Annales Henri Lebesgue
7 (2024) 823-839

ANNALES
HENRI LEBESGUE

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ON THE SIGNATURE OF A POSITIVE BRAID

SUR LA SIGNATURE D'UNE TRESSE POSITIVE

Abstract. — We show that the signature of a positive braid link is bounded from below by one-quarter of its first Betti number. This equates to one-half of the optimal bound conjectured by Feller, who previously provided a bound of one-eighth.

Résumé. — On montre que la signature d'un entrelacs représentable par une tresse positive est au moins un quart de son premier nombre de Betti. Cela correspond a la moitié de la borne optimale conjecturée par Feller, qui avait auparavant prouvé une borne d'un huitième.

1. Introduction

The signature $\sigma(L)$ of an oriented link L was introduced by Trotter [Tro62]. Its definition leads to the classic lower bound $\sigma(L) \leqslant b_1(L)$ on the minimum first Betti number of any Seifert surface for L. For certain classes of links, this bound can be reversed, up to scale[(1)]. The first such result is due to Feller, who proved

Keywords: Signature, positive braid, chessboard surface, Goeritz form.
2020 *Mathematics Subject Classification:* 57K10.
DOI: `https://doi.org/10.5802/ahl.212`
(*) JEG was supported by the National Science Foundation under Award No. DMS-2304856 and by a Simons Fellowship.

(1) While this does not coincide with the most commonly used definitions, we adopt Rudolph's convention that positive links have positive signature.

the lower bound $\sigma(L) \geqslant \frac{1}{100} b_1(L)$ for a positive braid closure L and conjectured an optimal slope of $\frac{1}{2}$ for these links [Fel15]. Notably, Rudolph had earlier shown that $\sigma(L) \geqslant 0$, and Stoimenow had shown that $\sigma(L) \geqslant b_1(L)^{1/3}$, for a positive braid closure L [Rud82, Sto08]. Subsequently, the slope was improved to $\frac{1}{24}$ for the more general class of positive links by Baader, Dehornoy and the second author [BDL18], and to $\frac{1}{8}$ by Feller for positive braid closures [Fel18].

Our goal is to establish the following result, striking within one-half of Feller's conjectured bound:

Theorem 1.1. — *For every positive braid closure L that is not an unlink, we have*

$$\sigma(L) \geqslant \frac{b_1(L)}{4} + \frac{1}{2}.$$

Our proof of Theorem 1.1 uses the signature formula of Gordon and Litherland for the Goeritz form of chessboard surfaces [GL78], as in [BDL18, Fel18]. Our advance in the case of a positive braid closure stems from a careful choice of subspaces on which we are able to tightly control the signature of the Goeritz form.

One-half of the signature of a knot is a lower bound for the topological four-genus [KT76, Mur65]. Theorem 1.1 immediately implies the following lower bound for the topological four-genus in terms of the usual Seifert genus.

Corollary 1.2. — *The topological four-genus of a positive braid knot is greater than one-quarter of the Seifert genus.*

For the consequences of Corollary 1.2 with respect to concordance, we refer to the discussions by Stoimenow [Sto08] as well as Baader, Dehornoy and the second author [BDL18].

Another application of Theorem 1.1 concerns the location of the zeroes of the Alexander polynomial of positive braid closures. Dehornoy noted that for random positive braids, these zeroes seem to accumulate on rather specific lines depending on the braid index and the probability of the braid generators, as in Figure 3.4 and the discussion surrounding it in [Deh15]. Since the signature is a lower bound for the number of zeroes of the Alexander polynomial that lie on the unit circle [GL16, Lie16], Theorem 1.1 at least explains why there is a substantial number of zeroes on the unit circle.

Corollary 1.3. — *For a positive braid link, more than one-quarter of the zeroes of the Alexander polynomial lie on the unit circle.*

Organisation. In Section 2, we introduce the necessary background on positive braids and their diagram combinatorics, as well as chessboard surfaces and the Goeritz form. In Section 3, we study the signature of specific integer matrices that we call trisum matrices. These matrices appear again as the matrices of the Goeritz form restricted to certain subspaces in Section 4, where we give a proof of Theorem 1.1. Finally, in Section 5, we discuss limitations of our approach.

Acknowledgements. We thank the anonymous referees for their comments on a first version of this article.

2. Background

2.1. Diagram combinatorics

Let β be a *positive braid* on $n+1$ strands, that is, a product of positive powers of the standard braid generators $\sigma_1, \ldots, \sigma_n$ of the braid group B_{n+1}. The *standard link diagram* D for the closure L of β contains a positive crossing for every occurrence of a braid generator σ_i in β. The *index* of a crossing is the index of the standard braid generator corresponding to it. An example of a standard diagram is shown in Figure 2.1.

