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# ON THE RANGE OF A TWO-DIMENSIONAL CONDITIONED SIMPLE RANDOM WALK

SUR L'AMPLITUDE D'UNE MARCHE  
ALÉATOIRE SIMPLE BIDIMENSIONNELLE  
CONDITIONNÉE

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**ABSTRACT.** — We consider the two-dimensional simple random walk conditioned on never hitting the origin. This process is a Markov chain, namely it is the Doob  $h$ -transform of the simple random walk with respect to the potential kernel. It is known to be transient and we show that it is “almost recurrent” in the sense that each infinite set is visited infinitely often, almost surely. We prove that, for a “large” set, the proportion of its sites visited by the conditioned walk is approximately a Uniform $[0, 1]$  random variable. Also, given a set  $G \subset \mathbb{R}^2$  that does not “surround” the origin, we prove that a.s. there is an infinite number of  $k$ 's such that  $kG \cap \mathbb{Z}^2$  is unvisited. These results suggest that the range of the conditioned walk has “fractal” behavior.

*Keywords:* random interlacements, range, transience, simple random walk, Doob's  $h$ -transform.  
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RÉSUMÉ. — Nous considérons une marche aléatoire simple en dimension 2 conditionnée à ne jamais atteindre l'origine. Ce processus est une chaîne de Markov, à savoir la transformation de Doob  $h$  de la marche aléatoire simple par rapport au noyau potentiel. Il est connu que ce processus est transient et nous montrons qu'il est « presque récurrent » en ce sens que chaque ensemble infini est visité infiniment souvent, presque sûrement. Nous prouvons que, pour un « grand » ensemble, la proportion des sites visités par la marche aléatoire conditionnée est approximativement une variable aléatoire uniforme dans  $[0, 1]$ . En outre, étant donné un ensemble  $G \subset \mathbb{R}^2$  qui « n'entoure » pas l'origine, nous prouvons que p.s., il existe un nombre infini d'entiers naturels  $k$  tels que  $kG \cap \mathbb{Z}^2$  ne soit pas visité. Ces résultats suggèrent que l'amplitude de la marche simple conditionnée a un comportement « fractal ».

## 1. Introduction and results

We start by introducing some basic notation and defining the “conditioned” random walk  $\widehat{S}$ , the main object of study in this paper. Besides being interesting on its own, this random walk is the main ingredient in the construction of the two-dimensional random interacements of [CP17, CPV16] (see also [CT12, DRS14, PT15, Szn10] for the higher-dimensional case).

Write  $x \sim y$  if  $x$  and  $y$  are neighbours in  $\mathbb{Z}^2$ . Let  $(S_n, n \geq 0)$  be two-dimensional simple random walk, i.e., the discrete-time Markov chain with state space  $\mathbb{Z}^2$  and transition probabilities defined in the following way:

$$(1.1) \quad P_{xy} = \begin{cases} \frac{1}{4}, & \text{if } x \sim y, \\ 0, & \text{otherwise.} \end{cases}$$

We assume that all random variables in this paper are constructed on a common probability space with probability measure  $\mathbb{P}$  and we denote by  $\mathbb{E}$  the corresponding expectation. When no confusion can arise, we will write  $\mathbb{P}_x$  and  $\mathbb{E}_x$  for the law and expectation of the<sup>(1)</sup> random walk started from  $x$ . Let

$$(1.2) \quad \tau_0(A) = \inf\{k \geq 0 : S_k \in A\},$$

$$(1.3) \quad \tau_1(A) = \inf\{k \geq 1 : S_k \in A\}$$

be the entrance and the hitting time of the set  $A$  by simple random walk  $S$  (we use the convention  $\inf \emptyset = +\infty$ ). For a singleton  $A = \{x\}$ , we will write  $\tau_i(A) = \tau_i(x)$ ,  $i = 0, 1$ , for short. One of the key objects needed to understand the two-dimensional simple random walk is the potential kernel  $a$ , defined by

$$(1.4) \quad a(x) = \sum_{k=0}^{\infty} (\mathbb{P}_0[S_k=0] - \mathbb{P}_x[S_k=0]).$$

It can be shown that the above series indeed converges and we have  $a(0) = 0$ ,  $a(x) > 0$  for  $x \neq 0$ . It is straightforward to check that the function  $a$  is harmonic outside the origin, i.e.,

$$(1.5) \quad \frac{1}{4} \sum_{y: y \sim x} a(y) = a(x) \quad \text{for all } x \neq 0.$$

<sup>(1)</sup>The simple one, or the conditioned one defined below.

Also, using (1.4) and the Markov property, one can easily obtain that  $\frac{1}{4} \sum_{x \sim 0} a(x) = 1$ , which implies by symmetry that

$$(1.6) \quad a(x) = 1 \text{ for all } x \sim 0.$$

Observe that (1.5) immediately implies that  $a(S_{k \wedge \tau_0(0)})$  is a martingale, we will repeatedly use this fact in the sequel. Further, one can show that (with  $\gamma = 0.5772156\dots$  the Euler–Mascheroni constant)

$$(1.7) \quad a(x) = \frac{2}{\pi} \ln \|x\| + \frac{2\gamma + 3 \ln 2}{\pi} + O(\|x\|^{-2})$$

as  $x \rightarrow \infty$ , cf. [LL10, Theorem 4.4.4].

Let us define another random walk  $(\widehat{S}_n, n \geq 0)$  on  $\mathbb{Z}^2 \setminus \{0\}$  in the following way: its transition probability matrix equals (compare to (1.1))

$$(1.8) \quad \widehat{P}_{xy} = \begin{cases} \frac{a(y)}{4a(x)}, & \text{if } x \sim y, x \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

It is immediate to see from (1.5) that the random walk  $\widehat{S}$  is indeed well defined.

The walk  $\widehat{S}$  is the Doob  $h$ -transform of the simple random walk, under the condition of not hitting the origin (see [CPV16, Lemma 3.3 and its proof]). Let  $\widehat{\tau}_0, \widehat{\tau}_1$  be defined as in (1.2)–(1.3), but with  $\widehat{S}$  in the place of  $S$ . We summarize the basic properties of the walk  $\widehat{S}$  in the following

**PROPOSITION 1.1.** — *The following statements hold:*

- (1) *The walk  $\widehat{S}$  is reversible, with the reversible measure  $\mu_x := a^2(x)$ .*
- (2) *In fact, it can be represented as a random walk on the two-dimensional lattice with conductances  $(a(x)a(y), x, y \in \mathbb{Z}^2, x \sim y)$ .*
- (3) *Let  $\mathcal{N}$  be the set of the four neighbours of the origin. Then the process  $1/a(\widehat{S}_{n \wedge \widehat{\tau}_0(\mathcal{N})})$  is a martingale.*
- (4) *The walk  $\widehat{S}$  is transient.*
- (5) *Moreover, for all  $x \neq 0$*

$$(1.9) \quad \mathbb{P}_x[\widehat{\tau}_1(x) < \infty] = 1 - \frac{1}{2a(x)},$$

and for all  $x \neq y, x, y \neq 0$

$$(1.10) \quad \mathbb{P}_x[\widehat{\tau}_0(y) < \infty] = \mathbb{P}_x[\widehat{\tau}_1(y) < \infty] = \frac{a(x) + a(y) - a(x - y)}{2a(x)}.$$

The statements of Proposition 1.1 are not novel (they appear already in [CPV16]), but we found it useful to collect them here for the sake of completeness and also for future reference. We will prove Proposition 1.1 in the next section. It is curious to observe that (1.10) implies that, for any  $x$ ,  $\mathbb{P}_x[\widehat{\tau}_1(y) < \infty]$  converges to  $\frac{1}{2}$  as  $y \rightarrow \infty$ . As noted in [CPV16], this is related to the remarkable fact that if one conditions on a very distant site being vacant, then this reduces the intensity “near the origin” of the two-dimensional random interlacement process by a factor of four.

