



BENOÎT LASLIER

FABIO TONINELLI

THE MIXING TIME OF THE LOZENGE TILING GLAUBER DYNAMICS

TEMPS DE MÉLANGE DE LA DYNAMIQUE
DE GLAUBER POUR LES PAVAGES
ALÉATOIRES

ABSTRACT. — The broad motivation of this work is a rigorous understanding of reversible, local Markov dynamics of interfaces, and in particular their speed of convergence to equilibrium, measured via the mixing time T_{mix} . In the $(d + 1)$ -dimensional setting, $d \geq 2$, this is to a large extent mathematically unexplored territory, especially for discrete interfaces. On the other hand, on the basis of a mean-curvature motion heuristics [Hen97, Spo93] and simulations (see [Des02] and the references in [Hen97, Wil04]), one expects convergence to equilibrium to occur on time-scales of order $\approx \delta^{-2}$ in any dimension, with $\delta \rightarrow 0$ the lattice mesh.

We study the single-flip Glauber dynamics for lozenge tilings of a finite domain of the plane, viewed as $(2 + 1)$ -dimensional surfaces. The stationary measure is the uniform measure on admissible tilings. At equilibrium, by the limit shape theorem [CKP01], the height function concentrates as $\delta \rightarrow 0$ around a deterministic profile ϕ , the unique minimizer of a surface tension functional. Despite some partial mathematical results [LT15a, LT15b, Wil04], the

Keywords: Mixing time, lozenge tilings, random interfaces, dimer model.

2020 Mathematics Subject Classification: 60K35, 82C20, 52C20.

DOI: <https://doi.org/10.5802/ahl.181>

(*) F. T. gratefully acknowledges financial support of the Austrian Science Fund (FWF), Project Number P 35428-N. Both authors acknowledge financial support of Agence Nationale de la Recherche (ANR), Project number ANR-18-CE40-0033.

conjecture $T_{mix} = \delta^{-2+o(1)}$ had been proven, so far, only in the situation where ϕ is an affine function [CMT12]. In this work, we prove the conjecture under the sole assumption that the limit shape ϕ contains no frozen regions (facets).

RÉSUMÉ. — La motivation de ce travail est la compréhension mathématique des dynamiques Markoviennes réversibles d'interfaces aléatoires, et de leur vitesse de convergence vers l'équilibre (temps de mélange T_{mix}). En dimension $(d + 1)$, $d \geq 2$, il s'agit de questions très ouvertes, en particulier pour des interfaces discrètes mais, sur la base d'arguments heuristiques et de simulations numériques, on conjecture que T_{mix} est d'ordre $\approx \delta^{-2}$ en toute dimension, si $\delta \rightarrow 0$ est le pas du réseau.

Nous étudions une dynamique de Glauber pour les pavages par losanges d'un domaine du plan, vues comme des surface $(2 + 1)$ -dimensionnelles. La mesure stationnaire est la mesure uniforme sur les pavages admissibles. À l'équilibre, la fonction de hauteur se concentre (pour $\delta \rightarrow 0$) autour d'un profil déterministe ϕ , l'unique minimiseur d'une fonctionnelle de tension de surface. Malgré certains résultats mathématiques partiels [LT15a, LT15b, Wil04], jusqu'ici la conjecture $T_{mix} = \delta^{-2+o(1)}$ n'avait été démontrée que dans le cas où ϕ est une fonction affine [CMT12]. Dans ce travail, nous prouvons la conjecture sous la seule hypothèse que la forme limite ϕ ne contienne pas de régions gelées.

1. Introduction

To define the problem that we study in this work, we start from \mathcal{T}_δ , the periodic triangular planar lattice where each face is an equilateral triangle of side δ . A *tile* will denote the lozenge-shaped polygon obtained by the union of two adjacent triangular faces. Note that tiles can have three different orientations, see Figure 1.1. We say that a domain D_δ (a bounded, connected union of faces of \mathcal{T}_δ) is *tilable* if it can be covered with tiles, in such a way that tiles do not intersect (except along their boundaries) and leave no hole. Call Ω_{D_δ} the set of possible tilings of D_δ .

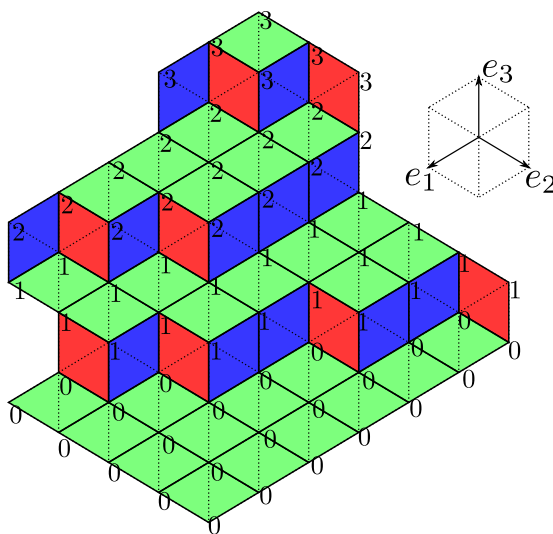


Figure 1.1. A tilable domain and the height function of a possible tiling. The value of the height function is given in units of δ .

A very natural, local, continuous-time Markov dynamics on the state space Ω_{D_δ} is defined by assigning a transition rate $1/2$ to the two elementary updates in Figure 1.2, which consist in rotating three lozenges by 60° around their common vertex.

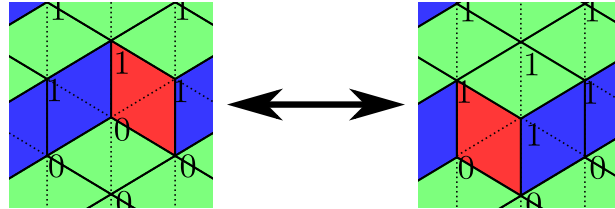


Figure 1.2. The elementary updates of the Glauber dynamics, each having transition rate $1/2$.

We refer to this as the “tiling Glauber dynamics” and in fact, as is well known (see e.g. [CMT12]), it can be seen as a zero-temperature limit of the Glauber dynamics of the three-dimensional Ising model with certain Dobrushin-type boundary conditions. The stationary (and reversible) measure of the process is the uniform distribution on Ω_{D_δ} . The process is easily seen to be ergodic, and a classical question in the domain of Markov chains [LP17] is to understand how fast equilibrium is reached, as measured via the mixing time T_{mix} . In [LRS01] Luby, Randall and Sinclair proved that if D_δ has diameter of order 1, so that it contains approximately δ^{-2} faces, then in the limit $\delta \rightarrow 0$ the mixing time grows at most like a polynomial: $T_{mix} \lesssim \delta^{-C}$ for some $C > 0$. This “fast mixing” result, based on a smart coupling idea, is already non-trivial in view of the fact that for most “reasonable” domains D_δ , the state space is of cardinality $\approx \exp(c/\delta^2)$. The main result of the present work is that, for a class of natural domains D_δ that approach as $\delta \rightarrow 0$ some bounded domain $D \subset \mathbb{R}^2$, the mixing time is $T_{mix} = \delta^{-2+o(1)}$.

In order to better explain our result, the expected picture and the motivations for our work, let us take a step back and put the problem into a more general context. The first important observation is that a lozenge tiling can be mapped into a discrete height function h , defined on the collection W_δ of the tile vertices, and taking values in $\delta\mathbb{Z}$ (see Figure 1.1). The tiling Glauber dynamics can then be seen as a continuous-time, reversible, Markov evolution of a discrete height function with local update rules, where elementary updates consist in adding $\pm\delta$ to the height of a single vertex. The equilibrium measure, that we call $\pi_{W_\delta, h|_{\partial W_\delta}}$, is the uniform measure on height functions satisfying a certain local Lipschitz constraint (see (2.2)) and with fixed boundary value $h|_{\partial W_\delta}$ on the boundary of W_δ , determined by the shape of the domain D_δ . In the scaling limit where the lattice mesh δ tends to 0, assuming that the domain W_δ tends to a continuous bounded domain $W \subset \mathbb{R}^2$ and the boundary height tends to some Lipschitz function $\phi|_{\partial W}$, the height function sampled from the measure $\pi_{W_\delta, h|_{\partial W_\delta}}$ tends in probability to a deterministic *limit shape* ϕ , that minimizes a surface energy functional [CKP01, CLP98]. Interestingly, for certain boundary conditions the limit shape has facets (or “frozen regions”) which, at the microscopic level, contain with overwhelming probability only one of the three types of tiles. The phenomenon of appearance of facets in the limit shape is called “arctic circle” or “frozen boundary” phenomenon [CKP01, CLP98, KO07].

On the basis of phenomenological arguments, one expects a hydrodynamic limit on the diffusive time scale $1/\delta^2$, i.e., the height profile $h_{t/\delta^2}(\cdot)$ should tend as $\delta \rightarrow 0$ to the solution $H(t, \cdot)$ of a parabolic PDE of the type

$$(1.1) \quad \partial_t H = \mathcal{L}H,$$

with \mathcal{L} an elliptic non-linear operator such that $\mathcal{L}\phi = 0$ if ϕ is the limit shape. Note that, since \mathcal{L} is elliptic, (1.1) is a variant of mean curvature motion; the main difference is that its stationary points minimize surface tension instead of surface area. On the basis of this picture, it is natural to conjecture that the time-scale for equilibration (mixing time) is of order $\delta^{-2+o(1)}$. The $o(1)$ hides unavoidable subdominant corrections: in fact, we point out that even in the much easier case of 1-dimensional symmetric simple exclusion process (SSEP) [Lac16, Wil04] or one-dimensional $\nabla\varphi$ interface dynamics with convex potential [CLL22], the mixing time turns out to be of order $\delta^{-2} \times |\log \delta|$.

The coupling argument of [LRS01] does not at all use this “mean curvature evolution” intuition and, not surprisingly, it does not allow to capture the conjectured exponent $C = 2$ in the $\delta^{-C+o(1)}$ behavior of the mixing time. The first step towards establishing the conjecture was performed in [CMT12], where it was proven that indeed $T_{mix} = \delta^{-2+o(1)}$ under the important restriction that the limit shape ϕ is an affine function (this is a non-trivial restriction on the domain D_δ). The simplifying feature of this case, in few words, is that the more the interfaces approaches equilibrium, the more its law resembles (locally) that of an infinite-volume, translation invariant Gibbs state, for which very sharp height fluctuations results are known [Ken09] (the role of such estimates will become clear later in the paper).

In the generic case where the limit shape ϕ is curved, the only available mathematical confirmation of the expected $\delta^{-2+o(1)}$ time-scale for equilibration is provided by [LT15b], which proves that at such time scales the height profile is with high probability within L^∞ distance ϵ from ϕ , for any fixed $\epsilon > 0$. We emphasize that this result, while suggestive of the expected behavior, has no implications on T_{mix} : it could very well be that the height profile is at some time at very small L^∞ distance from the limit shape, yet the *law* of the height function at the same time is at total variation distance essentially 1 from the equilibrium measure.

In the present work, as already mentioned, we prove the $T_{mix} = \delta^{-2+o(1)}$ conjecture, under the sole assumption that ϕ contains no facets. Our method builds on one hand on an iterative procedure first developed in [CMT12] (which itself is inspired by the mean-curvature heuristics), and on the other hand on sharp equilibrium fluctuation results on domains of mesoscopic size (much larger than the mesh size but much smaller than 1). Let us emphasize that, while for instance the recent [Agg19] provides a “local equilibrium” type of result on microscopic domains of order δ (where the equilibrium measure coincides asymptotically with a translation-invariant Gibbs measure), we really need to consider mesoscopic domains where the effect of the curvature of the height profile, that drives interface motion, is non-negligible. Technically, we rely in this respect on the recent work [Las21].

1.1. The broader context

To put our result in a broader context, let us point out that the $\delta^{-2+o(1)}$ behavior of the mixing time is expected for many natural, local, reversible interface dynamics in any dimension $(d+1)$, $d \geq 1$, and not just when $d = 2$ as in the case under consideration. The result, in its sharper form $T_{\text{mix}} \approx \delta^{-2} |\log \delta|$, is known to hold in several $(1+1)$ -dimensional reversible interface dynamics, most notably, the symmetric simple exclusion (SSEP) ([Lac16, Wil04], see also [KL98] for the hydrodynamic limit) and Ginzburg–Landau $\nabla\phi$ models with convex potential (see [CLL22], where a proof of the total variation cut-off phenomenon is also obtained). It is important to emphasize that the $(1+1)$ -dimensional case is easier for at least two reasons. First, by looking at interface gradients, the dynamics can be seen as an interacting particle system, whose equilibrium distributions are often of i.i.d. type (for instance, they are i.i.d. Bernoulli measures for SSEP), while interface gradients exhibit power-law decaying correlations in higher dimension. Secondly, limit shapes are affine in one dimension (by convexity of the surface tension), while they are generically curved in higher dimension.

On the other hand, we are not aware of a result comparable to ours in dimension $(d+1)$, $d \geq 2$, especially for discrete interfaces. A notable exception is [Wil04], which proves $\delta^{-2} |\log \delta|$ bounds for the mixing time of a *special, non-local* lozenge tiling dynamic (“tower-move dynamic”), where “non-local” refers to the fact that each update flips a random number, that can be as large as δ^{-1} , of tiles. Let us also mention that for continuous-height, *Gaussian* interface dynamics (that is, Ginzburg–Landau $\nabla\phi$ models with quadratic interaction potential, also known as the discrete GFF) the proof of $T_{\text{mix}} \approx \delta^{-2} |\log \delta|$ is rather easy, via an application of the method of [Wil04]. In the $(2+1)$ -dimensional case, for this Gaussian process, the total variation cut-off phenomenon has also been proven to hold [GG23].

A final remark: *biased* versions of (discrete) interface dynamics have been considered in the literature. In this case, the rate for updates that increase the height is p and that of updates that decrease it is q with, say, $p < q$. In infinite volume, these processes are irreversible and celebrated examples are the one-dimensional Asymmetric Simple Exclusion Process (ASEP) and its totally asymmetric counterpart, the TASEP (corresponding to $p = 0$). In finite volume, instead, these dynamics are reversible, and the stationary measures are just those of the symmetric processes, tilted by $(p/q)^V$, with V the volume below the interface. In this case, the phenomenology is *qualitatively very different* from the one studied in the present work. For one thing, macroscopic shapes in this case minimize not the free energy but the free energy with a volume constraint, and the mixing time turns out to scale like $\delta^{-1+o(1)}$ rather than $\delta^{-2+o(1)}$. We refer to [LL19] for the ASEP in an interval (the authors prove the sharp estimate $T_{\text{mix}} \sim c\delta^{-1}$, as well as the occurrence of the cut-off phenomenon) and to [CMT11, GPR09] for a dynamics of biased plane partitions, which is a lozenge tiling dynamics where the two updates of Figure 1.2 have different transition rates p and q . Also in the latter case, the result is that $T_{\text{mix}} = \delta^{-1+o(1)}$ (see [GPR09], where this is proven for small enough bias $\log(p/q)$, and [CMT11] for the general case of arbitrary non-zero bias).

Organization of the article

The rest of this work is organized as follows. In Section 2 we define precisely the problem and state the main results. In Section 3, we give some preliminary results and we present a sketch of the strategy of the proof of our main theorem. The proof of the mixing time bound is reduced in Section 4 to an inductive statement, Theorem 4.1. In Section 5 we prove Theorem 4.1 up to two sharp equilibrium results on mesoscopic scales, whose proofs are the contents of the final Section 6.