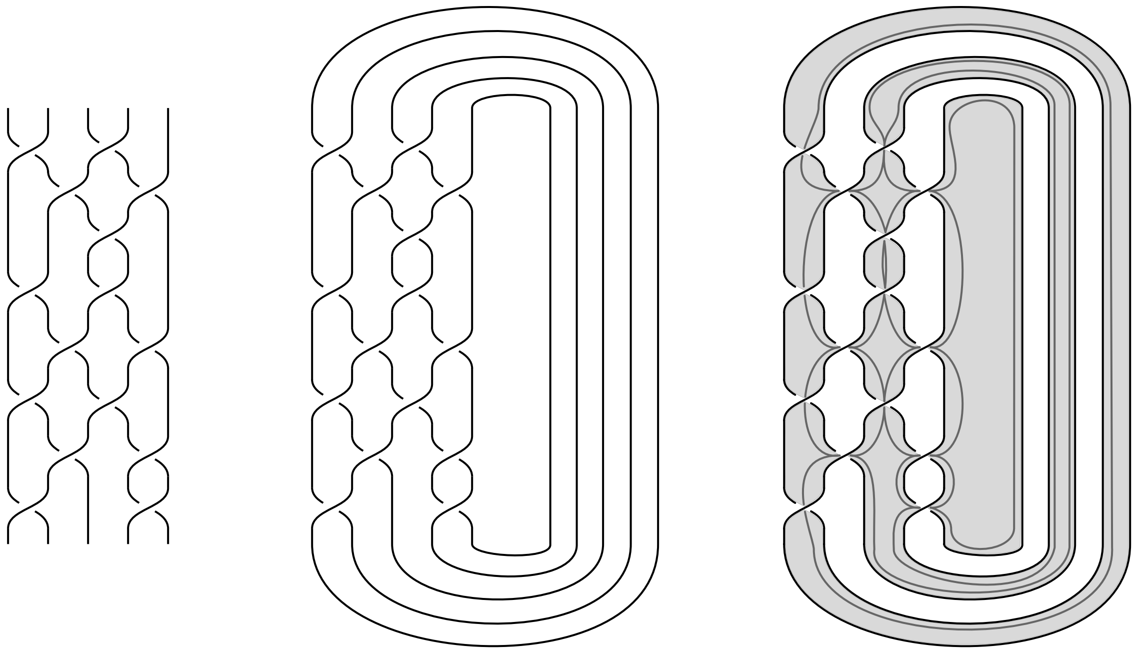

Figure 2.1. The geometric realization of the positive braid $\sigma_1\sigma_4^2\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_1\sigma_3^2\sigma_2\sigma_4\sigma_1\sigma_3$, its standard diagram and a chessboard coloring thereof. On the right, the grey curves indicate a basis of the first homology of the black chessboard surface.

Let $\mathrm{cr}(D)$ be the number of crossings of D. There are different types of faces in the diagram D. For each crossing of the diagram there is a face that starts above it. There are $\mathrm{cr}(D)$ faces of this type. Furthermore, there is an unbounded face F' and a face F containing the braid axis; compare with Figure 2.1. Among the $\mathrm{cr}(D)$ faces that belong to the first category, let f_i be the number of faces with i sides, for $i \geqslant 2$. Locally, every crossing is met by four faces, so we have

$$4\mathrm{cr}(D) = \sum_{i\geqslant 2} i f_i + s + s',$$

where s and s' are the number of sides of F and F', respectively.

Since

$$\sum_{i\geqslant 2} f_i = \mathrm{cr}(D),$$

subtracting $4\mathrm{cr}(D)$ from both sides yields

$$0 = \sum_{i \geqslant 2} (i-4) f_i + s + s',$$

and thus

$$2f_2 + f_3 = s + s' + \sum_{i \geqslant 5} (i-4) f_i. \tag{2.1}$$

We will use another fact about nonsplit positive braid links L on $n+1$ strands with standard diagram D, namely

$$b_1(L) = \mathrm{cr}(D) - n. \tag{2.2}$$

This is a consequence of Stallings's construction [Sta78] of a fibre surface (which must be genus-minimising) for nonsplit positive braid closures with first Betti number equal to $\mathrm{cr}(D) - n$.

2.2. Chessboard surfaces

Choose the chessboard colouring of the standard diagram D defined by the following property: the faces above odd index crossings are black, as in Figure 2.1. Let S_B be the (not necessarily orientable) surface defined by the black faces of this colouring, with $\partial S_B = L$, and let S_W be the (not necessarily orientable) surface defined by the white faces of this colouring. We have

$$\dim(H_1(S_B)) + \dim(H_1(S_W)) = \mathrm{cr}(D). \tag{2.3}$$

2.3. Goeritz form

The Goeritz form G is a symmetric bilinear form on the first homology $H_1(S)$ of a compact, not necessarily orientable surface S embedded in $\mathbb{S}^3$; see Goeritz's article [Goe33] or also Chapter 9 in Lickorish's book [Lic97]. For two simple closed curves γ_1 and γ_2 in S, the Goeritz form on the corresponding homology elements is defined as $G(\gamma_1, \gamma_2) = \mathrm{lk}(\gamma_1, \gamma_2^{\pm})$. Here, lk denotes the linking number and $\gamma_2^{\pm}$ denotes the two-sided push-off of γ_2 along the normal direction to S. In order to obtain positivity of the signature for positive links, we (nonstandardly) define the linking number to count one-half of the negative crossings minus one-half of the positive crossings between links.

If S is orientable, then by definition G equals the symmetrised Seifert form and thus only depends on ∂S. In particular, $\sigma(G) = \sigma(\partial S) = \sigma(L)$, the signature invariant of the boundary link. If S is nonorientable, a correction term is necessary, leading to the formula $\sigma(L) = \sigma(G) - \mu$ by Gordon and Litherland; see [GL78, Theorem 6] and the discussion following it, which extends the result to links. Here, μ is a correction term counting the number of positive minus the number of negative crossings among all crossings of a diagram where any local orientation of the chessboard surface fails to induce the correct link orientation on the boundary. Our nonstandard definition of the linking number results in a factor -1 for both $\sigma(L)$ and $\sigma(G)$ in the formula, so in our convention, the formula reads $\sigma(L) = \sigma(G) + \mu$. Since μ counts every crossing of a diagram for exactly one of the two chessboard surfaces, we sum both bounds to obtain the following result, which we use later on.

Theorem 2.1. — *Let D be a positive diagram of a link L, and let S_B and S_W be the two associated chessboard surfaces with Goeritz forms G_B and G_W, respectively. Then*

$$2\sigma(L) = \sigma(G_B) + \sigma(G_W) + \mathrm{cr}(D).$$

In order to use Theorem 2.1, we have to consider the Goeritz forms G_B and G_W of the chessboard surfaces S_B and S_W, respectively. For the respective first homologies, we pick a basis that consists of curves γ winding around white and black faces in the counterclockwise sense: see Figure 2.1.