Let  $\|\cdot\|$  be the Euclidean norm. Define the (discrete) ball

$$\mathbf{B}(x, r) = \{y \in \mathbb{Z}^2 : \|y - x\| \leq r\}$$

(note that this definition works for all  $x \in \mathbb{R}^2$  and  $r \in \mathbb{R}_+$ ), and abbreviate  $\mathbf{B}(r) := \mathbf{B}(0, r)$ . The (internal) boundary of  $A \subset \mathbb{Z}^2$  is defined by

$$\partial A = \{x \in A : \text{there exists } y \in \mathbb{Z}^2 \setminus A \text{ such that } x \sim y\}.$$

Now we introduce some more notation and state the main results. For a set  $T \subset \mathbb{Z}_+$  (thought of as a set of time moments) let

$$\widehat{S}_T = \bigcup_{m \in T} \{\widehat{S}_m\}$$

be the *range* of the walk  $\widehat{S}$  with respect to that set. For simplicity, we assume in the following that the walk  $\widehat{S}$  starts at a fixed neighbour  $x_0$  of the origin, and we write  $\mathbb{P}$  for  $\mathbb{P}_{x_0}$  (it is, however, clear that our results hold for any fixed starting position of the walk). For a nonempty and finite set  $A \subset \mathbb{Z}^2$ , let us consider random variables

$$\begin{aligned} \mathcal{R}(A) &= \frac{|A \cap \widehat{S}_{[0, \infty)}|}{|A|}, \\ \mathcal{V}(A) &= \frac{|A \setminus \widehat{S}_{[0, \infty)}|}{|A|} = 1 - \mathcal{R}(A); \end{aligned}$$

that is,  $\mathcal{R}(A)$  (respectively,  $\mathcal{V}(A)$ ) is the proportion of visited (respectively, unvisited) sites of  $A$  by the walk  $\widehat{S}$ . Let us also abbreviate, for  $M_0 > 0$ ,

$$(1.11) \quad \ell_A^{(n)} = |A|^{-1} \max_{y \in A} \left| A \cap \mathbf{B}\left(y, \frac{n}{\ln^{M_0} n}\right) \right|.$$

Our main result is the following

**THEOREM 1.2.** — *Let  $M_0 > 0$  be a fixed constant, and assume that  $A \subset \mathbf{B}(n) \setminus \mathbf{B}(n \ln^{-M_0} n)$ . Then, for all  $s \in [0, 1]$ , we have, with positive constants  $c_{1,2}$  depending only on  $M_0$ ,*

$$(1.12) \quad \left| \mathbb{P}[\mathcal{V}(A) \leq s] - s \right| \leq c_1 \left( \frac{\ln \ln n}{\ln n} \right)^{1/3} + c_2 \ell_A^{(n)} \left( \frac{\ln \ln n}{\ln n} \right)^{-2/3},$$

and the same result holds with  $\mathcal{R}$  on the place of  $\mathcal{V}$ .

The above result means that if  $A \subset \mathbf{B}(n) \setminus \mathbf{B}(n \ln^{-M_0} n)$  is “big enough and well distributed”, then the proportion of visited sites has approximately Uniform $[0, 1]$  distribution. In particular, one can obtain the following

**COROLLARY 1.3.** — *Assume that  $D \subset \mathbb{R}^2$  is a bounded open set. Then both sequences  $(\mathcal{R}(nD \cap \mathbb{Z}^2), n \geq 1)$  and  $(\mathcal{V}(nD \cap \mathbb{Z}^2), n \geq 1)$  converge in distribution to the Uniform $[0, 1]$  random variable.*

Indeed, it is straightforward to obtain it from Theorem 1.2 since  $|nD \cap \mathbb{Z}^2|$  is of order  $n^2$  as  $n \rightarrow \infty$  (note that  $D$  contains a disk), and so  $\ell_{nD \cap \mathbb{Z}^2}^{(n)}$  will be of order  $\ln^{-2M_0} n$ . Observe that we can cut out  $\mathbf{B}(n \ln^{-M_0} n)$  from  $nD$  without doing any harm to the limit theorem, since formally we need  $A \subset \mathbf{B}(n) \setminus \mathbf{B}(n \ln^{-M_0} n)$  in

order to apply Theorem 1.2. Then, we can choose  $M_0$  large enough such that the right-hand side of (1.12) goes to 0.

Also, we prove that the range of  $\widehat{S}$  contains many “big holes”. To formulate this result, we need the following

DEFINITION 1.4. — We say that a set  $G \subset \mathbb{R}^2$  does not surround the origin, if

- there exists  $c_1 > 0$  such that  $G \subset B(c_1)$ , i.e.,  $G$  is bounded;
- there exist  $c_2 > 0, c_3 > 0$ , and a function  $f = (f_1, f_2) : [0, 1] \mapsto \mathbb{R}^2$  such that  $f(0) = 0, \|f(1)\| = c_1, |f'_1(s)| + |f'_2(s)| \leq c_2$  for all  $s \in [0, 1]$ , and

$$\inf_{s \in [0, 1], y \in G} \|(f_1(s), f_2(s)) - y\| \geq c_3,$$

i.e., one can escape from the origin to infinity along a path which is uniformly away from  $G$ .

Then, we have

THEOREM 1.5. — Let  $G \subset \mathbb{R}^2$  be a set that does not surround the origin. Then,

$$(1.13) \quad \mathbb{P}[nG \cap \widehat{S}_{[0, \infty)} = \emptyset \text{ for infinitely many } n] = 1.$$

Theorem 1.5 invites the following

Remark 1.6. — A natural question to ask is whether there are also “big” completely filled subsets of  $\mathbb{Z}^2$ , that is, if a.s. there are infinitely many  $n$  such that  $(nG \cap \mathbb{Z}^2) \subset \widehat{S}_{[0, \infty)}$ , for  $G \subset \mathbb{R}^2$  being, say, a disk. It is not difficult to see that the answer to this question is “no”. We do not give all details, but the reason for this is that, informally, one  $\widehat{S}$ -trajectory corresponds to the two-dimensional random interlacements of [CPV16] “just above” the level  $\alpha = 0$ . Then, as in Theorem 2.5 (iii) (inequality (22)) of [CPV16], it is possible to show that, with any fixed  $\delta > 0$ ,

$$\mathbb{P}[(nG \cap \mathbb{Z}^2) \subset \widehat{S}_{[0, \infty)}] \leq n^{-2+\delta}$$

for all large enough  $n$ ; our claim then follows from the (first) Borel–Cantelli lemma.

We also establish some additional properties of the conditioned walk  $\widehat{S}$ , which will be important for the proof of Theorem 1.5 and are of independent interest. Consider an irreducible Markov chain. Recall that a set is called *recurrent* with respect to the Markov chain, if it is visited infinitely many times almost surely; a set is called *transient*, if it is visited only finitely many times almost surely. It is clear that any nonempty set is recurrent with respect to a recurrent Markov chain, and every finite set is transient with respect to a transient Markov chain. Note that, in general, a set can be neither recurrent nor transient (think e.g. of the simple random walk on a binary tree, fix a neighbour of the root and consider the set of vertices of the tree connected to the root through this fixed neighbour).

In many situations it is possible to characterize completely the recurrent and transient sets, as well as to answer the question if any set must be either recurrent or transient. For example, for the simple random walk in  $\mathbb{Z}^d, d \geq 3$ , each set is either recurrent or transient and the characterization is provided by the *Wiener’s test* (see e.g. [LL10, Corollary 6.5.9]), formulated in terms of capacities of intersections of the

set with exponentially growing annuli. Now, for the conditioned two-dimensional walk  $\widehat{S}$  the characterization of recurrent and transient sets is particularly simple:

**THEOREM 1.7.** — *A set  $A \subset \mathbb{Z}^2$  is recurrent with respect to  $\widehat{S}$  if and only if  $A$  is infinite.*

Next, we recall that a Markov chain has the *Liouville property*, see e.g. [Woe09, Chapter IV], if all bounded harmonic (with respect to that Markov chain) functions are constants. Since Theorem 1.7 implies that every set must be recurrent or transient, we obtain the following result as its corollary:

**THEOREM 1.8.** — *The conditioned two-dimensional walk  $\widehat{S}$  has the Liouville property.*

These two results, besides being of interest on their own, will also be operational in the proof of Theorem 1.5.

