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HYDRODYNAMIC LIMITS FOR  
KINETIC EQUATIONS  
PRESERVING MASS, MOMENTUM  
AND ENERGY: A SPECTRAL AND  
UNIFIED APPROACH IN THE  
PRESENCE OF A SPECTRAL GAP

LIMITES HYDRODYNAMIQUES POUR DES  
ÉQUATIONS CINÉTIQUES PRÉSERVANT LA  
MASSE, LA QUANTITÉ DE MOUVEMENT ET  
L'ÉNERGIE : UNE APPROCHE SPECTRALE  
UNIFIÉE EN PRÉSENCE D'UN TROU  
SPECTRAL

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ABSTRACT. — Triggered by the fact that, in the hydrodynamic limit, several different kinetic equations of physical interest all lead to the same Navier–Stokes–Fourier system, we develop in the paper an abstract framework which allows to explain this phenomenon. The method we develop can be seen as a significant improvement of known approaches for which we fully exploit some structural assumptions on the linear and nonlinear collision operators as well as a good knowledge of the Cauchy theory for the limiting equation. In particular, we fully exploit the fact that the collision operator is preserving both momentum and kinetic energy. We adopt a perturbative framework in a Hilbert space setting and first develop a general and fine spectral analysis of the linearized operator and its associated semigroup. Then, we introduce a splitting adapted to the various regimes (kinetic, acoustic, hydrodynamic) present in the kinetic equation which allows, by a fixed point argument, to construct a solution to the kinetic equation and prove the convergence towards suitable solutions to the Navier–Stokes–Fourier system. Our approach is robust enough to treat, in the same formalism, the case of the Boltzmann equation with hard and moderately soft potentials, with and without cut-off assumptions, as well as the Landau equation for hard and moderately soft potentials in presence of a spectral gap. New well-posedness and strong convergence results are obtained within this framework. In particular, for initial data with algebraic decay with respect to the velocity variable, our approach provides the first result concerning the strong Navier–Stokes limit from Boltzmann equation without Grad cut-off assumption or Landau equation. The method developed in the paper is also robust enough to apply, at least at the linear level, to quantum kinetic equations for Fermi–Dirac or Bose–Einstein particles.

RÉSUMÉ. — L'article développe un cadre abstrait pour expliquer le fait que, dans la limite hydrodynamique, plusieurs équations cinétiques différentes conduisent toutes au même système de Navier–Stokes–Fourier. La méthode que nous développons peut être considérée comme une amélioration significative des approches connues pour lesquelles nous exploitons pleinement certaines hypothèses structurelles sur les opérateurs de collision linéaires et non-linéaires ainsi qu'une bonne connaissance de la théorie de Cauchy pour l'équation limite. En particulier, nous exploitons pleinement le fait que l'opérateur de collision préserve à la fois la quantité de mouvement et l'énergie cinétique. Nous adoptons un cadre perturbatif dans un espace de Hilbert et développons d'abord une analyse spectrale fine de l'opérateur linéarisé et de son semigroupe. Ensuite, nous introduisons une décomposition du semigroupe adaptée aux différents régimes (cinétique, acoustique, hydrodynamique) présents dans l'équation cinétique qui permet, par un argument de point fixe, de construire une solution à l'équation cinétique et de prouver la convergence vers des solutions du système de Navier–Stokes–Fourier. Notre approche est suffisamment robuste pour traiter, dans le même formalisme, le cas de l'équation de Boltzmann avec des potentiels durs et (modérément) mous, avec et sans hypothèses de cutoff angulaire, ainsi que l'équation de Landau pour des potentiels durs et (modérément) mous en présence d'un trou spectral. De nouveaux résultats d'existence de solutions et de convergence sont obtenus dans ce cadre. En particulier, pour des données initiales avec une décroissance algébrique par rapport à la variable de vitesse, notre approche fournit le premier résultat concernant la limite forte de Navier–Stokes pour l'équation de Boltzmann sans hypothèse de cutoff angulaire de Grad ou pour l'équation de Landau. La méthode développée dans cet article est également suffisamment robuste pour s'appliquer, au moins au niveau linéaire, aux équations cinétiques quantiques pour les particules de Fermi–Dirac ou de Bose–Einstein.

# 1. Introduction

## 1.1. From nonlinear collisional model to Navier–Stokes–Fourier system

The connection between the Navier–Stokes and Boltzmann equations originates seemingly from the work [Hil12] regarding the mathematical treatment of the axioms of physics. Since this original idea, the derivation of suitable hydrodynamic equations from nonlinear kinetic equations has attracted a lot of attention in the recent years. We will review later in this introduction several of the main contributions in the field, illustrating in particular the large variety of models considered in the literature, but we wish to focus here on some striking universal features shared by several binary collisional models in the diffusive scaling.

Namely, for kinetic equations in adimensional form given by the evolution of a particles number density  $f(x, v, t)$  (with  $x \in \mathbb{R}^d$  denoting position,  $v \in \mathbb{R}^d$  the velocity,  $t > 0$  the time and  $\varepsilon > 0$  the mean free path between particles collisions)

$$(1.1) \quad \partial_t f + \frac{1}{\varepsilon} v \cdot \nabla_x f = \frac{1}{\varepsilon^2} Lf + \frac{1}{\varepsilon} Q(f, f), \quad f(x, v, 0) = f_{\text{in}}(x, v)$$

for some suitable *linear operator*  $L$  and *quadratic operator*  $Q$ , it has been shown in various contexts that, in the limit  $\varepsilon \rightarrow 0$ , the solution  $f$  converges (in some sense to determine) towards a “macroscopic” distribution  $f_{\text{NS}}(x, v, t)$  of the form

$$(1.2) \quad f_{\text{NS}}(x, v, t) = (\varrho(t, x) + u(t, x) \cdot v + C_0 \theta(t, x) (|v|^2 - E)) \mu(v)$$

where  $C_0 > 0, E > 0$  are depending only on the universal distribution  $\mu$  (independent of  $f_{\text{in}}$ ). More surprisingly, it is also known that the triple of functions

$$(\varrho(t, x), u(t, x), \theta(t, x)) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$$

associated to the macroscopic mass, mean velocity and temperature of the gas are suitable solutions to the Navier–Stokes–Fourier system

$$(1.3) \quad \begin{cases} \partial_t u - \kappa_{\text{inc}} \nabla_x u + \vartheta_{\text{inc}} u \cdot \nabla_x u = \nabla_x p, \\ \partial_t \theta - \kappa_{\text{Bou}} \nabla_x \theta + \vartheta_{\text{Bou}} u \cdot \nabla_x \theta = 0, \\ \nabla_x \cdot u = 0, \quad \nabla_x (\varrho + \theta) = 0, \end{cases}$$

where the third line describe respectively the *incompressibility* condition of the fluid and the *Boussinesq relation* between mass and temperature. The pressure of the fluid  $p$  is here above obtained implicitly as a Lagrange multiplier associated to the incompressibility constraint  $\nabla_x \cdot u = 0$ .

The striking phenomena we wish to discuss in this paper is the fact that a large variety of kinetic models described by (1.1) provide in the hydrodynamic limit the *same Navier–Stokes–Fourier system* (1.3), making that system a *universal hydrodynamic limit* for (1.1). The only memory of the original equation (1.1) kept in the system (1.3) is encapsulated in the various coefficients:

$$\kappa_{\text{inc}} > 0, \quad \kappa_{\text{Bou}} > 0,$$

which represent the viscosity and thermal conductivity, as well as  $\vartheta_{\text{inc}}, \vartheta_{\text{Bou}}$ , all of being defined explicitly in terms of the operators  $L$  and  $Q$  that encode the collision process. We refer to Section 2 for more details on those coefficients.

Recall that, in the kinetic equation (1.1), the unknown  $f(x, v, t)$  denotes typically the density of particles having position  $x \in \mathbb{R}^d$  and velocity  $v \in \mathbb{R}^d$  at time  $t > 0$  while the parameter  $\varepsilon$  represents the *Knudsen number* which is proportional to the mean free path between collisions. Typically, small values of  $\varepsilon$  correspond to a case in which particles suffer a very large number of collisions. The hydrodynamic limit  $\varepsilon \rightarrow 0$  consists in assuming that the mean free path is negligible when compared to the typical physical scale length. We refer to [Cer88, Son02] for details on the kinetic description of gases.

That kinetic equation (1.1) leads to (1.3) in the limit  $\varepsilon \rightarrow 0^+$  is a well-understood fact that have been proven, for several type of solutions and various mode of convergence, in the case of the classical Boltzmann equation for which

$$(1.4) \quad Q(f, f)(v)$$

$$= Q_{\text{Boltz}}(f, f) = \int_{\mathbb{R}^d \times S^{d-1}} B(|v - v'|, \sigma) [f(v)f(v') - f(v)f(v')] dv' d\sigma$$

where

$$v = \frac{v + v'}{2} + \frac{|v - v'|}{2} \sigma, \quad v' = \frac{v + v'}{2} - \frac{|v - v'|}{2} \sigma, \quad \sigma \in S^{d-1}$$

and the collision kernel  $B(|v - v'|, \sigma)$  is given by

$$B(|v - v'|, \sigma) = |v - v'| b(\cos \theta), \quad \cos \theta = \sigma \cdot \frac{v - v'}{|v - v'|}.$$

The method developed in the paper allows to consider *all kinds of collision kernel* of physical interest, covering the cases of hard and Maxwell potentials ( $\gamma > 0$ ) with and without cut-off assumptions as well as that of moderately soft potentials (without cut-off assumption) for which  $b(\cos \theta) \sim \theta^{-(d-1)-2s}$  and  $\gamma + 2s > 0$ . We refer to Appendix A for details. Besides this Boltzmann model, our approach is also robust enough to treat in the same formalism the case of the Landau equation

$$\begin{aligned} Q(f, f) &= Q_{\text{Landau}}(f, f) \\ &= \int_{\mathbb{R}^d} |v - v'|^{\gamma+2} \{ f(t, v) f(t, v') - f(t, v) f(t, v') \} dv' \end{aligned}$$

where  $\gamma > -d$  and

$$z = \text{Id} - \frac{z}{|z|^2}, \quad z \in \mathbb{R}^d \setminus \{0\}$$

denotes the projection in the direction orthogonal to  $z \in \mathbb{R}^d, z \neq 0$ . As before, our results cover the two cases of hard or Maxwell ( $\gamma > 0$ ) and moderately soft potentials ( $\gamma + 2 > 0$ ). For both these models, the solutions to (1.1) converges to a solution  $f$  given by (1.2) where

$$\mu(v) = (2\pi)^{-\frac{d}{2}} \exp \left( -\frac{|v|^2}{2} \right)$$

is a Maxwellian distribution with unit mass, unit energy and mean zero velocity, which is an equilibrium state of the collision operator  $Q$ , i.e.

$$Q(\mu, \mu) = 0$$

whereas  $L$  is the linearized operator around that equilibrium, i.e.

$$(1.5) \quad Lf = Q(\mu, f) + Q(f, \mu)$$

for any suitable  $f$  for which this makes sense.

In this paper, we introduce an abstract framework allowing to recover the above universal behaviour, as well as the well-posedness of (1.1) in a perturbative framework. Even though the Boltzmann and Landau equations are the two main models we have in mind as field of applications of our method, we wish again to point out that we are able to prove the convergence towards (1.3) for much more general models than those ones. In particular, we can handle general linear operator  $L$  and do not ask for the rest of the analysis that  $L$  and  $Q$  are related through (1.5).

The abstract framework developed in the paper is very general and robust and rely only on core assumptions about the linear part  $L$  and the quadratic part  $Q$ . In particular, our approach can also be adapted to handle the case of the Boltzmann equation with *relativistic velocities* and it is flexible enough to also encompass, at the price of some modifications, the case of quantum kinetic model (for which the collision operator is actually trilinear). Work is in progress in that direction in order to prove the strong convergence of solutions to the Boltzmann–Fermi–Dirac equation towards the above NSF system (1.3), see [GL].

## 1.2. Literature review

As said, the derivation of hydrodynamic limits from linear and nonlinear equation is an important problem which received a lot of attention since the pioneering work of [Hil12] and [Ens17]. We do not review here the vast literature on the problem of diffusion approximation for transport processes, just referring to the classical references [BLP79, BSS84] and the more recent contributions [BM22, GW17] and the references therein.

For nonlinear collisional models, we refer the reader to [Gol14, SR09] for a more exhaustive description of the mathematically relevant results in the field regarding the Boltzmann equation. Depending on the limiting equation and the type of convergence one is interested with, there are mainly three different approaches for the derivation of hydrodynamical limit from the Boltzmann equation: a first approach consists in justifying rigorously suitable (truncated) asymptotic expansions of the solution to the kinetic equation around some hydrodynamic solution

$$f(t, x, v) = f_0(t, x, v) + \sum_n \varepsilon^n F_n(t, x, v)$$

where, typically  $f_0(t, x, v)$  is a local Maxwellian whose macroscopic fields are required to satisfy the limiting fluid model. With such an approach, the works [Caf80]

and [DMEL89] obtained respectively the first rigorous justification of the compressible Euler limit up to the first singular time for the solution of the Euler system and a justification of the incompressible Navier–Stokes limit from Boltzmann equation. The work [Guo06] is another important reference on this line of research and we point out that, with such an approach, one is mainly interested with strong solutions for both the kinetic and fluid equations.

Regarding now *weak solutions* at both the kinetic and fluid models, a very important program has been introduced in [BGL91, BGL93] whose goal was to prove the convergence of the renormalized solutions to the Boltzmann equation towards weak solutions to the compressible Euler system or to the incompressible Navier–Stokes equations. This program has been continued exhaustively and the convergence have been obtained in several important results (see [GSR04, GSR09, JM17, LM10, LM01a, LM01b] to mention just a few).

The present contribution belongs to the third line of research which investigates *strong solutions close to equilibrium* and exploits a careful spectral analysis of the linearized kinetic equation. Strong solutions to the Boltzmann equation close to equilibrium have been obtained in a weighted  $L^2$ -framework in the work [Uka74] and the *local-in-time* convergence of these solutions towards solution to the compressible Euler equations have been derived in [Nis78]. For the limiting incompressible Navier–Stokes solution, a similar result have been carried out in [BU91] for smooth global solutions in  $\mathbb{R}^3$  with a small initial datum. The recent work [GT20] recently removed this smallness assumption, allowing to treat also non global in time solutions to the Navier–Stokes equation. A recent extension to less restrictive integrability conditions has been obtained in [Ger23]. Our work is falling into this framework and is closer in spirit to the work [GT20] than to [BU91] since it fully exploits the Cauchy theory of the limiting NSF system. This line of research, complemented for instance with [Bri15, BMAM19, CC23], exploits a very careful description of the spectrum of the linearized Boltzmann equation derived in [EP75]. We notice that they are framed in the space  $L^2(\mu^{-1})$  where the linearized Boltzmann operator is self-adjoint and coercive. The fact that the analysis of [EP75] has been extended recently in [Ger21] to larger functional spaces of the type  $L^2_\nu(\cdot^q)$  opens the gate to some refinements of several of the aforementioned results. We also mention here the work [JXZ18] which deals with an energy method in  $L^2(\mu^{-1})$  spaces (see also [GJJ10, Guo16] and [Rac21]) in order to prove the *strong convergence* of the solutions to the Boltzmann or Landau equation towards the incompressible Navier–Stokes equation without resorting to the work of [EP75].

Besides the above lines of research and contributions which are dealing mainly with Boltzmann or Landau equation, we wish to point out that other kinetic and fluid models have been considered in the literature. Exhaustive list of contributions to the field is out of reach and we just mention some recent works spanning from high friction regimes for kinetic models of swarming (see e.g. [FK19, KMT15] for the Cucker–Smale model) to the reaction-diffusion limit for Fitzhugh–Nagumo kinetic equations [CFF19]. For fluid-kinetic systems, the literature is even more important, we mention simply here the works [GJV04a, GJV04b] dealing with light or fine particles regimes for the Vlasov–Navier–Stokes system and refer to [HKM23] for the

more recent advances on the subject. We also mention the challenging study of gas of charged particles submitted to electro-magnetic forces (Vlasov–Maxwell–Boltzmann system) for which several incompressible fluid limits have been derived recently in the monograph [ASR19].

### 1.3. Objectives of the paper

The main scope of the paper is threefold:

- (I) First, we provide a unified framework which allows to capture a large variety of quadratic models and explain the emergence of the universal NSF system (1.3) in the hydrodynamic limit. To do so, we provide a general though seemingly minimal set of Assumptions under which the NSF would emerge. Those are structural assumptions on the collision operator  $Q$  as well as the linear operator  $L$ . They are related to physical properties of the kinetic equation: we assume in particular the rotational symmetry of  $L$  and  $Q$  due to the isotropy of the collision process as well as usual *local conservation laws* related to mass, bulk velocity and energy. This work can be considered as a quantitative version of the founding paper by [BGL91] in which general collision operators are considered. We refer to Section A.3 for more details.
- (II) Second, within the abstract framework considered here, we aim to provide a very fine spectral analysis of the linearized operator  $L - v \cdot \nabla_x$  as well as a thorough description of the decay and regularization properties of the associated semigroup. As in previous contributions to the field, such an analysis is performed in a Fourier-based formalism under which the linearized operator of peculiar interest becomes

$$\tilde{L} := L - i(\xi \cdot v)$$

where the transport term has been transformed in the more tractable multiplication operator by  $i(v \cdot \xi)$  in Fourier variable (see Section 2 for details). The advantage of working in this Fourier-based formalism is that it encompasses the various scales of frequencies according to

$$|\xi| \ll \varepsilon, \quad |\xi| \sim \varepsilon \quad \text{or} \quad \varepsilon \ll |\xi|$$

which let emerge the various (kinetic, hydrodynamic, dispersive) regimes of description at the linearized level. Under the structural assumptions on the linear part  $L$ , we give a full description of the spectrum of  $\tilde{L}$ , including the asymptotic expansion of both its leading eigenvalues and associated spectral projectors in the regime of small frequencies,  $|\xi| \ll 0$ . Such a spectral description yields to result similar to those obtained in the seminal work [EP75] but we provide here a *completely new and more direct approach to this question* in the unified and abstract framework. Our new approach is based upon a combination of Kato's perturbation theory [Kat66] and enlargement and factorization techniques from [GMM17].

- (III) Finally, we provide a *strong convergence* result from solutions  $f^\varepsilon$  to (1.1) towards the solution  $f$  given in (1.2) associated to (1.3). Moreover, the strong

convergence result is in essence *quantitative* since we carefully estimate the difference between the solution  $f$  and the solution  $f = f_{\text{NS}}$  by introducing a suitable splitting of  $f$  which, roughly speaking, can be given as

$$f = f_{\text{NS}} + h_{\text{err}}$$

where  $h_{\text{err}}$  is an error term that we aim to estimate as

$$\sup_{t > t_0} h_{\text{err}}(t) \leq \beta(\varepsilon), \quad \lim_0 \beta(\varepsilon) = 0$$

for any  $t > 0$  and some quantified error estimate  $\beta(\varepsilon)$ . Here above, the norm  $\|\cdot\|$  is quite involved and takes into account several phenomena that produce different convergence rates (e.g. acoustic waves, dissipation of entropy). The restriction  $t > t_0$  stems from the difficult task of estimating the initial layer and can be removed in the case of *well-prepared* initial datum (see Theorem 1.12 for a precise statement and a complete description of the difference  $f - f_{\text{NS}}$ ).

