (1.1) is that it has, at least formally, a gradient flow structure for the functional $\mathcal{E}_2$ with respect to the 2-Wasserstein distance in the space $\mathcal{P}_2(\Omega,\mathbb{R}^2)$ of probability measures with finite second moments, as pointed out in [LM13, ACCL19] for the thin film Muskat system. In particular, there is a natural variational structure associated with (1.1) which is suitable to construct weak solutions. However, the connection between this variational structure and the whole family $(\mathcal{E}_n)_{n\geqslant 2}$ of Liapunov functionals is yet unclear.

Notation 1.3. — For $p\in[1,\infty]$, we denote the L_p-norm in $L_p(\Omega)$ by $\|\cdot\|_p$ and set

$$L_p\left(\Omega,\mathbb{R}^2\right) := L_p(\Omega)\times L_p(\Omega)\,, \quad H^1\left(\Omega,\mathbb{R}^2\right) := H^1(\Omega)\times H^1(\Omega)\,.$$

The positive cone of a Banach lattice E is denoted by E_+. The space of 2×2 real-valued matrices is denoted by $\mathbf{M}_2(\mathbb{R})$, while $\mathbf{Sym}_2(\mathbb{R})$ is the subset of $\mathbf{M}_2(\mathbb{R})$ consisting of symmetric matrices and $\mathbf{SPD}_2(\mathbb{R})$ is the set of symmetric and positive definite matrices in $\mathbf{M}_2(\mathbb{R})$. The positive part of a real number $r\in\mathbb{R}$ is given by $r_+ := \max\{r,0\}$ and, for $X=(X_1,X_2)\in\mathbb{R}^2$, we define the positive part of X componentwise; that is, $X_+ := (X_{1,+},X_{2,+})$. Finally, $\langle\cdot,\cdot\rangle$ is the scalar product on $\mathbb{R}^2$.

2. A time discrete scheme

In order to construct bounded non-negative global weak solutions to the evolution problem (1.1), we employ a compactness approach, paying special attention to preserve as much as possible the structural properties (1.10), (1.11), and (1.12) in the design of the approximation. It turns out that implicit time discrete schemes are well-suited for that purpose and we thus establish in this section the existence of solutions to the implicit time discrete scheme associated with (1.1), see (2.1a)-(2.1b).

Proposition 2.1. — *Given $\tau>0$ and $U=(F,G)\in L_{\infty,+}(\Omega,\mathbb{R}^2)$, there is a solution*

$$u=(f,g)\in H^1\left(\Omega,\mathbb{R}^2\right)\cap L_{\infty,+}\left(\Omega,\mathbb{R}^2\right)$$

to

$$\int_\Omega \Big(f\varphi+\tau f\nabla\left[af+bg\right]\cdot\nabla\varphi\Big)\,\mathrm{d}x = \int_\Omega F\varphi\,\mathrm{d}x\,,\quad \varphi\in H^1(\Omega)\,, \tag{2.1a}$$

$$\int_\Omega \Big(g\psi+\tau g\nabla\left[cf+dg\right]\cdot\nabla\psi\Big)\,\mathrm{d}x = \int_\Omega G\psi\,\mathrm{d}x\,,\quad \psi\in H^1(\Omega)\,, \tag{2.1b}$$

which also satisfies

$$\mathcal{E}_n(u)\leqslant\mathcal{E}_n(U)\qquad \text{for } n\geqslant 2 \tag{2.2}$$

and

$$\mathcal{E}_1(u)+\frac{\tau}{a}\int_\Omega\Big[|\nabla(af+\Theta_1 g)|^2+\Theta_2|\nabla g|^2\Big]\,\mathrm{d}x\leqslant\mathcal{E}_1(U)\,, \tag{2.3}$$

recalling that, see (1.7),

$$\Theta_1=\frac{b(ad+bc)}{2ad}>0 \quad \text{and}\quad \Theta_2=\frac{b^2(ad-bc)(3ad+bc)}{4a^2d^2}>0\,.$$

As already mentioned, several steps are involved in the proof of Proposition 2.1. We begin with the existence of bounded weak solutions to an auxiliary elliptic system which shares the same structure with (2.1), but has bounded coefficients instead of linearly growing ones, see Section 2.1. As a next step, we introduce in Section 2.2 the approximation to (2.1) which is derived from (2.1) by replacing the matrix $M(\cdot)$ defined in (1.13) by a suitable invertible and bounded matrix $M_\varepsilon^\rho(\cdot)$ with $(\varepsilon,\rho) \in (0,1)\times(1,\infty)$. We emphasize here once more that the matrix $M_\varepsilon^\rho(\cdot)$ is designed in such a way that the inequalities (2.2) and (2.3) are not significantly altered. Passing to the limit, first as $\rho\to\infty$, and then as $\varepsilon\to 0$, is performed in Section 2.3 and Section 2.4, respectively, this last step completing the proof of Proposition 2.1.

Throughout this section, C and $(C_l)_{l\geqslant 0}$ denote various positive constants depending only on N, Ω, and (a, b, c, d). Dependence upon additional parameters will be indicated explicitly.

2.1. An auxiliary elliptic system

Let $A=(a_{jk})_{1\leqslant j,k\leqslant 2}$ and $B=(b_{jk})_{1\leqslant j,k\leqslant 2}$ be chosen such that $A\in\mathbf{SPD}_2(\mathbb{R})$ and $B\in BC(\mathbb{R}^2,\mathbf{M}_2(\mathbb{R}))$, with $AB(X)\in\mathbf{SPD}_2(\mathbb{R})$ for all $X\in\mathbb{R}^2$. Moreover, we assume that there is $\delta_1>0$ such that

$$\langle AB(X)\xi,\xi\rangle \geqslant \delta_1|\xi|^2\,, \qquad (X,\xi)\in\mathbb{R}^2\times\mathbb{R}^2\,. \tag{2.4}$$

Since $A\in\mathbf{SPD}_2(\mathbb{R})$, there is also $\delta_2>0$ such that

$$\langle A\xi,\xi\rangle \geqslant \delta_2|\xi|^2\,, \qquad \xi\in\mathbb{R}^2\,. \tag{2.5}$$

Lemma 2.2. — *Given $\tau>0$ and $U=(U_1,U_2)\in L_2(\Omega,\mathbb{R}^2)$, there exists a solution $u=(u_1,u_2)\in H^1(\Omega,\mathbb{R}^2)$ to the nonlinear equation*

$$\int_\Omega \left[\langle u,v\rangle + \tau\sum_{i=1}^N \langle B(u)\partial_i u,\partial_i v\rangle\right]\,\mathrm{d}x = \int_\Omega \langle U,v\rangle\,\mathrm{d}x\,, \qquad v\in H^1\left(\Omega,\mathbb{R}^2\right)\,. \tag{2.6}$$

Additionally:

(i) *If*

$$\begin{aligned} &b_{11}(X)\geqslant b_{12}(X)=0\,, && X\in(-\infty,0)\times\mathbb{R}\,,\\ &b_{22}(X)\geqslant b_{21}(X)=0\,, && X\in\mathbb{R}\times(-\infty,0)\,, \end{aligned} \tag{2.7}$$

and if $U(x)\in[0,\infty)^2$ for a.a. $x\in\Omega$, then $u(x)\in[0,\infty)^2$ for a.a. $x\in\Omega$.

(ii) *If there exists $\rho>0$ such that*

$$\begin{aligned} &b_{11}(X)\geqslant b_{12}(X)=0\,, && X\in(\rho,\infty)\times\mathbb{R}\,,\\ &b_{22}(X)\geqslant b_{21}(X)=0\,, && X\in\mathbb{R}\times(\rho,\infty)\,, \end{aligned} \tag{2.8}$$

and if $\max\{U_1,U_2\}\leqslant\rho$ a.e. in Ω, then $\max\{u_1,u_2\}\leqslant\rho$ a.e. in Ω.

Proof. — The proof of Lemma 2.2 is rather classical and it is actually similar to that of [LM22, Lemma B.1]. We nevertheless sketch it below for the sake of completeness.

Step 1. — To set up a fixed point scheme, we consider $u \in L_2(\Omega, \mathbb{R}^2)$ and define a bilinear form b_u on $H^1(\Omega, \mathbb{R}^2)$ by

$$b_u(v, w) := \int_\Omega \left[\langle Av, w \rangle + \tau \sum_{i=1}^N \langle AB(u)\partial_i v, \partial_i w \rangle \right] \, \mathrm{d}x$$

for $(v, w) \in H^1(\Omega, \mathbb{R}^2) \times H^1(\Omega, \mathbb{R}^2)$. Owing to (2.4) and (2.5),

$$b_u(v, v) \geqslant \delta_0 \|v\|_{H^1}^2 \,, \qquad v \in H^1\left(\Omega, \mathbb{R}^2\right) \,, \tag{2.9}$$

where $\delta_0 := \min\{\tau\delta_1, \delta_2\}$, while the boundedness of B guarantees that

$$|b_u(v, w)| \leqslant b_* \|v\|_{H^1} \|w\|_{H^1} \,, \qquad (v, w) \in H^1\left(\Omega, \mathbb{R}^2\right) \times H^1\left(\Omega, \mathbb{R}^2\right) \,,$$

with

$$b_* := 2 \max_{1 \leqslant j,k \leqslant 2} \{|a_{jk}|\} \left(1 + 2\tau \max_{1 \leqslant j,k \leqslant 2} \left\{\|b_{jk}\|_\infty\right\}\right) \,.$$

We then infer from Lax–Milgram's theorem that there is a unique $\mathcal{V}[u] \in H^1(\Omega, \mathbb{R}^2)$ such that

$$b_u(\mathcal{V}[u], w) = \int_\Omega \langle AU, w \rangle \, \mathrm{d}x \,, \qquad w \in H^1\left(\Omega, \mathbb{R}^2\right) \,. \tag{2.10}$$

An immediate consequence of (2.9), (2.10) (with $w = \mathcal{V}[u]$), and Hölder's inequality is the following estimate:

$$\delta_0 \|\mathcal{V}[u]\|_{H^1}^2 \leqslant b_u\left(\mathcal{V}[u], \mathcal{V}[u]\right) \leqslant \|AU\|_2 \|\mathcal{V}[u]\|_2 \leqslant \|AU\|_2 \|\mathcal{V}[u]\|_{H^1} \,.$$

Hence

$$\|\mathcal{V}[u]\|_{H^1} \leqslant \frac{\|AU\|_2}{\delta_0} \,. \tag{2.11}$$

We next argue as in the proof of [LM22, Lemma B.1] to show that the map $\mathcal{V}$ is continuous and compact from $L_2(\Omega, \mathbb{R}^2)$ to itself, the proof relying on (2.11), the compactness of the embedding of $H^1(\Omega, \mathbb{R}^2)$ in $L_2(\Omega, \mathbb{R}^2)$, and the continuity and boundedness of B.