## 2. Some auxiliary facts and proof of Proposition 1.1

For  $A \subset \mathbb{Z}^d$ , recall that  $\partial A$  denotes its internal boundary. We abbreviate  $\tau_1(R) = \tau_1(\partial \mathbb{B}(R))$ . We will consider, with a slight abuse of notation, the function

$$a(r) = \frac{2}{\pi} \ln r + \frac{2\gamma + 3 \ln 2}{\pi}$$

of a *real* argument  $r \geq 1$ . To explain why this notation is convenient, observe that, due to (1.7), we may write, for the case when (say)  $2\|x\| \leq r$  and as  $r \rightarrow \infty$ ,

$$(2.1) \quad \sum_{y \in \partial \mathbb{B}(x,r)} \nu(y) a(y) = a(r) + O\left(\frac{\|x\| \vee 1}{r}\right)$$

for *any* probability measure  $\nu$  on  $\partial \mathbb{B}(x, r)$ .

For all  $x \in \mathbb{Z}^2$  and  $R \geq 1$  such that  $x, y \in \mathbb{B}(R/2)$  and  $x \neq y$ , we have

$$(2.2) \quad \mathbb{P}_x[\tau_1(R) < \tau_1(y)] = \frac{a(x-y)}{a(R) + O(R^{-1}(\|y\| \vee 1))},$$

as  $R \rightarrow \infty$ . This is an easy consequence of the optional stopping theorem applied to the martingale  $a(S_{n \wedge \tau_0(y)} - y)$ , together with (2.1). Also, an application of the optional stopping theorem to the martingale  $1/a(\widehat{S}_{n \wedge \widehat{\tau}_0(\mathcal{N})})$  yields

$$(2.3) \quad \mathbb{P}_x[\widehat{\tau}_1(R) < \widehat{\tau}_1(r)] = \frac{(a(r))^{-1} - (a(x))^{-1} + O(R^{-1})}{(a(r))^{-1} - (a(R))^{-1} + O(r^{-1})},$$

for  $1 < r < \|x\| < R < \infty$ . Sending  $R$  to infinity in (2.3) we see that for  $1 \leq r \leq \|x\|$

$$(2.4) \quad \mathbb{P}_x[\widehat{\tau}_1(r) = \infty] = 1 - \frac{a(r) + O(r^{-1})}{a(x)}.$$

We need the fact that  $S$  conditioned on hitting  $\partial \mathbb{B}(R)$  before 0 is almost indistinguishable from  $\widehat{S}$ . For  $A \subset \mathbb{Z}^2$ , let  $\Gamma_A^{(x)}$  denote the set of all finite nearest-neighbour trajectories that start at  $x \in A \setminus \{0\}$  and end when entering  $\partial A$  for the first time.

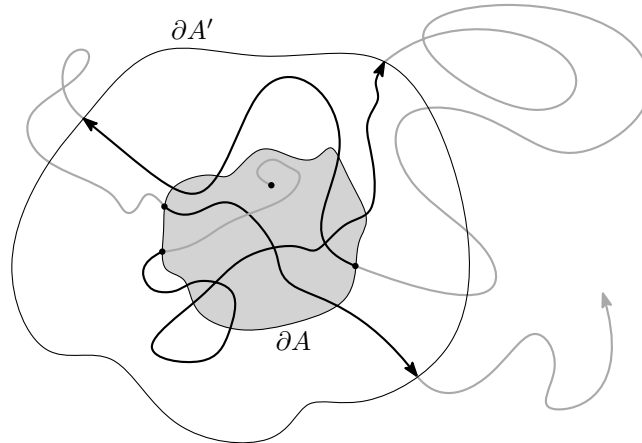


Figure 2.1. Excursions (pictured as bold pieces of trajectories) of random walks between  $\partial A$  and  $\partial A'$ .

For  $V \subset \Gamma_A^{(x)}$  write  $S \in V$  if there exists  $k$  such that  $(S_0, \dots, S_k) \in V$  (and the same for the conditioned walk  $\widehat{S}$ ). We write  $\Gamma_{0,R}^{(x)}$  for  $\Gamma_{B(R)}^{(x)}$ .

LEMMA 2.1. — Assume that  $V \subset \Gamma_{0,R}^{(x)}$ ; then we have

$$(2.5) \quad \mathbb{P}_x[S \in V \mid \tau_1(R) < \tau_1(0)] = \mathbb{P}_x[\widehat{S} \in V](1 + O((R \ln R)^{-1})).$$

*Proof.* — This is Lemma 3.3 (i) of [CPV16]. □

If  $A \subset A'$  are (finite) subsets of  $\mathbb{Z}^2$ , then the *excursions* between  $\partial A$  and  $\partial A'$  are pieces of nearest-neighbour trajectories that begin on  $\partial A$  and end on  $\partial A'$ , see Figure 2.1, which is, hopefully, self-explanatory. We refer to Section 3.4 of [CPV16] for formal definitions.

*Proof of Proposition 1.1.* It is straightforward to check (1)–(3) directly, we leave this task for the reader. Item (4) (the transience) follows from (3) and Theorem 2.5.8 of [MPW17].

As for (v), we first observe that (1.9) is a consequence of (1.10), although it is of course also possible to prove it directly, see [CPV16, Proposition 2.2]. Indeed, using (1.8) and then (1.10), (1.5) and (1.6), one can write

$$\begin{aligned} \mathbb{P}_x[\widehat{\tau}_1(x) < \infty] &= \frac{1}{4a(x)} \sum_{y \sim x} a(y) \mathbb{P}_y[\widehat{\tau}_1(x) < \infty] \\ &= \frac{1}{4a(x)} \sum_{y \sim x} \frac{1}{2} (a(y) + a(x) - a(y - x)) \\ &= 1 - \frac{1}{2a(x)}. \end{aligned}$$

Now, to prove (1.10), we essentially use the approach of Lemma 3.7 of [CPV16], although here the calculations are simpler. Let us define (note that all the probabilities

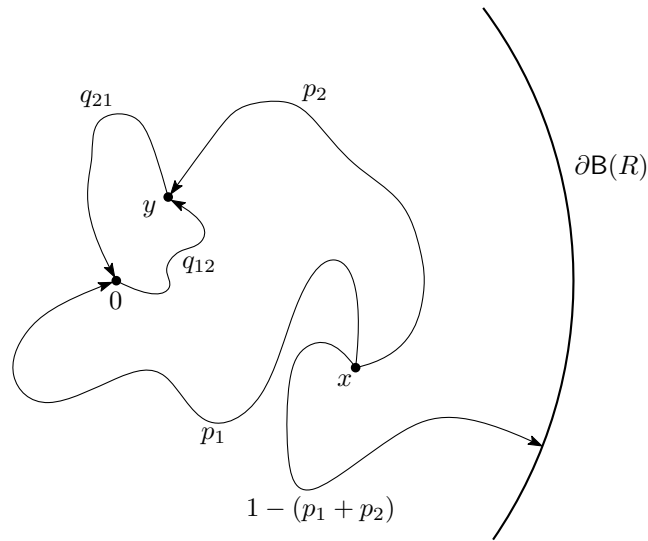


Figure 2.2. Trajectories for the probabilities of interest.

below are for the simple random walk  $S$ )

$$\begin{aligned}
 h_1 &= \mathbb{P}_x[\tau_1(0) < \tau_1(R)], \\
 h_2 &= \mathbb{P}_x[\tau_1(y) < \tau_1(R)], \\
 q_{12} &= \mathbb{P}_0[\tau_1(y) < \tau_1(R)], \\
 q_{21} &= \mathbb{P}_y[\tau_1(0) < \tau_1(R)], \\
 p_1 &= \mathbb{P}_x[\tau_1(0) < \tau_1(R) \wedge \tau_1(y)], \\
 p_2 &= \mathbb{P}_x[\tau_1(y) < \tau_1(R) \wedge \tau_1(0)],
 \end{aligned}$$

see Figure 2.2.