As a by-product of our third objective (III) here above, we show, for this variety of model, a close-to-equilibrium Cauchy theory for the kinetic equation (1.1) for suitably small value of  $\varepsilon$ . One of the main feature of our approach is that, inspired by the work [GT20], our methodology is “top-down” from the limit equation to the kinetic equation rather than “bottom-up” as usually done. This means that, as far as possible, we adapt our approach to the existing Cauchy theory for the limiting system (1.3) and deduce the Cauchy theory for the kinetic equation (1.1) by comparing it to the limiting equation (1.3) for small values of  $\varepsilon$ . This is achieved through a suitable fixed-point argument involving fluctuations around the solution  $f_{\text{NS}}$ . The fixed-point argument is based upon a simple use of Banach fixed point theorem or, for the more general case considered in the paper, by the convergence of a suitable scheme mimicking Picard iteration. Such an approach allows in particular to obtain well-posedness results *without* any smallness assumption on the initial datum  $f_{\text{in}}$  but only under some smallness assumption on the scaling parameter  $\varepsilon$  yielding several improvements of known results in the field.

Among the novelty of the paper, as just said, we adapt our approach to the existing Cauchy theory for the limiting system (1.3). A lot of efforts in the present paper are given to adapt several tools used in the estimates of the Navier–Stokes system and, in particular, we resort to several Fourier analysis tools as developed in [BCD11] to treat nonlinear terms. We in particular adapt the paraproduct estimates described in [BCD11] to handle  $x$ -estimates of products of the form  $Q(f, g)$  (see Appendix B.1 for more details). The case  $d = 2$  needs in particular a peculiar treatment for which we face several technical difficulties to handle nonlinear estimates.

Regarding the method used to achieve the above objectives, as in previous contributions to the field, we start by studying (1.1) without its non-linear part and in Fourier variables:

$$(1.6) \quad \partial_t \widehat{f}(\xi, v, t) + \frac{i}{\varepsilon} (\xi \cdot v) \widehat{f}(\xi, v, t) = \frac{1}{\varepsilon^2} L \widehat{f}(\xi, v, t)$$



where

$$\widehat{f}(\xi, v, t) = \int_{\mathbb{R}^d} e^{-i \cdot x} f(x, v, t) dx$$

is the Fourier transform with respect to the position variable  $x \in \mathbb{R}^d$  and we exploited the fact that  $L$  is local in  $x$ . In this framework, the linear operator of peculiar interest becomes

$$L := L - i(\xi \cdot v)$$

where the transport term has been transformed in the more tractable multiplication operator by  $i(v \cdot \xi)$  in Fourier variable. The idea of studying (1.6) originates from the seminal work [EP75] where a careful spectral analysis of the linearized Boltzmann operator was performed. It enforces somehow the study of both (1.1) and (1.3) in  $L^2_x$ -functional spaces. Here, we push forward this idea and try to extract from it minimal assumptions and optimal estimates for  $L$  and  $Q$ .

### 1.4. Notations

In all the sequel, given a closed densely defined linear operator on a Banach space  $Y$  of functions  $f : v \in \mathbb{R}^d \rightarrow f(v) \in \mathbb{C}$ ,

$$L : D(L) \subset Y \rightarrow Y$$

we denote, for any  $\xi \in \mathbb{R}^d$ , the operator  $L : D(L) \subset Y \rightarrow Y$  by

$$D(L)_\xi = \{f \in D(L) ; v f \in Y\} \quad L_\xi f = f - i(v \cdot \xi) f, \quad f \in D(L)_\xi.$$

The spectrum of  $L$  is denoted  $\mathfrak{S}(L)$  (or  $\mathfrak{S}_x(L)$  if it appears necessary to explicit the underlying Banach space) and, for  $z \in \mathbb{C} \setminus \mathfrak{S}(L)$ , the resolvent of  $L$  at  $z$  is denoted by

$$R(z, L) = (z - L)^{-1} \in B(Y)$$

where  $B(Y)$  is the space of all bounded linear operators on  $Y$  (with its usual norm  $\|\cdot\|_{B(Y)}$ ).

We introduce, for any  $a \in \mathbb{R}$  the right-half plane of the complex field  $\mathbb{C}$  as

$$\mathbb{H}_a := \{z \in \mathbb{C} ; \operatorname{Re} z > a\}.$$

To handle now functions depending on the position variable  $x \in \mathbb{R}^d$ , we define the inhomogeneous Sobolev spaces of order  $s \in \mathbb{R}$ ,

$$H_x^s(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) ; f \in H_x^s \iff \int_{\mathbb{R}^d} |\xi|^{2s} |\widehat{f}(\xi)|^2 d\xi < \infty\}$$

and the homogeneous Sobolev space

$$\dot{H}_x^s(\mathbb{R}^d) = \{f \in S_x(\mathbb{R}^d) ; f \in \dot{H}_x^s \iff \int_{\mathbb{R}^d} |\xi|^{2s} |\widehat{f}(\xi)|^2 d\xi < \infty\}$$

where  $S_x(\mathbb{R}^d)$  denotes the space of tempered distributions over  $\mathbb{R}^d$ . One can identify  $H_x^s(\mathbb{R}^d)$  as the space of tempered distribution  $f \in S_x(\mathbb{R}^d)$  such that

$$(\operatorname{Id} - \Delta_x)^{s/2} f \in L_x^2(\mathbb{R}^d)$$

whereas  $H_x^s(\mathbb{R}^d)$  is the space of mappings  $f \in S_x(\mathbb{R}^d)$  such that

$$(-x)^{s/2} f = |x|^s f \in L_x^2(\mathbb{R}^d).$$

We also introduce the homogeneous Besov spaces for  $p, q \in [1, \infty]$  and  $s \in \mathbb{R}$

$$\dot{B}_{p,q}^s(\mathbb{R}^d) = \{ f \in S_x(\mathbb{R}^d) ; \| f \|_{\dot{B}_{p,q}^s}^q = \sum_{n \in \mathbb{Z}} \left( 2^{ns} \| \dot{\Delta}_n f \|_{L_x^p} \right)^q < \infty$$

where the homogeneous dyadic projector  $\dot{\Delta}_n$  from Littlewood–Paley theory is recalled in Appendix B.1.

For a Banach space  $(Y_v, \| \cdot \|_{Y_v})$  of mappings depending on the variable  $v$ , the space  $H_x^s(Y_v)$  denotes the space of functions  $f : (x, v) \mapsto f(x, v)$  such that

$$\| f \|_{H_x^s(Y_v)} = \left\| \| f(x, \cdot) \|_{Y_v} \right\|_{H_x^s} < \infty.$$

Equivalently, one has

$$(1.7) \quad \| f \|_{H_x^s(Y_v)}^2 = \int_{\mathbb{R}^d} \xi^{2s} \| \widehat{f}(\xi) \|_{Y_v}^2 d\xi.$$

A similar definition applies to Besov spaces.

### 1.5. Assumptions

We work in a general setting of a perturbed kinetic equation of the form (1.1) which, for  $\varepsilon = 1$ , reads

$$(\partial_t + v \cdot \nabla_x) f = Lf + Q(f, f),$$

where  $L$  and  $Q$  are local in  $x$ , that is to say, they act on functions depending only on  $v$ . Their actions on functions  $f = f(x, v)$  depending on both  $x$  and  $v$  are naturally defined as

$$[Lf](x, v) = [Lf(x, \cdot)](v), \quad Q(f, f)(x, v) = \left( Q(f(x, \cdot), f(x, \cdot)) \right)(v).$$

At the linear level, we make the following assumptions on the linearized operator  $L$  in the space

$$H = L^2(\mu^{-1}(v)dv),$$

of functions depending only on the velocity variable where  $\mu : \mathbb{R}^d \rightarrow [0, \infty)$  is some measurable weight function.

**STRUCTURAL LINEAR ASSUMPTIONS 1.1.** — *The linear operator  $L : D(L) \rightarrow H$  satisfies the following.*

**(L1)** *The operator  $L$  is self-adjoint in  $H$  and commutes with orthogonal matrices:*

$$Lfg_H = f, Lg_H = L(f), \quad g \in H,$$

*for any  $f, g \in D(L)$  and orthogonal matrix  $M \in M_{d \times d}(\mathbb{R})$ , where  $[Mf](v) := f(M^{-1}v)$ .*

**(L2)** The weight function  $\mu$  is nonnegative, normalized, radial, and such that:

$$\mu = \mu(|v|) > 0, \quad \int_{\mathbb{R}^d} \mu(v) dv = 1,$$

$$E = \int_{\mathbb{R}^d} |v|^2 \mu(v) dv < \infty, \quad K = \frac{1}{E^2} \int_{\mathbb{R}^d} |v|^4 \mu(v) dv < \infty.$$

**(L3)** The null-space of  $L$  is given by

$$\text{Ker}(L) = \text{Span} \{ \mu, v_1 \mu, v_2 \mu, \dots, v_d \mu, |v|^2 \mu \}$$

and there exists a Hilbert space  $H^*$  such that

$$D(L) \subset H^* \subset H, \quad \cdot \subset H \subset \cdot \subset H^*,$$

and such that there holds

$$L f, f \subset H \subset -\lambda_L \|f\|_H^2 \quad \text{for any } f \subset D(L) \setminus \text{Ker}(L).$$

**(L4)** The operator  $L$  can be decomposed as

$$L = B + A, \quad D(B) = D(L), \quad A \subset B(H),$$

where the splitting is compatible with a hierarchy of Hilbert spaces  $(H_j)_{j=0}^2$  such that

(a) the spaces  $H_j$  continuously and densely embed into one another:

$$H_2 \subset H_1 \subset H_0 = H,$$

(b) the multiplication by  $v$  is bounded from  $H_{j+1}$  to  $H_j$ , i.e.

$$v f \subset H_j \subset f \subset H_{j+1} \subset f \subset H_{j+1}, \quad j = 0, 1,$$

(c) the operator  $A : H_j \subset H_{j+1}$  is bounded:

$$A \subset B(H_j, H_{j+1}), \quad j = 0, 1,$$

(d) the part  $B$  is hypo-dissipative on each space  $H_j$  uniformly in  $\xi \subset \mathbb{R}^d$ , that is to say there exists  $\lambda_B > \lambda_L$  such that, for  $j = 0, 1, 2$

$$\mathfrak{S}_{H_j}(B) \subset -\lambda_B = ?,$$

and

$$\sup_{\mathbb{R}^d} R(z, B) \subset B(H_j) \subset |\text{Re } z + \lambda_B|^{-1}, \quad z \subset -\lambda_B.$$

*Remark 1.2.* — Note that  $K > 1$  by a simple use of Jensen's inequality applied to the probability measure  $\mu(v)dv$ . Moreover, according to **(L1)**, Assumption **(L3)** can be formulated as follows:

$$f \subset D(L), \quad L f, \mu \subset H = L f, v \mu \subset H = \langle L f, |v|^2 \mu \rangle_H = 0,$$

that is to say  $\text{Ker}(L) = \text{Range}(L)$ , and the operator  $L$  has a spectral gap in  $H$ :

$$\mathfrak{S}_H(L) \subset -\lambda_L = \{0\}.$$

*Remark 1.3.* — Notice also that  $\text{Ker}(L) \subset H_2$ . Indeed, given  $f \in \text{Ker}(L)$ , with the splitting given in **(L4)**,  $Lf = 0$  implies

$$f = R(0, B)Af$$

and thanks to **(L4c)**,  $Af \in H_1$  which, with now **(L4d)**, yields  $R(0, B)Af \in H_1$ . Thus  $f \in H_1$  and one can repeat the argument to deduce that  $f \in H_2$ .

*Example 1.4.* — We show in Appendix A that the various Assumptions **(L1)**–**(L4)** hold for several models of physical interest, explicating for each of those models the precise definition of the various spaces  $H^*$  and  $H_j$  as well as the splitting  $L = A + B$ . Typically, Assumptions **(L1)**–**(L4)** apply to the Boltzmann equation with hard potentials with or without Grad’s cut-off assumptions or to Landau equation in spaces with Gaussian weights. To clarify right away the role of this set of Assumptions in our analysis, we illustrate here the form of the spaces  $H^*, H_j$  in the case of Boltzmann equation with hard-spheres interactions. This corresponds to (1.4) with the choice

$$B(|v - v'|, \sigma) = |v - v'|, \quad v, v' \in \mathbb{R}^d \times \mathbb{R}^d, \quad \sigma \in \mathbb{S}^{d-1}.$$

In such a case, as said,  $\mu$  is a Maxwellian distribution:

$$(1.8) \quad \mu(v) := (2\pi)^{-d/2} \exp\left(-\frac{|v|^2}{2}\right), \quad E = d, \quad K = 1 + \frac{2}{d},$$

and the usual linearized operator given by (1.5) is known to satisfy **(L1)**–**(L2)** with  $D(L) = L^2(v^2 \mu^{-1}(v) dv)$ . Moreover, assumption **(L3)** is met with the choice

$$H^* = L^2(v \mu^{-1}(v) dv)$$

Regarding assumption **(L4)**, one can choose the hierarchy of spaces  $H_j$  as

$$H_j := L^2(v^{2j} \mu^{-1}(v) dv),$$

for  $j = 0, 1, 2$ . The splitting is taken to be Grad’s splitting:

$$(Bf)(v) = -f(v) \int_{\mathbb{R}^d} |v - v'| \mu(v') dv, \quad Af = L - B.$$

Details are given in Appendix A. We point out that, in full generality,  $H^*$  maybe much more complicated than the above one and this is what motivated the introduction of the abstract framework **(L1)**–**(L4)**.

**DEFINITION 1.5.** — Under Assumption **(L2)**, we define the “dual” space  $H$  of the dissipation Hilbert space  $H^*$  as the completion of  $H$  for the norm

$$\|f\|_H := \sup_{\|\varphi\|_{H^*} \leq 1} \langle f, \varphi \rangle_{H^*}$$

*Remark 1.6.* — Since  $\|\cdot\|_H \leq \|\cdot\|_{H^*}$ , for any  $f \in H$  one has from Cauchy–Schwarz inequality

$$\|f\|_H \leq \sup_{\|\varphi\|_{H^*} \leq 1} \langle f, \varphi \rangle_H \leq \|f\|_{H^*}$$

we thus have the following comparison:

$$H^* \subset H \subset H, \quad \|\cdot\|_H \leq \|\cdot\|_H \leq \|\cdot\|_{H^*}.$$

At the nonlinear level, we make the following assumptions on  $Q$ .

STRUCTURAL QUADRATIC ASSUMPTIONS 1.7. — *The nonlinear operator  $Q$  is satisfying the following assumptions:*

(B1) *The bilinear operator is  $H$ -orthogonal to the null-space of  $L$ :*

$$Q(f, g), \mu_H = Q(f, g), v\mu_H = \langle Q(f, g), |v|^2 \mu \rangle_H = 0,$$

or, equivalently, in terms of integrals:

$$\int_{\mathbb{R}^d} Q(f, g)(v) dv = \int_{\mathbb{R}^d} v Q(f, g)(v) dv = \int_{\mathbb{R}^d} |v|^2 Q(f, g)(v) dv = 0.$$

(B2) *The bilinear operator commutes with orthogonal matrices:*

$$Q(f, g), h_H = \langle Q(f, g), h \rangle_H,$$

for any orthogonal matrix  $M_{d \times d}(\mathbb{R})$ .

(B3) *The bilinear operator satisfies the following dual estimate*

$$Q(f, g)_H \cdot f_H g_{H^*} + f_{H^*} g_H,$$

or, in other words, there holds

$$Q(f, g), h_H \cdot h_{H^*} (f_H g_{H^*} + f_{H^*} g_H).$$

### 1.6. Main results – first version

Under the above structural assumptions (L1)–(L4), the full description of the spectrum of  $L$  and the decay and regularization properties of the associated semigroup are made explicit in the following.

THEOREM 1.8 (Main spectral theorem). — *Assume (L1)–(L4), there exist explicitly computable constants  $C, \alpha_0, \lambda, \gamma, \sigma_0 > 0$  such that the following spectral and dynamical properties hold.*

(1) **Localization of the spectrum.** *The spectrum of  $L$  is localized as follows.*

- *If  $|\xi| > \alpha_0$ , the spectrum is at a positive distance from  $\{\text{Re } z > 0\}$ :*

$$\mathfrak{S}_H(L) \cap \{\text{Re } z > 0\} = \emptyset.$$

- *If  $|\xi| \leq \alpha_0$ , the spectrum is at a positive distance from  $\{\text{Re } z > 0\}$ , except for a finite number of small eigenvalues:*

$$\mathfrak{S}_H(L) \cap \{\text{Re } z > 0\} = \{\lambda_{\text{inc}}(\xi), \lambda_{\text{Bou}}(\xi), \lambda_{-\text{wave}}(\xi), \lambda_{+\text{wave}}(\xi)\},$$

and these eigenvalues  $\lambda(\xi)$  expand for  $\xi \rightarrow 0$  as

$$(1.9a) \quad \lambda_{\pm \text{wave}}(\xi) = \pm ic|\xi| - \kappa_{\text{wave}}|\xi|^2 + O(|\xi|^\beta),$$

$$(1.9b) \quad \lambda(\xi) = -\kappa|\xi|^2 + O(|\xi|^\beta), \quad * = \text{Bou, inc},$$

where the speed of sound is defined as

$$(1.10) \quad c := \frac{\sqrt{KE}}{d},$$

and the diffusion coefficients  $\kappa_\star(0, \cdot)$  are given by

$$(1.11) \quad \begin{aligned} \kappa_{\text{inc}} &:= -\frac{1}{(d-1)(d+1)} \langle L^{-1} \mathbf{A}, \mathbf{A} \rangle_H, & \kappa_{\text{Bou}} &:= -\frac{1}{d} \langle L^{-1} \mathbf{B}, \mathbf{B} \rangle_H, \\ \kappa_{\text{wave}} &:= \frac{d-1}{2d} \kappa_{\text{inc}} + \frac{E^2(K-1)}{2} \kappa_{\text{Bou}}, \end{aligned}$$

where the Burnett functions  $\mathbf{A}$  and  $\mathbf{B}$  are defined as

$$(1.12) \quad \begin{cases} \mathbf{A}(v) := \frac{1}{E} \left( v - \frac{|v|^2}{d} \text{Id} \right) \mu(v), \\ \mathbf{B}(v) := \frac{1}{\sqrt{K(K-1)}} \left( v - \frac{|v|^2}{E} \right) \mu(v). \end{cases}$$

(2) **Asymptotic behavior of the spectral projectors.** For any non-zero  $|\xi| \in \alpha_0$ , the spectral projectors associated with these eigenvalues expand in  $B(H; H^\star)$  as

$$(1.13) \quad \mathbf{P}_\star(\xi) = \mathbf{P}_\star^{(0)} \frac{\xi}{|\xi|} + i\xi \cdot \mathbf{P}_\star^{(1)} \frac{\xi}{|\xi|} + S_\star(\xi), \quad \star = \text{inc, Bou, } \pm\text{wave},$$

where  $S_\star(\xi) \in B(H; H^\star)$  with  $\|S_\star(\xi)\|_{B(H; H^\star)} \leq C|\xi|^\ell$ . The zeroth order coefficients are defined for any  $\omega \in S^{d-1}$  as

$$\begin{aligned} \mathbf{P}_{\text{inc}}^{(0)}(\omega) f(v) &= \frac{d}{E} \left( \langle f, v \mu \rangle_H \right) \cdot v \mu(v), \\ \mathbf{P}_\star^{(0)}(\omega) f &= \langle f, \psi_\star(\omega) \rangle_H \psi_\star(\omega), \quad \star = \text{Bou, } \pm\text{wave}, \end{aligned}$$

where we denoted  $\omega \cdot v = \langle \omega, v \rangle$  the orthogonal projection onto  $(\mathbb{R}\omega)^\perp$ , and the first order terms write explicitly for any  $f \in \ker(L)$  as

$$\begin{aligned} \mathbf{P}_{\text{inc}}^{(1)}(\omega) f(v) &= \frac{d}{E} \langle f, L^{-1} \mathbf{A} \rangle_H \omega \cdot v \mu, \\ \mathbf{P}_{\pm\text{wave}}^{(1)}(\omega) f &= \pm \frac{1}{2} \langle f, L^{-1} \mathbf{A} \omega \rangle_H + E \frac{K-1}{2} \langle f, L^{-1} \mathbf{B} \rangle_H \psi_{\pm\text{wave}}(\omega), \end{aligned}$$

and

$$(1.14) \quad \mathbf{P}_{\text{Bou}}^{(1)}(\omega) f = \langle f, L^{-1} \mathbf{B} \rangle_H \psi_{\text{Bou}},$$

where the zeroth order eigenfunctions  $\psi_{\pm\text{wave}}$  and  $\psi_{\text{Bou}}$  are defined as

$$(1.15) \quad \psi_{\pm\text{wave}}(\omega, v) := \frac{1}{2K} \left( 1 \pm \frac{dK}{E} \omega \cdot v + \frac{1}{E} (|v|^2 - E) \right) \mu(v),$$

$$(1.16) \quad \psi_{\text{Bou}}(v) := \frac{1}{\sqrt{K(K-1)}} \left( v - \frac{|v|^2}{E} \right) \mu(v).$$

