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VORONOI CELLS IN RANDOM SPLIT TREES

CELLULES DE VORONOÏ DANS LES
ARBRES ALÉATOIRES DE FRAGMENTATION

ABSTRACT. — We study the sizes of the Voronoi cells of k uniformly chosen vertices in a random split tree of size n . We prove that, for n large, the largest of these k Voronoi cells contains most of the vertices, while the sizes of the remaining ones are essentially all of order $n \exp(-\text{const}\sqrt{\log n})$. This “winner-takes-all” phenomenon persists if we modify the definition of the Voronoi cells by (a) introducing random edge lengths (with suitable moment assumptions), and (b) assigning different “influence” parameters (called “speeds” in the paper) to each of the k vertices. Our findings are in contrast to corresponding results on random uniform trees and on the continuum random tree, where it is known that the vector of the relative sizes of the k Voronoi cells is asymptotically uniformly distributed on the $(k - 1)$ -dimensional simplex.

Two intermediary steps in the proof of our main result may be of independent interest because of the information they give on the typical shape of large random split trees: we prove convergence in probability of their “profile”, and we prove asymptotic results for the size of fringe trees (trees rooted at an ancestor of a uniform random node).

Keywords: Random split trees, Voronoi cells in graphs, competition processes, profile, fringe trees.
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RÉSUMÉ. — Dans ce papier, nous étudions la taille des cellules de Voronoï de k nœuds choisis uniformément au hasard dans un arbre aléatoire de fragmentation de taille n . Nous montrons que, pour n grand, la plus grande de ces k cellules de Voronoï contient la plupart des nœuds de l'arbre, alors que les tailles des autres cellules sont d'ordre $n \exp(-\text{const}\sqrt{\log n})$. Cet effet du “gagnant qui décroche la timbale” persiste si l'on modifie la définition des cellules de Voronoï (a) en introduisant des longueurs d'arêtes aléatoires (sous une hypothèse de moment appropriée), et (b) en attribuant différents paramètres d’“influence” (appelés “vitesses” dans le papier) aux k différents nœuds choisis. Nos résultats sont à mettre en parallèle de ceux obtenus dans le cas d'un arbre aléatoire uniforme et dans le cas de l'arbre continu Brownien, pour lesquels il est connu que le vecteur des tailles relatives des k cellules de Voronoï est asymptotiquement uniforme sur le simplexe de dimension $(k - 1)$.

Deux étapes intermédiaires de notre preuve pourront être intéressantes en elles-mêmes pour les informations qu'elles donnent sur la forme typique d'un grand arbre aléatoire de fragmentation : la convergence en probabilité du profil de ces arbres, et un résultat asymptotique sur la taille de leurs arbres de frange (arbres enracinés en un ancêtre d'un nœud uniforme).

1. Introduction

Voronoi cells. Consider an n -node graph \mathcal{G}_n , from which we choose k vertices uniformly at random, and denote them by U_1, \dots, U_k . The *Voronoi cell* $\text{Vor}(U_j)$ of U_j consists of those vertices that are closer in graph distance to U_j than to any of the other chosen vertices $\{U_i : i = 1, \dots, k; i \neq j\}$, with an arbitrary rule to break ties. We are studying the vector of proportional sizes

$$\left(\frac{|\text{Vor}(U_1)|}{n}, \dots, \frac{|\text{Vor}(U_k)|}{n} \right),$$

in the limit as $n \rightarrow \infty$.

In recent work, Addario-Berry et al. [AACFG18] investigated this question for the case that \mathcal{G} is a uniform tree, and proved that the limiting vector is uniform on the $(k-1)$ -dimensional simplex. Indeed, they showed much more, namely that this is even true in a limiting sense on the Brownian continuum random tree (henceforth CRT), and thus for all graph models that converge to the Brownian CRT in the Gromov–Hausdorff–Prokhorov topology: rooted plane trees, rooted unembedded binary trees, stacked triangulations, and others. Guitter [Gui17] proved the same uniform limit for the case that \mathcal{G} is a random planar map of genus 0 and $k = 2$. Chapuy [Cha19] made the far-reaching conjecture that the uniform limit is true for *all random embedded graphs of fixed genus*.

Our first contribution. In this paper, we look at the distribution of the Voronoi cells of k uniform nodes in a random *split tree*. Split trees are a family of rooted trees introduced by Devroye [Dev98] and later extended by Janson [Jan19] who allowed trees of unbounded degrees: this family includes classical random trees such as the binary search tree, the random recursive tree, the preferential attachment tree (also called PORT for “plane oriented recursive tree”). In our first main result, we prove that the largest of the Voronoi cells of k uniform nodes in an n -node split tree contains a proportion $1/k$ of all nodes. We are also able to prove that the second, third, \dots , k^{th} largest Voronoi cells each contains an order $n \exp(-\text{const}\sqrt{\log n})$ of

all vertices. We show that this result also holds when edges of the tree are given random i.i.d. lengths (of finite variance, or heavy-tailed but with finite mean), and defining the Voronoi cells with respect to the distance induced by these edge lengths instead of the graph distance.

This result is in contrast with the findings of [AACFG18] for the uniform random tree equipped with the graph distance: the distribution of the sizes of Voronoi cells is balanced in the case of the uniform random tree (and other trees whose scaling limit is the CRT), while we show a “winner takes it all” behaviour in the case of split trees. This difference in behaviour should not be surprising: it is well-known that split trees have a very different shape from the uniform random trees (and other random trees whose scaling limit is the CRT): for example, the typical height of an n -node split tree is $\log n$, while the typical height of the uniform random tree is \sqrt{n} . In that sense, split trees belong to another universality class of random trees (as opposed to trees whose scaling limit is the CRT), and our first main result corresponds to the findings of [AACFG18] for this second universality class. Similarly to [AACFG18] conjecturing that their result generalises to maps that scale to the random Brownian map, one might expect that the behaviour we prove for random split trees might also be exhibited by other graphs such as preferential attachment graphs and other scale-free models such as the configuration model. However, our proofs cannot be straightforwardly generalised.

Extension to a competition/epidemics model. The Voronoi cells can be seen as the result of a *competition model* where k agents are claiming territory with uniform speed until they reach vertices that are already claimed by another agent. This procedure stops when all vertices are claimed by some agent; the final territories are the same as the Voronoi cells. As discussed above, our first result is that – unlike in the case of uniform trees – the final territories are rather unbalanced: while one agent will claim almost the entire tree, the rest has to live on a rather small territory. (This behaviour also persists when we introduce random edge lengths.)

One can also see this competition model as a competition between k mutually exclusive epidemics, which are started at k uniform vertices of a split tree, and which all spread at constant and equal speed. Our second main result is that, if the speed of transmission varies among the different epidemics, then the fastest epidemic spreads over order n of the vertices. We are also able to estimate precisely the number of nodes that get infected by each of the slower epidemics.

Note that this “winner takes it all” effect has already been observed in a competing first-passage percolation model on the configuration model with tail exponent $\tau \in (2, 3)$ (see [DH16]). The main difference with our model is that epidemics spread deterministically in our model, and randomly at a given rate in the competing first-passage percolation model. In both models, one epidemic occupies eventually almost all of the available territory. In the case of different speeds, this is the fastest one, but in the case of equal speed this is determined by the initial position (see [BHK15, HK15], where this is proved for the competing first-passage percolation model). The competing first-passage percolation model on random regular graphs exhibits similar behaviour [ADMP17]. This suggests that the uniform limiting proportion of Voronoi cells is not true on complex networks. Instead, our results support the belief that for

competing epidemics on networks with small distances (“small-world graphs”) there is one dominating epidemic.

Information on the typical shape of a random split tree. The asymptotic sizes of the Voronoi cells (or territories) of k nodes chosen uniformly at random in a tree gives information on the typical shape of a tree. In fact, to prove our main result, we prove two results that may be of independent interest because they give information of the typical shape of a random split tree: (1) in Proposition 1.10 we show the convergence in probability of the “profile” of a random split tree, and (2) in Proposition 2.7, we prove asymptotic results for the size of a typical “extended” fringe tree in a random split tree.

(1) The profile of a random tree is the distribution of the height⁽¹⁾ (distance to the root) of a node taken uniformly at random in the tree. If the tree is random then its profile is a random measure. In Proposition 1.10, we show that the profile of a random split tree behaves asymptotically (in probability) as a Gaussian centred around $\text{const} \cdot \log n$ and of standard deviation $\text{const} \cdot \log n$. Our framework includes the cases of the random binary and m -ary search trees, the random recursive tree and the preferential attachment trees, for which convergence of the profile is already known in the almost sure sense (see [CDJH01, MM17], and [Kat05], respectively).

(2) Fringe trees are subtrees that are rooted at an ancestor of a node taken uniformly at random in the tree (or at the uniform node itself). Oftentimes, this ancestor is chosen to be at constant distance of the uniform node (see, e.g. [HJ17] and the references therein). In Proposition 2.7, we extend this definition to allow the ancestor to be at distance to the uniform node that tends to infinity with n , the number of nodes in the whole split tree.

Other pieces of information on the typical shape of random split trees are available in the literature: see Devroye [Dev98] for results on their height and typical distances to the root, Broutin and Holmgren [BH12] for results on the total path length, Holmgren [Hol12] for results on the size of random split trees and typical distances to the root, Berzunza, Cai and Holmgren [BCH19, BH21] for results on connected components after performing Bernoulli percolation.

The main technical obstacles in our proofs come from the three levels of randomness: (a) the trees we consider are random split trees, (b) we then sample i.i.d. random edge-lengths, and (c) we finally sample k nodes uniformly at random in the tree. The advantage of our approach is that the framework we consider is very wide: the random split trees we consider include, among others, the binary and m -ary search trees, the random recursive tree, and the preferential attachment tree; our edge-length distribution can be of finite variance, or heavy-tailed with finite mean; we allow the different epidemics to have identical or different speeds.

In the rest of this section, we define our model (Section 1) and state our main results (Section 1.1).

⁽¹⁾Note that, in the computer science literature, the distance of a node to the root is sometimes called ‘depth’, whereas the ‘height’ of a node is the maximal distance from this node to a leaf.

Trees and random split trees. In this paper, we use the Ulam–Harris definition of m -ary trees: let $m \in \mathbb{N}$ and

$$\mathcal{D}_m = \{1, 2, \dots, m\}^* = \{\emptyset, 1, 2, \dots, m, 11, 12, \dots, 1m, \dots\},$$

be the set of all finite words on the alphabet $\{1, 2, \dots, m\}$. We further consider the case of infinitary trees, where $m = \infty$ and $\mathcal{D}_\infty = \mathbb{N}^*$. We henceforth formulate our results for finite and infinite m in a unified fashion (unless stated explicitly); finite tuples, such as in (1.1) below, should be interpreted as infinite sequence whenever $m = \infty$.

DEFINITION 1.1. — *An m -ary tree is a subset t of \mathcal{D}_m such that for all $w = w_1 \cdots w_\ell \in t$, all the prefixes of w are in t , i.e. for all $i \in \{0, \dots, \ell\}$ one has $w_1 \cdots w_i \in t$.*

See Figure 1.1 for an example of a 3-ary tree. In the following, we collect some standard vocabulary and notations; they reflect the fact that a tree is often seen as a genealogical structure:

- words are called “nodes”;
- the prefixes of a word are its “ancestors”: we write $v \prec w$ if v is an ancestor of w , and $v \preceq w$ if v is w or an ancestor of w ;
- the longest of the (strict) prefixes of a word w is its “parent”, which we denote by \hat{w} ;
- a node is a “child” of its parent, and it is a “descendant” of each of its ancestors;
- the “siblings” of a node v are all those nodes different from v that share the same parent with v ; its “left-siblings” (resp. “right-siblings”) are all its siblings that are smaller (resp. larger) in the lexicographic order;
- the word \emptyset is the “root” of the tree;
- the “height” of a node is the number of letters in the word (the root is at height 0);
- the “last common ancestor” of two nodes is the longest prefix shared by the two nodes: we write $u \wedge v$ for the last common ancestor of nodes u and v .

In particular, the definition of a tree can be immediately rephrased using this new vocabulary reflecting the genealogical point of view: a tree is a set of nodes such that if a node is in the tree, then all its ancestors must also be in the tree.

We now define a probability distribution on the set of m -ary trees: it is the distribution of “split trees” first introduced by Devroye [Dev98], but generalised to possibly infinite arity as in [Jan19]. Let ν be a probability distribution on the set

$$(1.1) \quad \Sigma_m = \left\{ (v_1, \dots, v_m) \in [0, 1]^m : \sum_{i=1}^m v_i = 1 \right\},$$

and $(\mathbf{Y}(w))_{w \in \mathcal{D}_m}$ be a family of i.i.d. ν -distributed random vectors. For each node $w = w_1 \cdots w_\ell \in \mathcal{D}_m$, we let $Z_w = Y_{w_\ell}(\hat{w})$, where \hat{w} is the parent of w , i.e. $\hat{w} = w_1 \cdots w_{\ell-1}$ and with $Y_{w_\ell}(\hat{w})$ denoting the w_ℓ^{th} coordinate of the vector $\mathbf{Y}(\hat{w})$ (see Figure 1.2 for an example: $\mathbf{Y}(3) = (.1, .4, .5)$ and thus $Z_{32} = .4$).

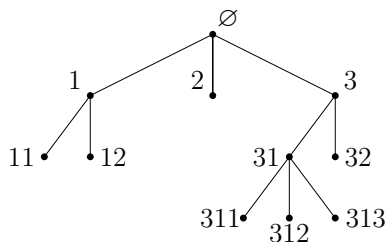


Figure 1.1. The 3-ary tree $\{\emptyset, 1, 2, 3, 11, 12, 31, 32, 311, 312, 313\}$. Node 312 is the “second child of the first child of the third child of the root”, its parent is node 31, its siblings are 311 and 313. The last common ancestor of 32 and 313 is 3.

We also let $(X_n)_{n \geq 0}$ be a sequence of i.i.d. random variables uniformly distributed on $[0, 1]$, and independent from the sequence $(\mathbf{Y}(w))_{w \in \mathcal{D}_m}$.

We need one last definition to define our sequence of random split trees: Given a tree t , we denote by ∂t the nodes of \mathcal{D}_m that are not in t but whose parent is in t , and we call the elements of this set the “leaves” of t . It is not hard to see that if t has n nodes, then ∂t has cardinality $(m - 1)n + 1$ (see Figure 1.2).

We can now define the sequence $(\tau_n)_{n \geq 1}$ of random trees recursively as follows.

- the tree τ_1 is defined to consist of the root only, i.e. $\tau_1 = \{\emptyset\}$.
- for $n \geq 1$ arbitrary, given τ_n , we define τ_{n+1} as the tree obtained by adding one node to τ_n as follows:
 - We subdivide the interval $[0, 1]$ in subintervals indexed by $\partial \tau_n$ of respective lengths $\prod_{\emptyset \neq v \preceq w} Z_v$, for all $w \in \partial \tau_n$. (Note that, by definition, $\sum_{w \in \partial \tau_n} \prod_{\emptyset \neq v \preceq w} Z_v = 1$; see Figure 1.2 for an example, and observe that some points form part of several intervals.)
 - We set $\xi(n + 1) = w$ if $X_{n+1} \in [0, 1]$ belongs to the part indexed by w of this partition of $[0, 1]$, and finally set $\tau_{n+1} = \tau_n \cup \{\xi(n + 1)\}$; note that this is well-defined almost surely.

The sequence of random trees $(\tau_n)_{n \geq 1}$ is called the *random split tree of split distribution* ν (which we recall is the distribution of the $\mathbf{Y}(w)$ ’s).