Using (2.2) (and in addition the Markov property and (1.5) for (2.8)) we have for  $x, y \neq 0, x \neq y$

$$(2.6) \quad h_1 = 1 - \frac{a(x)}{a(R) + O(R^{-1})},$$

$$(2.7) \quad h_2 = 1 - \frac{a(x-y)}{a(R) + O(R^{-1}\|y\|)},$$

$$(2.8) \quad q_{12} = 1 - \frac{a(y)}{a(R) + O(R^{-1}\|y\|)},$$

$$(2.9) \quad q_{21} = 1 - \frac{a(y)}{a(R) + O(R^{-1})},$$



which implies that

$$(2.10) \quad \lim_{R \rightarrow \infty} (1 - h_1)a(R) = a(x),$$

$$(2.11) \quad \lim_{R \rightarrow \infty} (1 - h_2)a(R) = a(x - y),$$

$$(2.12) \quad \lim_{R \rightarrow \infty} (1 - q_{12})a(R) = a(y),$$

$$(2.13) \quad \lim_{R \rightarrow \infty} (1 - q_{21})a(R) = a(y).$$

Observe that, due to the Markov property, it holds that

$$\begin{aligned} h_1 &= p_1 + p_2q_{21}, \\ h_2 &= p_2 + p_1q_{12}. \end{aligned}$$

Solving these equations with respect to  $p_1, p_2$ , we obtain

$$(2.14) \quad p_1 = \frac{h_1 - h_2q_{21}}{1 - q_{12}q_{21}},$$

$$(2.15) \quad p_2 = \frac{h_2 - h_1q_{12}}{1 - q_{12}q_{21}}.$$

Let us denote

$$(2.16) \quad \bar{h}_1 = 1 - h_1, \quad \bar{h}_2 = 1 - h_2, \quad \bar{q}_{12} = 1 - q_{12}, \quad \bar{q}_{21} = 1 - q_{21}.$$

Next, using Lemma 2.1, we have that

$$\begin{aligned} \mathbb{P}_x[\hat{\tau}_1(y) < \hat{\tau}_1(R)] &= \mathbb{P}_x[\tau_1(y) < \tau_1(R) \mid \tau_1(R) < \tau_1(0)](1 + o(R^{-1})) \\ &= \frac{\mathbb{P}_x[\tau_1(y) < \tau_1(R) < \tau_1(0)]}{\mathbb{P}_x[\tau_1(R) < \tau_1(0)]}(1 + o(R^{-1})) \\ &= \frac{p_2(1 - q_{21})}{1 - h_1}(1 + o(R^{-1})) \\ &= \frac{(h_2 - h_1q_{12})(1 - q_{21})}{(1 - q_{12}q_{21})(1 - h_1)}(1 + o(R^{-1})) \\ (2.17) \quad &= \frac{(\bar{h}_1 + \bar{q}_{12} - \bar{h}_2 - \bar{h}_1\bar{q}_{12})\bar{q}_{21}}{(\bar{q}_{12} + \bar{q}_{21} - \bar{q}_{12}\bar{q}_{21})\bar{h}_1}(1 + o(R^{-1})). \end{aligned}$$

Since  $\mathbb{P}_x[\hat{\tau}_1(y) < \infty] = \lim_{R \rightarrow \infty} \mathbb{P}_x[\hat{\tau}_1(y) < \hat{\tau}_1(R)]$ , using (2.10)–(2.13) we obtain (1.10) (observe that the “product” terms in (2.17) are of smaller order and will disappear in the limit). □

We now use the ideas contained in the last proof to obtain some refined bounds on the hitting probabilities for excursions of the conditioned walk.

Let us assume that  $\|x\| \geq n \ln^{-M_0} n$  and  $y \in A$ , where the set  $A$  is as in Theorem 1.2. Also, abbreviate  $R = n \ln^2 n$ .

LEMMA 2.2. — *In the above situation, we have*

$$(2.18) \quad \begin{aligned} \mathbb{P}_x[\hat{\tau}_1(y) < \hat{\tau}_1(R)] \\ = \left(1 + O(\ln^{-3} n)\right) \frac{a(x)a(R) + a(y)a(R) - a(x - y)a(R) - a(x)a(y)}{a(x)(2a(R) - a(y))}. \end{aligned}$$

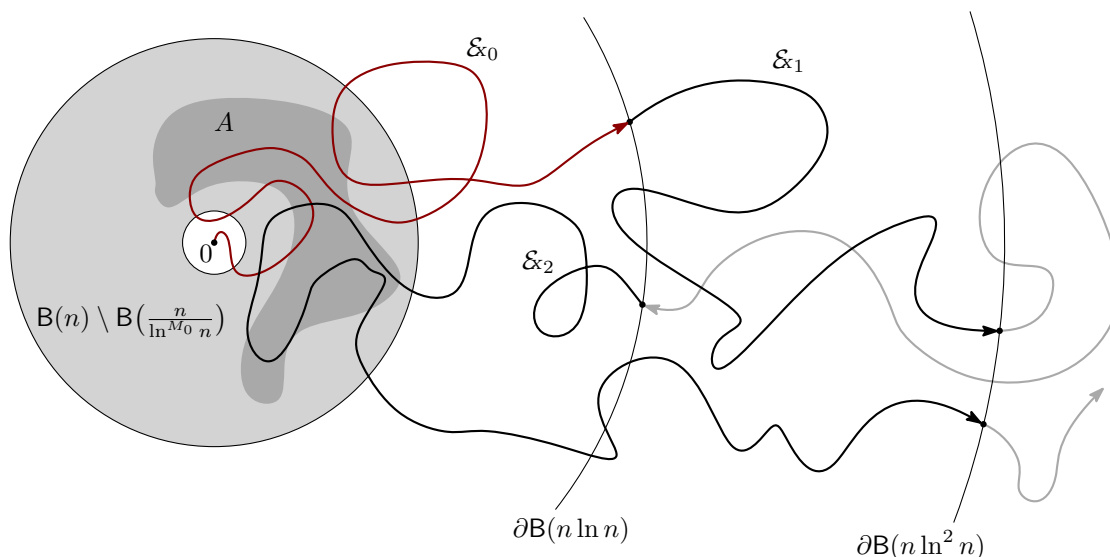


Figure 2.3. Excursions and their visits to  $A$

*Proof.* — This is essentially the same calculation as in the proof of (1.10), with the following difference: after arriving to the expression (2.17), instead of sending  $R$  to infinity (which conveniently “kills” many terms there), we need to carefully deal with all the  $O$ ’s. Specifically, we reuse notations (2.6)–(2.9) and (2.16), then write

$$\begin{aligned}
 \mathbb{P}_x[\hat{\tau}_1(y) < \hat{\tau}_1(R)] &= \mathbb{P}_x[\tau_1(y) < \tau_1(R) \mid \tau_1(R) < \tau_1(0)](1 + O(n^{-1})) \\
 (2.19) \qquad \qquad \qquad &= \frac{B_1}{B_2}(1 + o(n^{-1})),
 \end{aligned}$$

where (observe that, since  $\|y\| \leq n$  and  $R = n \ln^2 n$ , we have  $a(R) + O(R^{-1}\|y\|) = a(R) + O(\ln^{-2} n) = a(R)(1 + O(\ln^{-3} n))$ )

$$\begin{aligned}
 B_1 &= (\bar{h}_1 + \bar{q}_{12} - \bar{h}_2 - \bar{h}_1 \bar{q}_{12}) \bar{q}_{21} \\
 &= \frac{a(y)}{a(R) + O(R^{-1})} \left( \frac{a(x)}{a(R) + O(R^{-1})} + \frac{a(y)}{a(R) + O(R^{-1}\|y\|)} \right. \\
 &\quad \left. - \frac{a(x-y)}{a(R) + O(R^{-1}\|y\|)} - \frac{a(x)a(y)}{(a(R) + O(R^{-1}))(a(R) + O(R^{-1}\|y\|))} \right) \\
 &= (1 + O((R \ln R)^{-1})) \frac{a(y)}{a(R)} \cdot \frac{a(x)a(R) + a(y)a(R) - a(x-y)a(R) - a(x)a(y)}{(1 + O(\ln^{-3} n))a^2(R)} \\
 &= (1 + O(\ln^{-3} n)) \frac{a(y)}{a(R)} \cdot \frac{a(x)a(R) + a(y)a(R) - a(x-y)a(R) - a(x)a(y)}{a^2(R)},
 \end{aligned}$$

and

$$\begin{aligned}
 B_2 &= (\bar{q}_{12} + \bar{q}_{21} - \bar{q}_{12}\bar{q}_{21})\bar{h}_1 \\
 &= \frac{a(x)}{a(R) + O(R^{-1})} \left( \frac{a(y)}{a(R) + O(R^{-1}\|y\|)} + \frac{a(y)}{a(R) + O(R^{-1})} \right. \\
 &\quad \left. - \frac{a^2(y)}{(a(R) + O(R^{-1}))(a(R) + O(R^{-1}\|y\|))} \right) \\
 &= \left(1 + O((R \ln R)^{-1})\right) \frac{a(x)}{a(R)} \cdot \frac{2a(y)a(R) - a^2(y) + O(\ln^{-1} n)}{(1 + O(\ln^{-3} n))a^2(R)} \\
 &= \left(1 + O(\ln^{-3} n)\right) \frac{a(x)}{a(R)} \cdot \frac{2a(y)a(R) - a^2(y)}{a^2(R)}.
 \end{aligned}$$