Notice, in particular, that

$$(1.17) \quad \omega \cdot \mathbf{P}_{\text{inc}}^{(1)}(\omega) f = \frac{d}{E} \left( \langle f, L^{-1} \mathbf{A} \rangle_H \omega \right) \cdot v \mu.$$

(3) *Resolvent bounds and decay estimates. Setting*

$$(1.18) \quad P(\xi) = \mathbf{1}_{|\xi| \leq \alpha_0} (P_{\text{Bou}}(\xi) + P_{\text{inc}}(\xi) + P_{+\text{wave}}(\xi) + P_{-\text{wave}}(\xi))$$

the spectral projector associated to the part of the spectrum from point (1), the following resolvent bound holds

$$(1.19) \quad \sup_{z \in \mathbb{C} \setminus \Sigma} \left\| R(z, L) (\text{Id} - P(\xi)) \right\|_{B(H)} \leq C, \quad \xi \in \mathbb{R}^d,$$

where  $\sigma_0 := \min\{\lambda, \gamma\}$ . Finally, the  $C^0$ -semigroup  $(U(t))_{t>0}$  generated by  $(L, D(L))$  satisfies for any  $\sigma \in (0, \sigma_0)$ , any  $\xi \in \mathbb{R}^d$  and any  $f \in H$

$$(1.20a) \quad \sup_{t>0} e^{2\sigma t} \|U(t) (\text{Id} - P(\xi)) f\|_H^2 + \int_0^t e^{2\sigma s} \|U(s) (\text{Id} - P(\xi)) f\|_H^2 ds \leq C \|(\text{Id} - P(\xi)) f\|_H^2,$$

whereas, for any  $f \in H$ ,

$$(1.20b) \quad \int_0^t e^{2\sigma s} \|U(s) (\text{Id} - P(\xi)) f\|_H^2 ds \leq C \|(\text{Id} - P(\xi)) f\|_H^2.$$

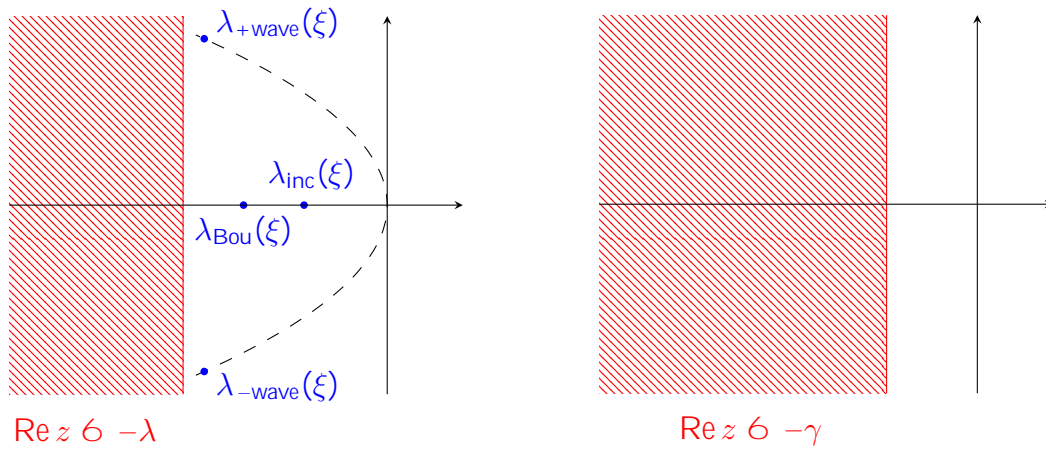


Figure 1.1. Localization of the spectrum of  $L - i(v \cdot \xi)$  for  $|\xi| \leq \alpha_0$  and for  $|\xi| > \alpha_0$

Remark 1.9. — Recall from Remark 1.3 that (L4) implies that  $|\cdot|^\beta \mu \in H_2$ . Using then (L4b) twice, we deduce that the mapping  $|\cdot|^\beta \mu$  belongs to  $H$ . Thus,

$$\int_{\mathbb{R}^d} |v|^\beta \mu(v) dv < \infty.$$

Consequently,  $\mathbf{A}, \mathbf{B} \in H$  and  $L^{-1}\mathbf{A}, L^{-1}\mathbf{B} \in H$ , and thus

$$\kappa < \infty, \quad \star = \text{Bou, inc, wave}.$$

We point out that, in a sense, we only assume (almost) enough integrability for the diffusion coefficients  $\kappa$  to be finite. This is to be contrasted with the work [MMM11]

in which they prove that if  $\kappa = \dots$ , then, under some appropriate scaling, one observes fractional diffusion in the limit  $\varepsilon \rightarrow 0$ . In this framework, a corresponding version of Theorem 1.8 was proved in [DP23]. We also refer to [BM22] for a unified spectral approach to the (fractional) diffusion limit for a large variety of linear collisional kinetic equations with a single conservation law. Finally, we point out that contrary to previous proofs of Theorem 1.8 for specific models, we do not assume that the weight  $\mu$  decays like a gaussian.

*Remark 1.10.* — Notice that  $R(z, L)(\text{Id} - P(\xi)) = R(z, L(\text{Id} - P(\xi)))$  and, by virtue of (1.13) the above resolvent bound (1.19) can be rewritten as

$$\sup_z \left\| R(z, L) - \sum_{\star = \text{inc}, \pm\text{Wave}, \text{Bou}} (z - \lambda(\xi))^{-1} P(\xi) \right\|_{B(H)} \leq C.$$

Note also that as  $L$  is self adjoint in  $H$ , the dual semigroup

$$\left( U(t)(\text{Id} - P(\xi)) \right)^* = U_-(t)(\text{Id} - P(-\xi)), \quad t > 0,$$

automatically satisfies the same estimates (1.20).

*Remark 1.11.* — The zeroth order terms in the expansions of the projectors are macroscopic in the sense that

$$PP^{(0)} = P^{(0)}P = P^{(0)}, \quad \star = \text{inc}, \text{Bou}, \pm\text{wave}.$$

As a consequence, they can be characterized in terms of the macroscopic components  $\varrho[\cdot]$ ,  $u[\cdot]$  and  $\theta[\cdot]$  where

$$(1.21) \quad \begin{cases} \varrho_f &= \varrho[f] := \langle f, \mu \rangle_H, \\ u_f &= u[f] := \frac{d}{E} \langle f, v \mu \rangle_H, \\ \theta_f &= \theta[f] := \frac{1}{E} \langle f, (|v|^2 - E) \mu \rangle_H \end{cases}$$

for any  $f \in H$ . We refer to Proposition 2.10 for a precise statement.

For the Boltzmann equation with hard potential interaction, the above theorem has been proven first in the seminal work [EP75] whose method has been adapted subsequently to encompass much more general models in the recent work [YY16] (see also [LY16, LY17, YY23]). The method in these contributions is based upon some compactness argument and a study of the eigenvalue problem through the use of the Implicit Function Theorem.

The approach we perform in the present paper appears much more direct and simpler. Any explicit computation relies solely on the isotropy of the operator  $L$ . To be more precise, we adapt here the perturbation theory of eigenvalues introduced in [Kat66] and exploit the structural assumption **(L4)** to prove the regularity and expansion of the eigen-projectors. Notice that, except for some peculiar cases (including the Boltzmann equation for hard-spheres interactions), our perturbative approach does not directly fall into the realm of the classical perturbation theory of



unbounded operators developed in [Kat66] since the multiplication operator  $i(v \cdot \xi)$  is *not*  $L$ -bounded in general. This induces some technical complications and requires to adapt the method of [Kat66] to the general situation we are dealing with here. This is done, borrowing and pushing further some ideas of [Tri16], by fully exploiting the splitting of  $L$  as

$$L = A + B$$

where  $B$  enjoys dissipative properties whereas  $A$  is a regularizing operator which compensates the unboundedness of the multiplication by  $v$  (see **(L4)**). Moreover, in contrast with existing results based upon [EP75], our method takes into account the role of the dissipation space  $H^*$  and its dual  $H$ . This allows us to emphasise and exploit regularizing effects of  $L$  in the scale of spaces  $H^* \subset H \subset H$ . A more detailed description of the approach we follow will be given in Section 3.1. We point out already that we strongly use here the fact that all functional spaces considered here are Hilbert spaces: this allows to use a suitable “diagonalization” of the transport operator thanks to Fourier transform and also permits to deduce spectral properties of the semigroup  $(U(t))_{t>0}$  through some of its generator  $L$  thanks to Gearhart–Pruss theorem.

We strongly believe that the new method we propose here to the fine spectral analysis of kinetic models is robust enough to be adapted to various contexts and can become a valid alternative to the technical approach of [EP75]. In our opinion, it replaces in an efficient way the compactness arguments introduced in [EP75] for the localization of the spectrum by a much modern and quantitative approach, combining enlargement techniques from [GMM17] to describe small frequencies  $\xi \rightarrow 0$  with hypocoercivity methods from [Dua11] for frequencies  $|\xi| \gg 1$ . Moreover, since it is based on the isotropic nature of  $L$  and  $Q$ , it can be directly adapted to more general equations of the type

$$\partial_t f + \mathbf{a}(v) \cdot \nabla_x f = Lf + Q(f, f)$$

where  $\mathbf{a}(\cdot)$  is a suitable smooth radial mapping and

$$\text{Ker } L = \text{Span} \{ \mu, v_1 \mu, \dots, v_d \mu, \mathbf{b}(v) \mu \}$$

for a suitable radial mapping  $\mathbf{b}(\cdot)$  such that

$$\int_{\mathbb{R}^d} Q(f, f) \mathbf{b}(v) dv = 0.$$

The relativistic Boltzmann and Landau equations both fall within the above framework with

$$\mathbf{a}(v) = \frac{1}{\mathbf{b}(v)}, \quad \mathbf{b}(v) = \frac{E}{1 + c_0^{-2}/v^2}, \quad \mu(v) = Z^{-1} e^{-\mathbf{b}(v)}, \quad c_0 > 0,$$

where  $Z > 0$  is a normalization constant so that the Juttner distribution  $\mu$  satisfies **(L1)**. Our structural assumptions **(L1)**–**(L4)** can then easily be modified to cover such a case. For instance, Assumption **L4b** should read now

$$\mathbf{a}(v) v f_{H_j} \cdot f_{H_{j+1}}.$$

We point out that several moments of the Juttner distribution  $\mu$  involving powers of  $\mathbf{a}(v) v$  and  $\mathbf{b}(v)$  would have to be considered in Assumption **(L2)**. In particular

the expressions of  $\psi$  and  $\kappa$  would be much more intricate. Thus, we do not pursue further this line of research since the present contribution is already quite technical and lengthy.

Besides the thorough description of the spectrum of  $L$  and the relevant eigenprojectors, Theorem 1.8 also describe the long-time behaviour of the associated linearized semigroup  $(U(t))_{t>0}$ . Our approach uses, as said enlargement techniques from [GMM17] as well as an abstract hypocoercivity result from [Dua11]. The decay of the linearized semigroup  $(U(t))_{t>0}$  in (1.20) is one of the fundamental brick on which it is possible to build the Cauchy theory for (1.1) whereas a comparison of  $(U(t))_{t>0}$  with the linearized semigroup associated to (1.3) is the main tool for the study of the hydrodynamic limit. This yields, in the Hilbert space setting  $H_x^s(H_v)$  to our main result as far as the above objective **(III)** is concerned:

**THEOREM 1.12 (Hydrodynamic limit theorem).** — *Let  $s > \frac{d}{2}$  be given as well as some initial data*

$$f_{\text{in}} \in H_x^s(H_v),$$

satisfying additionally, if  $d = 2$ ,  $f_{\text{in}} \in \dot{H}_x^\alpha(H_v)$  for some  $0 < \alpha < \frac{1}{2}$ .

Consider the solution of the Navier–Stokes–Fourier system (see Theorem 1.8 and Proposition 2.5 for the definitions of the coefficients, and Theorem C.1 for the existence of this solution)

$$(1.22) \quad \begin{cases} \partial_t u - \kappa_{\text{inc}} \Delta_x u + \vartheta_{\text{inc}} u \cdot \nabla_x u = \rho p, \\ \partial_t \theta - \kappa_{\text{Bou}} \Delta_x \theta + \vartheta_{\text{Bou}} u \cdot \nabla_x \theta = 0, \\ \nabla_x \cdot u = 0, \quad \nabla_x (\rho + \theta) = 0, \end{cases}$$

spanned by the initial conditions

$$u(0, x) = \text{Pu}[f_{\text{in}}](x), \quad \theta(0, x) = \frac{1}{K(K-1)}((K-1)\rho[f_{\text{in}}](x) - \theta[f_{\text{in}}](x)),$$

and which satisfies for some  $T \in (0, \infty]$

$$(\rho, u, \theta) \in C_b([0, T]; H_x^s), \quad (\nabla_x \rho, \nabla_x u, \nabla_x \theta) \in L^2([0, T]; H_x^s).$$

Introducing, for any  $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$  and  $t \in [0, T]$ ,

$$f_{\text{NS}}(t, x, v) = \rho(t, x) + u(t, x) \cdot v + \frac{\theta(t, x)}{E(K-1)} (|v|^2 - E) \mu(v),$$

the following holds.

**(1) Existence of a unique solution.** There exists some small  $c_0 > 0$  and  $\varepsilon_0 > 0$  such that the equation

$$\partial_t f = \frac{1}{\varepsilon^2} (L - \varepsilon v \cdot \nabla_x) f + \frac{1}{\varepsilon} Q(f, f), \quad f(0, x, v) = f_{\text{in}}(x, v)$$

admits for any  $\varepsilon \in (0, \varepsilon_0]$  a unique solution

$$f \in L_{\text{loc}}^2([0, T]; H_x^s(H_v^*)) \cap C([0, T]; H_x^s(H_v))$$

such that

$$\sup_{0 \leq t < T} \|f(t)\|_{H_x^s(H_v)} \leq \frac{C_0}{\varepsilon},$$

which satisfies furthermore the following uniform estimate:

$$\sup_{0 \leq t < T} \|f(t)\|_{H_x^s(H_v)}^2 + \int_0^T \left\| |x|^{\lambda} f(t) \right\|_{H_x^{s-(1-\alpha)}(H_v)}^2 dt \leq 1,$$

where we recall that  $0 < \alpha < \frac{1}{2}$  if  $d = 2$  and  $\alpha = 0$  if  $d > 3$ .

(2) **Decomposition and convergence of the solution.** The solution  $f$  splits as the sum of some limiting part  $f_{NS}$ , some initial layers  $(f_{disp}, f_{kin})$ , and a vanishing part  $f_{err}$ :

$$(1.23) \quad f = f_{NS} + f_{disp} + f_{kin} + f_{err}$$

where each part belongs to  $L^p([0, T]; H_x^s(H_v))$  uniformly in  $\varepsilon$  and

- The dispersive part  $f_{disp}$  vanishes in an averaged sense:

$$\lim_{\varepsilon \rightarrow 0} \int_0^t \|f_{disp}(\tau)\|_{L_x(H_v)}^p d\tau = 0, \quad 0 < t < T, \quad p > \frac{2}{d-1}$$

and uniformly away from  $t = 0$ :

$$\lim_{\varepsilon \rightarrow 0} \sup_{t_* \leq t < T} \|f_{disp}(t)\|_{L_x(H_v)} = 0, \quad 0 < t < T.$$

- The kinetic part  $f_{kin}$  satisfies for some universal  $\sigma > 0$

$$(1.24) \quad \sup_{0 \leq t < T} e^{2t/\varepsilon} \|f_{kin}(t)\|_{H_x^s(H_v)}^2 + \frac{1}{\varepsilon^2} \int_0^T e^{2t/\varepsilon} \|f_{kin}(t)\|_{H_x^s(H_v)}^2 dt \leq 1.$$

- The error term  $f_{err}$  vanishes uniformly:

$$\lim_{\varepsilon \rightarrow 0} \sup_{0 \leq t < T} \|f_{err}(t)\|_{H_x^s(H_v)} = 0.$$

**Remark 1.13.** — The rate of convergence of  $f_{disp}$  and  $f_{err}$  can be made explicit. Namely, the dispersive part  $f_{disp}$  satisfies:

$$\|f_{disp}\|_{L_x(H_v)} \leq 1 \left(\frac{\varepsilon}{t}\right)^{\frac{d-1}{2}} \left( P_{disp} f_{in} \Big|_{\dot{B}_{1,1}^{\frac{d+1}{2}}(H_v)} + P_{disp} f_{in} \Big|_{H_x^s(H_v)} \right), \quad \square$$

whereas the error term  $f_{err}$  is such that

$$\|f_{err}\|_{L_t H_x^s H_v} \leq \beta_{disp}(f_{in}, \varepsilon) + \beta_{NS}(f_{NS}, f_{in}, \varepsilon),$$

where  $P_{disp}$  is defined in Definition 2.6, and  $\beta_{disp}$  and  $\beta_{NS}$  are rates of convergence to zero described in Proposition 6.3.

**Remark 1.14.** — Notice that the initial datum here above  $f_{in}$  is not depending on  $\varepsilon$ . On this respect, the initial datum is *well-prepared* in a more restrictive sense than [BU91] (see in particular [BU91, Remark 1.5(ii)]). We however point out that it would be possible to choose a family of initial data  $f_{in}$  depending on  $\varepsilon$  provided with assume  $\lim_{\varepsilon \rightarrow 0} f_{in} = f_{in}$  in some explicit and quantitative sense which would allow to quantify the convergence of the error term as in the previous Remark. We also wish to emphasise that, aware of the general issue of sensitivity with respect

to initial data for both kinetic and hydrodynamic equations, the kind of solutions we are considering in the present contribution is much more regular than weak (renormalised) solutions considered for instance in [GSR04, GSR09] or, at the level of fluid-dynamical equation, Leray solutions to the Navier–Stokes–Fourier system. In particular, pathological issues as those exhibited in e.g. [ABC22, DLS12] are naturally excluded by our analysis.

*Example 1.15.* — Elaborating on Example 1.4, we can formulate Theorem 1.12 in the special case of the Boltzmann equation with hard spheres interactions. Recall that, in such a case, the collision operator  $Q$  is given by (1.4) with  $B(|v - v'|, \sigma) = |v - v'|$ . In such a case, considering for simplicity the physical dimension  $d = 3$ , we can consider functional spaces associated with Gaussian weight

$$H_\nu = L^2(\mu^{-1}(v)dv), \quad H^s = L^2(v \mu^{-1}(v)dv)$$

where  $\mu$  is given by (1.8) and, with an initial datum  $f_{\text{in}} \in H^s_x(H_\nu)$ ,  $s > \frac{3}{2}$ , Theorem 1.12 provides, for any  $\varepsilon > 0$  a unique solution  $f \in L^2_{\text{loc}}([0, T]; H^s_x(H^*_\nu)) \cap C([0, T]; H^s_x(H_\nu))$  to the Boltzmann equation with moreover the convergence of  $f$  to  $f_{\text{NS}}$  as  $\varepsilon \rightarrow 0$  (in some suitable sense, we refer to Example 1.17 for a more explicit statement).

