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CONNECTIVITY OF THE GROMOV BOUNDARY OF THE FREE FACTOR COMPLEX

CONNEXITÉ DE BORD DE GROMOV DU COMPLEXE DES FACTEURS LIBRES

Abstract. — We show that in large enough rank, the Gromov boundary of the free factor complex is path connected and locally path connected.

Résumé. — Nous montrons que, en rang suffisamment grand, le bord de Gromov du complexe des facteurs libres est connexe par arcs et localement connexe par arcs.

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1. Introduction

A prevailing theme in geometric group theory is to study groups using actions on suitable Gromov hyperbolic spaces. One of the most successful examples of this philosophy is the action of the mapping class group of a surface on the curve graph, which is hyperbolic by the seminal work of Masur and Minsky [MM99], and which has been used to give a hierarchical description of the geometry of mapping class groups.

One core tool when studying Gromov hyperbolic spaces is that they admit natural boundaries at infinity. In the case of the curve graph, this boundary can be explicitly described in terms of topological objects. Namely, Klarreich [Kla18] proved that the boundary is the space of ending laminations. That is, the boundary can be obtained from the sphere of projective measured laminations by removing all non-minimal laminations, and then identifying laminations with the same support. Although the Gromov boundaries of curve graphs are fairly complicated topological spaces (in the case of punctured spheres, they are Nöbeling spaces [Gab14]), the connection to laminations can be used to effectively study them. Maybe most relevant for our current work, Gabai [Gab09, Gab14] used this connection to show that the boundary is path-connected and locally path-connected.

In the setting of outer automorphism groups of free groups, there are several possible analogs of the curve graph. In this article, we focus on the *free factor complex* FF_n, which is hyperbolic by a result of Bestvina–Feighn [BF14]. Similar to Klarreich's theorem, the Gromov boundary has been identified as the space of arational trees modulo a suitable equivalence [BR15]. The role of the sphere of projective measured laminations is played by the boundary of Culler–Vogtmann's Outer space, which has much more complicated local topology than a sphere.

In this article, we nevertheless begin a study of connectivity properties of the boundary at infinity of FF_n. More specifically, we show

Theorem 1.1. — *The Gromov boundary ∂FF_n of the free factor complex is path connected and locally path connected for n large enough.*

As an immediate consequence we obtain the following coarse geometric property:

Corollary 1.2. — *The free factor complex FF_n is one-ended for n large enough.*

From this, one-endedness of various other combinatorial complexes (the free splitting complex, the cyclic splitting complex, and the maximal cyclic splitting complex) can also be concluded (Corollary 6.3).

In principle, one could extract a bound what "large enough" means, by tracing through the proofs – to the best knowledge of the authors 18 works – but since we do not think there is any reason to expect that this bound is anywhere close to optimal, we don't emphasise it. In fact, we expect that the result holds for much lower n (see below).

1.1. Outline of proof

Our strategy is motivated by the case of $\mathcal{PML}$ and ending lamination space, though it requires new ideas. The starting observation is that each identification of fundamental group of a surface with one boundary component with F_n yields a copy of $\mathcal{PML}$ in ∂CV_n. Furthermore, uniquely ergodic surface lamination correspond map to arational trees. Previous work of two of the authors [CH] (building on [LS09]) shows that the set of such laminations is path-connected, giving rise to many paths of arational trees in ∂CV_n.

Furthermore, the union of all such copies of $\mathcal{PML}$ (over all possible identifications) is readily seen to be path-connected and dense – however, the intersection of two copies never contains arational trees. This suggests the back-bone of our strategy: to start with a path traversing these copies of $\mathcal{PML}$ which fails to be arational only at the points where one switches from one copy to another, inductively improve this path to become "more arational", and take a limit to obtain the final path. We now give some more details.

For simplicity, lets start "upstairs" in ∂CV_n and assuming that two arational trees, T, T' lie on two different $\mathcal{PML}$s.

We have a chain of copies of $\mathcal{PML}$ that connects the $\mathcal{PML}$ containing T to the one containing T' and where each consecutive pair in the chain intersects in the $\mathcal{PML}$ of a subsurface. Using work from surface case, in particular [CH], which builds on [LS09], we can build a path, p_0 across this chain of $\mathcal{PML}$s so that every tree in it is arational except at the intersection of the consecutive $\mathcal{PML}$s and at these it is the stable lamination of a (partial) pseudo-Anosov supported on the subsurface. We wish to have a path entirely of arational trees, so we need to ensure that each free factor acts freely and discretely at each point. Enumerating the free factors in some way, this means that we have compact sets K_i (where a given free factor E_i fails to act freely or discretely), and we aim to make our path avoid the union $\cup K_i$. We iteratively improve our path p_j which by inductive hypothesis

- is a concatenation of paths in the union of copies of $\mathcal{PML}$,
- avoids $K_1, \ldots, K_j$ entirely,
- has that every foliation on it is minimal (under the corresponding identification with $\mathcal{PML}$), except for a finite number of points,
- these points are λ_ψ, the stable laminations of (partial) pseudo-Anosovs, ψ, supported on a subsurface

to a path p_{j+1} which avoids $K_1, \ldots, K_j, K_{j+1}$ entirely and has the same structure as above. Most of the work of this paper is in showing that one can avoid an additional K_{j+1}. (The fact that p_{j+1} avoids $K_1, \ldots, K_j$ is a consequence of the fact that we can make p_{j+1} as close as we want to p_j.)

We now discuss avoiding the K_{j+1}, i.e. improving the path so that a given free factor E_{j+1} acts freely and discretely. To guarantee this, it will suffice to show that our stable laminations of the partial pseudo-Anosov is supported on a subsurface whose fundamental group is not contained in E_{j+1} and that E_{j+1} trivially intersects the fundamental group of the complement of the support of partial pseudo-Anosov. See Lemma 2.21. (It is automatic that the rest of our path avoids K_j because these points are already arational.)

Conjecturally, there is a conceptual reason why this is possible. Namely, consider the graph dual to the configuration of copies of $\mathcal{PML}$, i.e. the graph whose vertices are identifications of F_n with the fundamental group of a surface, and edges if two identifications share a large genus subsurface. One would hope that it is one-ended, and the problems are "local" – and thus, paths that pass through a "problematic" partial pseudo-Anosov for E_{j+1} can be modified to avoid problems.

However, we could neither show this conjecture, nor find another soft argument. In fact, even for the most natural complex on which $\mathrm{Out}(F_n)$ acts – the free-factor complex – was not known to be one-ended (and this is in fact a consequence of our result). Hence, we resort to a much more ad-hoc approach, which relies on many explicit checks.

In Section 4 we describe how if one of our stable laminations λ_ψ is in K_{j+1} we can find a chain of $\mathcal{PML}$s that build a detour around it and moreover, the image of this path under a large power of ψ also avoids K_{j+1}. This involves explicitly constructing relations in $\mathrm{Out}(F_n)$, and is technically the most involved part of the paper.

In this way we can improve our path p_j on a small segment around λ_ψ to obtain p_{j+1}, which now avoids K_{j+1} and where the contraction properties of ψ guarantee that it still avoids $K_1, \ldots, K_j$. Our sequence of paths $p_0, p_1, \ldots$ is a Cauchy sequence and so we obtain in the limit p_∞ which is in $(\bigcup K_j)^c$. This describes an argument that the projection of the set of arational surface type trees to the ∂FF_n is path connected. To upgrade this to showing that ∂FF_n is path connected, we show (basically via Proposition 5.3 which uses folding paths) that we can choose sequences of surface type arational trees converging to our fixed (not necessarily surface type) tree so that the paths stay in a δ neighborhood of our arational tree. In the outline we ignored some subtleties, most notably how to construct the required relations in $\mathrm{Out}(F_n)$, and that our "paths" in ∂CV_n are not necessarily continuous at the end points, though the projection of these "paths" to ∂FF_n will be. This last point exploits contraction dynamics on the boundary of the hyperbolic space ∂FF_n.

We now briefly state the structure of the paper. Section 2 collects known results about $\mathcal{PML}$ and $\mathrm{Out}(F_n)$ and modifies them for our purposes. Section 3 relates ∂CV_n and $\mathrm{Out}(F_n)$. Section 4 is the technical heart of our paper, describing how to locally avoid a fixed K_j. Section 5 proves the main theorem. Section 6 uses the main theorem to establish one-endedness of some other combinatorial complexes. There are three appendices that treat issues related to non-orientable surfaces, which we use to address ∂FF_n when n is odd.

1.2. Previous work

Our work is a direct analogue of Gabai's work [Gab09] on the connectivity of the ending lamination space $\mathcal{EL}$, which is the Gromov boundary of the curve complex. He obtains optimal path connectivity results and establishes higher connectivity, where appropriate. He goes further and in [Gab14] identifies $\mathcal{EL}$ of punctured spheres with so called Nöbeling spaces. This had previously been done in the case of the 5-times punctured sphere by Hensel and Przytycki [HP11].

However, there is one crucial difference in approaches: throughout his arguments, Gabai homotopes paths in $\mathcal{PML}$, which is a sphere. It is unclear how to do this in our setting, because the topology of ∂CV_n is still poorly understood, and it is not even known if it is locally connected. In particular, our successive improvements of paths do not proceed by homotopy.

Perhaps a better analogy for our work is Leininger and Schleimer's proof that $\mathcal{EL}$ is connected [LS09]. They do this by using "point pushing" to find a dense path connected set of arational laminations upstairs in $\mathcal{PML}$. Building on this, the second and third named authors use contraction properties of the mapping class group on the curve complex to show that the subsets of uniquely ergodic and of cobounded foliations in $\mathcal{PML}$ are both path connected [CH]. This motivates our approach and especially Proposition 5.3. However, there is again an important difference between the paths built in [LS09] or [CH], and the ones we construct here: in the former sources, the paths are often obtained from lower complexity surfaces by lifting along (branched) covers. Here, we do not have this option, and instead need to construct the paths directly.

Finally, in our setting, this previous work is of little help in getting between adjacent $\mathcal{PML}$s, where the new ideas of this article are needed.

1.3. Questions

We end this introduction with a short list of further questions which this work suggests.

(1) Is the boundary ∂FF_N already path-connected for $N \geqslant 3$? The bound 18 used here certainly carries no special significance, and is an artifact of the proof.
(2) Do the boundaries ∂FF_N satisfy (for large enough rank N) higher connectivity properties?
(3) Are there topological models for the boundaries ∂FF_N? Most likely, this would involve showing that they satisfy other universal properties (dimension, locally finite n–disk properties)?
(4) Is the set of arational trees in ∂CV_n path-connected? We remark that the paths constructed between boundary points of the free factor complex do not yield paths in ∂CV_n, as continuity at the endpoints cannot be guaranteed. The corresponding question for $\mathcal{PML}$ is an open question of Gabai.
(5) Is the set of points in ∂FF_n which are not of surface type path-connected? The paths we construct contain subpaths each point of which is of surface type. Avoiding this seems to require new ideas. We were made aware of this question by Camille Horbez.

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2. Out(F_n) preliminaries

This section collects the necessary facts about (compactified) Culler–Vogtmann Outer space, related spaces where Out(F_n) acts, and geodesic laminations on surfaces.

2.1. Outer Space

Throughout this article, any *tree* is understood to be a tree together with an isometric action of F_n. Recall that a tree is *minimal* if does not contain a proper invariant subtree, and it is *nontrivial* if it does not have a global fixed point. Unless stated otherwise, all trees will be minimal and nontrivial. When T is an F_n-tree and a is an element or a conjugacy class in F_n, we write $\langle T, a\rangle$ for the translation length of a in T.

We denote by cv_n *(unprojectivized) Outer space* in rank n, and we denote by $CV_n = cv_n/(0,\infty)$ projectivised Outer space. Points in these spaces correspond to (projectivised) free, simplicial, minimal F_n–trees. Compare [CV86, Vog08] for details. One can think of the space CV_n as a free group analog of Teichmüller space. The function $T \mapsto (a \mapsto \langle T, a\rangle)$ defines an embedding of cv_n into the *space of length functions* $[0,\infty)^{F_n}$. We denote by $\overline{cv}_n$ the closure of cv_n in this space and by $\overline{CV}_n$ its projectivization. A point in $\overline{cv_n}$ determines a tree, unique up to equivariant isometry. Both $\overline{cv_n}$ and $\overline{CV_n}$ are metrizable ($\overline{cv}_n$ is a subspace of the metrizable space $[0,\infty)^{F_n}$ and $\overline{CV_n}$ embeds in $\overline{cv}_n$; see below). Moreover, $\overline{CV_n}$ is compact and $\partial CV_n = \overline{CV_n} \smallsetminus CV_n$ plays the role of $\mathcal{PML}$. For our later arguments we choose a distance dist on CV_n. As usual, this distance defines a (Hausdorff) distance on the set of compact subsets, and we keep the same notation for this distance. An element of $\overline{CV_n}$ is represented by a projective class of trees, but we follow the custom and talk about trees as points in $\overline{CV_n}$.

Recall the following characterisation of this compactification.

Definition 2.1. — *A nontrivial minimal tree T is* very small, *if arc stabilizers are trivial or maximal cyclic subgroups, and the fixed set of a nontrivial element does not contain a tripod.*

Proposition 2.2 ([BF, Hor17]). — *A nontrivial minimal tree T is contained in $\overline{CV_n}$ if and only if it is* very small.

2.2. Arational Trees

To describe the boundary of the free factor complex, we need the following notion.

Definition 2.3. — *A tree $T \in \partial CV_n$ is* arational *if for every proper free factor $A < F_n$ the induced action of A on T is free and discrete.*

Two arational trees are *topologically equivalent* if there is an equivariant homeomorphism in the observers' topology between them (we elaborate on observers' topology below). Equivalence classes of arational trees in $\overline{CV_n}$ can be naturally identified with simplices, analogously to simplices of projectivized transverse measures on geodesic laminations. Denote by $\mathcal{AT} \subset \partial CV_n$ the space of arational trees, and by $\mathcal{AT}/\sim$ the quotient space obtained by collapsing each equivalence class to a point.

Theorem 2.4 ([BR15, Ham14]). — *The Gromov boundary ∂FF_n of the free factor complex is homeomorphic to $\mathcal{AT}/\sim$.*

We will need a more precise version of this theorem. There is a function

$$\Phi : \overline{CV_n} \to \overline{FF_n}$$

(in [BR15] this is the map $\pi \cup \partial\pi$) with the following properties:

- The restriction of Φ to the space $\mathcal{AT}$ of arational trees maps it continuously onto ∂FF_n [BR15, Proposition 7.5], it is a closed map [BR15, Lemma 8.6], and the point inverses are exactly the simplices of equivalence classes of arational trees [BR15, Proposition 8.4].
- The restriction of Φ to the complement of $\mathcal{AT}$ has FF_n as its range and it is defined coarsely; it maps $T \in \partial CV_n \smallsetminus \mathcal{AT}$ to a free factor A such that the A-minimal subtree of T has dense orbits (or A is elliptic), and it maps $T \in CV_n$ to a free factor A realized as a subgraph of T/F_n. See [BR15, Lemma 5.1 and Corollary 5.3].
- Φ is coarsely continuous: if T_i is a sequence in $\overline{CV_n}$ and Δ is an equivalence class of arational trees, then $\Phi(T_i) \to \Phi(\Delta)$ if and only if the accumulation set of T_i is contained in Δ. See [BR15, Proposition 8.3 and Proposition 8.5].

We will have to regularly construct arational trees, and this subsection collects some tools to do so.

We let cv_n^+ be the union of cv_n together with a point 0 representing the trivial action. The space cv_n^+ is naturally a cone with $[0,\infty)$ acting by rescaling. In a similar way we define the space $\overline{cv_n}^+$ of very small trees, together with a point representing the trivial action.

There are many ways of realizing $\overline{CV_n}$ as a section of the cone $\overline{cv_n}^+$ (by which we mean a section of $\overline{cv_n}^+ \smallsetminus \{0\} \to \overline{CV_n}$). We choose the following: by Serre's lemma [Ser80], an action of F_n on a tree so that each element of length $\leqslant 2$ acts elliptically, in fact has a global fixed point. Thus, the sum of translation lengths of all elements of length $\leqslant 2$ is positive in each tree of $\overline{cv_n}$. We identify $\overline{CV_n}$ with the subset of $\overline{cv_n}$ where the sum of translation lengths for all elements of length $\leqslant 2$ is equal to 1.

Lemma 2.5. — *For every proper free factor $A < F_n$, restricting the action of F_n on a tree to a minimal subtree for the A–action yields a continuous map*

$$r_A : \overline{cv_n}^+ \to \overline{cv(A)}^+.$$

Proof. — First, observe that if A acts elliptically on T, then $r_A(T)$ is the cone point in $\overline{cv(A)}^+$.

Otherwise, we claim that the restriction is very small. Namely, consider any arc $a \subset T$. If its stabiliser is trivial, the same is obviously true for the restricted action. If the stabiliser is a maximal cyclic subgroup, then the same is true for the restricted action: since A is a free factor, if $1 \neq g \in A$ then the maximal cyclic subgroup in

F_n containing g is contained in A. Finally, suppose that g acts nontrivially on the restricted tree. If it would fix a tripod, the same would be true in T, violating that T is very small.

Continuity is clear, since translation lengths for the restriction are the same as translation lengths in T. □

We define

$$\rho_A : \partial CV_n \to \overline{cv(A)}^+$$

as the restriction of the maps r_A to the subset $\partial CV_n \subset \overline{cv_n}^+$ via the above normalization.

We now make the following definition, which will be crucial for our construction:

Definition 2.6. — *Let $\mathcal{F}$ be the countable set of conjugacy classes of proper nontrivial free factors of F_n. For any $A \in \mathcal{F}$ we define*

$$K_A = \partial CV_n \setminus \rho_A^{-1}(cv(A))$$

Thus K_A is the set of trees in ∂CV_n where A does not act freely and simplicially. The following is now clear from the above:

Proposition 2.7. — *The collection $\{K_A, A \in \mathcal{F}\}$ is a countable collection of closed subsets whose complement is the set of arational trees:*

$$\mathcal{AT} = \partial CV_n \setminus \left(\bigcup_{A \in \mathcal{F}} K_A \right)$$

2.3. Trees, currents, and the action of $\mathrm{Out}(F_n)$

Any $\phi \in \mathrm{Out}(F_n)$ acts naturally (on the left) on the set of conjugacy classes in F_n. To be consistent with our later constructions, we also define a left action of $\mathrm{Out}(F_n)$ on the set of trees defined by

$$\langle \phi T, a \rangle = \left\langle T, \phi^{-1}(a) \right\rangle$$

The length pairing can be extended from the set of conjugacy classes to the space $\mathcal{MC}_n$ of *measured geodesic currents* that contains positive multiples of conjugacy classes [Kap06, Mar95] (see below for the definition). It admits an action of $(0, \infty)$ by scaling and a left action of $\mathrm{Out}(F_n)$ that commutes with scaling and extends the action on conjugacy classes. The length pairing extends to a continuous function $\overline{cv_n} \times \mathcal{MC}_n \to [0, \infty)$ that commutes with scaling in each coordinate [KL09].

2.4. Dual laminations and currents

For more details on this section see [CHL07, CHL08a, CHL08b, CHL08c]. Denote by ∂F_n the Cantor set of ends of F_n. A *lamination* L is a closed subset of $\partial^2 F_n := \partial F_n \times \partial F_n \smallsetminus \Delta$ invariant under $(x, y) \mapsto (y, x)$ and under the left action of F_n, where Δ is the diagonal. To every $T \in \overline{cv_n}$ one associates the *dual lamination* $L(T)$, defined as

$$L(T) = \cap_{\epsilon > 0} L_\epsilon(T)$$

where $L_\epsilon(T)$ is the closure of set of pairs $(x, y) \in \partial^2 F_n$ which are endpoints in the Cayley graph of axes of elements with translation length $< \epsilon$ in T. It turns out that $L(T)$ is always *diagonally closed* i.e. $(a, b), (b, c) \in L(T)$ implies $(a, c) \in L(T)$ (if $a \neq c$), so it determines an equivalence relation on ∂F_n, and the equivalence classes form an upper semi-continuous decomposition of ∂F_n.

The precise definition of a measured geodesic current is that it is an F_n-invariant and $(x, y) \mapsto (y, x)$ invariant Radon measure on $\partial^2 F_n$ (i.e. a Borel measure which is finite on compact sets). For example, a conjugacy class in F_n determines a counting measure on $\partial^2 F_n$ and can be viewed as a current. The topology on the space $\mathcal{MC}_n$ of all currents is the weak* topology. The support $Supp(\mu)$ of a current is the smallest closed set such that μ is 0 in the complement; the support of a current is always a lamination. An important theorem relating currents and laminations is the following.

Theorem 2.8 ([KL10]). — *Let $T \in \overline{cv_n}$ and $\mu \in \mathcal{MC}_n$. Then $\langle T, \mu \rangle = 0$ if and only if $Supp(\mu) \subseteq L(T)$.*

2.5. Observers' topology

Let $T \in \overline{cv_n}$. Then T is a metric space, but it also admits a weaker topology, called *observers' topology*. A subbasis for this topology consists of complementary components of individual points in T. One can also form the metric completion of T and add the space of ends (i.e. the Gromov boundary) to form $\hat{T}$. The space $\hat{T}$ also has observers' topology, defined in the same way. In this topology $\hat{T}$ is always compact, and in fact it is a *dendrite* (uniquely arcwise connected Peano continuum). The following is a theorem of Coulbois–Hilion–Lustig. There is an alternative description of $L(T)$ in terms of the Q-map $Q : \partial F_n \to \hat{T}$ (see Levitt–Lustig [LL03]): $(a, b) \in L(T)$ if and only if $Q(a) = Q(b)$.

Theorem 2.9 ([CHL07]). — *Suppose T has dense orbits. Then $\hat{T}$ with observers' topology is equivariantly homeomorphic to $\partial F_n / L(T)$.*

In fact, the Q-map realizes this homeomorphism.

2.6. Surfaces, Laminations and $\overline{CV_n}$

Suppose that Σ is a compact oriented surface with one boundary component. Then the fundamental group $\pi_1(\Sigma)$ is the free group F_{2g}. If Σ is a nonorientable surface with a single boundary component, the fundamental group $\pi_1(\Sigma)$ is also free, of even or odd rank (depending on the parity of the Euler characteristic).

A lamination λ on Σ will always mean a *measured geodesic lamination*. For any surface Σ, we denote by $\mathcal{PML}(\Sigma)$ the sphere of projective measured laminations. See [FLP12] for a thorough treatment. We will also need measured laminations of nonorientable surfaces, but the only specific result we rely on is that for a nonorientable Σ, the lifting map

$$\mathcal{PML}(\Sigma) \to \mathcal{PML}(\Sigma')$$

is a topological embedding, where Σ' denotes the orientation double cover.

In our construction we will need paths in $\mathcal{PML}$ consisting only of minimal laminations. In the orientable case these will be provided by the following theorem (which is one of the reason why our connectivity proof only works in high enough rank).

Theorem 2.10 (Chaika and Hensel [CH]). — *For an orientable surface of genus at least 5 (with any number of marked points) the set of uniquely ergodic foliations is path-connected and locally path-connected in $\mathcal{PML}$. Furthermore, given any finite set B of laminations, the complement of B in $\mathcal{PML}$ is still path-connected.*

For non-orientable surfaces, we require a similar result. As we do not need the full strength of Theorem 2.10 for our argument, we will prove the following in the appendix (which follows quickly from methods developed in [LS09]):

Theorem 2.11. — *Suppose that Σ is a nonorientable surface with a single marked point p.*

Let $\mathcal{P} \subset \mathcal{PML}(\Sigma)$ be the set of minimal foliations which either

(1) *do not have an angle–π singularity at p, or*
(2) *are stable foliations of point-pushing pseudo-Anosovs.*

Then $\mathcal{P}$ is path-connected, and invariant under the mapping class group of Σ. In addition, if F is any finite set of laminations, the set $\mathcal{P} \setminus F$ is still path-connected.