Lemma 2.2. — *Let β be a positive braid on $n + 1$ strands that uses each generator σ_i at least twice, and let D be the standard diagram for its closure L. Let G_B and G_W be the Goeritz forms of the chessboard surfaces S_B and S_W, respectively. We have the following:*

(i) *if γ winds around an n-sided face that is not F or F', $G_*(\gamma, \gamma) = 4 - n$;*
(ii) *if γ_1 and γ_2 wind around adjacent faces of the same index whose boundaries meet in one crossing, $G_*(\gamma_1, \gamma_2) = -1$;*
(iii) *if γ_1 and γ_2 wind around faces with no common crossing in their boundaries, $G_*(\gamma_1, \gamma_2) = 0$;*
(iv) *if γ_1 and γ_2 wind around faces whose indices are apart by two and whose boundaries meet in one common crossing, $G_*(\gamma_1, \gamma_2) = 1$.*

Proof. — All four facts can be checked directly using the definition. Alternatively, one can use Lickorish's combinatorial description of the coefficients of a Goeritz matrix in [Lic97, Chapter 9]. Comparing with Lickorish's description, one should keep in mind that our nonstandard definition of the linking number results in a sign change for each coefficient. □

Remark 2.3. — We later change the orientation of certain curves γ_i in order to get particularly simple, nonnegative Goeritz matrices. We note that (ii) in Lemma 2.2 holds if the curves γ_1 and γ_2 are both oriented counterclockwise or if both are oriented clockwise. Otherwise, $G_*(\gamma_1, \gamma_2) = +1$. Similarly, (iv) in Lemma 2.2 holds if both curves are oriented counterclockwise or if both are oriented clockwise.

3. A signature bound for tridiagonal and trisum matrices

In this section, we discuss signature bounds for certain tridiagonal matrices. Further, we introduce and study a generalisation of them which we call trisum matrices. These matrices and their direct sums appear as the Gram matrices we study in the next section, when we restrict the Goeritz forms of the chessboard surfaces to appropriate subspaces.

3.1. Tridiagonal matrices.

Let $T(d_1, \ldots, d_n)$ denote the tridiagonal matrix with diagonal entries $d_1, \ldots, d_n \in \mathbb{Z}$ and with 1s on the secondary diagonals. We use the shorthand $d^{\underline{a}}$ to denote a string of a copies of d, where $a, d \in \mathbb{Z}$, $a \geqslant 0$.

The first lemma is easy and well-known, and we supply one short proof related to those that follow.

Lemma 3.1. — *If $\epsilon = \pm 1$ and $M = T((\epsilon \cdot 2)^{\underline{a}})$, then $\sigma(M) = \epsilon \cdot \dim(M)$.*

Proof. — The Gram-Schmidt algorithm shows that M is congruent over $\mathbb{Q}$ to the diagonal matrix with entries $\epsilon \cdot \frac{2}{1}, \epsilon \cdot \frac{3}{2}, \ldots, \epsilon \cdot \frac{a+1}{a}$. It follows that M is definite with sign ϵ, so $\sigma(M) = \epsilon \cdot \dim(M)$. □

The following result holds for general integer coefficients on the diagonal. Let $\underline{tr}(M)$ denote the sum of the negative entries on the diagonal of the matrix M.

Proposition 3.2. — *If $M = T(d_1, \ldots, d_n)$, then*

$$\sigma(M) \geqslant -\frac{1}{2} + \frac{1}{2}\underline{tr}(M).$$

Remark 3.3. —

(1) If $d_1, \ldots, d_n \geqslant 0$, then $\sigma(M) \geqslant 0$: for lowering diagonal entries does not raise the signature, and $\sigma(M) = 0$ when all d_i equal zero.
(2) The matrix $M = T(-1, -2, \ldots, -2)$ is negative definite and attains the bound in Proposition 3.2.
(3) The bound in Proposition 3.2 does not depend on the number or sum of positive diagonal coefficients. For example, if $M = T(0, N, 0, N, \ldots, 0, N, -1)$ with N any positive integer, then $\sigma(M) = -1$, which coincides with the bound in Proposition 3.2.
(4) Since $\sigma(-M) = -\sigma(M)$, Proposition 3.2 also implies the upper bound

$$\sigma(M) \leqslant \frac{1}{2} - \frac{1}{2}\underline{tr}(-M) = \frac{1}{2} + \overline{tr}(M),$$

where $\overline{tr}(M)$ denotes the sum of the positive diagonal coefficients of the matrix M. We shall not make use of this bound.

Proof of Proposition 3.2. — We proceed by induction on the dimension n of the matrix M. In the one-dimensional case $n = 1$, the statement clearly holds. We note that the summand $-\frac{1}{2}$ is necessary precisely for the matrix (-1).

We need a second base case $n = 2$ in case the first diagonal coefficient is $d_1 = 0$ or if the first diagonal coefficient is $d_1 = -1$ and the second is $d_2 \geqslant 0$. In the first case, $\sigma(M) = 0$, so the bound is satisfied. In the second case $\sigma(M) = 0$ and the bound is satisfied, even improved by a summand $+\frac{1}{2}$.

We now assume that $n \geqslant 2$, and $n \geqslant 3$ in case $d_1 = 0$ or if $d_1 = -1$ and $d_2 \geqslant 0$. We distinguish cases depending on the first diagonal coefficient d_1.

Case 1: $d_1 \geqslant 1$. — The signature of the matrix

$$M = \left(\begin{array}{c|ccc} d_1 & 1 & & \\ \hline 1 & d_2 & 1 & \\ & 1 & \ddots & \\ & & & \end{array}\right) \sim \left(\begin{array}{c|ccc} d_1 & & & \\ \hline & d_2 - \frac{1}{d_1} & 1 & \\ & 1 & \ddots & \\ & & & \end{array}\right)$$

is bounded from below by $1 + \sigma(M')$, where M' is the matrix obtained from M by deleting the first row and the first column and reducing the diagonal coefficient d_2 by 1. We have $\underline{tr}(M') \geqslant \underline{tr}(M) - 1$. By induction hypothesis, the statement is true for M', and we obtain

$$\sigma(M) \geqslant 1 + \sigma(M') \geqslant 1 - \frac{1}{2} + \frac{1}{2}\underline{tr}(M') \geqslant \frac{1}{2}\underline{tr}(M).$$

We see that an occurrence of this case actually improves the signature bound by at least $\frac{1}{2}$, and the signature bound is even improved by 1 if $d_2 > 0$.