We insert the above back to (2.19) and note that the factor  $\frac{a(y)}{a^3(R)}$  cancels to obtain (2.18). □

### 3. Proofs of the main results

We start with

*Proof of Theorem 1.2.* — First, we describe informally the idea of the proof. We consider the visits to the set  $A$  during excursions of the walk from  $\partial\mathbf{B}(n \ln n)$  to  $\partial\mathbf{B}(n \ln^2 n)$ , see Figure 2.3. The crucial argument is the following: the randomness of  $\mathcal{V}(A)$  comes from the *number* of excursions and not from the excursions themselves. If the number of excursions is around  $c \times \frac{\ln n}{\ln \ln n}$ , then it is possible to show (using a standard weak-LLN argument) that the proportion of uncovered sites in  $A$  is *concentrated* around  $e^{-c}$ . On the other hand, that number of excursions can be modeled roughly as  $Y \times \frac{\ln n}{\ln \ln n}$ , where  $Y$  is an Exponential(1) random variable. Then,  $\mathbb{P}[\mathcal{V}(A) \leq s] \approx \mathbb{P}[Y \geq \ln s^{-1}] = s$ , as required.

We now give a rigorous argument. Let  $\widehat{H}$  be the conditional entrance measure for the (conditioned) walk  $\widehat{S}$ , i.e.,

$$(3.1) \quad \widehat{H}_A(x, y) = \mathbb{P}_x \left[ \widehat{S}_{\widehat{\tau}_1(A)} = y \mid \widehat{\tau}_1(A) < \infty \right].$$

Let us first denote the initial piece of the trajectory by  $\mathfrak{E}_0 = \widehat{S}_{[0, \widehat{\tau}(n \ln n)]}$ . Then, we consider a *Markov chain*  $(\mathfrak{E}_k, k \geq 1)$  of excursions between  $\partial\mathbf{B}(n \ln n)$  and  $\partial\mathbf{B}(n \ln^2 n)$ , defined in the following way: for  $k \geq 2$  the initial site of  $\mathfrak{E}_k$  is chosen according to the measure  $\widehat{H}_{\mathbf{B}(n \ln n)}(z_{k-1}, \cdot)$ , where  $z_{k-1} \in \partial\mathbf{B}(n \ln^2 n)$  is the last site of the excursion  $\mathfrak{E}_{k-1}$ ; also, the initial site of  $\mathfrak{E}_1$  is the last site of  $\mathfrak{E}_0$ ; the weights of trajectories are chosen according to (1.8) (i.e., each excursion is an  $\widehat{S}$ -walk trajectory). It is important to observe that one may couple  $(\mathfrak{E}_k, k \geq 1)$  with the “true” excursions of the walk  $\widehat{S}$  in an obvious way: one just picks the excursions subsequently, each time tossing a coin to decide if the walk returns to  $\mathbf{B}(n \ln n)$ .

Let

$$\psi_n = \min_{x \in \partial\mathbf{B}(n \ln^2 n)} \mathbb{P}_x [\widehat{\tau}(n \ln n) = \infty]$$

be the *minimal* probability to avoid  $\mathbf{B}(n \ln n)$ , starting at sites of  $\partial\mathbf{B}(n \ln^2 n)$ . Using (2.4) it is straightforward to obtain that

$$\mathbb{P}_x[\widehat{\tau}_1(n \ln n) = \infty] = \frac{\ln \ln n}{\ln n + 2 \ln \ln n} (1 + O(n^{-1}))$$

for any  $x \in \partial\mathbf{B}(n \ln^2 n)$ , and so it also holds that

$$(3.2) \quad \psi_n = \frac{\ln \ln n}{\ln n + 2 \ln \ln n} (1 + O(n^{-1})).$$

Let us consider a sequence of i.i.d. random variables  $(\eta_k, k \geq 0)$  such that  $\mathbb{P}[\eta_k = 1] = 1 - \mathbb{P}[\eta_k = 0] = \psi_n$ . Let  $\widehat{N} = \min\{k : \eta_k = 1\}$ , so that  $\widehat{N}$  is a Geometric random variable with mean  $\psi_n^{-1}$ . Now, (3.2) implies that  $\mathbb{P}_x[\widehat{\tau}(n \ln n) = \infty] - \psi_n \leq O\left(\frac{\ln \ln n}{n \ln n}\right)$  for any  $x \in \partial\mathbf{B}(n \ln^2 n)$ , so it is clear<sup>(2)</sup> that  $\widehat{N}$  can be coupled with the actual number of excursions  $N$  in such a way that  $N \leq \widehat{N}$  a.s. and

$$(3.3) \quad \mathbb{P}[N \neq \widehat{N}] \leq O(n^{-1}).$$

Note that this construction preserves the independence of  $\widehat{N}$  from the excursion sequence  $(\mathfrak{X}_k, k \geq 1)$  itself.

Define

$$\mathcal{R}^{(k)} = \frac{|A \cap (\mathfrak{X}_0 \cup \mathfrak{X}_1 \cup \dots \cup \mathfrak{X}_k)|}{|A|},$$

and

$$\mathcal{V}^{(k)} = \frac{|A \setminus (\mathfrak{X}_0 \cup \mathfrak{X}_1 \cup \dots \cup \mathfrak{X}_k)|}{|A|} = 1 - \mathcal{R}^{(k)}$$

to be the proportions of visited and unvisited sites in  $A$  with respect to the first  $k$  excursions together with the initial piece  $\mathfrak{X}_0$ .

Now, it is straightforward to check that (2.18) implies that, for any  $x \in \partial\mathbf{B}(n \ln n)$  and  $y \in A$

$$(3.4) \quad \mathbb{P}_x[\widehat{\tau}_1(y) < \widehat{\tau}_1(n \ln^2 n)] = \frac{\ln \ln n}{\ln n} \left(1 + O\left(\frac{\ln \ln n}{\ln n}\right)\right),$$

and, for  $y, z \in \mathbf{B}(n) \setminus \mathbf{B}\left(\frac{n}{2 \ln^{M_0} n}\right)$  such that  $\|y - z\| = n/b$  with  $b \leq 2 \ln^{M_0} n$

$$(3.5) \quad \mathbb{P}_z[\widehat{\tau}_1(y) < \widehat{\tau}_1(n \ln^2 n)] = \frac{2 \ln \ln n + \ln b}{\ln n} \left(1 + O\left(\frac{\ln \ln n}{\ln n}\right)\right).$$

Indeed, first, observe that the factor  $B_2$  in (2.18) is, in both cases,

$$(3.6) \quad a(x)(2a(R) - a(y)) = \left(\frac{2}{\pi}\right)^2 \ln^2 n + O(\ln n \ln \ln n).$$

<sup>(2)</sup>Let  $(Z_n, n \geq 1)$  be a sequence of  $\{0, 1\}$ -valued random variables adapted to a filtration  $(\mathcal{F}_n, n \geq 1)$  and such that  $\mathbb{P}[Z_{n+1} = 1 \mid \mathcal{F}_n] \in [p, p + \varepsilon]$  a.s.. Then it is elementary to obtain that the total variation distance between the random variable  $\min\{k : Z_k = 1\}$  and the Geometric random variable with mean  $p^{-1}$  is bounded above by  $O(\varepsilon/p)$ .