2. Statement of the problem and results

The Glauber dynamics on lozenge tilings can be seen as a continuous-time Markov process on a certain set of discrete-valued height functions, and this is the point of view we adopt in the whole paper. The height function is defined on a portion of the planar triangular lattice. Since we are interested in large-scale behavior, we rescale the lattice mesh by $\delta > 0$ and we denote \mathcal{T}_δ the rescaled lattice. We let e_1, e_2, e_3 denote the elementary vectors of Figure 1.1.

We start with a few preliminary definitions.

2.1. Preliminary definitions

Given an open bounded domain D of \mathbb{R}^2 , a continuous function $f : D \mapsto \mathbb{R}$ and a real number $\delta > 0$, we let:

- ∂D be the boundary of D ;
- D^\sharp be the collection of vertices of \mathcal{T}_δ that are contained in D ;
- for $V \subset D$, $f|_V$ be the restriction of f to V .

Given $K > 0$, we say that $f^\sharp : D^\sharp \mapsto \delta\mathbb{Z}$ is a K -discretization of $f : D \mapsto \mathbb{R}$ if

$$(2.1) \quad \max_{x \in D^\sharp} |f^\sharp(x) - f(x)| \leq K\delta.$$

Finally, given a subset A of \mathcal{T}_δ (for instance, D^\sharp) we let ∂A be the collection of vertices in A that have a neighbor in $\mathcal{T}_\delta \setminus A$.

Given $A \subset \mathcal{T}_\delta$, a function $h : A \mapsto \delta\mathbb{Z}$ is said to be an admissible height function (or just “height function” for short) on A if whenever $x, y \in A$ are nearest neighbors, then

$$(2.2) \quad h(x) - h(y) \in \begin{cases} \{-\delta, 0\} & \text{if } x - y = e_1 \text{ or } x - y = e_2 \\ \{0, \delta\} & \text{if } x - y = e_3. \end{cases}$$

See Figure 1.1.

Remark 2.1 (The Newton Polygon \mathbb{T}). — Because of the conditions (2.2) imposed on the discrete gradients of h , not any continuous function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ admits a discretization that is an admissible height function on \mathcal{T}_δ . For that to be possible, f must be Lipschitz and, in addition, ∇f must belong to a certain polygon, called “Newton Polygon”. If we choose a coordinate system on \mathbb{R}^2 with coordinate

axes parallel to the vectors e_1, e_2 of Figure 1.1, then \mathbb{T} is the triangle of vertices $(0, 0), (-1, 0), (0, -1)$. If the function $f : D \mapsto \mathbb{R}$ is Lipschitz and $\nabla f \in \mathbb{T}$, then there exists a 1-discretization f^\sharp that is a height function on D^\sharp (see e.g. [LT15b, Sec. 2.1] and [LT15a, Sec. 2.5.4]).

Given $A \subset \mathcal{T}_\delta$ and a boundary height function $g : \partial A \mapsto \delta\mathbb{Z}$, a height function $h : A \mapsto \delta\mathbb{Z}$ is said to be compatible with the boundary value if it coincides with g on ∂A . We let $\Omega_{A,g}$ be the (possibly empty) set of height functions on A , compatible with g .

2.2. The dynamics, its stationary measure and the limit shape phenomenon

Given $A \subset \mathcal{T}_\delta$ and a boundary height function $g : \partial A \mapsto \delta\mathbb{Z}$, we define the dynamics as a continuous-time Markov process on $\Omega_{A,g}$, assumed to be non-empty. This can be described as follows: each vertex in $A \setminus \partial A$ has an i.i.d. mean-one Poisson clock. When the clock at x rings, the height function at x is resampled uniformly among the possible values of $h(x)$ given $\{h(y), y \neq x\}$. Note that, because of (2.2), the number of such possible values is either 1 (in which case we can equivalently say that the update does not occur) or 2. It is also elementary to see that the uniform measure $\pi_{A,g}$ on $\Omega_{A,g}$ is stationary and reversible.

Assume now that the domain A is the discretization D^\sharp of an open domain of \mathbb{R}^2 and the boundary height g is the restriction to ∂D^\sharp of the discretization f^\sharp of some $f : \mathbb{R}^2 \mapsto \mathbb{R}$, such that $\Omega_{D^\sharp, f^\sharp|_{\partial D^\sharp}} \neq \emptyset$. Then, the height function exhibits a *Law of Large Numbers* [CKP01, CLP98], better known as *Limit Shape Theorem* in this context: there exists a unique, continuous, deterministic function $\phi : D \mapsto \mathbb{R}$ such that, for every $\epsilon > 0$,

$$(2.3) \quad \pi_{D^\sharp, f^\sharp|_{\partial D^\sharp}} \left(\|h - \phi\|_\infty \geq \epsilon \right) \xrightarrow{\delta \rightarrow 0} 0,$$

where $\|h - \phi\|_\infty := \max_{x \in D^\sharp} |h(x) - \phi(x)|$ and ϕ satisfies the boundary condition $\phi|_{\partial D} = f|_{\partial D}$. The function ϕ is called *limit shape* and it is the unique minimizer of a *surface tension functional*

$$(2.4) \quad \Phi(\phi) = \int_D \sigma(\nabla \phi) dx$$

among all functions with Dirichlet boundary conditions $\phi|_{\partial D} = f|_{\partial D}$. The function σ is fully explicit [Ken09] but its specific form is not crucial for the present work. In the following remark we summarize all what we need to know about it.

Remark 2.2 (Surface tension and Euler–Lagrange equation). — The function σ is convex; it is finite in the Newton Polygon \mathbb{T} , equals zero on its boundary $\partial\mathbb{T}$ and equals $+\infty$ outside of it. In the interior of \mathbb{T} , σ is real analytic and strictly convex. More precisely, $\sigma_{ii} > 0$, where $\sigma_{ij}, 1 \leq i, j \leq 2$ denotes the derivative of σ with respect to its i^{th} and j^{th} arguments. In the regions of D where the limit shape ϕ is C^2 , it satisfies the Euler–Lagrange equation

$$(2.5) \quad \operatorname{div}(\nabla \sigma \circ \nabla \phi) := \sum_{i,j=1}^2 \sigma_{ij}(\nabla \phi) \partial_{x_i x_j}^2 \phi = 0,$$

which is a non-linear elliptic PDE.

A very interesting and well-known aspect of the limit shape ϕ of lozenge tilings is that it can exhibit facets. That is, there are domains D and boundary conditions f such that the minimizer ϕ contains both *liquid regions* where $\nabla\phi$ belongs to the interior of \mathbb{T} and (2.5) holds, and also *frozen regions* (with non-empty interior) where $\nabla\phi \in \partial\mathbb{T}$. The boundary between liquid and frozen regions is usually called *frozen boundary* and it consists in algebraic curves [ADPZ20, KO07].

In this work, we study the mixing time of the Glauber dynamics, under the assumption that the limit shape ϕ has no such frozen regions. Understanding the interplay between dynamics and frozen boundaries remains an interesting and challenging open problem.

2.3. The mixing time bounds

In order to define the precise setting of our results, we start from a simply connected, bounded, open domain U of \mathbb{R}^2 , whose boundary is a Jordan curve.

ASSUMPTION 2.3 (The limit shape). — We let $\phi : U \mapsto \mathbb{R}$ be a C^∞ function that satisfies the Euler–Lagrange equation (2.5) in U . We further require that ϕ contains no frozen regions or more precisely that there exists $a > 0$ such that

$$(2.6) \quad \inf_{x \in U} d(\nabla\phi(x), \partial\mathbb{T}) \geq a,$$

with d the Euclidean distance.

Actually, because the surface tension σ is C^∞ and real analytic in the interior of the Newton polygon, the weaker assumption

$$(2.7) \quad \phi \in C^1(U) \text{ and (2.6) holds}$$

implies that ϕ is C^∞ and real analytic in U [Gia83, Sec. VI.3]. In particular, in Assumption 2.3 we can replace the C^∞ by the C^2 requirement. We refer to [ADPZ20] for much deeper information on the regularity properties of limit shapes.

ASSUMPTION 2.4 (The boundary condition on the dynamics). — We let W be a simply connected, bounded, open domain of \mathbb{R}^2 , whose boundary is a Jordan curve, and such that $\overline{W} \subset U$. The dynamics is defined in the discrete set W^\sharp and the boundary condition is given by $\phi^\sharp|_{\partial W^\sharp}$, i.e., the restriction to the boundary of W^\sharp of some K -discretization of ϕ with $\Omega_{W^\sharp, \phi^\sharp|_{\partial W^\sharp}}$ non-empty.

The constants in the following theorem can depend implicitly on K, W, ϕ .

THEOREM 2.5. — For the Glauber dynamics in W^\sharp with boundary condition $\phi^\sharp|_{\partial W^\sharp}$ satisfying Assumption 2.4, there exists a constant $c_- > 0$ and, for every $\eta > 0$, a constant $c_+ < \infty$ such that the mixing time satisfies for all $\delta > 0$

$$(2.8) \quad c_- \delta^{-2} \leq T_{\text{mix}} \leq c_+ \delta^{-2-\eta}.$$

The reason why the upper bound is of the form $\delta^{-2+o(1)}$, instead of the more precise $\delta^{-2}|\log \delta|^C$ for some $C > 0$ obtained in [CMT12] when the limit shape is affine, is that Theorem 3.7 below (taken from [Las21]) provides a control on moments (of arbitrary order) of equilibrium height fluctuations, and not a stronger control on their tail.

Let us mention that the result of Theorem 2.5 would hold (with minor changes in the proof) under the weaker assumption that the boundary height $h|_{\partial W^\#}$ satisfies

$$(2.9) \quad \|h|_{\partial W^\#} - \phi|_{\partial W^\#}\|_\infty \stackrel{\delta \rightarrow 0}{\asymp} O(\delta^\eta)$$

for every $\eta \in (0, 1)$. In this case, the constant c_+ in the theorem would also depend on the constants implicit in the estimate (2.9).

Remark 2.6. — The example one should keep in mind is that U is the “natural liquid region” for some limit shape ϕ (i.e. the maximal domain where ϕ is smooth [KO07]), and W is obtained from U by removing an ε -neighborhood of ∂U , with $\varepsilon > 0$ arbitrarily small but independent of the lattice mesh δ . One reason why we cannot just take $\varepsilon = 0$ is that we will need some uniform control on the smoothness of ϕ in a neighborhood of ∂W and, in general, the derivatives of ϕ are singular at the natural boundary of liquid regions.

Remark 2.7. — The novelty in Theorem 2.5 is the upper bound. In fact, the argument for $T_{mix} \geq c_- \delta^{-2}$ given in [LT15a, Sec. 5.2] for the case where the limit shape ϕ is affine, works identically in the general case and we will not repeat it. We emphasize that, in contrast to the argument for the upper bound, the one for the lower bound is very soft and boils down to proving that at times δ^{-2} times a small constant, the height function is essentially unchanged (at the macroscopic scale) with respect to the initial condition.

3. Preliminaries and strategy of the proof

We start with a couple of useful general facts on continuous-time, irreducible Markov chains on a finite state space Ω (we refer the reader e.g. to [LP17]). First of all, the mixing time is defined as

$$(3.1) \quad T_{mix} := \inf \left\{ t \geq 0 : \max_{\eta \in \Omega} \|\mu_t^\eta - \pi\| \leq 1/4 \right\}$$

where μ_t^η denotes the law of the process at time t with initial condition η and π is the unique stationary probability measure. Here, $\|\cdot - \cdot\|$ denotes the total variation distance between probability measures:

$$(3.2) \quad \|\mu - \nu\| = \max_{B \subset \Omega} |\mu(B) - \nu(B)| = \max_{B \subset \Omega} \mu(B) - \nu(B).$$

We will use the following standard sub-multiplicativity property of total variation:

$$(3.3) \quad \max_{\eta \in \Omega} \|\mu_t^\eta - \pi\| \leq 2^{-\lfloor t/T_{mix} \rfloor}.$$

We will also need a version of the union-bound with respect to time, for continuous time Markov chains:

LEMMA 3.1. — Given $\eta \in \Omega$, let $c(\eta)$ be the sum of the rates of the transitions outgoing η and let $c^+ := \max\{c(\eta), \eta \in \Omega\}$. Then, for any $A \subset \Omega$ and $T > 0$,

$$(3.4) \quad \mathbb{P}_\pi(\exists t \leq T : X_t \in A) \leq 8Tc^+\pi(A)$$

where X_t denotes the state of the chain at time t and \mathbb{P}_π is the law of the stationary process (i.e., with initial condition distributed according to π).

Proof. — Let τ be the hitting time of A , so that the l.h.s. of (3.4) is $\mathbb{P}_\pi(\tau \leq T)$, and let $L_A(t)$ be the total time spent in A up to time t . We have

$$(3.5) \quad \mathbb{P}_\pi(\tau \leq T) = \mathbb{P}_\pi(\tau \leq T; L_A(2T) \leq 1/(2c^+)) + \mathbb{P}_\pi(\tau \leq T; L_A(2T) > 1/(2c^+)).$$

For the first term in the r.h.s. we note that the time spent in A just after τ is stochastically dominated below by an exponential random variable Y of parameter c^+ (this is a lower bound on the time it takes before the next update occurs). Therefore, by the strong Markov property that probability is upper bounded by

$$(3.6) \quad \mathbb{P}_\pi(\tau \leq T) \mathbb{P}(Y < 1/(2c^+)) \leq (1/2) \mathbb{P}_\pi(\tau \leq T).$$

As for the second term in the r.h.s. of (3.5), we upper bound it by

$$(3.7) \quad \begin{aligned} \mathbb{P}_\pi(L_A(2T) > 1/(2c^+)) &= \mathbb{P}_\pi\left(\int_0^{2T} \mathbf{1}_{X_t \in A} dt > 1/(2c^+)\right) \\ &\leq 2c^+ \mathbb{E}_\pi\left(\int_0^{2T} \mathbf{1}_{X_t \in A} dt\right) = 4c^+ T \pi(A) \end{aligned}$$

where we used Markov's inequality and stationarity of π . Putting everything together, (3.4) follows. \square

In the case of the Glauber dynamics on height functions that is the focus of the present work, we will use the following immediate corollary:

COROLLARY 3.2. — Consider the Glauber dynamics $\{h_t\}_{t \geq 0}$ in a domain $D^\sharp \subset \mathcal{T}_\delta$ of cardinality $|D^\sharp|$ and with any boundary condition $h|_{\partial D}$. Given a set $A \subset \Omega_{D^\sharp, h|_{\partial D^\sharp}}$ of height functions, one has

$$(3.8) \quad \mathbb{P}_\pi(\exists t < T : h_t \in A) \leq 8|D^\sharp|T\pi(A),$$

with $\pi = \pi_{D^\sharp, h|_{\partial D^\sharp}}$.

A crucial tool is monotonicity. Given two height functions h, h' on a domain D^\sharp , we say that $h \leq h'$ iff $h(x) \leq h'(x)$ for every $x \in D^\sharp$. The Glauber dynamics is well-known (and easily checked) to be monotone with respect to this partial order: we can couple all the dynamics started from different initial conditions in a way that if two initial conditions $h^{(1)}, h^{(2)}$ satisfy $h^{(1)} \leq h^{(2)}$, then almost surely $h_t^{(1)} \leq h_t^{(2)}$ for every $t \geq 0$, with obvious notations. Note that it is possible to have $h_{\partial D^\sharp}^{(1)} \neq h_{\partial D^\sharp}^{(2)}$, in which case the two dynamics evolve with different boundary conditions. Monotonicity is inherited by the equilibrium measure: if g, g' are boundary heights on ∂A and $g \leq g'$ pointwise, then one has the stochastic domination $\pi_{A, g} \preceq \pi_{A, g'}$.