To study both the kinetic equation (1.1) and the Navier–Stokes–Fourier system (1.3), we adopt a mild formulation which consists in writing the equations in Duhamel form

$$\begin{aligned} f(t) &= U(t)f_{\text{in}} + \frac{1}{\varepsilon} \int_0^t U(t-\tau)Q(f(\tau), f(\tau))d\tau \\ &=: U(t)f_{\text{in}} + [f, f](t) \end{aligned}$$

where  $(U(t))_{t>0}$  is the semigroup generated by  $-\varepsilon^{-2}L - \varepsilon^{-1}v \cdot \nabla_x$ .

Of course, the most obvious difficulty in establishing the hydrodynamic limit  $\varepsilon \rightarrow 0$  lies in the control of the stiff term  $\frac{1}{\varepsilon}Q(f, f)$ , however one first needs to construct a solution for any  $\varepsilon$ . In [BU91, GT20] in which the cut-off Boltzmann equation is considered, the authors only need to consider uniform in time estimates for the semigroup  $(U(t))_{t>0}$  to prove that  $[f, f]$  is well-defined. This means that, in such a case,  $[f, f]$  is bounded in a space of the type  $L_t H^s_x H_\nu$ . The study of the Boltzmann equation under cut-off assumption makes this approach possible since then, in the splitting

$$L = A + B$$

the dissipative part  $B$  is simply the multiplication by the collision frequency. In [KC22] in which the Landau equation is considered, the authors prove uniform in time regularization estimates for the semigroup and the kinetic solution which subsequently yield the boundedness of  $[f, f]$ . The short-time regularization effects are of course due to the elliptic-like nature of the Landau collision operator.

The abstract framework we are considering in this paper covers both cases, but the assumptions **(L1)**–**(L4)** or **LE** are not strong enough to directly deduce regularization effects and boundedness of  $[f, f]$ . To overcome this, we draw inspiration from suitable

energy methods used to construct close-to-equilibrium solutions for Boltzmann and Landau that leads us to a study of the solution in spaces of the form

$$L_t H \quad L_t^2 H^*$$

as expressed in (1.24). See Section 2 for more details of the functional setting and the mathematical difficulties encountered in the proof of Theorem 1.12.

We wish to point out also that the nonlinear effects induced by the collision and the lack of control for the  $L_x^2 H_v$  norm requires a refined inequality of quantity like  $f g \in H_x^s$  of the type

$$f g \in H_x^s \quad \left| \int_x f g \right| \in H^{s-r}, \quad g \in H_x^s, \quad 0 < r < \frac{d}{2} < s.$$

In the case of  $d > 3$ , one can take  $r = 1$  so that the dissipation of the  $L_x^2$ -norm and the energy control the nonlinearity. In the case  $d = 2$ , we cannot take  $r = 1$  so we consider  $r = 1 - \alpha$  where  $\alpha > 0$ . This explains why we require, in this specific case,

$$f_{in} \in \dot{H}^-$$

which is related to the control of the heat semigroup of the type

$$\left\| e^{t \Delta_x} \int_x f \right\|_{L_{t,x}^2} \in \dot{H}^{-\alpha}.$$

Note that in [Ger23, GT20], the stronger assumption  $f_{in} \in L_x^1(H_v)$  was made.

More details about the proof of Theorem 1.12 will be given in the next Section 2. However, let us already anticipate that the main relevant facts of our approach lie in the following

- (i) We insist here on the fact that, in the broad generality we are dealing with here, Theorem 1.12 is new even if results of similar flavors do exist in the literature for the Boltzmann or Landau equations. In particular, as already said, we do not assume here any special link between  $L$  and  $Q$  apart from the structure of  $\text{Ker}(L)$  and compatible nonlinear estimates. Precisely, *we do not require here* that  $L$  is the linearized version of  $Q$  around the equilibrium  $\mu$ .
- (ii) In dimension  $d = 2$ , one knows that solutions to the NSF system exist globally in time. When working in dimension  $d > 3$  one can prove that the solutions to the NSF systems are global assuming

$$(q_{in}, u_{in}, \theta_{in}) \in H_x^{\frac{d}{2}-1} \times 1$$

(see Appendix C for details), or equivalently, provided therefore that the corresponding parts of the initial datum  $f_{in}$  are small in  $H_x^{\frac{d}{2}-1}(H_v)$  norms. In both cases, solutions to (1.1) constructed in Theorem 1.12 are also global. This was already the case in the work [GT20] and this is an important contrast with respect to the result in [BU91] which assumed small  $f_{in}$  to generate global solutions.

- (iii) In the same spirit, contrary to [BU91, CC23, KC22], we do not work with small  $f_{in}$ , but as in [Ger23, GT20], we consider the a priori limit  $f_{NS}$  (which exists at least locally in time) and construct the kinetic solution in its neighborhood with the same lifespan. The smallness assumption we impose is transferred to

the physical parameter  $\varepsilon$ , i.e. assuming that a large number of collisions are experienced by the gas. This for instance extends the results of [CC23, KC22] to a larger class of initial data. Notice also that, since solutions to the NSF system (1.3) can be global depending on the properties of the initial data (such as symmetry, etc.), the kinetic solution to (1.1) we construct are also global.

- (iv) We also point out that our analysis is performed in the whole space  $\mathbb{R}_x^d$ . The strategy we adopt in the paper can be easily adapted to treat the case of a spatial torus  $\mathbb{T}_x^d$ . Furthermore, in such a case, assuming the initial datum  $\mathbb{P}f_{in}$  to be mean-free in space, i.e.

$$\int_{\mathbb{T}^d} \mathbb{P}f_{in}(x)dx = 0,$$

one can show the exponential trend to equilibrium for solutions to the kinetic equation (1.1). This is an easy consequence of Theorem 1.8 and this can be seen in the case of the Boltzmann equation (see [BMAM19, Ger23, GMM17, GT20]). The situation is much more delicate in the case of the whole space  $\mathbb{R}_x^d$  and the trend to equilibrium for solutions to (1.1) is not addressed in this paper.

Moreover, we wish to emphasize that, for *well-prepared* initial datum, i.e. in the case in which  $f_{in}$  is such that

$$x \cdot u[f_{in}] = 0, \quad x(\varrho[f_{in}] + \theta[f_{in}]) = 0$$

then no acoustic waves are produced:

$$f_{disp}(t) = 0 \quad t \in [0, T]$$

and, with the notations of Proposition 6.3, there holds  $\beta_{wave}(f_{in}, \varepsilon) = 0$ . In particular, for a smooth initial datum  $f_{in} \in H_x^{s+1}(H_v)$ , this yields to the convergence rate  $\beta_{NS} = O(\varepsilon)$  which is optimal (see [Guo06]). Let us state this clearly in the following corollary.

**COROLLARY 1.16 (*Optimal convergence rate*).** — *If the initial datum is smooth and well-prepared, in the sense that*

$$(1.25) \quad f_{in} \in H_x^{s+1}(H_v), \quad x \cdot u[f_{in}] = 0, \quad x(\varrho[f_{in}] + \theta[f_{in}]) = 0,$$

*then the conclusion of Theorem 1.12 holds with the decomposition*

$$f = f_{NS} + f_{kin} + f_{err},$$

*where, in this case, the error term is such that*

$$\sup_{0 \leq t < T} \|f_{err}(t)\|_{H_x^s(H_v)} \leq \varepsilon,$$

*and, in particular, away from  $t = 0$*

$$\sup_{0 < t < T} \|f(t) - f_{NS}(t)\|_{H_x^s(H_v)} \leq \varepsilon, \quad 0 < t < T.$$

*Example 1.17.* — With reference to Example 1.15, under the additional assumption (1.25), the solution  $f$  to the Boltzmann equation (1.4) with hard-spheres interactions is converging to  $f_{NS}$  with an explicit rate in, say  $L_x(H_\nu)$  with

$$\sup_{t \in [0, T]} \sup_x |f(t) - f_{NS}(t)|_{L^2(\mu^{-1}d\nu)} \leq \varepsilon$$

for any  $0 < t < T$ .

### 1.7. Main results – second version in larger functional spaces

We improve also the two main results here above by showing that the same conclusion still holds in a larger functional space  $X$  such that

$$H \subset X.$$

To do so, our analysis requires a new set of Assumption which complement **(L1)**–**(L4)**:

ASSUMPTIONS ON ENLARGED SPACES. — Besides Assumptions **(L1)**–**(L4)**, one assumes that  $L$  satisfies

**LE** Besides the splitting provided in **(L3)**, the operator  $L$  can be decomposed as

$$L = B^{(0)} + A^{(0)}, \quad D(B^{(0)}) = D(L), \quad A^{(0)} \in B(X, H)$$

where the splitting is compatible with a hierarchy of Hilbert spaces  $(X_j)_{j=0}^2$  such that

(a) the spaces  $X_j$  continuously and densely embed into one another:

$$X_2 \subset X_1 \subset X_0 = X, \quad H \subset X,$$

(b) the multiplication by  $v$  and its adjoint are bounded from  $X_{j+1}$  to  $X_j$ :

$$v f|_{X_j} \in X_{j+1}, \quad v^* f|_{X_j} \in X_{j+1}, \quad j = 0, 1,$$

(c) the part  $B^{(0)} = B^{(0)} - iv \cdot \xi$  is dissipative on each space  $X_j$  and  $H$  uniformly in  $\xi \in \mathbb{R}^d$ , that is, to say, for  $Y = X_0, X_1, X_2, H$

$$\Re \langle B^{(0)} f, f \rangle_Y \leq -\lambda_B \|f\|_Y^2$$

and

$$\sup_{\mathbb{R}^d} \left\| R_{z, B^{(0)}} \right\|_{B(Y)} \leq C / |\operatorname{Re} z + \lambda_B|^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Specifically, in the space  $X$ , there holds

$$\Re \langle B^{(0)} f, f \rangle_X \leq -\lambda_B \|f\|_X^2, \quad f \in X^*$$

for some dissipation Hilbert space  $X^*$  satisfying

$$H^* \subset X^* \subset X, \quad \cdot|_X \in \cdot|_{X^*},$$

(d) the operator  $A^{(0)}$  and its adjoint  $(A^{(0)})^*$  are bounded in the following spaces

$$A^{(0)} \in B(X; H) \cap B(X_j; X_{j+1}), \quad (A^{(0)})^* \in B(X_j; X_{j+1}), \quad j = 0, 1.$$

**BE** The corresponding nonlinear assumption is then the following:

$$Q(f, g), h_{X^*} \cdot h_{X^*} ( f_{X^*} g_{X^*} + f_{X^*} g_{X^*} ), \quad f, g, h \in X^*.$$

*Remark 1.18.* — As shown in [BMAM19, Ger23, GMM17], the operator  $L$  and  $Q$  satisfy **(LE)** and **(BE)** in the spaces

$$X_j = L^2 ( v^{k+2j} dv ), \quad X^* = L^2 ( v^{k+1} dv ), \quad D(L) = L^2 ( v^{k+2} dv )$$

for some  $k > 0$ .

As in Definition 1.5, one can define the dual space  $X$  of  $X^*$  as the completion of  $X$  for the norm

$$\|f\|_X := \sup_{\varphi \in X^*} | \langle f, \varphi \rangle |.$$

In that space  $X$ , combining suitable enlargement techniques introduced in [GMM17] with a bootstrap argument, we derive the following improvement of the spectral Theorem 1.8.

**THEOREM 1.19 (Enlarged spectral result).** — Assume **(L1)**–**(L4)** as well as **(LE)**. Then the results of Theorem 1.8 hold with  $(H, H^*, H)$  replaced by  $(X, X^*, X)$ . Furthermore the spectral projectors are regularizing in the sense that, in the decomposition

$$P(\xi) = P^{(0)} \tilde{S}_{\xi} + i\xi \cdot P^{(1)} \tilde{S}_{\xi} + S(\xi),$$

each term belongs to  $B(X; H^*)$  uniformly in  $|\xi| \leq \alpha_0$ , and  $S(\xi) \in B(X; H^*) \cdot |\xi|^\beta$ . Finally, the decay estimate (1.20) extends to  $X$  as follows: for any  $\xi \in \mathbb{R}^d$ , any  $\sigma \in (0, \sigma_0)$  and any  $f \in X$

$$(1.26a) \quad \sup_{t>0} e^{2\sigma t} \|U(t)(\text{Id} - P(\xi))f\|_X^2 + \int_0^t e^{2\sigma s} \|U(s)(\text{Id} - P(\xi))f\|_X^2 ds \leq C \|(\text{Id} - P(\xi))f\|_X^2,$$

whereas for any  $f \in X$

$$(1.26b) \quad \int_0^t e^{2\sigma s} \|U(s)(\text{Id} - P(\xi))f\|_X^2 ds \leq C \|(\text{Id} - P(\xi))f\|_X^2.$$

The extension of Theorem 1.8 to a larger Hilbert space  $X$  is done using the enlargement procedure developed in [GMM17], which inspired the subtle bootstrap argument leading to the regularity properties of  $P$ . Again, we refer to Section 3.1 for a description of the proof. The aforementioned regularity of  $P$  can be improved in the presence of yet another suitable splitting of  $L$ , this time in Banach spaces instead of just Hilbert spaces. This is done in Theorem 3.14, but, since such a result is not necessary for the derivation of the NSF system (1.3), we do not give the statement in this introduction.

We assume for this next theorem **(L1)**–**(L4)** and **(B1)**–**(B2)**, as well as **(LE)** and **(BE)**.



**THEOREM 1.20 (Enlarged hydrodynamic limit theorem).** — Under the assumptions of Theorem 1.12 on  $f_{\text{in}} \in H_x^s(X_v)$  and on the solution to the Navier–Stokes–Fourier system, the conclusion of Theorem 1.12 still holds with the following differences:

(1) **Existence of a unique solution.** There exists some small  $c_0 > 0$  and  $\varepsilon_0 > 0$  such that the equation

$$\partial_t f = \frac{1}{\varepsilon^2} (L - \varepsilon v \cdot \nabla_x) f + \frac{1}{\varepsilon} Q(f, f), \quad f(0, x, v) = f_{\text{in}}(x, v)$$

admits for any  $\varepsilon \in (0, \varepsilon_0]$  a unique solution among those satisfying

$$\sup_{0 \leq t < T} \|f(t)\|_{H_x^s(X_v)} \leq \frac{c_0}{\varepsilon}, \quad f \in L^2_{\text{loc}}([0, T]; H_x^s(X_v^*)),$$

and it satisfies furthermore the following uniform estimate:

$$\sup_{0 \leq t < T} \|f(t)\|_{H_x^s(H_v)}^2 + \int_0^T \left\| \langle \cdot \rangle^{1-\alpha} f(t) \right\|_{H_x^{s-1+\alpha}(H_v)}^2 dt \leq 1.$$

Moreover,

$$f \in C([0, T]; H_x^s(X_v)).$$

(2) **Decomposition and convergence of the solution.** The solution  $f$  splits as the sum of some limiting part  $f_{\text{NS}}$ , some initial layers  $(f_{\text{disp}}, f_{\text{kin}})$ , and a vanishing part  $f_{\text{err}}$ :

$$f = f_{\text{NS}} + f_{\text{disp}} + f_{\text{kin}} + f_{\text{err}}$$

where  $f_{\text{disp}}$  and  $f_{\text{err}}$  satisfy the same estimates, and this time the kinetic part  $f_{\text{kin}}$  satisfies

$$\sup_{0 \leq t < T} e^{2t/\varepsilon^2} \|f_{\text{kin}}(t)\|_{H_x^s(X_v)}^2 + \frac{1}{\varepsilon^2} \int_0^T e^{2t'/\varepsilon^2} \|f_{\text{kin}}(t')\|_{H_x^s(X_v^*)}^2 dt' \leq 1.$$

**Example 1.21.** — The above result provides a generalization of Theorem 1.12 to larger functional spaces. Keeping on elaborating on the Boltzmann equation with hard spheres interactions as in Example 1.4, we obtain now the existence, uniqueness of solutions  $f$  to (1.4) in the space

$$C([0, T]; H_x^s(X)) \cap L^2([0, T], H_x^s(X^*))$$

where now  $X$  and  $X^*$  are  $L^2$ -spaces with polynomial weights:

$$X := L^2(v^k dv), \quad X^* := L^2(\langle v \rangle^{k+\frac{1}{2}} dv), \quad k > 2.$$

Theorem 1.20 also provides the convergence in similar spaces of  $f$  to  $f_{\text{NS}}$  as  $\varepsilon \rightarrow 0$ .

Finally, drawing inspiration from works such as [CG24, CM17, CTW16, HTT20] which dealt with the Boltzmann equation without cutoff or the Landau equation, we present an alternative to the nonlinear assumption (BE) in which the two arguments of  $Q$  do not play symmetric roles.

**STRUCTURAL ASSUMPTIONS – NON SYMMETRIC CASE.** — Besides Assumptions (L1)–(L4), we assume the following:

**BED** We consider a hierarchy of spaces  $(\mathbf{X}_j)_{j=-2-s}^1$  for some  $s > 0$  such that

$$H \subset \mathbf{X}_1 \subset X = \mathbf{X}_0 \subset \dots \subset \mathbf{X}_{-2-s},$$

whose dissipation spaces are embedded in the same way:

$$H^\bullet \subset \mathbf{X}_1^\bullet \subset X^\bullet = \mathbf{X}_0^\bullet \subset \dots \subset \mathbf{X}_{-2-s}^\bullet,$$

and such that the following conditions hold:

(a) the assumption **(LE)** is satisfied in the larger spaces  $\mathbf{X}_{-2-s}, \dots, \mathbf{X}_{-2}$  (but not necessarily in  $\mathbf{X}_1$ ) and with a splitting of  $L$  that may be different in each space  $\mathbf{X}_j$ ,

(b) the nonlinear operator satisfies the following non-closed dual estimate:

$$(1.27) \quad Q(f, g)_{\mathbf{X}_j} \leq \|f\|_{\mathbf{X}_j} \|g\|_{\mathbf{X}_{j+1}^\bullet} + \|f\|_{\mathbf{X}_j^\bullet} \|g\|_{\mathbf{X}_{j+1}}, \quad j = -1-s, \dots, 0,$$

(c) the nonlinear operator satisfies the following closed dual estimates:

$$(1.28) \quad Q(f, g)_{\mathbf{X}_j} \leq \sum_{\{a,b,c\}=\{j,j,-1-s\}} \|f\|_{\mathbf{X}_a} \|g\|_{\mathbf{X}_b} \|g\|_{\mathbf{X}_c} + \|f\|_{\mathbf{X}_a^\bullet} \|g\|_{\mathbf{X}_b} \|g\|_{\mathbf{X}_c^\bullet}, \quad j = -1-s, \dots, 0,$$

(d)  $A$  sends  $\mathbf{X}_{j-1}$  to  $\mathbf{X}_j$  at the dual level, in the sense that

$$A f, f_{\mathbf{X}_j} \leq \|f\|_{\mathbf{X}_{j-1}}^2, \quad j = -1-s, \dots, 0.$$

**THEOREM 1.22 (Enlarged hydrodynamic limit theorem under (BED)).** Consider  $s \in \mathbb{N}$  such that  $s > \max\{3, d/2 + 1\}$  and denote for  $j = -1, 0$  the spaces  $\mathbf{X}^s$  and  $\mathbf{X}^{\bullet,s}$  defined by the norms

$$\|f\|_{\mathbf{X}_j^s} = \|f\|_{L_x^2(\mathbf{X}_j)} + \|x^s f\|_{L_x^2(\mathbf{X}_{j-s})}, \quad \|f\|_{\mathbf{X}_j^{\bullet,s}} = \|f\|_{L_x^2(\mathbf{X}_j)} + \|x^s f\|_{L_x^2(\mathbf{X}_{j-s}^\bullet)}.$$

For any  $f_{\text{in}} \in \mathbf{X}_0^s$ , and under the assumptions of Theorem 1.20 on the solution to the Navier–Stokes–Fourier system, the conclusion of Theorem 1.20 still holds with the following difference. There exists some small  $c_0 > 0$  and  $\varepsilon_0 > 0$  such that the equation

$$\partial_t f = \frac{1}{\varepsilon^2} (L - \varepsilon v \cdot \nabla_x) f + \frac{1}{\varepsilon} Q(f, f), \quad f(0, x, v) = f_{\text{in}}(x, v)$$

admits for any  $\varepsilon \in (0, \varepsilon_0]$  a unique solution among those satisfying

$$\sup_{0 \leq t < T} \|f(t)\|_{\mathbf{X}_0^s} \leq \frac{c_0}{\varepsilon}, \quad f \in L_{\text{loc}}^2([0, T]; \mathbf{X}_0^{\bullet,s}).$$

Moreover, it satisfies the following uniform estimate:

$$\sup_{0 \leq t < T} \|f(t)\|_{\mathbf{X}_0^s}^2 + \int_0^T \int |x| |f(t)|^2 (\mathbf{X}_0^{\bullet,s-}) dt \leq 1$$

and is continuous in the larger space  $\mathbf{X}_{-1}^s$ :

$$f \in C([0, T]; H_x^s(\mathbf{X}_{-1}^s)).$$

Finally, the kinetic initial layer is such that

$$\sup_{0 \leq t < T} e^{2t/\varepsilon^2} \|f_{\text{kin}}(t)\|_{\mathbf{X}_0^s}^2 + \frac{1}{\varepsilon^2} \int_0^T e^{2t/\varepsilon^2} \|f_{\text{kin}}(t)\|_{\mathbf{X}_0^{\bullet,s}}^2 dt \leq 1.$$

*Example 1.23.* — The above result is particularly well-suited to the study of the Landau equation. In such a case, the collision operator is given by

$$Q(f, f) = \int_{\mathbb{R}_v^3 \times S_\sigma} |v - v'|^2 (v - v') \left( f(v') - f(v) - f(v')f(v) \right) dv',$$

where  $P_z = \text{Id} - |z|^{-2}z \otimes z$  is the orthogonal projection onto  $z^\perp$ . We refer to Appendix A for more details.