Given a lamination on Σ, we can lift it to a lamination $\widetilde{\lambda}$ of the universal cover $\widetilde{\Sigma}$. Since $\partial_\infty\widetilde{\Sigma} = \partial_\infty\pi_1(\Sigma) = \partial_\infty F_n$, this allows us to interpret λ as a lamination on the free group. It is dual (in the sense above) to the dual tree of the lamination $\widetilde{\lambda}$ (in the geometric sense).

The following theorem, due to Skora, describes exactly which trees appear in this way.

Theorem 2.12 ([Sko96]). — *Suppose a closed surface group acts on a minimal $\mathbb{R}$-tree T with cyclic arc stabilizers. Then T is dual to a measured geodesic lamination on the surface. The same holds for surfaces with boundary if the fundamental group of each boundary component acts elliptically in T.*

For our purposes, we will need to be a bit more careful about how we identify surface laminations and free group laminations. Namely, for any identification $\sigma : \pi_1(\Sigma) \to F_n$ we obtain the corresponding map

$$\iota_\sigma : \mathcal{PML}(\Sigma) \to \overline{CV_n}$$

mapping a lamination to its dual tree. We denote by

$$\mathcal{PML}_\sigma = \iota_\sigma(\mathcal{PML}(\Sigma))$$

the image of ι_σ. In other words, $\mathcal{PML}_\sigma$ consists of those trees which are realisable as duals to geodesic laminations on Σ, given the identification of $F_n = \pi_1(\Sigma)$ via σ.

If ϕ is any outer automorphism, then the images of ι_σ and $\iota_{\sigma\circ\phi^{-1}}$ differ by applying the outer automorphism ϕ (acting on $\overline{CV_n}$). Core to our argument will be to use the intersection of "adjacent" such copies of $\mathcal{PML}$; see Section 3. A typical lamination contained in the intersection is one that fills a suitable subsurface of Σ.

2.7. Density of surface type arational trees

Arational trees come in two flavors: ones dual to a filling measured lamination on a compact surface with one boundary component (we will call them arational trees of *surface type*), and the “others” – these are free as F_n-trees (see [Rey12]). Recall that every arational tree T belongs to a canonical simplex $\Delta_T \subset \partial CV_n$ consisting of arational trees with the same dual lamination. We will need the following lemma.

Lemma 2.13. — *Let T be an arational tree and let U be a neighborhood of the simplex Δ_T in ∂CV_n. Then U contains an arational tree S of surface type.*

Proof. — All arational trees in rank 2 are of surface type (and all associated simplices are points) so we may assume $n \geqslant 3$. Vincent Guirardel showed in [Gui00a] that for $n \geqslant 3$ the boundary ∂CV_n contains a unique minimal $\mathrm{Out}(F_n)$-invariant closed set $\mathcal{M}_n$. In particular, $\mathrm{Out}(F_n)$ acts on $\mathcal{M}_n$ with dense orbits. He further showed that any arational tree (or indeed any tree with dense orbits) with ergodic Lebesgue measure belongs to $\mathcal{M}_n$. In our situation this means that the vertices of Δ_T belong to $\mathcal{M}_n$ and they can be approximated by points in the orbit of a fixed surface type arational tree that also belongs to $\mathcal{M}_n$. □

2.8. Dynamics of partial pseudo-Anosovs

In this section we will assemble the dynamical properties of partial pseudo-Anosov homeomorphism as they act on Outer space. The proofs are standard but we couldn’t find the statements we need in the literature.

The basic theorem is that of Levitt and Lustig. An outer automorphism of F_n is *fully irreducible* if all of its nontrivial powers are irreducible, i.e. don’t fix any proper free factors up to conjugation.

Theorem 2.14 ([LL03]). — *Let $f \in \mathrm{Out}(F_n)$ be a fully irreducible automorphism. Then f acts with north-south dynamics on $\overline{CV_n}$.*

We start with a pseudo-Anosov homeomorphism $f : S \to S$ of a compact surface S (with one or more boundary components) with $\pi_1(S) = F_n$. By $\lambda > 1$ denote the dilatation and by $\Lambda^\pm$ the stable and the unstable measured geodesic laminations, so $f(\Lambda^\pm) = \frac{1}{\lambda^{\pm 1}}\Lambda^\pm$. Let $T^\pm$ be the trees dual to $\Lambda^\pm$, and let $\mu^\pm$ be the measured currents corresponding to $\Lambda^\pm$. Thus

$$f_* T^\pm = \lambda^{\pm 1} T^\pm$$

and

$$f_*(\mu^\pm) = \lambda^{\pm 1}\mu^\pm$$

This implies $\langle T^\pm, \mu^\pm\rangle = 0$ and $\langle T^\pm, \mu^\mp\rangle > 0$.

Proposition 2.15. — *With notation as above, let $K \subset \overline{CV_n}$ be a compact set of trees such that $\langle T, \mu^+\rangle \neq 0$ for every $T \in K$. Then $f_*^m K$ converges to T^+ as $m \to \infty$.*

When S has more than one boundary component f_* is not fully irreducible. It is irreducible if it permutes the boundary components cyclically. The Levitt–Lustig argument can be used to prove the north-south dynamics for irreducible automorphisms, and this would suffice for our purposes since we could arrange that pseudo-Anosov homeomorphisms we use in our construction later have roots that cyclically permute the boundary components. However, we will give a direct argument.

Proof. — We will view $K \subset \overline{cv_n}$ as a compact set of unprojectivized trees as in Section 2.2. Let Y be the accumulation set of the scaled forward iterates $f_*^m K/\lambda^m$ of K. Note that if $T \in K$ then $\langle \frac{1}{\lambda^m} f_*^m T, \mu^- \rangle = \langle T, \frac{1}{\lambda^m} f_*^{-m}(\mu^-)\rangle = \langle T, \mu^- \rangle$, so the length of μ^-, being bounded below by some $\epsilon > 0$ over K, is also bounded below by ϵ on Y. Similarly, $\langle \frac{1}{\lambda^m} f_*^m T, \mu^+ \rangle = \langle T, \frac{1}{\lambda^m} f_*^{-m}(\mu^+)\rangle = \langle T, \frac{1}{\lambda^{2m}} \mu^+ \rangle \to 0$, so the length of μ^+ is 0 in all trees in Y. If a is any conjugacy class other than a power of a boundary component, then we have by surface theory $\frac{1}{\lambda^m} f_*^{-m}(a) \to C_a \mu^-$ for some $C_a > 0$, and a similar argument as above shows that $\langle T, a\rangle = C_a \langle T, \mu^- \rangle > 0$ for every $T \in Y$. However, when a represents a boundary component then $\langle T, a \rangle = 0$ since then $f_*^{-m}(a)$ is a boundary component for all m, and thus $\frac{1}{\lambda^m} f_*^{-m}(a) \to 0$. In particular, all scaled iterates $f_*^m K/\lambda^m$ are contained in a compact subset of $\overline{cv_n}$ and so the accumulation set in $\overline{CV_n}$ is the projectivization of Y. It follows from Skora's theorem that Y consists of trees dual to measured geodesic laminations on S. The only lamination where μ^+ has length 0 in the dual tree is Λ^+ and hence $Y = \{T^+\}$ (projectively). □

The next result gives a criterion to check the condition required in the previous proposition.

Proposition 2.16. — *With notation as in the paragraph before Proposition 2.15, suppose that R is a compact surface with marking induced by a homotopy equivalence $\phi : S \to R$ (which may not send boundary to boundary). Let Λ be a measured geodesic lamination on R and let T be the dual tree. If $\langle T, \mu^+ \rangle = 0$ then $T = T^+$ projectively.*

Proof. — The support $Supp(\mu^+)$ of μ^+ is Λ^+. The complementary component of Λ^+ in S that contains a boundary component is a "crown region" and adding diagonals amounts to adding infinitely many (non-embedded) lines that start and end in a cusp of the crown region and wind around the boundary component any number of times. These accumulate on the boundary, so each boundary component is in the diagonal closure of $Supp(\mu^+)$. Any other line will intersect the leaves of Λ^+ transversally, so it follows that the lamination $L(T^+)$ dual to the tree T^+ coincides with the diagonal closure of $Supp(\mu^+)$ (cf. [BR15, Proposition 4.2 (ii)]).

Similarly, the lamination $L(T)$ dual to T consists of leaves of Λ together with all lines not crossing Λ transversally. Since $\langle T, \mu^+ \rangle = 0$ we have $Supp(\mu^+) \subseteq L(T)$ by the Kapovich–Lustig Theorem 2.8, and hence $L(T^+) \subseteq L(T)$. We now argue that Λ must be a filling lamination. First, Λ cannot contain closed leaves, for otherwise $L(T)$ would be carried by an infinite index finitely generated subgroup, contradicting the fact that Λ^+ isn't. Now suppose that Λ contains a minimal component Λ_0 carried by a proper subsurface $R_0 \subset R$. If there is a leaf of $L(T^+)$ asymptotic to a leaf of Λ_0, then $L(T^+)$ contains Λ_0 as well as boundary components of R_0 and the diagonal leaves within R_0. It then follows that $L(T^+)$ doesn't contain any other leaves since it equals the diagonal closure of any non-closed leaf, and hence again it is carried by an infinite index finitely generated subgroup. Thus Λ must be filling and $L(T^+) = L(T)$. It now follows from the Coulbois–Hilion–Lustig Theorem 2.9 that T and T^+ are equivariantly homeomorphic in observers' topology. But since T^+ is uniquely ergodic, projectively $T = T^+$. □

Remark 2.17. — In fact, $\langle T, \mu^+\rangle = 0$ forces $T = T^+$ even without assuming that T is dual to a lamination on a surface. This can be proved by noting that $L(T^+) \subseteq L(T)$ (which is proved above for all T with $\langle T, \mu^+\rangle = 0$) forces $L(T) = L(T^+)$ by [BR15, Proposition 3.2] since T^+ is an indecomposable tree. It follows that f_* satisfies the north-south dynamics on all of $\overline{CV_n}$, not just on compact sets dual to surface laminations.

We now generalize this to partial pseudo-Anosov homeomorphisms.

Proposition 2.18. — *Let f be a homeomorphism of Σ, which restricts to a pseudo-Anosov on a π_1-injective subsurface $S \subset \Sigma$, and to the identity in the complement. Suppose that $K \subset \overline{CV_n}$ is compact such that every $T \in K$ satisfies $\langle T, \mu^+\rangle \neq 0$, where μ^+ is a current supported in $\pi_1(S)$ corresponding to the stable lamination Λ^+ of f.*

Then the sequence $f_*^i K$, $i \to \infty$, converges to the tree $T_f \in \partial CV_n$ dual to Λ^+ on Σ.

Note that the condition implies that $\pi_1(S)$ is not elliptic in any $T \in K$. To begin the proof of Proposition 2.18, denote by $Y \subset \overline{CV_n}$ the set of accumulation points of $f^i K$ as $i \to \infty$. We need to show that $Y = \{T_f\}$.

Lemma 2.19. — *Let $T \in Y$. Every conjugacy class represented by a (not necessarily simple) curve in $\Sigma \smallsetminus S$ is elliptic in T. The minimal $\pi_1(S)$-subtree of T is dual to Λ^+.*

Proof. — The set K is defined as a subset of $\overline{CV_n}$, but we can lift it to unprojectivized space $\overline{cv_n}$ as in Section 2.2. Now let γ be a curve in the complement of S. Since K is compact, the set $\{\ell_R(\gamma) \mid R \in Y\}$ is bounded. Since $f(\gamma) = \gamma$ we deduce that the set $\{\langle R, f_*^{-i}(\gamma)\rangle \mid R \in Y, i \in \mathbb{Z}\}$ is bounded as well. But $\langle f_*^i R, \gamma\rangle = \langle R, f_*^{-i}(\gamma)\rangle$, so the length of γ in $f_*^i(R)$ stays uniformly bounded for all i.

On the other hand, let δ be a curve in S. Applying Proposition 2.15 to the trees obtained from the trees in K by restricting to $\pi_1(S)$ we see that these restrictions converge to the tree dual to Λ^+. Moreover, the length of δ along $f_*^i K$ goes to infinity as $i \to \infty$ (in fact, the length grows like $(const)\lambda^i$ where λ is the dilatation). Thus projectively, the length of γ will go to 0 as $i \to \infty$. □

Proof of Proposition 2.18. — Let $T \in Y$. By Lemma 2.19 conjugacy classes of boundary components of Σ are all elliptic, so by Skora's theorem we deduce that T is dual to a measured geodesic lamination on Σ. Again by Lemma 2.19 this lamination is Λ^+ on S and possibly curves in $\partial S \smallsetminus \partial\Sigma$ with nonzero weight. Thus the accumulation set is contained in the simplex in $\mathcal{PML}(\Sigma)$ whose vertices are Λ^+ and these curves, it is compact, and disjoint from the face opposite the vertex Λ^+ (in particular, no point in the accumulation set is a curve in $\partial S \smallsetminus \partial\Sigma$). The only f-invariant compact set with this property is $\{\Lambda^+\}$ since f acts by attracting towards Λ^+ all compact sets disjoint from the opposite face. □

Corollary 2.20. — *In the setting of Proposition 2.18, let $A < F_n$ be a free factor such that the action of A on T^+ is free and discrete. Then for large $k > 0$ the action of $f_*^k(A)$ on every $T \in K$ is free and discrete.*

Proof. — The subgroup A will act freely and discretely in a neighborhood of T^+, and this includes Kf_*^k for large k. This is equivalent to the statement. □

There is a simple criterion for deciding if the action of A on T^+ is free and discrete.

Lemma 2.21. — *Let $A < F_n$ be a free factor and T^+ the tree dual to the stable lamination Λ_f of a partial pseudo-Anosov homeomorphism f supported on a subsurface $S \subset \Sigma$. The action of A on T^+ is free and discrete if and only if $\pi_1(S)$ is not conjugate into A and no nontrivial conjugacy class in A is represented by an immersed curve in $\Sigma \smallsetminus \Lambda^+$.*

Proof. — It is clear that these conditions are necessary. Assuming they hold, equip Σ with a complete hyperbolic metric of finite area and let $\widetilde{\Sigma} \to \Sigma$ be the covering space with $\pi_1(\widetilde{\Sigma}) = A$. Lift the hyperbolic metric to $\widetilde{\Sigma}$ and let $\widetilde{\Sigma}_C \subset \widetilde{\Sigma}$ be the convex core. Then $\widetilde{\Sigma}_C$ is compact, since any cusp would represent a boundary component of Σ whose conjugacy class is in A. Lift the lamination Λ^+ to $\widetilde{\Lambda} \subset \widetilde{\Sigma}$. Each leaf of $\widetilde{\Lambda}$ intersects $\widetilde{\Sigma}_C$ in an arc (or not at all) for otherwise A would carry Λ_f and would have to contain a finite index subgroup of $\pi_1(S)$. But A is root-closed, so it would contain $\pi_1(S)$, which we excluded. So the intersection of $\widetilde{\Lambda}$ with $\widetilde{\Sigma}_C$ consists of finitely many isotopy classes of arcs. These arcs are filling, for otherwise we would have a loop in the complement that would represent a nontrivial element of A whose image in Σ is disjoint from Λ^+. The minimal A-subtree of T_f is dual to this collection of arcs, and so is free and discrete. □

Finally, we need the following, which describes the dynamics of partial pseudo-Anosovs on free factors.

Proposition 2.22. — *Identify $\pi_1(\Sigma) = F_n$. Suppose that ψ is a partial pseudo-Anosov, supported on a subsurface $S^g \subset \Sigma$.*

Let E be any free factor, and B' either a free factor, or the fundamental group $\pi_1(S')$ of a subsurface $S' \subset \Sigma'$ under a (possibly different) identification $\pi_1(\Sigma') = F_n$.

Furthermore assume that B' does not contain $\pi_1(S^g)$ up to conjugacy.

Then one of the following holds:

1. *$\pi_1(S^g) \subset E$ up to conjugacy, or*
2. *there is some $k > 0$ so that for all large enough N the only conjugacy classes belonging to $\psi^{kN}E$ and B' are contained in $\pi_1(\Sigma - S^g)$.*

Before we begin the proof, we remark that the assumption on B' is not optimal. What we actually need is that B' is root-closed (i.e. if B' contains a power g^n of an element, then it contains g) and that if every element of a finitely generated group C is conjugate into B', then C itself is conjugate into B'. The latter is true for both groups of the type we consider: namely, these appear as stabilisers for suitable (Bass-Serre) trees on which F_n acts, and if every element of C fixes a vertex of a tree, then so does C itself.

Proof. — Assume that (1) fails, i.e. $\pi_1(S^g)$ is not conjugate into E.

Choose a hyperbolic metric on Σ. From now on, we assume that all curves and laminations are geodesic, and all covering maps are local isometries. By Scott's theorem, there is a finite cover $X \to \Sigma$ so that $E = \pi_1(X_E)$ where $X_E \subset X$ is a subsurface. We choose a power k so that ψ^k lifts to a partial pseudo-Anosov map $\hat{\psi}$ of X.

The mapping class $\hat{\psi}$ is supported on a (possibly disconnected) subsurface $Z_1 \cup \ldots \cup Z_k$ where the Z_i are the connected components of the preimage of S^g in the cover X. On each Z_i, the mapping class $\hat{\psi}$ restricts to a pseudo-Anosov mapping class.

Let $\hat{\lambda}^\pm$ be the lifts of the stable/unstable lamination of ψ to X; the restriction of $\hat{\lambda}^\pm$ to each Z_i is then the stable/unstable lamination of the pseudo-Anosov $\hat{\psi}|_{Z_i}$, and thus in particular minimal and filling (on that subsurface).

If any half-leaf ℓ of $\hat{\lambda}^\pm$ would be contained in X_E, then the same would therefore be true for the component Z_i containing ℓ. However, this would imply that a finite index subgroup of $\pi_1(S^g)$ is conjugate into E. Since free factors are root-closed, this would imply that every element of $\pi_1(S^g)$ is conjugate into E. Since E is a free factor, this would imply that $\pi_1(S^g)$ itself is conjugate into E, contradicting the assumption. Observe (for later use) that we only used the properties mentioned in the remark before the proof about E for this argument.

Now let Y be the smallest essential subsurface of X_E which contains the intersection $(\hat{\lambda}^+ \cup \hat{\lambda}^-) \cap X_E$. Essential here means that no boundary component of Y is trivial in X_E, and if a boundary component of Y is homotopic (in X) to a boundary of X, it is equal to that boundary (in other words, the subsurface of X_E filled by the intersection of the laminations with X_E). We let $Y' = X_E - Y$ be its complement. We emphasise that the complement could be empty, if the intersection of the laminations with X_E is a filling arc system. Also observe that any curve in Y' maps (under the covering map) into $\Sigma - S^g$.

By compactness and the fact that no leaf of $\hat{\lambda}^-$ is supported in X_E there is a number α with the following property: any geodesic starting in a point $p \in \hat{\lambda}^-$ and making angle $< \alpha$ with $\hat{\lambda}^-$ leaves X_E. In particular we conclude that any closed geodesic in X_E which intersects Y (and hence $\hat{\lambda}^-$) makes angle $\geqslant \alpha$ with $\hat{\lambda}^-$.

As a consequence, we have the following: for any L and ϵ there is a number $N(L,\epsilon)$ with the following property. If $\gamma \subset X_E$ is any geodesic intersecting Y, then $\hat{\psi}^n\gamma$ contains a segment of length $\geqslant L$ which ϵ–fellow-travels a leaf of $\hat{\lambda}^+$, for all $n > N(L,\epsilon)$.

We now claim that there are $L, \epsilon > 0$ with the following property: no geodesic γ in X, which represents a conjugacy class of B', contains a geodesic segment of length $\geqslant L$ which ϵ–fellow-travels a leaf of $\hat{\lambda}^+$.

To see this, we argue as above with B' in place of E. Namely, we consider a cover $\overline{X}$ of Σ containing a subsurface $\overline{X}_{B'}$ for the group B', and lifting a suitable power of ψ to $\overline{\psi}$. Using that B' does not contain a conjugate of $\pi_1(S^g)$, we again conclude that any half-leaf of the lifts $\overline{\lambda}^\pm$ of the stable/unstable foliation exits $\overline{X}_{B'}$. By compactness, this implies that no geodesic segment of length $\geqslant L$ (in $\overline{X}$) which

ϵ–fellow-travels a leaf of $\overline{\lambda}^+$ can be contained in $\overline{X}_{B'}$ – in particular, the (geodesic representative of) the lift of any element of B' cannot fellow-travel $\overline{\lambda}^+$ like this. Since all covering maps considered are local isometries, the claim follows.

We let $N = N(L, \epsilon)$ be the corresponding constant from above, applied to this pair of L, ϵ. Now, let $r > 0$ be a number so that ρ^r lifts to X for any loop ρ in Σ (this exists, since X is a finite cover). Let ρ be any element of E, which is not conjugate into $\pi_1(\Sigma - S^g)$. Since fundamental groups of subsurfaces are root-closed, the element ρ^r is then also not conjugate into $\pi_1(\Sigma - S^g)$. Then, ρ^r lifts to a geodesic γ in X_E, which intersects Y. Thus, $\hat{\psi}^n\gamma$ contains a segment with the property of the previous paragraph, showing that $\psi^{kn}\rho^r$ is not contained in B' for all $n > 0$.

In other words, if $\psi^{kn}\rho$ (and thus $\psi^{kn}\rho^r$) is contained in B', then ρ is contained in $\pi_1(\Sigma - S^g)$, showing (2). ☐

3. Basic Moves, Good and Bad Subsurfaces

In this section, we will study how to relate different identifications of a free group with the fundamental group of a surface. The basic situation will be to relate two identifications which differ by applying (certain) generators of $\mathrm{Out}(F_n)$.

3.1. Standard Geometric Bases

Let $\Sigma = S_{g,1}$ be a compact oriented surface of genus g with one boundary component. We think of Σ as a central disk with $2g$ bands attached (compare Figure 3.1). We pick a basepoint p, contained in the interior of the surface Σ, and once and for all choose a small disk D around p which is disjoint from the bands. If l is a (parametrised) simple loop based at p, we denote by l^+ (respectively l^-) an initial (respectively terminal) segment which is contained in D. We will later need this to compute the action of (certain) Dehn twists on the basis loops – and intersections with initial segments will lead to multiplications on the left, while intersections with terminal segments will lead to multiplications on the right.

A collection of simple closed (parametrised) loops $a_i, \hat{a}_i, 1 \leqslant i \leqslant g$ based at p is called a *standard geometric basis for $S_{g,1}$* if the following hold:

(1) The loops $a_i, \hat{a}_j$ generate $F_{2g} = \pi_1(\Sigma, p)$.
(2) Any two of the loops $\{a_i, \hat{a}_j, i, j = 1, \ldots, g\}$ intersect only in the basepoint p.
(3) The cyclic order of initial and terminal segments at p is

$$\hat{a}_1^+, a_1^+, \hat{a}_1^-, a_1^-, \hat{a}_2^+, \ldots, a_g^-.$$

Compare Figure 3.1 for an example of such a basis. Given a standard geometric basis, we say that $a_i, \hat{a}_i$ are an *intersecting pair* (of that basis). For ease of notation we define $\hat{\hat{a}}_i = a_i$. Observe that up to the action of the mapping class group of Σ there is a unique standard geometric basis.

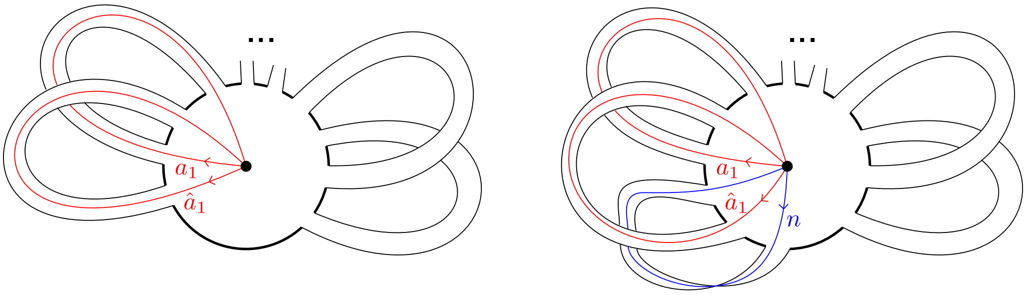

Figure 3.1. A basis for Σ of the type we use in this section.