Along the proof of Theorem 2.5, we will need to consider “constrained” or “censored” versions of the dynamics, where the height function is constrained to satisfy $h^-(x) \leq h_t(x) \leq h^+(x)$ for all times t in certain deterministic intervals, where h^\pm are two

fixed functions and “constrained” means that updates that violate these inequalities are discarded (censored). A very useful auxiliary result is a bound on the mixing time for this constrained dynamics:

LEMMA 3.3 ([CMT12, Th. 4.3]). — *The dynamics in a domain D^\sharp constrained between h^- and h^+ has a mixing time*

$$(3.9) \quad T_{\text{mix}} \leq c \operatorname{diam}(D)^2 \|h^+ - h^-\|_\infty^2 \delta^{-4} (\log \delta)^2$$

for some universal constant c , where $\operatorname{diam}(D)$ denotes the diameter of D .

Remark 3.4. — Note that the equilibrium measure of the dynamics constrained between h^- and h^+ is simply

$$(3.10) \quad \pi_{W^\sharp, \phi^\sharp|_{\partial W^\sharp}} \left(\cdot | h^- \leq h \leq h^+ \right).$$

From Lemma 3.3 it follows immediately:

COROLLARY 3.5. — *For the unconstrained Glauber dynamics in a domain D^\sharp , the mixing time satisfies*

$$(3.11) \quad T_{\text{mix}} \leq c \operatorname{diam}(D)^4 \delta^{-4} (\log \delta)^2$$

for some universal constant c that is independent of the boundary condition $h|_{\partial D^\sharp}$.

This follows simply by noting that the dynamics is trivially constrained between the minimal and maximal configurations in $\Omega_{D^\sharp, h|_{\partial D^\sharp}}$, whose height functions are 1-Lipschitz so that they are at sup-distance at most $\operatorname{diam}(D)$.

We also recall the main result of [LT15b]:

THEOREM 3.6. — *For the Glauber dynamics with domain and boundary condition satisfying Assumption 2.4, for every $\epsilon, \eta > 0$ there exists $c > 0$ such that for every $\delta > 0$ and for every $h \in \Omega_{W^\sharp, \phi^\sharp|_{\partial W^\sharp}}$ one has*

$$(3.12) \quad \mathbb{P}_h \left(\exists t \in \left[(c/\delta)^{2+\eta}, (1/\delta)^5 \right] : \max_{x \in W^\sharp} |h_t(x) - \phi(x)| \geq \epsilon \right) \leq \epsilon,$$

where \mathbb{P}_h denotes the law of the process started at h .

Actually, [LT15b, Th. 3.1] provides a similar statement but for any fixed $t \in [(c/\delta)^{2+\eta}, (1/\delta)^5]$. However, as easily seen from [LT15b, Claim 6.1], the actual proof of [LT15b, Th. 3.1] gives the stronger bound (3.12). Note that, since by Corollary 3.5 the mixing time satisfies $T_{\text{mix}} \ll \delta^{-5}$, by the limit shape theorem (2.3) the event $\max_{x \in W^\sharp} |h_t(x) - \phi(x)| \geq \epsilon$ has very small probability also for $t \geq \delta^{-5}$.

Finally, a key ingredient of the present work is the following sharp bound on height fluctuations in domains with smooth limit shape:

THEOREM 3.7 ([Las21, Prop. 1.2-1.4]). — *Let the domains U, W and the limit shape $\phi : U \mapsto \mathbb{R}$ satisfy Assumptions 2.3 and 2.4. There exists a sequence of discrete domains $W^\delta \subset \mathcal{T}_\delta$ and of boundary conditions $h_\delta : \partial W^\delta \rightarrow \delta\mathbb{Z}$ such that:*

- $d_{\mathcal{H}}(W^\delta, W^\sharp) = O(\delta)$, where W^\sharp is defined in Section 2.1 and $d_{\mathcal{H}}$ denotes Hausdorff distance;
- for all v in ∂W^\sharp , $|h_\delta(v) - \phi(v)| \leq C\delta$;

- for all v in W^\sharp and $n \geq 1$, $\pi_{W^\delta, h_\partial}(|h(v) - \phi(v)|^n) \leq C_n \delta^n |\log \delta|^{2n}$.

The constants C and C_n for $n \geq 1$ depend on the domain U and on the function $\phi : U \mapsto \mathbb{R}$. The dependence on ϕ is continuous for the topology induced by the Sobolev norm $\|\cdot\|_{k,p}$, $k = 33$, $p = 3$ in U , i.e. the L^p norm on all derivatives of ϕ up to order k in U .

We will actually use the following simple consequence, that follows immediately by monotonicity:

COROLLARY 3.8. — *In Theorem 3.7 one can take $W^\delta = W^\sharp$ and h_∂ as the restriction to ∂W^\sharp of any K -discretization of ϕ . In this case, the constants C_n depend also on K .*

3.1. Sketch of the proof of Theorem 2.5

Here, we sketch the strategy of the proof of the mixing time upper bound. The key point is to show that, with probability at least $1 - \epsilon$, for any initial condition and any time

$$(3.13) \quad t \in I := [\delta^{-2-\eta}, \delta^{-5}],$$

the height function satisfies

$$(3.14) \quad h^- \leq h_t \leq h^+ \text{ in the whole domain } W^\sharp,$$

for two well-chosen, time-independent functions $h^\pm : W^\sharp \mapsto \mathbb{R}$ such that $\|h^- - h^+\|_\infty \leq \delta^{1-\eta}$ (This should be compared with Theorem 3.6, where $\delta^{1-\eta}$ is replaced by the much bigger ϵ). This is the outcome of Theorem 4.1. Therefore, in the time interval I the true dynamics and the one constrained to stay between h^- and h^+ coincide with probability $1 - \epsilon$. On the other hand, the equilibrium measure $\pi_{W^\sharp, \phi^\sharp|_{\partial W^\sharp}}$ and the constrained one (recall Remark 3.10) have total variation distance $o(1)$ as $\delta \rightarrow 0$, as we will deduce from Corollary 3.8. Then, thanks to Lemma 3.3 (which can be used to estimate the mixing time of the constrained dynamics), this implies the mixing time upper bound of Theorem 2.5 (with η replaced by a constant times η , but η is anyway arbitrary); see Section 4 for details on this implication.

In order to show that (3.14) holds with probability $1 - \epsilon$, the idea is to introduce a sequence of deterministic upper and lower bounds $h_i^\pm : W^\sharp \mapsto \mathbb{R}$, $i \geq 0$ and to prove that, for an appropriately chosen sequence of deterministic times t_i , $i \geq 0$, one has $h_i^- \leq h_t \leq h_i^+$ for all $t \in [t_i, \delta^{-5}]$, with high probability. The distance $\|h_i^+ - h_i^-\|_\infty$ will decrease with i , starting from a value $O(1)$ for $i = 0$. The core of the proof is to carry out the induction over i until a final step i_{\max} such that $t_{i_{\max}} = O(\delta^{-2+O(\eta)})$ and $\|h_{i_{\max}}^- - h_{i_{\max}}^+\|_\infty \leq \delta^{1-\eta}$. See Theorem 4.1.

The general idea guiding the choice of h_i^\pm and the times t_i is as follows (by symmetry, we focus on the upper bounds on the height). First of all, recall that the time δ^{-5} is much larger than the mixing time of the Glauber dynamics on W^\sharp (by Corollary 3.5) so we will ignore the restriction $t \leq \delta^{-5}$ in this sketch. Suppose by induction that, for some i , with high probability, $h_t \leq h_i^+$ for all $t \geq t_i$. Fix some $x \in W^\sharp$ and assume

for simplicity that it is at a macroscopic (i.e. not vanishing as $\delta \rightarrow 0$) distance from the boundary ∂W^\sharp . Let us try to prove that with overwhelming probability

$$(3.15) \quad h_t(x) \leq h_{i+1}^+(x) := h_i^+(x) - \delta$$

for $t \geq t_{i+1}$, where t_{i+1} will be determined in a moment. (The actual relation between h_i^+ and h_{i+1}^+ will be somewhat less trivial than $h_i^+ - h_{i+1}^+ = \delta$, see (4.14), but for the purpose of this sketch we ignore this issue). By monotonicity and the induction hypothesis, it is enough to restart the dynamics at time t_i from (a discretization of) h_i^+ and to let it evolve with the constraint that $h_t \leq h_i^+$. One of the main ideas in the proof (which goes back to [CMT12]) is that running the dynamics only inside a ball B centered at x and of any radius r_i gives an upper bound on the height function⁽¹⁾. It turns out that the correct choice for r_i is such that $\|h_i^+ - h_i^-\|_\infty \times r_i^2 \approx \delta^{1-\eta}$, see more precisely (4.10). In particular, one can verify that $r_i \ll 1$ as $\delta \rightarrow 0$ (see (4.11)). This choice guarantees that the curvature of the height function plays a small but non-negligible effect on the evolution inside B .

An argument involving monotonicity and Lemma 3.3 (see Section 5.2) shows that the dynamics restricted to the ball B reaches equilibrium in a time of order $r_i^2 \delta^{2-2\eta} \delta^{-4}$. Taking $t_{i+1} - t_i \gtrsim r_i^2 \delta^{-2-2\eta}$ so that the dynamics is well mixed in B , we are reduced to the analysis of the equilibrium measure in the small region B , with boundary conditions $h_i^+|_{\partial B}$ on ∂B , which we view as a small perturbation of the limit shape ϕ . More precisely, we need for (3.15) to occur with high probability. For this, we will use corollary 3.8 to control separately the expectation and the fluctuations of $h(x)$ with these boundary conditions. Finally, we will easily verify (see Eq. (4.12)) that the final time $t_{i_{\max}} = O(\delta^{-2-O(\eta)})$ as desired.

This overall idea essentially goes back to [CMT12] and was adapted in [LT15b] to deal with non-affine limit shapes. However, in [LT15b] we were only able to carry out the strategy up to a step i_{\max} such that $\|h_{i_{\max}}^+ - h_{i_{\max}}^-\| \leq \epsilon$, with ϵ small but independent of δ . This allowed us to prove Theorem 3.6, which is much weaker than the statement of Theorem 2.5. In the present work, Theorem 3.6 is used instead as the first step of the new iteration which works up to the step i_{\max} where $\|h_{i_{\max}}^+ - h_{i_{\max}}^-\| \leq \delta^{1-\eta}$, and allows us to obtain the essentially sharp mixing time upper bound in (2.8). The crucial point limiting how far the iteration can be run is that we need to sharply control height fluctuations in balls B of diameter r_i for all $i \leq i_{\max}$, uniformly over the relevant boundary conditions on ∂B produced by the dynamics. In [LT15b], we were limited to working in domains B where the curvature of the limit shape plays a non-negligible role, but its derivatives of order at least 3 are negligible, so that a Taylor expansion of the limit shape up to order 2 is a good approximation; this restricted us to the regime $r_{i_{\max}} \sim \delta^{1/2-\eta}$. In contrast, here we will have $r_{i_{\max}} \sim \delta^\eta \gg \delta^{1/2-\eta}$; in domains of such size, the limit shape solves a fully non-linear problem and this is where the recent equilibrium estimates from Theorem 3.7 are crucial. It turns out that correctly applying Theorem 3.7 will require some delicate information on perturbations of the non-linear Euler–Lagrange

⁽¹⁾The case of x close to the boundary ∂W^\sharp requires special care at this step, because B does not necessarily fit inside W^\sharp . This is where it is important that the macroscopic shape ϕ is defined in a domain U that contains \overline{W} , as in Assumption 2.4.

PDE (2.5), which will be stated in Section 5.1 and proved in Section 6. The latter is the technical core of the proof.

4. Mixing time bounds

In this section, we provide a precise version of the inductive statement mentioned above and, assuming its validity, we conclude the proof of the mixing time upper bound. (Let us recall that the lower bound instead is proven exactly like in the case where the limit shape ϕ is affine, see [LT15a, Sec. 5.2].)

Our goal is to prove, say,

$$(4.1) \quad T_{mix} = O\left(\delta^{-2-10\eta}\right)$$

for every $\eta > 0$. Define the function $\psi : \mathbb{R}^2 \mapsto \mathbb{R}$ as

$$(4.2) \quad \psi(x) = \psi(x_1, x_2) = c - e^{x_1/\xi} - e^{x_2/\xi}$$

where $\xi > 0$ will be chosen small later and c is such that $\min_{x=(x_1, x_2) \in U} \psi(x_1, x_2) = 1$. Note that (with $\nabla^2 \psi$ the Hessian matrix of ψ)

$$(4.3) \quad \begin{aligned} \nabla \psi &= -\frac{1}{\xi} \left(e^{x_1/\xi}, e^{x_2/\xi} \right), \\ \nabla^2 \psi &= -\frac{1}{\xi^2} \begin{pmatrix} e^{x_1/\xi} & 0 \\ 0 & e^{x_2/\xi} \end{pmatrix} =: -\begin{pmatrix} m_1^2(x) & 0 \\ 0 & m_2^2(x) \end{pmatrix}. \end{aligned}$$

In particular, the second derivatives are negative and

$$(4.4) \quad \max \left(-\partial_{x_1}^2 \psi, -\partial_{x_2}^2 \psi \right) \geq \frac{1}{\xi} \max \left(|\partial_{x_1} \psi|, |\partial_{x_2} \psi| \right).$$

This specific choice of ψ will be crucial in the proof of Theorem 5.4 (see (6.59)), but until then the reader can think of ψ as any nice strictly concave function. Given $w \in W$, we define the positive quadratic form Q_w as

$$(4.5) \quad x \mapsto Q_w(x) := -\left\langle (x - w), \nabla^2 \psi(w), (x - w) \right\rangle \geq 0.$$

We let $t_0 = \delta^{-2-\eta}$ and we fix a constant $\epsilon_0 > 0$ that will be chosen small enough later. We define a set A_0 of height functions as

$$(4.6) \quad A_0 = \left\{ h \in \Omega_{W^\sharp, \phi^\sharp|_{\partial W^\sharp}} : \mathbb{P}_h \left(|h_t - \phi| < \epsilon_0 \psi \text{ for all } t \in [0, \delta^{-5} - t_0] \right) \geq 1 - \sqrt{\epsilon_0} \right\}$$

where \mathbb{P}_h denotes the law of the process started from h and the inequality $|h_t - \phi| \leq \epsilon_0 \psi$ is intended as holding pointwise in W^\sharp . Note that Theorem 3.6 plus the Markov property of the dynamics says that, for any initial condition h ,

$$(4.7) \quad \mathbb{E}_h \left[\mathbb{P}_{h_{t_0}} \left(\exists t \in [0, \delta^{-5} - t_0], x \in W^\sharp : |h_t(x) - \phi(x)| \geq \epsilon_0 \psi(x) \right) \right] \leq \epsilon_0.$$

Therefore, by Markov's inequality,

$$(4.8) \quad \begin{aligned} &\mathbb{P}_h(h_{t_0} \notin A_0) \\ &= \mathbb{P}_h \left[\mathbb{P}_{h_{t_0}} \left(\exists t \in [0, \delta^{-5} - t_0], x \in W^\sharp : |h_t(x) - \phi(x)| \geq \epsilon_0 \psi(x) \right) > \sqrt{\epsilon_0} \right] \leq \sqrt{\epsilon_0}, \end{aligned}$$

i.e, for any initial condition h

$$(4.9) \quad \mathbb{P}_h(h_{t_0} \in A_0) \geq 1 - \sqrt{\epsilon_0}.$$

For $0 \leq i \leq i_{max} := (\epsilon_0 - \delta^{1-4\eta})/\delta$,⁽²⁾ define ϵ_i , r_i and t_i by induction by

$$(4.10) \quad \epsilon_i := \epsilon_0 - \delta i, \quad r_i := \sqrt{\delta^{1-\eta}/\epsilon_i}, \quad t_{i+1} - t_i = \left(\frac{r_i}{\delta}\right)^2 \delta^{-5\eta} = \frac{\delta^{-1-6\eta}}{\epsilon_i}.$$