A few comments are in order regarding Theorems 1.20 and 1.22:

- Only the kinetic part is now living in the smaller space  $H_x^\xi(X_v)$ , whereas the dispersive parts  $f_{\text{disp}}$  and  $f_{\text{err}}$  satisfy the same properties as in Theorem 1.12.
- The method of proof of Theorem 1.20 is the same as for Theorem 1.12; the existence of solution to the kinetic equation can be deduced from the application of Banach fixed point theorem (and the proof is led independently of whether we assume **(BE)** or not). Concerning Theorem 1.22, the approach is more involved and a careful study of an approximating scheme (a variation of Picard iterations) has to be done in order to overcome the difficulty induced by the lack of symmetry of  $Q$ .
- Notice that, as for Theorem 1.20, we *do not require* here any smallness assumptions on  $f_{\text{in}}$  and the smallness is totally transferred in the parameter  $\varepsilon$ . Recalling that assumptions **(BED)** are suited for the study of the Boltzmann equation without cut-off assumption and for the Landau equation, this is an important improvement with respect to the previous results in the field which all require some additional restriction on the size of the initial datum to derive the hydrodynamical limit (see [KC22] for the Landau equation and [CC23] for the Boltzmann equation).
- Finally, we emphasize that this provides, up to our knowledge, the first result concerning the strong Navier–Stokes limit for initial data with algebraic decay with respect to the velocity variable in the case of Boltzmann equation without cut-off assumption or Landau equation (see Appendix A).

We will comment with more details the conclusion of Theorems 1.12, 1.20 and 1.22 in Section 2 where a detailed description of the proof and the role of the various assumptions will be illustrated.

## 1.8. Outline of the paper

In the next Section 2, we introduce the main ideas underlying the proofs of the hydrodynamic limit Theorems 1.12, 1.20 and 1.22. Notations and mathematical objects that are used in the rest of the analysis are also introduced in Section 2.

Section 3 gives the full proof of both the spectral Theorems 1.8 and 1.19. A detailed description of approach is given in Section 3.1 and the proof of Theorem 1.8 is then derived in various steps, together with its “regularized version” Theorem 3.14.

Section 4 establishes the main consequences of Theorem 1.8 on the semigroup  $(U(t))_{t>0}$  generated by the linear part  $\varepsilon^{-2}(L - \varepsilon v \cdot \nabla_x)$  of (1.1) in the various

regimes/time scales relevant for the hydrodynamic limit. In particular, the comparison between the linearized semigroups associated to (1.1) and (1.3) is given in Section 4.

The main bilinear estimates are then established in Section 5 as well as the main tools used for the hydrodynamic limit (and in particular the mild formulation of (1.3)). The proof of Theorems 1.12 and 1.20 under assumptions **BE**, is then given in Section 6, whereas the proof of Theorem 1.22 under Assumption **(BED)** is given in Section 7.

To make the paper self-contained, we end it with three different Appendices. In Appendix A, we discuss the general assumptions **(L1)–(L4)**, **(B1)–(B2)**, **(LE)–(BE)** and **(BED)** for an extensive list of physical models including, as said, the classical Boltzmann and Landau equations as well as their quantum counterpart covering, at the linearized level, both the Fermi–Dirac and Bose–Einstein descriptions. Appendix B gives the functional toolbox with particular emphasis to Littlewood–Paley theory (Section B.1 and other results relevant for our analysis). We also present in Section B.4 the bootstrap argument for projection operators which is a cornerstone of Theorem 1.19. The final Appendix C recalls the main properties of the Navier–Stokes–Fourier system (1.3) that are needed for our analysis and proves some results necessary for our framework.

## 2. Detailed description of our proofs

We give here a precise description of the main steps of our approach to prove the above two Theorems 1.12 and 1.20. We use repeatedly the spectral properties of  $L$  and the properties of the associated semigroup  $(U(t))_{t>0}$  as established in Theorems 1.19 and 1.19. Notations are those introduced in those two results.

### 2.1. The functional setting

The conclusion of the Theorem 1.12 and the splitting of  $f$  in (1.23) suggest to introduce the following definitions of position-velocity spaces and time-position-velocity spaces suited to the different regimes (kinetic, diffusive and mixed) we will consider in this work.

**DEFINITION 2.1.** — *Let  $s \in \mathbb{R}$  be given.*

(1) *For  $Y = H$  or  $Y = X$ , we define the position-velocity spaces*

$$Y = Y^s = H_x^s(Y_v),$$

*which we endow with their natural norms  $\|\cdot\|_Y$  defined in (1.7). We define in the same way the spaces*

$$Y^{*,s} = H_x^s(Y_v^*), \quad Y^{*,s} = H_x^s(Y_v^*)$$

*with  $Y_v^* = H^*$  or  $X^*$  and  $Y_v = H$  or  $X$ .*

(2) Given  $T \in (0, \infty]$  and  $\sigma \in (0, \sigma_0)$  (with  $\sigma_0$  defined in Theorems 1.8 or 1.19), we introduce here the kinetic-type time-position-velocity space

$$X^\sigma = X^\sigma(T, \sigma, \varepsilon) := \{f \in C_b([0, T]; X^s) ; \|f\|_{X^\sigma} < \infty\},$$

where the norm  $\|\cdot\|_{X^\sigma}$  is given by

$$\|f\|_{X^\sigma}^2 := \sup_{0 \leq t < T} e^{2\sigma t/2} \|f(t)\|_{X^s}^2 + \frac{1}{\varepsilon^2} \int_0^T e^{2\sigma t/2} \|f(t)\|_{X^{s,\sigma}}^2 dt.$$

(3) Given  $T \in (0, \infty]$ ,  $0 < \eta \leq 1$  and some mapping  $\phi$  such that

$$(2.1) \quad \|\phi\|_{L^2([0, T]; H^{s,\sigma})} < \infty$$

where  $\alpha \in (0, \frac{1}{2})$  if  $d = 2$  and  $\alpha = 0$  if  $d > 3$ , we introduce the parabolic-type time-position-velocity space

$$H^\sigma = H^\sigma(T, \phi, \eta) := \{f \in C_b([0, T]; H^s) ; \|f\|_{H^\sigma} < \infty\}$$

endowed with the norm

$$\|f\|_{H^\sigma}^2 := \sup_{0 \leq t < T} w_\eta(t)^2 \|f(t)\|_{H^{s,\sigma}}^2 + w_\eta(t)^2 \int_0^t \|\phi^{-1} f(\tau)\|_{H^{s,\sigma}}^2 d\tau,$$

where we set

$$(2.2) \quad w_\eta(t) = \exp\left\{-\frac{1}{2\eta^2} \int_0^t \|\phi^{-1} f(\tau)\|_{H^{s,\sigma}}^2 d\tau\right\}.$$

(4) Finally, with the notation of the previous points, we introduce the mixed-type time-position-velocity space

$$F^\sigma = F^\sigma(T, \phi, \eta, \varepsilon) := \{C_b([0, T]; H^s) ; \|f\|_{F^\sigma} < \infty\}$$

endowed with the norm

$$\|f\|_{F^\sigma}^2 := \sup_{0 \leq t < T} w_\eta(t)^2 \|f(t)\|_{H^s}^2 + \frac{1}{\varepsilon^2} w_\eta(t)^2 \int_0^t \|f(\tau)\|_{H^{s,\sigma}}^2 d\tau.$$

*Remark 2.2.* — Drawing inspiration from [GT20] or more precisely [Ger23], the above norm  $\|\cdot\|_{H^\sigma}$  and  $\|\cdot\|_{F^\sigma}$ , and more specifically the time weight  $w_\eta$ , are designed so that, no matter how big  $\phi$  is, there holds

$$(2.3) \quad \sup_{t \in [0, T]} w_\eta(t) \|\phi^{-1} f\|_{L^2([0, t]; H^{s,\sigma})} \leq \eta$$

as can be seen by a simple computation (see Proposition 5.4). Notice also that  $w_\eta$  is decreasing and satisfies the bounds  $0 < w_\eta(T) \leq w_\eta \leq 1$ , consequently

$$(2.4) \quad \|f\|_{H^\sigma}^2 \leq \sup_{0 \leq t \leq T} \|f(t)\|_{H^{s,\sigma}}^2 + \int_0^T \|\phi^{-1} f(\tau)\|_{H^{s,\sigma}}^2 d\tau \leq w_\eta(T)^{-2} \|f\|_{H^s}^2,$$

and

$$\|f\|_{F^\sigma}^2 \leq \sup_{0 \leq t \leq T} \|f(t)\|_{H^s}^2 + \frac{1}{\varepsilon^2} \int_0^T \|f(\tau)\|_{H^{s,\sigma}}^2 d\tau \leq w_\eta(T)^{-2} \|f\|_{F^\sigma}^2.$$

We point out already that we will work in the proof of Theorems 1.12, 1.20 and 1.22 with the choice  $\phi = f_{NS}$  in (2.1).

*Remark 2.3.* — Clearly, Theorems 1.20 and 1.22 suggest that, in the decomposition of the solution  $f$  in (1.23), we will look for the kinetic part  $f_{\text{kin}}$  in the space  $X$  in the sense that

$$f_{\text{kin}} \in X^s \quad , \quad s > \frac{d}{2},$$

and, although it is not as obvious, the error term  $f_{\text{err}}$  will be constructed as the sum of a part in  $H$  and one in  $F$ , which explains the global energy estimate satisfied by the solution.

The first two regimes  $X$  and  $H$  are preserved by the two corresponding parts  $U_{\text{kin}}(\cdot)$  and  $U_{\text{hydro}}(\cdot)$  of the linearized flow of the equation (1.1) (see Sections 2.2–2.3 for their definitions), and the mixed regime  $F$  is introduced to describe the interactions between the two different regimes (see Lemmas 4.1 and 4.9).

### 2.2. Reduction of the problem

We present in this section how we frame the problem of hydrodynamic limits. We start from the integral formulation

$$(2.5) \quad f(t) = U(t)f_{\text{in}} + [f, f](t),$$

where, denoting  $2Q^{\text{sym}}(f, g) := Q(f, g) + Q(g, f)$

$$[f, g](t) := \frac{1}{\varepsilon} \int_0^t U(t - \tau) Q^{\text{sym}}(f(\tau), g(\tau)) d\tau$$

and  $(U(t))_{t>0}$  is the  $C^0$ -semigroup in  $Y^s$  generated by the full linearized operator (in original variables):

$$Gf := -\frac{1}{\varepsilon} v \cdot \nabla_x f + \frac{1}{\varepsilon^2} Lf, \quad D(G) = \{f \in Y; Lf \in Y\}$$

where  $Y = H$  or  $Y = X$ . Notice that the semigroup  $(U(t))_{t>0}$  is related to the semigroup  $(U(t))_{t>0}$ ; for  $g \in Y$ , setting

$$\widehat{g}(\xi) = \widehat{g}(\xi, \cdot) = F_x[g(x, \cdot)](\xi) = \int_{\mathbb{R}^d} e^{i \cdot x} g(x, \cdot) dx \in Y$$

one has

$$F_x[Gg](\xi) = \varepsilon^{-2} L \widehat{g}(\xi, \cdot)$$

so that

$$F_x U(t) F_x^{-1} = U \left( \frac{t}{\varepsilon^2} \right), \quad t > 0.$$

We define now the linearized semigroup  $(U_{\text{NS}}(t))_{t>0}$ , adopting again a Fourier-based description which involves the projectors  $P_{\text{inc}}^{(0)}$  and  $P_{\text{Bou}}^{(0)}$  as defined in Theorem 1.8.

**DEFINITION 2.4.** — *We define the dispersive Navier–Stokes semigroup  $(U_{\text{NS}}(t))_{t>0}$  through its Fourier transform for any  $g \in Y = H$  or  $Y = X$  and  $t > 0$  as*

$$U_{NS}(t)g = F_x^{-1} \left[ \exp(-t\kappa_{inc}/\xi^{\rho}) P_{inc}^{(0)} \frac{\epsilon \tilde{\xi}}{\xi} \hat{g}(\xi) + \exp(-t\kappa_{Bou}/\xi^{\rho}) P_{Bou}^{(0)} \frac{\epsilon \tilde{\xi}}{\xi} \hat{g}(\xi) \right], \quad \tilde{\xi} := \frac{\xi}{|\xi|}.$$

We also define the one parameter family  $(V_{NS}(t))_{t>0}$  as

$$(2.6) \quad V_{NS}(t)g := F_x^{-1} \left[ \exp(-t\kappa_{inc}/\xi^{\rho}) P_{inc}^{(1)} \frac{\epsilon \tilde{\xi}}{\xi} \hat{g}(\xi) + \exp(-t\kappa_{Bou}/\xi^{\rho}) P_{Bou}^{(1)} \frac{\epsilon \tilde{\xi}}{\xi} \hat{g}(\xi) \right].$$

The link between the above objects and solutions to the NSF system (1.3) is given by the following Proposition whose proof is postponed to Appendix C.

PROPOSITION 2.5. — Consider  $T_0 > 0$  and

$$(\varrho, u, \theta) \in L^1([0, T_0]; H_x^s) \times L^2([0, T_0]; H_x^{s+1}),$$

and define the corresponding macroscopic distribution  $f$  as

$$(2.7) \quad f(t, x, v) = \varrho(t, x)\mu(v) + u(t, x) \cdot v\mu(v) + \frac{1}{E(K-1)}\theta(t, x)(|v|^{\rho} - E)\mu(v).$$

The macroscopic distribution  $f$  satisfies the integral equation

$$(2.8) \quad \begin{aligned} f(t) &= U_{NS}(t)f_{in} + \mathcal{NS}[f, f](t) \\ &= U_{NS}(t)f_{in} + \int_0^t \mathcal{V}_{NS}(t-\tau) Q^{sym}(f(\tau), f(\tau))(t). \end{aligned}$$

if and only if the coefficients  $(\varrho, u, \theta)$  satisfy the incompressible Navier–Stokes–Fourier equations:

$$\begin{cases} \partial_t u + \vartheta_{inc} u \cdot \nabla_x u = \kappa_{inc} \nabla_x u - \nabla_x p, & \nabla_x \cdot u = 0, \\ \partial_t \theta + \vartheta_{Bou} u \cdot \nabla_x \theta = \kappa_{Bou} \nabla_x \theta, & \nabla_x(\varrho + \theta) = 0, \end{cases}$$

where we denoted

$$\vartheta_{inc} = -\frac{\vartheta_1}{2} \frac{d^{\frac{3}{2}}}{E}, \quad \vartheta_{Bou} = -\frac{1}{K\sqrt{K(K-1)}} \left( 2\vartheta_2 + \frac{2\vartheta_3}{E(K-1)} \right)$$

with  $\vartheta_1, \vartheta_2$  and  $\vartheta_3$  defined in Lemma C.2.

With this at hands, one sees that (2.8) provides a *kinetic formulation* of the NSF system (1.3). As said in the Introduction, our approach is “top-down” so we start with solutions  $(\varrho, u, \theta)$  to the Navier–Stokes–Fourier system (1.3) to recover information about the kinetic equation (1.1) in its mild formulation (2.5). This allows in particular to define the “kinetic formulation” to the NSF system

$$(2.9) \quad f_{NS}(t) = U_{NS}(t)f_{in} + \mathcal{NS}[f_{NS}, f_{NS}](t).$$

Then, on the basis of the above Proposition, the hydrodynamic limit problem consists in proving

$$\lim_0 (U(t)f_{in} + [f, f](t)) = U_{NS}(t)f_{in} + [f_{NS}, f_{NS}](t) = f_{NS}(t)$$

in some precise sense. We point out already that using the representation (2.7), the solution  $f_{NS}$  to (2.9) actually belongs to  $H^1$  (see Lemma C.5). The key point will be therefore to split suitably  $U(\cdot)$  (and accordingly) in order to prove the convergence. The splitting will be based upon the different parts of the spectrum identified in Theorem 1.8:

$$U(t) = U_{NS}(t) + U_{wave}(t) + U_{kin}(t).$$

Here  $U_{NS}(t)$  is the leading order term which is expected to converge, as  $\varepsilon \rightarrow 0$  towards the linearized Navier–Stokes semigroup  $U_{NS}(t)$  whereas  $U_{wave}(t)$  contains the acoustic waves responsible for dispersive effects (which are absent if the initial data is well-prepared), and the combination of these two semigroups can be seen as a *pseudo-hydrodynamic semigroup* encapsulating the macroscopic behavior of the solution  $f$ . The part  $(U_{kin}(t))_{t>0}$  keeps track of the pseudo-kinetic (microscopic) behavior of the solution which is exponentially small in  $\frac{t}{\varepsilon}$  due to the dissipation of entropy, enhanced by the numerous collisions in this hydrodynamic scaling. Since

$$[f, f](t) = \frac{1}{\varepsilon} \int_0^t U(t - \tau) Q^{sym}(f(\tau), f(\tau)) d\tau$$

the above splitting of  $U(\cdot)$  induces a similar splitting of the nonlinear term as

$$[f, f] = [f, f]_{NS} + [f, f]_{wave} + [f, f]_{kin}.$$

Precise definitions of these objects are given in the next sections. Before this, we briefly describe the main difficulties faced in the proof of Theorem 1.12:

- (1) As said, the major difficulty in establishing the hydrodynamic limit  $\varepsilon \rightarrow 0$  lies in the control of the stiff term  $\frac{1}{\varepsilon} Q(f, f)$ . This requires a precise understanding of the asymptotic behavior when  $\varepsilon \rightarrow 0$  of both  $U(t)f$  and convolutions of the type

$$\frac{1}{\varepsilon} \int_0^t U(t - \tau) \varphi(\tau) d\tau$$

in various norms, having in mind that  $\varphi = Q(f, f)$ . Furthermore, the nonlinear operator  $Q$  induces a loss of regularity in the sense that  $Q(f, f) \in H^1$  when  $f \in H^s$ , where we recall

$$H^s \cdot H^s \subset H^{s-1}.$$

One of the difficulty is therefore to show that convolution by  $(U(t))_{t>0}$  is able to compensate this loss of regularity.