Consider now $\theta \in [0, 1]$ and a function $u \in L_2(\Omega, \mathbb{R}^2)$ satisfying $u = \theta\mathcal{V}[u]$. Then we have $u \in H^1(\Omega, \mathbb{R}^2)$ and, in view of (2.11),

$$\|u\|_2 = \theta \|\mathcal{V}[u]\|_2 \leqslant \|\mathcal{V}[u]\|_2 \leqslant \|\mathcal{V}[u]\|_{H^1} \leqslant \frac{\|AU\|_2}{\delta_0} \,.$$

Thanks to the above bound and the continuity and compactness properties of the map $\mathcal{V}$ in $L_2(\Omega, \mathbb{R}^2)$, we are in a position to apply Leray–Schauder's fixed point theorem, see [GT01, Theorem 11.3] for instance, and conclude that the map $\mathcal{V}$ has a fixed point $u \in L_2(\Omega, \mathbb{R}^2)$. Since $\mathcal{V}$ ranges in $H^1(\Omega, \mathbb{R}^2)$, the function u actually belongs to $H^1(\Omega, \mathbb{R}^2)$ and satisfies

$$b_u(u, w) = \int_\Omega \langle AU, w \rangle \, \mathrm{d}x \,, \qquad w \in H^1\left(\Omega, \mathbb{R}^2\right) \,.$$

Finally, given $v \in H^1(\Omega,\mathbb{R}^2)$, the function $w = A^{-1}v$ also belongs to $H^1(\Omega,\mathbb{R}^2)$ and we infer from the above identity and the symmetry of A that

$$\begin{aligned}\int_\Omega \langle U, v\rangle \,\mathrm{d}x = \int_\Omega \langle AU, w\rangle \,\mathrm{d}x = b_u(u,w) = b_u\left(u, A^{-1}v\right) \\ = \int_\Omega \left[\langle u, v\rangle + \tau \sum_{i=1}^N \langle B(u)\partial_i u, \partial_i v\rangle\right] \,\mathrm{d}x\,.\end{aligned}$$

We have thus constructed a weak solution $u \in H^1(\Omega,\mathbb{R}^2)$ to (2.6).

Step 2. — We now turn to the sign-preserving property (i) and assume $U(x) \in [0,\infty)^2$ for a.a. $x \in \Omega$. Let $u \in H^1(\Omega,\mathbb{R}^2)$ be a weak solution to (2.6) and set $\varphi := -u$. The function $(\varphi_{1,+},\varphi_{2,+})$ then belongs to $H^1(\Omega,\mathbb{R}^2)$ and it follows from (2.6) that

$$\begin{aligned}\int_\Omega \left[\varphi_1\varphi_{1,+} + \varphi_2\varphi_{2,+} + \tau\sum_{i=1}^N\sum_{j,k=1}^2 b_{jk}(u)\partial_i\varphi_k\partial_i(\varphi_{j,+})\right] \,\mathrm{d}x \\ = -\int_\Omega (U_1\varphi_{1,+} + U_2\varphi_{2,+}) \,\mathrm{d}x \leqslant 0\,.\end{aligned}\tag{2.12}$$

We now infer from (2.7) that, for $1 \leqslant i \leqslant N$,

$$\begin{aligned}b_{11}(u)\partial_i\varphi_1\partial_i\varphi_{1,+} &= b_{11}(u)\mathbf{1}_{(-\infty,0)}(u_1)|\partial_i u_1|^2 \geqslant 0\,,\\ b_{12}(u)\partial_i\varphi_2\partial_i\varphi_{1,+} &= b_{12}(u)\mathbf{1}_{(-\infty,0)}(u_1)\partial_i u_1\partial_i u_2 = 0\,,\\ b_{21}(u)\partial_i\varphi_1\partial_i\varphi_{2,+} &= b_{21}(u)\mathbf{1}_{(-\infty,0)}(u_2)\partial_i u_1\partial_i u_2 = 0\,,\\ b_{22}(u)\partial_i\varphi_2\partial_i\varphi_{2,+} &= b_{22}(u)\mathbf{1}_{(-\infty,0)}(u_2)|\partial_i u_2|^2 \geqslant 0\,,\end{aligned}$$

so that the second term on the left-hand side of (2.12) is non-negative. Consequently, (2.12) gives

$$\int_\Omega \left[|\varphi_{1,+}|^2 + |\varphi_{2,+}|^2\right] \,\mathrm{d}x \leqslant 0\,,$$

which implies that $\varphi_{1,+} = \varphi_{2,+} = 0$ a.e. in Ω. Hence, $u(x) \in [0,\infty)^2$ for a.a. $x \in \Omega$ as claimed.

Step 3. — It remains to prove (ii). We thus assume that $\max\{U_1,U_2\} \leqslant \rho$ a.e. in Ω and consider a weak solution $u \in H^1(\Omega,\mathbb{R}^2)$ to (2.6). As $v = ((u_1-\rho)_+,(u_2-\rho)_+)$ belongs to $H^1(\Omega,\mathbb{R}^2)$, we deduce from (2.6) that

$$\int_\Omega \left[\sum_{j=1}^2 (u_j - U_j)(u_j-\rho)_+ + \tau\sum_{i=1}^N\sum_{j,k=1}^2 b_{jk}(u)\partial_i u_k\partial_i(u_j-\rho)_+\right] \,\mathrm{d}x = 0\,.$$

On the one hand,

$$u_j - U_j \geqslant u_j - \rho \;\text{ a.e. in } \Omega\,, \qquad j = 1,2\,,$$

so that

$$(u_j - U_j)\,(u_j-\rho)_+ \geqslant (u_j-\rho)\,(u_j-\rho)_+ = (u_j-\rho)_+^2 \;\text{ a.e. in }\; \Omega\,, \qquad j=1,2\,.$$

On the other hand, we infer from (2.8) that, for $1 \leqslant i \leqslant N$,

$$\begin{aligned}
b_{11}(u)\partial_i u_1 \partial_i (u_1-\rho)_+ &= b_{11}(u)\mathbf{1}_{(\rho,\infty)}(u_1)|\partial_i u_1|^2 \geqslant 0\,, \\
b_{12}(u)\partial_i u_2 \partial_i (u_1-\rho)_+ &= b_{12}(u)\mathbf{1}_{(\rho,\infty)}(u_1)\partial_i u_1 \partial_i u_2 = 0\,, \\
b_{21}(u)\partial_i u_1 \partial_i (u_2-\rho)_+ &= b_{21}(u)\mathbf{1}_{(\rho,\infty)}(u_2)\partial_i u_1 \partial_i u_2 = 0\,, \\
b_{22}(u)\partial_i u_2 \partial_i (u_2-\rho)_+ &= b_{22}(u)\mathbf{1}_{(\rho,\infty)}(u_2)|\partial_i u_2|^2 \geqslant 0\,.
\end{aligned}$$

Therefore,

$$\sum_{j=1}^{2} \int_\Omega (u_j-\rho)_+^2 \,\mathrm{d}x \leqslant 0\,,$$

from which we deduce that $\max\{u_1,u_2\} \leqslant \rho$ a.e. in Ω. □

2.2. A regularised system

We now introduce the two-parameter approximation of (2.1) on which the subsequent analysis relies. Specifically, given $\rho > 1$, we define

$$\alpha_\rho(z) := \begin{cases} 0, & z \leqslant 0, \\ z, & 0 \leqslant z \leqslant \rho-1, \\ (\rho-1)(\rho-z), & \rho-1 \leqslant z \leqslant \rho, \\ 0, & z \geqslant \rho, \end{cases}$$

and observe that $\alpha_\rho \in BC(\mathbb{R})$ with

$$0 \leqslant \alpha_\rho(z) \leqslant \min\{\rho, z_+\}\,, \qquad z \in \mathbb{R}\,.$$

Next, for $\varepsilon \in (0,1)$ and $X \in \mathbb{R}^2$, we set

$$M_\varepsilon^\rho(X) = \left(m_{\varepsilon,jk}^\rho(X)\right)_{1\leqslant j,k\leqslant 2} := \varepsilon I_2 + \lambda_\varepsilon(X_+) M^\rho(X),$$

where

$$M^\rho(X) = \left(m_{jk}^\rho(X)\right)_{1\leqslant j,k\leqslant 2} := \begin{pmatrix} a\alpha_\rho(X_1) & b\alpha_\rho(X_1) \\ c\alpha_\rho(X_2) & d\alpha_\rho(X_2) \end{pmatrix}\,, \qquad X \in \mathbb{R}^2\,, \tag{2.13}$$

and

$$\lambda_\varepsilon(X) := \frac{2}{1+\exp\left[\varepsilon(X_1+X_2)\right]}\,, \qquad X \in \mathbb{R}^2\,.$$

Note that $(M^\rho)_{\rho>1}$ converges to M, defined in (1.13), locally uniformly in $[0,\infty)^2$ as $\rho \to \infty$, while $(\lambda_\varepsilon)_{\varepsilon\in(0,1)}$ converges to 1 locally uniformly in $\mathbb{R}^2$ as $\varepsilon \to 0$. In fact, for $R > 0$,

$$|\lambda_\varepsilon(X) - 1| \leqslant 2R\varepsilon\,, \qquad X \in [-R,R]^2\,. \tag{2.14}$$

The outcome of this section is that, given $\tau > 0$, $\varepsilon \in (0,1)$, $\varrho > 1$, and a function $U \in L_{\infty,+}(\Omega,\mathbb{R}^2)$, there is a weak solution $u_\varepsilon^\rho \in H^1(\Omega,\mathbb{R}^2) \cap L_{\infty,+}(\Omega,\mathbb{R}^2)$ to

$$u_\varepsilon^\rho - \tau \sum_{i=1}^{N} \partial_i\Big(M_\varepsilon^\rho(u_\varepsilon^\rho)\partial_i u_\varepsilon^\rho\Big) = U \quad \text{in} \quad \Omega\,,$$

which satisfies an appropriate weak version of (2.2), as stated below. The next lemma is actually the building block of the proof of Proposition 2.1.