To deal with free groups of odd rank, we need to also consider certain nonorientable surfaces. Namely, let $\Sigma = N_{2g+1}$ be the surface obtained from S_g by attaching a single twisted band as in Figure 3.1.

As before, we pick a basepoint p in the interior of Σ and a small disk D surrounding p so we can talk about initial and terminal segments (see above). A collection of simple closed curves $n, a_i, \hat{a}_i 1 \leqslant i \leqslant g$ is called a *standard geometric basis for* N_{2g+1} if the following hold:

(1) The curve n is one-sided,
(2) all $a_i, \hat{a}_i$ are two-sided,
(3) the elements $n, a_i, \hat{a}_i$ generate $F_{2g+1} = \pi_1(\Sigma, p)$,
(4) The $n, a_i, \hat{a}_i$ intersect only in p.
(5) The cyclic order of initial and terminal arcs at p is

$$n^+, \hat{a}_1^+, n^-, a_1^+, \hat{a}_1^-, a_1^-, \hat{a}_2^+, a_2^+, \hat{a}_2^-, a_2^- \ldots$$

As above, we say that $a_i, \hat{a}_i$ are an *intersecting pair*. In addition, we also say that $n, \hat{a}_1$ and n, a_1 are *intersecting pairs*[(1)].

Remark 3.1. — The reason for the somewhat asymmetric setup in the nonorientable setting is as follows. For later arguments, we will need to find two-sided curves which intersect the one-sided curve given by the basis letter in a single point. This forces at least one of the two-sided bands to be linked with the one-sided band. However, since we also need to be able to have an odd total number of bands, the described setup emerges.

Definition 3.2. — *Suppose that $x_1, \ldots x_n$ is a free basis for F_n. For any $x = x_i$, $y = x_j$ with $i \neq j$, we call an outer automorphism defined by the automorphism*

$$\rho_{x,y}(z) = \begin{cases} xy & z = x \\ z & z = x_k, k \neq i \end{cases}$$

or

$$\lambda_{x,y}(z) = \begin{cases} yx & z = x \\ z & z = x_k, k \neq i \end{cases}$$

[(1)] We use this terminology since, in the nonorientable setting, both for the pairs $n, \hat{a}_1$ and n, a_1 cases need to be distinguished, just like for the intersecting pairs on the orientable case – although topologically n, a_1 are not linked.

a basic (Nielsen) move. *For $x = x_i$, we call an outer automorphism defined by the automorphism*

$$\iota_x(z) = \begin{cases} x^{-1} & z = x \\ z & z = x_k, k \neq i \end{cases}$$

a basic (invert) move.

Lemma 3.3. — *Given any standard geometric basis,* $\mathrm{Out}(F_n)$ *is generated by the basic invert moves, and Nielsen moves $\phi_{x,y}$ for x, y not an intersecting pair.*

Proof. — It is well-known that for any basis (in particular, standard geometric bases) all basic moves of the form above generate $\mathrm{Out}(F_n)$ [Nie24]. So, to prove the lemma, we just need to show that a Nielsen move for an intersecting pair can be written in terms of nonintersecting pairs. This is clear, e.g. for an unrelated letter z we have:

$$\phi_{a_i,\hat{a}_i} = \phi^{-1}_{z,\hat{a}_i}\phi^{-1}_{a_i,z}\phi_{z,\hat{a}_i}\phi_{a_i,z}.$$

□

Definition 3.4. — *Suppose we have chosen an identification $\sigma : \pi_1(\Sigma) \to F_n$, and basic move ϕ. We say a subsurface $S \subset \Sigma$ is* good for ϕ *if there is a pseudo-Anosov ψ supported on S which commutes with ϕ on the level of fundamental groups, under the identification σ, i.e.*

$$\phi\sigma\psi_*\sigma^{-1} = \sigma\psi_*\sigma^{-1}\phi$$

We call such a ψ an associated partial pA. *We call the complement of a chosen good subsurface a* bad subsurface.

We need a bit more flexibility than basic moves, given by the following definition.

Definition 3.5. — *Suppose $\sigma : \pi_1(\Sigma) \to F_n$ is an identification, and $\mathcal{B}$ is a standard geometric basis for Σ. We then call a conjugate of a basic move (of $\mathcal{B}$) by a mapping class of Σ an* adjusted move.

We observe that good and bad subsurfaces of an adjusted move can be obtained from good and bad subsurfaces of the basic move by applying the conjugating mapping class.

Also observe that neither good nor bad subsurfaces are unique, but we will make an explicit choice below and keep it for the rest of the article.

The key reason why we are interested in good and bad subsurfaces is that we will try to apply Lemma 2.21 to the partial pseudo-Anosovs guaranteed to exist on the good subsurface. In order to do this, we will need to find relations in $\mathrm{Out}(F_n)$ avoiding the following two "problems" (corresponding to the two conditions in Lemma 2.21):

Definition 3.6. — *Let ϕ be a basic or adjusted move, and $E < F_n$ a free factor.*

1. *We say that E* is an overlap problem for ϕ (and a choice of good and bad subsurface) *if some nontrivial conjugacy class $w \in E$ is contained (up to conjugacy) in the fundamental group of the bad subsurface.*
2. *We say that E* is a containment problem for ϕ (and a choice of good and bad subsurface) *if the fundamental group of the good subsurface is contained (up to conjugacy) in E.*

Finally, recall that an identification $\sigma : \pi_1(\Sigma) \to F_n$ defines a copy $\mathcal{PML}_\sigma$ of $\mathcal{PML}(\Sigma)$ inside $\overline{CV_n}$. The following notion is central for our construction.

Definition 3.7. — *Given any identification $\sigma : \pi_1(\Sigma) \to F_n$, and adjusted move ϕ with respect to a standard geometric basis of Σ, we say that the copies*

$$\mathcal{PML}_\sigma \quad \text{and} \quad \phi\mathcal{PML}_\sigma = \mathcal{PML}_{\phi\sigma}$$

are adjacent.

3.2. Good Subsurfaces

To find good subsurfaces, we use the following two lemmas. The first constructs an "obvious" good subsurface (which is not large enough for our purposes). The second one will construct curves that yield additional commuting Dehn twists, which extend the obvious good subsurfaces to "improved" good subsurface (which will be large enough for our purposes).

Throughout this section, we fix a standard geometric basis $\mathcal{B}$, based at a point p.

Lemma 3.8 ("Obvious" good subsurfaces). — *Let x, y be two basis elements of $\mathcal{B}$ which are not an intersecting pair. Then there is a subsurface S_0 with the following properties.*

(1) *If x is two-sided and not linked with the one-sided loop, then $x, \hat{x}, y, \hat{y}$ are disjoint from S_0. If $x = n$ is one-sided and linked with a, or $x = a, \hat{a}$ is linked with the one-sided letter n, then $n, a, \hat{a}, y, \hat{y}$ are disjoint from S_0,*
(2) *Any other basis loop in $\mathcal{B}$ which does not correspond to one of the letters mentioned in the corresponding case of* (1) *is freely homotopic into S_0, and intersects ∂S_0 in two points.*
(3) *Any mapping class supported in S_0 commutes with ι_x, $\lambda_{x,y}$ and $\rho_{x,y}$. In other words, S_0 is a good subsurface for all three of these moves.*

We call the letters as in (1) *the* active *letters of the basis, and all others the* inactive.

Proof. — Let Y_1, Y_2 be the subsurfaces filled by the elements of the given standard geometric basis $\mathcal{B}^0$ which are between $x, \hat{x}$ and $y, \hat{y}$ in the cyclic ordering induced by the orientation of the surface. We homotope Y_1, Y_2 slightly off the basepoint so that they are disjoint from $x, \hat{x}, y, \hat{y}$. Depending on the configuration, one of the Y_i may be empty. If both Y_i are nonempty, choose an arc α connecting Y_1 to Y_2 disjoint from all $a_i, \hat{a}_i$, homotopic to the product $\hat{y}y\hat{y}^{-1}y^{-1}$. Compare Figure 3.2 for this setup. We let S_0 be the subsurface obtained as a band sum of Y_1, Y_2 along α, homotoped slightly so the basepoint is outside S_0.

Now observe that if F is any mapping class supported in S_0, then F acts trivially on all of $x, \hat{x}, y, \hat{y}$. Furthermore, by construction, any loop in S_0 can be written in the basis $\mathcal{B}$ without x. Together, these imply that F_* commutes with $\rho_{x,y}, \lambda_{x,y}$ and ι_x.

The argument in the nonorientable case is very similar, with the three letters $n, \hat{a}_1, a_1$ playing the role of $x, \hat{x}$. ☐

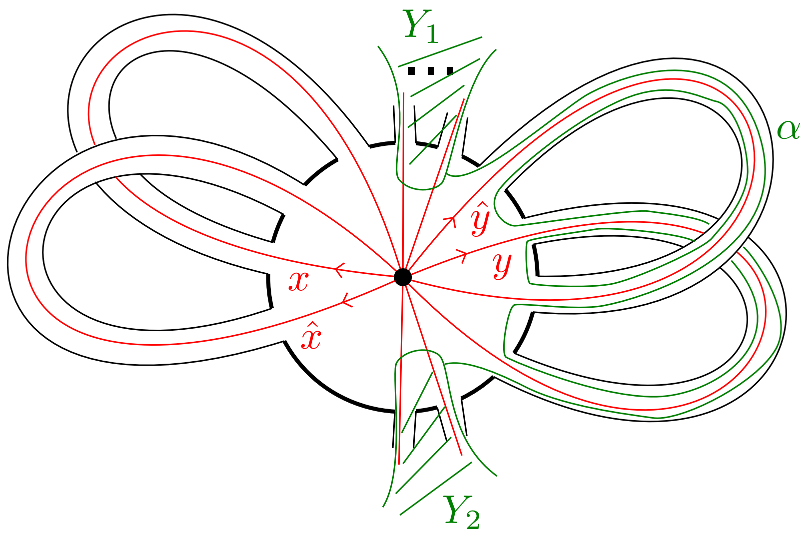

Figure 3.2. Standard geometric bases, and "obvious" good subsurfaces

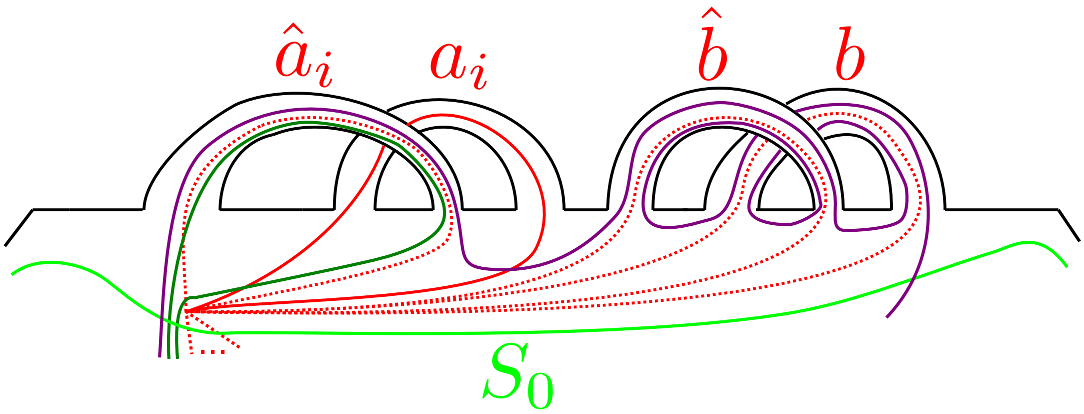

Figure 3.3. Constructing "extra twists" in Lemma 3.9. In this figure, x is a_i, and y is b.

Lemma 3.9. — *Let S_0 be the subsurface obtained by applying Lemma 3.8. Using the notation of that lemma, there are two-sided curves δ^+, δ^- with the following properties:*

(1) *δ^+ (or δ^-) intersect x in a single point on x^+ (or x^-)[(2)],*
(2) *the curves δ^+, δ^- do not cross the band corresponding to x.*
(3) *δ^+, δ^- are disjoint from y.*
(4) *δ^+, δ^- intersect S_0 essentially.*

If one of $y, \hat{y}$ is either one-sided or linked with the one-sided letter, then there is additionally a curve δ^0 intersecting ∂S_0 essentially, and which satisfies (2) and (3), but is disjoint from x.

[(2)] While it may seem at first that, up to isotopy, it is not well-defined where on x the intersection lies, together with the next condition this is indeed a nontrivial requirement.

Proof. — We construct the curves case-by-case, beginning with the orientable case. Here, we simply take γ to be an embedded arc in $\Sigma - S_0$ which intersects x^+ (or x^-) in a single point, is disjoint from the loops $y, \hat{y}$, and does not intersect the interior of the band corresponding to x (compare Figure 3.2 and 3.3). The desired curve is then obtained by concatenating γ with any nonseparating arc in S_0.

In the nonorientable case, we do exactly the same, making sure that the arc γ (and the arc in S_0) do not cross the band corresponding to the one-sided basis element n, and therefore their concatenation is two-sided. □

We now use the curves guaranteed by Lemma 3.9 to enlarge the "obvious" good subsurfaces defined above. These *improved good subsurfaces* will be guaranteed by the next lemma.

For some of the arguments in the sequel, we will need explicit descriptions of the curves produced by Lemma 3.9 and the fundamental groups of the complementary bad subsurfaces of the improved good subsurfaces. As this is a somewhat tedious exercise in constructing and analyzing explicit curves (and the proof follows the exact same strategy in all cases), we only discuss the orientable case here, and defer all further cases to Lemma A.1 in Appendix A.

LEMMA 3.10 (Improved Good Subsurfaces). — *Fix a standard geometric basis $\mathcal{B}$ of an orientable surface $\Sigma = \Sigma_{g,1}$, and use it to identify $\pi_1(\Sigma)$ with F_{2g}. We denote by ∂ the word representing the boundary of the surface, i.e.*

$$\partial = \prod_{i=1}^{g} \left[\hat{a}_i, a_i^{-1}\right],$$

and by ∂_w the cyclic permutation of ∂ starting with the element w.

Let x, y be two elements of $\mathcal{B}$ which are not linked. Then for the left and right multiplication moves $\rho_{x,y}, \lambda_{x,y}$ there are improved good subsurfaces, *whose complementary bad subsurfaces satisfy:*

- *If $x = a_i$, then the bad subsurface for the right multiplication move $\rho_{x,y}$ has fundamental group*

$$\pi_1(\Sigma - S^g) = \left\langle y, \hat{y}, x^{-1}\hat{x}x, \partial_{\hat{a}_{i+1}} \right\rangle.$$

  The bad subsurface for the left multiplication move $\lambda_{x,y}$ has fundamental group

$$\pi_1(\Sigma - S^g) = \langle y, \hat{y}, \hat{x}, \partial_{a_i} \rangle$$

- *If $x = \hat{a}_i$, then the bad subsurface for the right multiplication move $\rho_{x,y}$ has fundamental group*

$$\pi_1(\Sigma - S^g) = \left\langle y, \hat{y}, \hat{x}, \partial_{a_i^{-1}} \right\rangle$$

  The bad subsurface for the left multiplication move $\lambda_{x,y}$ has fundamental group

$$\pi_1(\Sigma - S^g) = \left\langle y, \hat{y}, x\hat{x}x^{-1}, \partial_{\hat{a}_i} \right\rangle$$

In both cases, every loop corresponding to a basis letter except $x, \hat{x}, y, \hat{y}$ is freely homotopic into the improved good subsurface.

Remark 3.11. — We remark that although one could describe the fundamental groups of the improved good subsurfaces explicitly as well, we elect not to do so. The reason is that their description is somewhat more unwieldy (depending on which basis letters lie between x and y in the cyclic order at p) – and we actually do not require this description anywhere in the sequel.

Proof. — We suppose that $x = a_i$, and begin with the subsurface S_0 from Lemma 3.8. Part (3) of that lemma shows that it is indeed good, and part (2) shows that it has the property claimed in the last sentence of the lemma.

We now use curves from Lemma 3.9 to find additional Dehn twists which commute with the basic moves. We begin with the case of $\phi = \rho_{x,y}$. Here, we use the curve δ^+ guaranteed by that lemma (shown in dark green in Figure 3.3). The action of the twist T_{δ^+} on $\mathcal{B}$ depends on the type of letter. However, the relevant properties for us are the following:

(1) $T_{\delta^+}(x) = wx$, where w is a word not involving x. Namely, by property (1) of Lemma 3.9, the twisted curve $T_{\delta^+}(x)$ is obtained by following x^+ until the intersection point, following around δ^+, and then continuing along x. By property (2), the curve δ^+ does not cross the band corresponding to x, yielding the desired property of w.
(2) $T_{\delta^+}(y) = y$. This follows since by (3) of Lemma 3.9, δ^+ and y are disjoint.
(3) For any other basis element z, the image $T_{\delta^+}(z)$ is a word in $\mathcal{B}$ not involving x. Again, this follows from Property (2), since the curve δ^+ does not cross the band corresponding to x.

These imply that T_{δ^+} commute with $\rho_{x,y}$:

(1) Since w does not involve x, we have
$$\rho_{x,y}T_{\delta^+}(x) = \rho_{x,y}(wx) = wxy.$$
Since, by (2) above, the twist fixes y, we also have:
$$T_{\delta^+}(\rho_{x,y}(x)) = T_{\delta^+}(xy) = wxy$$
(2) Since both T_{δ^+} and $\rho_{x,y}$ fix y, we have
$$\rho_{x,y}T_{\delta^+}(y) = y = T_{\delta^+}\left(\rho_{x,y}(y)\right)$$
(3) Finally, since for any other basis element z, the image $T_{\delta^+}(z)$ is a word in $\mathcal{B}$ not involving x by (3) above, we have
$$\rho_{x,y}T_{\delta^+}(z) = T_{\delta^+}(z) = T_{\delta^+}\left(\rho_{x,y}z\right).$$

Now, let S^g be a regular neighbourhood of $S_0 \cup \delta^+$. By property (4) of Lemma 3.9, this is strictly bigger than S_0. Observe that it is filled by δ^+ and curves contained in S_0. Since we have shown that such twists commute with $\rho_{x,y}$, and a suitable product of such twists is a pseudo-Anosov map of S^g, it is indeed a good subsurface for $\rho_{x,y}$. It remains to compute the fundamental group of the complementary bad subsurface.

For this, we refer again to Figure 3.3. One readily checks that the elements $y, \hat{y}, x^{-1}\hat{x}x$ are disjoint (up to homotopy) from both S_0 and the arc γ defining δ^+, and are thus contained in the bad subsurface. The same is clearly true for the boundary of the surface. The loop representing the boundary, starting at the first pair of bands after $\hat{a}_i, a_i$ (in the figure, this would be $\hat{b}$), i.e. $\partial_{\hat{a}_{i+1}}$ is thus another loop disjoint from S^g. It remains to see that these four elements indeed generate the fundamental group of $\Sigma - S^g$. To see this, note that the complement of S^g is a genus 2 surface with two boundary components. We obtain the complement of the improved good subsurface by further cutting at the loop γ defining δ^+ – which yields a torus with three boundary components. Two of these are $\partial_{\hat{a}_{i+1}}$ and $x^{-1}\hat{x}x$ (up to homotopy). Since $y, \hat{y}$ intersect once, the four elements therefore generate the fundamental group of that torus.

For the basic move $\lambda_{x,y}$ the proof is analogous, using that δ^- intersects x only in x^-, proving that $T_{\delta^-}(x) = xw'$.

The strategy for the case $x = \hat{a}_i$ is analogous. □

We also need the analog for invert moves. Here, the obvious good subsurface turn out to be already large enough. To keep notation consistent, we nevertheless phrase the following lemma about existence of improved good subsurfaces, so that these are defined for all moves, with uniform properties.

Lemma 3.12. — *With notation as in the previous lemma, consider a basic invert move $\phi = \iota_x$ (here, x is also allowed to be the non-orientable letter n). Then there is an* improved good subsurface *for ϕ whose complementary bad subsurface has fundamental group*

$$\pi_1(\Sigma - S^g) = \left\langle x, \hat{x}, \partial_{\hat{a}_{i+1}} \right\rangle$$

if $x = a_i$ or $x = \hat{a}_i$. Any basis letter except $x, \hat{x}$ is freely homotopic into the improved good subsurface.

Proof. — Here, only the first part of the proof of Lemma 3.10 is necessary; the desired good subsurface is the subsurface S_0 constructed in Lemma 3.8. □

3.3. BLASpaths

In this section we begin to discuss the paths we use as the basis for all of our constructions. The picture to have in mind is that we build paths by concatenating arational paths in *different* copies of $\mathcal{PML}$, joining them at points which fail to be arational in a very controlled way.

More formally, we say a path $p : [0,1] \to \partial CV_n$ is a BLASpath[(3)] if there is a finite set $B_p \subset p([0,1])$ so that

- Every point on $p([0,1]) \setminus B_p$ is an arational tree.
- If $T \in B_p$, then there is an identification $F_n = \pi_1(\Sigma)$ of the free group with a surface with one boundary component (where $\Sigma = \Sigma_g$ if $n = 2g$ is even, and $\Sigma = N_{2g+1}$ otherwise), and the following holds: T is the dual tree to the stable foliation of a pseudo-Anosov mapping class ψ_T supported on a subsurface S.

(3) These paths were heavily inspired by the clever construction of Leininger and Schleimer, who construct paths of arational foliations. Our paths aren't quite as good, since they have some points which are not arational. This lead, in an early iteration of this work, to the terminology Bad Leininger And Schleimer path, hence BLAS. Since it is pronouncable and short, the name stuck.

- For any $T \in B_p$ there is a neighborhood $\mathcal{N}(T)$ of T in ∂CV_n, so that $\mathcal{N}(T) \cap p([0,1]) \setminus T$ has two connected components, γ_1, γ_2.
- For each γ_i there is a path ξ_i so that $\xi_i \cup \psi_T \xi_i$ is a path and $\gamma_i = \cup_{k=0}^{\infty} \psi_T^k \xi_i$.

Using Theorem 2.10 and Theorem 2.11, we show:

Proposition 3.13. — *Let T_s, T_e in $\overline{CV_n}$ be two surface type arational trees, dual to uniquely ergodic laminations (or, in the nonorientable case, laminations in the set $\mathcal{P}$ from Theorem 2.11).*

Then T_s, T_e can be connected by a BLASpath. Moreover, if one prescribes a chain of adjacent $\mathcal{PML}$s connecting the $\mathcal{PML}$ on which T_s lies with the one on which T_e lies, we may assume the BLASpath travels exactly through that chain of $\mathcal{PML}$s in exactly the same order.

In addition, for any finite set F of arational trees not containing T_s, T_e, the path may be chosen to be disjoint from F.

Proof. — It suffices to show the proposition in the case where T_s, T_e lie in adjacent copies of $\mathcal{PML}$, say $\mathcal{PML}(\Sigma), \phi\mathcal{PML}(\Sigma)$. Furthermore, choose a partial pseudo-Anosov ψ on the improved good subsurface S of ϕ, and observe that its stable lamination λ^+ is not dual to any tree contained in the finite set F (since ψ is a partial pseudo-Anosov).

Observe that there is a neighbourhood U of λ^+ in $\mathcal{PML}(\Sigma)$ so that $\psi(U) \subset U$. We may assume further that no lamination in U is dual to any tree in F. Pick a uniquely ergodic surface lamination $\lambda_0 \in \mathcal{PML}$, and a path γ of uniquely ergodic laminations joining λ_0 to $\psi(\lambda_0)$ in $\mathcal{PML}$. This is possible by Theorem 2.10 or Theorem 2.11 (observing that, since the rank of the free group is assumed to be large enough, the genus of the surface is at least 5, and so the theorem applies). In particular, all points on γ correspond to laminations which fill the surface Σ, and therefore intersect the boundary ∂S of the improved good subsurface.