This definition incorporates a variety of different random trees that are classical in the literature:

- If $m = 2$ and ν is the distribution of $(Y, 1 - Y)$, where Y is uniform on $[0, 1]$, then $(\tau_n)_{n \geq 1}$ is the *random binary search tree* (see [Dev98, Table 1]).
- If ν is the uniform distribution on the simplex Σ_m for m finite, then $(\tau_n)_{n \geq 1}$ is the *m -ary increasing tree* (see [Dev98, Table 1]).
- If $m = \infty$ and ν is GEM(0, 1) (see Remark 1.2 below for the definition of the GEM distribution) on Σ_∞ , then $(\tau_n)_{n \geq 1}$ is the *random recursive tree* (see [Jan19, Corollary 1.2]).
- If $m = \infty$ and ν is GEM(1/2, 1/2), then $(\tau_n)_{n \geq 1}$ is the *random preferential attachment tree* (see [Jan19, Corollary 1.3]).

Remark 1.2. — For $\alpha \in [0, 1]$ and $\theta > 0$, the Griffiths–Engen–McCloskey distribution GEM(α, θ) is defined as the distribution of the sequence $(A_n)_{n \geq 1}$ defined as

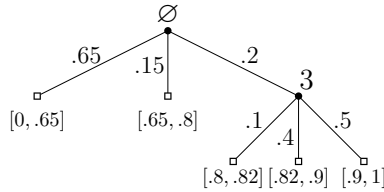


Figure 1.2. A realisation of the 3-ary split tree τ_2 , here we have $\tau_2 = \{\emptyset, 3\}$. The labels on the edges represent the values of $(\mathbf{Y}(w))_{w \in \tau_2}$: for example, $\mathbf{Y}(\emptyset) = (.65, .15, .2)$. The value of Z_w is thus the label on the edge from w to its parent: for example, $Z_{31} = .1$. The nodes that are marked by a square are the elements of $\partial\tau_2$, underneath each leaf is written the corresponding part in the partition used to build τ_3 . For example, the part corresponding to 32 is of length $Z_3 Z_{32} = .2 \times .4 = .08$.

follows: sample $(B_i)_{i \geq 1}$ a sequence of independent random variables of respective distributions $\text{Beta}(1 - \alpha, \theta + i\alpha)$, and, for all $n \geq 1$, set $A_n = B_n \prod_{i=1}^{n-1} (1 - B_i)$.

Note that our definition does not cover for the full class of split trees as introduced in Devroye [Dev98]: for example, the m -ary search tree does not fall into our framework although it is a split tree in the sense of Devroye. Using the notation of Devroye [Dev98], we assume that $s_0 = s = 1$ and $s_1 = 0$; we expect that our results as well as their proofs essentially carry through to the more general framework of Devroye; however, we choose not to include this generalisation so that our proofs remain easier to read, and because our framework already covers numerous interesting cases of random trees, as listed above.

1.1. Voronoi cells and final territories

In this paper, our aim is to investigate the sizes of the Voronoi cells corresponding to k nodes taken uniformly at random in the n -node random split tree τ_n defined in Section 1. In this context, we will also accommodate for the setting of having random edge lengths between the nodes: let ϖ be a probability distribution on $(0, \infty)$ and let $(L_w)_{w \in \mathcal{D}_m}$ be a sequence of i.i.d. random variables of distribution ϖ , and we define the distance between two nodes as the sum of the length of the edges on the unique shortest path between them; see Figure 1.3 for an example.

DEFINITION 1.3. — For all families $\ell = (\ell_w)_{w \in \mathcal{D}_m}$ of positive random variables we define a distance d_ℓ on \mathcal{D}_m as follows: for all pairs of nodes u and v in \mathcal{D}_m (for all $m \geq 2$), let

$$d_\ell(u, v) := \sum_{u \wedge v \prec w \preceq u} \ell_w + \sum_{u \wedge v \prec w \preceq v} \ell_w,$$

where we recall that $u \wedge v$ denotes the last common ancestor of nodes u and v . For all nodes $w \in \mathcal{D}_m$, we denote by $|w|_\ell := d_\ell(\emptyset, w)$.

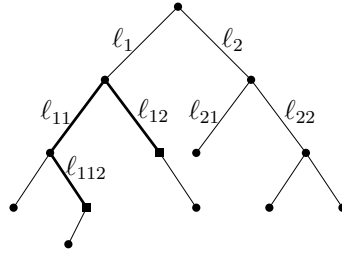


Figure 1.3. A binary tree. The distance between the nodes 112 and 12 (marked as squares on the picture) with respect to the sequence ℓ is $\ell_{11} + \ell_{112} + \ell_{12}$ (the sum of the length of the bold edges) because their last common ancestor is 1.

This definition holds for any fixed sequence ℓ of edge lengths: all along the paper, we use the distance $d_{\mathbf{L}}$, where $\mathbf{L} = (L_w)_{w \in \mathcal{D}_m}$ is the sequence of i.i.d. random edge lengths.

Also, note that if $\ell_w = 1$ for all $w \in \mathcal{D}_m$, then d_ℓ corresponds to the graph distance in the graph whose nodes are all elements of \mathcal{D}_m , and where there is an edge between two nodes if and only if one is the parent of the other.

DEFINITION 1.4. — Let u_1, \dots, u_k be k nodes in an m -ary tree t , and d a distance on \mathcal{D}_m . We define the Voronoi cells of u_1, \dots, u_k as follows: for all $1 \leq i \leq k$,

$$\text{Vor}_{t,d}^i(u_1, \dots, u_k) = \left\{ w \in t : d(w, u_i) \leq d(w, u_j) \text{ for } j = 1, \dots, i-1 \right. \\ \left. \text{and } d(w, u_i) < d(w, u_j) \text{ for } j = i+1, \dots, n \right\}.$$

We say that $\text{Vor}_{t,d}^i(u_1, \dots, u_k)$ is the Voronoi cell of u_i (with respect to u_1, \dots, u_k).

Remark 1.5. — The idea of Definition 1.4 is that $\text{Vor}_{t,d}^i(u_1, \dots, u_k)$ contains all the nodes that are closer to u_i than to any of the other u_j 's for distance d on t . The difference between ' $<$ ' and ' \leq ' induces a simple rule to break ties (in case of equal distances, the vertex with smaller index is preferred). However, since the number of boundary vertices is of constant order, the choice we make about how to break ties has no impact on our results.

In Section 1, we discussed a possible interpretation of Voronoi cells in terms of epidemics. From this point of view, it is natural to consider the case when the epidemics spread at different speeds:

DEFINITION 1.6. — Let t be an m -ary tree and denote by d be a distance on \mathcal{D}_m . Furthermore, let u_1, \dots, u_k be nodes in t and let $s_1, \dots, s_k \in (0, +\infty)$, the 'speeds of the epidemics'. We define the final territories of $(u_1, s_1), \dots, (u_k, s_k)$ as

$$\text{Ter}_{t,d}^i((u_1, s_1), \dots, (u_k, s_k)) \\ = \left\{ w \in t : d(w, u_i) \leq \frac{s_i + s_j}{s_i} d(w, u_j) \text{ for } j = 1, \dots, i-1 \right. \\ \left. \text{and } (w, u_i) < \frac{s_i + s_j}{s_i} d(w, u_j) \text{ for } j = i+1, \dots, n \right\}.$$

1.1.1. Main results

Our main result provides asymptotic statements on the sizes of k epidemics in the case when the epidemics have different speeds. We first state the result in the simpler case when all epidemics have the same speed (Theorem 1.7) and then extend it to the setting where different speeds are admissible (Theorem 1.9). Both theorems apply to finite ($m \in \{2, 3, \dots\}$) as well as infinite ($m = \infty$) arity. They hold under the following hypothesis on the split-vector distribution ν and the edge length distribution ϖ :

(A1) If $(Y_1, \dots, Y_m) \sim \nu$, U is a uniform random variable on $[0, 1]$, and⁽²⁾

$$(1.2) \quad \bar{Y} = \sum_{i=1}^m Y_i \mathbf{1} \left\{ \sum_{j=1}^{i-1} Y_j \leq U < \sum_{j=1}^i Y_j \right\}$$

is the size-biased version of the marginals of ν , then $\mu := \mathbb{E}[\log 1/\bar{Y}] > 0$ and $\sigma^2 := \text{Var}(\log \bar{Y}) < +\infty$.

(A2) If L is a random variable of distribution ϖ , then either $\text{Var}(L) \in [0, +\infty)$, in which case we set $\alpha := 2$, or there exists $\alpha \in (1, 2)$ and a non-negative function ℓ slowly varying at infinity, such that $\mathbb{P}(L > x) = x^{-\alpha} \ell(x)$. In particular, $\mathbb{E}L < \infty$ in this case.

The assumption that $\mu > 0$ excludes the trivial case when the n -node split tree is almost surely equal to a line of n nodes hanging under each other under the root. The assumption that $\sigma^2 < +\infty$ and (A2) give some control over the moments of respectively the split vectors and the edge lengths: these assumptions will be used when applying laws of large numbers and of the iterated logarithm, as well as central limit theorems to sum of independent copies of these random variables.

THEOREM 1.7. — *Let ϖ be a probability distribution on $(0, +\infty)$, and ν be a probability distribution on Σ_m . Let $(\tau_n)_{n \geq 1}$ be the random split tree of split distribution ν , and $\mathbf{L} = (L_w)_{w \in \mathcal{D}_m}$ be a sequence of i.i.d. random variables of distribution ϖ , independent of $(\tau_n)_{n \geq 1}$.*

For each $n \geq 1$, let $U_1(n), \dots, U_k(n)$ be k nodes taken uniformly at random among the n nodes of τ_n ; we let $V_{(1)}(n) \geq \dots \geq V_{(k)}(n)$ be the sizes of their Voronoi cells in τ_n with respect to the distance $d_{\mathbf{L}}$, ordered in decreasing order.

Under Assumptions (A1) and (A2), we have in distribution when $n \rightarrow +\infty$,

$$(1.3) \quad \frac{1}{(\log n)^{1/\alpha}} \left(\log(V_{(2)}(n)/n), \dots, \log(V_{(k)}(n)/n) \right) \Rightarrow \frac{\mathbf{v}}{2\mathbb{E}L} \left(\Psi_{(1)} - \Psi_{(2)}, \dots, \Psi_{(1)} - \Psi_{(k)} \right);$$

here, $\Psi_{(1)} \leq \dots \leq \Psi_{(k)}$ are the order statistics of k i.i.d. random variables whose distribution is $\mathcal{N}(0, \text{Var}(L) + \sigma^2(\mathbb{E}L)^2)$ if $\text{Var}(L) < +\infty$, and an α -stable distribution otherwise, and where

$$(1.4) \quad \mathbf{v} = \begin{cases} \mu^{-1/2} & \text{if } \text{Var}(L) < +\infty \\ \mu^{1-1/\alpha} & \text{otherwise.} \end{cases}$$

⁽²⁾By convention, we set $\sum_{i=1}^0 a_i = 0$ for each sequence $(a_i)_{i \geq 0}$ of real numbers.

Remark 1.8. — In Theorem 1.7, and elsewhere in this paper, we adopt the convention that $\mathcal{N}(0, 0)$ is interpreted as the Dirac measure at zero. With this convention, all our statements hold if, for example, $\sigma^2 = 0$ and/or $\text{Var}(L) = 0$.

In words, the above amounts to the fact that the second, third, \dots , k th largest component each occupies a proportion of roughly $\exp\{-\Psi(\log n)^{1/\alpha}\}$ of the vertices, where Ψ is some explicit positive random variable. This implies that asymptotically and in distribution, the entire mass is allocated to the largest component (which, by construction, belongs to the vertex closest to the root); this explains the asymmetry in (1.3). The allocation for split trees is therefore qualitatively very different from the allocation in the universality class of the continuum random tree, where the limit of the proportions of the masses is known to be uniform [AACFG18].

We now extend the results of the previous theorem to the case of different speeds at which the uniformly chosen vertices claim territory (use the same notation as in Theorem 1.7).

THEOREM 1.9. — *For all $s_1, \dots, s_k \in (0, +\infty)$, we let $W_{(1)}(n) \geq \dots \geq W_{(k)}(n)$ be the sizes (ordered in decreasing order) of the final territories in τ_n , equipped with the distance d_L , of k epidemics of respective speeds s_1, \dots, s_k and starting from $U_1(n), \dots, U_k(n)$, respectively. Without loss of generality, we assume that $s_1 = \dots = s_j > s_{j+1} \geq \dots \geq s_k$ for some $j \in \{1, \dots, k\}$.*

Then, under Assumptions (A1) and (A2), we have in distribution when $n \rightarrow +\infty$,

$$(1.5) \quad \frac{1}{(\log n)^{1/\alpha}} \left(\log(W_{(2)}(n)/n), \dots, \log(W_{(j)}(n)/n) \right) \Rightarrow \frac{\mathbf{v}}{2\mathbb{E}L} \left(\Psi_{(1)} - \Psi_{(2)}, \dots, \Psi_{(1)} - \Psi_{(j)} \right),$$

where \mathbf{v} is defined in (1.4), $\Psi_{(1)} \leq \dots \leq \Psi_{(j)}$ are the order statistics of j i.i.d. random variables whose distribution is $\mathcal{N}(0, \text{Var}(L) + \sigma^2(\mathbb{E}L)^2)$ if $\text{Var}(L) < +\infty$, and an α -stable distribution otherwise.

Furthermore, if $\mathbb{E}L^2 < +\infty$, then, for all $i \in \{j + 1, \dots, k\}$,

$$\frac{\log(W_{(i)}(n)/n) + \frac{s_1 - s_i}{s_1 + s_i} \log n}{\sqrt{\frac{s_1 - s_i}{s_1 + s_i} \log n}} \Rightarrow \mathcal{N}\left(0, \frac{\text{Var}(\log \bar{Y})\mathbb{E}L}{\mathbb{E}[\log \bar{Y}]^2} + \text{Var}(L)\right).$$

Otherwise, if $\mathbb{P}(L \geq x) = x^{-\alpha}\ell(x)$ for some function ℓ slowly varying at infinity and some $\alpha \in (1, 2)$, then, for all $i \in \{j + 1, \dots, k\}$,

$$\frac{\log(W_{(i)}(n)/n) + \frac{s_1 - s_i}{s_1 + s_i} \log n}{\left(\frac{s_1 - s_i}{s_1 + s_i} \log n\right)^{1/\alpha}} \Rightarrow \frac{\mu^{1-1/\alpha}}{\mathbb{E}L} \Upsilon(\alpha),$$

where $\Upsilon(\alpha)$ is an α -stable random variable. In particular, in both cases, we have

$$(1.6) \quad \frac{\log(W_{(i)}(n)/n)}{\log n} \rightarrow \frac{s_i - s_1}{s_i + s_1}$$

in probability when $n \rightarrow +\infty$, for each $i \in \{j + 1, \dots, k\}$.

Note that, given that the slower epidemics all have very small territories (cf. (1.6)), the j fastest territories behave as in Theorem 1.7, which – at least heuristically – entails (1.5).

It is also interesting to note that in their first asymptotic order given by (1.6), the sizes of the slow epidemics do not depend on the edge length. An intuitive indication towards this fact is that replacing L by cL for a positive constant c does not change the sizes of the territories. In a similar vein, the right-hand sides of (1.3) and (1.5) also remain unchanged upon replacing L by cL , as expected.

As a by-product of our proof of Theorems 1.7 and 1.9, we get the following result on the convergence of the profile of random split trees, which, as far as we are aware, is a new result in the context of split trees:

PROPOSITION 1.10. — *Let $(\tau_n)_{n \geq 1}$ be the random split tree of split distribution ν , and let, for all integer n , $\pi_n = \frac{1}{n} \sum_{i=1}^n \delta_{|\nu_i|}$ be the random profile of τ_n , where we recall that $|\nu_i|$ is the height of the node inserted at time i in $(\tau_n)_{n \geq 1}$. If ν satisfies Assumption (A1), then*

$$(1.7) \quad \pi_n \left(\cdot \sqrt{(\log n)/\mu^3} + (\log n)/\mu \right) \rightarrow \pi_\infty = \mathcal{N}(0, \sigma^2),$$

in probability as $n \rightarrow +\infty$, on the space of probability measures on \mathbb{R} equipped with the topology of weak convergence.