Lemma 2.3. — *Given $\tau > 0$, $U = (F,G) \in L_{\infty,+}(\Omega,\mathbb{R}^2)$, $\varepsilon \in (0,1)$, and a real number $\rho \geqslant \max\{1,\|F\|_\infty,\|G\|_\infty\}$, there exists a weak solution $u_\varepsilon^\rho = (u_{\varepsilon,1}^\rho, u_{\varepsilon,2}^\rho)$ in $H^1(\Omega,\mathbb{R}^2)\cap L_{\infty,+}(\Omega,\mathbb{R}^2)$ to*

$$\int_\Omega \left[\langle u_\varepsilon^\rho, v\rangle + \tau\sum_{i=1}^N \left\langle M_\varepsilon^\rho(u_\varepsilon^\rho)\partial_i u_\varepsilon^\rho, \partial_i v\right\rangle\right]\,\mathrm{d}x = \int_\Omega \langle U,v\rangle\,\mathrm{d}x\,, \quad v\in H^1\left(\Omega,\mathbb{R}^2\right)\,, \tag{2.15}$$

which additionally satisfies

$$\max\left\{\|u_{\varepsilon,1}^\rho\|_\infty,\ \|u_{\varepsilon,2}^\rho\|_\infty\right\} \leqslant \rho, \tag{2.16}$$

$$\|u_\varepsilon^\rho\|_2 \leqslant C_0\|U\|_2, \tag{2.17}$$

$$\|\nabla u_\varepsilon^\rho\|_2 \leqslant \frac{C_1}{\sqrt{\tau\varepsilon}}\|U\|_2\,. \tag{2.18}$$

Moreover, given $n \geqslant 2$, there exists a constant $C(n)$ such that

$$\mathcal{E}_n(u_\varepsilon^\rho) \leqslant \tau C(n)\frac{\rho^{n-1}}{e^{\varepsilon\rho}}\|\nabla u_\varepsilon^\rho\|_2^2 + \mathcal{E}_n(U)\,. \tag{2.19}$$

Proof. — Let $\varepsilon\in(0,1)$ and $\rho\geqslant\max\{1,\|F\|_\infty,\|G\|_\infty\}$. To deduce the existence result stated in Lemma 2.3 from the already established Lemma 2.2, we first recast (2.15) in the form (2.6). First, owing to the definition of the function α_ρ, the matrix M_ε^ρ lies in $BC(\mathbb{R}^2,\mathbf{M}_2(\mathbb{R}))$ and satisfies

$$0\leqslant m_{\varepsilon,jk}^\rho(X)\leqslant \varepsilon + 2\rho\max\{a,\,b,\,c,\,d\}\,,\qquad 1\leqslant j,k\leqslant 2\,,\ X\in\mathbb{R}^2\,, \tag{2.20a}$$

as well as

$$\begin{aligned} m_{\varepsilon,11}^\rho(X) &\geqslant m_{\varepsilon,12}^\rho(X) = 0\,, && X\in(-\infty,0)\times\mathbb{R}\,,\\ m_{\varepsilon,22}^\rho(X) &\geqslant m_{\varepsilon,21}^\rho(X) = 0\,, && X\in\mathbb{R}\times(-\infty,0)\,. \end{aligned} \tag{2.20b}$$

and

$$\begin{aligned} m_{\varepsilon,11}^\rho(X) &\geqslant m_{\varepsilon,12}^\rho(X) = 0\,, && X\in(\rho,\infty)\times\mathbb{R}\,,\\ m_{\varepsilon,22}^\rho(X) &\geqslant m_{\varepsilon,21}^\rho(X) = 0\,, && X\in\mathbb{R}\times(\rho,\infty)\,. \end{aligned} \tag{2.20c}$$

Next, according to [DGJ97], it is natural to use the Hessian matrix of the convex function Φ_2 to symmetrize (2.15). We thus set

$$S := \frac{bd}{2}D^2\Phi_2 = \begin{pmatrix} ac & bc\\ bc & bd\end{pmatrix}$$

and observe that S is symmetric and positive definite by (1.2). In addition, for all $X\in\mathbb{R}^2$,

$$SM_\varepsilon^\rho(X) = \varepsilon S + \lambda_\varepsilon(X_+)SM^\rho(X)$$

with

$$SM^\rho(X) = \begin{pmatrix} a^2c\alpha_\rho(X_1) + bc^2\alpha_\rho(X_2) & abc\alpha_\rho(X_1) + bcd\alpha_\rho(X_2)\\ abc\alpha_\rho(X_1)+bcd\alpha_\rho(X_2) & b^2c\alpha_\rho(X_1) + bd^2\alpha_\rho(X_2)\end{pmatrix} \in \mathbf{Sym}_2(\mathbb{R})\,.$$

Since $\mathrm{tr}(SM^\rho(X)) \geqslant 0$ and

$$\det\left(SM^\rho(X)\right) = \det(S)\det\left(M^\rho(X)\right) = bc(ad-bc)^2\alpha_\rho(X_1)\alpha_\rho(X_2) \geqslant 0$$

by (1.2), the matrix $SM^\rho(X)$ is positive semidefinite, so that the matrix $SM_\varepsilon^\rho(X)$ belongs to $\mathbf{SPD}_2(\mathbb{R})$ for all $X \in \mathbb{R}^2$ with

$$\text{(2.20d)} \quad \langle SM_\varepsilon^\rho(X)\xi,\xi\rangle \geqslant \varepsilon\langle S\xi,\xi\rangle \geqslant \varepsilon\frac{\det(S)}{\mathrm{tr}(S)}|\xi|^2 = \varepsilon\frac{bc(ad-bc)}{ac+bd}|\xi|^2\,, \qquad \xi \in \mathbb{R}^2\,.$$

According to the properties (2.20), we are now in a position to apply Lemma 2.2 (with $A = S$ and $B = M_\varepsilon^\rho$) and deduce that there is $u_\varepsilon^\rho \in H^1(\Omega,\mathbb{R}^2) \cap L_{\infty,+}(\Omega,\mathbb{R}^2)$ which solves (2.15) and satisfies (2.16). Moreover, it follows from the integral identity (2.15) (with $v = Su_\varepsilon^\rho \in H^1(\Omega,\mathbb{R}^2)$), (2.20d), and the positive definiteness of S,

$$\langle S\xi,\xi\rangle \geqslant \frac{bc(ad-bc)}{ac+bd}|\xi|^2\,, \qquad \xi \in \mathbb{R}^2\,,$$

that

$$\begin{aligned}
\|SU\|_2\|u_\varepsilon^\rho\|_2 \geqslant \int_\Omega \langle SU, u_\varepsilon^\rho\rangle\ \mathrm{d}x &= \int_\Omega \left[\langle u_\varepsilon^\rho, Su_\varepsilon^\rho\rangle + \tau\sum_{i=1}^N \left\langle M_\varepsilon^\rho\left(u_\varepsilon^\rho\right)\partial_i u_\varepsilon^\rho, \partial_i S u_\varepsilon^\rho\right\rangle\right]\ \mathrm{d}x \\
&= \int_\Omega \left[\langle Su_\varepsilon^\rho, u_\varepsilon^\rho\rangle + \tau\sum_{i=1}^N \left\langle SM_\varepsilon^\rho\left(u_\varepsilon^\rho\right)\partial_i u_\varepsilon^\rho, \partial_i u_\varepsilon^\rho\right\rangle\right]\ \mathrm{d}x \\
&\geqslant \frac{bc(ad-bc)}{ac+bd}\left(\|u_\varepsilon^\rho\|_2^2 + \tau\varepsilon\,\|\nabla u_\varepsilon^\rho\|_2^2\right)\,.
\end{aligned}$$

Owing to (1.2), we conclude that the estimates (2.17) and (2.18) are satisfied.

It remains to establish the estimate (2.19). Let therefore $n \geqslant 2$. Since u_ε^ρ belongs to $H^1(\Omega,\mathbb{R}^2) \cap L_\infty(\Omega,\mathbb{R}^2)$, the vector field $D\Phi_n(u_\varepsilon^\rho)$ lies in $H^1(\Omega,\mathbb{R}^2)$ and we infer from (2.15) (with $v = D\Phi_n(u_\varepsilon^\rho)$) that

$$\text{(2.21)} \qquad \int_\Omega \left[\left\langle u_\varepsilon^\rho - U, D\Phi_n(u_\varepsilon^\rho)\right\rangle + \tau\sum_{i=1}^N \left\langle M_\varepsilon^\rho(u_\varepsilon^\rho)\partial_i u_\varepsilon^\rho, \partial_i D\Phi_n(u_\varepsilon^\rho)\right\rangle\right]\ \mathrm{d}x = 0\,.$$

On the one hand, the convexity of Φ_n implies that

$$\text{(2.22)} \quad \int_\Omega \left\langle u_\varepsilon^\rho - U, D\Phi_n(u_\varepsilon^\rho)\right\rangle\ \mathrm{d}x \geqslant \int_\Omega \left[\Phi_n(u_\varepsilon^\rho) - \Phi_n(U)\right]\ \mathrm{d}x = \mathcal{E}_n(u_\varepsilon^\rho) - \mathcal{E}_n(U)\,.$$

On the other hand, using the symmetry and the positive semidefiniteness of the matrix $D^2\Phi_n(u_\varepsilon^\rho)$, see Lemma A.2, we have

$$\begin{aligned}
&\tau\sum_{i=1}^N\int_\Omega\Big\langle M_\varepsilon^\rho(u_\varepsilon^\rho)\partial_i u_\varepsilon^\rho, \partial_i D\Phi_n(u_\varepsilon^\rho)\Big\rangle\,\mathrm{d}x\\
&\quad= \tau\sum_{i=1}^N\int_\Omega\Big\langle M_\varepsilon^\rho(u_\varepsilon^\rho)\partial_i u_\varepsilon^\rho, D^2\Phi_n(u_\varepsilon^\rho)\partial_i u_\varepsilon^\rho\Big\rangle\,\mathrm{d}x\\
&\quad= \tau\sum_{i=1}^N\int_\Omega\Big\langle D^2\Phi_n(u_\varepsilon^\rho)M_\varepsilon^\rho(u_\varepsilon^\rho)\partial_i u_\varepsilon^\rho, \partial_i u_\varepsilon^\rho\Big\rangle\,\mathrm{d}x\\
&\quad= \tau\varepsilon\sum_{i=1}^N\int_\Omega\Big\langle D^2\Phi_n(u_\varepsilon^\rho)\partial_i u_\varepsilon^\rho, \partial_i u_\varepsilon^\rho\Big\rangle\,\mathrm{d}x\\
&\qquad+\tau\sum_{i=1}^N\int_\Omega\lambda_\varepsilon(u_\varepsilon^\rho)\Big\langle D^2\Phi_n(u_\varepsilon^\rho)M^\rho(u_\varepsilon^\rho)\partial_i u_\varepsilon^\rho, \partial_i u_\varepsilon^\rho\Big\rangle\,\mathrm{d}x\\
&\quad\geqslant \tau\sum_{i=1}^N\int_\Omega\lambda_\varepsilon(u_\varepsilon^\rho)\Big\langle D^2\Phi_n(u_\varepsilon^\rho)M^\rho(u_\varepsilon^\rho)\partial_i u_\varepsilon^\rho, \partial_i u_\varepsilon^\rho\Big\rangle\,\mathrm{d}x\,.
\end{aligned}\tag{2.23}$$

Since $S_n(u_\varepsilon^\rho) := D^2\Phi_n(u_\varepsilon^\rho)M(u_\varepsilon^\rho)$ is positive semidefinite by Lemma A.3, we further have

$$\begin{aligned}
&\tau\sum_{i=1}^N\int_\Omega\lambda_\varepsilon(u_\varepsilon^\rho)\Big\langle D^2\Phi_n(u_\varepsilon^\rho)M^\rho(u_\varepsilon^\rho)\partial_i u_\varepsilon^\rho, \partial_i u_\varepsilon^\rho\Big\rangle\,\mathrm{d}x\\
&\quad= \tau\sum_{i=1}^N\int_\Omega\lambda_\varepsilon(u_\varepsilon^\rho)\Big\langle D^2\Phi_n(u_\varepsilon^\rho)M(u_\varepsilon^\rho)\partial_i u_\varepsilon^\rho, \partial_i u_\varepsilon^\rho\Big\rangle\,\mathrm{d}x\\
&\qquad+\tau\sum_{i=1}^N\int_\Omega\lambda_\varepsilon(u_\varepsilon^\rho)\Big\langle D^2\Phi_n(u_\varepsilon^\rho)\Big[M^\rho(u_\varepsilon^\rho)-M(u_\varepsilon^\rho)\Big]\partial_i u_\varepsilon^\rho, \partial_i u_\varepsilon^\rho\Big\rangle\,\mathrm{d}x\\
&\quad\geqslant \tau\sum_{i=1}^N\int_\Omega\lambda_\varepsilon(u_\varepsilon^\rho)\Big\langle D^2\Phi_n(u_\varepsilon^\rho)\Big[M^\rho(u_\varepsilon^\rho)-M(u_\varepsilon^\rho)\Big]\partial_i u_\varepsilon^\rho, \partial_i u_\varepsilon^\rho\Big\rangle\,\mathrm{d}x\,.
\end{aligned}\tag{2.24}$$