Case 2: $d_1 = 0$. — The signature of the matrix

$$M = \left(\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & \cdots \\ 1 & d_2 & 1 & 0 & \cdots \\ \hline 0 & 1 & & & \\ 0 & 0 & & M' & \\ \vdots & \vdots & & & \end{array}\right) \sim \left(\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & \cdots \\ 1 & d_2 & 0 & 0 & \cdots \\ \hline 0 & 0 & & & \\ 0 & 0 & & M' & \\ \vdots & \vdots & & & \end{array}\right)$$

equals the signature of the matrix M'. Since $\underline{tr}(M') \geqslant \underline{tr}(M)$, we obtain the desired bound for $\sigma(M)$ by induction hypothesis.

Case 3: $d_1 = -1$. — The signature of the matrix

$$M = \left(\begin{array}{c|ccc} -1 & 1 & & \\ \hline 1 & d_2 & 1 & \\ & 1 & \ddots & \\ & & & \end{array}\right) \sim \left(\begin{array}{c|ccc} -1 & & & \\ \hline & d_2 + 1 & 1 & \\ & 1 & \ddots & \\ & & & \end{array}\right)$$

equals $-1 + \sigma(M')$, where M' is the matrix obtained from M by deleting the first row and the first column and replacing the diagonal coefficient d_2 by $d_2 + 1$. We distinguish two cases. If $d_2 < 0$, then $\underline{tr}(M') = \underline{tr}(M) + 2$. In particular, the desired lower bound for $\sigma(M)$ follows from the induction hypothesis on M'. If $d_2 \geqslant 0$, then $\underline{tr}(M') = \underline{tr}(M) + 1$. This is not enough to prove the desired inequality by the induction hypothesis on M'. However, we notice that after such a case, the matrix M' is as in Case 1 of our case distinctions, which in turn improves the signature bound for M' by at least $\frac{1}{2}$. This exactly compensates for the loss in the present case and the desired bound is attained after two steps.

Case 4: $d_1 \leqslant -2$. — The signature of the matrix

$$M = \left(\begin{array}{c|ccc} d_1 & 1 & & \\ \hline 1 & d_2 & 1 & \\ & 1 & \ddots & \\ & & & \end{array}\right) \sim \left(\begin{array}{c|ccc} d_1 & & & \\ \hline & d_2 - \frac{1}{d_1} & 1 & \\ & 1 & \ddots & \\ & & & \end{array}\right)$$

is bounded from below by $-1 + \sigma(M')$, where M' is the matrix obtained from M by deleting the first row and the first column. We have $\underline{tr}(M') \geqslant \underline{tr}(M) + 2$. Again, the desired inequality follows from the induction hypothesis on M'.

□

3.2. Trisum matrices

We will need an extension of the lower bound of Proposition 3.2 to a slightly wider class of matrices.

Let $N = T(d_1, \ldots, d_r)$ and let $C_i = T(2^{\underline{a}_i}, 1, 2^{\underline{b}_i})$, $i = 1, \ldots, k$. Let $\widetilde{M}$ be the direct sum of matrices $\widetilde{M} = N \oplus C_1 \oplus \cdots \oplus C_k$. Furthermore, for each $i = 1, \ldots, k$, let $1 \leqslant g(i) \leqslant r$ be an index with $d_{g(i)} \leqslant 0$, and let $h(i)$ be the index of the column of $\widetilde{M}$ that contains the diagonal coefficient 1 of the block C_i. Let M be the matrix obtained from $\widetilde{M}$ by letting $M_{g(i),h(i)} = M_{h(i),g(i)} = 1$ for each $i = 1, \ldots, k$ and keeping all other entries the same. We call M a *trisum* matrix with *core* N and *blocks* C_i. Let $\underline{tr}(M)$ denote the sum of the negative diagonal coefficients of its core, and let $b(M)$ denote the sum of the dimensions of its blocks.

Proposition 3.4. — *If M is a trisum matrix, then*

$$\sigma(M) \geqslant -\frac{1}{2} + \frac{1}{2}\underline{tr}(M) + \frac{1}{2}b(M).$$

Proof. — We proceed by induction on the number k of blocks of M.

If $k = 0$, then the bound coincides with the bound from Proposition 3.2.

Now suppose the bound holds for matrices with k blocks C_i, and let M be a matrix with $k+1$ blocks C_i. We make a case distinction depending on the block C_{k+1}.

Case 1: $C_{k+1} = T(1)$. — In this case, $\sigma(M) = 1 + \sigma(M')$, where M' is obtained from M by deleting the last row and column and replacing the diagonal coefficient $d_{g(k+1)}$ by $d_{g(k+1)} - 1$. Thus, $\underline{tr}(M') = \underline{tr}(M) - 1$ and $b(M') = b(M) - 1$. Applying the induction hypothesis for M', we get

$$\begin{aligned}\sigma(M) = 1 + \sigma(M') &\geqslant 1 - \frac{1}{2} + \frac{1}{2}\underline{tr}(M') + \frac{1}{2}b(M') \\ &= -\frac{1}{2} + \frac{1}{2}\underline{tr}(M) + \frac{1}{2}b(M).\end{aligned}$$

Case 2: $C_{k+1} = T(2^{\underline{a}}, 1, 2^{\underline{b}})$ for $a = 0$ or $b = 0$. — Note that the matrix $T(2^{\underline{a}}, 1)$ is congruent to the diagonal matrix with diagonal coefficients $\frac{2}{1}, \frac{3}{2}, \ldots, \frac{a+1}{a}, \frac{1}{a+1}$. In particular, we get that $\sigma(M) = a + 1 + \sigma(M')$, where M' is obtained from M by deleting the last $a+1$ rows and columns and replacing the diagonal coefficient $d_{g(k+1)}$

by $d_{g(k+1)} - (a+1)$. As in Case 1, applying the induction hypothesis for M' gives the desired bound.