- (2) As explained in the introduction, in the abstract framework considered here, our minimal assumptions on  $L$  and  $Q$  are not sufficient to deduce in a direct way regularization estimates or direct boundedness of  $[f, f]$  as it is the case for



the Boltzmann equation under cut-off assumptions in [BU91, GT20] or for the Landau equation [KC22]. In a more explicit way, our splitting

$$L = A + B$$

does not induce, in full generality, regularization estimates of the form

$$\| \exp(tB) \|_{B(H;H)} + \| \exp(tB) \|_{B(H;H^*)} \leq L_{loc}^1((0, T))$$

which would allow to compensate the unboundedness of  $Q$  in the Duhamel nonlinear term. Inspired by known energy methods introduced for instance in [Guo04] and which rely on a suitable dissipation in  $L^2$ -norm, the abstract functional setting which is adapted to our framework is the one involving spaces of the type

$$L_t^2 H \cap L_t^2 H^*.$$

Such spaces correspond to the above defined space  $X$ ,  $H$  and  $F$ .

### 2.3. The pseudo-hydrodynamic and pseudo-kinetic projectors

In this section, we denote by  $(Y, Y, Y^*, Y)$  the spaces  $(H, H, H^*, H)$  under assumption **(L1)**–**(L4)**, as well as  $(X, X, X^*, X)$  under the extra assumption **(LE)**. We introduce the *pseudo-hydrodynamic projector*, denoted  $P_{hydro}$ , corresponding to the small eigenvalues identified in Theorem 1.8, and defined as a Fourier multiplier:

$$P_{hydro}g := F^{-1} \left[ P(\varepsilon\xi) \widehat{g}(\xi) \right].$$

Using the splitting of  $P$  in (1.18), one sees that it is made up of two parts; one corresponding to the acoustic modes, denoted  $P_{wave}$ , and another one corresponding to the Navier–Stokes–Fourier modes, denoted  $P_{NS}$ :

$$P_{hydro} = P_{wave} + P_{NS}$$

defined in the following.

**DEFINITION 2.6.** — *The projectors  $P_{wave}$  and  $P_{NS}$  are defined through their Fourier transform, namely for any  $g \in Y$*

$$P_{wave}g := F^{-1} \left[ P_{+wave}(\varepsilon\xi) \widehat{g}(\xi) + P_{-wave}(\varepsilon\xi) \widehat{g}(\xi) \right],$$

$$P_{NS}g := F^{-1} \left[ P_{inc}(\varepsilon\xi) \widehat{g}(\xi) + P_{Bou}(\varepsilon\xi) \widehat{g}(\xi) \right].$$

We also define the limit of the first one as  $\varepsilon \rightarrow 0$  provided by the expansions of the projectors in Theorem 1.8:

$$P_{disp}g := F^{-1} \left[ P_{+wave}^{(0)} \frac{\varepsilon \widetilde{\xi}}{\xi} \widehat{g}(\xi) + P_{-wave}^{(0)} \frac{\varepsilon \widetilde{\xi}}{\xi} \widehat{g}(\xi) \right], \quad \widetilde{\xi} := \frac{\xi}{|\xi|}.$$

Using these projectors, we define the corresponding partial semigroups:

$$U(\cdot) := P U(\cdot) = U(\cdot) P, \quad \star = \text{hydro, NS, wave,}$$

which gives

$$U_{hydro}(\cdot) := U_{NS}(\cdot) + U_{wave}(\cdot).$$

More precisely, the above semigroups are defined as follows.

DEFINITION 2.7. — *The pseudo-Navier–Stokes (divisive) part  $(U_{\text{NS}}(t))_{t>0}$  is defined through its Fourier transform for any  $g \in Y$  and  $t > 0$  as*

$$(2.10) \quad U_{\text{NS}}(t)g = F^{-1} \left[ \exp(-\varepsilon^{-2}t\lambda_{\text{Bou}}(\varepsilon\xi)) P_{\text{Bou}}(\varepsilon\xi)\widehat{g}(\xi) + \exp(-\varepsilon^{-2}t\lambda_{\text{inc}}(\varepsilon\xi)) P_{\text{inc}}(\varepsilon\xi)\widehat{g}(\xi) \right],$$

whereas the pseudo-acoustic (dispersive) part  $(U_{\text{wave}}(t))_{t>0}$  is defined as

$$(2.11) \quad U_{\text{wave}}(t)g = F^{-1} \left[ \exp(\varepsilon^{-2}t\lambda_{+\text{wave}}(\varepsilon\xi)) P_{+\text{wave}}(\varepsilon\xi)\widehat{g}(\xi) + \exp(\varepsilon^{-2}t\lambda_{-\text{wave}}(\varepsilon\xi)) P_{-\text{wave}}(\varepsilon\xi)\widehat{g}(\xi) \right].$$

Because  $P_{\text{wave}} \sim P_{\text{disp}}$  as  $\varepsilon \rightarrow 0$ , the leading order terms of  $U_{\text{wave}}(t)$  denoted respectively  $U_{\text{disp}}(t)$  will play also a crucial roles in the study of hydrodynamic limits:

DEFINITION 2.8. — *The dispersive semigroup  $(U_{\text{disp}}(t))_{t>0}$  is defined as*

$$U_{\text{disp}}(t)g = F^{-1} \left[ \exp(i c \varepsilon^{-1} t / |\xi| - t \kappa_{\text{wave}} / |\xi|^2) P_{+\text{wave}}^{(0)} \widehat{g}(\xi) + \exp(-i c \varepsilon^{-1} t / |\xi| - t \kappa_{\text{wave}} / |\xi|^2) P_{-\text{wave}}^{(0)} \widehat{g}(\xi) \right].$$

Remark 2.9. — Recall that  $U_{\text{NS}}(t)$  and  $V_{\text{NS}}(t)$  were introduced in Definition 2.4. Observe that  $U_{\text{NS}}(t)$  is the leading order term in the expansion of  $U_{\text{NS}}(t)$  while  $x \cdot V_{\text{NS}}(t)$  is the leading order term of  $U_{\text{NS}}(t)$  on  $\text{Ker}(L)$ , that is to say

$$U_{\text{NS}}(t) \sim U_{\text{NS}}(t), \quad (U_{\text{NS}}(t))|_{\text{Ker}(L)} \sim \varepsilon x \cdot V_{\text{NS}}(t)$$

as will be exploited in Lemma 4.4.

Note that the projectors  $P_{\text{inc}}^{(0)}$ ,  $P_{\text{Bou}}^{(0)}$  and  $P_{\pm\text{wave}}^{(0)}$  are macroscopic in the sense that they take values in

$$\text{Ker}(L) = \{ (\varrho + u \cdot v + \theta (|v|^2 - E)) \mu; \varrho, u, \theta \in L^2(\mathbb{R}^d) \}$$

and vanish on its orthogonal, thus they can be characterized using the macroscopic components  $\varrho, u$  and  $\theta$  (see Remark 1.11). Similarly, the first order projectors  $P_{\text{inc}}^{(1)}$  and  $P_{\text{Bou}}^{(1)}$  restricted to  $\text{Ker}(L)$  can be characterized in such a way, which will be useful for describing  $V_{\text{NS}}$ .

PROPOSITION 2.10. — *The zeroth order projector related to the Navier–Stokes (incompressible) mode is characterized for  $f = f(x, v) \in L^2_x(H_v)$  by*

$$\varrho P_{\text{inc}}^{(0)} f = \theta P_{\text{inc}}^{(0)} f = 0, \quad u P_{\text{inc}}^{(0)} f = \frac{E}{d} P u[f],$$

the one related to the Fourier (Boussinesq) mode for  $f = f(x, v)$  by

$$u P_{\text{Bou}}^{(0)} f = \varrho P_{\text{Bou}}^{(0)} f + \theta P_{\text{Bou}}^{(0)} f = 0, \\ \sqrt{K(K-1)} \varrho P_{\text{Bou}}^{(0)} f - \theta P_{\text{Bou}}^{(0)} f = (K-1)\varrho[f] - \theta[f],$$

and the ones related to the acoustic modes (recall that  $(\text{Id} - P)_{\pm} = P_{\pm}$ ) for  $f = f(x, v)$  by

$$\begin{aligned} (K - 1)\varrho P_{\pm\text{wave}}^{(0)} f - \theta P_{\pm\text{wave}}^{(0)} f &= 0, \\ \frac{2K}{2K} u P_{\pm\text{wave}}^{(0)} f &= (-x)^{-\frac{1}{2}} x (\varrho[f] + \theta[f]) \pm c(\text{Id} - P) u[f], \\ \frac{1}{2K} \left( 1 - \frac{1}{K} \right) \varrho P_{\pm\text{wave}}^{(0)} f + \theta P_{\pm\text{wave}}^{(0)} f &= (\varrho[f] + \theta[f]) \pm c(-x)^{-\frac{1}{2}} x \cdot u[f]. \end{aligned}$$

The first order projectors related to the Navier–Stokes (incompressible) mode satisfy the identities for  $f(x, \cdot) \in \text{Ker}(L)$

$$\begin{aligned} \varrho P_{\text{inc}}^{(1)} f &= \theta P_{\text{inc}}^{(1)} f = 0, \\ \frac{E}{d} \langle u \cdot x \cdot P_{\text{inc}}^{(1)} f \rangle &= P(x \cdot f, L^{-1} \mathbf{A}_H), \end{aligned}$$

and the first order coefficient related to the Fourier (Boussinesq) mode for  $f(x, \cdot) \in \text{Ker}(L)$

$$\begin{aligned} u P_{\text{Bou}}^{(1)} f &= \varrho P_{\text{Bou}}^{(1)} f + \theta P_{\text{Bou}}^{(1)} f = 0, \\ (K - 1)\varrho P_{\text{Bou}}^{(1)} f - \theta P_{\text{Bou}}^{(1)} f &= \frac{1}{\sqrt{K(K - 1)}} \langle f, L^{-1} \mathbf{B} \rangle_H. \end{aligned}$$

We end this section by defining, in a similar way, the pseudo-kinetic part of the whole linearized semigroup

**DEFINITION 2.11.** — We define the pseudo-kinetic projector  $P_{\text{kin}}$  through its Fourier transform for any  $g \in Y$

$$P_{\text{kin}} g := F^{-1} \left[ (\text{Id} - P(\varepsilon\xi)) \widehat{g}(\xi) \right] = (\text{Id} - P_{\text{hydro}}) g,$$

as well as the corresponding semigroup  $(U_{\text{kin}}(t))_{t>0}$

$$U_{\text{kin}}(t)g := F^{-1} \left[ U(\varepsilon^{-2}t) (\text{Id} - P(\varepsilon\xi)) \widehat{g}(\xi) \right] = F^{-1} \left[ (\text{Id} - P(\varepsilon\xi)) U(\varepsilon^{-2}t) \widehat{g}(\xi) \right].$$

### 2.4. Decomposition of the solution

With the above definitions, we obtain the following decomposition of the semigroup  $U(t)$  as

$$(2.12) \quad U(t) = U_{\text{hydro}}(t) + U_{\text{kin}}(t) = U_{\text{NS}}(t) + U_{\text{wave}}(t) + U_{\text{kin}}(t) \quad t > 0$$

and we split the nonlinear integral operator accordingly, that is to say as a hydrodynamic part and a kinetic part:

$$[f, g](t) = {}_{\text{hydro}}[f, g](t) + {}_{\text{kin}}[f, g](t),$$

with

$$[f, g](t) := P \int_0^t U(t - \tau) Q^{\text{sym}}(f(\tau), g(\tau)) d\tau.$$

The main idea behind the proof of Theorems 1.12 or 1.20 or 1.22 is to consider an a priori decomposition of the unknown  $f$  in  $X + F + H$ :

$$(2.13) \quad f = f_{\text{kin}} + f_{\text{mix}} + f_{\text{hydro}}, \quad f_{\text{kin}} \in X, f_{\text{mix}} \in F, f_{\text{hydro}} \in H$$

which will enable us to reduce the construction of a solution of (2.5) to that of a solution to some appropriate system of equations for all the new unknowns

$$(f_{\text{kin}}, f_{\text{mix}}, f_{\text{hydro}}) \in X \times F \times H.$$

The term  $f_{\text{mix}}$  is a coupling term between the purely kinetic  $f_{\text{kin}}$  and macroscopic  $f_{\text{hydro}}$  parts and which need to be studied separately.

Let us dive more deeply in such a strategy, aiming to determine the system solved by  $(f_{\text{kin}}, f_{\text{mix}}, f_{\text{hydro}})$ . The splitting (2.13) induces the a priori decomposition of the kinetic part of the non-linear term:

$$\text{kin}[f, f] = \text{kin}[f_{\text{kin}}, f_{\text{kin}}] + 2 \text{kin}[f_{\text{kin}}, f_{\text{hydro}} + f_{\text{mix}}] + A_{\text{mix}}$$

where we expect the first two terms to belong to  $X$  and the third one

$$A_{\text{mix}} := \text{kin}[f_{\text{mix}} + f_{\text{hydro}}, f_{\text{mix}} + f_{\text{hydro}}] \in F.$$

In the same way, we introduce the following a priori decomposition of the hydrodynamic part of the non-linearity, which will only be used to make the following presentation more compact:

$$\text{hydro}[f, f] = \text{hydro}[f_{\text{hydro}}, f_{\text{hydro}}] + A_{\text{hydro}}[f_{\text{hydro}}] + B_{\text{hydro}}$$

where we denoted

$$\begin{aligned} A_{\text{hydro}}[f_{\text{hydro}}] &= 2 \text{hydro}[f_{\text{hydro}}, f_{\text{mix}} + f_{\text{kin}}], \\ B_{\text{hydro}} &:= \text{hydro}[f_{\text{mix}} + f_{\text{kin}}, f_{\text{mix}} + f_{\text{kin}}]. \end{aligned}$$

We consider an *arbitrary* system of equations for each part, where  $A_{\text{mix}}$  is assigned to the equation for  $f_{\text{mix}}$ :

$$\begin{cases} f_{\text{kin}}(t) = U_{\text{kin}}(t)f_{\text{in}} + \text{kin}[f_{\text{kin}}, f_{\text{kin}}](t) + 2 \text{kin}[f_{\text{kin}}, f_{\text{hydro}} + f_{\text{mix}}], \\ f_{\text{mix}}(t) = A_{\text{mix}}(t), \\ f_{\text{hydro}}(t) = U_{\text{hydro}}(t)f_{\text{in}} + \text{hydro}[f_{\text{hydro}}, f_{\text{hydro}}](t) + A_{\text{hydro}}[f_{\text{hydro}}](t) + B_{\text{hydro}}(t). \end{cases}$$

We see that, under the *ansatz* (2.13), solving (2.5) amounts to solve the above system for

$$(f_{\text{kin}}, f_{\text{mix}}, f_{\text{hydro}}) \in X \times F \times H,$$

as well as proving the uniqueness of solutions to the original equation (1.1) since our system is arbitrary.

In the hydrodynamic limit, we moreover expect  $f_{\text{hydro}}$  to be the leading term of  $f$  converging to  $f_{\text{NS}}$ , the other two terms being expected to converge to zero. Notice that we look for the solution  $f_{\text{hydro}} \in H$  and, as observed already, this is the functional space to which  $f_{\text{NS}}$  actually belongs. In other words, we expect

$$\lim_0 \left\| f_{\text{hydro}} - f_{\text{NS}} \right\|_H = 0.$$

Moreover, we need to prove that, in the above system, all terms are well-defined and belong to the desired spaces. This is one of the main technical difficulties of the work and, as explained in the introduction, will follow from a careful study of the behaviour of the various semigroups  $U(\cdot)$  as well as their action on convolutions. See Section 4 for full proofs.

**2.5. Removing the acoustic initial layer and the hydrodynamic limit**

To justify the convergence of  $f_{\text{hydro}}$  towards  $f_{\text{NS}}$ , we actually will need to rewrite the equation for  $f_{\text{hydro}}(t)$  by removing its leading order terms, namely a Navier–Stokes part and an acoustic part. The construction of those leading order terms rely on already existing theory for the Navier–Stokes equations and the wave equation.

More precisely, we split the hydrodynamic part  $f_{\text{hydro}}$  into an oscillating one  $f_{\text{disp}}(t)$  (which is explicit) and another one  $f_{\text{NS}}(t)$  that will be shown to be an approximation of the hydrodynamic limit  $f_{\text{NS}}(t)$ :

$$(2.14) \quad f_{\text{hydro}}(t) = f_{\text{disp}}(t) + f_{\text{NS}}(t), \quad f_{\text{disp}}(t) := U_{\text{disp}}(t)f_{\text{in}}.$$

Inserting this into the equation solved by  $f_{\text{hydro}}$ , we see that  $f_{\text{NS}}(t)$  satisfies

$$\begin{aligned} f_{\text{NS}}(t) = & (U_{\text{hydro}}(t)f_{\text{in}} - U_{\text{disp}}(t)f_{\text{in}}) + \mathcal{H}_{\text{hydro}}[f_{\text{NS}}, f_{\text{NS}}](t) \\ & + 2 \mathcal{H}_{\text{hydro}}[f_{\text{disp}}, f_{\text{NS}}](t) + A_{\text{hydro}}[f_{\text{NS}}](t) \\ & + A_{\text{hydro}}[f_{\text{disp}}](t) + B_{\text{hydro}}(t) + \mathcal{H}_{\text{hydro}}[f_{\text{disp}}, f_{\text{disp}}](t). \end{aligned}$$

We further split  $f_{\text{NS}}$  into its a priori limit  $f_{\text{NS}}(t)$  and an error term  $g(t)$ :

$$f_{\text{NS}}(t) = f_{\text{NS}}(t) + g(t),$$

so that, using (2.9), the part  $g(t)$  satisfies the equation

$$\begin{aligned} g(t) = & (U_{\text{hydro}}(t)f_{\text{in}} - U_{\text{NS}}(t)f_{\text{in}} - U_{\text{disp}}(t)f_{\text{in}}) \\ & + (\mathcal{H}_{\text{hydro}}[f_{\text{NS}}, f_{\text{NS}}](t) - \mathcal{H}_{\text{NS}}[f_{\text{NS}}, f_{\text{NS}}](t)) \\ & + 2 \mathcal{H}_{\text{hydro}}[f_{\text{NS}}, g](t) + 2 \mathcal{H}_{\text{hydro}}[f_{\text{disp}}, g](t) \\ & + A_{\text{hydro}}[g](t) + \mathcal{H}_{\text{hydro}}[g, g](t) \\ & + 2 \mathcal{H}_{\text{hydro}}[f_{\text{disp}}, f_{\text{NS}}](t) + \mathcal{H}_{\text{hydro}}[f_{\text{disp}}, f_{\text{disp}}](t) \\ & + A_{\text{hydro}}[f_{\text{NS}}](t) + A_{\text{hydro}}[f_{\text{disp}}](t) + B_{\text{hydro}}(t) \\ = & 2 \mathcal{H}_{\text{hydro}}[g, f_{\text{NS}} + f_{\text{disp}} + f_{\text{mix}} + f_{\text{kin}}](t) + \mathcal{H}_{\text{hydro}}[g, g](t) + S(t) \end{aligned}$$

Here, we have denoted the vanishing non-linear source term (which depends on  $f_{\text{kin}}$  and  $f_{\text{mix}}$  but not on  $g$ ), as

$$(2.15a) \quad S(t) = S_1(t) + S_2(t) + S_3[f_{\text{kin}}, f_{\text{mix}}](t)$$

where the first two parts depend only on  $f_{\text{disp}}$  through  $f_{\text{in}}$  and  $f_{\text{NS}}$ , which are considered given

$$(2.15b) \quad \begin{aligned} S_1(t) = & (U_{\text{hydro}}(t)f_{\text{in}} - U_{\text{NS}}(t)f_{\text{in}} - U_{\text{disp}}(t)f_{\text{in}}) \\ & + (\mathcal{H}_{\text{hydro}}[f_{\text{NS}}, f_{\text{NS}}](t) - \mathcal{H}_{\text{NS}}[f_{\text{NS}}, f_{\text{NS}}](t)) \\ S_2(t) = & \mathcal{H}_{\text{hydro}}[f_{\text{disp}}, 2f_{\text{NS}} + f_{\text{disp}}](t), \end{aligned}$$

and the third one also depends on the partial solutions  $f_{kin}$  and  $f_{mix}$

$$(2.15c) \quad \begin{aligned} S_3[f_{kin}, f_{mix}](t) &= A_{hydro} [f_{NS} + f_{disp}](t) + B_{hydro}(t) \\ &= hydro [f_{kin} + f_{mix}, f_{kin} + f_{mix} + f_{NS} + f_{disp}](t). \end{aligned}$$

### 2.6. Summary of the proof

The above technical splitting allows us to consider the solution to (2.5) we aim to construct in the form

$$\begin{aligned} f(t) &= f_{kin}(t) + f_{mix}(t) + f_{hydro}(t) \\ &= f_{kin}(t) + f_{mix}(t) + f_{disp}(t) + f_{NS}(t) + g(t) \end{aligned}$$

where  $f_{NS}(\cdot)$  as well as  $f_{in}$  (and thus  $f_{disp}(t) = U_{disp}(t)f_{in}$ ) are functions to be considered as fixed parameters since they depend only on the initial datum  $f_{in}$  (and  $\varepsilon$ ).