Stronger results are already known for certain cases of split trees: in particular, it is known that (1.7) holds almost surely in the case of the binary search tree [CDJH01], the random recursive tree [MM17], and the preferential attachment tree [Kat05]. The profile of the uniform random tree (considered by [AACFG18] in the context of Voronoi cells) converges in distribution to the local time of a Brownian excursion (see [DG97]).

Remark 1.11. — Note that Theorem 1.7 holds in an averaged sense (or with respect to the joint law). One could imagine two quenched versions by (i) conditioning on the random split tree $(\tau_n)_{n \geq 1}$ or (ii) conditioning additionally also on the sequence of edge lengths \mathbf{L} . Since, in our proof, we use the central limit theorem for the sequence \mathbf{L} , our current methods do not provide with a possible version of Theorem 1.7 quenched with respect to \mathbf{L} . However, for the split distributions ν for which (1.7) holds almost surely, Theorem 1.7 would hold almost surely given $(\tau_n)_{n \geq 1}$.

The remainder of the paper is organised as follows. In Section 2, we establish a central limit theorem for the joint law of the height of uniform vertices and derive Proposition 1.10. Furthermore, we prove Theorem 1.7. In Section 3, we extend these arguments to the case of different speeds thereby proving Theorem 1.9.

2. Proof of Theorem 1.7

In this section, we use the same notation, and place ourselves under the assumptions of Theorem 1.7. The idea of the proof is as follows: if $\varpi = \delta_1$ (i.e. all edge lengths are equal to 1 almost surely, i.e. $\mathbf{L} \equiv 1$) then among the nodes $U_1(n), \dots, U_k(n)$, the one closest to the root belongs to the Voronoi cell containing the root, and this Voronoi

cell typically is the largest of all Voronoi cells. As a consequence, it is important to understand the heights of k uniform nodes in a random split tree. Recall that for a graph node $v \in \tau$, we write $|v|$ for that graph distance between the root \emptyset and v .

LEMMA 2.1 (CLT for heights of uniform vertices). — *Let $k \in \mathbb{N}$ and \bar{Y} be distributed as the size-biased version of the marginals of ν (see (1.2)), and denote $\mu = \mathbb{E}[\log(1/\bar{Y})]$ as well as $\sigma^2 = \text{Var}(\log \bar{Y})$.*

Then, in distribution as $n \rightarrow +\infty$, we have

$$\left(\frac{|U_1(n)| - (\log n)/\mu}{\sqrt{(\log n)/\mu^3}}, \dots, \frac{|U_k(n)| - (\log n)/\mu}{\sqrt{(\log n)/\mu^3}} \right) \Rightarrow (\Lambda_1, \dots, \Lambda_k),$$

where the $\Lambda_1, \dots, \Lambda_k$ are independent centred Gaussian random variables of variance σ^2 .

This lemma straightforwardly implies Proposition 1.10.

Proof of Proposition 1.10. — We use [MM17, Lemma 3.1], which states that for a sequence of random measures $(\pi_n)_{n \geq 0}$ to converge in probability to a limiting measure π_∞ , it is enough to show, for two random variables A_n and B_n sampled independently according to the random measure π_n , that $(A_n, B_n) \rightarrow (A, B)$ in distribution, where A and B are π_∞ -distributed and independent. (Note that, on the left-hand side, A_n and B_n are independent conditionally on π_n , but not without this conditioning.) As a direct consequence, we conclude the proof using [MM17, Lemma 3.1] in combination with Lemma 2.1 for the particular case $k = 2$ in order to ensure the required convergence conditions of [MM17, Lemma 3.1] to be fulfilled. \square

The proof of Lemma 2.1 uses two main results, which we state as separate lemmas: convergence of the marginals (see Lemma 2.2) and asymptotic independence (see Lemma 2.3).

LEMMA 2.2. — *Recall that, for all $n \geq 1$, $\xi(n)$ denotes the nodes added to the split tree at time n ; i.e. the unique node that belongs to τ_n but not to τ_{n-1} . In distribution when $n \rightarrow +\infty$,*

$$\frac{|\xi(n)| - (\log n)/\mu}{\sqrt{(\log n)/\mu^3}} \Rightarrow \mathcal{N}(0, \sigma^2).$$

Proof. — We follow arguments in [Dev98, Theorem 2] and extend them to the cases $m = \infty$ and $\sigma^2 = 0$. For any node $u \in \mathcal{D}_m$, we let $N_n(u)$ be the number of nodes of τ_n whose ancestor is u (if $u \notin \tau_n$, then $N_n(u) = 0$).

First sample a sequence of nodes $(u_i)_{i \geq 0}$ in \mathcal{D}_m as follows: $u_0 = \emptyset$ and, for all $i \geq 0$, given u_i , let $u_{i+1} = u_i j$ with probability $Y_j(u_i)$, for all $j \geq 1$. Finally, let $u^*(n)$ be the node with maximal height in $(u_i)_{i \geq 0} \cap \tau_n$. By definition, in distribution, $|\xi(n)| = |u^*(n)|$.

Indeed, from our definition of the random split tree $(\tau_n)_{n \geq 0}$, one can see that an equivalent way of constructing τ_n given τ_{n-1} is as follows: Drop a ball at the root of τ_{n-1} . If this root is empty, then insert the ball here, i.e. set $\xi(n) = \emptyset$. If the root is not empty, then, for all $1 \leq j \leq m$, with probability V_j , insert the ball in the j^{th}

subtree of the root of τ_{n-1} , and repeat the insertion procedure in this subtree. From this equivalent definition, one can see that, indeed, $|\xi(n)| = |u^*(n)|$ in distribution.

Step 1: We first fix $u \in \mathbb{R}$ and $k = \lfloor (\log n)/\mu + u\sqrt{(\log n)/\mu^3} \rfloor$. We also set $\ell = \log \log n$. If $|u^*(n)| \geq k + \ell$, then the subtree of τ_n rooted at u_k contains at least ℓ nodes; in other words, $\{|u^*(n)| \geq k + \ell\} \subseteq \{N_n(u_k) \geq \ell\}$. Thus,

$$\mathbb{P}(|\xi(n)| \geq k + \ell) \leq \mathbb{P}(N_n(u_k) \geq \ell).$$

Now note that the random variable $N_n(u_k)$ is stochastically dominated by a binomial of parameters n and $\prod_{i=1}^k \bar{Y}_i$. Hence, we can use due to [Dev98, Lemma 4] to upper bound

$$\begin{aligned} \mathbb{P}(|\xi(n)| \geq k + \ell) &\leq \mathbb{P}\left(\text{Bin}\left(n, \prod_{i=1}^k \bar{Y}_i\right) \geq \ell\right) \\ &\leq \mathbb{P}\left(\sum_{i=1}^k \log(\bar{Y}_i) > \log(\ell/2n)\right) + \left(\frac{e}{4}\right)^{\ell/2}. \end{aligned}$$

Therefore, as $n \rightarrow +\infty$,

$$\mathbb{P}(|\xi(n)| \geq k + \ell) \leq \mathbb{P}\left(\frac{\sum_{i=1}^k \log \bar{Y}_i + k\mu}{\sqrt{k}} > \frac{\log(\ell/2n) + k\mu}{\sqrt{k}}\right) + o(1).$$

Now note that, by the central limit theorem, $(\sum_{i=1}^k \log \bar{Y}_i + k\mu)/\sqrt{k} \Rightarrow \mathcal{N}(0, \sigma^2)$ and, by definition of k and ℓ , $(\log(\ell/2n) + k\mu)/\sqrt{k} \rightarrow u$ as $n \rightarrow +\infty$, as we have

$$\begin{aligned} \frac{\log(\ell/2n) + k\mu}{\sqrt{k}} &\sim \frac{\log \log \log n - \log 2 - \log n + \mu(\log n)/\mu + u\sqrt{(\log n)/\mu^3}}{\sqrt{(\log n)/\mu}} \\ &\sim \frac{u\mu^{1-3/2}}{\mu^{-1/2}} \sim u. \end{aligned}$$

We thus get

$$\mathbb{P}(|\xi(n)| \geq k + \ell) \leq \mathbb{P}(\mathcal{N}(0, \sigma^2) \geq u) + o(1).$$

Replacing k and ℓ by their respective values, we get

$$\begin{aligned} \mathbb{P}\left(\frac{|\xi(n)| - (\log n)/\mu}{\sqrt{(\log n)/\mu^3}} - \frac{\log \log n}{\sqrt{(\log n)/\mu^3}} \geq u\right) \\ &= \mathbb{P}\left(|\xi(n)| \geq \frac{1}{\mu} \log n + u\sqrt{\frac{\log n}{\mu^3}} + \log \log n\right) \\ &\leq \mathbb{P}\left(|\xi(n)| \geq \left\lfloor \frac{1}{\mu} \log n + u\sqrt{\frac{\log n}{\mu^3}} \right\rfloor + \log \log n\right) \\ &= \mathbb{P}(|\xi(n)| \geq k + \ell). \end{aligned}$$

Therefore,

$$\mathbb{P}\left(\frac{|\xi(n)| - (\log n)/\mu}{\sqrt{(\log n)/\mu^3}} - \frac{\log \log n}{\sqrt{(\log n)/\mu^3}} \geq u\right) \leq \mathbb{P}(\mathcal{N}(0, \sigma^2) \geq u) + o(1),$$

and because $\log \log n / \sqrt{(\log n) / \mu^3} \rightarrow 0$ as $n \rightarrow +\infty$, this implies

$$(2.1) \quad \mathbb{P} \left(\frac{|\xi(n)| - (\log n) / \mu}{\sqrt{(\log n) / \mu^3}} \geq u \right) \leq \mathbb{P}(\mathcal{N}(0, \sigma^2) \geq u) + o(1).$$

Step 2: We now fix $u \in \mathbb{R}$, set $k = \lceil (\log n) / \mu + u \sqrt{(\log n) / \mu^3} \rceil$, and look at the probability that $|\xi(n)| \leq k$; first note that, if $|u^*(n)| \leq k$, then the subtree of τ_n rooted at u_k is empty, i.e. $\{|u^*(n)| \leq k\} \subseteq \{N_n(u_k) = 0\}$. Thus,

$$(2.2) \quad \mathbb{P}(|\xi(n)| \leq k) \leq \mathbb{P}(N_n(u_k) = 0).$$

If $N_n(u_k) \geq 1$, then there exists an increasing subsequence of length k of (X_1, \dots, X_n) such that X_{i_1} falls in the interval associated to u_1 , X_{i_2} falls into the interval associated to u_2 , etc. Since the intervals associated to u_0, \dots, u_1 are all of length at least $\prod_{i=1}^k \bar{Y}_i$, this happens with probability at least $\mathbb{P}(\text{Bin}(n, \prod_{i=1}^k \bar{Y}_i) \geq k)$, and thus

$$\mathbb{P}(N_n(u_k) \geq 1) \geq \mathbb{P} \left(\text{Bin} \left(n, \prod_{i=1}^k \bar{Y}_i \right) \geq k \right),$$

which is equivalent to

$$\mathbb{P}(N_n(u_k) = 0) \leq \mathbb{P} \left(\text{Bin} \left(n, \prod_{i=1}^k \bar{Y}_i \right) < k \right).$$

We now use [Dev98, Lemma 4] again to get

$$\begin{aligned} \mathbb{P}(N_n(u_k) = 0) &\leq \mathbb{P} \left(\prod_{i=1}^k \bar{Y}_i \leq \frac{2k}{n} \right) + \left(\frac{2}{e} \right)^k = \mathbb{P} \left(\sum_{i=1}^k \log \bar{Y}_i \leq \log \left(\frac{2k}{n} \right) \right) + o(1) \\ &= \mathbb{P} \left(\frac{\sum_{i=1}^k \log \bar{Y}_i + k\mu}{\sqrt{k}} \leq \frac{\log(2k/n) + k\mu}{\sqrt{k}} \right) + o(1) \\ &= \mathbb{P}(\mathcal{N}(0, \sigma^2) \leq u) + o(1), \end{aligned}$$

by the central limit theorem, and because $(\log(2k/n) + k\mu) / \sqrt{k} \rightarrow u$ as $n \rightarrow +\infty$. Recalling (2.2), this gives

$$\mathbb{P} \left(\frac{|\xi(n)| - (\log n) / \mu}{\sqrt{(\log n) / \mu^3}} \leq u \right) = \mathbb{P}(|\xi(n)| \leq k) \leq \mathbb{P}(\mathcal{N}(0, \sigma^2) \leq u) + o(1).$$

Together with (2.1), this concludes the proof. \square

LEMMA 2.3. — For all $h \in \mathbb{N}$, let us denote by $U_1^{(h)}(n), \dots, U_k^{(h)}(n)$ the respective ancestors of $U_1(n), \dots, U_k(n)$ that have height h (if $h > |U_i(n)|$, we set $U_i^{(h)}(n) = U_i(n)$). Let $H_n = \max_{1 \leq i < j \leq k} |U_i(n) \wedge U_j(n)|$ be the height of the most recent common ancestor of $U_1(n), \dots, U_k(n)$ and $S_1(n), \dots, S_k(n)$ be the sizes of the subtrees of τ_n rooted at $U_1^{(H_n)}(n), \dots, U_k^{(H_n)}(n)$ respectively. In distribution when $n \rightarrow +\infty$,

$$\left(\frac{S_1(n)}{n}, \dots, \frac{S_k(n)}{n}, H_n \right) \Rightarrow (\alpha_1, \dots, \alpha_k, H),$$

where H is an almost surely finite random variable, and $\alpha_1, \dots, \alpha_k$ are almost surely positive random variables.

Proof. — We first prove that $H_n \Rightarrow H$, where H is an almost surely finite random variable. We first look at the last common ancestor of $U_1(n)$ and $U_2(n)$: for all words $w \in \mathcal{D}_m$, we have

$$\mathbb{P}(U_1(n) \wedge U_2(n) = w \mid \tau_n) = \sum_{1 \leq i \neq j \leq m} \frac{s_{wi}(n)}{n} \cdot \frac{s_{wj}(n)}{n},$$

where $s_v(n)$ is the size of the subtree of τ_n rooted at v (in particular, this is equal to zero if $v \notin \tau_n$). By the definition of the model and the strong law of large numbers we know that, conditionally on the sequence $\mathbf{Y} = (\mathbf{Y}(v))_{v \in \mathcal{D}_m}$, for all $v \in \mathcal{D}_m$, almost surely when $n \rightarrow +\infty$,

$$\frac{s_v(n)}{n} \rightarrow \prod_{u \preceq v} Z_u,$$

where we recall that $Z_{w\ell} = Y_\ell(w)$, for all $w \in \mathcal{D}_m$ and $\ell \in \{1, \dots, m\}$. Therefore, conditionally on \mathbf{Y} and almost surely when $n \rightarrow +\infty$, we have

$$\mathbb{P}(U_1(n) \wedge U_2(n) = w \mid \tau_n, \mathbf{Y}) \rightarrow \sum_{1 \leq i \neq j \leq m} \left(\prod_{u \preceq wi} Z_u \right) \left(\prod_{u \preceq wj} Z_u \right),$$

which implies, using dominated convergence,

$$\mathbb{P}(U_1(n) \wedge U_2(n) = w) \rightarrow \mathbb{E} \left[\sum_{1 \leq i \neq j \leq m} \left(\prod_{u \preceq wi} Z_u \right) \left(\prod_{u \preceq wj} Z_u \right) \right].$$

To prove that this implies convergence in distribution of $U_1(n) \wedge U_2(n)$ to an almost surely finite random variable $K_{1,2} \in \mathcal{D}_m$, we need to show that

$$(2.3) \quad \sum_{w \in \mathcal{D}_m} \mathbb{E} \left[\sum_{1 \leq i \neq j \leq m} \left(\prod_{u \preceq wi} Z_u \right) \left(\prod_{u \preceq wj} Z_u \right) \right] = 1.$$

To prove (2.3), we first note that, by independence of the Z_u 's (except among siblings), for all $w \in \mathcal{D}_m$,

$$\begin{aligned} \mathbb{E} \left[\sum_{1 \leq i \neq j \leq m} \left(\prod_{u \preceq wi} Z_u \right) \left(\prod_{u \preceq wj} Z_u \right) \right] &= \mathbb{E} \left[\prod_{u \preceq w} Z_u^2 \right] \sum_{1 \leq i \neq j \leq m} \mathbb{E}[Z_{wi} Z_{wj}] \\ &= \beta \mathbb{E} \left[\prod_{u \preceq w} Z_u^2 \right], \end{aligned}$$

where we have introduced the shorthand $\beta = \sum_{1 \leq i \neq j \leq m} \mathbb{E}[Y_i Y_j]$, with Y a random vector of distribution ν . By Assumption (A1), $\mathbb{E}[\log 1/\bar{Y}] > 0$, which implies that Y is not equal to a standard basis vector with positive probability. In particular, with positive probability, there exists $1 \leq i \neq j \leq m$ such that $Y_i Y_j \neq 0$, and thus $\beta \neq 0$.