Case 3: $C_{k+1} = T(2,1,2)$. — The matrix $T(2,1,2)$ is congruent to the diagonal matrix with diagonal coefficients $2,0,2$. It follows that $\sigma(M) = 2+\sigma(M')$, where M' is obtained from M by deleting the last three rows and columns, but also the row and column of index $g(k+1)$. In order to see this, first apply the base change to M that effectuates the congruence of the block $T(2,1,2)$ to the diagonal matrix with diagonal coefficients $2,0,2$. The row and the column of index $h(k+1)$ now have one nonzero coefficient, namely $M_{g(k+1),h(k+1)} = M_{h(k+1),g(k+1)} = 1$. As a next change of base, the row/column of index $h(k+1)$ can be subtracted as often as necessary from other rows/columns in order for the row and the column of index $g(k+1)$ to have only their diagonal coefficient nonzero, as well as the coefficients $M_{g(k+1),h(k+1)} = M_{h(k+1),g(k+1)} = 1$. After this change of base, the last three indices together with index $g(k+1)$ become a direct summand of the matrix M, with signature 2.

We remark that deleting the row and column of index $g(k+1)$ makes the blocks C_i with $g(i) = g(k+1)$ into direct summands of the matrix M'. Therefore M' decomposes as $M'_1 \oplus M'_2 \oplus C$ where each M'_j has at most k of the blocks C_i and is of the form so that we can apply the induction hypothesis, and C consists of direct summands of the type C_i. For a block of type C_i, we have $\sigma(C_i) \geqslant \frac{1}{2}\dim(C_i)$. Applying the induction hypothesis to M'_j, we obtain

$$\begin{aligned}
\sigma(M) &= 2+\sigma(M') \\
&= 2+\sigma(M'_1)+\sigma(M'_2)+\sigma(C) \\
&\geqslant 2-\frac{1}{2}+\frac{1}{2}\underline{tr}(M'_1)+\frac{1}{2}b(M'_1)-\frac{1}{2}+\frac{1}{2}\underline{tr}(M'_2)+\frac{1}{2}b(M'_2)+\frac{1}{2}\dim(C) \\
&\geqslant 1+\frac{1}{2}\underline{tr}(M)+\frac{1}{2}\left(b(M'_1)+b(M'_2)+\dim(C)\right) \\
&= -\frac{1}{2}+\frac{1}{2}\underline{tr}(M)+b(M),
\end{aligned}$$

where the first inequality uses the induction hypothesis for both M'_1 and M'_2, and the second inequality uses the fact that $d_{g(k+1)} \leqslant 0$, which in particular means that $\underline{tr}(M'_1)+\underline{tr}(M'_2) \geqslant \underline{tr}(M)$.The final equality follows from $\dim(C_{k+1}) = 3$.

Case 4: $C_{k+1} = T(2^{\underline{a}},1,2^{\underline{b}})$ for $a \geqslant 1$ and $b \geqslant 1$. — Case 3 covers the case $a = b = 1$, so we can suppose that $a > 1$ or $b > 1$. The matrix $T(2^{\underline{a}},1,2^{\underline{b}})$ is congruent to the diagonal matrix with coefficients $\frac{2}{1}, \ldots, \frac{a+1}{a}, x, \frac{b+1}{b}, \ldots, \frac{2}{1}$, where $x = 1-\frac{a}{a+1}-\frac{b}{b+1} < 0$. In particular, $\sigma(M) \geqslant a+b-1+\sigma(M')$, where M' is obtained from M by deleting the last $a+b+1$ rows and columns. Applying the induction hypothesis for M' yields the desired bound, since $a+b \geqslant 3$ and therefore the improvement $a+b-1$ is at least half as large as the deficit $a+b+1$ for the bound we have when we apply the induction hypothesis to M'. □

The congruences noted in the proof of Proposition 3.4 establish the following useful result.

Porism 3.5. — *If $M = T(2^{\underline{a}}, 1, 2^{\underline{b}})$, then*

$$\sigma(M) = \begin{cases} \dim(M), & \text{if } \min\{a,b\} = 0, \\ \dim(M) - 1, & \text{if } a = b = 1, \text{ and} \\ \dim(M) - 2, & \text{otherwise.} \end{cases}$$

In particular, $\sigma(M) \geqslant \frac{1}{2}\dim(M)$. □

4. Proof of Theorem 1.1

4.1. Preliminaries.

Let L be a positive braid link, and let β be a positive braid whose closure is L such that

(a) β has minimal index $n \geqslant 1$ among all positive braids whose closure is L and
(b) the sum of the indices of the generators appearing in β is minimal subject to (a).

We quickly dispense with some trivial cases:

- If $n = 1$, then L is the unknot, and $\sigma(L) = b_1(L) = 0$.
- If $n \geqslant 2$ and β does not use the generator σ_i for some $1 \leqslant i \leqslant n-1$, then L is the split union $L_1 \sqcup L_2$, where each of L_1 and L_2 is a positive braid link. In this case, $\sigma(L) = \sigma(L_1) + \sigma(L_2)$ and $b_1(L) = b_1(L_1) + b_1(L_2)$.
- If $n \geqslant 2$ and β uses a single generator σ_i for some $1 \leqslant i \leqslant n-1$, then L is the connected sum $L_1 \# L_2$, where each of L_1 and L_2 is a positive braid link. Once more, in this case, we have $\sigma(L) = \sigma(L_1) + \sigma(L_2)$ and $b_1(L) = b_1(L_1) + b_1(L_2)$.
- If $n = 2$ and β uses σ_1 at least twice, then L is a $(2,k)$-torus link, $k \geqslant 2$, and $k - 1 = \sigma(L) = b_1(L) \geqslant \frac{1}{4} b_1(L) + \frac{1}{2}$.