As for the factor  $B_1$ , we have

$$\begin{aligned} B_1 &= a(x)a(R) + a(y)a(R) - a(x-y)a(R) - a(x)a(y) \\ &= (a(x) - a(x-y))a(R) - (a(R) - a(x))a(y) \\ &= O((\ln n)^{-1}) \times O(\ln n) + \left(\frac{2}{\pi} \ln \ln n + o(n^{-2})\right) \times \left(\frac{2}{\pi} \ln n + O(\ln \ln n)\right) \\ &= \left(\frac{2}{\pi}\right)^2 \ln n \ln \ln n + O((\ln \ln n)^2) \end{aligned}$$

in the case of (3.4), and (writing also  $\|z\| = n/c$  with  $(2 \ln^{M_0} n)^{-1} \leq c \leq 1$ )

$$\begin{aligned} B_1 &= a(z)a(R) + a(y)a(R) - a(z-y)a(R) - a(z)a(y) \\ &= (a(z) - a(z-y))a(R) - (a(R) - a(z))a(y) \\ &= \frac{2}{\pi}(-\ln c + \ln b + o(n^{-1})) \times \frac{2}{\pi}(\ln n + O(\ln \ln n)) \\ &\quad + \frac{2}{\pi}(2 \ln \ln n + \ln c + o(n^{-1})) \times \left(\frac{2}{\pi} \ln n + O(\ln \ln n)\right) \\ &= \left(\frac{2}{\pi}\right)^2 \ln n \times (2 \ln \ln n + \ln b) + O((\ln \ln n)^2) \end{aligned}$$

in the case of (3.5); with (3.6) we then obtain (3.4)–(3.5).

For  $y \in A$  and a fixed  $k \geq 1$  consider the random variable

$$\xi_y^{(k)} = \mathbf{1}\{y \notin \mathfrak{E}_0 \cup \mathfrak{E}_1 \cup \dots \cup \mathfrak{E}_k\},$$

so that  $\mathcal{V}^{(k)} = |A|^{-1} \sum_{y \in A} \xi_y^{(k)}$ . Now (3.4) implies that, for all  $j \geq 1$ ,

$$\mathbb{P}[y \notin \mathfrak{E}_j] = 1 - \frac{\ln \ln n}{\ln n} \left(1 + O\left(\frac{\ln \ln n}{\ln n}\right)\right),$$

and (3.5) implies that

$$\mathbb{P}[y \notin \mathfrak{E}_0 \cup \mathfrak{E}_1] = 1 - O\left(\frac{\ln \ln n}{\ln n}\right)$$

for any  $y \in A$ . Let  $\mu_y^{(k)} = \mathbb{E}\xi_y^{(k)}$ . Then we have

$$\begin{aligned} \mu_y^{(k)} &= \mathbb{P}[y \notin \mathfrak{E}_0 \cup \mathfrak{E}_1 \cup \dots \cup \mathfrak{E}_k] \\ &= \left(1 - O\left(\frac{\ln \ln n}{\ln n}\right)\right) \times \left(\left(1 - \frac{\ln \ln n}{\ln n} \left(1 + O\left(\frac{\ln \ln n}{\ln n}\right)\right)\right)\right)^{k-1} \\ (3.7) \quad &= \exp\left(-k \frac{\ln \ln n}{\ln n} \left(1 + O\left(k^{-1} + \frac{\ln \ln n}{\ln n}\right)\right)\right). \end{aligned}$$

Next, we need to estimate the covariance of  $\xi_y^{(k)}$  and  $\xi_z^{(k)}$  in case  $\|y - z\| \geq n \ln^{-M_0} n$ . First note that, for any  $x \in \partial B(n \ln n)$

$$\begin{aligned} \mathbb{P}_x[\{y, z\} \cap \mathfrak{E}_1 = \emptyset] &= 1 - \mathbb{P}_x[y \in \mathfrak{E}_1] - \mathbb{P}_x[z \in \mathfrak{E}_1] + \mathbb{P}_x[\{y, z\} \subset \mathfrak{E}_1] \\ &= 1 - 2 \frac{\ln \ln n}{\ln n} \left(1 + O\left(\frac{\ln \ln n}{\ln n}\right)\right) + \mathbb{P}_x[\{y, z\} \subset \mathfrak{E}_1] \end{aligned}$$

by (3.4); also, since

$$\begin{aligned} & \left\{ \hat{\tau}_1(y) < \hat{\tau}_1(z) < \hat{\tau}_1(n \ln^2 n) \right\} \\ & \subset \left\{ \hat{\tau}_1(y) < \hat{\tau}_1(n \ln^2 n), \hat{S}_k = z \text{ for some } \hat{\tau}_1(y) < k < \hat{\tau}_1(n \ln^2 n) \right\} \end{aligned}$$

from (3.4)–(3.5) we obtain

$$\begin{aligned} \mathbb{P}_x[\{y, z\} \subset \mathfrak{X}_1] &= \mathbb{P}_x[\max\{\hat{\tau}_1(y), \hat{\tau}_1(z)\} < \hat{\tau}_1(n \ln^2 n)] \\ &= \mathbb{P}_x[\hat{\tau}_1(y) < \hat{\tau}_1(z) < \hat{\tau}_1(n \ln^2 n)] + \mathbb{P}_x[\hat{\tau}_1(z) < \hat{\tau}_1(y) < \hat{\tau}_1(n \ln^2 n)] \\ &\leq \mathbb{P}_x[\hat{\tau}_1(y) < \hat{\tau}_1(n \ln^2 n)] \mathbb{P}_y[\hat{\tau}_1(z) < \hat{\tau}_1(n \ln^2 n)] \\ &\quad + \mathbb{P}_x[\hat{\tau}_1(z) < \hat{\tau}_1(n \ln^2 n)] \mathbb{P}_z[\hat{\tau}_1(y) < \hat{\tau}_1(n \ln^2 n)] \\ &\leq 2 \frac{\ln \ln n}{\ln n} \times \frac{(2 + M_0) \ln \ln n}{\ln n} \left( 1 + O\left(\frac{\ln \ln n}{\ln n}\right) \right) \\ &= O\left(\left(\frac{\ln \ln n}{\ln n}\right)^2\right). \end{aligned}$$

Therefore, similarly to (3.7) we obtain

$$\mathbb{E}(\xi_y^{(k)} \xi_z^{(k)}) = \exp\left(-2k \frac{\ln \ln n}{\ln n} \left(1 + O\left(k^{-1} + \frac{\ln \ln n}{\ln n}\right)\right)\right),$$

which, together with (3.7), implies after some elementary calculations that, for all  $y, z \in A$  such that  $\|y - z\| \geq n \ln^{-M_0} n$

$$(3.8) \quad \text{cov}(\xi_y^{(k)}, \xi_z^{(k)}) = O\left(\frac{\ln \ln n}{\ln n}\right)$$

uniformly in  $k$ , since

$$\left(\frac{\ln \ln n}{\ln n} + k \left(\frac{\ln \ln n}{\ln n}\right)^2\right) \exp\left(-2k \frac{\ln \ln n}{\ln n}\right) = O\left(\frac{\ln \ln n}{\ln n}\right)$$

uniformly in  $k$ . Recall the notation  $\ell_A^{(n)}$  from (1.11). Now, using Chebyshev's inequality, we write

$$\begin{aligned}
 & \mathbb{P} \left[ \left| |A|^{-1} \sum_{y \in A} (\xi_y^{(k)} - \mu_y^{(k)}) \right| > \varepsilon \right] \\
 & \leq (\varepsilon |A|)^{-2} \text{Var} \left( \sum_{y \in A} \xi_y^{(k)} \right) \\
 & = (\varepsilon |A|)^{-2} \sum_{y, z \in A} \text{cov}(\xi_y^{(k)}, \xi_z^{(k)}) \\
 & = (\varepsilon |A|)^{-2} \left( \sum_{\substack{y, z \in A, \\ \|y-z\| < \frac{n}{\ln^{M_0} n}}} \text{cov}(\xi_y^{(k)}, \xi_z^{(k)}) + \sum_{\substack{y, z \in A, \\ \|y-z\| \geq \frac{n}{\ln^{M_0} n}}} \text{cov}(\xi_y^{(k)}, \xi_z^{(k)}) \right) \\
 & \leq (\varepsilon |A|)^{-2} \left( \sum_{y \in A} |A \cap \mathbf{B}(y, \frac{n}{\ln^{M_0} n})| + |A|^2 O\left(\frac{\ln \ln n}{\ln n}\right) \right) \\
 (3.9) \quad & \leq \varepsilon^{-2} \ell_A^{(n)} + \varepsilon^{-2} O\left(\frac{\ln \ln n}{\ln n}\right).
 \end{aligned}$$