This entails that

$$(2.4) \quad \begin{aligned} \sum_{w \in \mathcal{D}_m} \mathbb{E} \left[\sum_{1 \leq i \neq j \leq m} \left(\prod_{u \preceq wi} Z_u \right) \left(\prod_{u \preceq wj} Z_u \right) \right] &= \beta \sum_{w \in \mathcal{D}_m} \mathbb{E} \left[\prod_{u \preceq w} Z_u^2 \right] \\ &= \beta \sum_{h \geq 0} \sum_{|w|=h} \mathbb{E} \left[\prod_{u \preceq w} Z_u^2 \right], \end{aligned}$$

where we recall that $|w|$ is the height of w . For all $h \geq 0$, using again the independence of the Z_u 's, we infer that

$$(2.5) \quad \sum_{|w|=h+1} \mathbb{E} \left[\prod_{u \preceq w} Z_u^2 \right] = \sum_{|w|=h} \sum_{i=1}^m \mathbb{E} \left[\prod_{u \preceq wi} Z_u^2 \right] = \sum_{|w|=h} \mathbb{E} \left[\prod_{u \preceq w} Z_u^2 \right] \sum_{i=1}^m \mathbb{E} [Z_{wi}^2].$$

Since, by definition, $\sum_{i=1}^m Z_{wi} = \sum_{i=1}^n Y_i(w) = 1$, we get that

$$\sum_{i=1}^m \mathbb{E} [Z_{wi}^2] = \left(\sum_{i=1}^m \mathbb{E} Z_{wi} \right)^2 - \sum_{1 \leq i \neq j \leq m} \mathbb{E} [Z_{wi} Z_{wj}] = 1 - \beta,$$

by definition of β . Plugging the last equality into (2.5), this amounts to

$$\sum_{|w|=h+1} \mathbb{E} \left[\prod_{u \preceq w} Z_u^2 \right] = (1 - \beta) \sum_{|w|=h} \mathbb{E} \left[\prod_{u \preceq w} Z_u^2 \right] = \dots = (1 - \beta)^{h+1}$$

by iteration, and further, using (2.4) and the fact that by Assumption (A1), $\beta \neq 0$, we get

$$\mathbb{E} \left[\sum_{1 \leq i \neq j \leq m} \left(\prod_{u \preceq wi} Z_u \right) \left(\prod_{u \preceq wj} Z_u \right) \right] = \beta \sum_{h=0}^{\infty} (1 - \beta)^h = 1.$$

This concludes the proof of (2.3), and thus of the fact that $U_1(n) \wedge U_2(n)$ converges in distribution to an almost surely finite random variable $K_{1,2}$. Consequently,

$$H_n = \max_{1 \leq i < j \leq k} |U_i(n) \wedge U_j(n)| \Rightarrow \max_{1 \leq i < j \leq k} |K_{i,j}| =: H,$$

where each of the $K_{i,j}$ (which are not independent) has the same distribution as $K_{1,2}$. The random variable H is almost surely finite since all the $K_{i,j}$'s are.

Finally, for all $n \geq 1$, $x_1, \dots, x_k \in [0, \infty)$ and $h \in \{0, 1, \dots\}$,

$$\begin{aligned} \mathbb{P} \left(\frac{S_1(n)}{n} \geq x_1, \dots, \frac{S_k(n)}{n} \geq x_k, H_n = h \mid \tau_n, \mathbf{Y} \right) \\ = \sum_{w_1, \dots, w_k \in \mathcal{D}_m^{(h)}} \prod_{i=1}^k \frac{s_{w_i}(n)}{n} \mathbf{1}_{\{s_{w_i}(n) \geq x_i n\}}, \end{aligned}$$

where $\mathcal{D}_m^{(h)}$ is the set of all distinct $w_1, \dots, w_k \in \{1, \dots, m\}^h$ such that the cardinality of the set $\{\tilde{w}_1, \dots, \tilde{w}_k\}$ is at most $k - 1$ (where we recall that \tilde{w} denotes the parent of a node w). By the strong law of large numbers, we thus get

$$\begin{aligned} \mathbb{P}\left(\frac{S_1(n)}{n} \geq x_1, \dots, \frac{S_k(n)}{n} \geq x_k, H_n = h \mid \tau_n, \mathbf{Y}\right) \\ \rightarrow \sum_{w_1, \dots, w_k \in \mathcal{D}_m^{(h)}} \prod_{i=1}^k \left(\prod_{u \preceq w_i} Z_u \right) \mathbf{1} \left\{ \prod_{u \preceq w_i} Z_u \geq x_i \right\}, \end{aligned}$$

and by dominated convergence,

$$\begin{aligned} \mathbb{P}\left(\frac{S_1(n)}{n} \geq x_1, \dots, \frac{S_k(n)}{n} \geq x_k, H_n = h\right) \\ \rightarrow \mathbb{E} \left[\sum_{w_1, \dots, w_k \in \mathcal{D}_m^{(h)}} \prod_{i=1}^k \left(\prod_{u \preceq w_i} Z_u \right) \mathbf{1} \left\{ \prod_{u \preceq w_i} Z_u \geq x_i \right\} \right]. \end{aligned}$$

To check that the right-hand side is indeed a probability distribution, take $x_1 = \dots = x_k = 0$ on both sides: The left-hand side is equal to $\mathbb{P}(H_n = h)$, which, by the first half of the proof, converges almost surely to H , an almost surely finite random variable. The right-hand side is thus equal to $\mathbb{P}(H = h)$, which satisfies $\sum_{h \geq 0} \mathbb{P}(H = h) = 1$. This concludes the proof that

$$\left(\frac{S_1(n)}{n}, \dots, \frac{S_k(n)}{n}, H_n\right) \Rightarrow (\alpha_1, \dots, \alpha_k, H),$$

where the joint distribution of $(\alpha_1, \dots, \alpha_k, H)$ is given by

$$\begin{aligned} \mathbb{P}(\alpha_1 \geq x_1, \dots, \alpha_k \geq x_k, H = h) \\ = \mathbb{E} \left[\sum_{w_1, \dots, w_k \in \mathcal{D}_m^{(h)}} \prod_{i=1}^k \left(\prod_{u \preceq w_i} Z_u \right) \mathbf{1} \left\{ \prod_{u \preceq w_i} Z_u \geq x_i \right\} \right], \end{aligned}$$

for all $x_1, \dots, x_k \geq 0$ and $h \in \{0, 1, 2, \dots\}$.

To see that $\alpha_j > 0$ almost surely for all $j \in \{1, \dots, k\}$, note that

$$\mathbb{P}(\alpha_j = 0) = \sum_{h \geq 0} \mathbb{E} \left[\sum_{w_1, \dots, w_k \in \mathcal{D}_m^{(h)}} \mathbf{1} \left\{ \prod_{u \preceq w_j} Z_u = 0 \right\} \prod_{i=1}^k \left(\prod_{u \preceq w_i} Z_u \right) \right] = 0,$$

because, if $\prod_{u \preceq w_j} Z_u = 0$ then $\prod_{i=1}^k (\prod_{u \preceq w_i} Z_u) = 0$ and thus

$$\mathbf{1} \left\{ \prod_{u \preceq w_j} Z_u = 0 \right\} \prod_{i=1}^k \left(\prod_{u \preceq w_i} Z_u \right) = 0.$$

Since this is true for all $1 \leq j \leq k$, we indeed have that the α_j 's are almost surely positive. □

Proof of Lemma 2.1. — We first prove the convergence of the marginals: let k_n be an integer chosen uniformly at random in $\{1, \dots, n\}$, then $\xi(k_n) = U_1(n)$ in

distribution; recall that by definition, for all $n \geq 1$, $\xi(n)$ is the unique node that belongs to τ_n but not to τ_{n-1} . By Lemma 2.2, in distribution when $n \rightarrow +\infty$,

$$\frac{|\xi(n)| - (\log n)/\mu}{\sqrt{(\log n)/\mu^3}} \Rightarrow \mathcal{N}(0, \sigma^2).$$

Therefore, since $\log(k_n/n)$ converges to $\log U$ in distribution as $n \rightarrow +\infty$, where U is uniform on $[0, 1]$, we get

$$(2.6) \quad \frac{|\xi(k_n)| - (\log n)/\mu}{\sqrt{(\log n)/\mu^3}} \Rightarrow \mathcal{N}(0, \sigma^2),$$

which immediately entails the convergence of the marginals to the desired limit. To show that the limits are independent, we use Lemma 2.3, and the fact that, by definition of the model, given $H_n, S_1(n), \dots, S_k(n)$, the trees rooted at $U_1^{(H_n)}(n), \dots, U_k^{(H_n)}(n)$ are independent split trees of split distribution ν and of respective sizes $S_1(n), \dots, S_k(n)$. Moreover, for all $1 \leq i \leq k$, the node $U_i(n)$ is distributed uniformly at random among the nodes of the split tree rooted at $U_i^{(H_n)}(n)$. Therefore, given $H_n, S_1(n), \dots, S_k(n)$, we have, in distribution and jointly for all $1 \leq i \leq k$,

$$|U_i(n)| = H_n + |\widehat{U}^{(i)}(S_i(n))|,$$

where the $\widehat{U}^{(i)}$'s are independent, and for all i , $\widehat{U}^{(i)}(S_i(n))$ is a node taken uniformly at random in a split tree of size $S_i(n)$. As a consequence, applying (2.6) to each of the k independent split trees, we get that, in distribution and jointly for all $1 \leq i \leq k$,

$$\begin{aligned} \frac{|U_i(n)| - (\log n)/\mu}{\sqrt{(\log n)/\mu^3}} &= \frac{H_n + |\widehat{U}^{(i)}(S_i(n))| - (\log n)/\mu}{\sqrt{(\log n)/\mu^3}} \\ &= \frac{|\widehat{U}^{(i)}(S_i(n))| - (\log S_i(n))/\mu}{\sqrt{(\log S_i(n))/\mu^3}} \cdot \sqrt{\frac{\log S_i(n)}{\log n}} + \frac{\log S_i(n) - \log n}{\sqrt{\mu \log n}} + o_{\mathbb{P}}(1) \\ &\Rightarrow \Lambda_i, \end{aligned}$$

where $(\Lambda_1, \dots, \Lambda_k)$ are k independent centred Gaussians of variance σ^2 ; we have used the fact that by Lemma 2.3, $\log S_i(n) = \log n + \mathcal{O}_{\mathbb{P}}(1)$ when $n \rightarrow +\infty$. \square

Applying the law of large numbers to the i.i.d. edge lengths, and using the fact that, by Lemma 2.3, the height H_n of the last common ancestor of $U_1(n), \dots, U_k(n)$ converges in distribution to an almost surely finite random variable, Lemma 2.1 entails a similar result for the distances of $U_1(n), \dots, U_k(n)$ to the root (for the distance $d_{\mathbf{L}}$). In the following, L denotes a random variable distributed according to ϖ .

LEMMA 2.4 (CLT for distances to the root). — *Under the assumptions of Theorem 1.7, if $\text{Var}(L) < +\infty$, then, in distribution when $n \rightarrow +\infty$, we have*

$$\left(\frac{|U_1(n)|_{\mathbf{L}} - (\log n)\mathbb{E}L/\mu}{\sqrt{(\log n)/\mu^3}}, \dots, \frac{|U_k(n)|_{\mathbf{L}} - (\log n)\mathbb{E}L/\mu}{\sqrt{(\log n)/\mu^3}} \right) \Rightarrow (\Xi_1, \dots, \Xi_k),$$

where the Ξ_1, \dots, Ξ_k are independent centred Gaussian random variables of variance $\text{Var}(L) + \sigma^2(\mathbb{E}L)^2$.

Proof. — In this proof, we set $U_i[H_n] = U_i^{(H_n)}(n)$, i.e. the ancestor of $U_i(n)$ at height H_n , where H_n is defined in Lemma 2.3. For all $1 \leq i \leq k$, we have

$$(2.7) \quad |U_i(n)|_{\mathbf{L}} = \sum_{\emptyset \neq u \preceq U_i(n)} L_u = \sum_{\emptyset \neq u \preceq U_i[H_n]} (L_u - L_u^{(i)}) + A_i(n),$$

where the $\mathbf{L}^{(i)} = (L_u^{(i)})_{u \in \mathcal{D}_m}$ are i.i.d. copies of \mathbf{L} , and where

$$A_i(n) := \sum_{\emptyset \neq u \preceq U_i[H_n]} L_u^{(i)} + \sum_{U_i[H_n] \prec u \preceq U_i(n)} L_u.$$

Since, by Lemma 2.3, H_n converges in distribution to an almost surely finite random variable H , we have

$$(2.8) \quad \frac{1}{\sqrt{\log n}} \sum_{\emptyset \neq u \preceq U_i[H_n]} (L_u - L_u^{(i)}) \rightarrow 0,$$

in distribution when $n \rightarrow +\infty$. Note that, given H_n , the random variables $A_1(n), \dots, A_k(n)$ are independent, because the L_u 's are independent, and the sums in $A_1(n), \dots, A_k(n)$ that involve the sequence \mathbf{L} (as opposed to its i.i.d. copies $\mathbf{L}^{(1)}, \dots, \mathbf{L}^{(k)}$) range over distinct nodes u . Therefore, in distribution, we have, jointly for all $1 \leq i \leq k$,

$$A_i(n) = \sum_{\emptyset \neq u \preceq U_i(n)} L_u^{(i)} = \sum_{j=1}^{|U_i(n)|} \tilde{L}_j^{(i)},$$

where $\mathbf{L}^{(i)} = (\tilde{L}_j^{(i)})_{j \geq 1}$ is a sequence of i.i.d. copies of the $L_u^{(i)}$'s, and the k sequences $(\mathbf{L}^{(i)})_{1 \leq i \leq k}$ are independent of each other. By the central limit theorem, we have, jointly for all $1 \leq i \leq k$,

$$\frac{\sum_{j=1}^m \tilde{L}_j^{(i)} - m\mathbb{E}L}{\sqrt{m}} \Rightarrow \Theta_i,$$

in distribution when $n \rightarrow +\infty$, where $\Theta_1, \dots, \Theta_k$ are independent centred Gaussian random variables of variance $\text{Var}(L)$. Since $|U_i(n)| \rightarrow +\infty$ in probability when $n \rightarrow +\infty$, and since $(|U_i(n)|)_{1 \leq i \leq k}$ is independent from $(L_j^{(i)} : j \geq 1)_{1 \leq i \leq k}$, this implies that, jointly for all $1 \leq i \leq k$,

$$(2.9) \quad \frac{A_i(n) - |U_i(n)|\mathbb{E}L}{\sqrt{|U_i(n)|}} \Rightarrow \Theta_i.$$