Taking now advantage of the fact that $0\leqslant u_{\varepsilon,j}^\rho\leqslant\rho$ a.e. in Ω for $j=1,\,2$ by (2.16), we further have

$$\begin{aligned}
&\left|\tau\sum_{i=1}^N\int_\Omega\lambda_\varepsilon(u_\varepsilon^\rho)\Big\langle D^2\Phi_n(u_\varepsilon^\rho)\Big[M^\rho(u_\varepsilon^\rho)-M(u_\varepsilon^\rho)\Big]\partial_i u_\varepsilon^\rho, \partial_i u_\varepsilon^\rho\Big\rangle\,\mathrm{d}x\right|\\
&\quad\leqslant 2\tau\max\{a,\,b,\,c,\,d\}\left\|D^2\Phi_n\right\|_{L_\infty((0,\rho)^2)}\sum_{j=1}^2\int_\Omega\lambda_\varepsilon(u_\varepsilon^\rho)\left|\alpha_\rho\left(u_{\varepsilon,j}^\rho\right)-u_{\varepsilon,j}^\rho\right|\left|\nabla u_\varepsilon^\rho\right|^2\,\mathrm{d}x\\
&\quad\leqslant 8\tau\max\{a,\,b,\,c,\,d\}\kappa_n\rho^{n-2}\sum_{j=1}^2\int_{\left\{\rho-1\leqslant u_{\varepsilon,j}^\rho\leqslant\rho\right\}}\frac{\left|\alpha_\rho\left(u_{\varepsilon,j}^\rho\right)-u_{\varepsilon,j}^\rho\right|}{1+\exp\left(\varepsilon u_{\varepsilon,j}^\rho\right)}\left|\nabla u_\varepsilon^\rho\right|^2\,\mathrm{d}x\,,
\end{aligned}$$

where $\kappa_n \in \mathbb{R}$ is a positive constant such that

$$\left|D^2\Phi_n(X)\right| \leqslant \kappa_n \left(X_1^{n-2} + X_2^{n-2}\right) \qquad \text{for all } X \in [0,\infty)^2.$$

Owing to the definition of α_ρ, we further obtain

$$\begin{aligned}(2.25)\quad &\left|\tau \sum_{i=1}^N \int_\Omega \lambda_\varepsilon(u_\varepsilon^\rho) \left\langle D^2\Phi_n(u_\varepsilon^\rho)\Big[M^\rho(u_\varepsilon^\rho) - M(u_\varepsilon^\rho)\Big]\partial_i u_\varepsilon^\rho, \partial_i u_\varepsilon^\rho\right\rangle\ \mathrm{d}x\right| \\ &\qquad \leqslant 8\tau \max\{a,\, b,\, c,\, d\}\kappa_n \rho^{n-2} \sum_{j=1}^2 \int_{\left\{\rho-1 \leqslant u_{\varepsilon,j}^\rho \leqslant \rho\right\}} \frac{\rho}{1+e^{\varepsilon(\rho-1)}} |\nabla u_\varepsilon^\rho|^2\ \mathrm{d}x \\ &\qquad \leqslant 16e\tau \max\{a,\, b,\, c,\, d\}\kappa_n \rho^{n-1} e^{-\varepsilon\rho} \left\|\nabla u_\varepsilon^\rho\right\|_2^2 .\end{aligned}$$

The desired estimate (2.19) is now a straightforward consequence of (2.21)-(2.25). □

2.3. A regularised system: $\rho \to \infty$

We next study the cluster points of the family $\{u_\varepsilon^\rho : \rho \geqslant \max\{1, \|F\|_\infty, \|G\|_\infty\}\}$ provided in Lemma 2.3, as $\rho \to \infty$, the parameter $\varepsilon \in (0,1)$ being held fixed.

Lemma 2.4. — *Given $\tau > 0$, $U = (F,G) \in L_{\infty,+}(\Omega,\mathbb{R}^2)$, and $\varepsilon \in (0,1)$, there exist a sequence $(\rho_l)_{l\geqslant 1}$ and a function $u_\varepsilon = (u_{\varepsilon,1}, u_{\varepsilon,2}) \in H^1(\Omega,\mathbb{R}^2) \cap L_{\infty,+}(\Omega,\mathbb{R}^2)$ such that $\rho_l \to \infty$ and*

$$(2.26)\qquad u_\varepsilon^{\rho_l} \to u_\varepsilon \qquad \text{in } L_p\left(\Omega,\mathbb{R}^2\right) \text{ for all } p \in [1,\infty) \text{ and pointwise a.e. in } \Omega\,,$$

$$(2.27)\qquad \nabla u_\varepsilon^{\rho_l} \rightharpoonup \nabla u_\varepsilon \quad \text{in } L_2\left(\Omega,\mathbb{R}^{2N}\right) .$$

Moreover, u_ε solves the equation

$$\begin{aligned}(2.28)\qquad \int_\Omega \left[\langle u_\varepsilon, v\rangle + \tau\sum_{i=1}^N \Big\langle M_\varepsilon(u_\varepsilon)\partial_i u_\varepsilon, \partial_i v\Big\rangle\right] \mathrm{d}x \\ = \int_\Omega \langle U, v\rangle\ \mathrm{d}x\,, \quad v \in H^1\left(\Omega,\mathbb{R}^2\right) ,\end{aligned}$$

where

$$M_\varepsilon(X) = (m_{\varepsilon,jk}(X))_{1\leqslant j,k\leqslant 2} := \varepsilon I_2 + \lambda_\varepsilon(X_+)M(X),$$

with $M(X)$ defined in (1.13), and, for each $n \geqslant 2$, we have

$$(2.29)\qquad \mathcal{E}_n(u_\varepsilon) \leqslant \mathcal{E}_n(U)\,.$$

Furthermore,

$$(2.30)\qquad \min\left\{1, \frac{c}{d}\right\} \|u_{\varepsilon,1} + u_{\varepsilon,2}\|_\infty \leqslant \max\left\{1, \frac{a}{b}\right\} \|F+G\|_\infty\,.$$

Proof. — Recalling (2.17)-(2.18), we deduce that $(u_\varepsilon^\rho)_\rho$ is bounded in $H^1(\Omega,\mathbb{R}^2)$. Moreover, since

$$(2.31)\qquad \frac{\varepsilon^n z^n}{n!} \leqslant e^{\varepsilon z}\,, \qquad z \in [0,\infty)\,, \quad n \geqslant 1\,,$$

the estimates (2.18) and (2.19), along with Lemma A.4, ensure that $(u_\varepsilon^\rho)_\rho$ is bounded in $L_n(\Omega,\mathbb{R}^2)$ for any integer $n \geqslant 2$ (with an ε-dependent bound). We may then use a Cantor diagonal process, together with Rellich–Kondrachov' theorem and an interpolation argument, to deduce the convergences (2.26) and (2.27) along a sequence $\rho_l \to \infty$, as well as the componentwise non-negativity of u_ε.

Since Φ_n is convex on $[0,\infty)^2$ for all $n \geqslant 2$, see Lemma A.2, it follows from the relations (2.18), (2.19), (2.26), and (2.31) that (2.29) holds true. Using once more Lemma A.4, we infer from (2.29) that

$$\|cu_{\varepsilon,1} + du_{\varepsilon,2}\|_n \leqslant \frac{d}{b}\|aF + bG\|_n$$

for all $n \geqslant 2$. Passing to the limit $n \to \infty$ in the above inequality, we deduce that the function $u_\varepsilon \in L_\infty(\Omega,\mathbb{R}^2)$ satisfies (2.30).

Let us now consider $v \in H^1(\Omega,\mathbb{R}^2)$. Since (2.26) and (2.27) imply that

$$\lim_{l\to\infty}\int_\Omega \langle u_\varepsilon^{\rho_l}, v\rangle \,\mathrm{d}x = \int_\Omega \langle u_\varepsilon, v\rangle \,\mathrm{d}x \quad \text{and} \quad \lim_{l\to\infty}\int_\Omega \langle \partial_i u_\varepsilon^{\rho_l}, \partial_i v\rangle \,\mathrm{d}x = \int_\Omega \langle \partial_i u_\varepsilon, \partial_i v\rangle \,\mathrm{d}x$$

for $1 \leqslant i \leqslant N$, the identity (2.28) is satisfied provided that

$$\lim_{l\to\infty}\int_\Omega \lambda_\varepsilon(u_\varepsilon^{\rho_l})\Big\langle M^{\rho_l}(u_\varepsilon^{\rho_l})\partial_i u_\varepsilon^{\rho_l}, \partial_i v\Big\rangle \,\mathrm{d}x = \int_\Omega \lambda_\varepsilon(u_\varepsilon)\,\langle M(u_\varepsilon)\partial_i u_\varepsilon, \partial_i v\rangle \,\mathrm{d}x \tag{2.32}$$

for each $1 \leqslant i \leqslant N$. To prove (2.32), we observe that, for $1 \leqslant i \leqslant N$ and $j \in \{1,2\}$,

$$\int_\Omega \lambda_\varepsilon(u_\varepsilon^{\rho_l})\Big\langle M^{\rho_l}(u_\varepsilon^{\rho_l})\partial_i u_\varepsilon^{\rho_l}, \partial_i v\Big\rangle \,\mathrm{d}x = \int_\Omega \lambda_\varepsilon(u_\varepsilon^{\rho_l})\,\Big\langle M^{\rho_l}(u_\varepsilon^{\rho_l})^t\partial_i v, \partial_i u_\varepsilon^{\rho_l}\Big\rangle \,\mathrm{d}x \tag{2.33}$$

with

$$\begin{aligned}\left|\lambda_\varepsilon(u_\varepsilon^{\rho_l})\sum_{k=1}^2 m_{kj}^{\rho_l}(u_\varepsilon^{\rho_l})\partial_i v_k\right| &\leqslant 2\max\{a,b,c,d\}\frac{u_{\varepsilon,1}^{\rho_l}+u_{\varepsilon,2}^{\rho_l}}{1+\exp\left[\varepsilon\left(u_{\varepsilon,1}^{\rho_l}+u_{\varepsilon,2}^{\rho_l}\right)\right]}|\partial_i v| \\ &\leqslant \frac{2\max\{a,b,c,d\}}{\varepsilon}|\partial_i v| \qquad \text{a.e. in } \Omega\,,\end{aligned}$$

by the definition of λ_ε and (2.31) (with $n=1$), and

$$\lim_{l\to\infty}\lambda_\varepsilon(u_\varepsilon^{\rho_l})\sum_{k=1}^2 m_{kj}^{\rho_l}(u_\varepsilon^{\rho_l})\partial_i v_k = \lambda_\varepsilon(u_\varepsilon)\sum_{k=1}^2 m_{kj}(u_\varepsilon)\partial_i v_k \quad \text{a.e. in } \Omega\,,$$

by (2.13), the pointwise almost everywhere convergence in Ω established in (2.26), and the properties of α_{ρ_l}. Lebesgue's dominated convergence theorem then guarantees that

$$\lim_{l\to\infty}\left\|\lambda_\varepsilon(u_\varepsilon^{\rho_l})\sum_{k=1}^2 m_{kj}^{\rho_l}(u_\varepsilon^{\rho_l})\partial_i v_k - \lambda_\varepsilon(u_\varepsilon)\sum_{k=1}^2 m_{kj}(u_\varepsilon)\partial_i v_k\right\|_2 = 0\,.$$