Let

$$\Phi^{(s)} = \min \{ k : \mathcal{V}^{(k)} \leq s \}$$

be the number of excursions necessary to make the unvisited proportion of  $A$  at most  $s$ . We have

$$\begin{aligned}
 \mathbb{P}[\mathcal{V}(A) \leq s] &= \mathbb{P}[\Phi^{(s)} \leq N] \\
 &= \mathbb{P}[\Phi^{(s)} \leq N, N = \widehat{N}] + \mathbb{P}[\Phi^{(s)} \leq N, N \neq \widehat{N}] \\
 &= \mathbb{P}[\Phi^{(s)} \leq \widehat{N}] + \mathbb{P}[\Phi^{(s)} \leq N, N \neq \widehat{N}] - \mathbb{P}[\Phi^{(s)} \leq \widehat{N}, N \neq \widehat{N}],
 \end{aligned}$$

so, recalling (3.3),

$$(3.10) \quad \left| \mathbb{P}[\mathcal{V}(A) \leq s] - \mathbb{P}[\Phi^{(s)} \leq \widehat{N}] \right| \leq \mathbb{P}[N \neq \widehat{N}] \leq O(n^{-1}).$$

Next, we write

$$\begin{aligned}
 \mathbb{P}[\Phi^{(s)} \leq \widehat{N}] &= \mathbb{E} \left( \mathbb{P}[\widehat{N} \geq \Phi^{(s)} \mid \Phi^{(s)}] \right) \\
 (3.11) \quad &= \mathbb{E}(1 - \psi_n)^{\Phi^{(s)}},
 \end{aligned}$$

(here we used the independence property stated below (3.3)) and concentrate on obtaining lower and upper bounds on the expectation in the right-hand side of (3.11). For this, assume that  $s \in (0, 1)$  is fixed and abbreviate

$$\begin{aligned}
 \delta_n &= \left( \frac{\ln \ln n}{\ln n} \right)^{1/3} \\
 k_n^- &= \left\lfloor (1 - \delta_n) \ln s^{-1} \frac{\ln n}{\ln \ln n} \right\rfloor, \\
 k_n^+ &= \left\lceil (1 + \delta_n) \ln s^{-1} \frac{\ln n}{\ln \ln n} \right\rceil;
 \end{aligned}$$

we also assume that  $n$  is sufficiently large so that  $\delta_n \in (0, \frac{1}{2})$  and  $1 < k_n^- < k_n^+$ . Now, according to (3.7),

$$\begin{aligned} \mu_y^{(k_n^\pm)} &= \exp\left(- (1 \pm \delta_n) \ln s^{-1} \left(1 + O\left((k_n^\pm)^{-1} + \frac{\ln \ln n}{\ln n}\right)\right)\right) \\ &= s \exp\left(- \ln s^{-1} \left(\pm \delta_n + O\left((k_n^\pm)^{-1} + \frac{\ln \ln n}{\ln n}\right)\right)\right) \\ &= s \left(1 + O\left(\delta_n \ln s^{-1} + \frac{\ln \ln n}{\ln n} (1 + \ln s^{-1})\right)\right), \end{aligned}$$

so in both cases it holds that (observe that  $s \ln s^{-1} \leq 1/e$  for all  $s \in [0, 1]$ )

$$(3.12) \quad \mu_y^{(k_n^\pm)} = s + O\left(\delta_n + \frac{\ln \ln n}{\ln n}\right) = s + O(\delta_n).$$

With a similar calculation, one can also observe that

$$(3.13) \quad (1 - \psi_n)^{(k_n^\pm)} = s + O(\delta_n).$$

We then write, using (3.12)

$$\begin{aligned} \mathbb{P}[\Phi^{(s)} > k_n^+] &= \mathbb{P}[\mathcal{V}^{(k_n^+)} > s] \\ &= \mathbb{P}\left[|A|^{-1} \sum_{y \in A} \xi_y^{(k_n^+)} > s\right] \\ &= \mathbb{P}\left[|A|^{-1} \sum_{y \in A} (\xi_y^{(k_n^+)} - \mu_y^{(k_n^+)}) > s - |A|^{-1} \sum_{y \in A} \mu_y^{(k_n^+)}\right] \\ (3.14) \quad &= \mathbb{P}\left[|A|^{-1} \sum_{y \in A} (\xi_y^{(k_n^+)} - \mu_y^{(k_n^+)}) > O(\delta_n)\right]. \end{aligned}$$

Then, (3.9) implies that

$$(3.15) \quad \mathbb{P}[\Phi^{(s)} > k_n^+] \leq O\left(\ell_A^{(n)} \left(\frac{\ln \ln n}{\ln n}\right)^{-2/3} + \left(\frac{\ln \ln n}{\ln n}\right)^{1/3}\right).$$

Quite analogously, one can also obtain that

$$(3.16) \quad \mathbb{P}[\Phi^{(s)} < k_n^-] \leq O\left(\ell_A^{(n)} \left(\frac{\ln \ln n}{\ln n}\right)^{-2/3} + \left(\frac{\ln \ln n}{\ln n}\right)^{1/3}\right).$$

Using (3.13) and (3.15), we then write

$$\begin{aligned} \mathbb{E}(1 - \psi_n)^{\Phi^{(s)}} &\geq \mathbb{E}\left((1 - \psi_n)^{\Phi^{(s)}} \mathbf{1}\{\Phi^{(s)} \leq k_n^+\}\right) \\ &\geq (1 - \psi_n)^{k_n^+} \mathbb{P}[\Phi^{(s)} \leq k_n^+] \\ (3.17) \quad &\geq \left(s - O\left(\left(\frac{\ln \ln n}{\ln n}\right)^{1/3}\right)\right) \left(1 - O\left(\ell_A^{(n)} \left(\frac{\ln \ln n}{\ln n}\right)^{-2/3} + \left(\frac{\ln \ln n}{\ln n}\right)^{1/3}\right)\right), \end{aligned}$$



and, using (3.13) and (3.16),

$$\begin{aligned}
 \mathbb{E}(1 - \psi_n)^{\Phi^{(s)}} &= \mathbb{E}\left((1 - \psi_n)^{\Phi^{(s)}} \mathbf{1}\{\Phi^{(s)} \geq k_n^-\}\right) + \mathbb{E}\left((1 - \psi_n)^{\Phi^{(s)}} \mathbf{1}\{\Phi^{(s)} < k_n^-\}\right) \\
 &\leq (1 - \psi_n)^{k_n^-} + \mathbb{P}[\Phi^{(s)} < k_n^-] \\
 &\leq \left(s + O\left(\left(\frac{\ln \ln n}{\ln n}\right)^{1/3}\right)\right) \\
 (3.18) \qquad &\quad + \left(1 - O\left(\ell_A^{(n)}\left(\frac{\ln \ln n}{\ln n}\right)^{-2/3} + \left(\frac{\ln \ln n}{\ln n}\right)^{1/3}\right)\right).
 \end{aligned}$$

Therefore, using also (3.10)–(3.11), we obtain (1.12), thus concluding the proof of Theorem 1.2.  $\square$

Next, we will prove Theorems 1.7 and 1.8, since the latter will be needed in the course of the proof of Theorem 1.5.

*Proof of Theorem 1.7.* — Clearly, we only need to prove that every infinite subset of  $\mathbb{Z}^d$  is recurrent for  $\widehat{S}$ . Basically, this is a consequence of the fact that, due to (1.10),

$$(3.19) \qquad \lim_{y \rightarrow \infty} \mathbb{P}_{x_0}[\widehat{\tau}_1(y) < \infty] = \frac{1}{2}$$

for any  $x_0 \in \mathbb{Z}^2$ . Indeed, let  $\widehat{S}_0 = x_0$ ; since  $A$  is infinite, by (3.19) one can find  $y_0 \in A$  and  $R_0$  such that  $\{x_0, y_0\} \subset \mathbf{B}(R_0)$  and

$$\mathbb{P}_{x_0}[\widehat{\tau}_1(y_0) < \widehat{\tau}_1(R_0)] \geq \frac{1}{3}.$$

Then, for any  $x_1 \in \partial\mathbf{B}(R_0)$ , we can find  $y_1 \in A$  and  $R_1 > R_0$  such that  $y_1 \in \mathbf{B}(R_1) \setminus \mathbf{B}(R_0)$  and

$$\mathbb{P}_{x_1}[\widehat{\tau}_1(y_1) < \widehat{\tau}_1(R_1)] \geq \frac{1}{3}.$$