As a consequence, for $k \to \infty$, the paths $\psi^k(\gamma)$ converge to λ^+. Thus, we can choose an number $K > 0$ so that $\psi^k(\gamma) \subset U$ for all $k \geqslant K$. In particular,

$$\psi^K(\gamma) * \psi^{K+1}(\gamma) * \cdots$$

can be completed to a path p joining $\psi^K(\lambda_0)$ to λ_+ contained in U. In particular, no point on p is dual to a tree in F.

Now, by Theorem 2.10 or Theorem 2.11, there is a path joining λ_0 to $\psi^K(\lambda_0)$, and so that no point on it is dual to a tree in F. The concatenation of these two paths is the desired path. □

4. Avoiding problems

Recall from Section 2.2 that K_E is the set of trees in ∂CV_n where a free factor E does not act freely and simplicially. The main point of this section is to prove the following local result:

Theorem 4.1. — *Suppose that the rank of the free group F_n is at least* 18. *Let $E \in \mathcal{F}$ be any proper free factor. Assume that*

- *$\mathcal{PML}_{\sigma_1}$, $\mathcal{PML}_{\sigma_2}$ are adjacent copies of $\mathcal{PML}$ (i.e. differ by applying an adjusted move ϕ),*
- *$\lambda_1 \in \mathcal{PML}_{\sigma_1}$, $\lambda_2 \in \mathcal{PML}_{\sigma_2}$ are uniquely ergodic (or, if the rank is odd, in the set $\mathcal{P}$ from Theorem 2.11).*
- *ψ is a partial pseudo-Anosov mapping class on Σ supported on the improved good subsurface $S \subset \Sigma_1$ of the adjusted move ϕ as in Section 3.*

Then there exists a BLASpath $q : [0,1] \to \partial CV_N$ joining the dual trees T_1, T_2 of λ_1, λ_2, so that

(1) *There is a number $m > 0$ so that for all $k = mr \geqslant 0$ we have $\psi^k(q[0,1]) \cap K_E = \emptyset$. In particular, $q[0,1] \cap K_E = \emptyset$.*
(2) *$\psi^k(q[0,1])$ converges to the stable lamination of ψ as k goes to infinity.*

Also observe that $\psi^k(q[0,1])$ still connect $\mathcal{PML}_{\sigma_1}$, $\mathcal{PML}_{\sigma_2}$, since ψ commutes with the adjusted move.

We begin by describing how to construct a relation in $\mathrm{Out}(F_n)$ (which will serve as a “combinatorial skeleton” for a BLASpath), so that overlap and containment problems of A can be avoided at each step. It is important for the strategy that containment problems can be solved first; see below.

Before we begin in earnest, we want to briefly discuss the setup for the rest of this section. We begin by fixing once and for all an identification $\pi_1(\Sigma) \simeq F_n$, a corresponding “basepoint copy” $\mathcal{PML}(\Sigma)$, and a standard geometric basis $\mathcal{B}$ (as in Section 3.1). In building BLASpaths, we will always use $\mathrm{Out}(F_n)$ to move the current copy of $\mathcal{PML}$ to this basepoint copy, and work there.

When doing this, one has to be careful with the order of multiplication. To avoid confusion, we discuss this here first. Suppose, e.g., that we have a relation of basic or adjusted moves

$$\phi = \phi_1 \circ \cdots \circ \phi_l$$

in $\mathrm{Out}(F_n)$. Associated to this we have a sequence of consecutively adjacent copies of $\mathcal{PML}$

$$\mathcal{PML}(\Sigma), \phi_1\mathcal{PML}(\Sigma), \phi_1\phi_2\mathcal{PML}(\Sigma), \ldots, \phi_1 \circ \cdots \circ \phi_l\mathcal{PML}(\Sigma) = \phi\mathcal{PML}(\Sigma).$$

The i^{th} adjacency of this sequence, i.e. between $\phi_1 \circ \cdots \circ \phi_{i-1}\mathcal{PML}(\Sigma)$ and $\phi_1 \circ \cdots \circ \phi_l\mathcal{PML}(\Sigma)$ is the image under $\phi_1 \circ \cdots \circ \phi_{i-1}$ of the adjacency given by the adjusted move ϕ_i. This motivates the following

Definition 4.2. — *Let*

$$\phi = \phi_1 \circ \cdots \circ \phi_l$$

be a relation in $\mathrm{Out}(F_n)$. *We say that E* is a overlap or containment problem at the i^{th} step of the relation, *if $(\phi_1 \circ \cdots \circ \phi_{i-1})^{-1}(E)$ is an overlap or containment problem for the adjusted move ϕ_i.*

Our strategy will be to replace adjusted moves ϕ by such relations, so that a given factor E is not an overlap or containment problem at any stage of the relation. Recall that later, such relations will guide the construction of BLASpaths in which E will no longer be an obstruction to arationality at any point.

The details of this approach are involved and so we now state the two main ingredients, eliminating containment problems (Proposition 4.3) and eliminating overlap problems (Proposition 4.4), and prove Theorem 4.1 conditional on these results. (We will prove Proposition 4.3 in the next subsection, prove Proposition 4.4 in the orientable case in the following subsection and prove Proposition 4.4 in the non-orientable case in Appendix B.)

Proposition 4.3. — *Suppose that $E < F_n$ is a proper free factor, and ϕ is an adjusted move. If E is a containment problem for ϕ, then there is a relation*

$$\phi = \phi_1 \cdots \phi_l$$

with the property that E is not a containment problem at any stage of the relation.

Proposition 4.4. — *Suppose that ϕ is an adjusted move, and E is a free factor which is not a containment problem for ϕ. Then there is a relation*

$$\phi = \phi_1 \cdots \phi_l,$$

where each ϕ_i is an adjusted move, and so that E is neither a containment nor an overlap problem at any stage of the relation.

If ψ is any partial pseudo-Anosov supported on the improved good subsurface of the basic move ϕ then there is an number $k > 0$ so that for any $n \geqslant 0$ the conjugated relation

$$\phi = \psi^{-kn}\phi_1 \cdots \phi_l \psi^{kn}$$

has the same property.

Proof of Theorem 4.1 assuming Propositions 4.3 and 4.4. — Let ϕ be the adjusted move by which the adjacent $\mathcal{PML}$s differ. Using first Proposition 4.3, and then Proposition 4.4 (to each adjusted move appearing in that first relation), we can replace ϕ by a relation

$$\phi = \phi_1 \cdots \phi_l$$

so that in each stage E is neither an overlap nor containment problem.

Now we will use Proposition 3.13 to find a BLASpath q going through the chain of $\mathcal{PML}$s defined by the relation, that is

$$\mathcal{PML}(\Sigma_1), \phi_1\mathcal{PML}(\Sigma_1), \ldots, \phi_1 \cdots \phi_{l-1}\mathcal{PML}(\Sigma_1), \phi\mathcal{PML}(\Sigma_1) = \mathcal{PML}(\Sigma_2)$$

We claim that this path q itself is disjoint from K_E. Observe that it suffices to check this at all of the points of the BLASpath which are not minimal, and therefore dual to the stable lamination of a partial pseudo-Anosov. For these finitely many points, Lemma 2.21 applies (exactly because we have guaranteed that E is not an overlap or containment problem at any stage of the relation by Proposition 4.4), and shows that they are outside K_E as well.

To prove Property (1), i.e. that $\psi^k q$ is disjoint from K_E, observe that $\psi^k q$ can be thought of as a BLASpath guided by the conjugated relation

$$\phi = \psi^{-k}\phi_1 \cdots \phi_l \psi^k,$$

to which (by the last sentence of Proposition 4.4) the same argument applies.

Finally, Property (2) is implied by Proposition 2.18, assuming that the BLASpath is constructed to never intersect the unstable lamination of ψ – which can be done by avoiding a single lamination in the construction of the BLASpath, and is therefore clearly possible. □

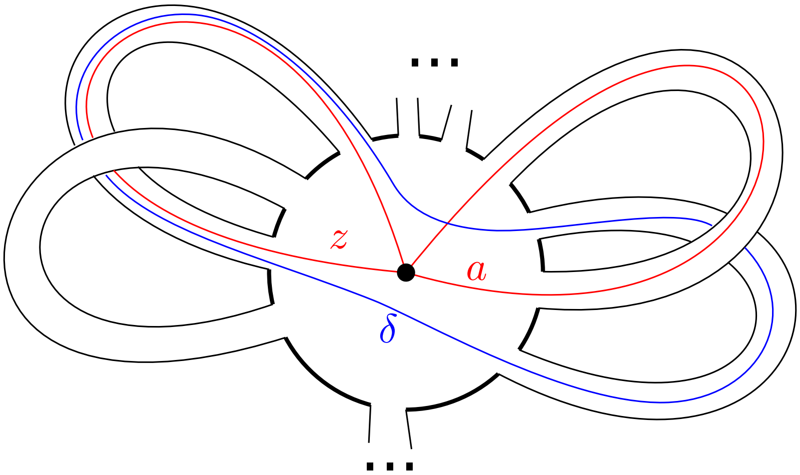

Figure 4.1. The curve from Lemma 4.5

4.1. Containment problems: Proof of Proposition 4.3

The proof of this proposition relies on the construction of a curve with certain properties.

LEMMA 4.5. — *Suppose we are given elements z, w, a of our chosen standard geometric basis $\mathcal{B}$, all of which are two-sided, and no two of which are linked. Then there is a two-sided curve δ with the following properties:*

(i) *δ intersects a in a single point,*
(ii) *δ does not cross the band corresponding to w (i.e. in the fundamental group, δ can be written without the letter w),*
(iii) *There is another, unrelated two-sided letter e, so that δ does not cross the band corresponding to e (i.e. δ can be written without e).*
(iv) *δ intersects each basis loop of $\mathcal{B}$ at most two points, one on an initial and one on a terminal segment (with terminology as in Section 3),*
(v) *in homology we have $[\delta] = \pm[\hat{a}] \pm [z]$.*

Indeed, the desired curve can be found as the concatenation of $\hat{a}$ and z as in Figure 4.1.

Proof of Proposition 4.3. — We begin by considering the basic move $\phi = \lambda_{x,y}$ or $\phi = \rho_{x,y}$, and assume that E is a containment problem for the basic move. Let z be a basis letter so that $[z] \notin H_1(E)$ (and thus, in particular, z is not conjugate into E). Such a letter exists, since E is a proper free factor.

Also observe that since we assume that E is a containment problem (and thus the fundamental group of the improved good subsurface is contained in E up to conjugacy), the basis letter z is one of $x, \hat{x}, y, \hat{y}$ or the one-sided letter n. Namely, according to Lemma 3.10 (or Lemma A.1), these are the only basis letters which are not conjugate into the fundamental group of the improved good subsurface.

Next, choose w an unrelated, two-sided letter, i.e. different from all of $x, \hat{x}, y, \hat{y}$, and not linked with z; in particular it is *good for* ϕ (i.e. intersects the improved good subsurface) by Lemma A.1. Let a be a two-sided basis element so that $a, \hat{a}$ are good for ϕ and $\lambda_{z,w}$, and are distinct from any of the previously chosen letters.

Now, let δ be the curve guaranteed by Lemma 4.5. Define an auxiliary adjusted move

$$\theta = T_\delta \lambda_{w,z} T_\delta^{-1}.$$

We observe that θ has the following properties:

(1) θ fixes every basis element, except possibly w.
Namely, by property (iv) of the curve δ, the Dehn twist T_δ acts on each basis element by conjugation, left, or right multiplication by a word obtained by tracing δ starting at a suitable point. By property (ii) , none of these words involve the letter w. Thus, the letter w appears only in the image of w in $T_\delta(\mathcal{B})$. Since $\lambda_{w,z}$ fixes all letters except w, the claim follows,

(2) The word $\theta(w)$ does not involve the letter e from Lemma 4.5.
This follows from the description of the action of T_δ above, together with property (iii) of δ.

(3) $[\theta(w)] = [w] + [z] \notin H_1(E)$.
Namely, by property (v) of δ, we have $[T_\delta(w)] = [w]$ (as the algebraic intersection between δ and w is zero). Thus, $[\lambda_{w,z} T_\delta^{-1}(w)] = [w] + [z]$. Since the algebraic intersection number between δ and z is also zero, we therefore have $[\theta(w)] = [w] + [z]$. Since w is good for $\lambda_{x,y}$, and we assume that E is a containment problem for ϕ, we have $[w] \in H_1(E)$. Thus, since $[z] \notin H_1(E)$, the claim follows.

(4) E is not a containment problem for θ.
Since a and $\hat{a}$ are good for ϕ, and we assume that E is a containment problem, it follows that $[a] + [\hat{a}] \in H_1(E)$. On the other hand, the improved good subsurface of θ can be obtained from the improved good subsurface of $\lambda_{w,z}$ by applying T_δ. Recall that a is good for $\lambda_{w,z}$, and thus $T_\delta(a)$ is good for θ. In homology we have $[T_\delta(a)] = [a] + [\hat{a}] + [z]$. Since $[a] + [\hat{a}] \in H_1(E)$ but $[z] \notin H_1(E)$ this implies $[T_\delta(a)] \notin H_1(E)$.

(5) The automorphism $\phi^{-1}\theta^{-1}\phi\theta$ fixes all basis elements except possibly w.
This is an immediate consequence of the fact that θ fixes all letters except w, which is distinct from x, y (which are the only letters involved in ϕ).

This shows that there is a relation of the form

$$\phi = \theta \phi D_w \theta^{-1}, \tag{4.1}$$

where D_w is a product of basic moves of the form $\phi^{\pm}_{w,q}$ acting on the letter w, and no q is equal to e. Hence, D_w commutes with $\lambda_{e,z}$, and we obtain a relation

$$\phi = \theta \phi \lambda_{e,z} D_w \lambda_{e,z}^{-1} \theta^{-1}. \tag{4.2}$$

Observe that D_w may be identity; in which case we also remove the $\lambda_{e,z}$–terms from this relation.

We now check that this relation has no containment problems, as claimed. Recall that we have to check this left-to-right.

- *The initial θ move:* This is (4) above.

- *The move ϕ:* Here, we observe that w is good for the two possibilities $\lambda_{x,y}$ and $\rho_{x,y}$ that ϕ can be. On the other hand, we have that $\theta(w) \notin E$ by (3) above. Thus, $w \notin \theta^{-1}E$, and the claim follows.
- *The auxiliary move $\lambda_{e,z}$:* This move has w as a good letter, and $\theta\lambda_{x,y}(w) = \theta(w) \notin E$ just like above.
- *The moves in D_w:* These basic moves all have e as a good letter. Now, we have $\theta\phi\lambda_{e,z}(e) = \theta(ez) = ez$, and $[e] + [z] \notin H_1(E)$.
- *Undoing the auxiliary move $\lambda_{e,z}$:* This move has w as a good letter, and $\theta\phi\lambda_{e,z}D_w(w) = \theta\phi D_w(w) = \phi\theta(w)$ by the fact that D_w commutes with $\lambda_{e,z}$ and Equation (4.1). But, $[\phi\theta(w)] = [z] + [w] \notin H_1(E)$. Thus, $\theta\phi\lambda_{e,z}D_w(w) \notin E$.
- *The final θ^{-1} move:* Here, we use again (as in (4) above) that $T_\delta(a)$ is good for θ. We have $[T_\delta(a)] = [a] + [\hat{a}] + [z]$, and thus

$$\left[\theta\phi\lambda_{e,z}D_w\lambda_{e,z}^{-1}T_\delta(a)\right] = [a] + [\hat{a}] + [z]$$

  which does not lie in $H_1(E)$. Thus, we do not have a containment problem.

Finally, we need to deal with a basic inversion move $\phi = \iota_x$. Again, assume that E is a containment problem. As above, there is a basis letter whose homology class does not lie in $H_1(E)$, and thus not in the homology of the improved good subsurface. Comparing with Lemma 3.12 we see that the only possible such basis letters are $x, \hat{x}$.

Thus, either $[x] \notin H_1(E)$ or $[\hat{x}] \notin H_1(E)$. In the former case, we start with the relation

$$\iota_x = \lambda_{x,z}\iota_x\rho_{x,z},$$

and in the latter case, with a relation of the type

$$\iota_x = \rho_{\hat{x},z}\iota_x\rho_{\hat{x},z}^{-1}$$

for an unrelated letter z. Observe that $\lambda_{x,z}^{-1}E$ (respectively $\rho_{\hat{x},z}^{-1}E$) is then not a containment problem for ι_x: namely, since z is good for ι_x, but $\lambda_{x,z}z = xz$ (respectively $\hat{x}z$) is not conjugate into E, since $[z]+[x]$ (respectively $[z]+[\hat{x}]$) do not lie in $H_1(E)$ by the choice above.

Now, suppose we are in the former case (the other one is completely analogous). Then, by the first part of the proof, we can find a relation

$$\lambda_{x,z} = \phi_1 \circ \cdots \circ \phi_i,$$

so that E is not containment problem at any step of this relation. Hence, in the relation

$$\iota_x = \phi_1 \circ \cdots \circ \phi_i\iota_x\rho_{x,z},$$

the factor E is now not a containment problem at the first $i+1$ steps. Now, appealing to the first part of the proof again, we can find a relation

$$\rho_{x,z} = \phi'_1 \circ \cdots \circ \phi'_j,$$

so that $\lambda_{x,z}\iota_x(E)$ is not a containment problem at any step. Then, the relation

$$\iota_x = \phi_1 \circ \cdots \circ \phi_i\iota_x\phi'_1 \circ \cdots \circ \phi'_j,$$

has the desired properties.

Finally, we discuss adjusted moves. Suppose $\phi = \varphi^{-1}\phi_0\varphi$ is the conjugate of a basic move by a mapping class group element φ of Σ. We then apply the Proposition 4.3 to the basic move ϕ_0 and the factor $\varphi^{-1}E$, and conjugate the resulting by φ. This resulting relation (of adjusted moves) then has the desired property. □

4.2. Overlap Problems: Proof of Proposition 4.4

The proof of Proposition 4.4 is technically very involved, and the details vary depending on the nature of the move ϕ. First, observe that exactly as in the last paragraph of the proof of Proposition 4.3, the case of adjusted moves can be reduced to the case of basic moves. The rest of this section is therefore only concerned with basic moves.

For basic moves, we will (again, similar to the proof of Proposition 4.3), reduce the case of invert moves to the case of Nielsen moves. For Nielsen moves the relation claimed in the proposition will be constructed using the following two lemmas, which construct a "preliminary relation", and "short relations":

LEMMA 4.6 (Preliminary Relation). — *Suppose that ϕ is an adjusted move, and E is a free factor which is not a containment problem for ϕ.*

If ϕ is not an invert move, there is a relation

$$\phi = \phi_1 \cdots \phi_r,$$

so that if B_i is the bad subgroup for ϕ_i and B is the bad subgroup of ϕ, then (up to conjugation) the intersection

$$\phi_1 \cdots \phi_{i-1} B_i \cap B$$

is trivial or (up to conjugacy) contained in a "problematic" group of the form $\langle\partial\rangle$,$\langle x_i\rangle$ or $\langle x_i, \partial\rangle$ for some element x_i, where ∂ is the boundary component of the surface (compare Section 3.8)[4].

LEMMA 4.7 (Short Relations). — *For all indices i in Lemma 4.6 where there is a nontrivial problematic group, there is a relation*

$$\phi_i = \rho_i \phi_i \rho_i^{-1},$$

with the properties

(1) *No conjugacy class of the problematic group $\langle\partial\rangle$,$\langle x_i\rangle$ or $\langle x_i, \partial\rangle$ is contained in the bad subsurface of ρ_i and in E,*
(2) *and also $E \cap \phi_1 \cdots \phi_{i-1}\rho_i B_i$ is trivial.*

The proofs of these lemmas construct the desired relations fairly explicitly, and involve lengthy checks. Before we begin with these proofs, we explain how to use the lemmas in the proof of Proposition 4.4. We need two more tools: first, the following immediate consequence of Proposition 2.22 (this corollary is the reason why in our strategy, containment problems need to be solved before overlap problems).

[4] We remark that, in general, the intersection of two subgroups up to conjugacy could be a collection of conjugacy classes of subgroups. Here, in turns out (from the explicit computations in the proof) that the intersection always consists of at most one such conjugacy class.

Observe that the Proposition 2.22 may be used, since the fundamental group of improved good subsurfaces are free factors of large rank (corank at most 5), while the fundamental groups of the complements of improved good subsurfaces have rank at most 5 – and so the former can never be contained in the latter up to conjugacy, since free factors intersect other subgroups in factors.

Corollary 4.8. — *Suppose ϕ is a basic move with bad subgroup $B = \pi_1(\Sigma - S^g)$, ψ the commuting partial pseudo-Anosov supported on the improved good subsurface on ϕ, and E any free factor. Suppose that E is not a containment problem for ϕ. Then there is a number k with the following property.*

If

$$\phi = \phi_1 \cdots \phi_l$$

is a relation, and $M = km$ is large enough, then the conjugated relation

$$\phi = \left(\psi^M \phi_1 \psi^{-M}\right) \left(\psi^M \cdots \psi^{-M}\right) \left(\psi^M \phi_l \psi^{-M}\right),$$

has the following property: at every step of the relation, an overlap problems with E occurs exactly if E contains nontrivial elements conjugate into $B \cap B_i$, the intersection of the bad subgroups of the original relation.

Second, we need the following lemma, guaranteeing that within a relation which replaces a move without containment problem, no new containment problems are created.

Lemma 4.9. — *Assume that E is not a containment problem for ϕ, and that*

$$\phi = \phi'_1 \circ \cdots \circ \phi'_R$$

is a relation. Then, there is a number $k > 0$ so that conjugating the relation by any large power kN of a partial pseudo-Anosov ψ supported on the improved good subsurface of ϕ, we can guarantee that E is not a containment problem at any stage of the resulting relation

$$\phi = \psi^{kN} \circ \phi'_1 \circ \cdots \circ \phi'_R \circ \psi^{-kN}.$$

Proof. — Let G_i be the fundamental group of the improved good subsurface at the i^{th} step of the relation, and note that it is a free factor. After conjugating the relation by ψ^{-n}, this good free factor becomes $\psi^{-n} G_i$.

Now, recall that the intersection of the free factor G_i with the bad subgroup B of ϕ is a free factor of B. In particular, since the rank of the bad subgroups is at most 5, but the good free factor G_i has rank strictly larger than 5, it cannot be completely contained in B. In other words, there is some element $g_i \in G_i$ which intersects the subsurface in which ψ is supported.

Now, apply Proposition 2.22 for the factor E, and $B' = G_i$. Since Conclusion1 of that proposition is impossible here (as E is not a containment problem for ϕ), we see that for large $n = kN$, the only classes contained in $\psi^n E$ and G_i are contained in the bad subsurface fundamental group B. Since g_i is not contained in B, this shows that E is not a containment problem. □

We are now ready for the proof of the central result of the section.

Proof of Proposition 4.4. — First, we prove the proposition for basic Nielsen moves. We first apply Lemma 4.6 to obtain the preliminary relation

$$\phi = \phi_1 \cdots \phi_r \tag{4.3}$$

of some length r. Next, we apply Lemma 4.7 to each index $i = 1, \ldots, r$ it applies to in Relation (4.3) (i.e. where the problematic subgroup is nontrivial); say there are k such indices. We then have a relation of length $r + 2k$

$$\phi = \phi_1 \cdots \phi_{i-1}\rho_i\phi_i\rho_i^{-1}\phi_{i+1} \cdots \phi_r \tag{4.4}$$

where we replace each factor ϕ_i of Equation (4.3) to which Lemma 4.7 applied (because there was originally a problematic subgroup) has been replaced by the corresponding "short relation" of length 3.