Note that ϵ_i is decreasing, r_i is increasing,

$$(4.11) \quad r_0 = \sqrt{\delta^{1-\eta}/\epsilon_0}, \quad \epsilon_{i_{max}} = \delta^{1-4\eta}, \quad r_{i_{max}} = \delta^{3\eta/2} \ll 1,$$

and

$$(4.12) \quad t_{i_{max}} = t_0 + \delta^{-2-6\eta} \sum_{k=\delta^{-4\eta}}^{\epsilon_0/\delta} \frac{1}{k} = O\left(\delta^{-2-6\eta} \log(\epsilon_0/\delta)\right) = O(\delta^{-2-7\eta}).$$

Similarly to Eq. (4.6), we also define the following i -dependent sets of height functions $A_i, i \geq 1$:

$$(4.13) \quad A_i = \left\{ h \in \Omega_{W^\sharp, \phi^\sharp|_{\partial W^\sharp}} : \mathbb{P}_h\left(|h_t - \phi| < \epsilon_i \psi \text{ for all } t \in [0, \delta^{-5} - t_i]\right) \geq 1 - 2i\delta^3 \right\}.$$

Note that this definition effectively corresponds to choosing the functions h_i^+ mentioned in Section 3.1 as

$$(4.14) \quad h_i^\pm = \phi \pm \epsilon_i \psi.$$

The inductive statement underlying the core of the proof is the following:

THEOREM 4.1. — *There exists $\delta_0 > 0$ such that for every $0 \leq i < i_{max}$, for every $h \in A_i$ and if the lattice mesh satisfies $\delta < \delta_0$, one has*

$$(4.15) \quad \mathbb{P}_h(h_{t_{i+1}-t_i} \in A_{i+1}) \geq 1 - \delta^2.$$

Proof of (2.8) (upper bound) given Proposition 4.1. — Like in the argument leading from Theorem 3.6 to (4.9), we apply successively at all times t_i , $0 \leq i < i_{max}$ the Markov property and Markov's inequality. Thanks to a union bound on $i \leq i_{max} \leq \delta^{-1}$, we see that, for any initial condition h ,

$$(4.16) \quad \mathbb{P}_h(h_{t_{i_{max}}} \in A_{i_{max}}) \geq 1 - \sqrt{\epsilon_0} - \delta^2 i_{max} \geq 1 - 2\sqrt{\epsilon_0}$$

if δ is small enough. Let $T = \delta^{-2-10\eta}$. We have then, with μ_T^h denoting the law of the process h_T at time T with initial condition h and $\pi := \pi_{W^\sharp, \phi^\sharp|_{\partial W^\sharp}}$,

$$(4.17) \quad \begin{aligned} \|\mu_T^h - \pi\| &= \max_B \left(\mathbb{P}_h(h_T \in B) - \pi(B) \right) \\ &\leq \max_B \left(\mathbb{P}_h(h_T \in B, h_{t_{i_{max}}} \in A_{i_{max}}) - \pi(B) \right) + 2\sqrt{\epsilon_0}. \end{aligned}$$

Next, note that by the Markov property

$$(4.18) \quad \mathbb{P}_h(h_T \in B, h_{t_{i_{max}}} \in A_{i_{max}}) \leq \max_{h' \in A_{i_{max}}} \mathbb{P}_{h'}(h_{T-t_{i_{max}}} \in B).$$

⁽²⁾It is understood that i_{max} is the integer part of this value, but we drop all “integer parts” here and in the following, for lightness of notation.

Let τ be the stopping time

$$(4.19) \quad \tau := \inf \left\{ t \geq 0 : |h_t(x) - \phi(x)| \geq \epsilon_{i_{max}} \psi(x) \text{ for some } x \in W^\sharp \right\}.$$

For $h' \in A_{i_{max}}$, the probability that $\tau < \delta^{-5} - t_{i_{max}}$ is at most $2i_{max}\delta^3 \leq \delta^{3/2}$. On the event $\tau \geq \delta^{-5} - t_{i_{max}}$, instead, the evolution in the time interval $[0, \delta^{-5} - t_{i_{max}}]$ can be perfectly coupled with the dynamics constrained between $\phi - \delta^{1-4\eta}\psi$ and $\phi + \delta^{1-4\eta}\psi$, whose law we denote here by $\hat{\mathbb{P}}_{h'}$ (here we have used that $\epsilon_{i_{max}} = \delta^{1-4\eta}$). Noting that $T - t_{i_{max}} \leq \delta^{-5} - t_{i_{max}}$, this implies

$$(4.20) \quad \mathbb{P}_h \left(h_T \in B, h_{t_{i_{max}}} \in A_{i_{max}} \right) \leq \max_{h' \in A_{i_{max}}} \left[\mathbb{P}_{h'} \left(h_{T-t_{i_{max}}} \in B; \tau < \delta^{-5} - t_{i_{max}} \right) + \mathbb{P}_{h'} \left(h_{T-t_{i_{max}}} \in B; \tau \geq \delta^{-5} - t_{i_{max}} \right) \right] \leq \delta^{3/2} + \max_{h' \in A_{i_{max}}} \hat{\mathbb{P}}_{h'} \left(h_{T-t_{i_{max}}} \in B \right).$$

By Lemma 3.3, the constrained dynamics has mixing time upper bounded by $\delta^{-2-9\eta}$, for δ small. We note also that

$$(4.21) \quad \delta^{-2-9\eta} \ll (1/2)\delta^{-2-10\eta} \leq T - t_{i_{max}} \ll \delta^{-5} - t_{i_{max}}$$

as $\delta \rightarrow 0$. Therefore, using the sub-multiplicative property (3.3), the r.h.s. of (4.20) is upper bounded by

$$(4.22) \quad \hat{\pi}(B) + \delta^{3/2} + \epsilon_0$$

for δ small enough, uniformly in h' , with $\hat{\pi}$ the equilibrium measure of the constrained dynamics. To conclude, we recall (see (3.10)) that $\hat{\pi}$ is simply $\pi_{W^\sharp, \phi^\sharp|_{\partial W^\sharp}}$ conditioned to the event $\{\phi - \delta^{1-4\eta}\psi \leq h \leq \phi + \delta^{1-4\eta}\psi\}$ and that, by Corollary 3.8, this event occurs under $\pi_{W^\sharp, \phi^\sharp|_{\partial W^\sharp}}$ with probability $1 + o(1)$ as $\delta \rightarrow 0$, and in particular with probability at least $1 - \epsilon_0$. Wrapping up, we have obtained that

$$(4.23) \quad \left\| \mu_T^h - \pi \right\| \leq \delta^{3/2} + 2\epsilon_0 + 2\sqrt{\epsilon_0} \leq \frac{1}{4}$$

if δ and ϵ_0 are small enough, which implies (4.1), by definition of mixing time and the choice $T = \delta^{-2-10\eta}$. \square

5. Proof of Theorem 4.1

In this section, we consider a fixed $i \leq i_{max}$ and we reduce the inductive statement of Theorem 4.1 to two sharp statements on height fluctuations for some equilibrium measures with suitable non-affine boundary height, as mentioned in Section 3.1.

Note that by symmetry it is enough to prove the analogous statement (4.15) for the event

$$(5.1) \quad A'_{i+1} = \left\{ h : \mathbb{P}_h \left(h_t - \phi < \epsilon_{i+1}\psi \text{ for all } t \in [0, \delta^{-5} - t_{i+1}] \right) \geq 1 - 2(i+1)\delta^3 \right\}$$

with δ^2 replaced by $\delta^2/2$ in the right-hand side of (4.15), since the probability of deviations of h_t below $\phi - \epsilon_{i+1}\psi$ can be estimated analogously.

In analogy with (4.19), define the stopping time

$$(5.2) \quad \tau_i := \inf \left\{ t \geq 0 : |h_t(x) - \phi(x)| \geq \epsilon_i \psi(x) \text{ for some } x \in W^\sharp \right\}$$

and note that on the event $\{\tau_i \geq \delta^{-5} - t_i \geq t_{i+1} - t_i\}$, we can perfectly couple the dynamics on $[0, \delta^{-5} - t_i]$ with the one constrained between $\phi - \epsilon_i \psi$ and $\phi + \epsilon_i \psi$, whose law we denote $\widehat{\mathbb{P}}_h$. For $h \in A_i$ we have

$$\begin{aligned} \mathbb{P}_h(h_{t_{i+1}-t_i} \in A'_{i+1}) &\geq \mathbb{P}_h(h_{t_{i+1}-t_i} \in A'_{i+1}; \tau_i \geq \delta^{-5} - t_i) \\ (5.3) \quad &\geq \widehat{\mathbb{P}}_h(h_{t_{i+1}-t_i} \in A'_{i+1}) - \mathbb{P}_h(\tau_i < \delta^{-5} - t_i) \\ &\geq \widehat{\mathbb{P}}_h(h_{t_{i+1}-t_i} \in A'_{i+1}) - 2i\delta^3 \end{aligned}$$

because, by definition of A_i , $\mathbb{P}_h(\tau_i < \delta^{-5} - t_i) \leq 2i\delta^3$. Since $2i\delta^3 \leq 2i_{\max}\delta^3 \leq 2\epsilon_0\delta^2 \leq \delta^2/4$ for ϵ_0 small, it is enough to prove that, for the constrained dynamics, $h_{t_{i+1}-t_i} \in A'_{i+1}$ with $\widehat{\mathbb{P}}_h$ -probability at least $1 - \delta^2/4$. By monotonicity, the worst case for the constrained dynamics is to start from the highest configuration $h_{\max}^{(i)} \in \Omega_{W^\sharp, \phi^\sharp|_{\partial W^\sharp}}^{(i)}$ lower than $\phi + \epsilon_i \psi$, i.e., we need

$$(5.4) \quad \widehat{\mathbb{P}}_{h_{\max}^{(i)}}(h_{t_{i+1}-t_i} \in A'_{i+1}) \geq 1 - \frac{\delta^2}{4}.$$

Assume that we can prove

$$(5.5) \quad \widehat{\mathbb{P}}_{h_{\max}^{(i)}}(h_t < \phi + \epsilon_{i+1}\psi \text{ for every } t \in [t_{i+1} - t_i, \delta^{-5} - t_i]) \geq 1 - \delta^6.$$

Applying the Markov property as for A_0 above, this is equivalent to

$$(5.6) \quad \widehat{\mathbb{E}}_{h_{\max}^{(i)}} \widehat{\mathbb{P}}_{h_{t_{i+1}-t_i}}(h_t(x) \geq \phi(x) + \epsilon_{i+1}\psi(x) \text{ for some } x \in W^\sharp, t \leq \delta^{-5} - t_{i+1}) \leq \delta^6.$$

Via Markov's inequality, this gives

$$(5.7) \quad \widehat{\mathbb{P}}_{h_{\max}^{(i)}} \left[\widehat{\mathbb{P}}_{h_{t_{i+1}-t_i}}(h_t < \phi + \epsilon_{i+1}\psi \text{ for all } t \leq \delta^{-5} - t_{i+1}) \geq 1 - \delta^3 \right] \geq 1 - \delta^3,$$

that is, since $\delta^3 < 2(i+1)\delta^3$,

$$(5.8) \quad \widehat{\mathbb{P}}_{h_{\max}^{(i)}}(h_{t_{i+1}-t_i} \in A'_{i+1}) \geq 1 - \delta^3 \geq 1 - \frac{\delta^2}{4}$$

and (5.4) follows for δ small. Therefore, we have reduced our task to proving (5.5).

5.1. Some sharp equilibrium results on mesoscopic scales

The crucial ingredients for the proof of Eq. (5.5) are two sharp equilibrium fluctuation statements on mesoscopic scales.

DEFINITION 5.1. — *Given $w \in W^\sharp$, let $E_{i,w}$ be the ellipse centered at w and determined by the equation*

$$(5.9) \quad x \in E_{i,w} \Leftrightarrow Q_w(x) \leq r_i^2$$

with Q_w the quadratic form in (4.5). Note that the horizontal (resp. vertical) axes of the ellipse are of length $r_i/m_1(w)$ (resp. $r_i/m_2(w)$).

Remark 5.2. — The family of ellipses $\{E_{i,w}\}_{i \leq i_{\max}, w \in W}$ has aspect ratio uniformly bounded away from zero and infinity (once ξ and W are given). Note also that the ellipses have axes parallel to the Cartesian axes, because the matrix $\nabla^2 \psi$ is diagonal.

Note that by our choice of r_i we have that, for $x \in E_{i,w}$,

$$(5.10) \quad \phi(x) + \epsilon_i \psi(x) = \phi(x) + \epsilon_i \psi(w) + \epsilon_i \langle \nabla \psi(w), x - w \rangle - \frac{\epsilon_i}{2} Q_w(x) + o(\delta)$$

where $o(\delta)$ is in fact $O(r_i^3 \epsilon_i) = O(\delta^{1+\eta/2})$ uniformly in i, w (the constants implicit in the error terms can depend on the parameter ξ that enters the definition of ψ). By construction of the ellipse,

$$(5.11) \quad \epsilon_i \psi(x) = \epsilon_i \left(C_{i,w} + \langle \nabla \psi(w), x - w \rangle \right) + o(\delta) \text{ for } x \in \partial E_{i,w}$$

where

$$(5.12) \quad C_{i,w} = \psi(w) - \frac{1}{2} r_i^2.$$

We emphasize that even for w close to ∂W^\sharp , both ϕ and ψ are well defined over the whole ellipse $E_{i,w}$ because $r_i \ll 1$ for all $i \leq i_{\max}$ and therefore $E_{i,w}$ fits in U .

DEFINITION 5.3. — We let $\phi_{i,w} : E_{i,w} \mapsto \mathbb{R}$ denote the limit shape in $E_{i,w}$, that is the solution of the PDE (2.5) in $E_{i,w}$ with boundary condition on $\partial E_{i,w}$ given by

$$(5.13) \quad f_{i,w}(x) := \phi(x) + \epsilon_i \left(C_{i,w} + \langle \nabla \psi(w), x - w \rangle \right), \quad x \in \partial E_{i,w}.$$

Comparing with (5.11), we see that the b.c. is $o(\delta)$ away from $[\phi + \epsilon_i \psi]|_{\partial E_{i,w}}$. Note also that the boundary condition of $\phi_{i,w}$ is just $\phi|_{\partial E_{i,w}}$, up to an additive constant and a linear function. The additive constant has a trivial effect on the limit shape $\phi_{i,w}$; the linear does not, because the PDE (2.5) is non-linear.

Theorems 5.4 and 5.7 below summarize the information we need on the local equilibrium in $E_{i,w}$, respectively on the limit shape $\phi_{i,w}$ itself and on fluctuations around it.

THEOREM 5.4. — The limit shape $\phi_{i,w}$ in $E_{i,w}$ satisfies

$$(5.14) \quad \phi_{i,w}(x) = \epsilon_i C'_{i,w} + \phi(x) + \epsilon_i \langle \nabla \psi(w), x - w \rangle - a \epsilon_i Q_w(x) + o(\delta) + O\left(\epsilon_i^2 r_i^2\right)$$

where Q_w is the positive quadratic form (4.5) and

$$(5.15) \quad C'_{i,w} = \psi(w) - \left(\frac{1}{2} - a\right) r_i^2$$

is independent of x , the constant a is smaller than $1/4$ for ξ small (uniformly in $\epsilon_0, w, i \leq i_{\max}, \delta$) and the constants implicit in the error terms are uniform w.r.t. $w, i \leq i_{\max}, x \in E_{i,w}$.