According to the analysis performed in Section 4 (see Lemma 4.8, Lemma 4.1 and Lemma C.5 respectively) that

$$U_{kin}(\cdot)f_{in} \times \cdot 1, \quad \|f_{disp}\|_{H^1} \cdot 1, \quad f_{NS} \in H^1 \cdot 1.$$

Since  $f_{NS}$  is entirely determined by the solutions  $(\varrho, u, \theta)$  to the NSF system (1.22), we point out that, in the definition of the space  $H^1$ , we choose the function  $\phi$  to be *exactly* the solution  $f_{NS}$ . This corresponds, in Eq. (2.2), to the choice of the weight function

$$w_{f_{NS}}(t) = \exp \left( -\frac{1}{2\eta^2} \int_0^t \| |x|^{1-\alpha} f_{NS}(\tau) \|_{H^1, s}^2 d\tau \right) \quad t > 0,$$

with  $\eta > 0$  is a parameter which is still to free to be chosen suitably small for the upcoming fixed point argument to work. The above system considered in Section 2.2 writes now:

$$(2.16) \quad \begin{cases} f_{kin}(t) &= U_{kin}(t)f_{in} + kin [f_{kin}, f_{kin}](t) + 2 kin [f_{kin}, f_{hydro} + f_{mix}](t), \\ f_{mix}(t) &= kin [(f_{NS} + f_{disp}) + f_{mix} + g, (f_{NS} + f_{disp}) + f_{mix} + g](t), \\ g(t) &= [f_{kin}, f_{mix}](t)g(t) + hydro [g, g](t) + S(t). \end{cases}$$

We will construct a solution  $(f_{kin}, f_{mix}, g)$  of this system in the space  $X \times F \times H$  and more specifically, in a product of suitable balls in such spaces. This is achieved through a suitable use of Banach fixed point Theorem in the case of Assumptions **(BE)** whereas, under Assumptions **(BED)**, the situation is much more involved and we adapt a Picard-like scheme to construct our solution  $(f_{kin}, f_{mix}, g)$ .

As said already, in order to study the system (2.16), we first need to prove that all the various terms make sense under the *ansatz* (2.13), that is we need to show that the various bilinear terms  $kin[f_{kin}, f_{kin}]$ ,  $kin[f_{kin}, f_{hydro} + f_{mix}]$  are defined and belong to  $X$  if  $(f_{kin}, f_{mix}, g) \in X \times F \times H$ , that the bilinear term appearing in the equation for  $f_{mix}$  is well defined and belong to  $F$  while the bilinear terms involved

in the equation for  $g$  are well-defined and belong to  $H$ . This is done in Section 5 which is based on the thorough analysis led in Section 4 of the various semigroups  $U_{\text{kin}}(\cdot)$ ,  $U_{\text{hydro}}(\cdot)$  and convolutions of the type

$$\frac{1}{\varepsilon}U_{\text{kin}}(\cdot) \varphi \quad \text{and} \quad \frac{1}{\varepsilon}U_{\text{hydro}}(\cdot) \varphi.$$

### 3. Spectral analysis of the linearized operator

This section is mainly devoted to the proof of the main spectral result Theorem 1.8 in the Introduction about the linearized operator  $L_\varepsilon = L - i(v \cdot \xi)$ .

#### 3.1. Description of the novel spectral approach

We give here a precise description of the main steps of our approach to prove the above two main results.

In order to prove our main spectral result, we adopt a “direct method” which appears much simpler than the original method of [EP75]. More precisely, their approach relies on the Lyapunov–Schmidt reduction process, which consists roughly in projecting the eigenvalue problem on the unperturbed eigenspace, whereas ours relies on Kato’s reduction process, which consists roughly in rectifying the perturbed operator as another one defined on the unperturbed space. To some extent, we believe our approach to be somehow more natural and direct, fully exploiting the symmetry properties of the collision operators  $Q$  and  $L$ . It is for sure of a more “functional analytic flavour” than the one of [EP75].

First, to study the spectrum of  $L_\varepsilon$ , we use the fact that, for  $\xi = 0$ , the spectrum of  $L_0 = L$  is explicit thanks to Assumption **(L3)** and show that, for  $|\xi|$  small enough, the structure of  $\mathfrak{S}(L_\varepsilon)$  is similar to that of  $\mathfrak{S}(L)$ , i.e. there exists some explicit value  $\alpha_0$  and  $\gamma > 0$  such that

$$\mathfrak{S}(L_\varepsilon) \subset \{z \in \mathbb{C}_+ ; \text{Re } z > -\gamma\}, \quad |\xi| \leq \alpha_0$$

consists in a finite number of eigenvalues. Such a localization of the spectrum is obtained here *without resorting to any compactness argument*. This is the main contrast with respect to the original work [EP75] whose approach disseminates in the literature.

The localization of the spectrum is not enough for the purpose of the paper and we also need to compute explicitly the asymptotic spectrum  $\mathfrak{S}(L_\varepsilon) \subset \{z \in \mathbb{C}_+ ; \text{Re } z > -\gamma\}$  and the associated spectral projector. This is done in a quantitative way, using Kato’s perturbation theory as developed in [Kat66]. Typically, one observes that, since for any  $\xi \in \mathbb{R}^d$ ,  $L_\varepsilon$  is a perturbation of  $L$  by the multiplication operator with  $-i(v \cdot \xi)$ , for any  $z \notin \mathfrak{S}(L_\varepsilon) \cap \mathfrak{S}(L)$ , the following expansion formulae are valid for  $N > 0$ :

$$(3.1) \quad R(z, L_\varepsilon) = \sum_{n=0}^{N-1} R(z, L) \left( (-iv \cdot \xi) R(z, L) \right)^n + R(z, L) \left( (-iv \cdot \xi) R(z, L) \right)^N,$$

as well as

$$(3.2) \quad R(z, L) = \sum_{n=0}^{N-1} R(z, L) \left( (-iv \cdot \xi) R(z, L) \right)^n + R(z, L) \left( (-iv \cdot \xi) R(z, L) \right)^N.$$

Various choice of the parameter  $N > 1$  would allow us to recover estimates on  $R(z, L)$  from known result on  $R(z, L)$  and provide the asymptotic expansion of the eigenvalue and eigen-projectors. In a more specific way, the proof of Theorem 1.8 is done according to the following roadmap:

- In Lemma 3.3, we show that the spectrum of  $L := L - iv \cdot \xi$  contained in some right half plane is confined in a ball of radius of order  $\xi$  centered around the origin, and establish some bounds on the resolvent.
- In Lemma 3.4, we prove that the spectral projector associated with this part of the spectrum has a first order expansion as  $\xi \rightarrow 0$ . To study the aforementioned part of the spectrum, we then introduce  $L$  which is a matrix conjugated to the restriction of  $L$  to the corresponding stable subspace (sum of eigenspaces), thus allowing to rely on perturbation theory in finite dimension.
- We establish in Lemma 3.5 some invariance (*isotropy*) properties satisfied by  $L$  and give its first order expansion.
- A block matrix representation of  $L$  is presented in Lemma 3.7, thus identifying its only multiple eigenvalue, and isolating it from the three simple remaining ones as is shown in Lemma 3.8.
- From that point on, we use finite dimensional perturbation theory and show that  $L$  is diagonalizable and establish a second order expansion of its eigenvalues as well as a first order expansion of its spectral projectors in Lemma 3.9, from which we deduce the spectral decomposition of the original operator  $L$ , as well as expansions of the projectors in Lemma 3.10.

Finally, we combine the resolvent bounds for  $|\xi| \leq 1$  from the previous lemmas, and use a hypocoercivity theorem from [Dua11] for  $|\xi| \geq \delta$  to obtain an uniform exponential decay in  $H$  of the semigroup generated by  $L$  on the stable subspace associated with the rest of the spectrum. We then improve this uniform decay estimate in  $H$  as an integral regularization and decay in  $H - H^*$  and  $H^* - H$  by combining it with an energy method.

### 3.2. The spatially homogeneous setting

Before undertaking the program described here above, it is important to recall the spectral picture in the spatially homogeneous setting corresponding to  $\xi = 0$ . Assumptions **(L1)**–**(L4)** directly give the localization of the spectrum and the fact that 0 is a semi-simple eigenvalue of  $L$  with  $d+2$ -dimensional (geometric) multiplicity. Associated to such an eigenvalue, the spectral projection

$$P := \frac{1}{2i\pi} \oint_{|z|=r} R(z, L) dz$$

has the following properties:



PROPOSITION 3.1 (**Representation formulae involving P**). — We recall the macroscopic (fluctuations of) mass  $\varrho \in \mathbb{R}$ , velocity  $u \in \mathbb{R}^d$  and temperature  $\theta \in \mathbb{R}$  as defined in (1.21). Under Assumptions (L1)-(L2), the spectral projector P on the null-space  $\text{Ker}(L)$  is  $H$ -orthogonal and has the following explicit representation:

$$(3.3) \quad Pf(v) = \varrho_f + u_f \cdot v + \frac{\theta_f}{E(K-1)} (|v|^2 - E) \mu(v),$$

as well as, denoting  $\omega = \text{Id} - \omega$  the orthogonal projection on  $\omega = \{u \in \mathbb{R}^d / u \cdot \omega = 0\}$  for any  $\omega \in S^{d-1}$ :

$$Pf(v) = (u_f \cdot v) \mu(v) + \frac{1}{K(K-1)} \left( (K-1)\varrho_f - \theta_f \right) K - \frac{|v|^2}{E} \mu(v) + \frac{1}{dc^2} (\varrho_f + \theta_f) |v|^2 \mu(v) + ((\text{Id} - \omega) u_f) \cdot v \mu(v)$$

where we introduced the speed of sound  $c$  in (1.10). More compactly, in terms of the eigenfunctions  $\psi_{\pm\text{wave}}$  and  $\psi_{\text{Bou}}$  defined in (1.15)–(1.16)

$$(3.4) \quad Pf(v) = \frac{d}{E} \left( f, v \mu \right)_H v \mu + \langle f, \psi_{-\text{wave}}(\omega) \rangle_H \psi_{-\text{wave}}(\omega) + \langle f, \psi_{+\text{wave}}(\omega) \rangle_H \psi_{+\text{wave}}(\omega) + \langle f, \psi_{\text{Bou}} \rangle_H \psi_{\text{Bou}}.$$

We finally notice that the Burnett functions introduced in (1.12) are related to  $v \mu \in \text{Ker}(L)$  and  $\psi_{\text{Bou}} \in \text{Ker}(L)$  through

$$A(v) = (\text{Id} - P)[v \cdot v \mu] \quad \text{and} \quad B(v) = (\text{Id} - P)[v \psi_{\text{Bou}}].$$

Remark 3.2. — Note that, if a given function  $\mu$  satisfies (L2), and if one defines P as the  $H$ -orthogonal projection on  $\text{Span}\{\mu, v_1, \dots, v_d \mu, |v|^2 \mu\}$ , i.e. as (3.3), then Proposition 3.1 still holds.

### 3.3. Proof of Theorem 1.8

We are now ready to attack the full proof of Theorem 1.8. We begin with the localization of the spectrum.

LEMMA 3.3 (**Localization of the spectrum**). — For any gap size

$$0 < \lambda < \lambda_L,$$

there exists some  $C_0 = C_0(\lambda) > 0$  and  $\alpha_0 = \alpha_0(\lambda) > 0$  that can be assumed small, such that the spectrum is localized as follows:

$$\mathfrak{S}(L) \subset \{ |z| \leq C_0 |\xi| \}, \quad |\xi| \leq \alpha_0.$$

Moreover, there exist  $C_1 = C_1(\alpha_0, \lambda) > 0$  such that, for any  $|\xi| \leq \alpha_0$

$$(3.5) \quad \sup_{|z|=r} R(z, L)_{B(H; H^*)} + \sup_{|z|=r} R(z, L)_{B(H; H)} + \sup_z R(z, L)_{B(H)} \leq C_1$$

where  $r = C_0 \alpha_0 > 0$  and  $\Omega := \{ |z| > r \}$ .

Proof. — In all the proof, we assume  $0 < \lambda < \lambda_L$  to be fixed.

*Step 1: Resolvent properties in the spaces  $H_j$ .* — We first observe that,  $B - \lambda$  being dissipative in the spaces  $H_j$  by hypothesis **(L4d)**, it holds for any  $j = 0, 1, 2$

$$(3.6) \quad R(z, B) \in B(H_j) \cdot 1, \quad z \in \mathbb{C} \setminus \sigma(B),$$

uniformly in  $\xi \in \mathbb{R}^d$ . In particular, the above is true for  $\xi = 0$ , that is to say for  $R(z, B)$ . Using that  $\mathfrak{G}_H(L) \setminus \{0\}$  from **(L3)**, we have the factorization formula for any  $N > 0$ :

$$(3.7) \quad R(z, L) = \sum_{n=0}^{N-1} R(z, B) \left( AR(z, B) \right)^n + \left( R(z, B)A \right)^N R(z, L).$$

holds for any  $z \in \mathbb{C} \setminus \{0\}$ . Furthermore, since  $L$  is self-adjoint in  $H$  by hypothesis **(L1)**, it is well-known that the zero eigenvalue is semi-simple so that

$$(3.8) \quad R(z, L) \in B(H) \cdot \frac{1}{|z|}, \quad z \in \mathbb{C} \setminus \{0\}.$$

Thus, using the factorization (3.7) with  $N = 1$ , we deduce from a repeated use of (3.6) that, for any  $f \in H_1$  and any  $z \in \mathbb{C} \setminus \{0\}$ ,

$$\begin{aligned} R(z, L)f \in H_1 &\subset R(z, B)f \in H_1 + R(z, B)AR(z, L)f \in H_1 \\ &\subset f \in H_1 + AR(z, L)f \in H_1, \end{aligned}$$

and using the boundedness of  $A : H_j \rightarrow H_{j+1}$  from **(L4c)** as well as (3.8)

$$\begin{aligned} R(z, L)f \in H_1 &\subset f \in H_1 + A \in B(H_1, H) \cdot R(z, L)f \in H \\ &\subset f \in H_1 + |z|^{-1} f \in H \cdot \left( 1 + \frac{1}{|z|} \right) f \in H_1 \end{aligned}$$

where we used  $H_1 \subset H$  in the last inequality. Using this estimate and proceeding in the same way for  $j = 2$ , we deduce that

$$(3.9) \quad R(z, L) \in B(H_j) \cdot \left( 1 + \frac{1}{|z|} \right), \quad z \in \mathbb{C} \setminus \{0\}, \quad j = 0, 1, 2.$$

We conclude that, in each space  $H_j$ , the eigenvalue 0 is semi-simple (i.e. a simple pole of  $R(\cdot, L)$ ) and the resolvent writes in  $B(H_j)$  as the sum of a singular part and a holomorphic part (see [Kat66, Chapter 3, Section 6.5]):

$$(3.10) \quad R(z, L) = z^{-1}P + R(z), \quad z \in \mathbb{C} \setminus \{0\}.$$

with the regular part defined as

$$(3.11) \quad R(z) = \sum_{n=1}^{\infty} z^n R_0^{n+1}, \quad R_0 := R(0, L) (Id - P),$$

where we notice that  $R_0 = -L^{-1} (Id - P) \in B(H_j)$ .

*Step 2: Localization of the spectrum and resolvent bound in  $B(H)$ .* — We draw inspiration from the proof of [Tri16, Lemma 2.16]. Let us start with the following factorization formula permitted by the dissipativity hypothesis from **(L4d)** for some large enough  $a > 0$ :

$$R(z, L) = R(z, B) + R(z, L)AR(z, B) \quad z \in \mathbb{C} \setminus \sigma(B)$$

and, expanding the term  $R(z, L)$  using (3.1) with  $N = 1$ , we deduce now

$$(3.12) \quad R(z, L) = R(z, B) + R(z, L)AR(z, B) + R(z, L)(-iv \cdot \xi)R(z, L)AR(z, B).$$

This allows to localize the spectrum using the bounds (3.9) and the regularization hypothesis for  $A$  coming from **(L4c)**. Indeed, according to **(L4b)**

$$vR(z, L)AR(z, B) \in B(H) \cdot R(z, L)AR(z, B) \in B(H, H_1)$$

and, using now (3.9) and the fact that  $A \in B(H, H_1)$  from **(L4c)**, we deduce that

$$vR(z, L)AR(z, B) \in B(H) \cdot \left(1 + \frac{1}{|z|}\right) R(z, B) \in B(H) \cdot \left(1 + \frac{1}{|z|}\right),$$

thanks to (3.6). In particular, for any  $c_0 > 0$ ,

$$(\xi \cdot v) R(z, L)AR(z, B) \in B(H) \cdot |\xi| + c_0, \quad |z| > \frac{|\xi|}{c_0},$$

thus, considering  $c_0, \alpha_0 > 0$  small enough, we deduce that

$$\xi \cdot v R(z, L)AR(z, B) \in B(H) \subset \frac{1}{2}, \quad |z| > \frac{|\xi|}{c_0}, \quad |\xi| \leq \alpha_0,$$

and in particular, for such a choice of  $(\xi, z)$ , the operator  $\text{Id} + (iv \cdot \xi)R(z, L)AR(z, B)$  is invertible in  $B(H)$  with

$$(3.13) \quad \left\| \left( \text{Id} + (iv \cdot \xi)R(z, L)AR(z, B) \right)^{-1} \right\|_{B(H)} \leq 2, \quad \text{for any } |z| > \frac{|\xi|}{c_0}, \quad |\xi| \leq \alpha_0,$$

thus it follows from (3.12) that

$$R(z, L) = \left( R(z, B) + R(z, L)AR(z, B) \right) \left( \text{Id} + (iv \cdot \xi)R(z, L)AR(z, B) \right)^{-1}.$$

Each term on the right hand side belongs to  $B(H)$  for  $z \in \{ |z| > c_0^{-1}|\xi| \}$ , thus the following localization of the spectrum holds:

$$\mathfrak{S}_H(L) \subset \left\{ z \in \mathbb{C} ; |z| \leq c_0^{-1}|\xi| \right\}, \quad |\xi| \leq \alpha_0.$$

More precisely, using the bounds on  $R(z, B)$  from **(L4)** and (3.9) together with (3.13) we have

$$(3.14) \quad R(z, L) \in B(H) \cdot \left(1 + \frac{1}{|z|}\right)$$

for any  $z \in \mathbb{C}, |z| > \frac{|\xi|}{c_0}$ , with  $|\xi| \leq \alpha_0$ . This proves the  $B(H)$ -bound in (3.5). This concludes this step.