Indeed, for all $u_1, \dots, u_k \in \mathbb{R}$, and $\varepsilon > 0$, there exists $m_0 > 0$ such that, for all $m_1, \dots, m_k \geq m_0$,

$$\left| \mathbb{P} \left(\frac{\sum_{j=1}^{m_i} \tilde{L}_j^{(i)} - m_i\mathbb{E}L}{\sqrt{m_i}} \leq u_i, \forall 1 \leq i \leq k \right) - \mathbb{P}(\Theta_i \leq u_i, \forall 1 \leq i \leq k) \right| \leq \varepsilon/2.$$

Because $U_i(n) \rightarrow +\infty$ in probability, there exists n_0 such that, for all $n \geq n_0$, $\mathbb{P}(\inf_{1 \leq i \leq k} |U_i(n)| \geq m_0) \geq 1 - \varepsilon/2$. Therefore, for all $u_1, \dots, u_k \in \mathbb{R}$, $\varepsilon > 0$, and $n \geq n_0$,

$$\begin{aligned}
& \left| \mathbb{P} \left(\frac{A_i(n) - |U_i(n)|\mathbb{E}L}{\sqrt{|U_i(n)|}} \leq u_i, \forall 1 \leq i \leq k \right) - \mathbb{P}(\Theta_i \leq u_i, \forall 1 \leq i \leq k) \right| \\
& \leq \left| \mathbb{P} \left(\frac{A_i(n) - |U_i(n)|\mathbb{E}L}{\sqrt{|U_i(n)|}} \leq u_i, \forall 1 \leq i \leq k \text{ and } \inf_{1 \leq i \leq k} |U_i(n)| \geq m_0 \right) \right. \\
& \qquad \qquad \qquad \left. - \mathbb{P}(\Theta_i \leq u_i, \forall 1 \leq i \leq k) \right| + \frac{\varepsilon}{2} \\
& \leq \sum_{\ell=m_0}^{\infty} \left| \mathbb{P} \left(\frac{A_i(n) - \ell\mathbb{E}L}{\sqrt{\ell}} \leq u_i, \forall 1 \leq i \leq k \right) - \mathbb{P}(\Theta_i \leq u_i, \forall 1 \leq i \leq k) \right| \\
& \qquad \qquad \qquad \times \mathbb{P} \left(\inf_{1 \leq i \leq k} |U_i(n)| = \ell \right) + \frac{\varepsilon}{2} \leq \varepsilon,
\end{aligned}$$

where, in the second inequality, we have conditioned on the different possible values of $\inf_{1 \leq i \leq k} |U_i(n)|$, and used the triangular inequality. This concludes the proof of (2.9).

We thus get that, jointly for all $1 \leq i \leq k$,

$$\begin{aligned}
& \frac{|U_i(n)|_{\mathbf{L}} - (\log n)\mathbb{E}L/\mu}{\sqrt{(\log n)/\mu^3}} \\
& = \frac{|U_i(n)|_{\mathbf{L}} - |U_i(n)|\mathbb{E}L}{\sqrt{\text{Var}(L)|U_i(n)|}} \cdot \sqrt{\frac{\text{Var}(L)|U_i(n)|}{(\log n)/\mu^3}} + \frac{|U_i(n)|\mathbb{E}L - (\log n)\mathbb{E}L/\mu}{\sqrt{(\log n)/\mu^3}} \\
& \Rightarrow \Theta_i \sqrt{\text{Var}(L)} + \Lambda_i \mathbb{E}L,
\end{aligned}$$

where we have used Lemma 2.1 (where $\Lambda_1, \dots, \Lambda_k$ are defined). Note that, by definition, Θ_i is independent from Λ_i for all $1 \leq i \leq k$; indeed, Θ_i is $\mathbf{L}^{(i)}$ -measurable, while Λ_i is $(\tau_n)_{n \geq 1}$ -measurable. Therefore, $\Xi_i := \Theta_i \sqrt{\text{Var}(L)} + \Lambda_i \mathbb{E}L$ is a centred Gaussian of variance $\text{Var}(L) + \sigma^2 \mathbb{E}(L)^2$, and Ξ_1, \dots, Ξ_k are independent, as claimed. \square

We now look at the respective version of Lemma 2.4 when the edge lengths have heavier tails:

LEMMA 2.5. — *Under the assumptions of Theorem 1.7, if there exists $\alpha \in (1, 2)$ and a function $\ell: [0, +\infty) \rightarrow [0, +\infty)$ slowly varying at infinity such that $\mathbb{P}(L \geq x) = x^{-\alpha} \ell(x)$ for all $x \geq 0$, then, in distribution when $n \rightarrow +\infty$,*

$$\left(\frac{|U_1(n)|_{\mathbf{L}} - (\log n)\mathbb{E}L/\mu}{((\log n)/\mu)^{1/\alpha}}, \dots, \frac{|U_k(n)|_{\mathbf{L}} - (\log n)\mathbb{E}L/\mu}{((\log n)/\mu)^{1/\alpha}} \right) \Rightarrow (\Upsilon_1(\alpha), \dots, \Upsilon_k(\alpha)),$$

where the $\Upsilon_1(\alpha), \dots, \Upsilon_k(\alpha)$ are i.i.d. copies of a centred α -stable random variable.

Proof. — We proceed as in the proof of Lemma 2.4 and employ the same notation: in particular, using that our assumptions on L entail its expectation being finite, (2.7) and (2.8) give that, in distribution when $n \rightarrow +\infty$,

$$(2.10) \quad |U_i(n)|_{\mathbf{L}} = A_i(n) + o_{\mathbb{P}}\left(\sqrt{\log n}\right) = A_i(n) + o_{\mathbb{P}}\left((\log n)^{1/\alpha}\right).$$

We apply the functional limit theorem for sums of i.i.d. heavy-tailed random variables (see, e.g., [GK54, § 35, Theorem 2]): to do this, it is enough to check that

- (i) $\frac{\mathbb{P}(L \leq -x)}{1 - \mathbb{P}(L \leq x)} \rightarrow 0$ as $x \rightarrow +\infty$, and
- (ii) for all constant $c > 0$, $\frac{1 - \mathbb{P}(L \leq x) + \mathbb{P}(L \leq -x)}{1 - \mathbb{P}(L \leq cx) + \mathbb{P}(L \leq -cx)} \rightarrow c^\alpha$ as $x \rightarrow +\infty$.

Since L is almost surely non-negative, we have $\mathbb{P}(L \leq -x) = 0$, which straightforwardly implies (i). For (ii), we use the fact that, by assumption, $\mathbb{P}(L > x) = x^{-\alpha} \ell(x)$, where ℓ is slowly varying at infinity: this implies

$$\frac{1 - \mathbb{P}(L \leq x) + \mathbb{P}(L \leq -x)}{1 - \mathbb{P}(L \leq cx) + \mathbb{P}(L \leq -cx)} = \frac{\mathbb{P}(L > x)}{\mathbb{P}(L > cx)} = c^\alpha \frac{\ell(x)}{\ell(cx)} \rightarrow c^\alpha \quad \text{as } x \rightarrow +\infty,$$

because ℓ is slowly varying at infinity. Thus, [GK54, § 25, Theorem 2] applies, which implies that

$$\frac{A_i(n) - |U_i(n)|_{\mathbb{E}L}}{|U_i(n)|^{1/\alpha}} \Rightarrow \Upsilon_i(\alpha),$$

where $\Upsilon_1(\alpha), \dots, \Upsilon_k(\alpha)$ are i.i.d. copies of an α -stable random variable. We thus get that

$$\begin{aligned} & \frac{|U_i(n)|_{\mathbf{L}} - |U_i(n)|_{\mathbb{E}L}}{|U_i(n)|^{1/\alpha}} \\ &= \frac{A_i(n) - |U_i(n)|_{\mathbb{E}L}}{|U_i(n)|^{1/\alpha}} + \frac{|U_i(n)|_{\mathbf{L}} - A_i(n)}{((\log n)/\mu)^{1/\alpha}} \cdot \frac{((\log n)/\mu)^{1/\alpha}}{|U_i(n)|^{1/\alpha}} \Rightarrow \Upsilon_i(\alpha), \end{aligned}$$

in distribution when $n \rightarrow +\infty$, where we have used (2.10) and the fact that, by Lemma 2.1, $\frac{|U_i(n)|}{(\log n)/\mu} \rightarrow 1$ in probability when $n \rightarrow +\infty$. We thus get that, jointly for all $1 \leq i \leq k$,

$$(2.11) \quad \begin{aligned} & \frac{|U_i(n)|_{\mathbf{L}} - (\log n)\mathbb{E}L/\mu}{((\log n)/\mu)^{1/\alpha}} \\ &= \frac{|U_i(n)|_{\mathbf{L}} - |U_i(n)|_{\mathbb{E}L}}{|U_i(n)|^{1/\alpha}} \cdot \left(\frac{|U_i(n)|}{(\log n)/\mu}\right)^{1/\alpha} + \frac{|U_i(n)|_{\mathbb{E}L} - (\log n)\mathbb{E}L/\mu}{((\log n)/\mu)^{1/\alpha}} \\ &\Rightarrow \Upsilon_i(\alpha), \end{aligned}$$

in distribution when $n \rightarrow +\infty$; here, we have used Lemma 2.1 and the fact that $\alpha < 2$, which implies $1/\alpha > 1/2$ (and $\alpha > 1$ again to get the finiteness of $\mathbb{E}L$). \square

Remark 2.6. — Note that in the α -stable case, the second summand on the right-hand side of (2.11) is negligible compared to the first summand, i.e. the fluctuations coming from the height of $U_i(n)$ are asymptotically negligible in front of the fluctuations coming from the edge-lengths.

Next we control the sizes of subtrees rooted at certain nodes within the tree. For this purpose, imagine that $U_1(n)$ is the closest to the root (in graph distance) among $U_1(n), \dots, U_k(n)$. Then the Voronoi cell of $U_2(n)$ is the subtree rooted at the ancestor of $U_2(n)$ that has height

$$|U_1(n) \wedge U_2(n)|_{\mathbf{L}} + \left| |U_1(n)| - |U_2(n)| \right|_{\mathbf{L}} / 2.$$

From Lemma 2.3, we already know that $|U_1(n) \wedge U_2(n)|$ converges in distribution to an almost surely finite random variable. The following lemma gives a limiting result for the size of the subtree rooted at an ancestor of $U_2(n)$ at height $h(n)$ for some function $h(n) = o(\log n)$ when $n \rightarrow +\infty$.

For $f: \mathbb{N} \rightarrow \mathbb{R}$, $x \in [0, \infty)$, $n \geq 1$, and $1 \leq i \leq k$, we write $D_n^{(i)}(xf)$ (or $D_n^{(i)}(xf(n))$) for the size of the subtree of τ_n rooted at the ancestor of $U_i(n)$ closest to the root whose \mathbf{L} -distance to the root is at least

$$(2.12) \quad xF_i(n) := \min(xf(n), |U_i(n)|_{\mathbf{L}}).$$

By Lemmas 2.4 and 2.5, $|U_i(n)|_{\mathbf{L}}$ grows logarithmically in n ; therefore, if $f(n) = o(\log n)$, then for large n we typically have $F_i(n) = f(n)$.

PROPOSITION 2.7 (Convergence of (extended) fringe trees). — *Let $f: \mathbb{N} \rightarrow \mathbb{N}$ be a function such that $\lim_{n \rightarrow +\infty} f(n) = +\infty$, and $x > 0$. We assume that either $f(n) = o(\log n)$ when $n \rightarrow +\infty$, or $f(n) = \log n$ for all $n \geq 1$ and set*

$$C(f) = \begin{cases} +\infty & \text{if } f(n) = o(\log n), \\ \mathbb{E}L/\mu & \text{if } f(n) = \log n. \end{cases}$$

Then, under the assumptions of Theorem 1.7, for all $1 \leq i \leq k$ and for all $0 < a \leq b < C(f)$,

$$(2.13) \quad \sup_{x \in [a, b]} \left| \frac{\log(D_n^{(i)}(xf)/n)}{xf(n)} - \frac{\mathbb{E}[\log \bar{Y}]}{\mathbb{E}L} \right| \rightarrow 0 \quad \text{in probability when } n \rightarrow +\infty.$$

This lemma is at the heart of the proof of Theorem 1.7: it establishes a law of large numbers for the logarithm of the size of fringe trees. A fringe tree, as defined in [HJ17] (see also the references therein for a literature review on the subject), is the subtree rooted at a node taken uniformly at random among the n -nodes of a random tree (in our case the n -node split tree of split distribution ν). An extended fringe tree (still following [HJ17]) is the subtree rooted at one of the ancestors of this randomly chosen node, under the assumption that this ancestor is at a *fixed* graph distance of the randomly chosen node. In Proposition 2.7, however, we also consider subtrees rooted at an ancestor of a node $U(n) = U_i(n)$ chosen uniformly at random in our tree, but this ancestor can be at a distance that grows with n .

Proof of Proposition 2.7. — We defined the split tree $(\tau_n)_{n \geq 1}$ as a sequence of random trees, with $\xi(n+1)$ denoting the unique node in τ_{n+1} but not in τ_n . Now let \tilde{U} be a uniform random variable on $[0, 1]$, set $k_n = \lceil \tilde{U}n \rceil \in \{1, \dots, n\}$ for all $n \geq 1$, and note that

$$(2.14) \quad n - k_n \rightarrow +\infty \quad \text{almost surely as } n \rightarrow +\infty.$$

We fix $i \in \{1, \dots, k\}$ throughout the proof. Letting $U_i(n)$ be the node of index k_n , we observe that $U_i(n)$ is indeed uniformly distributed in τ_n , as required.

We fix $0 < a < b < C(f)$. For $x \in [a, b]$, we define $h = h(n)$ so that $xh(n)$ is the height of the ancestor of $U_i(n)$ closest to the root whose \mathbf{L} -distance to the root is at least $x F_i(n) = \min(xf(n), |U_i(n)|_{\mathbf{L}})$ (recall (2.12)). As a consequence of Lemmas 2.5 and 2.4, respectively, $F_i(n) \rightarrow +\infty$ in probability when $n \rightarrow +\infty$, therefore

$$(2.15) \quad \frac{F_i(n)}{h(n)} \rightarrow \mathbb{E}L \quad \text{in probability when } n \rightarrow +\infty$$

by the law of large numbers.

Recall that $U_i^{(xh(n))}(n)$ denotes the ancestor of $U_i(n)$ closest to the root whose height is at least $xh(n)$; we denote by $k_n(x)$ the integer such that $U_i^{(xh(n))}(n) = \xi(k_n(x))$. By definition, we have $k_n(x) \leq k_n$. We next derive a law of large numbers for the width of the split interval associated to the node of index $k_n(x)$. Recall that, by definition of $(\tau_n)_{n \geq 1}$, to each node $w \in \tau_n$ (among which $\xi(k_n)$) is associated a sub-interval of $[0, 1]$, whose length is given by $\prod_{\emptyset \neq u \preceq w} Z_u$. We let

$$(2.16) \quad Q_n(x) := \prod_{\emptyset \neq u \preceq \xi(k_n(x))} Z_u$$

be the length of the interval associated to node $\xi(k_n(x))$. We claim the following law of large numbers for $Q_n(x)$:

$$(2.17) \quad \frac{\log Q_n(x)}{xh(n)} \rightarrow \mathbb{E}[\log \bar{Y}] = -\mu \quad \text{in probability when } n \rightarrow +\infty.$$

Indeed, first note that the random variables Z_u are sized-biased since we condition on the event $u \preceq \xi(k_n)$; more precisely, we condition the intervals associated to the nodes u occurring in the product of (2.16) to contain X_{k_n} . In other words, conditionally on $u \preceq \xi(k_n)$, we have $Z_u = \bar{Y}$ in distribution, where \bar{Y} as in (1.2) and $\mathbf{Y} = (Y_1, \dots, Y_m)$ a random vector of distribution ν . Therefore, by the law of large numbers, since $h(n) \rightarrow +\infty$ in probability (cf. (2.15)), we get

$$\frac{\log Q_n(x)}{xh(n)} = \frac{1}{xh(n)} \log \left(\prod_{\emptyset \neq u \preceq \xi(k_n(x))} Z_u \right) = \frac{1}{xh(n)} \sum_{\emptyset \neq u \preceq \xi(k_n(x))} \log Z_u \rightarrow \mathbb{E}[\log \bar{Y}],$$

in probability when $n \rightarrow +\infty$, which concludes the proof of (2.17).