Remark 5.5. — For any ellipse $E_{i,w}^\rho$ with the same center and aspect ratio as $E_{i,w}$, just shrunk by a factor $\rho < 1$, Theorem 5.4 implies that

$$(5.16) \quad [\phi_{i,w}]|_{\partial E_{i,w}^\rho} - [\phi + \epsilon_i \psi]|_{\partial E_{i,w}^\rho} = -\epsilon_i r_i^2 \left(\frac{1}{2} - a\right) (1 - \rho^2) + o(\delta) + O\left(\epsilon_i^2 r_i^2\right)$$

and the r.h.s. is strictly negative for $\rho < 1$: the limit shape $\phi_{i,w}$ is strictly lower than $\phi + \epsilon_i \psi$, in the interior of $E_{i,w}$.

DEFINITION 5.6. — We let $\pi_{i,w}$ be the uniform distribution on height functions on the discretized domain $(E_{i,w}^{1/2})^\sharp$ with boundary condition given by (any 1-discretization of) $\phi_{i,w}$ on $\partial(E_{i,w}^{1/2})^\sharp$.

THEOREM 5.7. — For every $n \geq 1$ there exists a constant $C_n > 0$ such that for all $w \in W^\sharp, i \leq i_{\max}$, all x in $(E_{i,w}^{1/2})^\sharp$,

$$(5.17) \quad \pi_{i,w} \left[\left| h(x) - \phi_{i,w}(x) \right|^n \right] \leq C_n \delta^n |\log \delta|^{2n}.$$

Let us emphasize here that compared to Theorem 3.7, the main difference in the above statement is the uniformity of the constants C_n with respect to w and i .

The reason why we introduce the smaller ellipse $E_{i,w}^{1/2}$ is that, as in Theorem 3.7, we want the limit shape $\phi_{i,w}|_{E_{i,w}^{1/2}}$ to be the restriction of the limit shape in a larger domain (in this case, $E_{i,w}$).

Theorems 5.4 and 5.7 will be proven in Section 6. For the moment, let us assume they hold and let us complete the proof of (5.5).

5.2. Proof of Equation (5.5)

We will prove (5.5) by proving that, for $\delta < \delta_0$ for some small but positive δ_0 ,

$$(5.18) \quad \widehat{\mathbb{P}}_{h_{\max}^{(i)}} \left(h_t(w) < \phi(w) + \epsilon_{i+1}\psi(w) \text{ for every } t \in [t_{i+1} - t_i, \delta^{-5} - t_i] \right) \geq 1 - \delta^9$$

separately for each $w \in W^\sharp$ and then applying a union bound, since the cardinality of W^\sharp is $O(\delta^{-2})$. For the rest of this section, δ_0 can change from line to line but it is always a small constant that is independent of i, w .

Recall that $\widehat{\mathbb{P}}_{h_{\max}^{(i)}}$ denotes the law of the process in the whole domain W^\sharp , with height constrained between $\phi - \epsilon_i\psi$ and $\phi + \epsilon_i\psi$, and with initial condition $h_{\max}^{(i)}$ that is the maximal height function in $\Omega_{W^\sharp, \phi^\sharp|_{\partial W^\sharp}}$ that is lower than $\phi + \epsilon_i\psi$.

First we observe that, since the dynamics starts from the maximal condition allowed by the constraints, by monotonicity one can censor any update outside of $W^\sharp \cap (E_{i,w}^{1/2})^\sharp$. In order to treat on the same footing the points w close and far from the boundary of W^\sharp , it is convenient to have the height function evolving on the whole discretized ellipse $(E_{i,w}^{1/2})^\sharp$, even when this does not entirely fit in W^\sharp . This is done via a chain of monotonicity arguments in Proposition 5.9 below. For this, we need a couple of auxiliary definitions:

DEFINITION 5.8. — Let $h_{\max}^{(i,w)} : (E_{i,w}^{1/2})^\sharp \mapsto \delta\mathbb{Z}$ denote the maximal height function on $(E_{i,w}^{1/2})^\sharp$ lower than $\phi + \epsilon_i\psi$. Let $\widetilde{\mathbb{P}}_{i,w}$ be the law of the Glauber dynamics in $(E_{i,w}^{1/2})^\sharp$ started from the initial condition $h_{\max}^{(i,w)}$, evolving with height function constrained between $\phi_{i,w} - \delta^{1-2\eta}$ and $\phi + \epsilon_i\psi$, and with boundary height given by the restriction $h_{\max}^{(i,w)}|_{\partial(E_{i,w}^{1/2})^\sharp}$.

PROPOSITION 5.9. — The probability in the l.h.s. of (5.18) is lower bounded by

$$(5.19) \quad \widetilde{\mathbb{P}}_{i,w} \left(h_t(w) < \phi(w) + \epsilon_{i+1}\psi(w) \text{ for every } t \in [t_{i+1} - t_i, \delta^{-5} - t_i] \right).$$

We emphasize that the main difference between the processes of law $\widehat{\mathbb{P}}_{h_{max}}^{(i)}$ and $\widetilde{\mathbb{P}}_{i,w}$ is that in the former, the height evolves in W^\sharp while in the latter it evolves in the discrete ellipse $(E_{i,w}^{1/2})^\sharp$. For most points w one has $W^\sharp \supset (E_{i,w}^{1/2})^\sharp$ but, for w close to the boundary, this is not true and the implication of Proposition 5.9 requires extra care.

Proof of Proposition 5.9. — Let us note first of all that

$$(5.20) \quad \phi < \phi_{i,w} - \delta^{1-2\eta} \text{ on } E_{i,w}^{1/2}.$$

To see this, start by observing that on $\partial E_{i,w}$ one has

$$(5.21) \quad \phi_{i,w} - \delta^{1-2\eta} = \phi + \epsilon_i \psi - \delta^{1-2\eta} \geq \phi + \delta^{1-4\eta} - \delta^{1-2\eta} > \phi$$

because $\psi \geq 1$ (as observed just after (4.2)) and $\epsilon_i \geq \epsilon_{i_{max}} = \delta^{1-4\eta}$. Since both $\phi_{i,w}$ and ϕ are solutions of (2.5) in $E_{i,w}$ and the former has higher boundary condition than the latter, by the maximum principle for elliptic PDE the inequality is preserved in the interior of the domain, and in particular on $E_{i,w}^{1/2}$.

If $(E_{i,w}^{1/2})^\sharp \subset W^\sharp$, the statement of Proposition 5.9 is an immediate consequence of the following three observations:

- (i) one can lower bound the probability in (5.18) by censoring the updates outside $(E_{i,w}^{1/2})^\sharp$ (because the initial condition is maximal). In order to avoid introducing a new notation, we still call $\widehat{\mathbb{P}}_{h_{max}}^{(i)}$ the law of the censored process. Now, both under $\widehat{\mathbb{P}}_{h_{max}}^{(i)}$ and $\widetilde{\mathbb{P}}_{i,w}$, the height function evolves only inside the domain $(E_{i,w}^{1/2})^\sharp$;
- (ii) the lower constraint $h_t \geq \phi_{i,w} - \delta^{1-2\eta}$ of the dynamics with law $\widetilde{\mathbb{P}}_{i,w}$ is more stringent than the constraint $h_t \geq \phi - \epsilon_i \psi$ of the dynamics $\widehat{\mathbb{P}}_{h_{max}}^{(i)}$, in view of (5.20) and of $\psi \geq 0$;
- (iii) the two dynamics have the same upper constraint $h_t \leq \phi + \epsilon_i \psi$;
- (iv) the initial condition $h_{max}^{(i)}|_{(E_{i,w}^{1/2})^\sharp}$ of the dynamics $\widehat{\mathbb{P}}_{h_{max}}^{(i)}$ is lower (or equal) to that of the dynamics $\widetilde{\mathbb{P}}_{i,w}$, because for the latter one takes the highest configuration lower than $\phi + \epsilon_i \psi$, while for the former one additionally requires that $h_{max}^{(i)} \in \Omega_{W^\sharp, \phi^\sharp|_{\partial W^\sharp}}$. The same observation holds for the boundary conditions on $\partial(E_{i,w}^{1/2})^\sharp$: the one of the dynamics $\widehat{\mathbb{P}}_{h_{max}}^{(i)}$ is lower or equal to that of the dynamics $\widetilde{\mathbb{P}}_{i,w}$.

The argument for w sufficiently close to the boundary of W^\sharp , so that $(E_{i,w}^{1/2})^\sharp$ does not fit in it, is slightly more involved. In this case, under $\widehat{\mathbb{P}}_{h_{max}}^{(i)}$ the height in $(E_{i,w}^{1/2})^\sharp \setminus W^\sharp$ is time-independent. In fact, the height in $(E_{i,w}^{1/2})^\sharp \setminus W^\sharp$ can be imagined to be fixed to any configuration compatible with the actual boundary condition ϕ^\sharp on ∂W^\sharp ; in particular, it is convenient to imagine that h is fixed to ϕ^\sharp on the whole $(E_{i,w}^{1/2})^\sharp \setminus W^\sharp$. In this case, one sees easily that the steps (i) to (iv) above again imply the statement of the proposition, once one adds the extra observation that the height in $(E_{i,w}^{1/2})^\sharp \setminus W^\sharp$ of the process with law $\widetilde{\mathbb{P}}_{i,w}$ is deterministically higher

than the (time-independent) one of the process with law $\widehat{\mathbb{P}}_{h_{max}^{(i)}}$, because by (5.21) ϕ^\sharp is lower than $\phi_{i,w} - \delta^{1-2\eta}$. \square

We are then left with the task of showing:

$$(5.22) \quad \text{for } \delta < \delta_0, \text{ the l.h.s. of (5.19) is lower bounded by } 1 - \delta^9.$$

A first observation in this respect is that the mixing time of the dynamics with law $\mathbb{P}_{i,w}$ is upper bounded by

$$(5.23) \quad c_0 \delta^{-4} (\log \delta)^2 r_i^2 \left(\epsilon_i r_i^2 + \delta^{1-2\eta} \right)^2$$

for some absolute constant c_0 that can change from line to line in the rest of the proof. To see this, it is sufficient to recall Lemma 3.3, together with the fact that the dynamics is constrained between $\phi_{i,w} - \delta^{1-2\eta}$ and $\phi + \epsilon_i \psi$ and to observe that on $E_{i,w}^{1/2}$,

$$(5.24) \quad \left| (\phi_{i,w} - \delta^{1-2\eta}) - (\phi + \epsilon_i \psi) \right| \leq c_0 \left(\delta^{1-2\eta} + \epsilon_i r_i^2 \right),$$

as follows from (5.16). We emphasize that the constant c_0 is uniform with respect to $i \leq i_{max}, w \in W^\sharp$. Recalling that by definition $\epsilon_i r_i^2 = \delta^{1-\eta}$ (see (4.10)), we see that the mixing time is upper bounded by

$$(5.25) \quad c_0 \left(\frac{r_i}{\delta} \right)^2 \delta^{-4\eta} = c_0 \delta^\eta (t_{i+1} - t_i).$$

Therefore, by the sub-multiplicative property (3.3), at times $t \geq t_{i+1} - t_i$ the chain is at total variation distance $2^{-1/(c_0 \delta^\eta)}$ from its equilibrium measure $\tilde{\pi}_{i,w}$. We have then that the probability in (5.19) is lower bounded by

$$(5.26) \quad \tilde{\mathbb{P}}_{i,w}^{eq} \left(h_t(w) < \phi(w) + \epsilon_{i+1} \psi(w) \text{ for every } t \in [0, \delta^{-5}] \right) - 2^{-1/(c_0 \delta^\eta)},$$

where $\tilde{\mathbb{P}}_{i,w}^{eq}$ denotes the law of the equilibrium process. We claim that under $\tilde{\pi}_{i,w}$ one has

$$(5.27) \quad \tilde{\pi}_{i,w} \left(h(w) < \phi(w) + \epsilon_{i+1} \psi(w) \right) \geq 1 - \delta^{18}$$

for $\delta \leq \delta_0$, where the exponent 18 has been chosen simply so that we will get δ^9 in (5.22), but it could be replaced by any other positive number, at the price of changing δ_0 .

Assume for a moment that (5.27) holds; since the cardinality of $(E_{i,w}^{1/2})^\sharp$ is smaller than δ^{-2} we deduce from corollary 3.2 that

$$(5.28) \quad \tilde{\mathbb{P}}_{i,w}^{eq} \left(h_t(w) < \phi(w) + \epsilon_{i+1} \psi(w) \text{ for every } t \in [0, \delta^{-5}] \right) \geq 1 - \delta^{10}.$$

Putting everything together, (5.22) follows, provided we prove (5.27).

Recall from eq. (3.10) that

$$(5.29) \quad \tilde{\pi}_{i,w} = \pi_{i,w} \left(\cdot \mid \phi_{i,w} - \delta^{1-2\eta} \leq h \leq \phi + \epsilon_i \psi \right),$$

where $\pi_{i,w}$ is given in Definition 5.6. Since the event in (5.27) is decreasing, in (5.29) we can drop the conditioning on $h \leq \phi + \epsilon_i \psi$. The conditioning on $h \geq \phi_{i,w} - \delta^{1-2\eta}$, instead, cannot be dropped by monotonicity. However, here Theorem 5.7 enters into

play. Choosing $n = n(\eta)$ large enough in (5.17) and applying Tchebyshev's inequality, we see that

$$(5.30) \quad \pi_{i,w} \left(h \geq \phi_{i,w} - \delta^{1-2\eta} \right) \geq 1 - \delta^{18},$$

so that the entire conditioning can be dropped in the definition of $\tilde{\pi}_{i,w}$ and it is enough to prove (5.27) for the unconditional measure $\pi_{i,w}$.

To this end, note first of all that (5.16) taken at $\rho = 0$ implies that

$$(5.31) \quad \phi_{i,w}(w) = \phi(w) + \epsilon_i \psi(w) - \epsilon_i r_i^2 (1/2 - a) + o(\delta) + O(\epsilon_i^2 r_i^2).$$

Recall that $\epsilon_i r_i^2 = \delta^{1-\eta}$ and that a can be assumed to be smaller than $1/4$, see Theorem 5.4. We have then

$$(5.32) \quad \phi_{i,w}(w) < \phi(w) + \epsilon_i \psi(w) - \delta^{1-\eta} c_0$$

for some constant $c_0 > 0$ uniform with respect to w, i . Finally, recalling that $\epsilon_i - \epsilon_{i+1} = \delta$ and that ψ is bounded, we see that also

$$(5.33) \quad \phi_{i,w}(w) < \phi(w) + \epsilon_{i+1} \psi(w) - \delta^{1-\eta} \frac{c_0}{2}.$$

Therefore, we have

$$(5.34) \quad \pi_{i,w} \left(h(w) < \phi(w) + \epsilon_{i+1} \psi(w) \right) \geq \pi_{i,w} \left(h(w) < \phi_{i,w}(w) + (c_0/2) \delta^{1-\eta} \right)$$

and it follows from Theorem 5.7 (choosing $n = n(\eta)$ large enough) that the latter probability is larger than $1 - \delta^{18}$, as desired.

6. Proof of Theorems 5.4 and 5.7

In this section, we use arguments from the theory of elliptic PDEs to control the effect on the limit shape, i.e. on the solution of (2.5), of a small perturbation of its boundary conditions that amounts to adding a linear tilt to the boundary datum. The estimates are a bit delicate because we need to control also the higher derivatives of the perturbed solution, uniformly with respect to the domains and the boundary conditions we consider.