We emphasise that this relation (4.4) may still have overlap problems (in particular, since ρ_i may have other, "new" overlap problems. However, from construction, we know that these new overlap problems will be guaranteed to be outside the intersection $E \cap \phi_1 \cdots \phi_{i-1}B_i$).

Now, for all $N > 0$, we apply Lemma 4.9 to the relation (4.4) to obtain a conjugated relation

$$\phi = \psi^{kN} \circ \phi_1 \cdots \phi_{i-1}\rho_i\phi_i\rho_i^{-1}\phi_{i+1} \cdots \phi_r \circ \psi^{-kN} \tag{4.5}$$

Further, in this conjugated relation, the factor E is now not a containment problem at any stage of this relation by Lemma 4.9.

Hence, we can apply Corollary 4.8 to the each of the inserted "small relations" of length 3 in Relation (4.5), further replacing them by conjugates of suitable powers of the associated pseudo-Anosov of ϕ_i, yielding a relation of the form

$$\phi = \psi^{kN} \circ \phi_1 \cdots \phi_{i-1}\left(\psi_i^{-M_i}\rho_i\psi_i^{M_i}\phi_i\psi_i^{-M_i}\rho_i^{-1}\psi_i^{M_i}\right)\phi_{i+1} \cdots \phi_r \circ \psi^{-kN} \tag{4.6}$$

Since Relation (4.5) had no containment problems at any stage, Lemma 4.9 can again be applied to guarantee that there replacements also do not have containment problems.

Furthermore, Corollary 4.8 implies that for this relation any overlap problems can only occur within the intersection of the bad factor B_i of ϕ_i and the bad factor of the move ρ_i. Now, by construction, there are no conjugacy classes that both of those factors have in common with $(\phi_1 \cdots \phi_{i-1})^{-1}E$. Hence, this final relation indeed solves all containment and overlap problems.

If we conjugate this relation by a further power of ψ^k, then Lemma 4.9 shows that in the resulting relation E is still no containment problem at any stage, and Corollary 4.8 shows the same for overlap problems. This shows the proposition for basic Nielsen moves.

Now, let $\phi = \iota_x$ be a basic invert move. Since the subgroup generated by basic Nielsen moves is normal, for any product α of basic Nielsen moves there is a product of basic Nielsen moves β, so that

$$\iota_x = \alpha\iota_x\beta.$$

By choosing α to be a large power of a pseudo-Anosov mapping class, we may assume that $\alpha^{-1}E$ is not a containment or overlap problem for ι_x.

Now (similar to the proof of Proposition 4.3), by applying the current proposition for basic Nielsen moves, we can write

$$\alpha = \alpha_1 \circ \cdots \circ \alpha_r, \beta = \beta_1 \circ \cdots \circ \beta_s$$

so that E is not an overlap or containment problem at any stage of the first relation, and so that $(\alpha\iota_x)^{-1}E$ is not an overlap or containment problem at any stage of the second. The resulting relation

$$\iota_x = \alpha_1 \circ \cdots \circ \alpha_r \iota_x \beta_1 \circ \cdots \circ \beta_s$$

then has the desired property. □

To prove Lemmas 4.6 and 4.7 which construct relations, we need to collect some results on controlling the intersections between finitely generated subgroups of free groups. These results are basically standard (see [Sta83]), but we present them in a form useful for the checks below. Throughout, we denote by R_n the rose labelled by the elements of out chosen standard geometric basis $\mathcal{B}$. We identify edge-paths in R_n with words in $\mathcal{B}$.

Suppose we are given a subgroup

$$A = \langle \alpha_1, \ldots, \alpha_r \rangle$$

where each α_i is a reduced word in our fixed basis $\mathcal{B}$. We denote by R_A the subdivided rose labelled by the α_i, and by $f : R_A \to R_n$ the graph morphism inducing the inclusion of A as a subgroup of F_n (recall that graph morphisms map vertices to vertices, and edges to edges).

Let Γ_A be a graph obtained by folding from R_A, so that f factors as

$$R_A \overset{p_A}{\to} \Gamma_A \overset{g_A}{\to} R_n$$

where g_A is an immersion.

Definition 4.10. —

(1) *A subword w of one of the α_i is called a* certificate in α_i, *if there is an embedded path $\gamma_w \subset \Gamma_A$, which lifts to a path $\tilde{\gamma}_i$ in R_A, which is contained in the geodesic representative of the free homotopy class of the petal corresponding to α_i, and representing w.*
(2) *We say that a certificate is* uncancellable *if γ_w is disjoint from the images of all other petals $\alpha_j, j \neq i$ of R_A under p_A.*
(3) *A reduced word w in $\mathcal{B}$ is* impossible in A, *if the corresponding path in R_n does not lift to Γ_A (equivalently, there is no path in Γ_A labelled by w)*

At this point, observe that if w is an uncancellable certificate in α_i, then the edge e of Γ_A containing γ_w is nonseparating, and no image of a petal except α_i crosses it. Since the α_i form a basis of the group A, this implies that α_i crosses the edge e exactly once.

Lemma 4.11 (Dropping Generators – Impossible Certificates). — *Suppose that*

$$A = \langle \alpha_1, \ldots, \alpha_r \rangle,$$
$$B = \langle \beta_1, \ldots, \beta_s \rangle$$

are two subgroups (where the α_i, β_j are words in a common basis $\mathcal{B}$).

Suppose that τ is an uncancellable certificate in α_1, which is impossible in B. Then any conjugacy class contained in A and B is also contained in $\langle\alpha_2, \ldots, \alpha_r\rangle$.

Proof. — Let $x \in A$ be an element which is not conjugate into $\langle\alpha_2, \ldots, \alpha_r\rangle$. Then, let $\gamma \subset \Gamma_A$ be a geodesic representing x.

By definition of uncancellable certificate, this geodesic crosses the path γ_τ, and so contains a subpath labelled by τ. Namely, we can collapse a maximal tree in Γ_A disjoint from the edge e containing γ_τ to obtain a rose R' with fundamental group A, and the images of the α_i give a basis of B so that exactly α_1 crosses the edge e, and exactly once. Now it is clear that an element which does not have a conjugacy class which does not use α_1 needs to cross the edge e.

Thus, the geodesic $g_A(\gamma)$ contains a subpath $g_A(\tau)$ which, as τ is impossible for B, is in the image of no loop $\gamma' \subset \Gamma_B$ under g_B. This shows the claim. □

We need a version of the dropping letters lemma which applies when A and B share a generator.

LEMMA 4.12 (Dropping Generators – Impossible Unique Followup). — *We are given two subgroups*

$$A = \langle\alpha_1, \ldots, \alpha_r, \delta_A\rangle$$
$$B = \langle\beta_1, \ldots, \beta_s, \delta_B\rangle,$$

where the $\alpha_i, \delta_A, \beta_j, \delta_B$ are words in a fixed basis $\mathcal{B}$.

Suppose that

1. *β_1 contains an uncancelable certificate τ,*
2. *the only path τ_A in Γ_A which lifts to τ is contained within the geodesic representative $\overline{\delta_A}$ of the image of δ_A in the immersed graph Γ_A,*
3. *there is an uncancellable certificate τ' in δ_A whose image immediately follows τ_A,*
4. *no path corresponding to a reduced word $\beta_1 b$ (for $b \in B$). lifts to a path starting with $\tau\tau'$.*

Then any conjugacy class in A and B is also contained in

$$\langle\beta_2, \ldots, \beta_s, \delta_B\rangle.$$

Proof. — The proof is very similar to the previous one. Let $x \in B$ be an element which is not conjugate into $\langle\beta_2, \ldots, \beta_s, \delta_B\rangle$. Then, let $\gamma \subset \Gamma_B$ be a geodesic representing x. Since any loop representing x in R_B has to involve β_1, and by definition of uncancellable certificate, γ contains a subpath labelled by τ. Now, suppose $\gamma' \subset \Gamma_A$ is a geodesic representing the same conjugacy class x. Then, γ' contains a subpath labelled by τ, and by (2) this occurs in $\overline{\delta_A}$ and is followed by τ'. By uncancelability, τ' also follows τ in the loop γ – which contradicts (4). □

Finally, we need the following well-known fact. For the proof (and for later use), we recall the notion of *Whitehead graph*. Namely, suppose that $\mathcal{B} = \{x_1, \ldots, x_n\}$ is a free generating set of F_n, and that w is a cyclically reduced word in the x_i. Then the Whitehead graph of w is the graph whose vertex set is $\{x_i^\pm, i = 1, \ldots, n\}$, and vertices x, y are joined by an edge if xy^{-1} is a subword of a cyclic permutation of w.

A classical result by Whitehead [Whi36] shows that if w represents an element which is contained in a free factor, then the Whitehead graph (with respect to any generating set) is either disconnected, or contains a cut vertex (i.e. a vertex whose removal disconnects the graph).

Lemma 4.13 (Intersecting with factors). — *Let*

$$\partial = \prod_{i=1}^{g} \left[\hat{a}_i, a_i^{-1}\right]$$

or

$$\partial = \left(n\hat{a}a\hat{a}^{-1}na\right) \prod_{i=1}^{g} \left[\hat{a}_i, a_i^{-1}\right]$$

be the boundary of the surface (in the nonorientable case, $a, \hat{a}$ are the letters linked with the one-sided n, and the indexed letters are the other generators). Then ∂ is contained in no proper free factor of the free group.

Proof. — The claim follow since the Whitehead graph for ∂ is a single loop in both cases, and therefore is connected without a cut point. □

We are now ready to prove the lemmas.

Proof of Lemma 4.6. — The construction depends on the nature of the involved letters (one- or two-sided, linked with the one-sided or not; as in Section 3) of the basic move ϕ.

Here, we discuss the case of $\phi = \rho_{x,y}$ on an orientable surface in detail. The computations for the other cases follow the same general approach; we have collected the details in Appendix B.

For ease of notation in this construction, we assume that the order of loops in the basis is

$$\hat{x}, x, \hat{y}, y, \hat{a}_3, a_3, \ldots$$

Thus, by Lemma 3.10, the bad subgroup is $B = \langle y, \hat{y}, x^{-1}\hat{x}x, \partial_{\hat{y}}\rangle$, where $\partial_{\hat{y}}$ denotes the cyclic permutation of the boundary word

$$\partial = \left[\hat{x}, x^{-1}\right] \left[\hat{y}, y^{-1}\right] \prod_{i>2}^{g} \left[\hat{a}_i, a_i^{-1}\right]$$

starting at $\hat{y}$.

We use the relation

$$\rho_{x,y} = \rho_{\hat{y},u}^{-1}\rho_{y,z}^{-1}\rho_{x,y}\rho_{x,z}\rho_{y,z}\rho_{\hat{y},u},$$

where $u = a_4, z = a_6$ and $g \geqslant 7$.

(a) By Lemma 3.10, $\rho_{\hat{y},u}^{-1}$ has bad subgroup

$$A = \left\langle u, \hat{u}, y, \partial_{y^{-1}} \right\rangle,$$

where $\partial_{y^{-1}}$ is the cyclic permutation of ∂ beginning with y^{-1}. We need to intersect this subgroup with

$$B = \left\langle y, \hat{y}, x^{-1}\hat{x}x, \partial_{\hat{y}} \right\rangle$$

We begin by finding graphs which immerse into the rose with petals corresponding to the basis $\mathcal{B}$, and which represent A and B.

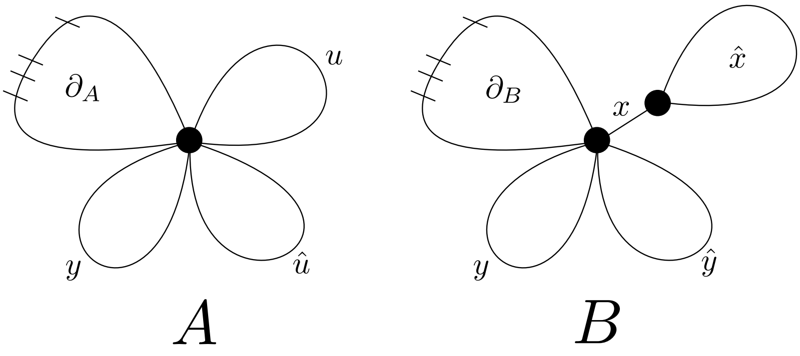

Figure 4.2. The immersed graphs for the intersection in step a)

We begin with A. Here, the starting point is a rose with four petals corresponding to the four generators $u, \hat{u}, y, \partial_{y^{-1}}$. This is not yet immersed, as the petal corresponding to $\partial_{y^{-1}}$ begins with a segment $y^{-1}\hat{y}^{-1}y$ whose first part can be folded over the petal corresponding to y. The resulting folded petal ∂_A now starts with $\hat{y}^{-1}$ (and still ends with a_g).

Hence, this resulting graph immerses (compare the left side of Figure 4.2).

The immersed graph for B is similarly obtained by first folding the first and last segment of the petal labeled by $x^{-1}\hat{x}x$ together, and then folding the initial commutator $[\hat{y}, y^{-1}]$ and last segment $x^{-1}\hat{x}x$ of $\partial_{\hat{y}}$ over the rest. We denote by ∂_B the image of this folded petal; note that it is still based at the same point (compare the right side of Figure 4.2).

To compute the conjugacy classes in the intersection of these groups, we begin by using Lemma 4.12 with $\tau = u$ as the input path. Observe that it is indeed uncancellable in A, and appears in B only in the petal ∂_B.

Since $u = a_4$ and the rank is at least 6, the petal ∂_B will contain a subpath labelled $[\hat{a}_4, a_4^{-1}][\hat{a}_5, a_5^{-1}]$. We let τ' be the (uncancellable) path $a_4[\hat{a}_5, a_5^{-1}]$ following u in this subpath.

Observe that this it is impossible to achieve such a path in A starting with u, since a_5 appears only in the interior ∂_A. Hence, Lemma 4.12 applies, and any conjugacy class contained in A and B is in fact also contained in

$$A' = \langle \hat{u}, y, \partial_{y^{-1}} \rangle .$$

Hence, we now aim to compute the intersection of A' and B using the same method. The immersed graph for A' is obtained by simply deleting the petal labeled u from the graph for A. We can then argue exactly as above (with the input path $\tau = \hat{u}$) to also drop the generator $\hat{u}$, and find that any conjugacy class common to A and B is also contained in

$$A'' = \langle y, \partial_{y^{-1}} \rangle .$$

Observe that this rank-2 group is indeed contained in both A and B, and so it is the full intersection. Since it has the desired form, we are done with this step.

(b) $\rho_{y,z}^{-1}$ has bad subgroup $A_2 = \langle z, \hat{z}, y^{-1}\hat{y}y, \partial_{\hat{a}_3}\rangle$, which we need to intersect with

$$\rho_{\hat{y},u}B = \left\langle y, \hat{y}u, x^{-1}\hat{x}x, \rho_{\hat{y},u}\partial_{\hat{y}} \right\rangle .$$

For this intersection, we need to take some care of the order of simplifications. We begin by observing that the path $\hat{y}u$, which corresponds to a petal of the immersed graph of $\rho_{\hat{y},u}B$ is impossible in A_2 – the only generator which contains $u = a_4$ at all is $\partial_{\hat{a}_3}$, and there it is never directly adjacent to y. Hence, by Lemma 4.11 we may replace $\rho_{\hat{y},u}B$ by

$$\left\langle y, x^{-1}\hat{x}x, \rho_{\hat{y},u}\partial_{\hat{y}} \right\rangle .$$

Now, we can further remove $\rho_{\hat{y},u}\partial_{\hat{y}}$, as it also contains $\hat{y}u$ as a subword (observe that this would have been impossible as the first step, since this subword was folded over the petal $\hat{y}u$ in the original immersed graph). Now, we need to compare

$$\left\langle y, x^{-1}\hat{x}x \right\rangle \quad \text{and } A_2 = \left\langle z, \hat{z}, y^{-1}\hat{y}y, \partial_{\hat{a}_3} \right\rangle .$$

From the latter, we can drop $\partial_{\hat{a}_3}$ since it clearly contains uncancellable subwords which are impossible in the former (again, using Lemma 4.11). Then, it is easy to see that the remaining groups have no conjugacy classes in common (by drawing immersed graphs representing them, or further applying Lemma 4.11).

(c) $\rho_{x,y}$ has bad subgroup $B = \langle y, \hat{y}, x^{-1}\hat{x}x, \partial_{\hat{y}}\rangle$ and we need to intersect with

$$\rho_{y,z}\rho_{\hat{y},u}B = \left\langle yz, \hat{y}u, x^{-1}\hat{x}x, \rho_{y,z}\rho_{\hat{y},u}\partial_{\hat{y}} \right\rangle .$$

The argument is similar to (b). We first focus on the generators $yz, \hat{y}u$ of $\rho_{y,z}\rho_{\hat{y},u}B$. Using Lemma 4.11 we can drop these in order to compute the intersection (as these certificates are impossible in B). After that is done, we can then also further drop $\rho_{y,z}\rho_{\hat{y},u}\partial_{\hat{y}}$ from $\rho_{y,z}\rho_{\hat{y},u}B$ using Lemma 4.11 again, as yz or $\hat{y}u$ are now certificates (after the previous step, these survive in the immersed graph), which are impossible in B. Hence, the intersection is $\langle \hat{x}\rangle$.

(d) $\rho_{x,z}$ has bad subgroup $A_3 = \langle z, \hat{z}, x^{-1}\hat{x}x, \partial_{\hat{y}}\rangle$, and we need to intersect with

$$\rho_{x,y}^{-1}\rho_{y,z}\rho_{\hat{y},u}B = \left\langle yz, \hat{y}u, yx^{-1}\hat{x}xy^{-1}, \rho_{x,y}^{-1}\rho_{y,z}\rho_{\hat{y},u}\partial_{\hat{y}} \right\rangle .$$

We begin by dropping $\hat{y}u$ from the latter, since it is impossible in A_3. Afterwards, we can also drop $\rho_{x,y}^{-1}\rho_{y,z}\rho_{\hat{y},u}\partial_{\hat{y}}$ (since it also contains $\hat{y}u$, and this subpath is now certainly not folded over anymore, as above). After this, we can remove $\partial_{\hat{y}}$ from A_3 since it contains (many) subpaths which are impossible in the other group. At this stage, we need to compare

$$\left\langle z, \hat{z}, x^{-1}\hat{x}x \right\rangle \quad \text{and} \quad \left\langle yz, yx^{-1}\hat{x}xy^{-1} \right\rangle ,$$

whose intersection is clearly $\langle \hat{x}\rangle$.

(e) $\rho_{y,z}$ has bad subgroup $A_2 = \langle z, \hat{z}, y^{-1}\hat{y}y, \partial_{\hat{a}_3}\rangle$ and we need to intersect with

$$\rho_{x,z}^{-1}\rho_{x,y}^{-1}\rho_{y,z}\rho_{\hat{y},u}B = \left\langle yz, \hat{y}u, x^{-1}\hat{x}x, \rho_{x,z}^{-1}\rho_{x,y}^{-1}\rho_{y,z}\rho_{\hat{y},u}\partial_{\hat{y}} \right\rangle .$$

As before, we start by removing $\hat{y}u$ from the latter, then the boundary word from both. The remaining intersection between

$$\left\langle z, \hat{z}, y^{-1}\hat{y}y \right\rangle \quad \text{and} \quad \left\langle yz, x^{-1}\hat{x}x \right\rangle .$$

is trivial.

(f) Finally, $\rho_{\hat{y},u}$ has bad subgroup $A = \langle u, \hat{u}, y, \partial_{y^{-1}} \rangle$, which we intersect with

$$\rho_{y,z}^{-1}\rho_{x,z}^{-1}\rho_{x,y}^{-1}\rho_{y,z}\rho_{\hat{y},u}B = \left\langle y, \hat{y}u, x^{-1}\hat{x}x, \rho_{y,z}^{-1}\rho_{x,z}^{-1}\rho_{x,y}^{-1}\rho_{y,z}\rho_{\hat{y},u}\partial_{\hat{y}} \right\rangle$$

to find in $\langle y \rangle$ (arguing as before).

The relation for $\lambda_{x,y}$ is similar, with ρ changed to λ. The case where either x, y or both are "hatted letters" is also analogous. □

Proof of Lemma 4.7. — As in the previous lemma, the details vary depending on the nature of ϕ, and the construction is explicit. In contrast to the previous lemma, the arguments are straightforward here, and we only give the details for the case discussed in the proof of Lemma 4.6. The letters below indicate the terms in the relation constructed in that proof.

(a) We perform $\lambda_{y,w}$ before this move and $\lambda_{y,w}^{-1}$ after. Note that these moves indeed commute with $\rho_{\hat{y},u}^{-1}$.

The bad subgroup of $\lambda_{y,w}$ is $\langle \hat{y}, w, \hat{w}, \partial_y \rangle$. We want to compute the intersection with the rank 2 intersection group from step (a) of the previous lemma, i.e. with $\langle y, \partial_{y^{-1}} \rangle$. Using e.g. Lemma 4.12 we can see that the intersection of these two is in fact $\langle \partial_{y^{-1}} \rangle$. By Lemma 4.13 $\langle \partial_{y^{-1}} \rangle$ intersects E trivially, and so (1) holds as claimed.

Finally, as yw is not bad for $\rho_{\hat{y},u}^{-1}$, claim (2) holds.

(b) No need

(c) We perform $\rho_{\hat{x},w}$ before this move and $\rho_{\hat{x},w}^{-1}$ after. Note they commute with $\rho_{x,y}$, and that $\hat{x}$ is not bad for $\rho_{\hat{x},w}$. As $\hat{x}w$ is not bad for $\rho_{x,y}$, the conclusion holds.

(d) No need

(e) No need

(f) This is analogous to (a).

□

5. Proof of Theorem 1.1

Before proving the main theorem, we establish the main ingredient, that the set of arational surface type elements of $\overline{CV_n}$ (even in different copies of $\mathcal{PML}$) is path connected. Note that the last sentence of Theorem 5.1 is used with Proposition 5.3 to prove Theorem 1.1.

Theorem 5.1. — *If x_s and x_e are dual to uniquely ergodic (or, in the nonorientable case, elements of $\mathcal{P}$) surface type elements of $\overline{CV_n}$ then there exists $p : [0,1] \to \overline{CV_n}$ continuous joining x_s to x_e, so that $p(t)$ is arational for all $t \in [0,1]$. Moreover, for any $\epsilon > 0$ and combinatorial chain of $\mathcal{PML}$s from x_s to x_e we may assume that this path is contained in an ϵ neighborhood of that chain.*

Proposition 5.2. — *Let $p : [0,1] \to \overline{CV_n}$ be a BLASpath, K_E as in Definition 2.6, and $\epsilon > 0$ be given. There exists $p' : [0,1] \to \overline{CV_n}$ so that*

(1) *the distance from $p(x)$ to $p'(x)$ is at most ϵ for all $x \in [0,1]$,*
(2) *$p'([0,1]) \cap K_E = \emptyset$.*

Proof. — It suffices to prove the proposition in the case where there is exactly one point in $p([0,1])$ which is not arational, call that point σ. Recall, from the definition of BLASpaths, that in that case σ is the dual tree to a stable lamination of a partial pseudo-Anosov ψ_σ (for some identification with a surface). Also recall that in a neighbourhood of σ the path p has the form $\cup_k \psi_\sigma^k \xi_i$ for $i = 1, 2$. Let x_1 be the starting point of ξ_1 (which means $\psi_\sigma x_1$ is the ending point) and similarly for x_2 and ξ_2. Let q be the path as in Theorem 4.1 with x_1, x_2, K_i and ψ_σ. By Theorem 4.1 (2) there exists k_0 so that for all $k \geqslant k_0$, the distance from $\psi_\sigma^k(q([0,1])$ to σ is at most $\frac{\epsilon}{2}$. Let $k_1 \geqslant k_0$ so that $\psi_\sigma^{k_1}(q([0,1])) \cap K_i = \emptyset$ and the Hausdorff distance from $\psi_\sigma^r \xi_1$ and $\psi_\sigma^r \xi_2$ to σ is at most $\frac{\epsilon}{2}$. This exists by Theorem 4.1 (1). Let $p' = p$ outside of $\cup_{i=k_1}^\infty \gamma_i$ and let $p'(t) = \psi_\sigma^{k_1} q$ on $p \setminus \cup_{i=k_1}^\infty \gamma_i$. Condition (1) is clear for the x so that $p(x) = p'(x)$. All other x have that the distance from both $p(x)$ and $p'(x)$ to σ is at most $\frac{\epsilon}{2}$. Condition (2) is obvious for the points in p' that are arational. The other points are contained in $\psi_\sigma^{k_1} q$, which was constructed to avoid K_i. ☐

Proof of Theorem 5.1. — Enumerate the set of proper free factors in some way as $\mathcal{F} = \{E_i, i \in \mathbb{N}\}$, and denote by $K_i = K_{E_i}$. By Proposition 3.13 there exists a BLASpath from x_s to x_e.