Thus, Theorem 1.1 follows if we can prove it under the additional assumptions that

(c) $n \geqslant 3$ and
(d) β uses each generator σ_i at least twice,

which we assume in the remainder of the section. In particular, we have $s + s' \geqslant 4$ in the notation of Section 2.1, so Equation (2.1) entails

$$2f_2 + f_3 \geqslant 4 + \sum_{i \geqslant 5} (i-4) f_i. \tag{4.1}$$

Lemma 4.1. — *In every column of β except possibly the first, there exists a face with at least four sides.*

Proof. — Let $i > 1$ denote the index of a column of β. Select an occurrence of σ_{i-1} in β and let f denote the face in the i^{th} column incident to it. By assumption (d), the top and bottom crossings incident with f exist and are distinct from one another. Suppose for a contradiction that f is not incident with any other crossings. Then a cyclic permutation and a sequence of distant braid moves converts β into a minimal index positive braid representation β' of L with the same index sum as β and which contains the subword $\sigma_i \sigma_{i-1} \sigma_i$. Then the braid move $\sigma_i \sigma_{i-1} \sigma_i \to \sigma_{i-1} \sigma_i \sigma_{i-1}$

converts β' into a minimal index positive braid representation β'' of L with lower index sum than β', violating assumption (b). Hence f is incident with at least one additional crossing, so it has at least four sides. □

4.2. Subsets and subspaces

In this section, we describe our choice of subspaces on which to examine the Goeritz form in the case of a positive braid closure. The subspaces are chosen so that the restrictions of the Goeritz forms to them are presented by trisum matrices. Then we can estimate their signatures using the results of Section 3 towards the end of proving Theorem 1.1.

4.2.1. The black surface

We define two subsets $B_0, B_2 \subset H_1(S_B)$ of the first homology of the black chessboard surface, obtained by selecting homology classes of certain curves γ winding around white faces.

Let B_0 denote the set of homology classes of the following curves in S_B. First, we take all curves γ corresponding to white faces above crossings of index $i \equiv 0 \,(\mathrm{mod} 4)$, with the following exception: for each index, we omit one curve corresponding to a face with the most sides. Note that this number of sides is at least four, by Lemma 4.1. Second, we take all the curves corresponding to white faces with two sides above crossings of index $i \equiv 2 \,(\mathrm{mod} 4)$. Third, we take some curves corresponding to white faces with three sides above crossings of index $i \equiv 2 \,(\mathrm{mod}\, 4)$. More precisely, we identify a face with at least four sides in this column. Proceeding upwards from it, we take every other three-sided face we encounter, beginning with the first one. In this way, we collect at least half of the white faces with three sides in each column of index $\equiv 2 \,(\mathrm{mod}\, 4)$.

Similarly, define B_2 just like B_0, but with the role of the indices $i \equiv 0 \,(\mathrm{mod}\, 4)$ and $i \equiv 2 \,(\mathrm{mod}\, 4)$ interchanged: see Figure 4.1 for an example.

4.2.2. The white surface

Now, we define two subsets $B_1, B_3 \subset H_1(S_W)$ of the first homology of the white chessboard surface S_W, as follows.

Let B_1 denote the set of homology classes of the following curves in S_W. First, we take all curves γ corresponding to black faces above crossings of index $i \equiv 1 \,(\mathrm{mod}\, 4)$, with the following exception: for each index, we omit one curve corresponding to a face with the most sides. Once more, Lemma 4.1 implies that this face must have at least four sides, unless $i = 1$ and every face in the first column has three or fewer sides; then assumption (d) applies to show that this face has three sides. Furthermore, we add the curves corresponding to black faces with two sides above crossings of index $i \equiv 3 \,(\mathrm{mod}\, 4)$. Finally, we add every other curve corresponding to white faces with three sides above crossings of index $i \equiv 3 \,(\mathrm{mod}\, 4)$, exactly as in the definition of B_0.

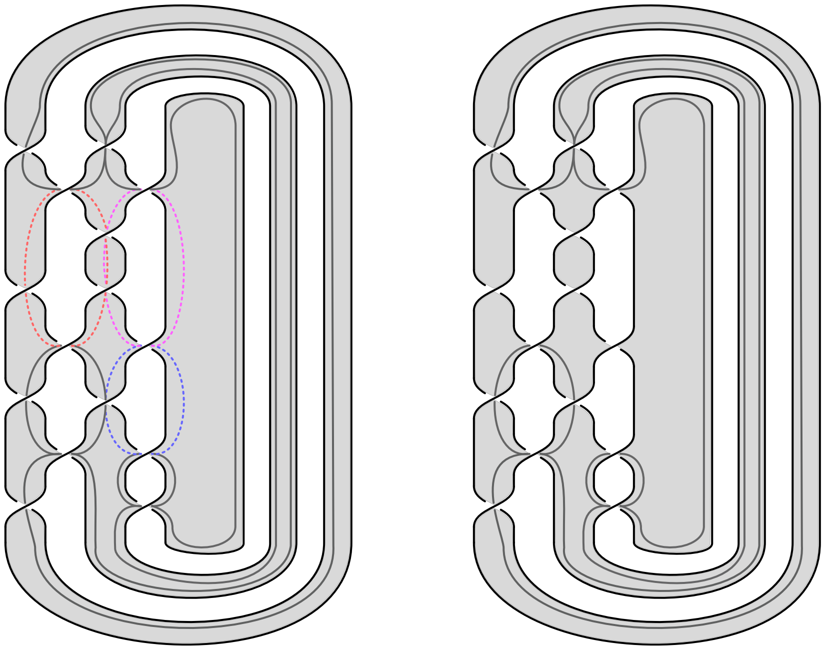

Figure 4.1. An example of selecting $B_2 \subset H_1(S_B)$: the red dashed curve is omitted since it runs around a white face with the most sides among all faces of index two. The purple dashed curve is omitted since its corresponding face has four crossings. Finally, the blue dashed curve is omitted since it is the second curve running around a face with three sides when counting upwards from the purple curve.

Finally, define B_3 just like B_1, but with the role of the indices $i \equiv 1 \pmod 4$ and $i \equiv 3 \pmod 4$ interchanged. The exception for the leftmost column is now the following: when we add every other face with three sides, we round down in case the number of such faces is odd and there is no face with more than three sides.

4.2.3. Subspaces and Gram matrices

Define X_j to be the subspace of homology generated by B_j, $j = 0, 1, 2, 3$. Note that B_j is a basis of X_j for each j. Define G_j to be the Gram matrix of B_j with respect to the relevant Goeritz form (i.e. G_B, if j is even, and G_W, if j is odd).

The first lemma in this section is more or less immediate from the construction.