Our first goal is Theorem 5.4, that is a description of the limit shape $\phi_{i,w}$ in the elliptical domains $E_{i,w}$ of Definition 5.1, with boundary condition given by a perturbation of $\phi|_{\partial E_{i,w}}$, with ϕ the limit shape of Assumption 2.3. The first step is to rescale the problem in order to work in a fixed domain, namely the disk $B(0, 1)$, instead of the ellipses $E_{i,w}$.

DEFINITION 6.1. — *For $w \in W$, let $t_w : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be the affine map*

$$(6.1) \quad t_w(x) = w + T_w x, \quad T_w = \left(-\nabla^2 \psi(w) \right)^{-1/2},$$

see (4.3). Note that $t_w(B(0, r_i)) = E_{i,w}$ for all i , with $E_{i,w}$ the ellipse of Definition 5.1. Given $w \in W$, $i \geq 1$ we also define rescaled versions $\Phi_{i,w} : B(0, 1) \mapsto \mathbb{R}$ of the limit shapes $\phi_{i,w} : E_{i,w} \mapsto \mathbb{R}$ of Theorem 5.4 by

$$(6.2) \quad \Phi_{i,w}(x) = \frac{1}{r_i} \left(\phi_{i,w}(t_w(r_i x)) - \phi_{i,w}(w) \right).$$

Note that the $\Phi_{i,w}$ has been normalized so that $\Phi_{i,w}(0) = 0$ and that, since $\phi_{i,w}$ solves the Euler–Lagrange equation (2.5) in $E_{i,w}$, $\Phi_{i,w}$ is a solution $u : B(0, 1) \mapsto \mathbb{R}$ of the modified PDE

$$(6.3) \quad \sum_{a,b=1}^2 \sigma_{ab}^{(w)} (\nabla u(x)) \partial_{x_a x_b}^2 u(x) = 0$$

where, denoting Σ and $\Sigma^{(w)}$ the 2×2 matrices of elements σ_{ab} and $\sigma_{ab}^{(w)}$ respectively, we have

$$(6.4) \quad \Sigma^{(w)}(\cdot) = (T_w)^{-1} \Sigma \left((T_w)^{-1} \cdot \right) (T_w)^{-1}.$$

Note that, like (2.5), Eq. (6.3) is a non-linear elliptic PDE.

In analogy with the rescaled, perturbed limit shapes $\Phi_{i,w}$ just defined, we introduce also the rescaled, unperturbed limit shapes. Namely, with ϕ denoting the limit shape in Assumption 2.3:

DEFINITION 6.2. — *Given $v_0 > 0$ we let \mathcal{W} be the following set of pairs consisting of a point and a C^∞ function from $B(0, 1)$ to \mathbb{R} :*

$$(6.5) \quad \mathcal{W} = \left\{ (w, \Phi) : w \in \overline{W}, \Phi : B(0, 1) \mapsto \mathbb{R}, \Phi(\cdot) \right. \\ \left. = \frac{1}{r} \left(\phi(t_w(r \cdot)) - \phi(w) \right) \text{ for some } r \in [0, v_0] \right\},$$

where for $r = 0$, by convention the equation for Φ denotes the linear map

$$(6.6) \quad \Phi(\cdot) := \langle \nabla \phi(w), t_w(\cdot) - w \rangle.$$

The constant v_0 will be taken sufficiently small later, see remark 6.7 and the proof of Corollary 6.12.

Remark 6.3. — In the rest of this section, we use the notation $W^{k,p}(U)$ and $W_0^{k,p}(U)$ to denote the Sobolev spaces of functions with derivatives of order up to k belonging to L^p in the domain U , and the subscript zero indicates that functions have “zero boundary conditions” at ∂U . When no confusion arises, we omit the argument U . For details on Sobolev spaces see [Kry08, Chapt. 8], where these spaces are denoted W_p^k and W_p^k , respectively.

In the following, whenever Sobolev spaces $W^{k,p}$ and Sobolev norms $\|\cdot\|_{k,p}$ appear, one should keep in mind that, in view of Theorem 3.7, we need the statements for $k = 33, p = 3$. We write most statements for generic exponents $k, p > 2$ in order to emphasize that the specific values just mentioned play no particular role in the proofs of this section (see also remark 6.11).

Note that, given $r > 0$ and w , the function Φ in Definition 6.2 is simply the limit shape ϕ , restricted to an ellipse centered at w and of size r , up to an affine transformation of space that turns its domain into the unit disk $B(0, 1)$. As is the case for $\Phi_{i,w}$, Φ is normalized to equal zero at the center of the ball $B(0, 1)$, and it satisfies the PDE (6.3).

LEMMA 6.4. — Consider the natural topology on \mathcal{W} induced by the Euclidean distance in \mathbb{R}^2 for the coordinate w , and the $\|\cdot\|_{k,p}$ Sobolev norm on functions in $B(0,1)$ for the coordinate Φ . With this topology, \mathcal{W} is a compact set.

Proof. — By construction, \mathcal{W} is the image of $\overline{W} \times [0, v_0]$ via the map

$$(6.7) \quad (w, r) \rightarrow \left(w, \Phi \right), \quad \Phi(\cdot) := \frac{1}{r}(\phi(t_w(r \cdot)) - \phi(w))$$

so it is enough to show that this function is continuous. For any $r > 0$ this is trivial. For the continuity for $r = 0$, note that any derivative of order 2 or more of Φ converges to 0 as $r \rightarrow 0$ uniformly with respect to w (because the limit shape ϕ is C^∞ in \overline{W} , thanks to Assumption 2.3), so that Φ converges to the linear map (6.6) in $\|\cdot\|_{k,p}$ topology uniformly as $r \rightarrow 0$. This linear map depends continuously on w and this completes the proof. \square

We need two more definitions before entering the heart of the proofs.

DEFINITION 6.5. — Given $v_0 > 0$, let

$$(6.8) \quad \mathcal{L} = \left\{ \ell : \mathbb{R}^2 \rightarrow \mathbb{R} \text{ linear}, \|\ell\| \leq v_0 \right\}$$

be the compact set of linear maps on \mathbb{R}^2 of norm at most v_0 (the choice of norm is arbitrary and unimportant for the following).

DEFINITION 6.6. — For any $(w, \Phi) \in \mathcal{W}$, we let $F_{w,\Phi}$ be the following map that takes as input a pair $(\ell, f) \in \mathcal{L} \times C_0^2(B(0,1))$ (with $C_0^2(B(0,1))$ the set of C^2 functions $f : B(0,1) \mapsto \mathbb{R}$ that vanish on $\partial B(0,1)$) and outputs the function $F_{w,\Phi}(\ell, f) : B(0,1) \mapsto \mathbb{R}$:

$$(6.9) \quad F_{w,\Phi}(\ell, f)(x) = \sum_{i,j=1}^2 \sigma_{ij}^{(w)}(\nabla(\Phi + \ell + f)(x)) \partial_{x_i x_j}^2(\Phi + f)(x), \quad x \in B(0,1),$$

with $\sigma^{(w)}(\cdot)$ defined as in (6.3).

Remark 6.7. — Recall that σ_{ab} is well defined only provided that its argument is in the Newton polygon \mathbb{T} , i.e., $\sigma_{ab}^{(w)}(u)$ is well defined only when $(T_w^{-1})u \in \mathbb{T}$, recall (6.4). On the other hand, $\nabla\Phi(x) = T_w \nabla\phi(t_w(rx))$ and, by Assumption 2.3, $\nabla\phi$ is in the Newton polygon, at distance at least $a > 0$ from its boundary. In the following, therefore, we assume that the constant v_0 in the definition of \mathcal{L} and the sup-norm of ∇f on $B(0,1)$ are small enough (as a function of the constant a) so that $\sigma_{ab}^{(w)}$ is well-defined.

Remark 6.8. — Note that, by construction, $F_{w,\Phi}(0,0) = 0$ (the identically zero function on $B(0,1)$), because, as observed above, Φ is just a rescaled version of the limit shape and therefore solves the PDE (6.3). Also, if $0 \leq i \leq i_{max}$, letting

$$(6.10) \quad \Phi(\cdot) := \frac{1}{r_i}(\phi(t_w(r_i \cdot)) - \phi(w))$$

and

$$(6.11) \quad \ell(\cdot) := \epsilon_i \langle \nabla\psi(w), t_w(\cdot) - w \rangle,$$

then

$$(6.12) \quad F_{w,\Phi}(\ell, \Phi_{i,w} - \Phi - \ell) = 0,$$

with $\Phi_{i,w}$ defined in (6.2). This is just a way of rewriting that $\Phi_{i,w}$ is a solution of (6.3).

In the following, we will let the linear maps $d_1 F_{w,\Phi}(\ell, f)$, $d_2 F_{w,\Phi}(\ell, f)$ denote the differential maps of $F_{w,\Phi}$ with respect to the first or second argument, computed at (ℓ, f) . When the differentials are computed at $(0, 0)$, we write simply $d_1 F_{w,\Phi}$, $d_2 F_{w,\Phi}$ for brevity. A direct computation gives that $d_2 F_{w,\Phi}(\ell, 0)$ is a linear elliptic operator with non-constant but smooth coefficients given by

$$(6.13) \quad d_2 F_{w,\Phi}(\ell, 0) \circ f = \sum_{i,j=1}^2 \sigma_{ij}^{(w)}(\nabla(\Phi + \ell)) \partial_{x_i x_j}^2 f + \sum_{k=1}^2 \partial_{x_k} f \sum_{i,j=1}^2 \sigma_{ijk}^{(w)}(\nabla(\Phi + \ell)) \partial_{x_i x_j}^2 \Phi,$$

with $\sigma_{ijk}^{(w)}$ the derivative of $\sigma_{ij}^{(w)}$ with respect to its k^{th} argument. As for $d_1 F_{w,\Phi}$, it transforms a linear map ℓ into

$$(6.14) \quad d_1 F_{w,\Phi} \circ \ell = \sum_{k=1}^2 \ell_k \sum_{i,j=1}^2 \sigma_{ijk}^{(w)}(\nabla \Phi) \partial_{x_i x_j}^2 \Phi, \quad \ell_k := \partial_{x_k} \ell(x)$$

which is a function from $B(0, 1)$ to \mathbb{R} .

Let us now start the proof of Theorems 5.4, 5.7 using the above definitions. The first step is based on a classical result on linear elliptic PDEs.

PROPOSITION 6.9. — *For every pair of integers $k, p > 2$, for every $(w, \Phi) \in \mathcal{W}$ and every $\ell \in \mathcal{L}$, the map $f \mapsto d_2 F_{w,\Phi}(\ell, 0) \circ f$ is a continuous invertible map from $W_0^{k,p} := W_0^{k,p}(B(0, 1))$ to $W^{k-2,p} := W^{k-2,p}(B(0, 1))$. Furthermore, there exists a constant C such that for all $f \in W_0^{k,p}$, $\|f\|_{k,p} \leq C \|d_2 F_{w,\Phi}(\ell, 0) \circ f\|_{k-2,p}$. The constant C is uniform over admissible choices $(w, \Phi) \in \mathcal{W}$, $\ell \in \mathcal{L}$.*

Proof of Proposition 6.9. — The map $f \mapsto d_2 F_{w,\Phi}(\ell, 0) \circ f$ is trivially continuous, see (6.13), so the only question is the existence and continuity of the inverse map. Since the disk $B(0, 1)$ is a smooth bounded domain and the coefficients of the operator $d_2 F_{w,\Phi}(\ell, 0)$ are C^∞ , [Kry08, Th. 11.3.2 and Th. 9.2.3] imply that $d_2 F_{w,\Phi}(\ell, 0)$ is invertible and that $[d_2 F_{w,\Phi}(\ell, 0)]^{-1}$ is continuous (and therefore bounded) from $W_0^{k,p}$ to $W^{k-2,p}$.

For the uniform bound on the norm of $[d_2 F_{w,\Phi}(\ell, 0)]^{-1}$, first note that the map $(w, \Phi, \ell) \rightarrow d_2 F_{w,\Phi}(\ell, 0)$ is continuous when considering the topology described in Lemma 6.4 on the domain $\mathcal{W} \times \mathcal{L}$ and the operator norm from $W_0^{k,p}$ to $W^{k-2,p}$ on the codomain. Since the inverse is defined everywhere, this implies that $(w, \Phi, \ell) \rightarrow [d_2 F_{w,\Phi}(\ell, 0)]^{-1}$ is also continuous. We conclude because $\mathcal{W} \times \mathcal{L}$ is compact by Lemma 6.4. \square

The main point in the proof of Theorem 5.4 is the following implicit function theorem. For $(w, \Phi) \in \mathcal{W}$ and $\ell \in \mathcal{L}$, let

$$(6.15) \quad \chi = -(d_2 F_{w,\Phi})^{-1} \circ d_1 F_{w,\Phi} \circ \ell,$$

i.e $\chi \in W_0^{k,p}$ is the function $\chi : B(0,1) \mapsto \mathbb{R}$ which solves the linear elliptic PDE

$$(6.16) \quad d_2 F_{w,\Phi} \circ \chi = -d_1 F_{w,\Phi} \circ \ell.$$

PROPOSITION 6.10. — *If the constant v_0 in the definition of \mathcal{W}, \mathcal{L} is chosen small enough, then for every $(w, \Phi) \in \mathcal{W}$ and every $\ell \in \mathcal{L}$, there exists a unique $f \in W_0^{k,p}(B(0,1))$ such that $F_{w,\Phi}(\ell, f) = 0$. Furthermore the map $(w, \Phi, \ell) \rightarrow f$ is continuous and satisfies*

$$(6.17) \quad \|f - \chi\|_{k,p} \leq C \|F_{w,\Phi}(\ell, \chi)\|_{k-2,p}$$

where the constant C is uniform with respect to $(w, \Phi) \in \mathcal{W}$ and $\ell \in \mathcal{L}$.

Remark 6.11. — We recall also (see for instance [Kry08, Th. 10.4.10]) that for integer $k \geq 1$ and $p > 2$, $W^{k,p}(B(0,1))$ is continuously embedded into

$$(6.18) \quad C^{k-1,1-2/p}(B(0,1)) \subset W^{k-1,\infty}(B(0,1))$$

where $C^{k-1,1-2/p}(B(0,1))$ is the Hölder space of functions whose derivatives up to order $k-1$ are $(1-2/p)$ -Hölder continuous. In particular,

$$(6.19) \quad \|u\|_{C^{k-1,1-2/p}(B(0,1))} \leq N \|u\|_{k,p}$$

for some constant $N = N(k, p)$. Therefore, taking the integers $k, p > 2$ we have that f in Proposition 6.10 is actually twice continuously differentiable on $B(0,1)$ and therefore is a classical solution of the PDE $F_{w,\Phi}(\ell, f) = 0$, see (6.9).

COROLLARY 6.12. — *The family $\{\Phi_{i,w}\}_{i \leq i_{max}, w \in W}$ is precompact in $W^{k,p}(B(0,1))$.*

Proof of Corollary 6.12. — For this, recall Remark 6.8. One has $r_i \leq r_{i_{max}} = \delta^{3\eta/2} \leq v_0$, with v_0 the constant that enters the definition of \mathcal{L}, \mathcal{W} , so that the pair (w, Φ) belongs to \mathcal{W} . Similarly, the linear map ℓ satisfies $\ell \in \mathcal{L}$ provided that the constant ϵ_0 introduced just before (4.6) is small enough (because v_0 in the definition of \mathcal{L} is fixed, while $\epsilon_i \leq \epsilon_0$.) The claim of the corollary then follows from Proposition 6.10 together with Eq. (6.10), which says that f of Proposition 6.10 is in this case just $\Phi_{i,w} - \Phi - \ell$. \square

Proof of proposition 6.10. — In view of $F_{w,\Phi}(0,0) = 0$ and of (6.16), Proposition 6.10 is almost the usual statement of the implicit function theorem applied to $F_{w,\Phi}$, but with a precise estimate on the error bound in (6.17) which is a bit more delicate than usual and which will be important later.