Continuing in this way, we can construct a sequence  $R_0 < R_1 < R_2 < \dots$  (depending on the set  $A$ ) such that, for each  $k \geq 0$ , the walk  $\widehat{S}$  hits  $A$  on its way from  $\partial\mathbf{B}(R_k)$  to  $\partial\mathbf{B}(R_{k+1})$  with probability at least  $\frac{1}{3}$ , regardless of the past. This clearly implies that  $A$  is a recurrent set.  $\square$

*Proof of Theorem 1.8.* — Indeed, Theorem 1.7 implies that every subset of  $\mathbb{Z}^2$  must be either recurrent or transient, and then Proposition 3.8 in [Rev84, Chapter 2] implies the Liouville property. Still, for the reader’s convenience, we include the proof here. Assume that  $h : \mathbb{Z}^2 \setminus \{0\} \rightarrow \mathbb{R}$  is a bounded harmonic function for  $\widehat{S}$ . Let us prove that

$$(3.20) \qquad \liminf_{y \rightarrow \infty} h(y) = \limsup_{y \rightarrow \infty} h(y),$$

that is,  $h$  must have a limit at infinity. Indeed, assume that (3.20) does not hold, which means that there exist two constants  $b_1 < b_2$  and two infinite sets  $B_1, B_2 \subset \mathbb{Z}^2$  such that  $h(y) \leq b_1$  for all  $y \in B_1$  and  $h(y) \geq b_2$  for all  $y \in B_2$ . Now, on one hand  $h(\widehat{S}_n)$  is a bounded martingale, so it must a.s. converge to some limit; on the other hand, Theorem 1.7 implies that both  $B_1$  and  $B_2$  will be visited infinitely often by  $\widehat{S}$ , and so  $h(\widehat{S}_n)$  cannot converge to any limit, thus yielding a contradiction. This proves (3.20).

Now, if  $\lim_{y \rightarrow \infty} h(y) = c$ , then it is easy to obtain from the Maximum Principle that  $h(x) = c$  for any  $x$ . This concludes the proof of Theorem 1.8.  $\square$

Finally, we are able to prove that there are “big holes” in the range of  $\widehat{S}$ :

*Proof of Theorem 1.5.* — Clearly, if  $G$  does not surround the origin in the sense of Definition 1.4, then  $G \subset \mathbf{B}(c_1) \setminus \mathbf{B}(c_3)$ . For the sake of simplicity, let us assume that  $G \subset \mathbf{B}(1) \setminus \mathbf{B}(1/2)$ ; the general case can be treated in a completely analogous way.

Consider the two sequences of events

$$E_n = \left\{ \widehat{\tau}_1(2^{3n-1}G) > \widehat{\tau}_1(2^{3n}), \|\widehat{S}_j\| > 2^{3n-1} \text{ for all } j \geq \widehat{\tau}_1(2^{3n}) \right\},$$

$$E'_n = \left\{ \|\widehat{S}_j\| > 2^{3n-1} \text{ for all } j \geq \widehat{\tau}_1(2^{3n}) \right\}$$

and note that  $E_n \subset E'_n$  and  $2^{3n-1}G \cap \widehat{S}_{[0,\infty)} = \emptyset$  on  $E_n$ . Our goal is to show that a.s. an infinite number of events  $(E_n, n \geq 1)$  occurs. Observe, however, that the events in each of the above two sequences are *not* independent, so the “basic” second Borel–Cantelli lemma will not work.

In the following, we use a generalization of the second Borel–Cantelli lemma, known as the Kochen–Stone theorem [KS64]: it holds that

$$(3.21) \quad \mathbb{P} \left[ \sum_{k=1}^{\infty} \mathbf{1}\{E_k\} = \infty \right] \geq \limsup_{k \rightarrow \infty} \frac{\left( \sum_{i=1}^k \mathbb{P}[E_i] \right)^2}{\sum_{i,j=1}^k \mathbb{P}[E_i \cap E_j]}.$$

We will now prove that there exists a positive constant  $c_4$  such that

$$(3.22) \quad \mathbb{P}[E_n] \geq \frac{c_4}{n} \quad \text{for all } n \geq 1.$$

Indeed, since  $G \subset \mathbf{B}(1) \setminus \mathbf{B}(1/2)$  does not surround the origin, by comparison with Brownian motion it is elementary to obtain that, for some  $c_5 > 0$ ,

$$\mathbb{P}_x \left[ \tau_1(2^{3n-1}G) > \tau_1(2^{3n}), \tau_1(0) > \tau_1(2^{3n}) \right] > c_5$$

for all  $x \in \partial\mathbf{B}(2^{3(n-1)})$ . Lemma 2.1 then implies that, for some  $c_6 > 0$ ,

$$(3.23) \quad \begin{aligned} & \mathbb{P}_x \left[ \widehat{\tau}_1(2^{3n-1}G) > \widehat{\tau}_1(2^{3n}) \right] \\ &= \left( 1 + o(2^{-3n}) \right) \mathbb{P}_x \left[ \tau_1(2^{3n-1}G) > \tau_1(2^{3n}) \mid \tau_1(0) > \tau_1(2^{3n}) \right] \\ &= \left( 1 + o(2^{-3n}) \right) \mathbb{P}_x \left[ \tau_1(2^{3n-1}G) > \tau_1(2^{3n}), \tau_1(0) > \tau_1(2^{3n}) \right] > c_6 \end{aligned}$$

for all  $x \in \partial\mathbf{B}(2^{3(n-1)})$ . Let us denote, recalling (1.7),  $\gamma^* = \frac{\pi}{2} \times \frac{1}{\ln 2} \times \frac{2\gamma+3\ln 2}{\pi} = \frac{2\gamma+3\ln 2}{2\ln 2}$ . Using (2.4), we then obtain

$$(3.24) \quad \begin{aligned} \mathbb{P}_z \left[ \|\widehat{S}_j\| > 2^{3n-1} \text{ for all } j \geq 0 \right] &= 1 - \frac{a(2^{3n-1}) + O(2^{-3n})}{a(2^{3n}) + O(2^{-3n})} \\ &= \frac{1}{3n + \gamma^*} \left( 1 + o(2^{-3n}) \right). \end{aligned}$$

for any  $z \in \partial\mathbf{B}(2^{3n})$ . The inequality (3.22) follows from (3.23) and (3.24).

Now, we need an upper bound for  $\mathbb{P}[E_m \cap E_n]$ ,  $m \leq n$ . Clearly,  $E_m \cap E_n \subset E'_m \cap E'_n$ , and note that the event  $E'_m \cap E'_n$  means that the particle hits  $\partial\mathbf{B}(2^{3m})$  before  $\partial\mathbf{B}(2^{3m-1})$

starting from a site on  $\partial\mathcal{B}(2^{3m})$ , and then never hits  $\partial\mathcal{B}(2^{3n-1})$  starting from a site on  $\partial\mathcal{B}(2^{3n})$ . So, again using (2.4) and Lemma 2.1, we write analogously to (3.24) (and also omitting a couple of lines of elementary calculations)

$$\begin{aligned}
 \mathbb{P}[E_m \cap E_n] &\leq \mathbb{P}[E'_m \cap E'_n] \\
 &= \frac{(a(2^{3m-1}))^{-1} - (a(2^{3m}))^{-1} + O(2^{-3m})}{(a(2^{3m-1}))^{-1} - (a(2^{3n}))^{-1} + O(2^{-3m})} \times \left(1 - \frac{a(2^{3n-1}) + O(2^{-3n})}{a(2^{3n}) + O(2^{-3n})}\right) \\
 (3.25) \quad &= \frac{1}{(3(n-m)+1)(3m+\gamma^*)} (1 + o(2^{-3m})).
 \end{aligned}$$

Now, (3.22) implies that  $\sum_{i=1}^k \mathbb{P}[E_i] \geq c_9 \ln k$ , and (3.25) implies (again, after some elementary calculations) that  $\sum_{i,j=1}^k \mathbb{P}[E_i \cap E_j] \leq c_{10} \ln^2 k$ . So, using (3.21), we obtain that

$$\mathbb{P}\left[\sum_{k=1}^{\infty} \mathbf{1}\{E_k\} = \infty\right] \geq c_{11} > 0.$$

Now, note that, again due to Proposition 3.8 in [Rev84, Chapter 2], the Liouville property implies that every tail event must have probability 0 or 1, and so the probability in the above display must be equal to 1. This concludes the proof of Theorem 1.5.  $\square$

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