Let $\epsilon' > 0$ be given. By Proposition 5.2, with $\epsilon_0 := \epsilon = \frac{\epsilon'}{4}$ we may assume $p([0,1]) \cap K_1 = \emptyset$. Since K_1 is closed and $p([0,1])$ is compact, there exists $\epsilon_1 > 0$ so that $\operatorname{dist}(p([0,1)], K_1) > \epsilon_1$. Inductively we assume that we are given a $BLAS$ path p_i and a $\epsilon_1, \dots \epsilon_i > 0$ so that

$$\operatorname{dist}(p([0,1)], K_j) > \left(1 - \sum_{\ell=j+1}^{i} 3^{j-\ell}\right) \epsilon_j > \frac{1}{2}\epsilon_j \tag{5.1}$$

for all $j \leqslant i$. By Proposition 5.2 with $\epsilon = \epsilon_{i+1} = \frac{1}{3}^{i+1} \min\{\epsilon_j\}_{j=1}^i$ and $p = p_i$ and $K = K_{i+1}$ there exists a BLASpath, p_{i+1} from x_s to x_e satisfying equation (5.1) for all $j \leqslant i+1$. Let p_∞ be the limit of the p_i. By our inductive procedure our sequence of function p_i converges. By (5.1) we have $\operatorname{dist}(p_\infty([0,1]), K_i) \geqslant \frac{1}{2}\epsilon_i > 0$ for all i. Thus by Proposition 2.7 we have a path from x_s to x_e so that every $p_\infty(t)$ is arational for all $t \in [0,1]$, establishing Theorem 5.1. ☐

To complete the proof of Theorem 1.1 we need the following result:

Proposition 5.3. — *For every neighborhood U of Δ in $\overline{CV_n}$ there is a smaller neighborhood V with the following property. If $x, y \in V$ are arational and dual to surface laminations (possibly on different surfaces) then they are joined by a chain of consecutively adjacent $\mathcal{PML}$'s, each of which is contained in U.*

The proof of this proposition requires a variant of [BR15, Theorem 4.4]. In its statement we denote by $L(T)$ the dual lamination to a tree T. Given a lamination L we denote by L' the sublamination formed by all non-isolated leaves of L.

Proposition 5.4. — *Let $T \in \partial CV_n$ be an arational tree. If μ is a current so that*

$$\langle T, \mu \rangle = 0,$$

and $U \in \partial CV_n$ is another tree with

$$\langle U, \mu \rangle = 0,$$

then either,

(1) *the dual laminations of T, U agree: $L(U) = L(T)$, or*
(2) *T is dual to a lamination on a surface S, and the support of μ is a multiple of the boundary current $\mu_{\partial S}$ of that surface.*

Proof. — By the assumption on T, μ, [KL10, Theorem 1.1] yields

$$\mathrm{Supp}(\mu) \subset L(T).$$

We begin with the case where T is not dual to a surface lamination. In this case, [BR15, Proposition 4.2 (i)] applies, and shows that $L(T)$ is obtained from the minimal lamination $L'(T)$ by adding isolated leaves, each of which is diagonal and not periodic. On the other hand, the support of a current cannot contain non-periodic isolated leaves. Thus, we then have $\mathrm{Supp}(\mu) \subset L'(T)$, hence $\mathrm{Supp}(\mu) = L'(T)$ by minimality.

Applying [KL10, Theorem 1.1] to U, μ yields

$$L'''(T) \subset L'(T) = \mathrm{Supp}(\mu) \subset L(U).$$

In this case, [BR15, Corollary 4.3] shows that $L(T) = L(U)$, and we are in case (1).

Now suppose that T is dual to a surface lamination. In this case we need to describe the dual lamination of T more precisely (see also the proof of [BR15, Proposition 4.2 (ii)]). Let S be a hyperbolic surface with one boundary component which is totally geodesic and let Λ be a minimal filling measured geodesic lamination on S, so that T is the $\mathbb{R}$-tree dual to Λ.

Consider the universal cover $\widetilde{S}$ and the preimage $\widetilde{\Lambda}$ of Λ. The complementary components of $\widetilde{\Lambda}$ are ideal polygons and regions containing the lifts of the boundary (these are universal covers of hyperbolic crowns and are bounded by a lift of ∂S and a chain of leaves with consecutive leaves cobounding a cusp) and these, along with non-boundary leaves of $\widetilde{\Lambda}$, are in 1-1 correspondence with the points of T. The lamination $L(T)$ dual to T consists of pairs of distinct ends of $\widetilde{S}$ that are joined by geodesics with 0 measure. Thus the leaves of $L(T)$ are as follows:

(i) leaves of $\widetilde{\Lambda}$,
(ii) diagonal leaves in the complementary components that are ideal polygons,
(iii) leaves in the crown regions connecting distinct cusps,
(iv) leaves in the crown regions connecting a cusp with an end corresponding to a lift of ∂S,
(v) lifts of ∂S.

Recall that $\mathrm{Supp}(\mu) \subset L(T)$. Since the leaves of type (ii) and (iii) are isolated and accumulate on leaves of type (i), the measure μ must assign zero measure to them. Thus the support of μ is contained in the sublamination of $L(T)$ consisting of leaves of type (i), (iv) and (v). In this sublamination, the leaves of type (iv) are isolated and accumulate on the leaves of both types (i) and (v), so μ is supported on the disjoint union of $\widetilde{\Lambda}$ and the lamination Δ consisting of the lifts of ∂S. Thus

$$\mu = \nu_1 + \nu_2,$$

where ν_1 is supported on $\widetilde{\Lambda}$ and ν_2 supported on Δ. If ν_2 assigns $\alpha \geqslant 0$ to a lift of ∂S then $\nu_2 = \alpha\mu_{\partial S}$.

Now, if $\nu_1 \neq 0$, then since

$$\langle U, \nu_1\rangle = 0,$$

we can apply [KL10, Theorem 1.1] to U, ν_1 to obtain

$$L'''(T) = \widetilde{\Lambda} = \mathrm{Supp}(\nu_1) \subset L(U),$$

and [BR15, Corollary 4.3] again shows that $L(T) = L(U)$, hence we are in case (1).

Otherwise, $\mu = \nu_2$ and we are in case (2). □

We also require the following:

Lemma 5.5. — *For every neighborhood U of Δ in $\overline{CV_n}$ there is a smaller neighborhood V with the following property. If $\mathcal{PML}(\Sigma)$ intersects V then it is contained in U.*

Proof. — Suppose such V does not exist. Then we have a sequence of pairwise distinct surfaces Σ_i and points $x_i, y_i \in \mathcal{PML}(\Sigma_i)$ such that $x_i \to x \in \Delta$ and $y_i \to y \notin U$. The boundary curve γ_i of Σ_i is elliptic in both x_i and y_i. After a subsequence, γ_i projectively converges to a current μ, and by the continuity of the length pairing we have

$$\langle x, \mu\rangle = \langle y, \mu\rangle = 0.$$

Now, apply Proposition 5.4 to x, y, μ. If we are in case (1) of that proposition, then y has the same dual lamination as x (equivalently, T), i.e. $y \in \Delta$. This is a contradiction.

In case (2), we instead conclude that the boundary curves γ_i of the Σ_i converge (as currents) to the boundary γ of the surface Σ supporting the dual lamination of T. Fix a basis of F_n in which γ has length $2n$ with each letter appearing exactly twice. The conjugacy classes of γ_i are all distinct, and they are distinct from the conjugacy class of γ, and convergence $\gamma_i \to \gamma$ as projective currents implies that arbitrarily high (positive or negative) powers of γ appear as subwords of γ_i (viewed as a reduced cyclic word) for large i. For each i there is an automorphism $\phi_i \in Aut(F_n)$ taking γ to a word conjugate to γ_i. If x is a basis element, after perhaps conjugating ϕ_i, we may assume that $\phi_i(x)$ is a cyclically reduced word. Therefore $\phi_i(x)$ does not contain $\gamma^{\pm 2}$ as a subword, by Whitehead's theorem [Whi36] (see the discussion before Lemma 4.13). Now let y be another basis element. This time we are allowed to conjugate ϕ_i only by powers of $C = \phi_i(x)$. Write $\phi_i(y)$ as a reduced word ABA^{-1} where B is cyclically reduced and A does not start with C or C^{-1}. As above, B does not contain $\gamma^{\pm 2}$ as a subword. Now consider $\phi_i(xy) = CABA^{-1}$. Again by Whitehead, after cyclic reduction this does not contain $\gamma^{\pm 2}$. Since C is cyclically reduced, the cyclic reduction of $CABA^{-1}$ can occur only with CA or with $A^{-1}C$ but not both, and in either case C cancels only partially. We conclude that A, and therefore $\phi_i(y)$, contains only a bounded power of γ as a subword. Now note that we didn't have to conjugate by powers of ϕ_i, since without conjugation we would have $\phi_i(y) = C^m D C^{-m}$ where D contains bounded powers of γ and so does C^m by Whitehead, so the same is true for $\phi_i(y)$. This analysis therefore holds for any other basis element and we deduce that reduced words representing images of the basis elements, and therefore of γ, have bounded powers of γ as subwords. This contradicts the assumption that projectively as currents these conjugacy classes converge to γ. □

Lemma 5.6. — *Let U be a neighborhood of Δ in $\overline{CV_n}$. There exists a neighborhood V of Δ in $\overline{CV_n}$ so that if $x', y' \in CV_n \cap V$ then any folding path from x' to y' is contained in U.*

Proof. — Recall from Section 2.2 that there is a coarsely continuous function $\Phi : \overline{CV_n} \to \overline{FF_n}$ that restricted to arational trees gives a quotient map to ∂CV_n. This map takes folding paths in CV_n to reparametrized quasigeodesics with uniform constants in FF_n [BF14] and it takes Δ to a point $[\Delta] \in \partial FF_n$. By the coarse continuity, there is a neighborhood U' of $[\Delta] \in \overline{FF_n}$ such that $\Phi^{-1}(U') \subset U$. By hyperbolic geometry there is a neighborhood $V' \subset U'$ of $[\Delta]$ such that any quasigeodesic with above constants with endpoints in V' is contained in U'. Finally, let V be a neighborhood of Δ such that $\Phi(V) \subset V'$ (V exists by the coarse continuity). □

Before we can prove Proposition 5.3, we need one more definition. Namely, given an identification σ of the free group with $\pi_1(\Sigma)$, we define the *extended projective measured lamination sphere* $\widetilde{\mathcal{PML}_\sigma}$ to be the union of $\mathcal{PML}_\sigma$ and the subset of CV_n consisting of graphs where the boundary curve of Σ crosses every edge exactly twice (alternatively, the graph can be embedded in the surface with the correct marking).

Proof of Proposition 5.3. — Let $U = U_0$ be a given neighborhood of Δ. For a large (for now unspecified) integer N find neighborhoods

$$U_0 \supset U_1 \supset U_2 \supset \cdots \supset U_N$$

of Δ so that each pair (U_i, U_{i+1}) satisfies Lemmas 5.5 and 5.6. We then set $V = U_N$. To see that this works, let $x, y \in V$ be arational and dual to surface laminations. Let P_x and P_y be the extended PML's containing x, y respectively. Thus $P_x, P_y \subset U_{N-1}$. Choose roses $x' \in P_x \cap CV_n$ and $y' \in P_y \cap CV_n$. After adjusting the lengths of edges of x' there will be a folding path from x' to y' which is then contained in U_{N-2}. We can assume that the folding process folds one edge at a time. We can choose a finite sequence of graphs along the path, starting with x' and ending with y', so that the change in topology in consecutive graphs is a simple fold. It follows that the extended PML's can be chosen so that the surfaces share a subsurface of small cogenus. Further, in each graph we can collapse a maximal tree so we get a rose. Consecutive roses will differ by the composition of boundedly many Whitehead automorphisms and each Whitehead automorphism is a composition of boundedly many basic moves. We can then insert a bounded chain of extended PML's between any two in our sequence so that in this expanded chain any two consecutive PML's differ by a basic move. If N is sufficiently large this new chain will be contained in $U = U_0$. □

Proof of Theorem 1.1. — To prove that ∂FF_n is path-connected, it suffices to join by a path points $\Phi(T), \Phi(S) \in \partial FF_n$ where both $T, S \in \partial CV_n$ are arational trees and S is dual to a surface lamination. Let $\Delta \subset \partial CV_n$ be the simplex of arational trees equivalent to T, and let $U_1 \supset U_2 \supset \cdots$ be a nested sequence of smaller and smaller neighborhoods of Δ so that each pair (U_i, U_{i+1}) satisfies Proposition 5.3. Choose $S_i \in U_i$ to be arational of surface type, for $i \geqslant 1$ (see Lemma 2.13). By Theorem 5.1 there is a path p from S to S_1 in ∂CV_n consisting of arational trees, and likewise there is such a path p_i from S_i to S_{i+1}. By our choice of the U_i and the last sentence in Theorem 5.1 we can arrange that each p_i is contained in U_{i-1} for $i \geqslant 2$. The concatenation $q = p * p_1 * p_2 * \cdots$ is a path parametrized by a half-open interval that accumulates on Δ since it is eventually contained in U_i for every i. It may not converge in ∂CV_n unless Δ is a point (that is, T is uniquely ergometric, see [CHL07]) but $\Phi(q)$ converges to $\Phi(T)$, proving path connectivity.

Local path connectivity is similar. If $S \in U_i$ the whole path may be taken to be in U_{i-1} (recall that a space X is locally path connected at $x \in X$ if for every neighborhood U of x there is a smaller neighborhood V of x so that any two points in V are connected by a path in U; this implies the ostensibly stronger property that x has a path connected neighborhood contained in U, namely take the path component of x in U). To finish, we need to argue that for every i there is $j > i$ so that if $T' \in U_j$ is arational, there is a path of arationals defined on an open interval accumulating to the associated simplices Δ, Δ' on the two ends. Thus if T' is of surface type we can take $j = i + 1$. In general there is $j > i$ so that when $T' \in U_j$ is arational then its simplex Δ' is contained in U_{i+1} because $\Phi(U_{i+1})$ contain neighbourhoods of $\Phi(T)$. Choose a surface type arational tree $S \in U_{i+1}$ close to Δ'. By the surface case we have paths from S to Δ' and from S to Δ, both in U_i. Putting them together gives the desired path. ☐

6. One-endedness of other combinatorial complexes

In this section, we discuss one-endedness of various combinatorial complexes. To this end, we use the following criterion.

Proposition 6.1. — *Let X, Y be δ-hyperbolic spaces, G a group acting coboundedly by isometries on X and Y, and let $\pi : X \to Y$ be an equivariant Lipschitz map which is alignment preserving. Suppose there is some $g \in G$ which is loxodromic in Y (and therefore also in X).*

If Y is 1-ended, so is X.

Recall [Gui00b, KR14] that a map π is *alignment preserving* if there is a constant $C \geqslant 0$ such that the image of any geodesic segment is contained in the C-neighborhood of any geodesic joining the images of the endpoints.

Remark 6.2. — We want to remark that [KR14] use only the apparently weaker property that $\pi([x, y])$ is bounded whenever $\pi(x), \pi(y)$ are close, rather than alignment preserving. However, they also show that a map between hyperbolic metric spaces with this weaker property is alignment preserving in the stronger sense.

Proof of Proposition 6.1. — Let K_X be a metric ball in X. We define L_X to be the Hausdorff N-neighborhood of K_X, with N sufficiently large. Let $x_1, x_2 \in X \setminus L_X$. We will connect x_1, x_2 by a path in the complement of K_X.

Fix an axis ℓ in X of g (i.e. a quasi-geodesic line where g acts by translation). Since the action of G on X is cobounded, there is a translate ℓ_1 of ℓ that passes within a bounded distance from x_1. Let r be the ray starting at x_1, having a bounded initial segment joining x_1 with ℓ_1, and the rest is one of the two half-lines in ℓ_1. Since K_X is quasi-convex, there is a choice of a half-line so that if N is sufficiently large, r is disjoint from K_X. The image of r in Y follows an axis of a conjugate of g, so it goes to infinity in Y. We can thus join x_1 by a path missing K_X to a point x_1' whose image in Y misses a bounded set P such that points in the complement of P can be joined by paths missing the N-neighborhood of $\pi(K_X)$. In the same way we can join x_2 to a point x_2'. It now remains to join x_1' to x_2'.

Join $\pi(x_1'), \pi(x_2')$ by a path missing the N-neighborhood of $\pi(K_X)$. We will now coarsely lift this path to the desired path. Let $\pi(x_1) = y_1, y_2, \cdots, y_s = \pi(x_2)$ be points along the path at distance $\leqslant 1$. For each y_i choose a point $\widetilde{y}_i \in X$ whose image in Y is at a bounded distance from y_i (this is possible since π is coarsely onto), and so that $\widetilde{y}_1 = x_1$ and $\widetilde{y}_s = x_2$. The desired path is the concatenation of geodesic segments joining the consecutive $\widetilde{y}_i$. Since π is alignment preserving, the images in Y of these geodesic segments are uniformly bounded, so when N is large they will miss $\pi(K_X)$, and the path between x_1 and x_2 will miss K_X. □

Corollary 6.3. — *For $n \geqslant 18$ the free splitting complex FS_n, the cyclic splitting complex FZ_n, and the maximal cyclic splitting complex FZ_n^{max} are all 1-ended.*

Proof. — There are natural coarse maps

$$CV_n \to FS_n \to FZ_n^{max} \to FZ_n \to FF_n$$

and they are equivariant with respect to the action of $Out(F_n)$. Except on CV_n, all these spaces are hyperbolic and the $Out(F_n)$ action is cobounded. Proofs of hyperbolicity show that images of folding paths in CV_n are reparametrized quasi-geodesics with uniform constants. This implies that all the maps starting from FS_n are alignment preserving. Fully irreducible automorphisms are loxodromic in all four complexes. □

Appendix A. Explicit Constructions of Curves and Subsurfaces

In this section we collect the constructions of curves and subsurfaces claimed in Lemma 3.9.

We begin with the construction of curves in Lemma 3.9. In Figure A.1, the curves for the right multiplication moves are shown in green; the curves for the left multiplication moves are shown in purple. Dashed lines entering a group of bands are understood to follow around the boundary of the surface, not intersecting the basis loops corresponding to the loops.

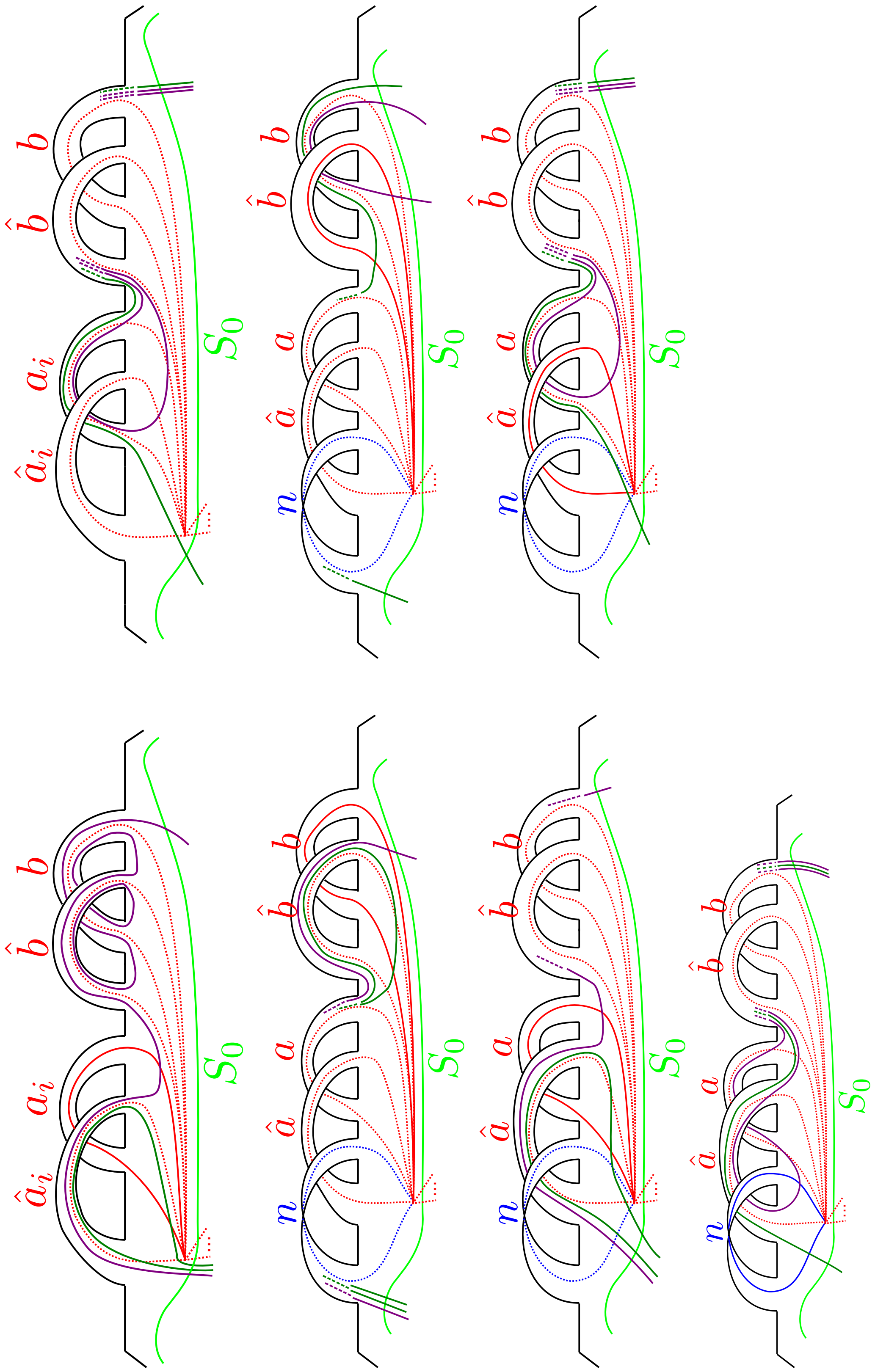

Figure A.1. Constructing "extra twists"

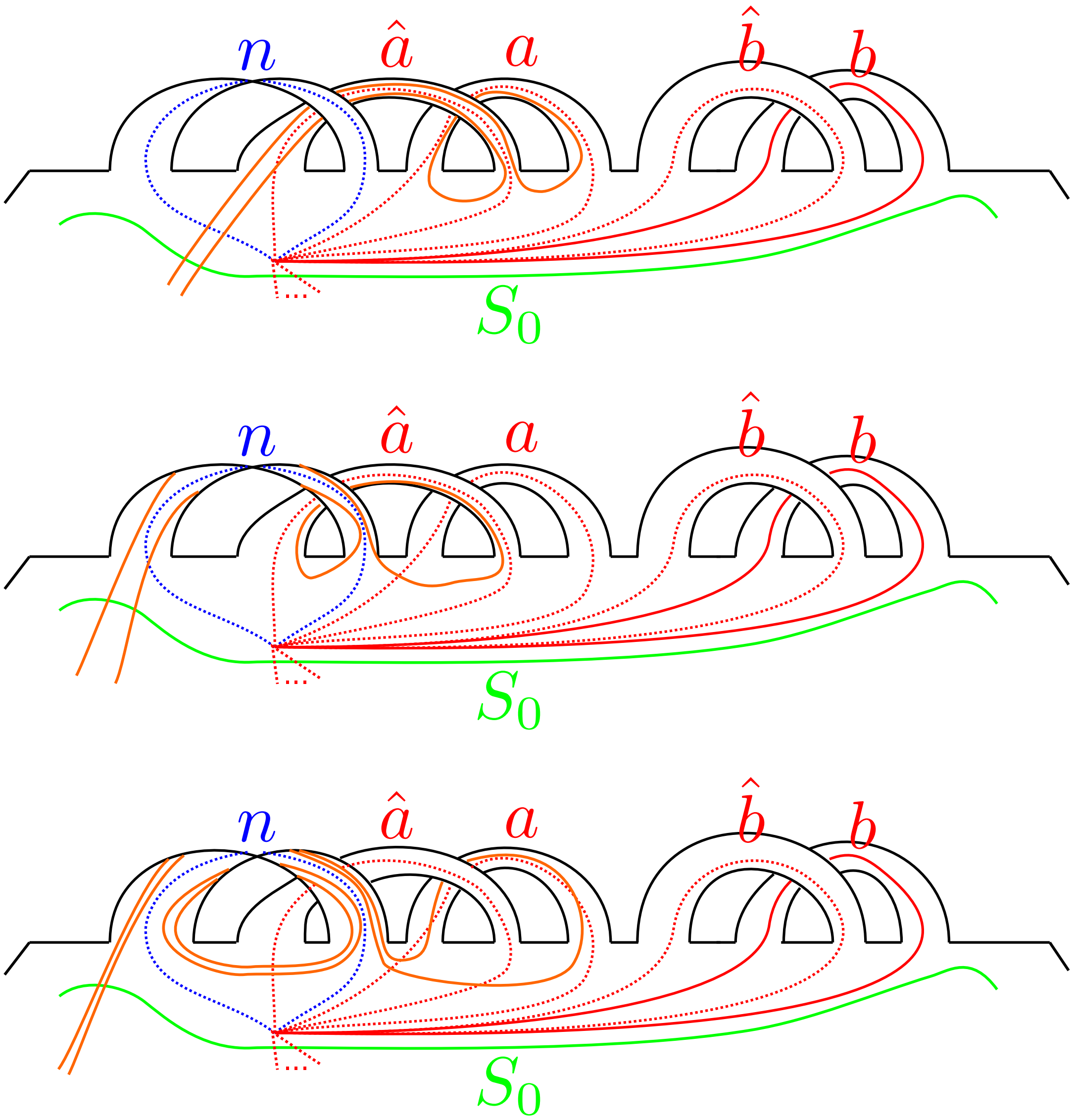

Figure A.2. Constructing even more "extra twists"

Finally, in Figure A.2, the additional curves for the last claim of the lemma are shown.