Combining the above convergence with (2.27) allows us to pass to the limit $l \to \infty$ in (2.33) and find

$$\begin{aligned}\lim_{l\to\infty}\int_\Omega \lambda_\varepsilon(u_\varepsilon^{\rho_l})\,\langle M^{\rho_l}(u_\varepsilon^{\rho_l})\partial_i u_\varepsilon^{\rho_l}, \partial_i v\rangle \,\mathrm{d}x &= \int_\Omega \lambda_\varepsilon(u_\varepsilon)\,\Big\langle M(u_\varepsilon)^t\partial_i v, \partial_i u_\varepsilon\Big\rangle \,\mathrm{d}x \\ &= \int_\Omega \lambda_\varepsilon(u_\varepsilon)\,\langle M(u_\varepsilon)\partial_i u_\varepsilon, \partial_i v\rangle \,\mathrm{d}x\end{aligned}$$

for $1 \leqslant i \leqslant N$, which proves (2.32). We have thus shown that u_ε solves (2.28) and thereby completed the proof of Lemma 2.4. □

We next show that the entropy functional $\mathcal{E}_1$ evaluated at the function u_ε identified in Lemma 2.4 is dominated by $\mathcal{E}_1(U)$ and that the associated dissipation term $\mathcal{E}_1(U) - \mathcal{E}_1(u_\varepsilon)$ provides a control on the gradient of u_ε which is essential when considering the limit $\varepsilon \to 0$.

Lemma 2.5. — *Let $\tau > 0$, $U = (F, G) \in L_{\infty,+}(\Omega, \mathbb{R}^2)$, and $\varepsilon \in (0,1)$. The function*

$$u_\varepsilon = (u_{\varepsilon,1}, u_{\varepsilon,2}) \in H^1\left(\Omega, \mathbb{R}^2\right) \cap L_{\infty,+}\left(\Omega, \mathbb{R}^2\right)$$

identified in Lemma 2.4 satisfies

$$\mathcal{E}_1(u_\varepsilon) + \frac{\tau}{a}\int_\Omega \lambda_\varepsilon(u_\varepsilon)\Big[\left|\nabla(au_{\varepsilon,1} + \Theta_1 u_{\varepsilon,2})\right|^2 + \Theta_2\left|\nabla u_{\varepsilon,2}\right|^2\Big]\,\mathrm{d}x \leqslant \mathcal{E}_1(U)\,.$$

Proof. — Let $\eta \in (0,1)$. Then $(\ln(u_{\varepsilon,1}+\eta), (b^2/ad)\ln(u_{\varepsilon,2}+\eta)) \in H^1(\Omega,\mathbb{R}^2)$ and we infer from (2.28) that

$$0 = \int_\Omega \left[(u_{\varepsilon,1} - U_1)\ln(u_{\varepsilon,1}+\eta) + \frac{b^2}{ad}(u_{\varepsilon,2} - U_2)\ln(u_{\varepsilon,2}+\eta)\right]\,\mathrm{d}x + D(\eta)\,, \tag{2.34}$$

where

$$\begin{aligned} D(\eta) &:= \tau\int_\Omega \sum_{i=1}^N \Big(m_{\varepsilon,11}(u_\varepsilon)\partial_i u_{\varepsilon,1} + m_{\varepsilon,12}(u_\varepsilon)\partial_i u_{\varepsilon,2}\Big)\frac{\partial_i u_{\varepsilon,1}}{u_{\varepsilon,1}+\eta}\,\mathrm{d}x \\ &\quad + \frac{\tau b^2}{ad}\int_\Omega \sum_{i=1}^N \Big(m_{\varepsilon,21}(u_\varepsilon)\partial_i u_{\varepsilon,1} + m_{\varepsilon,22}(u_\varepsilon)\partial_i u_{\varepsilon,2}\Big)\frac{\partial_i u_{\varepsilon,2}}{u_{\varepsilon,2}+\eta}\,\mathrm{d}x\,. \end{aligned}$$

Since $L(r) = r\ln r - r + 1$ is convex on $[0,\infty)$ with $L'(r) = \ln r$, the first term on the right-hand side of (2.34) can be estimated as follows:

$$\begin{aligned} &\int_\Omega \left[(u_{\varepsilon,1} - U_1)\ln(u_{\varepsilon,1}+\eta) + \frac{b^2}{ad}(u_{\varepsilon,2} - U_2)\ln(u_{\varepsilon,2}+\eta)\right]\,\mathrm{d}x \\ &\qquad \geqslant \int_\Omega \left[\Big(L(u_{\varepsilon,1}+\eta) - L(U_1+\eta)\Big) + \frac{b^2}{ad}\Big(L(u_{\varepsilon,2}+\eta) - L(U_2+\eta)\Big)\right]\,\mathrm{d}x \\ &\qquad = \mathcal{E}_1((u_{\varepsilon,1}+\eta, u_{\varepsilon,2}+\eta)) - \mathcal{E}_1((U_1+\eta, U_2+\eta))\,. \end{aligned}$$

Using the continuity of Φ_1 and the boundedness of u_ε, see (2.30), we deduce that

$$\begin{aligned} \liminf_{\eta\to 0} \int_\Omega \left[(u_{\varepsilon,1} - U_1)\ln(u_{\varepsilon,1}+\eta) + \frac{b^2}{ad}(u_{\varepsilon,2} - U_2)\ln(u_{\varepsilon,2}+\eta)\right]\,\mathrm{d}x& \\ \geqslant \mathcal{E}_1(u_\varepsilon) - \mathcal{E}_1(U)\,.& \end{aligned} \tag{2.35}$$

Next, recalling the definition of the matrix M_ε, see Lemma 2.4, we have

$$\begin{aligned}
D(\eta) &= \tau\varepsilon \int_\Omega \left(\frac{|\nabla u_{\varepsilon,1}|^2}{u_{\varepsilon,1}+\eta} + \frac{b^2}{ad}\frac{|\nabla u_{\varepsilon,2}|^2}{u_{\varepsilon,2}+\eta} \right)\, \mathrm{d}x \\
&\quad + \tau \int_\Omega \lambda_\varepsilon(u_\varepsilon) \left(\frac{u_{\varepsilon,1}}{u_{\varepsilon,1}+\eta} - 1 + 1 \right) \nabla u_{\varepsilon,1} \cdot \nabla(a u_{\varepsilon,1} + b u_{\varepsilon,2})\, \mathrm{d}x \\
&\quad + \frac{\tau b^2}{ad} \int_\Omega \lambda_\varepsilon(u_\varepsilon) \left(\frac{u_{\varepsilon,2}}{u_{\varepsilon,2}+\eta} - 1 + 1 \right) \nabla u_{\varepsilon,2} \cdot \nabla(c u_{\varepsilon,1} + d u_{\varepsilon,2})\, \mathrm{d}x \\
&= \tau\varepsilon \int_\Omega \left(\frac{|\nabla u_{\varepsilon,1}|^2}{u_{\varepsilon,1}+\eta} + \frac{b^2}{ad}\frac{|\nabla u_{\varepsilon,2}|^2}{u_{\varepsilon,2}+\eta} \right)\, \mathrm{d}x \\
&\quad + \frac{\tau}{a} \int_\Omega \lambda_\varepsilon(u_\varepsilon) \Big[|\nabla(a u_{\varepsilon,1} + \Theta_1 u_{\varepsilon,2})|^2 + \Theta_2 |\nabla u_{\varepsilon,2}|^2 \Big]\, \mathrm{d}x \\
&\quad - J_1(\eta) - J_2(\eta)\,,
\end{aligned}$$

where

$$\begin{aligned}
J_1(\eta) &:= \tau \int_\Omega \frac{\eta \lambda_\varepsilon(u_\varepsilon)}{u_{\varepsilon,1}+\eta} \nabla u_{\varepsilon,1} \cdot \nabla(a u_{\varepsilon,1} + b u_{\varepsilon,2})\, \mathrm{d}x\,, \\
J_2(\eta) &:= \frac{\tau b^2}{ad} \int_\Omega \frac{\eta \lambda_\varepsilon(u_\varepsilon)}{u_{\varepsilon,2}+\eta} \nabla u_{\varepsilon,2} \cdot \nabla(c u_{\varepsilon,1} + d u_{\varepsilon,2})\, \mathrm{d}x\,.
\end{aligned}$$

Since $u_\varepsilon \in H^1(\Omega,\mathbb{R}^2)$ satisfies $\nabla u_{\varepsilon,j} = 0$ a.e. on the level set $\{x \in \Omega \ :\ u_{\varepsilon,j} = 0\}$ for $j \in \{1,2\}$, we have

$$\begin{aligned}
&\lim_{\eta\to 0} \frac{\eta \lambda_\varepsilon(u_\varepsilon)}{u_{\varepsilon,j}+\eta} \nabla u_{\varepsilon,j} = 0 \qquad \text{a.e. in } \Omega\,, \\
&\left| \frac{\eta \lambda_\varepsilon(u_\varepsilon)}{u_{\varepsilon,j}+\eta} \nabla u_{\varepsilon,j} \right| \leqslant |\nabla u_{\varepsilon,j}| \qquad \text{a.e. in } \Omega\,.
\end{aligned}$$

Lebesgue's dominated convergence theorem ensures now that

$$\lim_{\eta\to 0} \left(J_1(\eta) + J_2(\eta) \right) = 0\,.$$

This shows that

$$\liminf_{\eta\to 0} D(\eta) \geqslant \frac{\tau}{a} \int_\Omega \lambda_\varepsilon(u_\varepsilon) \Big[\left|\nabla\left(a u_{\varepsilon,1} + \Theta_1 u_{\varepsilon,2}\right)\right|^2 + \Theta_2 \left|\nabla u_{\varepsilon,2}\right|^2 \Big]\, \mathrm{d}x\,. \tag{2.36}$$

Passing to the limit $\eta \to 0$ in (2.34), we get the desired estimate in view of (2.35) and (2.36). $\square$

2.4. A regularised system: $\varepsilon \to 0$

We complete this section with the proof of Proposition 2.1.