First, existence and uniqueness in the statement comes from the corresponding statement for the limit shape PDE, equation (2.5). Indeed, $u := \Phi + f + \ell$ solves (6.3) with Dirichlet-type boundary conditions on $\partial B(0,1)$, which is equivalent to solving (2.5) in an ellipse. However, we will prove existence of f constructively, by a fixed point argument; this will give as byproduct the claimed continuity and the estimate on the error bound.

For the moment, let us fix $(\Phi, w) \in \mathcal{W}, \ell \in \mathcal{L}$. Since the function $F_{w,\Phi}(\cdot, \cdot)$ is smooth with respect to both arguments it is easy to see that, given any constant

$\mathcal{C} > 0$, we can find ϵ small enough such that, for any two functions g and g' in $W_0^{k,p}$ with $\|g\|_{k,p}, \|g'\|_{k,p} \leq \epsilon$, we have

$$(6.20) \quad \left\| F_{w,\Phi}(\ell, g') - F_{w,\Phi}(\ell, g) - d_2 F_{w,\Phi}(\ell, g) \circ (g' - g) \right\|_{k-2,p} \leq \mathcal{C} \|g - g'\|_{k,p}.$$

In particular, it turns out to be convenient to choose

$$(6.21) \quad \mathcal{C} := \frac{1}{5 \sup_{(w,\Phi) \in \mathcal{W}} \|(d_2 F_{w,\Phi})^{-1}\|}$$

where here and later in this proof, $\|d_2 F_{w,\Phi}^{-1}\|$ denotes the operator norm of $(d_2 F_{w,\Phi})^{-1}$ from $W^{k-2,p}$ to $W_0^{k,p}$. Thanks to Proposition 6.9, we have $\mathcal{C} > 0$. By continuity of $d_2 F_{w,\Phi}(\cdot, \cdot)$ (the continuity is uniform w.r.t. $(w, \Phi) \in \mathcal{W}$ because \mathcal{W} is compact by Lemma 6.4), for any $g \in W_0^{k,p}$ with $\|g\|_{k,p} \leq \epsilon$ with ϵ small enough, we also have

$$(6.22) \quad \|d_2 F_{w,\Phi}(\ell, g) - d_2 F_{w,\Phi}(0, 0)\| \leq \mathcal{C},$$

(here the norm is the operator norm from $W_0^{k,p}$ to $W^{k-2,p}$), provided that v_0 in the definition of \mathcal{L} is small enough, so that the linear map ℓ has small norm. Once more, we choose \mathcal{C} as in (6.21).

We define $g_0 := \chi$ and, by induction,

$$(6.23) \quad g_{n+1} - g_n := -(d_2 F_{w,\Phi})^{-1} \circ F_{w,\Phi}(\ell, g_n),$$

i.e the sequence $(g_n)_{n \geq 0}$ is a Newton approximation sequence, except that we keep constant the point where the differential d_2 is computed. We will show that g_n converges to the desired solution f , which in addition satisfies the desired estimates.

To this purpose, we will prove by induction that

$$(6.24) \quad \|F_{w,\Phi}(\ell, g_n)\|_{k-2,p} \leq 2^{-n} \|F_{w,\Phi}(\ell, g_0)\|_{k-2,p}$$

$$(6.25) \quad \|g_{n+1} - g_n\|_{k,p} \leq 2^{-n} \|(d_2 F_{w,\Phi})^{-1}\| \|F_{w,\Phi}(\ell, g_0)\|_{k-2,p}$$

for all n and, along the way, we will make sure that

$$(6.26) \quad \|g_n\|_{k,p} \leq \epsilon,$$

so that (6.20) and (6.22) can be applied. If this holds, then $\{g_n\}$ is a Cauchy sequence and the limit f satisfies $F_{w,\Phi}(\ell, f) = 0$ and (6.17) with $C = 2 \sup_{\mathcal{W}} \|(d_2 F_{w,\Phi})^{-1}\|$ which is finite because $(w, \Phi) \mapsto (d_2 F_{w,\Phi})^{-1}$ is continuous and \mathcal{W} is compact.

For $n = 0$, (6.24) and (6.25) are trivial. Also, we choose v_0 small enough in the definition of \mathcal{W}, \mathcal{L} so that

$$(6.27) \quad \|g_0\|_{k,p} \leq \frac{\epsilon}{2}, \quad \|F_{w,\Phi}(\ell, g_0)\|_{k-2,p} \|(d_2 F_{w,\Phi})^{-1}\| \leq \frac{\epsilon}{4}$$

so that in particular also (6.26) holds for $n = 0$. Assume that the three claims hold up to some step $n - 1$. From (6.27) and (6.25) (for all $k \leq n - 1$) one easily sees that

$$(6.28) \quad \|g_n\|_{k,p} \leq \frac{\epsilon}{2} + 2 \|(d_2 F_{w,\Phi})^{-1}\| \|F_{w,\Phi}(\ell, g_0)\|_{k-2,p} \leq \epsilon$$

so that (6.26) follows at step n .

As for (6.24), we write

$$\begin{aligned}
 (6.29) \quad & \|F_{w,\Phi}(\ell, g_n)\|_{k-2,p} \\
 & \leq \|F_{w,\Phi}(\ell, g_n) - F_{w,\Phi}(\ell, g_{n-1}) - d_2 F_{w,\Phi}(\ell, g_{n-1}) \circ (g_n - g_{n-1})\|_{k-2,p} \\
 & \quad + \|F_{w,\Phi}(\ell, g_{n-1}) + d_2 F_{w,\Phi} \circ (g_n - g_{n-1})\|_{k-2,p} \\
 & \quad + \|(d_2 F_{w,\Phi}(\ell, g_{n-1}) - d_2 F_{w,\Phi}) \circ (g_n - g_{n-1})\|_{k-2,p}.
 \end{aligned}$$

The second term in the last expression is zero by (6.23). The sum of the first and third terms is upper bounded, thanks to (6.20) and (6.22), by

$$(6.30) \quad 2\mathcal{C}\|g_n - g_{n-1}\|_{k,p} \leq 4\mathcal{C} 2^{-n} \|(d_2 F_{w,\Phi})^{-1}\| \|F_{w,\Phi}(\ell, g_0)\|_{k-2,p}$$

$$(6.31) \quad \leq 2^{-n} \|F_{w,\Phi}(\ell, g_0)\|_{k-2,p},$$

where in the first inequality we used (6.25) for $n-1$ and in the second one the definition of \mathcal{C} . Then, (6.24) at step n follows. Given this, (6.25) at step n is proven immediately.

Concerning the continuity statement, first observe that the map

$$(6.32) \quad (w, \Phi, \ell, g) \rightarrow F_{w,\Phi}(\ell, g)$$

is continuous for the $W^{k,p}$ topology on Φ and g and the $W^{k-2,p}$ topology on the target space. Indeed, the map $(\Phi, \ell, g) \rightarrow \nabla(\Phi + \ell + g)$ is by definition continuous for the $W^{k-1,p}$ topology and in particular (by Remark 6.11) also for the $W^{k-2,\infty}$ topology. The maps

$$(6.33) \quad (u \in \mathbb{T}, w \in W) \rightarrow \sigma_{ij}^{(w)}(u), 1 \leq i, j \leq 2$$

are smooth and bounded if $T_w^{-1}u$ is uniformly bounded away from the boundary of the Newton polygon \mathbb{T} , which is the case if $u = \nabla(\Phi + \ell + g)$ because the limit shape satisfies Assumption 2.3 and ℓ, g are small in the appropriate norms. By composition,

$$(6.34) \quad (w, \Phi, \ell, g) \rightarrow \sigma_{ij}^{(w)}(\nabla(\Phi + \ell + g))$$

is continuous in $W^{k-2,\infty}$. Overall $F_{w,\Phi}(\ell, g)$ is a product of functions continuous in $W^{k-2,\infty}$ and functions continuous in $W^{k-2,p}$, which is still continuous in $W^{k-2,p}$. This concludes the continuity of $(w, \Phi, \ell, g) \rightarrow F_{w,\Phi}(\ell, g)$. A similar statement holds for $d_1 F_{w,\Phi}$ and $d_2 F_{w,\Phi}$. Recalling that the g_n are defined through compositions of $F_{w,\Phi}$, $d_2 F_{w,\Phi}$ and $d_1 F_{w,\Phi}$ (the latter enters the definition of g_0), we conclude that the functions g_n are continuous in terms of (w, Φ, ℓ) . Together with the uniform control

$$(6.35) \quad \|f - g_n\|_{k,p} \leq C 2^{-n} \|F_{w,\Phi}(\ell, \chi)\|_{k-2,p} \leq C' 2^{-n},$$

this implies that f is also a continuous function of (w, Φ, ℓ) . \square

We can now conclude the proof of our equilibrium estimates.

Proof of Theorem 5.7. — Our aim is to prove (5.17) about the equilibrium measure $\pi_{i,w}$ in the discrete ellipse $(E_{i,w}^{1/2})^\sharp$. First of all, in order to apply Corollary 3.8, it is convenient to work on a domain with diameter of order 1 centered at zero, rather than with diameter of order r_i and centered at w . This can be achieved via a trivial translation that maps w to the origin and a rescaling of lengths by a factor $1/r_i$. This

way, $(E_{i,w}^{1/2})^\sharp$ is replaced by the discretization $(\mathcal{E}_{i,w}^{1/2})^\sharp$ (with lattice mesh $\delta' = \delta/r_i$ instead of δ) of an ellipse $\mathcal{E}_{i,w}^{1/2}$ centered at zero, with horizontal axis of length $1/(2m_1(w))$ and of the same aspect ratio as $E_{i,w}^{1/2}$. At the same time, we rescale the height function (both the discrete one, h , and the limit shape, $\phi_{i,w}$) by multiplying it by $1/r_i$, and by setting it to zero at the origin. That is, calling \check{h} (resp. $\check{\phi}_{i,w}$) the height functions thus obtained, we have

$$(6.36) \quad \check{h}(x) = \frac{1}{r_i} (h(w + xr_i) - h(w)), \quad x \in (\mathcal{E}_{i,w}^{1/2})^\sharp,$$

and similarly for $\check{\phi}_{i,w}$. Note that the discrete gradients of \check{h} between neighboring vertices of $(\mathcal{E}_{i,w}^{1/2})^\sharp$ are of order δ' instead of δ , and that $\check{\phi}_{i,w}$ is a limit shape (i.e. a solution of (2.5)) in $\mathcal{E}_{i,w}^{1/2}$. We call $\check{\pi}_{i,w}$ the uniform measure on height functions \check{h} in $(\mathcal{E}_{i,w}^{1/2})^\sharp$, with boundary condition given as in Definition 5.6, up to the rescaling just introduced.

After this trivial rescaling, the inequality (5.17) to be proven becomes

$$(6.37) \quad r_i^n \check{\pi}_{i,w} \left[\left| \check{h}(x) - \check{\phi}_{i,w}(x) \right|^n \right] \leq C_n \delta^n |\log \delta|^n.$$

Since (recall Definition 5.3) the limit shape $\check{\phi}_{i,w}$ is defined on an ellipse $\mathcal{E}_{i,w}$ that is $\mathcal{E}_{i,w}^{1/2}$ expanded by a factor 2, so that in particular $\mathcal{E}_{i,w}$ contains the closure of $\mathcal{E}_{i,w}^{1/2}$, applying Corollary 3.8 we see that the l.h.s. of (6.37) is upper bounded by

$$(6.38) \quad r_i^n C_n (\delta')^n |\log \delta'|^n = C_n \delta^n |\log \delta'|^n.$$

Next we remark that, in view of (4.11), one has

$$(6.39) \quad \delta^{1-3\eta/2} \leq \delta' = \frac{\delta}{r_i} \leq \delta^{(1+\eta)/2}, \quad i \leq i_{\max},$$

so that (6.37) holds, up to changing the definition of C_n by a multiplicative factor that depends only on η and n . However, as pointed out just after the statement of Theorem 5.7, there is still an important and delicate point to be checked, that is that the constants C_n can be chosen uniform with respect to $i \leq i_{\max}$ and $w \in W$. For this, we use a compactness argument based on Proposition 6.9.

From Theorem 3.7 we know that, if $\check{\phi}_{i,w}$ can be extended to some open domain U with $U \supset \overline{\mathcal{E}_{i,w}^{1/2}}$, then the constants C_n can be chosen as a function $c_n(U, \check{\phi}_{i,w}|_U)$ and that, given U , the map $f \mapsto c_n(U, f)$ is continuous with respect to the Sobolev norm $W^{k,p}(U)$. A first observation is that, while by construction the natural domain of definition of $\check{\phi}_{i,w}$ is the ellipse $\mathcal{E}_{i,w}$, we are not forced to take $U = \mathcal{E}_{i,w}$ or any other choice of U that is different for each (i, w) ; actually a finite number of such domains U suffices. In fact, since the aspect ratio of the ellipses is uniformly bounded w.r.t. $i \leq i_{\max}$, $w \in W$, we can find an integer k and a collection $\{U_s\}_{s \leq k}$ of open domains of \mathbb{R}^2 such that for every $i \leq i_{\max}$, $w \in W$,

$$(6.40) \quad \mathcal{E}_{i,w}^{1/2} \subset U_{s(i,w)} \subset \mathcal{E}_{i,w}$$

for some $s(i, w) \leq k$. Therefore, we can take C_n as some function

$$(6.41) \quad C_n = c'_n \left(s(i, w), \check{\phi}_{i,w}|_{U_{s(i,w)}} \right),$$

continuous in its second argument. Since $s(i, w)$ takes finitely many values, for the issue of the uniform bound on the constants we can disregard the dependence of c'_n on the argument s . Secondly, note that the functions $\check{\phi}_{i,w} : \mathcal{E}_{i,w} \mapsto \mathbb{R}$ and $\Phi_{i,w} : B(0, 1) \mapsto \mathbb{R}$ of (6.2) are immediately related by

$$(6.42) \quad \check{\phi}_{i,w}(T_w x) = \Phi_{i,w}(x), \quad T_w := \left(-\nabla^2 \psi(w) \right)^{-1/2}$$

see (6.1). Since both T_w and its inverse are bounded uniformly in $w \in W$, and the restriction map $\check{\phi}_{i,w}|_{\mathcal{E}_{i,w}} \mapsto \check{\phi}_{i,w}|_{U_{s(i,w)}}$ is obviously a continuous map, we can take C_n as $C_n = c''_n(\Phi_{i,w})$, with $f \mapsto c''_n(f)$ continuous. Finally, the family $\{\Phi_{i,w}\}_{i,w}$ is precompact in $W^{k,p}(B(0, 1))$ by Corollary 6.12. This implies that the supremum over i, w of $c''_n(\Phi_{i,w})$ is finite. \square

Proof of Theorem 5.4. — In the course of this proof, C denotes a constant whose value can change from line to line. We work with $\Phi_{i,w}$, which is obtained from $\phi_{i,w}$ via the rescaling (6.2) and at the end of the proof we go back to $\phi_{i,w}$ to prove (5.14). Fix $i \leq i_{max}$ and $w \in W$ and, with the notations of Remark 6.8, let $\Phi(\cdot) : B(0, 1) \mapsto \mathbb{R}$ be the rescaled limit shape around w and ℓ the linear map defined in (6.11). If χ is defined as in (6.15), by Proposition 6.10 and the definition of $\Phi_{i,w}$, we have

$$(6.43) \quad \|\Phi_{i,w} - (\Phi + \ell + \chi)\|_{k,p} \leq C \|F_{w,\Phi}(\ell, \chi)\|_{k-2,p}.$$

This should be read as saying that $\Phi_{i,w} = \Phi + \ell + \chi$ plus an error term. We will first prove that χ is essentially a quadratic function and then show that $\|F_{w,\Phi}(\ell, \chi)\|_{k-2,p}$ is very small.