From these explicit descriptions, fundamental groups of the bad subsurfaces can be read off. We collect the results in the following lemma.

Lemma A.1. — *For a move $\phi = \lambda_{x,y}$ or $\rho_{x,y}$, the bad subsurfaces have the following fundamental groups. We denote by ∂ the word representing the boundary of the surface, i.e.*

$$\partial = \prod_{i=1}^{g} \left[\hat{a}_i, a_i^{-1}\right],$$

if the surface is orientable, and

$$\partial = \left(n\hat{a}a\hat{a}^{-1}na\right)\prod_{i=2}^{g}\left[\hat{a}_i, a_i^{-1}\right],$$

otherwise (here, $\hat{a}, a$ are linked with the nonorientable letter n, and $\hat{a}_2, \dots$ are the following letters). We denote by ∂_w the cyclic permutation of ∂ starting with the letter w.

x **two-sided, not linked with one-sided:** *Here, three possibilities for y exist.*

y **two-sided, not linked with one-sided:** *For $x = a_i$ (not a hatted letter), and the right multiplication move $\rho_{x,y}$ we have*

$$\pi_1(\Sigma - S^g) = \left\langle y, \hat{y}, x^{-1}\hat{x}x, \partial_{\hat{a}_{i+1}}\right\rangle$$

For $x = a_i$ (not a hatted letter), and the left multiplication move $\lambda_{x,y}$ we have

$$\pi_1(\Sigma - S^g) = \langle y, \hat{y}, \hat{x}, \partial_{a_i}\rangle$$

For $x = \hat{a}_i$ (a hatted letter), and the right multiplication move $\rho_{x,y}$ we have

$$\pi_1(\Sigma - S^g) = \left\langle y, \hat{y}, \hat{x}, \partial_{a_i^{-1}}\right\rangle$$

For $x = \hat{a}_i$ (a hatted letter), and the left multiplication move $\lambda_{x,y}$ we have

$$\pi_1(\Sigma - S^g) = \left\langle y, \hat{y}, x\hat{x}x^{-1}, \partial_{\hat{a}_i}\right\rangle$$

y **one-sided or linked with one-sided:** *Here, we call the one-sided letter n, the linked two-sided letters $a, \hat{a}$ (i.e. y is one of these three), and we assume that x is one of the adjacent $b, \hat{b}$ (this is enough due to the previous normalisation). In this case, the fundamental group of the bad subsurface $\Sigma - S^g$ has rank three, with two generators g_1, g_2 depending solely on y, and the final one on x and the type of move. Namely, put*

$y = a$**:**

$$g_1 = a, \quad g_2 = n\hat{a}a\hat{a}^{-1}$$

$y = \hat{a}$**:**

$$g_1 = \hat{a}, \quad g_2 = na$$

$y = n$**:**

$$g_1 = n, \quad g_2 = \left(a^{-1}n^{-1}\hat{a}\right)a\left(a^{-1}n^{-1}\hat{a}\right)^{-1}$$

For $x = b$, and the right multiplication move $\rho_{x,y}$ we have

$$\pi_1(\Sigma - S^g) = \left\langle g_1, g_2, b^{-1}\hat{b}b, \partial_{b^{-1}}\right\rangle$$

For $x = b$, and the left multiplication move $\lambda_{x,y}$ we have

$$\pi_1(\Sigma - S^g) = \left\langle g_1, g_2, \hat{b}, \partial_{\hat{b}^{-1}}\right\rangle$$

For $x = \hat{b}$, and the right multiplication move $\rho_{x,y}$ we have

$$\pi_1(\Sigma - S^g) = \langle g_1, g_2, b, \partial_{b^{-1}}\rangle$$

For $x = \hat{b}$, and the left multiplication move $\lambda_{x,y}$ we have

$$\pi_1(\Sigma - S^g) = \left\langle g_1, g_2, \hat{b}b\hat{b}^{-1}, \partial_{\hat{b}} \right\rangle$$

x **two-sided, linked with one-sided:** *Again, we call the one-sided letter n and the linked two-sided letters $a, \hat{a}$ (of which x is one), and we assume that y is one of the adjacent $b, \hat{b}$ (this is enough due to the previous normalisation).*

For $x = a$, and the right multiplication move $\rho_{x,y}$ we have

$$\pi_1(\Sigma - S^g) = \left\langle b, \hat{b}, a^{-1}n^{-1}\hat{a}, a^{-1}\hat{a}a, \partial_{\hat{b}} \right\rangle$$

For $x = a$, and the left multiplication move $\lambda_{x,y}$ we have

$$\pi_1(\Sigma - S^g) = \left\langle \hat{a}, b, \hat{b}, a\hat{a}^{-1}n, \partial_a \right\rangle$$

For $x = \hat{a}$, and the right multiplication move $\rho_{x,y}$ we have

$$\pi_1(\Sigma - S^g) = \left\langle b, \hat{b}, a, \hat{a}^{-1}n\hat{a}, \partial_a \right\rangle$$

For $x = \hat{a}$, and the left multiplication move $\lambda_{x,y}$ we have

$$\pi_1(\Sigma - S^g) = \left\langle b, \hat{b}, n, \hat{a}a\hat{a}^{-1}, \partial_n \right\rangle$$

x **one-sided:** *Again, we call the one-sided letter $n = x$, the linked two-sided letters $a, \hat{a}$, and we assume that y is one of the adjacent $b, \hat{b}$ (this is enough due to the previous normalisation).*

For $x = n$, and the right multiplication move $\rho_{x,y}$ we have

$$\pi_1(\Sigma - S^g) = \left\langle b, \hat{b}, \hat{a}, a\hat{a}^{-1}n, \partial_{\hat{a}} \right\rangle$$

For $x = n$, and the left multiplication move $\lambda_{x,y}$ we have

$$\pi_1(\Sigma - S^g) = \left\langle b, \hat{b}, a\hat{a}^{-1}, n\hat{a}, \partial_n \right\rangle$$

In all cases, any loop corresponding to a basis element except for $x, \hat{x}, y, \hat{y}$ (and possibly n, if one of x, y is linked) can be homotoped into the improved good subsurface.

Appendix B. Proof of Lemma 4.6

Throughout, we call the argument in the proof of Proposition 4.4.

B.1. Case 1

First, we observe that the argument of Case 1 extends to the case nonorientable surface. The only difference in this case is that the boundary word ∂ has a slightly different form (see Section 3). However, as we may assume that all the auxiliary letters used above are two-sided, and the boundary word of the nonorientable surface also contains commutators of all the two-sided letters which are not linked with the one-sided letter, the argument works completely analogously.

It remains to discuss the remaining cases in the case of a non-orientable surface, where either x or y is one-sided or linked with the one-sided. Unjustified claims about intersections between subgroups are proved using the arguments in the proof of Proposition 4.4. We make use of the following notation and assumptions throughout:

(1) We denote by ∂_x the cyclic permutation of the boundary word ∂ starting at (the first occurrence of) the (signed) letter x.
(2) All "auxiliary letters" are chosen to be separated by at least one index from all active letters and from each other (so that the subword detection arguments from Case 1 apply).
(3) If x is a chosen, two-sided letter (i.e. $x = a_i$ or $\hat{a}_i$), then we denote by x_+ the *next* letter of the same type (i.e. $x_+ = a_{i+1}$ or $x_+ = \hat{a}_{i+1}$) and $\hat{x}_+$ the next letter of opposite type (i.e. $\hat{x}_+ = \hat{a}_{i+1}$ or $\hat{x}_+ = a_{i+1}$).

B.2. Case 2

This case concerns y general x two-sided, linked to one-sided. Let n be the one-sided letter. The fundamental group of the bad surface is

$$B = \left\langle y, \hat{y}, x^{-1}n^{-1}\hat{x}, x^{-1}\hat{x}x, \partial_{\hat{x}_+} \right\rangle .$$

The relation we will use is:

$$\rho_{x,y} = \rho_{\hat{y},u}^{-1}\rho_{y,z}^{-1}\rho_{n,w}^{-1}\rho_{x,y}\rho_{x,z}\rho_{n,w}\rho_{y,z}\rho_{\hat{y},u}.$$

Check:

(a) $\rho_{\hat{y},u}^{-1}$ has bad subgroup $\langle u, \hat{u}, y, \partial_{y^{-1}}\rangle$ which intersects B in $\langle y, \partial_{y^{-1}}\rangle$.
(b) $\rho_{y,z}^{-1}$ has bad subgroup $\langle z, \hat{z}, y^{-1}\hat{y}y, \partial_{\hat{y}_+}\rangle$ which intersects

$$\rho_{\hat{y},u}B = \left\langle y, \hat{y}u, x^{-1}n^{-1}\hat{x}, x^{-1}\hat{x}x, \rho_{\hat{y},u}\partial_{\hat{x}_+} \right\rangle$$

trivially. Indeed, by Lemma 4.11 applied to the path $\hat{y}y\hat{y}^{-1}y^{-1}$ we may drop $\partial_{\hat{y}_+}$ from $\langle z, \hat{z}, y^{-1}\hat{y}y, \partial_{\hat{y}_+}\rangle$. Having done this, applying Lemma 4.11 (to a number of paths) we may drop $\rho_{\hat{y},u}\partial_{\hat{x}_+}$ from $\rho_{\hat{y},u}B$. Having done this we may apply Lemma 4.11 to $\hat{y}u$ and $y^{-1}\hat{y}y$ we may drop $\hat{y}u$ from $\rho_{\hat{y},u}\langle y, \hat{y}, x^{-1}n^{-1}\hat{x}, x^{-1}\hat{x}x\rangle$ and $y^{-1}\hat{y}y$ from $\langle u, \hat{u}, y^{-1}\hat{y}y\rangle$. The rest of this case is straightforward.
(c) $\lambda_{n,w}^{-1}$ has bad subgroup $\langle n\hat{x}, x^{-1}\hat{x}^{-1}, w, \hat{w}^{-1}, \partial_n\rangle$ which intersects $\rho_{y,z}\rho_{\hat{y},u}B = \langle yz, \hat{y}u, x^{-1}n^{-1}\hat{x}, x^{-1}\hat{x}x, \rho_{y,z}\rho_{\hat{y},u}\partial_{\hat{x}_+}\rangle$ in $\langle \hat{x}\rangle$ up to conjugation.

Indeed, as in the previous step, by Lemma 4.11 applied to the path $\hat{y}y\hat{y}^{-1}y^{-1}$ we may drop ∂_n from $\langle n\hat{x}, x^{-1}\hat{x}^{-1}, w, \hat{w}^{-1}, \partial_n\rangle$. Having done this we may drop $\rho_{y,z}\rho_{\hat{y},u}\partial_{\hat{x}_+}$ and then yz and $\hat{y}u$ from $\rho_{y,z}\rho_{\hat{y},u}B$. So it suffices to consider the intersection of $\langle n\hat{x}, x^{-1}\hat{x}^{-1}\rangle$ and $\langle x^{-1}n^{-1}\hat{x}, x^{-1}\hat{x}x\rangle$. Since these are free factors we consider the abelianization of these which are isomorphic to $\mathbb{Z}^3$ where the vector (a, b, c) represents $n^a\hat{x}^bx^c$. The claim follows from the fact that the subspace spanned by $\{(1, 1, 0), (0, -1, -1)\}$ intersects the subspace spanned by $\{(-1, 1, -1), (0, 1, 0)\}$ trivially.

(d) $\rho_{x,y}$ has bad subgroup B which intersects

$$\lambda_{n,w}\rho_{y,z}\rho_{\hat{y},u}B = \left\langle yz, \hat{y}u, n^{-1}w^{-1}\hat{x}, x^{-1}\hat{x}x, \lambda_{n,w}\rho_{y,z}\rho_{\hat{y},u}\partial_{\hat{x}_+}\right\rangle$$

in $\langle x^{-1}\hat{x}x\rangle$.

(e) $\rho_{x,z}$ has bad subgroup $\langle z, \hat{z}, xn^{-1}\hat{x}, x^{-1}\hat{x}x, \partial_{\hat{x}_+}\rangle$ which intersects

$$\rho_{x,y}^{-1}\lambda_{n,w}\rho_{y,z}\rho_{\hat{y},u}B = \left\langle yz, \hat{y}u, yx^{-1}n^{-1}w^{-1}\hat{x}, yx^{-1}\hat{x}xy^{-1}, \rho_{x,y}^{-1}\lambda_{n,w}\rho_{y,z}\rho_{\hat{y},u}\partial_{\hat{x}_+}\right\rangle$$

in $\langle \hat{x}\rangle$ up to conjugation.

(f) $\lambda_{n,w}$ has bad subgroup $\langle n\hat{x}, x\hat{x}^{-1}, w, \hat{w}^{-1}, \partial_x\rangle$ which intersects

$$\begin{aligned}\rho_{x,z}^{-1}\rho_{x,y}^{-1}\lambda_{n,w}\rho_{y,z}\rho_{\hat{y},u}B &\\ &= \left\langle yz, \hat{y}u, yzx^{-1}n^{-1}w^{-1}\hat{x}, yzx^{-1}\hat{x}xz^{-1}y^{-1}, \rho_{x,z}^{-1}\rho_{x,y}^{-1}\lambda_{n,w}\rho_{y,z}\rho_{\hat{y},u}\partial_{\hat{x}_+}\right\rangle \\ &= \left\langle yz, \hat{y}u, x^{-1}w^{-1}n^{-1}\hat{x}, x^{-1}\hat{x}x, \rho_{x,z}^{-1}\rho_{x,y}^{-1}\rho_{n,w}\rho_{y,z}\rho_{\hat{y},u}\partial_{\hat{x}_+}\right\rangle\end{aligned}$$

trivially. Similarly to in previous cases we apply Lemma 4.11 to first drop ∂_x and then $\rho_{x,y}^{-1}\rho_{n,w}\rho_{y,z}\rho_{\hat{y},u}\partial_{\hat{x}_+}$ from their respective subgroups. It is now clear that we can restrict our consideration to possible intersections of $\langle n\hat{x}, x\hat{x}^{-1}, w\rangle$ and $\langle x\hat{x}^{-1}nw, \hat{x}\rangle$. Any nontrivial reduced word in the latter (except $\hat{x}$) contains the subword nw (or its inverse) without cancellation. As neither $\hat{x}$ nor any word containing the subword nw is contained in the former, the claim follows.

(g) $\rho_{y,z}$ has bad subgroup $\langle z, \hat{z}, y^{-1}\hat{y}y, \partial_{\hat{y}_+}\rangle$ which intersects

$$\begin{aligned}\lambda_{n,w}^{-1}\rho_{x,z}^{-1}\rho_{x,y}^{-1}\lambda_{n,w}\rho_{y,z}\rho_{\hat{y},u}B &\\ &= \left\langle yz, \hat{y}u, yzx^{-1}w^{-1}w^{-1}n^{-1}\hat{x}, yzx^{-1}\hat{x}xz^{-1}y^{-1}, \rho_{x,z}^{-1}\rho_{x,y}^{-1}\rho_{y,z}\rho_{\hat{y},u}\partial_{\hat{x}_+}\right\rangle\end{aligned}$$

trivially.

(h) $\rho_{\hat{y},u}$ has bad subgroup $\langle u, \hat{u}, y, \partial_{y^{-1}}\rangle$ which intersects

$$\rho_{y,z}^{-1}\lambda_{n,w}^{-1}\rho_{x,z}^{-1}\rho_{x,y}^{-1}\lambda_{n,w}\rho_{y,z}\rho_{\hat{y},u}B = \left\langle y, \hat{y}u, yx^{-1}w^{-1}w^{-1}n^{-1}\hat{x}, yx^{-1}\hat{x}xy^{-1}, \rho_{x,y}^{-1}\rho_{\hat{y},u}\partial_{\hat{x}_+}\right\rangle$$

in $\langle y\rangle$.

This completes the checks for the preliminary relation.

We now collect some variants on this case. First is *Case 2'* of the left multiplication move $\lambda_{x,y}$. Here, the bad subgroup is $B = \langle \hat{x}, y, \hat{y}, x\hat{x}^{-1}n, \partial_x\rangle$. We use the relation

$$\lambda_{x,y} = \rho_{\hat{y},u}^{-1}\rho_{y,z}^{-1}\lambda_{n,w}^{-1}\lambda_{x,y}\lambda_{x,z}\lambda_{n,w}\rho_{y,z}\rho_{\hat{y},u}.$$

We only indicate how the checks above need to be amended in this case.

(a), (b), (c) are similar to case 2.

(d) $\lambda_{x,z}$ which has bad subgroup $\langle \hat{x}, z, \hat{z}, x\hat{x}^{-1}n, \partial_x\rangle$ which intersects $\rho_{n,w}\rho_{y,z}\rho_{\hat{y},u}B$ in $\langle \hat{x}\rangle$.

(e) $\lambda_{x,y}$ which has bad subgroup B which intersects

$$\lambda_{x,z}\rho_{n,w}\rho_{y,z}\rho_{\hat{y},u}B = \left\langle \hat{x}, yz, \hat{y}u, z^{-1}x\hat{x}^{-1}x^{-1}x, \lambda_{x,z}\rho_{n,w}\rho_{y,z}\rho_{\hat{y},u}\partial_x\right\rangle$$

in $\langle \hat{x}\rangle$.

(f), (g) and (h) are similar to case 2.

Finally, *Case 2"* and *Case 2"'*: with x hatted for both ρ and λ are similar.

B.3. Case 3

x general unhatted, y unhatted two-sided and linked to one-sided. Let n be the one-sided letter and u, z, w, v be general.

$$B = \left\langle y, n\hat{y}y\hat{y}^{-1}, x^{-1}\hat{x}x, \partial_{x^{-1}} \right\rangle$$
$$\rho_{x,y} = \rho_{\hat{x},z}^{-1}\rho_{n,u}^{-1}\rho_{y,v}^{-1}\rho_{x,y}\rho_{y,v}\rho_{x,v}\rho_{n,u}\rho_{\hat{x},z}$$

Check:

(a) $\rho_{\hat{x},z}^{-1}$ has bad subgroup $\langle z, \hat{z}, x, \partial_{x^{-1}}\rangle$ which intersects B at most in $\langle x, \partial_{x^{-1}}\rangle$. In fact, by considering immersed graphs representing the subgroups, one can show that the intersection is $\langle \partial_{x^{-1}}\rangle$, but we do not need this fact.

(b) $\lambda_{n,u}^{-1}$ has bad subgroup $\langle n\hat{y}, y\hat{y}^{-1}, u, \hat{u}, \partial_n\rangle$ which intersects $\rho_{\hat{x},z}B$ in $\langle n\hat{y}y\hat{y}^{-1}\rangle$. Indeed, by applying Lemma 4.11 as above we may drop ∂_n, $\rho_{\hat{x},z}\partial_{x^{-1}}$, and $u, \hat{u}, x^{-1}\hat{x}zx$ in sequence. So it suffices to consider the intersection of $\langle y, n\hat{y}y\hat{y}^{-1}\rangle$ and $\langle n\hat{y}, y\hat{y}^{-1}\rangle$. As both of these are free factors, the intersection is again a free factor. In particular, either the two factors are equal, or the intersection is of rank at most 1. Since neither is contained in the other (e.g. by considering Abelianisations), the intersection is at most cyclic. As $\langle n\hat{y}y\hat{y}^{-1}\rangle$ is contained in both, the claim follows.

(c) $\rho_{y,v}^{-1}$ has bad subgroup $\langle v, \hat{v}, y^{-1}n^{-1}\hat{y}, y^{-1}\hat{y}y, \partial_{\hat{y}_+}\rangle$ which intersects $\lambda_{n,u}\rho_{\hat{x},z}B = \langle y, un\hat{y}y\hat{y}^{-1}, x^{-1}\hat{x}zx, \lambda_{n,u}\rho_{\hat{x},z}\partial_{x^{-1}}\rangle$ trivially.

(d) $\rho_{x,y}$ has bad subgroup B which intersects

$$\rho_{y,v}\lambda_{n,u}\rho_{\hat{x},z}B = \left\langle yv, un\hat{y}yv\hat{y}^{-1}, x^{-1}\hat{x}zx, \rho_{y,v}\lambda_{n,u}\rho_{\hat{x},z}\partial_{x^{-1}} \right\rangle$$

trivially.

(e) $\rho_{y,v}$ has bad subgroup $\langle v, \hat{v}, y^{-1}n^{-1}\hat{y}, y^{-1}\hat{y}y, \partial_{\hat{y}_+}\rangle$ which intersects

$$\rho_{x,y}^{-1}\rho_{y,v}\lambda_{n,u}\rho_{\hat{x},z}B = \left\langle yv, un\hat{y}yv\hat{y}^{-1}, yx^{-1}\hat{x}zxy^{-1}, \rho_{x,y}^{-1}\rho_{y,v}\lambda_{n,u}\rho_{\hat{x},z}\partial_{x^{-1}} \right\rangle$$

trivially.