Proof of Proposition 2.1. — Consider $\tau > 0$ and $U = (F,G) \in L_{\infty,+}(\Omega,\mathbb{R}^2)$. Given $\varepsilon \in (0,1)$, let

$$u_\varepsilon = (u_{\varepsilon,1}, u_{\varepsilon,2}) \in H^1\left(\Omega,\mathbb{R}^2\right) \cap L_{\infty,+}\left(\Omega,\mathbb{R}^2\right)$$

denote the weak solution to (2.28) provided by Lemma 2.4. According to (2.30),

$$\max\{\|u_{\varepsilon,1}\|_\infty, \|u_{\varepsilon,2}\|_\infty\} \leqslant \|u_{\varepsilon,1} + u_{\varepsilon,2}\|_\infty \leqslant R_0 := \frac{d}{b}\frac{\max\{a,b\}}{\min\{c,d\}}\|F+G\|_\infty \,. \tag{2.37}$$

Hence,

$$\lambda_\varepsilon(u_\varepsilon) \geqslant \frac{2}{1+e^{R_0}}\,,$$

a lower bound which, together with Lemma 2.5 and the non-negativity of $\mathcal{E}_1$, ensures that

$$(\nabla u_\varepsilon)_{\varepsilon\,\in\,(0,1)} \text{ is bounded in } L_2\left(\Omega, \mathbb{R}^{2N}\right). \tag{2.38}$$

We now infer from (2.37), (2.38), Rellich–Kondrachov' theorem, an interpolation argument, and a Cantor diagonal process that there exist a function

$$u = (f,g) \in H^1\left(\Omega, \mathbb{R}^2\right) \cap L_{\infty,+}\left(\Omega, \mathbb{R}^2\right)$$

and a sequence $(\varepsilon_l)_{l\geqslant 1}$, with $\varepsilon_l \to 0$, such that

$$u_{\varepsilon_l} \to u \qquad \text{in } L_p\left(\Omega, \mathbb{R}^2\right) \text{ for all } p \in [1,\infty)\,, \tag{2.39}$$

$$u_{\varepsilon_l} \overset{*}{\rightharpoonup} u \qquad \text{in } L_\infty\left(\Omega, \mathbb{R}^2\right)\,, \tag{2.40}$$

$$\nabla u_{\varepsilon_l} \rightharpoonup \nabla u \quad \text{in } L_2\left(\Omega, \mathbb{R}^{2N}\right)\,. \tag{2.41}$$

An immediate consequence of (2.29) and (2.39) is the estimate (2.2). As $\sqrt{\lambda_{\varepsilon_l}(u_{\varepsilon_l})} \to 1$ in $L_\infty(\Omega)$ by (2.14) and (2.37), we conclude together with (2.41) that

$$\begin{aligned}
\sqrt{\lambda_{\varepsilon_l}(u_{\varepsilon_l})}\nabla\Big(au_{\varepsilon_l,1} + \Theta_1 u_{\varepsilon_l,2}\Big) &\rightharpoonup \nabla\Big(au_1 + \Theta_1 u_2\Big) \qquad && \text{in } L_2\left(\Omega, \mathbb{R}^N\right)\,,\\
\sqrt{\Theta_2\lambda_{\varepsilon_l}(u_{\varepsilon_l})}\nabla u_{\varepsilon_l,2} &\rightharpoonup \sqrt{\Theta_2}\nabla u_2 && \text{in } L_2\left(\Omega, \mathbb{R}^N\right)\,.
\end{aligned}$$

Moreover, the L_∞-bound (2.37) and the convergence (2.39) imply that

$$\liminf_{l\to\infty} \mathcal{E}_1(u_{\varepsilon_l}) \geqslant \mathcal{E}_1(u)\,,$$

and the estimate (2.3) is now obtained by passing to lim inf in the inequality reported in Lemma 2.5 (with ε replaced by ε_l).

Finally, (2.39), along with (2.37) and the convergence property

$$\lim_{\varepsilon\to 0} |m_{\varepsilon,jk}(X) - m_{jk}(X)| = 0\,,$$

which is uniform with respect to $X \in [0,R_0]^2$ and $1 \leqslant j,k \leqslant 2$, enables us to use Lebesgue's dominated convergence theorem to show that, for $v = (\varphi,\psi) \in H^1(\Omega,\mathbb{R}^2)$,

$$\lim_{l\to\infty} \left\| M_{\varepsilon_l}(u_{\varepsilon_l})^t \partial_i v - M(u)^t \partial_i v \right\|_2 = 0\,, \qquad 1 \leqslant i \leqslant N\,.$$

Together with (2.39) and (2.41), the above convergence allows us to let $\varepsilon_l \to 0$ in (2.28) and conclude that $u = (f,g)$ satisfies (2.1). This completes the proof of Proposition 2.1. □

3. Existence of bounded weak solutions

This section is devoted to the proof of Theorem 1.1, which relies on rather classical arguments, besides the estimates derived in Proposition 2.1, and proceeds along the lines of the proof of [LM22, Theorem 1.2]. As a first step, we use Proposition 2.1 to construct a family of piecewise constant functions $(u^\tau)_{\tau\in(0,1)}$ starting from the initial condition $(f^{in}, g^{in}) \in L_{\infty,+}(\Omega, \mathbb{R}^2)$. More precisely, for $\tau \in (0,1)$, we set $u^\tau(0) := u_0^\tau$ and

$$u^\tau(t) = u_l^\tau\,, \qquad t \in ((l-1)\tau, l\tau]\,, \qquad l \in \mathbb{N}\setminus\{0\}\,, \tag{3.1}$$

where the sequence $(u_l^\tau)_{l\geqslant 0}$ is defined as follows:

$$\begin{aligned} &u_0^\tau = u^{in} := (f^{in}, g^{in}) \in L_{\infty,+}\left(\Omega, \mathbb{R}^2\right)\,, \\ &u_{l+1}^\tau = (f_{l+1}^\tau, g_{l+1}^\tau) \in H^1\left(\Omega, \mathbb{R}^2\right) \cap L_{\infty,+}\left(\Omega, \mathbb{R}^2\right) \text{ is the solution to (2.1)} \\ &\text{with } U = u_l^\tau = (f_l^\tau, g_l^\tau) \text{ constructed in Proposition 2.1 for } l \geqslant 0\,. \end{aligned} \tag{3.2}$$

In order to establish Theorem 1.1, we show that the family $(u^\tau)_{\tau\in(0,1)}$ defined in (3.2) converges along a subsequence $\tau_j \to 0$ towards a pair $u = (f,g)$ which fulfills all the requirements of Theorem 1.1.

Below, C and $(C_l)_{l\geqslant 0}$ denote various positive constants depending only on (a,b,c,d) and u^{in}. Dependence upon additional parameters will be indicated explicitly.

Proof of Theorem 1.1. — Let $\tau \in (0,1)$ and let u^τ be defined in (3.1)-(3.2). Given $l \geqslant 0$, we infer from Proposition 2.1 that

$$\begin{aligned} &\int_\Omega \left(f_{l+1}^\tau \varphi + \tau f_{l+1}^\tau \nabla\left[a f_{l+1}^\tau + b g_{l+1}^\tau\right]\cdot\nabla\varphi\right)\,\mathrm{d}x \\ &\qquad\qquad = \int_\Omega f_l^\tau \varphi\,\mathrm{d}x\,, \quad \varphi \in H^1(\Omega)\,, \end{aligned} \tag{3.3a}$$

$$\begin{aligned} &\int_\Omega \left(g_{l+1}^\tau \psi + \tau g_{l+1}^\tau \nabla\left[c f_{l+1}^\tau + d g_{l+1}^\tau\right]\cdot\nabla\psi\right)\,\mathrm{d}x \\ &\qquad\qquad = \int_\Omega g_l^\tau \psi\,\mathrm{d}x\,, \quad \psi \in H^1(\Omega)\,. \end{aligned} \tag{3.3b}$$

Moreover,

$$\mathcal{E}_n(u_{l+1}^\tau) \leqslant \mathcal{E}_n(u_l^\tau) \qquad \text{for } n \geqslant 2, \tag{3.4}$$

and we also have

$$\mathcal{E}_1(u_{l+1}^\tau) + \frac{\tau}{a}\int_\Omega \left[\left|\nabla\left(a f_{l+1}^\tau + \Theta_1 g_{l+1}^\tau\right)\right|^2 + \Theta_2\left|\nabla g_{l+1}^\tau\right|^2\right]\,\mathrm{d}x \leqslant \mathcal{E}_1(u_l^\tau)\,. \tag{3.5}$$

It readily follows from (3.1), (3.2), (3.4), and (3.5) that, for $t > 0$,

$$\mathcal{E}_n(u^\tau(t)) \leqslant \mathcal{E}_n(u^{in})\,, \quad n \geqslant 2\,, \tag{3.6}$$

and

$$\mathcal{E}_1(u^\tau(t)) + \frac{1}{a}\int_0^t\int_\Omega \left[\,\left|\nabla\left(a f^\tau + \Theta_1 g^\tau\right)\right|^2 + \Theta_2\left|\nabla g^\tau\right|^2\right]\,\mathrm{d}x\mathrm{d}s \leqslant \mathcal{E}_1(u^{in})\,. \tag{3.7}$$

An immediate consequence of (3.6) and Lemma A.4 is the estimate

$$\|f^\tau(t)+g^\tau(t)\|_n \leqslant \frac{d}{b}\frac{\max\{a,\, b\}}{\min\{c,\, d\}} \left\|f^{in}+g^{in}\right\|_n\,, \quad n\geqslant 2\,,\ t>0\,.$$

Letting $n\to\infty$ in the above inequality gives

$$\|f^\tau(t)+g^\tau(t)\|_\infty \leqslant C_2 := \frac{d}{b}\frac{\max\{a,\, b\}}{\min\{c,\, d\}} \left\|f^{in}+g^{in}\right\|_\infty\,, \quad t>0\,. \tag{3.8}$$

Also, taking advantage of the non-negativity of $\mathcal{E}_1$, we deduce from (3.7) that

$$\int_0^t \left[\|\nabla f^\tau(s)\|_2^2 + \|\nabla g^\tau(s)\|_2^2\right]\,\mathrm{d}s \leqslant C_3 := \frac{a^2+2\left(\Theta_2+\Theta_1^2\right)}{a\Theta_2}\mathcal{E}_1(u^{in}) \tag{3.9}$$

for $t>0$.