The main step now is to show

$$(2.18) \quad \sup_{x \in [a, b]} \left| \frac{\log(D_n^{(i)}(xf)/n)}{xh(n)} - \mathbb{E}[\log \bar{Y}] \right| \rightarrow 0 \quad \text{in probability when } n \rightarrow +\infty,$$

At the end of the proof, we show that this implies (2.13).

To prove (2.18), we rewrite its left-hand side as a sum of several terms to which we will apply various concentration inequalities. First note that, for all $x \in [a, b]$, for all $n \geq 1$,

$$\begin{aligned}
(2.19) \quad & \frac{1}{xh(n)} \left| \log \left(\frac{D_n^{(i)}(xf)}{n} \right) - xh(n) \mathbb{E} [\log \bar{Y}] \right| \\
& \leq \frac{1}{ah(n)} \left| \log \frac{D_n^{(i)}(xf)}{(n - k_n(x))Q_n(x)} \right| + \frac{1}{ah(n)} \left| \log \left(1 - \frac{k_n(x)}{n} \right) \right| \\
& \quad + \left| \frac{\log Q_n(x)}{xh(n)} - \mathbb{E} [\log \bar{Y}] \right|.
\end{aligned}$$

We show that the right hand side converges to zero in probability as $n \rightarrow +\infty$ uniformly for $x \in [a, b]$ by treating each of the three summands separately.

We start with the second term, which is the easiest. Recall that $k_n(x) \leq k_n$ and that $k_n = \lceil \tilde{U}n \rceil$, so

$$(2.20) \quad \sup_{x \in [a, b]} \left| \log \left(1 - \frac{k_n(x)}{n} \right) \right| \leq \left| \log \left(1 - \frac{k_n}{n} \right) \right| \rightarrow -\log(1 - \tilde{U}),$$

almost surely as $n \rightarrow +\infty$.

For the third term on the right-hand side of (2.19), we proceed as follows. In distribution, $(Q_n(x))_{x \in [a, b]} = (\sum_{\ell \leq xh(n)} \log \tilde{Z}_\ell)_{x \in [a, b]}$, where $(\tilde{Z}_\ell)_{\ell \geq 1}$ is a sequence of i.i.d. copies of \bar{Y} . We apply the law of the iterated logarithm to the sequence of i.i.d. random variables $(\log \tilde{Z}_\ell)_{\ell \geq 1}$, this gives

$$\limsup_{m \rightarrow +\infty} \frac{\left| \sum_{\ell=1}^m \log \tilde{Z}_\ell - m \mathbb{E} [\log \bar{Y}] \right|}{\sqrt{m \log \log m}} = \text{Var} (\log \bar{Y})^{1/2} = \sigma \quad \text{almost surely.}$$

This implies in particular that there exists an almost surely finite random number m_0 such that almost surely,

$$\sup_{m \geq m_0} \frac{\left| \sum_{\ell=1}^m \log \tilde{Z}_\ell - m \mathbb{E} [\log \bar{Y}] \right|}{\sqrt{m \log \log m}} \leq 2\sigma.$$

We fix $\varepsilon > 0$ and choose $m_1 \geq m_0$ such that $2\sigma \sqrt{\log \log(m_1)/m_1} \leq \varepsilon$. We have, almost surely

$$\begin{aligned}
\sup_{m \geq m_1} \left| \frac{\sum_{\ell=1}^m \log \tilde{Z}_\ell}{m} - \mathbb{E} [\log \bar{Y}] \right| & \leq \sqrt{\frac{\log \log m_1}{m_1}} \sup_{m \geq m_1} \frac{\left| \sum_{\ell=1}^m \log \tilde{Z}_\ell - m \mathbb{E} [\log \bar{Y}] \right|}{\sqrt{m \log \log m}} \\
& \leq \sqrt{\frac{\log \log m_1}{m_1}} \sup_{m \geq m_0} \frac{\left| \sum_{\ell=1}^m \log \tilde{Z}_\ell - m \mathbb{E} [\log \bar{Y}] \right|}{\sqrt{m \log \log m}} \leq \varepsilon.
\end{aligned}$$

Consequently, we have for all $n \geq 1$ that

$$\begin{aligned} & \mathbb{P} \left(\sup_{x \in [a,b]} \left| \frac{\log Q_n(x)}{xh(n)} - \mathbb{E} [\log \bar{Y}] \right| \geq \varepsilon \right) \\ &= \mathbb{P} \left(\sup_{x \in [a,b]} \left| \frac{\sum_{\ell=1}^{xh(n)} \log \tilde{Z}_\ell}{xh(n)} - \mathbb{E} [\log \bar{Y}] \right| \geq \varepsilon \right) \\ &\leq \mathbb{P} \left(\sup_{x \in [a,b]} \left| \frac{\sum_{\ell=1}^{xh(n)} \log \tilde{Z}_\ell}{xh(n)} - \mathbb{E} [\log \bar{Y}] \right| \geq \varepsilon \text{ and } ah(n) \geq m_1 \right) \\ &\qquad\qquad\qquad + \mathbb{P}(ah(n) < m_1) \\ &\leq \mathbb{P} \left(\sup_{m \geq m_1} \left| \frac{\sum_{\ell=1}^m \log \tilde{Z}_\ell}{m} - \mathbb{E} [\log \bar{Y}] \right| \geq \varepsilon \right) + \mathbb{P}(ah(n) < m_1) \\ &= \mathbb{P}(ah(n) < m_1) \rightarrow 0, \end{aligned}$$

as $n \rightarrow +\infty$, because $h(n) \rightarrow +\infty$ in probability with n . In other words,

$$(2.21) \quad \sup_{x \in [a,b]} \left| \frac{\log Q_n(x)}{xh(n)} - \mathbb{E} [\log \bar{Y}] \right| \rightarrow 0 \quad \text{in probability as } n \rightarrow +\infty.$$

Finally, we deal with the first term in the right-hand side of (2.19) and aim to prove

$$(2.22) \quad \sup_{x \in [a,b]} \left| \log \frac{D_n^{(i)}(xf)}{(n - k_n(x)) Q_n(x)} \right| \rightarrow 0 \quad \text{in probability as } n \rightarrow +\infty,$$

Mind that inserting (2.20), (2.21) and (2.22) into (2.19) implies (2.18). In order to prove (2.22), we first recall that, for all $x \in [a, b]$ conditionally on $k_n(x)$ and $\mathbf{Z} = (Z_u)_{u \in \mathcal{D}_m}$, we have, in distribution,

$$D_n^{(i)}(xf) = \sum_{\ell=k_n(x)+1}^n \mathbf{1}_{X_\ell < Q_n(x)}.$$

Thus, for all $x \in [a, b]$, $\varepsilon \in (0, 1)$, $\lambda \geq 0$, the exponential Chebychev inequality yields

$$\begin{aligned} & \mathbb{P} \left(\sum_{\ell=k_n(x)}^n \mathbf{1}_{X_\ell < Q_n(x)} \geq (1 + \varepsilon) Q_n(x) (n - k_n(x)) \mid k_n(x), \mathbf{Z} \right) \\ &\leq e^{-\lambda(1+\varepsilon)Q_n(x)(n-k_n(x))} \prod_{\ell=k_n(x)}^n (1 + Q_n(x) (e^\lambda - 1)) \\ &= \exp \left(- Q_n(x)(n - k_n(x)) \left((1 + \varepsilon)\lambda - e^\lambda + 1 \right) \right). \end{aligned}$$

Taking $\lambda = \log(1 + \varepsilon)$, this yields the upper bound

$$\begin{aligned} \mathbb{P} \left(\sum_{\ell=k_n(x)}^n \mathbf{1}_{X_\ell < Q_n(x)} \geq (1 + \varepsilon)Q_n(x)(n - k_n(x)) \middle| k_n(x), \mathbf{Z} \right) \\ \leq \exp \left(- Q_n(x)(n - k_n(x))\varepsilon' \right), \end{aligned}$$

where we have set $\varepsilon' := (1 + \varepsilon) \log(1 + \varepsilon) - \varepsilon > 0$. Using the fact that, for all $x \in [a, b]$, $Q_n(x) \geq Q_n(b)$, and $k_n(x) \leq k_n$, taking expectations on both sides, and then a supremum over $x \in [a, b]$, we infer that

$$\begin{aligned} \sup_{x \in [a, b]} \mathbb{P} \left(\sum_{\ell=k_n(x)}^n \mathbf{1}_{X_\ell < Q_n(x)} \geq (1 + \varepsilon)Q_n(x)(n - k_n(x)) \right) \\ \leq \mathbb{E} \left[\exp \left(- Q_n(b)(n - k_n)\varepsilon' \right) \right] \end{aligned}$$

In a similar vein, one deduces

$$\begin{aligned} \sup_{x \in [a, b]} \mathbb{P} \left(\sum_{\ell=k_n(x)}^n \mathbf{1}_{X_\ell < Q_n(x)} \leq (1 - \varepsilon)Q_n(x)(n - k_n(x)) \right) \\ \leq \mathbb{E} \left[\exp \left(- Q_n(b)(n - k_n)\varepsilon'' \right) \right], \end{aligned}$$

where $\varepsilon'' = (1 - \varepsilon) \log(1 - \varepsilon) + \varepsilon$. In total, we thus get that, for all $\varepsilon > 0$,

$$\begin{aligned} (2.23) \quad \sup_{x \in [a, b]} \mathbb{P} \left(|D_n^{(i)}(xf) - (n - k_n(x))Q_n(x)| > \varepsilon Q_n(x)(n - k_n(x)) \right) \\ \leq 2\mathbb{E} \left[\exp \left(- c Q_n(b)(n - k_n) \right) \right], \end{aligned}$$

where $c := \min(\varepsilon', \varepsilon'') > 0$.

We now recall that (see (2.15)) $h(n) \sim F_i(n)/\mathbb{E}L$ in probability as $n \rightarrow +\infty$. In the case when $f(n) = o(\log n)$, since $|U_i(n)|_{\mathbf{L}}/\log n$ converges to $\mathbb{E}L/\mu$ in probability when $n \rightarrow +\infty$, and since $x F_i(n) = \min(xf(n), |U_i(n)|_{\mathbf{L}})$, we get

$$(2.24) \quad \frac{F_i(n)}{f(n)} \rightarrow 1 \quad \text{in probability as } n \rightarrow +\infty.$$

In the case when $f(n) = \log n$, since $|U_i(n)|_{\mathbf{L}}/\log n$ converges to $\mathbb{E}L/\mu$ in probability when $n \rightarrow +\infty$, and since $x \leq b < \mathbb{E}L/\mu$, we get $F_i(n) \sim \log n$ in probability when $n \rightarrow +\infty$. In both cases,

$$(2.25) \quad \frac{h(n)}{f(n)} \rightarrow \mathbf{c} := \frac{1}{\mathbb{E}L} \quad \text{in probability as } n \rightarrow +\infty.$$

For all $\delta > 0$ and for all $\eta > 0$ small enough such that $\sup_{x \in [1-\eta, 1+\eta]} |\log x| \leq \delta$, we have

$$\begin{aligned}
 (2.26) \quad & \mathbb{P} \left(\sup_{x \in [a, b]} \left| \log \frac{D_n^{(i)}(xf)}{(n - k_n(x))Q_n(x)} \right| \geq \delta \right) \\
 & \leq \mathbb{P} \left(\sup_{x \in [a, b]} \left| \frac{D_n^{(i)}(xf)}{(n - k_n(x))Q_n(x)} - 1 \right| \geq \eta \text{ and } \mathbf{c} - \eta \leq \frac{h(n)}{f(n)} \leq \mathbf{c} + \eta \right) \\
 & \quad + \mathbb{P} \left(\left| \mathbf{c} - \frac{h(n)}{f(n)} \right| > \eta \right),
 \end{aligned}$$

where we have set $\mathbf{c} = 1/\mathbb{E}L$. Because of (2.25), the second summand converges to 0 as $n \rightarrow +\infty$. Note that, as x increases between a and b , $D_n^{(i)}(xf)/(n - k_n(x))Q_n(x)$ only changes value when $xh(n)$ crosses an integer value.

Thus, the event inside the first probability on the right-hand side implies that there exists $\ell \in \mathbb{N} \cap [a(\mathbf{c} - \eta)f(n), b(\mathbf{c} + \eta)f(n)]$ such that $|D_n^{(i)}(xf)/(n - k_n(x))Q_n(x) - 1| \geq \eta$ for $x = \ell/h(n)$. We thus get via a union bound

$$\begin{aligned}
 (2.27) \quad & \mathbb{P} \left(\sup_{x \in [a, b]} \left| \frac{D_n^{(i)}(xf)}{(n - k_n(x))Q_n(x)} - 1 \right| \geq \eta \text{ and } \mathbf{c} - \eta \leq \frac{h(n)}{f(n)} \leq \mathbf{c} + \eta \right) \\
 & \leq \sum_{\ell = \lceil a(1-\eta)f(n) \rceil}^{\lfloor b(1+\eta)f(n) \rfloor} \mathbb{P} \left(\left| \frac{D_n^{(i)}(xf)}{(n - k_n(x))Q_n(x)} - 1 \right| \geq \eta \text{ for } x = \ell/h(n) \right. \\
 & \quad \left. \text{and } \mathbf{c} - \eta \leq \frac{h(n)}{f(n)} \leq \mathbf{c} + \eta \right) \\
 & \leq ([b(\mathbf{c} + \eta) - a(\mathbf{c} - \eta)]f(n) + 1) \sup_{x \in [\hat{a}, \hat{b}]} \mathbb{P} \left(\left| \frac{D_n^{(i)}(xf)}{(n - k_n(x))Q_n(x)} - 1 \right| \geq \eta \right) \\
 & \leq Kf(n)\mathbb{E} \left[\exp \left(-cQ_n(\hat{b})(n - k_n) \right) \right],
 \end{aligned}$$

where $\hat{a} = a(\mathbf{c} - \eta)/(\mathbf{c} - \eta)$ and $\hat{b} = b(\mathbf{c} + \eta)/(\mathbf{c} - \eta)$, and we used (2.23) in the last inequality. We have also set K large enough so that, for all $n \geq 1$, $([b(\mathbf{c} + \eta) - a(\mathbf{c} - \eta)]f(n) + 1) \leq Kf(n)$, which is possible since $f(n) \rightarrow +\infty$ with n .

Using the law of large numbers in (2.17), we get that, in probability as $n \rightarrow \infty$,

$$-cQ_n(\hat{b})(n - k_n) + \log f(n) = -ce^{(-\mu\hat{b}/\mathbb{E}L + o(1))f(n)}(n - k_n) + \log f(n) \rightarrow -\infty,$$

because either $f(n) = o(\log n)$, or $f(n) = \log n$ and $\eta > 0$ can be chosen small enough so that $\hat{b} < \mathbb{E}L/\mu$ (because, by assumption, $b < \mathbb{E}L/\mu$, and $\hat{b} = b(\mathbf{c} + \eta)/(\mathbf{c} - \eta)$). This implies that the right-hand side in (2.27) converges to zero as $n \rightarrow +\infty$. Using again that $h(n) \sim f(n)$ in probability when $n \rightarrow +\infty$, we get that the right-hand side of (2.26) also tends to zero with n , and thus (2.22) is true, which concludes the proof of (2.18).