Recall that the function χ solves the linear equation

$$(6.44) \quad d_2 F_{w,\Phi} \circ \chi = -d_1 F_{w,\Phi} \circ \ell,$$

where $d_2 F_{w,\Phi}$ and $d_1 F_{w,\Phi}$ are given by (6.13) and (6.14), so that the latter equation can be written as

$$(6.45) \quad \sum_{r,s=1}^2 \left(\partial_{x_r x_s}^2 \Phi \right) \sum_{k=1}^2 (\ell_k + \partial_{x_k} \chi) \sigma_{rsk}^{(w)}(\nabla \Phi) + \sum_{r,s=1}^2 \sigma_{rs}^{(w)}(\nabla \Phi) \partial_{x_r x_s}^2 \chi = 0.$$

Because of the rescaling implicit in the definition of Φ , one has $\|\partial_{x_r x_s}^2 \Phi\|_{k-2,p} \leq C r_i$ and the coefficients ℓ_k of the linear map ℓ are of order ϵ_i , so altogether $\|d_1 F_{w,\Phi} \circ \ell\|_{k-2,p} \leq C \epsilon_i r_i$. Since $[d_2 F_{w,\Phi}]^{-1}$ is a bounded linear map, we get $\|\chi\|_{k,p} \leq C \epsilon_i r_i$. Thanks to Remark 6.11, the L^∞ norm of the second derivatives of χ is also bounded by $C \epsilon_i r_i$.

In order to show that χ is well approximated by a quadratic function vanishing at the boundary of $B(0, 1)$, let $Q(x) := (1 - \|x\|^2)$ (the additive constant is there so that Q indeed vanishes on $\partial B(0, 1)$). For any constant b , we view $\chi - bQ$ as the

solution of the equation

$$(6.46) \quad d_2 F_{w,\Phi} \circ (\chi - bQ)(x) = -[d_1 F_{w,\Phi} \circ \ell](x) \\ - 2b \sum_{k=1}^2 \sigma_{kk}^{(w)}(\nabla \Phi) - 2b \sum_{r,s,k=1}^2 x_k \sigma_{rsk}^{(w)}(\nabla \Phi) \partial_{x_r x_s}^2 \Phi.$$

Since the surface tension σ is strictly convex, or more precisely $\sigma_{kk}^{(w)} > 0$, we can choose b so that the right hand side vanishes at the point $x = 0$. Reasoning as for the above bound on $\|\chi\|_{k,p}$, we see that this choice satisfies $|b| \leq C\epsilon_i r_i$. Next, we show that, with this choice of b ,

$$(6.47) \quad \left| \text{r.h.s. of (6.46)} \right| \leq C\epsilon_i r_i^2, \text{ uniformly on } B(0, 1).$$

In fact, note that the space derivatives of the right hand side of (6.46) are linear combinations of terms of the form $\partial^3 \Phi \cdot (\ell + bx)\sigma(\nabla \Phi)$, $\partial^2 \Phi \cdot (\ell + bx)\partial(\sigma(\nabla \Phi))$, $b\sigma(\nabla \Phi)\partial^2 \Phi$ and $b\partial(\sigma(\nabla \Phi))$, where we omitted all indices to simplify notations. Thanks to the higher order in the derivatives and to the estimate on b , all these terms are of order at most $\epsilon_i r_i^2$. Since the r.h.s. of (6.46) is zero for $x = 0$ by the choice of b , (6.47) holds in the entire disk $B(0, 1)$. Higher derivatives of the r.h.s. of (6.46) are also bounded by $C\epsilon_i r_i^2$ or even smaller, since each derivative acting on Φ brings a factor r_i :

$$(6.48) \quad \left| \partial_{x_{i_1} \dots x_{i_n}}^n \Phi \right| \leq C_n r_i^{n-1}.$$

In conclusion,

$$(6.49) \quad \|d_2 F_{w,\Phi} \circ (\chi - bQ)\|_{k-2,p} \leq C\epsilon_i r_i^2$$

and, by Proposition 6.9,

$$(6.50) \quad \|\chi - bQ\|_{k,p} \leq C\epsilon_i r_i^2.$$

We now turn to the “error term”

$$(6.51) \quad F_{w,\Phi}(\ell, \chi) = \sum_{r,s=1}^2 \sigma_{rs}^{(w)}(\nabla(\Phi + \ell + \chi)) \partial_{rs}^2(\Phi + \chi).$$

Comparing this equation to (6.45) and recalling that $F_{w,\Phi}(0, 0) = 0$, we see that

$$(6.52) \quad F_{w,\Phi}(\ell, \chi) = \sum_{r,s,k=1}^2 (\ell_k + \partial_{x_k} \chi) \sigma_{rsk}^{(w)}(\nabla \Phi) \partial_{x_r x_s}^2 \chi \\ + \sum_{r,s=1}^2 \partial_{x_r x_s}^2 (\Phi + \chi) \left[\sigma_{rs}^{(w)}(\nabla(\Phi + \ell + \chi)) - \sigma_{rs}^{(w)}(\nabla \Phi) - \sum_{k=1}^2 \sigma_{rsk}^{(w)}(\nabla \Phi) (\ell_k + \partial_{x_k} \chi) \right]$$

and note that the last expression contains a second order Taylor expansions of $\sigma_{rs}^{(w)}$. Using the apriori bound $\|\chi\|_{k,p} \leq C\epsilon_i r_i$, $\|\ell\| \leq C\epsilon_i$ and the uniform bound (6.48), we see that

$$(6.53) \quad \|F_{w,\Phi}(\ell, \chi)\|_{k-2,p} \leq C\epsilon_i^2 r_i.$$

Altogether, putting together (6.43), (6.50) and (6.53) we have shown that

$$(6.54) \quad \|\Phi_{i,w} - (\Phi + \ell + bQ)\|_{k,p} \leq C(\epsilon_i r_i^2 + \epsilon_i^2 r_i)$$

if the constant b is precisely chosen as above.

Finally we need to undo the scaling and show that (6.54) indeed gives (5.14) and the other statements of the Theorem. We have for $y \in E_{i,w}$, up to an additive constant,

$$(6.55) \quad \phi_{i,w}(y) = r_i \Phi_{i,w} \left(\frac{1}{r_i} t_w^{-1}(y) \right) = \phi(y) + \ell \left(t_w^{-1}(y) \right) - \frac{b}{r_i} \left\| t_w^{-1}(y) \right\|^2 + R(y)$$

$$R(y) = r_i \left[\Phi_{i,w} - (\Phi + \ell + bQ) \right] \left(\frac{1}{r_i} t_w^{-1}(y) \right).$$

By construction, one has

$$(6.56) \quad \frac{b}{r_i} \left\| t_w^{-1}(y) \right\|^2 = a \epsilon_i Q_w(y), \quad a := \frac{b}{\epsilon_i r_i}$$

with Q_w the quadratic form defined in (4.5) and, from (6.11), we see that

$$(6.57) \quad \ell \left(t_w^{-1}(y) \right) = \epsilon_i \langle \nabla \psi(w), y - w \rangle.$$

Note also that R vanishes on the boundary of $E_{i,w}$ since $Q|_{\partial B(0,1)} = 0$ while the boundary condition of $\Phi_{i,w}$ and $\Phi + \ell$ are the same. Also, by (6.54), the first derivatives of R are bounded by $C(\epsilon_i r_i^2 + \epsilon_i^2 r_i)$. Since the ellipse $E_{i,w}$ has diameter of order r_i , we conclude that $|R(x)| \leq C(\epsilon_i^2 r_i^2 + \epsilon_i r_i^3)$. Altogether, up to an additive constant,

$$(6.58) \quad \phi_{i,w}(y) = \phi(y) + \epsilon_i \langle \nabla \psi(w), y - w \rangle - a \epsilon_i Q_w(y) + O \left(\epsilon_i^2 r_i^2 + \epsilon_i r_i^3 \right)$$

where the $O(\epsilon_i^2 r_i^2 + \epsilon_i r_i^3)$ term is uniform in $y \in E_{i,w}$. By (4.10) and $r_i \leq \delta^{3\eta/2}$, we see that $\epsilon_i r_i^3 = o(\delta)$ uniformly in i , which completes the proof of (5.14). The claimed uniformity of the error terms comes from the uniform bound on the constant in Proposition 6.10 and on all operator norms involved, which are continuous functions over the compact set $\mathcal{W} \times \mathcal{L}$.

The validity of (5.15) is simply due to the fact that $\phi_{i,w}$ has to satisfy the correct boundary condition on $\partial E_{i,w}$, which imposes uniquely the additive constant $\epsilon_i C'_{i,w}$ in (5.14).

Finally, to see that $a = b/(\epsilon_i r_i)$ can be taken smaller than $1/4$ by taking ξ small enough, let us go back to the definition of b as the value such that the r.h.s. of (6.46) computed for $x = 0$ vanishes. Recalling (6.2) and (6.4), this can be rewritten as

$$(6.59) \quad 2a \sum_{k=1}^2 \partial_{x_k}^2 \psi(w) \sigma_{kk}(\nabla \phi_{i,w}(w))$$

$$= -\epsilon_i r_i \sum_{u,v,k=1}^2 \sigma_{uvk}(\nabla \phi_{i,w}(w)) \partial_{x_k} \psi(w) \partial_{x_u x_v}^2 \phi_{i,w}(w),$$

with σ_{uvk} the derivative of σ_{uv} with respect to its k^{th} argument. Using (4.4) and the fact that σ_{kk} is strictly positive (by strict convexity of the surface tension), one sees that $a \leq C\xi \leq 1/4$ for ξ small enough. \square

BIBLIOGRAPHY

- [ADPZ20] Kari Astala, Erik Duse, István Prause, and Xiao Zhong, *Dimer models and conformal structures*, <https://arxiv.org/abs/2004.02599>, 2020. ↑914
- [Agg19] Amol Aggarwal, *Universality of tiling local statistics*, <https://arxiv.org/abs/1907.09991>, to appear in *Annals of Mathematics*, 2019. ↑910
- [CKP01] Henry Cohn, Richard Kenyon, and James Propp, *A variational principle for domino tilings*, J. Am. Math. Soc. **14** (2001), 297–346. ↑907, 909, 913
- [CLL22] Pietro Caputo, Cyril Labbé, and Hubert Lacoin, *Spectral gap and cutoff phenomenon for the Gibbs sampler of $\nabla\varphi$ interfaces with convex potential*, Ann. Inst. Henri Poincaré, Probab. Stat. **58** (2022), no. 2, 794–826. ↑910, 911
- [CLP98] Henry Cohn, Michael Larsen, and James Propp, *The Shape of a Typical Boxed Plane Partition*, New York J. Math. **4** (1998), 137–165. ↑909, 913
- [CMT11] Pietro Caputo, Fabio Martinelli, and Fabio L. Toninelli, *Convergence to equilibrium of biased plane partitions*, Random Struct. Algorithms **39** (2011), 83–114. ↑911
- [CMT12] ———, *Mixing times of monotone surfaces and SOS interfaces: a mean curvature approach*, Commun. Math. Phys. **311** (2012), 157–189. ↑908, 909, 910, 915, 917, 919
- [Des02] Nicolas Destainville, *Flip dynamics in octagonal rhombus tiling sets*, Phys. Rev. Lett. **88** (2002), no. 3, article no. 030601. ↑907
- [GG23] Shirshendu Ganguly and Reza Gheissari, *Cutoff for the Glauber dynamics of the lattice free field*, Probab. Math. Phys. **4** (2023), 433–475. ↑911
- [Gia83] Mariano Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Annals of Mathematics Studies, vol. 105, Princeton University Press, 1983. ↑914
- [GPR09] Sam Greenberg, Amanda Pascoe, and Dana Randall, *Sampling biased lattice configurations using exponential metrics*, Proc. of the 20th Annual ACM-SIAM Symposium on Discrete Algorithms, ACM Press, 2009, pp. 76–85. ↑911
- [Hen97] Christopher L. Henley, *Relaxation time for a dimer covering with height representation*, J. Stat. Phys. **89** (1997), no. 3–4, 483–507. ↑907
- [Ken09] Richard Kenyon, *Lectures on dimers*, <https://arxiv.org/abs/0910.3129>, 2009. ↑910, 913
- [KL98] Claude Kipnis and Claudio Landim, *Scaling limits of interacting particle systems*, Grundlehren der Mathematischen Wissenschaften, vol. 320, Springer, 1998. ↑911
- [KO07] Richard Kenyon and Andrei Okounkov, *Limit shapes and the complex Burgers equation*, Acta Math. **199** (2007), no. 2, 263–302. ↑909, 914, 915
- [Kry08] Nicolaï V. Krylov, *Lectures on elliptic and parabolic equations in Sobolev spaces*, Graduate Studies in Mathematics, vol. 96, American Mathematical Society, 2008. ↑929, 931, 932
- [Lac16] Hubert Lacoin, *Mixing time and cutoff for the adjacent transposition shuffle and the simple exclusion*, Ann. Probab. **44** (2016), no. 2, 1426–1487. ↑910, 911
- [Las21] Benoît Laslier, *Central limit theorem for lozenge tilings with curved limit shape*, <https://arxiv.org/abs/2102.05544>, 2021. ↑910, 915, 917
- [LL19] Cyril Labbé and Hubert Lacoin, *Cutoff phenomenon for the asymmetric simple exclusion process and the biased card shuffling*, Ann. Probab. **47** (2019), no. 3, 1541–1586. ↑911
- [LP17] David A. Levin and Yuval Peres, *Markov chains and mixing times*, second ed., American Mathematical Society, 2017. ↑909, 915
- [LRS01] Michael Luby, Dana Randall, and Alistair Sinclair, *Markov Chain Algorithms for Planar Lattice Structures*, SIAM J. Comput. **31** (2001), 167–192. ↑909, 910

- [LT15a] Benoît Laslier and Fabio L. Toninelli, *How quickly can we sample a uniform domino tiling of the $2L \times 2L$ square?*, Probab. Theory Relat. Fields **161** (2015), no. 3-4, 509–559. ↑907, 908, 913, 915, 920
- [LT15b] ———, *Lozenge tilings, Glauber dynamics and macroscopic shape*, Commun. Math. Phys. **338** (2015), no. 3, 1287–1326. ↑907, 908, 910, 913, 917, 919
- [Spo93] Herbert Spohn, *Interface motion in models with stochastic dynamics*, J. Stat. Phys. **71** (1993), 1081–1132. ↑907
- [Wil04] David B. Wilson, *Mixing times of Lozenge tiling and card shuffling Markov chains*, Ann. Appl. Probab. **14** (2004), 274–325. ↑907, 908, 910, 911

Manuscript received on 18th July 2022,
 revised on 13th April 2023,
 accepted on 15th May 2023.

Recommended by Editors S. Gouëzel and H. Lacoïn.
 Published under license CC BY 4.0.



eISSN: 2644–9463

This journal is a member of Centre Mersenne.



Benoît LASLIER
 Université Paris Cité,
 UFR de Mathématiques,
 Bâtiment Sophie Germain,
 8 place Aurélie Nemour,
 75205 Paris CEDEX 13, France
 laslier@lpms.paris

Fabio TONINELLI
 Technical University of Vienna,
 Institut für Stochastik
 und Wirtschaftsmathematik,
 Wiedner Hauptstraße 8-10,
 A-1040 Vienna, Austria
 fabio.toninelli@tuwien.ac.at