Next, for $l\geqslant 1$ and $t\in((l-1)\tau, l\tau]$, we deduce from (3.3a), (3.8), and Hölder's inequality that, for $\varphi\in H^1(\Omega)$,

$$\begin{aligned}\left|\int_\Omega \Big(f^\tau(t+\tau)-f^\tau(t)\Big)\varphi\,\mathrm{d}x\right| &= \left|\int_{l\tau}^{(l+1)\tau}\int_\Omega f_{l+1}^\tau \nabla\left[af_{l+1}^\tau + bg_{l+1}^\tau\right]\cdot\nabla\varphi\,\mathrm{d}x\mathrm{d}s\right| \\ &\leqslant \int_{l\tau}^{(l+1)\tau} \|f^\tau(s)\|_\infty \|\nabla\left[af^\tau(s)+bg^\tau(s)\right]\|_2 \|\nabla\varphi\|_2\,\mathrm{d}s \\ &\leqslant C_2\|\nabla\varphi\|_2 \int_{l\tau}^{(l+1)\tau} \|\nabla\left[af^\tau(s)+bg^\tau(s)\right]\|_2\,\mathrm{d}s\,.\end{aligned}$$

A duality argument then gives

$$\|f^\tau(t+\tau)-f^\tau(t)\|_{(H^1)'} \leqslant C_2\int_{l\tau}^{(l+1)\tau} \|\nabla\left[af^\tau(s)+bg^\tau(s)\right]\|_2\,\mathrm{d}s$$

for $t\in((l-1)\tau, l\tau]$ and $l\geqslant 1$. Now, for $l_0\geqslant 2$ and $T\in((l_0-1)\tau, l_0\tau]$, the above inequality, along with Hölder's inequality, entails that

$$\begin{aligned}\int_0^{T-\tau} \|f^\tau(t+\tau)-f^\tau(t)\|_{(H^1)'}^2\,\mathrm{d}t &\leqslant \int_0^{(l_0-1)\tau} \|f^\tau(t+\tau)-f^\tau(t)\|_{(H^1)'}^2\,\mathrm{d}t \\ &= \sum_{l=1}^{l_0-1}\int_{(l-1)\tau}^{l\tau} \|f^\tau(t+\tau)-f^\tau(t)\|_{(H^1)'}^2\,\mathrm{d}t \\ &\leqslant C_2^2\tau\sum_{l=1}^{l_0-1}\left(\int_{l\tau}^{(l+1)\tau} \|\nabla\left[af^\tau(s)+bg^\tau(s)\right]\|_2\,\mathrm{d}s\right)^2 \\ &\leqslant C_2^2\tau^2\sum_{l=1}^{l_0-1}\int_{l\tau}^{(l+1)\tau} \|\nabla\left[af^\tau(s)+bg^\tau(s)\right]\|_2^2\,\mathrm{d}s \\ &\leqslant C_2^2\tau^2\int_0^{l_0\tau} \|\nabla\left[af^\tau(s)+bg^\tau(s)\right]\|_2^2\,\mathrm{d}s\,.\end{aligned}$$

We then use (3.9) (with $t = l_0\tau$) and Young's inequality to obtain

$$\begin{aligned}(3.10)\quad \int_0^{T-\tau} \|f^\tau(t+\tau) - f^\tau(t)\|_{(H^1)'}^2 \ \mathrm{d}t &\\ &\leqslant C_2^2\tau^2 \int_0^{l_0\tau} \left(2a^2 \|\nabla f^\tau(s)\|_2^2 + 2b^2 \|\nabla g^\tau(s)\|_2^2\right) \ \mathrm{d}s \\ &\leqslant C_4\tau^2\,,\end{aligned}$$

with $C_4 := 2(a^2+b^2)^2 C_2^2 C_3$. Similarly,

$$\int_0^{T-\tau} \|g^\tau(t+\tau) - g^\tau(t)\|_{(H^1)'}^2 \ \mathrm{d}t \leqslant C_5\tau^2\,, \tag{3.11}$$

with $C_5 := 2(c^2+d^2)C_2^2 C_3$.

According to Rellich–Kondrachov' theorem, $H^1(\Omega,\mathbb{R}^2)$ is compactly embedded in $L_2(\Omega,\mathbb{R}^2)$, while $L_2(\Omega,\mathbb{R}^2)$ is continuously and compactly embedded in $H^1(\Omega,\mathbb{R}^2)'$. Gathering (3.8)-(3.11), we infer from [DJ12, Theorem 1] that, for any $T>0$,

$$(u^\tau)_{\tau\in(0,1)} \ \text{ is relatively compact in } \ L_2\left((0,T)\times\Omega,\mathbb{R}^2\right)\,. \tag{3.12}$$

Owing to (3.8), (3.9), and (3.12), we may use a Cantor diagonal argument to find a function

$$u = (f,g) \in L_{\infty,+}\left((0,\infty)\times\Omega,\mathbb{R}^2\right)$$

and a sequence $(\tau_m)_{m\geqslant 1}$, $\tau_m \to 0$, such that, for any $T>0$ and $p\in[1,\infty)$,

$$\begin{aligned} u^{\tau_m} &\to u \quad \text{in} \quad L_p\left((0,T)\times\Omega,\mathbb{R}^2\right)\,,\\ u^{\tau_m} &\overset{*}{\rightharpoonup} u \quad \text{in} \quad L_\infty\left((0,T)\times\Omega,\mathbb{R}^2\right)\,,\\ u^{\tau_m} &\rightharpoonup u \quad \text{in} \quad L_2\left((0,T),H^1\left(\Omega,\mathbb{R}^2\right)\right)\,.\end{aligned} \tag{3.13}$$

In addition, the compact embedding of $L_2(\Omega,\mathbb{R}^2)$ in $H^1(\Omega,\mathbb{R}^2)'$, along with (3.6) for $n=2$, (3.10), and (3.11), allows us to apply once more [DJ12, Theorem 1] to conclude that

$$u \in C\left([0,\infty), H^1\left(\Omega,\mathbb{R}^2\right)'\right)\,. \tag{3.14}$$

Let us now identify the equations solved by the components f and g of u. To this end, let $\chi\in W_\infty^1([0,\infty))$ be a compactly supported function and $\varphi\in C^1(\overline{\Omega})$. In view of (3.3a), classical computations give

$$\begin{aligned}\int_0^\infty \int_\Omega \frac{\chi(t+\tau)-\chi(t)}{\tau} f^\tau(t)\varphi \ \mathrm{d}x\mathrm{d}t + \left(\frac{1}{\tau}\int_0^\tau \chi(t)\ \mathrm{d}t\right)\int_\Omega f^{in}\varphi\ \mathrm{d}x &\\ = \int_0^\infty \int_\Omega \chi(t) f^\tau(t)\nabla\left[af^\tau(t)+bg^\tau(t)\right]\cdot\nabla\varphi \ \mathrm{d}x\mathrm{d}t\,.&\end{aligned}$$

Taking $\tau = \tau_m$ in the above identity, it readily follows from (3.13) and the regularity of χ and φ that we may pass to the limit as $m \to \infty$ and conclude that

$$\begin{aligned}(3.15)\quad \int_0^\infty \int_\Omega \frac{d\chi}{dt}(t) f(t,x)\varphi(x)\ \mathrm{d}x\mathrm{d}t + \chi(0)\int_\Omega f^{in}(x)\varphi(x)\ \mathrm{d}x \\ = \int_0^\infty \int_\Omega \chi(t) f(t,x) \nabla\left[af+bg\right](t,x)\cdot\nabla\varphi(x)\ \mathrm{d}x\mathrm{d}t\,.\end{aligned}$$

Since $f\nabla f$ and $f\nabla g$ belong to $L_2((0,T)\times\Omega)$ for all $T>0$ by (3.13), a density argument ensures that the identity (3.15) is valid for any $\varphi \in H^1(\Omega)$. We next use the time continuity (3.14) of f and a classical approximation argument to show that f solves (1.9a). A similar argument allows us to derive (1.9b) from (3.3b).

Finally, combining (3.13), (3.14), and a weak lower semicontinuity argument, we may let $m \to \infty$ in (3.6), (3.7), and (3.8) with $\tau = \tau_m$ to show that $u = (f,g)$ satisfies (1.10), (1.11), and (1.12), thereby completing the proof of Theorem 1.1. □

Appendix A. The polynomials Φ_n, $n \geqslant 2$

Let $n \geqslant 2$. According to the discussion in the introduction, we look for an homogeneous polynomial Φ_n of degree n such that:

- (P1) Φ_n is convex on $[0,\infty)^2$;
- (P2) the matrix $S_n(X) := D^2\Phi_n(X)M(X)$ is symmetric and positive semidefinite for $X \in [0,\infty)^2$.

We recall that the mobility matrix $M(X)$ is given by

$$M(X) = (m_{jk}(X))_{1\leqslant j,k\leqslant 2} := \begin{pmatrix} aX_1 & bX_1 \\ cX_2 & dX_2 \end{pmatrix}, \qquad X \in \mathbb{R}^2,$$

see (1.13). Specifically, we set

$$(\mathrm{A}.1)\qquad \Phi_n(X) := \sum_{j=0}^{n} a_{j,n} X_1^j X_2^{n-j}, \qquad X = (X_1,X_2) \in \mathbb{R}^2,$$

with $a_{j,n}$, $0 \leqslant j \leqslant n$, to be determined in order for properties (P1)-(P2) to be satisfied. We recall that the parameters $(a,\, b,\, c,\, d)$ are assumed to satisfy (1.2).

Lemma A.1. — *Set $a_{0,n} := 1$ and*

$$(\mathrm{A}.2)\qquad a_{j,n} := \prod_{k=0}^{j-1} \frac{(n-k)[ak+c(n-k-1)]}{(k+1)[bk+d(n-k-1)]} = \binom{n}{j} \prod_{k=0}^{j-1} \frac{ak+c(n-k-1)}{bk+d(n-k-1)}$$

for $1 \leqslant j \leqslant n$. Then $a_{j,n} > 0$ for $0 \leqslant j \leqslant n$ and $S_n(X) = D^2\Phi_n(X)M(X) \in \mathbf{Sym}_2(\mathbb{R})$ for $X \in \mathbb{R}^2$.

Proof. — Given $X \in \mathbb{R}^2$, we compute

$$\begin{aligned}
\partial_1^2 \Phi_n(X) &= \sum_{j=1}^{n-1} j(j+1) a_{j+1,n} X_1^{j-1} X_2^{n-j-1} = \sum_{j=0}^{n-2} (j+1)(j+2) a_{j+2,n} X_1^j X_2^{n-j-2}, \\
\partial_1\partial_2 \Phi_n(X) &= \sum_{j=1}^{n-1} j(n-j) a_{j,n} X_1^{j-1} X_2^{n-j-1} = \sum_{j=0}^{n-2} (j+1)(n-j-1) a_{j+1,n} X_1^j X_2^{n-j-2}, \\
\partial_2^2 \Phi_n(X) &= \sum_{j=0}^{n-2} (n-j)(n-j-1) a_{j,n} X_1^j X_2^{n-j-2}.
\end{aligned}$$

It then follows that

$$\begin{aligned}
[S_n(X)]_{11} &= a X_1 \partial_1^2 \Phi_n(X) + c X_2 \partial_1 \partial_2 \Phi_n(X) \\
&= a \sum_{j=1}^{n-1} j(j+1) a_{j+1,n} X_1^j X_2^{n-j-1} + c \sum_{j=0}^{n-2} (j+1)(n-j-1) a_{j+1,n} X_1^j X_2^{n-j-1}, \\
[S_n(X)]_{12} &= b X_1 \partial_1^2 \Phi_n(X) + d X_2 \partial_1 \partial_2 \Phi_n(X) \\
&= b \sum_{j=1}^{n-1} j(j+1) a_{j+1,n} X_1^j X_2^{n-j-1} + d \sum_{j=0}^{n-2} (j+1)(n-j-1) a_{j+1,n} X_1^j X_2^{n-j-1} \\
&= bn(n-1) a_{n,n} X_1^{n-1} + \sum_{j=1}^{n-2} (j+1)[bj + d(n-j-1)] a_{j+1,n} X_1^j X_2^{n-j-1} \\
&\qquad + d(n-1) a_{1,n} X_2^{n-1}, \\
[S_n(X)]_{21} &= a X_1 \partial_1 \partial_2 \Phi_n(X) + c X_2 \partial_2^2 \Phi_n(X) \\
&= a \sum_{j=1}^{n-1} j(n-j) a_{j,n} X_1^j X_2^{n-j-1} + c \sum_{j=0}^{n-2} (n-j)(n-j-1) a_{j,n} X_1^j X_2^{n-j-1} \\
&= a(n-1) a_{n-1,n} X_1^{n-1} + \sum_{j=1}^{n-2} (n-j)[aj + c(n-j-1)] a_{j,n} X_1^j X_2^{n-j-1} \\
&\qquad + cn(n-1) a_{0,n} X_2^{n-1}, \\
[S_n(X)]_{22} &= b X_1 \partial_1 \partial_2 \Phi_n(X) + d X_2 \partial_2^2 \Phi_n(X) \\
&= b \sum_{j=1}^{n-1} j(n-j) a_{j,n} X_1^j X_2^{n-j-1} + d \sum_{j=0}^{n-2} (n-j)(n-j-1) a_{j,n} X_1^j X_2^{n-j-1}.
\end{aligned}$$

Hence, $S_n(X)$ is symmetric provided that

$$(j+1)[bj + d(n-j-1)] a_{j+1,n} = (n-j)[aj + c(n-j-1)] a_{j,n}, \quad 0 \leqslant j \leqslant n-1\,,$$

or, equivalently,

$$a_{j+1,n} = \frac{(n-j)[aj + c(n-j-1)]}{(j+1)[bj + d(n-j-1)]} a_{j,n}\,, \quad 0 \leqslant j \leqslant n-1\,. \tag{A.3}$$

Since $a_{0,n} = 1$, the closed form formula (A.2) readily follows from (A.3) and we deduce from (A.2) and the positivity of (a, b, c, d) that $a_{j,n} > 0$ for all $0 \leqslant j \leqslant n$. □

We next show that $D^2\Phi_n(X)$ is positive definite for $X \in [0,\infty)^2 \setminus \{(0,0)\}$. This property implies in particular that $D^2\Phi_n(X)$ is positive semidefinite for $X \in [0,\infty)^2$.