It only remains to show that (2.18) implies (2.13); intuitively, this is true because of (2.25). Indeed, we have

$$\begin{aligned} & \sup_{x \in [a,b]} \left| \frac{\log(D_n^{(i)}(xf)/n)}{xf(n)} - \frac{\mathbb{E}[\log \bar{Y}]}{\mathbb{E}L} \right| \\ & \leq \sup_{x \in [a,b]} \left| \frac{\log(D_n^{(i)}(xf)/n)}{xh(n)\mathbb{E}L} - \frac{\mathbb{E}[\log \bar{Y}]}{\mathbb{E}L} \right| + \sup_{x \in [a,b]} \left| \frac{\log(D_n^{(i)}(xf)/n)}{xh(n)\mathbb{E}L} \left(1 - \frac{h(n)\mathbb{E}L}{f(n)}\right) \right| \\ & = \frac{o_{\mathbb{P}}(1)}{\mathbb{E}L} + \left|1 - \frac{h(n)\mathbb{E}L}{f(n)}\right| \sup_{x \in [a,b]} \left| \frac{\log(D_n^{(i)}(xf)/n)}{xh(n)\mathbb{E}L} \right| \\ & = o_{\mathbb{P}} \left(1 + \sup_{x \in [a,b]} \left| \frac{\log(D_n^{(i)}(xf)/n)}{xh(n)} \right| \right), \end{aligned}$$

where we have used (2.18), in the first equality, and (2.25) in the second one. By (2.18) and the triangular inequality, this last supremum goes to zero in probability when $n \rightarrow +\infty$, which concludes the proof of Proposition 2.7. \square

We are now ready to prove Theorem 1.7.

Proof of Theorem 1.7. — We let $U_{(1)}(n), \dots, U_{(k)}(n)$ be the nodes $U_1(n), \dots, U_k(n)$ ordered in increasing \mathbf{L} -distance to the root, that is, $|U_{(1)}(n)|_{\mathbf{L}} \leq \dots \leq |U_{(k)}(n)|_{\mathbf{L}}$, and let $V_1(n), \dots, V_k(n)$ be the sizes of their respective Voronoi cells (in that order, i.e. the Voronoi cell of $U_{(i)}$ is $V_i(n)$). We set $\mathbf{m} = \mathbb{E}[\log \bar{Y}]/\mathbb{E}L$ and start by showing that, in distribution when $n \rightarrow +\infty$,

$$(2.28) \quad \left(\frac{\log(V_2(n)/n)}{v_n}, \dots, \frac{\log(V_k(n)/n)}{v_n} \right) \Rightarrow \frac{\mathbf{m}}{2} (\Psi_{(2)} - \Psi_{(1)}, \dots, \Psi_{(k)} - \Psi_{(1)}),$$

where $\Psi_{(1)} \leq \dots \leq \Psi_{(k)}$ is the increasing order statistics of Ψ_1, \dots, Ψ_k , and with

$$(2.29) \quad v_n = \begin{cases} \sqrt{(\log n)/\mu^3} & \text{if } \text{Var}(L) < +\infty, \\ ((\log n)/\mu)^{1/\alpha} & \text{otherwise.} \end{cases}$$

Note that, because $\mathbf{m} = -\mathbb{E}[\log(1/\mathbb{E}\bar{Y})]/\mathbb{E}L = -\mu/\mathbb{E}L$, this is equivalent to

$$(2.30) \quad \left(\frac{\log(V_2(n)/n)}{(\log n)^{1/\alpha}}, \dots, \frac{\log(V_k(n)/n)}{(\log n)^{1/\alpha}} \right) \Rightarrow \frac{\mathbf{v}}{2\mathbb{E}L} (\Psi_{(1)} - \Psi_{(2)}, \dots, \Psi_{(1)} - \Psi_{(k)}),$$

where we recall that $\alpha := 2$ when $\text{Var}(L) < +\infty$, and where we have set

$$\mathbf{v} = \begin{cases} \mu^{-1/2} & \text{if } \text{Var}(L) < +\infty, \\ \mu^{1-1/\alpha} & \text{otherwise.} \end{cases}$$

We now show that (2.30) implies (1.3): the only difference between the two is that the entries in the left-hand side of (2.30) are ordered in increasing distance of the respective $U_i(n)$'s to the root, while those in the left-hand side of (1.3) are ordered in decreasing sizes of the Voronoi cells.

However, the convergence in (2.30) implies in particular that

$$\mathbb{P}(V_1(n) \geq V_2(n) \geq \dots \geq V_k(n)) = \mathbb{P}(V_i(n) = V_{(i)}(n), \forall i \in \{1, \dots, k\}) \rightarrow 1,$$

where we recall that, by definition, $V_{(i)}(n)$ is the i -th largest of the k Voronoi cells. We now let \mathcal{C}_n denote the event that $V_1(n) \geq V_2(n) \geq \dots \geq V_k(n)$: we have, for all $x_2, \dots, x_k < 0$,

$$\begin{aligned} \mathbb{P}\left(\forall i \in \{2, \dots, k\} : \frac{\log(V_{(i)}(n)/n)}{(\log n)^{1/\alpha}} \geq x_i\right) \\ = \mathbb{P}\left(\forall i \in \{2, \dots, k\} : \frac{\log(V_i(n)/n)}{(\log n)^{1/\alpha}} \geq x_i \text{ and } \mathcal{C}_n\right) + o(1) \end{aligned}$$

when $n \rightarrow +\infty$, because $\mathbb{P}(\mathcal{C}_n) \rightarrow 1$. Thus, (2.28) entails (1.3), and due to the above it is sufficient to establish (2.30).

For this purpose, for each $n \geq 1$ set

$$(2.31) \quad K_n := \max_{1 \leq i < j \leq k} \{|U_i(n) \wedge U_j(n)|_{\mathbf{L}}\}.$$

On the event

$$(2.32) \quad \mathcal{E}_n = \bigcap_{1 \leq i < j \leq k} \left\{ |U_i(n) \wedge U_j(n)|_{\mathbf{L}} + \left| |U_i(n)|_{\mathbf{L}} - |U_j(n)|_{\mathbf{L}} \right| / 2 \geq K_n \right\},$$

for all $1 \leq i \neq j \leq k$, the \mathbf{L} -distance to the root of the point where the Voronoi cells of $U_i(n)$ and $U_j(n)$ would meet if we ignored all other $k - 2$ points would be at least K_n . Thus, for all $i \neq \ell := \arg \min_{1 \leq j \leq k} |U_j(n)|_{\mathbf{L}}$, the Voronoi cell of $U_i(n)$ meets the Voronoi cell of $U_\ell(n)$ at \mathbf{L} -distance to the root exceeding K_n , implying that the Voronoi cell of $U_i(n)$ for all $i \neq \ell$ is the subtree rooted at the ancestor of $U_i(n)$ closest to the root among all ancestors of $U_i(n)$ whose \mathbf{L} -distance to the root exceeds

$$(2.33) \quad |U_i(n) \wedge U_\ell(n)|_{\mathbf{L}} + \frac{\left| |U_i(n)|_{\mathbf{L}} - |U_\ell(n)|_{\mathbf{L}} \right|}{2}.$$

By Lemmas 2.3, 2.4 and 2.5, we have $\mathbb{P}(\mathcal{E}_n) \rightarrow 1$; thus, in the rest of the proof, we work on the event \mathcal{E}_n .

For all $1 \leq i \neq j \leq k$, we set

$$X_n^{(i,j)} = \frac{|U_i(n) \wedge U_j(n)|_{\mathbf{L}} + \left| |U_i(n)|_{\mathbf{L}} - |U_j(n)|_{\mathbf{L}} \right| / 2}{v_n},$$

where v_n is as in (2.29). Note that, by symmetry, the $X_n^{(i,j)}$ all have the same distribution. Moreover, by Lemmas 2.3, 2.4 and 2.5 (see also for notation), in distribution when $n \rightarrow +\infty$, jointly for all $1 \leq i \neq j \leq k$, we have

$$(2.34) \quad X_n^{(i,j)} \Rightarrow |\Psi_i - \Psi_j| / 2,$$

where we have set, for all $1 \leq i \leq k$,

$$(2.35) \quad \Psi_i := \begin{cases} \Xi_i & \text{if } \text{Var}(L) < +\infty, \\ \Upsilon_i(\alpha) & \text{otherwise,} \end{cases}$$

where the random variables (Ξ_1, \dots, Ξ_k) and $(\Upsilon_1(\alpha), \dots, \Upsilon_d(\alpha))$ are defined in Lemmas 2.4 and 2.5, respectively.

For all $a, b \in (0, \infty)$, we define the event

$$\mathcal{B}_n(a, b) := \left\{ \forall 1 \leq i < j \leq k : X_n^{(i,j)} \in [a, b] \right\}.$$

Note that, due to (2.34),

$$(2.36) \quad \lim_n \mathbb{P}(\mathcal{B}_n(a, b)^c) \rightarrow 0 \quad \text{as } a \rightarrow 0 \text{ and } b \rightarrow +\infty.$$

To prove (2.28), we start by setting, for any permutation $\sigma \in \mathfrak{S}_k$,

$$\mathcal{A}_n(\sigma) = \{|U_{\sigma_1}(n)|_{\mathcal{L}} \leq |U_{\sigma_2}(n)|_{\mathcal{L}} \leq \dots \leq |U_{\sigma_k}(n)|_{\mathcal{L}}\}.$$

For all $\varepsilon > 0$ and $\sigma \in \mathfrak{S}_k$, we have, using the notation introduced before as well as in Proposition 2.7,

$$(2.37) \quad \begin{aligned} & \mathbb{P}\left(\exists i \in \{2, \dots, k\} : \log(V_i(n)/n) \right. \\ & \quad \geq (\mathbf{m} + \varepsilon)X_n^{(\sigma_1, \sigma_i)}v_n \text{ and } \mathcal{B}_n(a, b) \cap \mathcal{A}_n(\sigma)\big) \\ & = \mathbb{P}\left(\exists i \in \{2, \dots, k\} : \log\left(\frac{D_n^{(\sigma_i)}(X_n^{(\sigma_1, \sigma_i)}v_n)}{n}\right) \right. \\ & \quad \geq (\mathbf{m} + \varepsilon)X_n^{(\sigma_1, \sigma_i)}v_n \text{ and } \mathcal{B}_n(a, b) \cap \mathcal{A}_n(\sigma)\big) \\ & \leq \sum_{i=2}^k \mathbb{P}\left(\sup_{x \in [a, b]} \frac{\log(D_n^{(i)}(xv_n)/n)}{xv_n} - \mathbf{m} \geq \varepsilon\right) \\ & \leq (k-1)\mathbb{P}\left(\sup_{x \in [a, b]} \frac{\log(D_n^{(i)}(xv_n)/n)}{xv_n} - \mathbf{m} \geq \varepsilon\right) \rightarrow 0, \end{aligned}$$

when $n \rightarrow +\infty$ by Proposition 2.7. Since $\lim_n \mathbb{P}(\mathcal{B}_n(a, b)^c) \rightarrow 0$ when $a \rightarrow 0$ and $b \rightarrow +\infty$ due to (2.36), we get

$$\mathbb{P}\left(\exists i \in \{2, \dots, k\} : \frac{\log(V_i(n)/n)}{X_n^{(\sigma_1, \sigma_i)}v_n} \geq \mathbf{m} + \varepsilon \text{ and } \mathcal{A}_n(\sigma)\right) \rightarrow 0,$$

when $n \rightarrow +\infty$. Therefore, for all $\varepsilon \in (0, -\mathbf{m})$, for all $x_i < 0$, we have, asymptotically when $n \rightarrow +\infty$ that

$$\begin{aligned} & \mathbb{P}\left(\forall i \in \{2, \dots, k\} : \frac{\log(V_i(n)/n)}{v_n} \geq x_i\right) \\ & = (1 + o(1)) \sum_{\sigma \in \mathfrak{S}_k} \mathbb{P}\left(\forall i \in \{2, \dots, k\} : \frac{\log(V_i(n)/n)}{v_n} \geq x_i \right. \\ & \quad \left. \text{and } \frac{\log(V_i(n)/n)}{X_n^{(\sigma_1, \sigma_i)}v_n} \leq \mathbf{m} + \varepsilon \text{ and } \mathcal{A}_n(\sigma)\right) \\ & \leq (1 + o(1)) \sum_{\sigma \in \mathfrak{S}_k} \mathbb{P}\left(\forall i \in \{2, \dots, k\} : x_i \leq X_n^{(\sigma_1, \sigma_i)}(\mathbf{m} + \varepsilon) \text{ and } \mathcal{A}_n(\sigma)\right) \\ & = (1 + o(1)) \sum_{\sigma \in \mathfrak{S}_k} \mathbb{P}\left(\forall i \in \{2, \dots, k\} : X_n^{(\sigma_1, \sigma_i)} \leq \frac{x_i}{\mathbf{m} + \varepsilon} \text{ and } \mathcal{A}_n(\sigma)\right), \end{aligned}$$

where we have used that $\mathbf{m} + \varepsilon < 0$. Since this is true for all $\varepsilon \in (0, -\mathbf{m})$, we get

$$\begin{aligned} & \mathbb{P}\left(\forall i \in \{2, \dots, k\} : \frac{\log(V_i(n)/n)}{v_n} \geq x_i\right) \\ & \leq (1 + o(1)) \sum_{\sigma \in \mathfrak{S}_k} \mathbb{P}\left(\forall i \in \{2, \dots, k\} : X_n^{(\sigma_1, \sigma_i)} \leq \frac{x_i}{\mathbf{m}} \text{ and } \mathcal{A}_n(\sigma)\right). \end{aligned}$$

By definition of $\mathcal{A}_n(\sigma)$ and due to (2.34), we thus get

$$\begin{aligned} & \mathbb{P}\left(\forall i \in \{2, \dots, k\} : \frac{\log(V_i(n)/n)}{v_n} \geq x_i\right) \\ & \leq (1 + o(1)) \sum_{\sigma \in \mathfrak{S}_k} \mathbb{P}\left(\forall i \in \{2, \dots, k\} : |\Psi_{\sigma_1} - \Psi_{\sigma_i}| \leq \frac{2x_i}{\mathbf{m}} \text{ and } \Psi_{\sigma_1} \leq \dots \leq \Psi_{\sigma_k}\right) \\ & = (1 + o(1)) \mathbb{P}\left(\forall i \in \{2, \dots, k\} : \Psi_{(i)} - \Psi_{(1)} \leq \frac{2x_i}{\mathbf{m}}\right), \end{aligned}$$

which concludes the proof of (2.28) and thus of (2.30). □

3. Proof of Theorem 1.9

Before proving Theorem 1.9, we prove a central limit theorem that extends the weak law of large numbers proved in Proposition 2.7:

LEMMA 3.1. — *Using the same notation as in Proposition 2.7, assume that $(x_n)_{n \geq 1}$ is a sequence of positive random variables converging in probability as $n \rightarrow +\infty$ to a positive constant $x < C(f)$. Then, under the assumptions of Theorem 1.9 the following hold true.*

(i) *If $\text{Var}(L) < +\infty$, then, in distribution when $n \rightarrow +\infty$,*

$$\begin{aligned} & \frac{\log(D_n^{(i)}(x_n f)/n) - x f(n) \mathbb{E}[\log \bar{Y}]/\mathbb{E}L}{\sqrt{x f(n)}} \\ & \Rightarrow \mathcal{N}\left(0, \frac{\text{Var}(\log \bar{Y})}{\mathbb{E}L} + \frac{\mathbb{E}[\log \bar{Y}]^2 \text{Var}(L)}{(\mathbb{E}L)^2}\right). \end{aligned}$$

(ii) *If $\mathbb{P}(L \geq x) = \ell(x)x^{-\alpha}$ for $\alpha \in (1, 2)$ and ℓ slowly varying at infinity, then, in distribution when $n \rightarrow +\infty$,*

$$\frac{\log(D_n^{(i)}(x_n f)/n) - x f(n) \mathbb{E}[\log \bar{Y}]/\mathbb{E}L}{(x f(n))^{1/\alpha}} \Rightarrow \frac{\mathbb{E}[\log(1/\bar{Y})]}{(\mathbb{E}L)^{1+1/\alpha}} \Upsilon(\alpha),$$

where $\Upsilon(\alpha)$ is an α -stable distribution.