Lemma 4.2. — *For each $j = 0, 1, 2, 3$, the matrix G_j is a direct sum of tridiagonal matrices $T(2^{\underline{a}})$, tridiagonal matrices $T(2^{\underline{a}}, 1, 2^{\underline{b}})$, and trisum matrices M_i, for indices $1 \leqslant i \leqslant n$, $i \equiv j \pmod 4$.*

Proof. — Fix a value $j = 0, 1, 2, 3$ and an index $i \equiv j \pmod 4$, $1 \leqslant i \leqslant n$. As in the proof of Lemma 4.1, assumption (b) implies that there is no appearance of $\sigma_i \sigma_{i-1} \sigma_i$ in β, even after distant braid moves. It follows that a 3-sided face in column $i-2$ shares two crossings with faces in the same column and one crossing with a face in column i; hence no 3-sided face in column $i+2$ shares a crossing with a face in column i.This shows that the subspace of homology generated by the curves we selected for the base B_j from column $i-2$ and i gives rise to a direct summand of G_j. We now show that this direct summand is in turn a direct sum of tridiagonal matrices $T(2^{\underline{a}})$, tridiagonal matrices $T(2^{\underline{a}},1,2^{\underline{b}})$, and a trisum matrix M_i.

The core of the trisum matrix M_i corresponds to the faces selected for B_j from the i^{th} column. It is a tridiagonal matrix by Lemma 2.2. In fact, Lemma 2.2 gives off-diagonal coefficients -1, but following Remark 2.3 we may change the orientation of every other curve γ winding around a face from the i^{th} column in order to have all off-diagonal coefficients 1. Further, the trisum matrix M_i has one block for each 3-sided face in column $i-2$ which shares a crossing with one of the selected faces in the i^{th} column. This block is of the form $T(2^{\underline{a}},1,2^{\underline{b}})$, where $a,b\geqslant 0$. This follows Lemma 2.2 as well as the care we took in selecting every other 3-sided face in column $i-2$: at most one 3-sided face can appear in any chain of faces[(2)] we add from the column $i-2$ during the construction of the base B_j. The off-diagonal coefficient 1 we need for each block of the trisum matrix is given by (iv) in Lemma 2.2. Again, we might need to invoke Remark 2.3 and adjust the orientations of the curves winding around the faces in column $i-2$ in order to have all off-diagonal coefficients 1.

A 3-sided face in column $i-2$ might share its crossing with the face from the i^{th} column we did not select in the construction of B_j. Such a 3-sided face is responsible for a direct summand $T(2^{\underline{a}},1,2^{\underline{b}})$. Finally, a chain of 2-sided faces in column $i-2$ we added during the construction of the base B_j is responsible for a direct summand $T(2^{\underline{a}})$. One more time, we might need to invoke Remark 2.3 and adjust the orientations of the curves winding around faces in column $i-2$ for off-diagonal coefficients 1. ☐

The second lemma follows directly from counting the sizes of the bases B_j.

Lemma 4.3. —

$$\sum_{j=1}^{4}\dim(X_j)\geqslant f_2+\frac{f_3-1}{2}+\mathrm{cr}(D)-n.$$

Proof. — For each index $i=1,\ldots,n$, every 2-sided face in the i^{th} column contributes a basis element to B_{j+2}, where $0\leqslant j+2\leqslant 3$ and $i+2\equiv j+2 \pmod 4$. The total number of these elements, summed over $i=1,\ldots,n$, is f_2. At least half of the 3-sided faces in the i^{th} column contribute a basis element to B_{j+2}, with the exception of $i=1$, where at least one less than half of them contribute. The total number of these elements is thus $\geqslant (f_3-1)/2$. Lastly, all but one face in the i^{th} column contributes a basis element to B_j, for each index $0\leqslant j\leqslant 3$ and $i\equiv j \pmod 4$. The total number of these elements is $\mathrm{cr}(D)-n$. The stated inequality now issues directly. ☐

The final lemma concerns the signature of the matrices G_j.

(2) By a chain of faces we mean a linearly ordered set of faces in the same column such that two faces share a crossing exactly if they are adjacent in the linear order.

Lemma 4.4. — *We have*

$$\sum_{j=0}^{3}\sigma(G_j) \geqslant -\frac{1}{2}(f_2+f_3)+2.$$

Proof. — By Lemma 4.2,

$$\bigoplus_{j=0}^{3} G_j = \left(\bigoplus_{i=1}^{n} M_i\right)\oplus T, \tag{4.2}$$

where T is the direct sum of some tridiagonal matrices of the form $T(\underline{2^a})$, $a \geqslant 1$, and $T(\underline{2^a}, 1, \underline{2^b})$, $a, b \geqslant 0$. By Lemma 3.1 and Porism 3.5, we have

$$\sigma(T) \geqslant \frac{1}{2}\dim(T). \tag{4.3}$$

We also have

$$\sum_{i=1}^{n} b(M_i) + \dim(T) = f_2 + f_3', \tag{4.4}$$

where f_3' denotes the number of 3-sided faces used in the construction of the bases B_j.

Let p denote the number of indices $1 \leqslant i \leqslant n$ such that $\underline{tr}(M_i) < 0$. The negative diagonal coefficients in the core of the matrix M_i are the values of the form $4-s$, where $s \geqslant 5$ is the number of sides in a face in the i^{th} column. Since we omit at least one face with $\geqslant 5$ sides in forming the core of M_i for each of p indices, it follows that

$$\sum_{i=1}^{n} \underline{tr}(M_i) \geqslant p + \sum_{i\geqslant 5}(4-i)f_i.$$

Invoking Inequality (4.1) leads to the bound

$$\sum_{i=1}^{n} \underline{tr}(M_i) \geqslant p - (2f_2 + f_3 - 4). \tag{4.5}$$

Let q denote the number of indices $1 \leqslant i \leqslant n$ such that $\underline{tr}(M_i) = b(M_i) = 0$. For each such index, (1) of Remark 3.3 gives

$$\sigma(M_i) \geqslant 0 = \frac{1}{2}\underline{tr}(M_i) + \frac{1}{2}b(M_i).$$