Lemma A.2. — *Let Φ_n be the polynomial defined by* (A.1) *and* (A.2). *Then we have $D^2\Phi_n(X) \in \mathbf{SPD}_2(\mathbb{R})$ for $X \in [0,\infty)^2 \setminus \{(0,0)\}$.*

Proof. — Given $X \in [0,\infty)^2$, the positivity of the coefficients $a_{j,n}$, $0 \leqslant j \leqslant n$, of Φ_n ensures that

$$\mathrm{tr}(D^2\Phi_n(X)) := \partial_1^2\Phi_n(X) + \partial_2^2\Phi_n(X) \geqslant 0\,, \qquad X \in [0,\infty)^2\,.$$

It remains to show that the determinant $\det(D^2\Phi_n(X))$ is also non-negative. To this end we compute

$$\begin{aligned}\det(D^2\Phi_n(X)) &= \partial_1^2\Phi_n(X)\partial_2^2\Phi_n(X) - [\partial_1\partial_2\Phi_n(X)]^2 \\ &= \sum_{j=0}^{n-2}\sum_{k=0}^{n-2}(j+1)(n-k-1)A_{j,k}X_1^{j+k}X_2^{2n-j-k-4},\end{aligned} \tag{A.4}$$

where

$$A_{j,k} := (j+2)(n-k)a_{j+2,n}a_{k,n} - (n-j-1)(k+1)a_{j+1,n}a_{k+1,n}\,, \qquad 0 \leqslant j,k \leqslant n-2\,.$$

Using (A.3), we express $a_{j+2,n}$ and $a_{k+1,n}$ in terms of $a_{j+1,n}$ and $a_{k,n}$, respectively, to arrive at the following formula

$$\begin{aligned}A_{j,k} &= (n-k)(n-j-1)\left[\frac{a(j+1)+c(n-j-2)}{b(j+1)+d(n-j-2)} - \frac{ak+c(n-k-1)}{bk+d(n-k-1)}\right]a_{j+1,n}a_{k,n} \\ &= (ad-bc)(n-k)(n-j-1) \\ &\quad \times \frac{(j+1)(n-k-1)-k(n-j-2)}{[b(j+1)+d(n-j-2)]\,[bk+d(n-k-1)]}a_{j+1,n}a_{k,n} \\ &= (ad-bc)\frac{(n-1)(n-k)(n-j-1)(j+1-k)}{\alpha_{j+1,n}\alpha_{k,n}}a_{j+1,n}a_{k,n}\,,\end{aligned} \tag{A.5}$$

where $\alpha_{k,n}$ denotes the positive number

$$\alpha_{k,n} := bk + d(n-k-1)\,, \quad 0 \leqslant k \leqslant n-1\,.$$

In particular,

$$A_{k-1,j+1} = -A_{j,k}\,, \qquad 0 \leqslant j \leqslant n-3\,,\ 1 \leqslant k \leqslant n-2\,. \tag{A.6}$$

It then follows from (A.4) that

$$\begin{aligned}2\det(D^2\Phi_n(X)) &= \sum_{j=0}^{n-2}\sum_{k=0}^{n-2}(j+1)(n-k-1)A_{j,k}X_1^{j+k}X_2^{2n-j-k-4} \\ &\quad + \sum_{l=1}^{n-1}\sum_{i=-1}^{n-3} l(n-i-2)A_{l-1,i+1}X_1^{i+l}X_2^{2n-i-l-4} \\ &= \sum_{j=0}^{n-2}\sum_{k=0}^{n-2}(j+1)(n-k-1)A_{j,k}X_1^{j+k}X_2^{2n-j-k-4}\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{j=-1}^{n-3} \sum_{k=1}^{n-1} k(n-j-2) A_{k-1,j+1} X_1^{j+k} X_2^{2n-j-k-4} \\
&= \sum_{j=0}^{n-3} \sum_{k=1}^{n-2} (j+1)(n-k-1) A_{j,k} X_1^{j+k} X_2^{2n-j-k-4} \\
&\quad + \sum_{k=0}^{n-2} (n-1)(n-k-1) A_{n-2,k} X_1^{n-2+k} X_2^{n-k-2} \\
&\quad + \sum_{j=0}^{n-3} (j+1)(n-1) A_{j,0} X_1^{j} X_2^{2n-j-4} \\
&\quad + \sum_{j=0}^{n-3} \sum_{k=1}^{n-2} k(n-j-2) A_{k-1,j+1} X_1^{j+k} X_2^{2n-j-k-4} \\
&\quad + \sum_{k=1}^{n-1} k(n-1) A_{k-1,0} X_1^{k-1} X_2^{2n-k-3} \\
&\quad + \sum_{j=0}^{n-3} (n-1)(n-j-2) A_{n-2,j+1} X_1^{j+n-1} X_2^{n-j-3}.
\end{aligned}
$$

According to (1.2) and (A.5),

$$
\begin{aligned}
A_{l,0} &= (ad-bc) \frac{n(n-1)(n-1-l)(l+1)}{\alpha_{0,n}\alpha_{l+1,n}} > 0\,, \qquad 0 \leqslant l \leqslant n-2\,, \\
A_{n-2,l} &= (ad-bc) \frac{(n-1)(n-l)(n-1-l)}{\alpha_{n-1,n}\alpha_{l,n}} > 0\,, \qquad 0 \leqslant l \leqslant n-2\,.
\end{aligned}
$$

In particular, all the terms in the above identity involving a single sum are non-negative. Therefore, using the symmetry property (A.6) and retaining in the last two sums only the terms corresponding to $k=1$ and $j=n-3$, respectively, we get

$$
\begin{aligned}
2\det\Big(D^2\Phi_n(X)\Big) &\geqslant \sum_{j=0}^{n-3} \sum_{k=1}^{n-2} \left[(j+1)(n-k-1) - k(n-j-2)\right] A_{j,k} X_1^{j+k} X_2^{2n-j-k-4} \\
&\qquad + (n-1) A_{n-2,n-2} X_1^{2n-4} + (n-1) A_{0,0} X_2^{2n-4} \\
&= \sum_{j=0}^{n-3} \sum_{k=1}^{n-2} (n-1)(j+1-k) A_{j,k} X_1^{j+k} X_2^{2n-j-k-4} \\
&\qquad + (n-1) A_{n-2,n-2} X_1^{2n-4} + (n-1) A_{0,0} X_2^{2n-4}.
\end{aligned}
$$

Observing that

$$
(n-1)(j+1-k) A_{j,k} = (ad-bc) \frac{(n-1)^2(n-k)(n-j-1)(j+1-k)^2}{\alpha_{j+1,n}\alpha_{k,n}} a_{j+1,n} a_{k,n} \geqslant 0
$$

for $0 \leqslant j,k \leqslant n-2$, we conclude that

$$
2\det\Big(D^2\Phi_n(X)\Big) \geqslant (n-1) A_{n-2,n-2} X_1^{2n-4} + (n-1) A_{0,0} X_2^{2n-4} \tag{A.7}
$$

for $X \in [0,\infty)^2$. Since $A_{0,0} > 0$ and $A_{n-2,n-2} > 0$, we have thus established that the symmetric matrix $D^2\Phi_n(X)$ has non-negative trace and positive determinant, so that it is positive definite for each $X \in [0,\infty)^2 \setminus \{(0,0)\}$. □

We next turn to the positive definiteness of $S_n = D^2\Phi_n M$.

LEMMA A.3. — *Let Φ_n be defined by* (A.1) *and* (A.2)*. Then*

$$S_n(X) = D^2\Phi_n(X)M(X) \in \mathbf{SPD}_2(\mathbb{R}) \text{ for } X \in (0,\infty)^2 .$$

Proof. — Let $X \in (0,\infty)^2$. On the one hand, by (1.2), (A.7), and the positivity of $A_{0,0}$ and $A_{n-2,n-2}$,

$$\begin{aligned} 2\det(S_n(X)) &= 2(ad-bc)X_1X_2 \det\left(D^2\Phi_n(X)\right) \\ &\geqslant (ad-bc)X_1X_2(n-1)\left[A_{n-2,n-2}X_1^{2n-4} + A_{0,0}X_2^{2n-4}\right] > 0 . \end{aligned}$$

On the other hand, the positivity of $a_{j,n}$ for $0 \leqslant j \leqslant n$ and (1.2) imply that

$$\operatorname{tr}(S_n(X)) = [S_n(X)]_{11} + [S_n(X)]_{22} > 0 .$$

Consequently, $S_n(X)$ has positive trace and positive determinant, and is thus positive definite as claimed. □

We end up this section with useful upper and lower bounds for Φ_n.

LEMMA A.4. — *Let Φ_n be defined by* (A.1) *and* (A.2)*. Then*

$$\frac{(cX_1+dX_2)^n}{d^n} \leqslant \Phi_n(X) \leqslant \frac{(aX_1+bX_2)^n}{b^n}, \qquad X \in [0,\infty)^2 . \tag{A.8}$$

Proof. — Since the function

$$\chi(z) := \frac{(a-c)z+c}{(b-d)z+d}, \qquad z \in [0,1] ,$$

is increasing and positive, we deduce from (A.2) that, for $1 \leqslant j \leqslant n$,

$$a_{j,n} = \binom{n}{j}\prod_{k=0}^{j-1}\chi\left(\frac{k}{n-1}\right) \leqslant \binom{n}{j}[\chi(1)]^j = \binom{n}{j}\left(\frac{a}{b}\right)^j$$

and

$$a_{j,n} = \binom{n}{j}\prod_{k=0}^{j-1}\chi\left(\frac{k}{n-1}\right) \geqslant \binom{n}{j}[\chi(0)]^j = \binom{n}{j}\left(\frac{c}{d}\right)^j .$$

The upper and lower bounds in (A.8) are direct consequences of the above inequalities. □

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