Proof. — In this proof, we use the notation introduced in the proof of Proposition 2.7.

(i) Fix $\varepsilon > 0$ such that $x + \varepsilon < C(f)$ and $x - \varepsilon > 0$. By assumption, the probability of the good events $\mathcal{G}_n := \{x_n \in [x - \varepsilon, x + \varepsilon]\}$ tends to 1 as $n \rightarrow +\infty$. On this event, we have

$$\begin{aligned} \left| \log \left(\frac{D_n^{(i)}(x_n f)}{n} \right) - \log Q_n(x_n) \right| &= \left| \log \left(\frac{D_n^{(i)}(x_n f)}{(n - k_n(x_n))Q_n(x_n)} \right) + \log \left(1 - \frac{k_n(x)}{n} \right) \right| \\ &\leq \left| \log \left(\frac{D_n^{(i)}(x_n f)}{(n - k_n(x_n))Q_n(x_n)} \right) \right| + \left| \log \left(1 - \frac{k_n}{n} \right) \right|, \end{aligned}$$

because $k_n(x_n) \leq k_n$, by definition. By (2.22), we have on \mathcal{G}_n that

$$\left| \log \left(\frac{D_n^{(i)}(x_n f)}{(n - k_n(x_n))Q_n(x_n)} \right) \right| \leq \sup_{x_n \in [x - \varepsilon, x + \varepsilon]} \left| \log \left(\frac{D_n^{(i)}(x f)}{(n - k_n(x))Q_n(x)} \right) \right| = o_{\mathbb{P}}(1).$$

We thus get that on \mathcal{G}_n ,

$$\left| \log \left(\frac{D_n^{(i)}(x_n f)}{n} \right) - \log Q_n(x_n) \right| = \log(1 - \tilde{U}) + o_{\mathbb{P}}(1),$$

which implies that

$$(3.1) \quad \left| \log \left(\frac{D_n^{(i)}(x_n f)}{n} \right) - \log Q_n(x_n) \right| = \mathcal{O}_{\mathbb{P}}(1) \quad \text{as } n \rightarrow +\infty.$$

Now recall that

$$\log Q_n(x_n) = \sum_{\emptyset \neq u \leq \xi(k_n(x_n))} \log Z_u$$

is a sum of $x_n h(n)$ i.i.d. random variables with finite variance, since $\text{Var}(\log \bar{Y}) < +\infty$ by assumption. By definition, for all $y > 0$, the node $\xi(k_n(y))$ is the ancestor of $U_i(n)$ closest to the root whose \mathbf{L} -distance to the root is at least $yh(n)$. Therefore, almost surely, for all $y \leq z$, $\xi(k_n(y))$ is an ancestor of $\xi(k_n(z))$ (which includes the case when both nodes are equal). In other words, as y increases from $x - \varepsilon$ to $x + \varepsilon$, $\xi(k_n(y))$ goes through the ancestors of $U_i(n)$ at \mathbf{L} -distance to the root between $(x - \varepsilon)h(n)$ and $(x + \varepsilon)h(n)$, in that order. Therefore, in distribution, we have, jointly for all $y \in [x - \varepsilon, x + \varepsilon]$,

$$\log Q_n(y) = \sum_{i=1}^{|\xi(k_n(y))|} \log \tilde{Z}_i = \sum_{i=1}^{yh(n)} \log \tilde{Z}_i,$$

where the sequence $(\tilde{Z}_i)_{i \geq 1}$ is a sequence of i.i.d. random copies of \bar{Y} , because, by definition, $|\xi(k_n(y))| = yh(n)$ for all $y > 0$. Thus, by the central limit theorem with random index (see, e.g., [Gut05, Theorem 3.2, Chapter 7]), we get

$$(3.2) \quad \frac{\log Q_n(x_n) - xh(n)\mathbb{E}[\log \bar{Y}]}{\sqrt{xh(n)}} \Rightarrow \mathcal{N}(0, \text{Var}(\log \bar{Y})).$$

Also recall that, by definition, $x F_i(n) = \sum_{i=1}^{xh(n)} L_i$ in distribution, and thus applying the central limit theorem, but this time to the sequence \mathbf{L} , we get that, if $\text{Var}(L) < +\infty$, then

$$(3.3) \quad \frac{x F_i(n) - x h(n) \mathbb{E} L}{\sqrt{x h(n)}} \Rightarrow \mathcal{N}(0, \text{Var}(L)).$$

Note that, by the independence of the sequence \mathbf{L} and the rest of the process, the two limits in (3.2) and (3.3) hold jointly, and the two Gaussians are independent. Combining (3.1) to (3.3), the above yields

$$\begin{aligned} \sqrt{x F_i(n)} \left(\frac{\log(D_n^{(i)}(x_n f)/n)}{x F_i(n)} - \frac{\mathbb{E}[\log \bar{Y}]}{\mathbb{E} L} \right) &= \\ \sqrt{\frac{h(n)}{F_i(n)}} \left(\frac{\log Q_n(x_n) + \mathcal{O}_{\mathbb{P}}(1) - x h(n) \mathbb{E}[\log \bar{Y}]}{\sqrt{x h(n)}} + \frac{\mathbb{E}[\log \bar{Y}]}{\mathbb{E} L} \cdot \frac{x h(n) \mathbb{E} L - x_n F_i(n)}{\sqrt{x h(n)}} \right) &= \\ \Rightarrow \mathcal{N} \left(0, \frac{\text{Var}(\log \bar{Y})}{\mathbb{E} L} + \frac{\mathbb{E}[\log \bar{Y}]^2 \text{Var}(L)}{(\mathbb{E} L)^2} \right). \end{aligned}$$

In (2.24) we have proved that $F_i(n) \sim f(n)$ in probability as $n \rightarrow +\infty$, but in fact, this statement can be made stronger: we have $F_i(n) = f(n)$ as soon as $|U_i(n)|_{\mathbf{L}} \geq x f(n)$, an event whose probability tends to 1 with n because either $f(n) = o(\log n)$ or $f(n) = \log n$ and $x < \mathbb{E} L / \mu$. This implies (i).

(ii) Under the assumption that $\mathbb{P}(L \geq x) = \ell(x)x^{-\alpha}$ with $\alpha \in (1, 2)$, the limit in (3.3) does not hold, instead we have that

$$\frac{x F_i(n) - x h(n) \mathbb{E} L}{(x h(n))^{1/\alpha}} \Rightarrow \Upsilon(\alpha),$$

where $\Upsilon(\alpha)$ is an α -stable distribution. Thus

$$\begin{aligned} \frac{\log(D_n^{(i)}(x_n f)/n) - x F_i(n) \mathbb{E}[\log \bar{Y}] / \mathbb{E} L}{(x F_i(n))^{1/\alpha}} &= \left(\frac{h(n)}{F_i(n)} \right)^{1/\alpha} \\ \left(\frac{\mathbb{E}[\log \bar{Y}]}{\mathbb{E} L} \cdot \frac{x h(n) \mathbb{E} L - x F_i(n)}{(x h(n))^{1/\alpha}} + \frac{\log Q_n(x) + \mathcal{O}_{\mathbb{P}}(1) - x h(n) \mathbb{E}[\log \bar{Y}]}{(x h(n))^{1/\alpha}} \right) &= \\ \Rightarrow \frac{\mathbb{E}[\log(1/\bar{Y})]}{(\mathbb{E} L)^{1+1/\alpha}} \Upsilon(\alpha), \end{aligned}$$

where the second summand now vanishes in the limit. This concludes the proof of (ii) because $\mu = \mathbb{E}[\log(1/\bar{Y})]$, and because $F_i(n) = f(n)$ with probability tending to 1 when n tends to infinity. \square

Proof of Theorem 1.9. — For this proof, we consider that the infections “creep along edges” between the times at which they infect vertices: if two infections of respective speeds s and s' start at two neighbouring vertices v and v' (respectively), then they meet at distance from v proportional to $\frac{s}{s+s'}$.

For any two infections i and ℓ , the \mathbf{L} -distance between $U_i(n)$ and $U_\ell(n)$ is equal to

$$\Delta_{i,\ell} := |U_i(n)|_{\mathbf{L}} + |U_\ell(n)|_{\mathbf{L}} - 2|U_i(n) \wedge U_\ell(n)|_{\mathbf{L}}.$$

Therefore, the time $t_{i,\ell}$ at which epidemics i and ℓ would meet if there were no other infection at play is equal to the time it would take for an infection of speed $s_i + s_\ell$ to cross a distance $\Delta_{i,\ell}$, i.e.

$$(3.4) \quad t_{i,\ell} = \frac{\Delta_{i,\ell}}{s_i + s_\ell} = \frac{|U_i(n)|_{\mathbf{L}} + |U_\ell(n)|_{\mathbf{L}} - 2|U_i(n) \wedge U_\ell(n)|_{\mathbf{L}}}{s_i + s_\ell}.$$

Therefore, in the absence of the other $k - 2$ infections, the i -th and ℓ -th infections would meet at \mathbf{L} -distance to the root equal to the maximum of

$$(3.5) \quad |U_i(n)|_{\mathbf{L}} - s_i t_{i,\ell} \quad \text{and} \quad |U_\ell(n)|_{\mathbf{L}} - s_\ell t_{i,\ell}.$$

Thus, on the event (recall K_n from (2.31)),

$$\mathcal{A}_n = \bigcap_{1 \leq i < \ell \leq k} \left\{ |U_\ell(n)|_{\mathbf{L}} - s_\ell t_{i,\ell} \geq K_n \right\},$$

for all $1 \leq i < \ell \leq k$, if we ignored all other $k - 2$ epidemics, the epidemics started respectively at $U_i(n)$ and $U_\ell(n)$ would meet at \mathbf{L} -distance to the root at least K_n . Also note that the probability of \mathcal{A}_n goes to one when $n \rightarrow +\infty$ by Lemmas 2.3 and 2.4. Thus it is enough to restrict ourselves to the set where \mathcal{A}_n holds.

We let $\kappa = \kappa(n) = \arg \min\{1 \leq \ell \leq j : |U_\ell(n)|_{\mathbf{L}}\}$. On the event \mathcal{A}_n , for all $i \neq \kappa$, the territory of the i^{th} infection neighbours a unique other territory, and this neighbouring territory is the territory of the κ^{th} epidemic. We let $d_i(n)$ denote the \mathbf{L} -distance from the root to the point where they meet (this point can be in the middle of an edge). On \mathcal{A}_n , the territory of the i^{th} infection is the subtree of τ_n rooted at the ancestor of $U_i(n)$ closest to the root whose \mathbf{L} -distance to the root is at least $d_i(n)$. We now show that

$$(3.6) \quad \frac{d_i(n)}{\log n} \rightarrow \frac{s_1 - s_i}{s_1 + s_i} \cdot \frac{\mathbb{E}L}{\mu} \quad \text{in probability as } n \rightarrow +\infty.$$

Indeed, first note that, by (3.4) and (3.5), under \mathcal{A}_n , the i^{th} and κ^{th} infections meet at \mathbf{L} -distance to the root equal to $|U_i(n)|_{\mathbf{L}} - s_i t_{i,\kappa}$, and thus

$$d_i(n) = |U_i(n)|_{\mathbf{L}} - s_i t_{i,\kappa}.$$

By Lemma 2.3, and using the notation $H_n = \max_{1 \leq i < k \leq n} |U_i(n) \wedge U_k(n)|$, we get

$$(3.7) \quad 0 \leq \frac{|U_i(n) \wedge U_\kappa(n)|_{\mathbf{L}}}{\log n} \leq \frac{H_n}{\log n} \rightarrow 0 \quad \text{in probability as } n \rightarrow +\infty,$$

implying that, as $n \rightarrow +\infty$,

$$(3.8) \quad t_{i,\kappa} = \frac{|U_i(n)|_{\mathbf{L}} + |U_\kappa(n)|_{\mathbf{L}}}{s_1 + s_i} + o_{\mathbb{P}}(\log n),$$

which yields

$$d_i(n) = \frac{s_1 |U_i(n)|_{\mathbf{L}} - s_i |U_\kappa(n)|_{\mathbf{L}}}{s_1 + s_i} + o_{\mathbb{P}}(\log n), \quad \text{as } n \rightarrow +\infty.$$

Since, by Lemmas 2.4 and 2.5, $U_i(n) = (\mathbb{E}L/\mu + o_{\mathbb{P}}(1)) \log n$ as $n \rightarrow +\infty$ (and similarly for $U_{\kappa}(n)$), we get (3.6).

As argued above, on \mathcal{A}_n , $\text{Ter}_{t,d}^{(i)}((U_1, s_1), \dots, (U_k, s_k))$ is the subtree of τ_n rooted at the ancestor of $U_i(n)$ closest to the root whose L -distance is at least $x_n^{(i)} \log n$, where, by (3.6),

$$x_n^{(i)} := \frac{d_i(n)}{\log n} = \frac{s_1 - s_i}{s_1 + s_i} \cdot \frac{\mathbb{E}L}{\mu} + o_{\mathbb{P}}(1).$$

We set $x^{(i)} = \lim_{n \rightarrow +\infty} x_n^{(i)}$ (with the limit holding in probability). Thus, by Lemma 3.1 (i), and because $W_{(i)}(n) = D_n^{(i)}(x_n^{(i)} f)$, we get

$$\begin{aligned} \sqrt{x^{(i)} \log n} \left(\frac{\log(W_{(i)}(n)/n)}{x^{(i)} \log n} - \frac{\mathbb{E}[\log \bar{Y}]}{\mathbb{E}L} \right) \\ \Rightarrow \mathcal{N} \left(0, \frac{\text{Var}(\log \bar{Y})}{\mathbb{E}L} + \frac{\mathbb{E}[\log \bar{Y}]^2 \text{Var}(L)}{(\mathbb{E}L)^2} \right), \end{aligned}$$

if the random variable L has finite variance. This implies

$$\frac{\log(W_{(i)}(n)/n) + \frac{s_1 - s_i}{s_1 + s_i} \log n}{\sqrt{\frac{s_1 - s_i}{s_1 + s_i} \log n}} \Rightarrow \mathcal{N} \left(0, \frac{\text{Var}(\log \bar{Y}) \mathbb{E}L}{\mathbb{E}[\log \bar{Y}]^2} + \text{Var}(L) \right),$$

as claimed. In the case when the edge lengths are heavy-tailed, we get from Lemma 3.1 (ii) that

$$\frac{\log(W_{(i)}(n)/n) - x^{(i)}(\log n) \mathbb{E}[\log \bar{Y}]/\mathbb{E}L}{(x^{(i)} \log n)^{1/\alpha}} \Rightarrow \frac{\mu}{(\mathbb{E}L)^{1+1/\alpha}} \Upsilon(\alpha),$$

which implies

$$\frac{\log(W_{(i)}(n)/n) + \frac{s_1 - s_i}{s_1 + s_i} \log n}{\left(\frac{s_1 - s_i}{s_1 + s_i} \log n\right)^{1/\alpha}} \Rightarrow \frac{\mu^{1-1/\alpha}}{\mathbb{E}L} \Upsilon(\alpha).$$

Because the probability of \mathcal{A}_n converges to 1 as $n \rightarrow +\infty$, this concludes the proof of (1.6).

To prove the convergence in (1.5), note that it follows from Theorem 1.7 and (1.6) that the sizes $W_{(i)}(n)$, $j + 1 \leq i \leq k$ are $o(W_{(\ell)}(n))$ for all $\ell \in \{1, \dots, j\}$ with high probability. This implies (1.5). \square

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