For the other $n-q$ indices, we bound $\sigma(M_i)$ from below by Proposition 3.4. Summing all of these bounds, we obtain

$$\sum_{i=1}^{n}\sigma(M_i) \geqslant -\frac{n-q}{2} + \frac{1}{2}\sum_{i=1}^{n}\underline{tr}(M_i) + \frac{1}{2}\sum_{i=1}^{n} b(M_i). \tag{4.6}$$

It follows that the number of indices $1 \leqslant i \leqslant n$ for which $\underline{tr}(M_i) = 0$ and $b(M_i) \geqslant 1$ is equal to $n-p-q$. Thus, $f_3' \geqslant n-p-q$, which rearranges to

$$p + q - n + f_3' \geqslant 0. \tag{4.7}$$

Consequently,

$$\begin{aligned}\sum_{j=0}^{3}\sigma(G_j) &\overset{(4.2)}{=} \sum_{i=1}^{n}\sigma(M_i)+\sigma(T)\\ &\overset{(4.3)+\ (4.6)}{\geqslant} \frac{q-n}{2}+\frac{1}{2}\sum_{i=1}^{n}\underline{tr}(M_i)+\frac{1}{2}\sum_{i=1}^{n}b(M_i)+\frac{1}{2}\dim(T)\\ &\overset{(4.4)+\ (4.5)}{\geqslant} \frac{q-n}{2}+\frac{1}{2}(p-(2f_2+f_3-4))+\frac{1}{2}(f_2+f_3')\\ &\overset{(4.7)}{\geqslant} -\frac{1}{2}(f_2+f_3)+2,\end{aligned}$$

as desired. ☐

4.3. The combined signature bound

We are nearly ready to put everything together to prove Theorem 1.1. The last fact we shall use is that if $G : H\times H\to\mathbb{Z}$ is a symmetric, bilinear form, and $X\subset H$ is a subspace, then

$$\sigma(G)+\dim H\geqslant\sigma(G|_{X\times X})+\dim X. \tag{4.8}$$

That is because the left side equals two times the dimension of the largest subspace of H on which G is positive definite, while the right side equals two times the dimension of the largest subspace of $X\subset H$ on which G is positive definite.

Proof of Theorem 1.1. — As argued at the outset of the section, we may assume that hypotheses (a)-(d) hold for L, so the results of this section pertain to it. Using Theorem 2.1 twice $(*)$, then (4.8) four times $(**)$ and finally Lemma 4.4, Lemma 4.3 and (2.3) $(***)$, we obtain

$$\begin{aligned}4\sigma(L)-2\mathrm{cr}(D) &\overset{(*)}{=} 2\sigma(G_B)+2\sigma(G_W)\\ &\overset{(**)}{\geqslant} \sum_{j=0}^{3}(\sigma(G_j)+\dim(X_j))-2\left(\dim(H_1(S_B))+\dim(H_1(S_W)\right)\\ &\overset{(***)}{\geqslant} \left(-\frac{1}{2}(f_2+f_3)+2\right)+\left(f_2+\frac{f_3-1}{2}+\mathrm{cr}(D)-n\right)-2\mathrm{cr}(D)\\ &=\frac{3}{2}+\frac{f_2}{2}-\mathrm{cr}(D)-n.\end{aligned}$$

Thus,

$$4\sigma(L)\geqslant\frac{3}{2}+\frac{f_2}{2}+\mathrm{cr}(D)-n\geqslant\frac{3}{2}+b_1(L), \tag{4.9}$$

using (2.2) in the second inequality. Both values $\sigma(L)$ and $b_1(L)$ are integers. Thus, we obtain the improvement $4\sigma(L)\geqslant 2+b_1(L)$, and dividing through by 4 gives the bound in Theorem 1.1. ☐

5. Discussion

The proof of Theorem 1.1 suggests room for improvement in the direction of Feller's conjecture. However, we could not quickly see how to do more, and examples suggest the need for more ideas. On the one hand, Proposition 3.2 is clearly suboptimal if, say, $\underline{tr}(M) < -2n$, which suggests improving it towards the end of improving Theorem 1.1. On the other hand, it is easy to give examples for which the relevant tridiagonal matrices have only -1s and -2s on the diagonal. Then the bound in Proposition 3.2 seems to be sharp, and it correspondingly seems hard to improve on Theorem 1.1 for these examples. In another direction, Equation (4.9) shows that the bound of Theorem 1.1 can be improved in case the proportion of faces with two sides to the total number of faces is large. However, this proportion can be arbitrarily small, as witnessed by the following example.

Example 5.1. — Consider the positive braid $\beta = (\sigma_1 \cdots \sigma_n \sigma_n \cdots \sigma_1)^2 \in B_{n+1}$. Even up to conjugation, no braid moves can be applied to this braid. Furthermore, its closure L is a nontrivial and nonsplit link. Finally β is of minimal index representing L, since the link L has $n+1$ components, so a braid representing it must have at least $n+1$ strands. Independently of n, its standard diagram D has $f_2 = 4$, $f_4 = \mathrm{cr}(D) - 4$, and $f_k = 0$ for $k \neq 2, 4$. The signature of such links L was computed by the second author in [Lie20, Remark 15]. It equals approximately two-thirds of the first Betti number. More precisely, $\sigma(L) = 2n+1$ and $\mathrm{null}(L) = n-1$.

It seems that in order to obtain a bound for the signature of positive braids via Goeritz forms that is better than the quarter from Theorem 1.1, one would have to deal with subspaces of the first homology groups of the chessboard surfaces that contain more faces with four or more sides. In our proof, the specific almost tridiagonal form of the matrices for the Goeritz forms restricted to our subspaces X_j played a crucial role. Since faces with four sides can share crossings with faces to both the right *and* to the left, a division of the matrix for the Goeritz form into blocks that correspond to columns two indices apart as in Lemma 4.2 should be impossible in general. Perhaps more complicated partitions or also a new and more global approach would be necessary for further improvements.

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Manuscript received on 10th August 2023,
revised on 23rd March 2024,
accepted on 10th May 2024.

Recommended by Editors C. Sorger and V. Colin.

eISSN: 2644-9463
This journal is a member of Centre Mersenne.

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