(f) $\rho_{x,v}$ has bad subgroup $\langle v, \hat{v}, x^{-1}\hat{x}x, \partial_{\hat{x}_+}\rangle$ which intersects

$$\rho_{y,v}^{-1}\rho_{x,y}^{-1}\rho_{y,v}\lambda_{n,u}\rho_{\hat{x},z}B = \left\langle y, un\hat{y}y\hat{y}^{-1}, yv^{-1}x^{-1}\hat{x}zxvy^{-1}, \rho_{y,v}^{-1}\rho_{x,y}^{-1}\rho_{y,v}\lambda_{n,u}\rho_{\hat{x},z}\partial_{x^{-1}} \right\rangle$$

in the conjugacy class $\langle \hat{x}\rangle$.

(g) $\lambda_{n,u}$ has bad subgroup $\langle n\hat{y}, y\hat{y}^{-1}, u, \hat{u}, \partial_n\rangle$ which intersects

$$\rho_{x,v}^{-1}\rho_{y,v}^{-1}\rho_{x,y}^{-1}\rho_{y,v}\lambda_{n,u}\rho_{\hat{x},z}B = \left\langle y, un\hat{y}y\hat{y}^{-1}, yx^{-1}\hat{x}zxy^{-1}, \rho_{x,y}^{-1}\lambda_{n,u}\rho_{\hat{x},z}\partial_{x^{-1}} \right\rangle$$

trivially.

(h) $\rho_{\hat{x},z}$ has bad subgroup $\langle z, \hat{z}, x, \partial_{x^{-1}}\rangle$ which intersects

$$\lambda_{n,u}^{-1}\rho_{x,v}^{-1}\rho_{y,v}^{-1}\rho_{x,y}^{-1}\rho_{y,v}\lambda_{n,u}\rho_{\hat{x},z}B = \left\langle y, n\hat{y}y\hat{y}^{-1}, yx^{-1}\hat{x}zxy^{-1}, \rho_{x,y}^{-1}\rho_{\hat{x},z}\partial_{x^{-1}} \right\rangle$$

at most in $\langle x, \partial_{x^{-1}}\rangle$. In fact, by considering immersed graphs representing the subgroups, one can show that the intersection is trivial, but we do not need this fact.

This completes the checks for the preliminary relation. We now collect some variants on this case. The case of $\lambda_{x,y}$ is analogous using $\lambda_{x,y} = \rho_{\hat{x},z}^{-1}\rho_{n,u}^{-1}\lambda_{x,y}\rho_{n,u}\rho_{\hat{x},z}$. Indeed the bad subgroup is the same except $x^{-1}\hat{x}x$ is replaced by x, and $\partial_{x^{-1}}$ by $\partial_{\hat{x}^{-1}}$.
Case 3' is the case of x general, y hatted two-sided and linked to one-sided. The bad subgroup now is $B = \langle \hat{y}, ny, x^{-1}\hat{x}x, \partial_{x^{-1}}\rangle$.

We use the relation $\rho_{x,y} = \rho_{\hat{x},z}^{-1}\rho_{n,u}^{-1}\rho_{x,y}\rho_{n,u}\rho_{\hat{x},z}$ and the steps are the same except the overlap of the bad factor for $\rho_{x,y}$ and $\rho_{n,u}\rho_{\hat{x},z}B$ is $\langle ny\rangle$.
Case 3" is x general hatted, y unhatted two sided and linked to one-sided. The bad subgroup is $B = \langle y, n\hat{y}y\hat{y}^{-1}, \hat{x}, \partial_{\hat{x}^{-1}}\rangle$. This is similar.
Finally, *Case 3"'* is x general hatted, y hatted two-sided and linked to one-sided. The bad subgroup is $B = \langle \hat{a}, na, b, \partial_{\hat{x}^{-1}}\rangle$. Again, this is similar.

B.4. Case 4

x general unhatted, y one-sided, $\rho_{x,y}$ and let $a, \hat{a}$ denote the letters that are linked with y.

The bad subgroup is $B = \langle y, (a^{-1}y^{-1}\hat{a})a(a^{-1}y^{-1}\hat{a})^{-1}, x^{-1}\hat{x}x, \partial_{x^{-1}}\rangle$, and we use the relation

$$\rho_{x,y} = \rho_{y,u}^{-1}\rho_{\hat{x},z}^{-1}\rho_{x,y}\rho_{x,u}\rho_{\hat{x},z}\rho_{y,u}$$

Check:

(a) $\rho_{y,u}^{-1}$ has bad subgroup $\langle \hat{a}, a\hat{a}^{-1}y, u, \hat{u}, \partial_{\hat{a}}\rangle$ which intersects B in $\langle \partial_{\hat{a}}\rangle$.
Indeed, as in previous cases by Lemma 4.12 we may drop u, $\hat{u}$ and $x^{-1}\hat{x}x$. We now consider $\langle y, (a^{-1}y^{-1}\hat{a})a(a^{-1}y^{-1}\hat{a})^{-1}, \partial_{x^{-1}}\rangle$ and $\langle \hat{a}, a\hat{a}^{-1}y, \partial_{\hat{a}}\rangle$. By considering $y^{-1}\hat{a}aay$, a subword of $(a^{-1}y^{-1}\hat{a})a(a^{-1}y^{-1}\hat{a})^{-1}$ which can not occur in $\langle \hat{a}, a\hat{a}^{-1}y, \partial_{\hat{a}}\rangle$ we reduce to $\langle \hat{a}, a\hat{a}^{-1}y, \partial_{x^{-1}}\rangle$ and $\langle y, \partial_{\hat{a}}\rangle$. Similarly we may remove $a\hat{a}^{-1}y$ and then y and $\hat{a}$.

(b) $\rho_{\hat{x},z}^{-1}$ has bad subgroup $\langle z, \hat{z}, x, \partial_{x^{-1}}\rangle$ which intersects

$$\rho_{y,u}B = \left\langle yu, \left(a^{-1}u^{-1}y^{-1}\hat{a}\right)a\left(a^{-1}u^{-1}y^{-1}\hat{a}\right)^{-1}, x^{-1}\hat{x}x, \rho_{y,u}\partial_{x^{-1}}\right\rangle$$

trivially.

(c) $\rho_{x,y}$ has bad subgroup B which intersects

$$\rho_{\hat{x},z}\rho_{y,u}B = \left\langle yu, \left(a^{-1}u^{-1}y^{-1}\hat{a}\right)a(a^{-1}u^{-1}y^{-1}\hat{a})^{-1}, x^{-1}\hat{x}zx, \rho_{\hat{x},z}\rho_{y,u}\partial_{x^{-1}}\right\rangle$$

trivially.

(d) $\rho_{x,u}$ has bad subgroup $\langle u, \hat{u}, x^{-1}\hat{x}x, \partial_{\hat{x}_+}\rangle$ which intersects

$$\rho_{x,y}^{-1}\rho_{\hat{x},z}\rho_{y,u}B = \left\langle yu, (a^{-1}u^{-1}y^{-1}\hat{a})a\left(a^{-1}u^{-1}y^{-1}\hat{a}\right)^{-1}, yx^{-1}\hat{x}zxy^{-1}, \rho_{x,y}^{-1}\rho_{\hat{x},z}\rho_{y,u}\partial_{x^{-1}}\right\rangle$$

trivially.

(e) $\rho_{\hat{x},z}$ has bad subgroup $\langle z, \hat{z}, x, \partial_{x^{-1}}\rangle$ which intersects

$$\begin{aligned}&\rho_{x,u}^{-1}\rho_{x,y}^{-1}\rho_{\hat{x},z}\rho_{y,u}B\\ &= \left\langle yu, \left(a^{-1}u^{-1}y^{-1}\hat{a}\right)a\left(a^{-1}u^{-1}y^{-1}\hat{a}\right)^{-1}, yux^{-1}\hat{x}zxu^{-1}y^{-1}, \rho_{x,u}^{-1}\rho_{x,y}^{-1}\rho_{\hat{x},z}\rho_{y,u}\partial_{x^{-1}}\right\rangle\end{aligned}$$

trivially.

(f) $\rho_{y,u}$ has bad subgroup $\langle u, \hat{u}, a\hat{a}^{-1}y, \hat{a}, \partial_{\hat{a}}\rangle$ which intersects

$$\rho_{\hat{x},z}^{-1}\rho_{x,u}^{-1}\rho_{x,y}^{-1}\rho_{\hat{x},z}\rho_{y,u}B$$
$$= \left\langle yu, \left(a^{-1}u^{-1}y^{-1}\hat{a}\right) a \left(a^{-1}u^{-1}y^{-1}\hat{a}\right)^{-1}, yux^{-1}\hat{x}xu^{-1}y^{-1}, \rho_{\hat{x},z}^{-1}\rho_{x,u}^{-1}\rho_{x,y}^{-1}\rho_{\hat{x},z}\rho_{y,u}\partial_{x^{-1}}\right\rangle$$

trivially.

Namely, as before, we can drop the (modified) boundary words, as well as $yux^{-1}\hat{x}xu^{-1}y^{-1}, \hat{a}$. We now need to control the intersection of $\langle u, \hat{u}, a\hat{a}^{-1}y\rangle$ and $\langle yu, (a^{-1}u^{-1}y^{-1}\hat{a})a(a^{-1}u^{-1}y^{-1}\hat{a})^{-1}\rangle$. Since both are free factors, and their Abelianisations do not intersect, the claim follows.

The case of x general hatted and the relevant λ cases are similar.

B.5. Case 5

x one sided, y general. As before, we denote by $a, \hat{a}$ the linked two-sided letters. Here, we consider $\rho_{x,y}$ which has bad subgroup

$$B = \langle y, \hat{y}, \hat{a}, a\hat{a}^{-1}x, \partial_{\hat{a}}\rangle.$$

We use the relation

$$\rho_{x,y} = \rho_{\hat{y},u}^{-1}\rho_{y,z}^{-1}\lambda_{\hat{a},w}^{-1}\rho_{x,y}\rho_{x,z}\lambda_{\hat{a},w}\rho_{y,z}\rho_{\hat{y},u}.$$

(a) $\rho_{\hat{y},u}^{-1}$ has bad subgroup $\langle u, \hat{u}, y, \partial_{y^{-1}}\rangle$. This intersects B in $\langle y, \partial_{y^{-1}}\rangle$.

(b) $\rho_{y,z}^{-1}$ has bad subgroup $\langle z, \hat{z}, y^{-1}\hat{y}y, \partial_{\hat{y}_+}\rangle$ and this intersects $\rho_{\hat{y},u}B = \langle y, \hat{y}u, \hat{a}, a\hat{a}^{-1}x, \rho_{\hat{y},u}\partial_{\hat{a}}\rangle$ trivially.

(c) $\lambda_{\hat{a},w}^{-1}$ has bad subgroup $\langle \hat{a}a\hat{a}^{-1}, x, w, \hat{w}, \partial_n\rangle$ and this intersects

$$\rho_{y,z}\rho_{\hat{y},u}B = \left\langle yz, \hat{y}u, \hat{a}, a\hat{a}^{-1}x, \rho_{y,z}\rho_{\hat{y},u}\partial_{\hat{a}}\right\rangle$$

in $\langle \hat{a}a\hat{a}^{-1}x\rangle$. Namely, after dropping the boundary terms as usual, we can also drop $yz, \hat{y}u, w, \hat{w}$. The resulting rank 2 free factors $\langle \hat{a}a\hat{a}^{-1}, x\rangle$ and $\langle \hat{a}, a\hat{a}^{-1}x\rangle$ have Abelianisations that intersect in a rank 1 submodule. The intersection is therefore at most a rank 1 free factor, hence it is the one claimed.

(d) $\rho_{x,y}$ has bad subgroup B which intersects

$$\lambda_{\hat{a},w}\rho_{y,z}\rho_{\hat{y},u}B = \left\langle yz, \hat{y}u, w\hat{a}, a\hat{a}^{-1}w^{-1}x, \lambda_{\hat{a},w}\rho_{y,z}\rho_{\hat{y},u}\partial_{\hat{a}}\right\rangle$$

trivially.

(e) $\rho_{x,z}$ has bad subgroup $\langle \hat{a}, a\hat{a}^{-1}x, z, \hat{z}, \partial_{\hat{a}}\rangle$ which intersects $\rho_{x,y}^{-1}\lambda_{\hat{a},w}\rho_{y,z}\rho_{\hat{y},u}B = \langle yz, \hat{y}u, w\hat{a}, a\hat{a}^{-1}w^{-1}xy^{-1}, \rho_{x,y}^{-1}\lambda_{\hat{a},w}\rho_{y,z}\rho_{\hat{y},u}\partial_{\hat{a}}\rangle$ trivially.

(f) $\lambda_{\hat{a},w}$ has bad subgroup $\langle \hat{a}a\hat{a}^{-1}, x, w, \hat{w}, \partial_n\rangle$ which we need to intersect with $\rho_{x,z}^{-1}\rho_{x,y}^{-1}\lambda_{\hat{a},w}\rho_{y,z}\rho_{\hat{y},u}B = \langle yz, \hat{y}u, w\hat{a}, a\hat{a}^{-1}w^{-1}xz^{-1}y^{-1}, \rho_{x,z}^{-1}\rho_{x,y}^{-1}\lambda_{\hat{a},w}\rho_{y,z}\rho_{\hat{y},u}\partial_{\hat{a}}\rangle$. As usual, we can discard the boundary word terms, and clean up generators to compare $\langle \hat{a}a\hat{a}^{-1}, x, w, \hat{w}\rangle$ and $\langle yz, \hat{y}u, w\hat{a}, a\hat{a}^{-1}w^{-1}x\rangle$. We can drop $yz, \hat{y}u$ from the latter, replacing it with $\langle w\hat{a}, a\hat{a}^{-1}w^{-1}x\rangle$. Since the $w\hat{a}$ is not homologous into the former factor, the intersection is at most rank 1. Thus, the intersection is $\langle w\hat{a}a\hat{a}^{-1}w^{-1}x\rangle$.

(g) $\rho_{y,z}$ has bad subgroup $\langle z, \hat{z}, y^{-1}\hat{y}y, \partial_{\hat{y}_+}\rangle$ which intersects

$$\lambda^{-1}_{\hat{a},w}\rho^{-1}_{x,z}\rho^{-1}_{x,y}\lambda_{\hat{a},w}\rho_{y,z}\rho_{\hat{y},u}B = \left\langle yz, \hat{y}u, \hat{a}, a\hat{a}^{-1}xz^{-1}y^{-1}, \lambda^{-1}_{\hat{a},w}\rho^{-1}_{x,z}\rho^{-1}_{x,y}\lambda_{\hat{a},w}\rho_{y,z}\rho_{\hat{y},u}\partial_{\hat{a}}\right\rangle$$

trivially.

(h) $\rho_{\hat{y},u}$ has bad subgroup $\langle u, \hat{u}, y, \partial_{y^{-1}}\rangle$ which intersects

$$\rho^{-1}_{y,z}\lambda^{-1}_{\hat{a},w}\rho^{-1}_{x,z}\rho^{-1}_{x,y}\lambda_{\hat{a},w}\rho_{y,z}\rho_{\hat{y},u}B = \left\langle y, \hat{y}u, \hat{a}, a\hat{a}^{-1}xy^{-1}, \rho^{-1}_{y,z}\lambda^{-1}_{\hat{a},w}\rho^{-1}_{x,z}\rho^{-1}_{x,y}\lambda_{\hat{a},w}\rho_{y,z}\rho_{\hat{y},u}\partial_{\hat{a}}\right\rangle \text{ in } \langle y\rangle.$$

B.6. Case 5'

This is the analogous left-multiplication move $\lambda_{x,y}$ with notation as in Case 5, and thus the bad subgroup is

$$B = \left\langle x\hat{a}, a\hat{a}^{-1}, y, \hat{y}, \partial_n\right\rangle$$

where $a, \hat{a}$ is linked to x. Let z, w, u be general. We use the relation

$$\lambda_{x,y} = \rho^{-1}_{\hat{y},u}\rho^{-1}_{y,z}\rho^{-1}_{\hat{a},w}\lambda_{x,z}\lambda_{x,y}\rho_{\hat{a},w}\rho_{y,z}\rho_{\hat{y},u}.$$

The checks here are similar to Case 5. Indeed, *Check:*

(a) $\rho^{-1}_{\hat{y},u}$ has bad subgroup $\langle u, \hat{u}, y, \partial_{y^{-1}}\rangle$. This intersects B in $\langle y, \partial_{y^{-1}}\rangle$.

(b) $\rho^{-1}_{y,z}$ has bad subgroup $\langle z, \hat{z}, y^{-1}\hat{y}y, \partial_{\hat{y}_+}\rangle$ and this intersects

$$\rho_{\hat{y},u}B = \left\langle x\hat{a}, a\hat{a}^{-1}, y, \hat{y}u, \rho_{\hat{y},u}\partial_n\right\rangle$$

trivially.

(c) $\rho^{-1}_{\hat{a},w}$ has bad subgroup $\langle w, \hat{w}, a, \hat{a}^{-1}x\hat{a}, \partial_a\rangle$ and this intersects

$$\rho_{y,z}\rho_{\hat{y},u}B = \left\langle x\hat{a}, a\hat{a}^{-1}, yz, \hat{y}u, \rho_{y,z}\rho_{\hat{y},u}\partial_n\right\rangle$$

in $\langle a\hat{a}^{-1}x\hat{a}\rangle$.

(d) $\lambda_{x,z}$ has bad subgroup $\langle x\hat{a}, a\hat{a}^{-1}, z, \hat{z}, \partial_n\rangle$ which intersect

$$\rho_{\hat{a},w}\rho_{y,z}\rho_{\hat{y},u}B = \left\langle x\hat{a}w, aw^{-1}\hat{a}^{-1}, yz, \hat{y}u, \rho_{\hat{a},w}\rho_{y,z}\rho_{\hat{y},u}\partial_n\right\rangle$$

trivially.

(e) $\lambda_{x,y}$ has bad subgroup B which intersects

$$\lambda^{-1}_{x,z}\rho_{\hat{a},w}\rho_{y,z}\rho_{\hat{y},u}B = \left\langle z^{-1}x\hat{a}w, aw^{-1}\hat{a}^{-1}, yz, \hat{y}u, \lambda^{-1}_{x,z}\rho_{\hat{a},w}\rho_{y,z}\rho_{\hat{y},u}\partial_n\right\rangle$$

trivially.

(f) $\rho_{\hat{a},w}$ has bad subgroup $\langle w, \hat{w}, a, \hat{a}^{-1}x\hat{a}, \partial_a\rangle$ which intersects

$$\lambda^{-1}_{x,y}\lambda^{-1}_{x,z}\rho_{\hat{a},w}\rho_{y,z}\rho_{\hat{y},u}B = \left\langle z^{-1}y^{-1}x\hat{a}w, aw^{-1}\hat{a}^{-1}, yz, \hat{y}u, \lambda^{-1}_{x,y}\lambda^{-1}_{x,z}\rho_{\hat{a},w}\rho_{y,z}\rho_{\hat{y},u}\partial_n\right\rangle$$

in $\langle aw^{-1}\hat{a}^{-1}x\hat{a}w\rangle$.

(g) $\rho_{y,z}$ has bad subgroup $\langle z, \hat{z}, y^{-1}\hat{y}y, \partial_{\hat{y}_+}\rangle$ which intersects

$$\rho^{-1}_{\hat{a},w}\lambda^{-1}_{x,y}\lambda^{-1}_{x,z}\rho_{\hat{a},w}\rho_{y,z}\rho_{\hat{y},u}B = \left\langle z^{-1}y^{-1}x\hat{a}, a\hat{a}^{-1}, yz, \hat{y}u, \rho^{-1}_{\hat{a},w}\lambda^{-1}_{x,y}\lambda^{-1}_{x,z}\rho_{\hat{a},w}\rho_{y,z}\rho_{\hat{y},u}\partial_n\right\rangle$$

trivially.

(h) $\rho_{\hat{y},u}$ has bad subgroup $\langle u, \hat{u}, y, \partial_{y^{-1}}\rangle$ which intersects

$$\rho_{y,z}^{-1}\rho_{\hat{a},w}^{-1}\lambda_{x,y}^{-1}\lambda_{x,z}^{-1}\rho_{\hat{a},w}\rho_{y,z}\rho_{\hat{y},u}B = \left\langle y^{-1}x\hat{a}, a\hat{a}^{-1}, y, \hat{y}u, \rho_{y,z}^{-1}\rho_{\hat{a},w}^{-1}\lambda_{x,y}^{-1}\lambda_{x,z}^{-1}\rho_{\hat{a},w}\rho_{y,z}\rho_{\hat{y},u}\partial_n\right\rangle \text{ in } \langle y\rangle.$$

Appendix C. Minimal foliations for nonorientable surfaces

The purpose of this appendix is to show the following result, which was stated as Theorem 2.11 above.

Theorem C.1. — *Suppose that Σ is a nonorientable surface with a single boundary component or marked point. Then, there is a path-connected subset*

$$\mathcal{P} \subset \mathcal{M}(\Sigma) \subset \mathcal{PML}(\Sigma)$$

consisting of minimal measured foliations, which is invariant under the mapping class group of Σ. In addition, if F is any finite set of laminations, the set $\mathcal{P} \setminus F$ is still path-connected.

The proof uses methods established in [LS09]. We consider throughout the case of a surface $\Sigma = (S, p)$ with a marked point; the other claim is equivalent.

We begin by observing that any foliation on S defines a foliation on Σ, and the resulting foliations of Σ are exactly those which do not have an angle-π singularity at p.

Lemma C.2. — *A foliation $\mathcal{F}$ on S is minimal (as a foliation on S) if and only if it is minimal as a foliation of Σ.*

Proof. — This follows, since any essential simple closed curve on Σ defines an essential simple closed curve on S (i.e. after forgetting the marked point). ☐

Definition C.3. — *We define $\mathcal{P} \subset \mathcal{PML}(\Sigma)$ to be the set of minimal foliations which either*

(1) *do not have an angle–π singularity at p, or*
(2) *are stable foliations of point-pushing pseudo-Anosovs.*

It is clear from construction that $\mathcal{P}$ is invariant under the mapping class group of Σ. We aim to show that any foliation in $\mathcal{P}$ of the first type can be connected by a path to any foliation of the second type, which will prove Theorem 2.11, as we have full flexibility which point-pushes to use.

To do so, we need to recall some facts about point-pushing maps; compare [CH, LS09]. Let $\gamma : [0, 1] \to S$ be an immersed smooth loop based at p. We let

$$D_\gamma : [0, 1] \to \mathrm{Diff}(S)$$

be a smooth isotopy starting in the identity, so that $D_\gamma(t)(p) = \gamma(t)$. By definition, the endpoint $D_\gamma(1)$ is then a representative of the point-pushing mapping class Ψ_γ defined by γ.

Suppose that $\mathcal{F}$ be a foliation of S which is minimal. Then, the same is true for $D_\gamma(t)\mathcal{F}$ (as they are indeed isotopic). When seen as minimal foliations of Σ, the assignment

$$t \mapsto D_\gamma(t)\mathcal{F}$$

is a continuous path of minimal foliations joining $\mathcal{F}$ to $\Psi_\gamma\mathcal{F}$: minimality follows by Lemma C.2, and continuity since D_γ is smooth and intersections with $\mathcal{F}$ vary continuously with the curve.

Now, we use the following:

Lemma C.4. — *Let Ψ be a point-pushing pseudo-Anosov of Σ. Then Ψ acts on $\mathcal{PML}(\Sigma)$ with north-south dynamics, and both fixpoints have an angle-π–singularity.*

Proof. — Let $X \to \Sigma$ be the orientation double cover. Then Ψ lifts to a pseudo-Anosov of X, and the first claim follows. The second claim follows since point-pushes have angle–π singularities at the marked point [LS09]. ☐

By the lemma, the path $\{D_\gamma(t)\mathcal{F}, t \in [0,1]\}$ is disjoint from the repelling fixed point of Ψ, and thus

$$\bigcup_{n \in \mathbb{N}} \Psi^n \{D_\gamma(t)\mathcal{F}, t \in [0,1]\}$$

is the desired path joining $\mathcal{F}$ to the stable foliation of Ψ.

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