**Step 3: Resolvent bound in  $B(H; H^*) \subset B(H; H)$ .** — First of all, note that the following identity for bounded operator  $T : H \rightarrow H^*$  is proved in Appendix B.3 (where the adjoint  $T^*$  is considered for the inner product of  $H$ ):

$$T \in B(H; H) = T^* \in B(H; H^*),$$

furthermore, using that  $L = L^*$  since  $L$  is self adjoint, one has

$$R(z, L) = R(\bar{z}, L^*) = \xi \in \mathbb{R}^d, z \in \mathfrak{S}(L),$$

thus it is enough to prove the bound in  $B(H; H^*)$ .

Using the dissipativity estimate for  $L$  from **(L3)** and the fact that the multiplication by  $iv \cdot \xi$  is skew-adjoint, we have for any  $z > 0$

$$(L - z)f, f_H \leq -\lambda_L (\text{Id} - P)f^2_{H^*} - z f^2_H.$$

Furthermore, using that  $P$  is  $H$ -orthogonal as well as the fact that  $P^2 = P$  and  $P \in B(H; H^*) \subset M$  for some  $M > 0$

$$\begin{aligned} (L - z)f, f_H &\leq -\lambda_L (\text{Id} - P)f^2_{H^*} - z Pf^2_H \\ &\leq -\lambda_L (\text{Id} - P)f^2_{H^*} - zM^{-2} Pf^2_{H^*}. \end{aligned}$$

The term  $(\text{Id} - P)f^2_{H^*}$  can be estimated using the polar identity and Young's inequality (note that  $P$  may not be  $H^*$ -orthogonal):

$$\begin{aligned} (3.15) \quad (\text{Id} - P)f^2_{H^*} &= \frac{1}{2} (\text{Id} - P)f^2_{H^*} \\ &+ \frac{1}{2} \left( f^2_{H^*} - Pf^2_{H^*} - 2\langle (\text{Id} - P)f, Pf \rangle_{H^*} \right) > \frac{1}{2} f^2_{H^*} - Pf^2_{H^*}, \end{aligned}$$

therefore we have

$$\langle (L - z)f, f \rangle_H \leq -\frac{\lambda_L}{2} f^2_{H^*} - (zM^{-2} - \lambda_L) Pf^2_{H^*}.$$

We conclude that for  $z_0 = \lambda_L M^2$ , we have

$$f \in D(L), \quad \langle (L - z_0)f, f \rangle_H \leq -\frac{\lambda_L}{2} f^2_{H^*}.$$

This, together with the comparison  $\cdot_H \leq \cdot_{H^*}$ , implies the resolvent bound

$$(3.16) \quad R(z_0, L) \in B(H; H^*) \leq \frac{2}{\lambda_L}.$$

Using the resolvent identity

$$R(z, L) = R(z_0, L) + (z_0 - z)R(z_0, L)R(z, L),$$

we can combine (3.14) and (3.16) so as to obtain, for  $|\xi| \leq \alpha_0$ ,

$$(3.17) \quad R(z, L) \in B(H; H^*) \leq 1 + |z| + \frac{1}{|z|}, \quad z \in \mathbb{R}, \quad |z| > c_0^{-1}|\xi|,$$

from which we deduce the  $B(H; H^*)$ -bound of (3.5). This concludes the proof of Lemma 3.3.

**LEMMA 3.4 (Expansion of the total projector).** — *With the notations of Lemma 3.3, considering some  $0 < \lambda < \lambda_L$ , for any  $|\xi| \leq \alpha_0$ , the spectral projector  $P(\xi)$  associated with the 0-group of Lemma 3.3 and defined as*

$$P(\xi) = \frac{1}{2i\pi} \oint_{|z|=r} R(z, L) dz, \quad |\xi| \leq \alpha_0,$$

where the integration along the circle  $\{|z|=r\}$  is counterclockwise, has the following first order expansion in  $B(H; H^*)$ :

$$P(\xi) = P + i\xi \cdot (PvR_0 + R_0vP) + S(\xi), \quad R_0 := R(0, L)(\text{Id} - P)$$

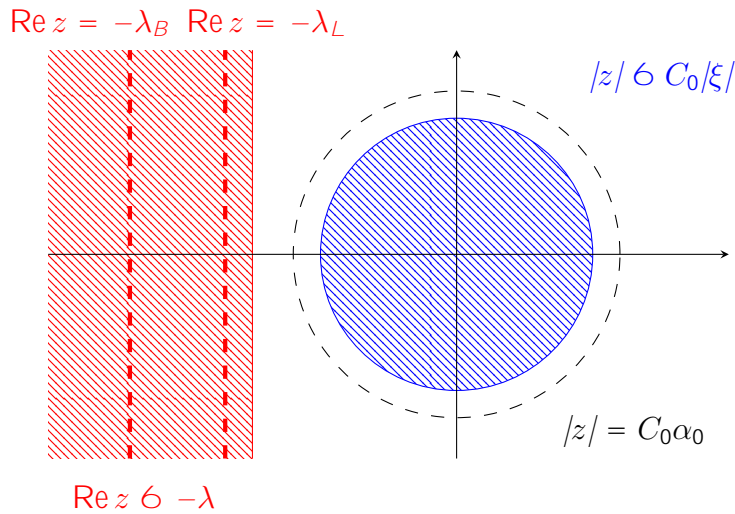


Figure 3.1. Localization of the spectrum from Lemma 3.3. The hatched blue part contains the “pseudo hydrodynamic part” of the spectrum of  $L$  (i.e. the perturbation of the macroscopic eigenvalue of  $L$ , that is to say 0). The hatched red part contains the “pseudo kinetic part” of the spectrum of  $L$  (i.e. the perturbation of the microscopic part of the spectrum of  $L$ , that is to say  $\mathfrak{S}(L) \setminus \{0\}$ ).

where  $S(\xi) \in B(H; H^*)$  with  $\|S(\xi)\|_{B(H; H^*)} \leq |\xi|^2$ .

*Proof.* — For a fixed  $0 < \lambda < \lambda_L$ , let  $\alpha_0 = \alpha_0(\lambda)$  be provided by Lemma 3.3. Since

$$P(\xi) = \frac{1}{2i\pi} \oint_{|z|=r} R(z, L) dz,$$

we deduce directly from the bound (3.5) that

$$\|P(\xi)\|_{B(H; H^*)} \leq \|P(\xi)\|_{B(H; H)} \leq 1,$$

and using that  $P(\xi)^2 = P(\xi)$ , we actually deduce

$$(3.18) \quad \|P(\xi)\|_{B(H; H^*)} \leq 1, \quad P(0) = P \in B(H; H^*).$$

We will refine this information and prove a second order Taylor expansion using the bootstrap formulae from Appendix B.4. Before doing so, we observe that the following factorization holds true

$$R(z, L) = R(z, B) + (R(z, B)A)^2 R(z, B) + (R(z, B)A)^2 R(z, L),$$

where the first two terms are actually  $B(H)$ -valued holomorphic in  $z \in \mathbb{C}$ . Therefore,  $P(\xi)$  can be written equivalently as

$$P(\xi) = \frac{1}{2i\pi} \oint_{|z|=r} (R(z, B)A)^2 R(z, L) dz.$$

Since, according to the regularization properties (L4c) of  $A$  together with the resolvent bounds (3.5) and (3.6) for  $L$  and  $B$  respectively, it holds, uniformly in  $|z| = r$  and  $|\xi| \leq \alpha_0$

$$R(z, B)A_{B(H; H_1)} + R(z, B)A_{B(H_1; H_2)} + R(z, L)_{B(H; H)} \cdot 1,$$

and thus, using  $H_2 \subset H_1$

$$\|(R(z, B)A)^2 R(z, L)\|_{B(H; H_2)} \cdot 1,$$

which once integrated along  $|z| = r$ , gives

$$(3.19) \quad P(\xi)_{B(H; H_j)} \cdot 1, \quad j = 0, 1, 2, \quad |\xi| \leq \alpha_0.$$

*Step 1: First order expansion.* — A representation of the first order Taylor expansion is given by integrating (3.1) or (3.2) with  $N = 1$  yielding

$$P(\xi) = P + \xi \cdot P^{(1)}(\xi), \quad |\xi| \leq \alpha_0,$$

where  $P^{(1)}(\xi)$  has the integral representation

$$P^{(1)}(\xi) := \frac{1}{2i\pi} \oint_{|z|=r} R(z, L)(-iv)R(z, L)dz = \frac{1}{2i\pi} \oint_{|z|=r} R(z, L)(-iv)R(z, L)dz.$$

We know that  $P^{(1)}(\xi)_{B(H; H^*)}$  (since  $\xi \cdot P^{(1)}(\xi) = P(\xi) - P$ ) let us prove that this holds uniformly in  $\xi$ . We begin with the following estimate in  $B(H_1; H^*)$  which is made possible since the multiplication by  $v$  is bounded from  $H_1$  to  $H$ . Namely, thanks to (L4b) and the resolvent bounds (3.5) and (3.9), we have uniformly in  $|\xi| \leq \alpha_0$

$$(3.20) \quad \begin{aligned} \|P^{(1)}(\xi)\|_{B(H_1; H^*)} &\leq \frac{1}{2\pi} \oint_{|z|=r} R(z, L)vR(z, L)_{B(H_1; H^*)}d|z| \\ &\cdot \oint_{|z|=r} R(z, L)_{B(H; H^*)} R(z, L)_{B(H_1)}d|z| \cdot 1. \end{aligned}$$

Let us now explain how to extend such an estimate to  $B(H; H)$ . Using the formula (B.6), we have, for  $|\xi| \leq \alpha_0$

$$P^{(1)}(\xi) = P(\xi)P^{(1)}(\xi) + P^{(1)}(\xi)P,$$

and therefore

$$\begin{aligned} \|P^{(1)}(\xi)\|_{B(H; H^*)} &\leq \|P(\xi)P^{(1)}(\xi)\|_{B(H; H^*)} + \|P^{(1)}(\xi)P\|_{B(H; H^*)} \\ &\leq \|P^{(1)}(\xi)P(\xi)\|_{B(H; H^*)} + \|P^{(1)}(\xi)P\|_{B(H; H^*)} \end{aligned}$$

where we used the adjoint identity (B.1) (where the adjoint is considered for  $\cdot, \cdot_H$ ). Since we have  $P(\xi) = P(-\xi)$  and  $P = P$ , one has  $P^{(1)}(\xi) = -P^{(1)}(-\xi)$ . Therefore

$$\begin{aligned} \|P^{(1)}(\xi)\|_{B(H; H^*)} &\leq \|P^{(1)}(-\xi)P(-\xi)\|_{B(H; H^*)} + \|P^{(1)}(\xi)P\|_{B(H; H^*)} \\ &\leq \|P^{(1)}(-\xi)\|_{B(H_1; H^*)} \|P(-\xi)_{B(H; H_1)} + \|P^{(1)}(\xi)\|_{B(H_1; H^*)} \|P_{B(H; H_1)} \\ &\cdot \|P^{(1)}(-\xi)\|_{B(H_1; H^*)} + \|P^{(1)}(\xi)\|_{B(H_1; H^*)}, \end{aligned}$$

where we used the regularizing property (3.19) of  $P(\xi)$  in the last line (recall that  $P = P(0)$ ). Using the above estimates (3.20), we deduce that

$$\|P^{(1)}(\xi)\|_{B(H; H^*)} \leq 1, \quad |\xi| \leq \alpha_0,$$

which concludes this step. Notice that a clear implication is that, for any  $|\xi| \leq \alpha_0$

$$P(\xi) - P \in B(H; H^*) \leq |\xi|.$$

*Step 2: Second order expansion.* — In a similar fashion, we obtain a second order expansion integrating once again (3.1) or (3.2) with  $N = 2$ :

$$P(\xi) = P + i\xi \cdot P_1 + \xi \otimes \xi : P^{(2)}(\xi), \quad |\xi| \leq \alpha_0,$$

where the first and second order terms are defined as

$$\begin{aligned} P_1 &= -\frac{1}{2i\pi} \oint_{|z|=r} R(z, L)vR(z, L)dz, \\ P^{(2)}(\xi) &:= -\frac{1}{2i\pi} \oint_{|z|=r} \left( R(z, L)v \right)^2 R(z, L) dz \\ &= -\frac{1}{2i\pi} \oint_{|z|=r} R(z, L) \left( vR(z, L) \right)^2 dz. \end{aligned}$$

The first order term is explicitly computable using the Laurent series expansion (3.10):

$$P_1 = -\frac{1}{2i\pi} \oint_{|z|=r} R(z, L)vR(z, L)dz = R_0vP + PvR_0,$$

so in particular, one checks directly that

$$P_1 = PP_1 + P_1P.$$

We use now the formula (B.8), which gives

$$\begin{aligned} \|P^{(2)}(\xi)\|_{B(H; H^*)} &\leq \|P(\xi)P^{(2)}(\xi)\|_{B(H; H^*)} + \|P^{(1)}(\xi) \cdot P_1\|_{B(H; H^*)} + \|P^{(2)}(\xi)P\|_{B(H; H^*)}. \end{aligned}$$

We use the bound from Step 1 for the second term, and, after using duality as in Step 1 (for the first term), we use the regularization property (3.19) of  $P(\xi)$ :

$$\begin{aligned} \|P^{(2)}(\xi)\|_{B(H; H^*)} &\leq 1 + \|P^{(2)}(-\xi)P(-\xi)\|_{B(H; H^*)} + \|P^{(2)}(\xi)P\|_{B(H; H^*)} \\ &\leq 1 + \|P^{(2)}(-\xi)\|_{B(H_2; H^*)} + \|P^{(2)}(\xi)\|_{B(H_2; H^*)}. \end{aligned}$$

As previously, using the fact that the multiplication by  $v$  is bounded from  $H_j$  to  $H_{j+1}$  for  $j = 0, 1$ , we show using the resolvent bounds (3.5) for  $L$  and (3.9) for  $L$  that

$$\begin{aligned} \|P^{(1)}(\xi)\|_{B(H_2; H^*)} &\leq \frac{1}{2\pi} \oint_{|z|=r} \left\| R(z, L) \left( vR(z, L) \right)^2 \right\|_{B(H_2; H^*)} d|z| \\ &\leq \oint_{|z|=r} \|R(z, L)\|_{B(H; H^*)} \|R(z, L)\|_{B(H_1)} \|R(z, L)\|_{B(H_2)} d|z| \leq 1. \end{aligned}$$

This concludes the proof, setting  $S(\xi) := \xi \otimes \xi : P^{(2)}(\xi)$ .

Following Kato's reduction process from [Kat66, Section I-4.6, pp. 32–34], we introduce for any  $|\xi| \leq \alpha_0$  a "rectified" version of  $L$  in which we cut off any spectral points that is not a small eigenvalue. Precisely, following [Kat66, Section I-4.6] and since

$$P(\xi) - P \in B(H, H^*) \cdot |\xi| \quad |\xi| \leq \alpha_0,$$

according to the previous Lemma and the injection  $H^* \hookrightarrow H \hookrightarrow H$ , we deduce that for  $\alpha_0$  small enough

$$\|(P(\xi) - P)^2\|_{B(H)} < 1, \quad |\xi| \leq \alpha_0,$$

where we recall that  $H^* \hookrightarrow H \hookrightarrow H$ . In particular (see [Kat66, Eqs. (4.36)–(4.39), p. 33]), we can define

$$\begin{aligned} U &= \left( P(\xi)P + (\text{Id} - P(\xi))(\text{Id} - P) \right) \left( \text{Id} - (P(\xi) - P)^2 \right)^{-\frac{1}{2}} \\ &= \left( \text{Id} - (P(\xi) - P)^2 \right)^{-\frac{1}{2}} \left( P(\xi)P + (\text{Id} - P(\xi))(\text{Id} - P) \right), \quad |\xi| \leq \alpha_0 \end{aligned}$$

where we used the definition  $(\text{Id} - T)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-T)^n$  for any  $T \in B(H)$ . With such a definition,  $U$  mapping isomorphically the null-space of  $L$  onto the eigenspaces corresponding to the small eigenvalues of  $L$ :

$$\text{Ker}(L) = \text{Range}(P) \stackrel{U_\xi}{=} \text{Range}(P(\xi)),$$

we define the *finite dimensional* rectified operator on  $\text{Ker}(L)$ :

$$L_\xi := \left( U^{-1} L U \right)_{|\text{Ker}(L)} \in B(\text{Ker}(L)), \quad |\xi| \leq \alpha_0,$$

whose spectrum is related to that of  $L$  by  $\mathfrak{S}(L_\xi) - \xi = \mathfrak{S}(L)$ . In the rest of this section, we will study the structure of  $L_\xi$  so as to reduce its diagonalization to the perturbation of a diagonalizable matrix with simple eigenvalues. Using (Kato's) classical perturbation theory for matrices, this will provide expansion of the eigenvalues and eigenfunctions of  $L_\xi$ , and in turn those of  $L$ .

**LEMMA 3.5 (*Properties of the rectified operator*).** — *The rectified operator  $L_\xi$  has the following properties:*

(1)  $L_\xi$  is compatible with any orthogonal  $d \times d$  matrix:

$$L_\xi = L_\xi^*,$$

in particular,  $L_\xi$  commutes with orthogonal matrices that fix  $\xi$ , and it preserves evenness and oddity in directions orthogonal to  $\xi$ :

$$\xi = \xi \quad L_\xi = L_\xi^*,$$

$$\left( u \perp \xi \text{ and } \varphi(-u) = \pm \varphi(u) \right) \implies \left( L_\xi \varphi \right)(-u) = \pm \left( L_\xi \varphi \right)(u).$$

(2)  $L_\xi$  has the following second order expansion in  $B(\text{Ker}(L))$

$$L_\xi = -iP(v \cdot \xi) + P(v \cdot \xi)R_0(v \cdot \xi) + \mathbf{r}_0(\xi)$$

where  $\mathbf{r}_0(\xi) \in B(\text{Ker}(L))$  is such that  $\|\mathbf{r}_0(\xi)\|_{B(\text{Ker}(L))} \leq |\xi|^\beta$ .



*Proof.* — As  $L = L$  holds for any orthogonal matrix, it is clear that the resolvent satisfies the same relation  $R(z, L) = R(z, L)$ , and thus  $P(\xi) = P(\xi)$ . Using the above definition of  $U$ , this implies  $U = U$  and thus the first point of this lemma. We only need to check the expansion for  $L$ . According to Lemma 3.4, there holds

$$\begin{aligned} P(\xi)P &= P + iR_0(v \cdot \xi)P + r_1(\xi), \\ (\text{Id} - P(\xi))(\text{Id} - P) &= \text{Id} - P - iP(v \cdot \xi)R_0 + r_2(\xi), \end{aligned}$$

and

$$(P(\xi) - P)^2 = r_3(\xi),$$

where the *remainder operators*  $r_k$  are such that

$$r_k(\xi) \in B(H; H^*) \cdot |\xi|^k, \quad k = 1, 2, 3,$$

for  $|\xi| \leq \alpha_0$  with  $\alpha_0$  small enough. Inserting this in the definition of  $U$ , there exist additional remainder operators  $r_k(\xi) \in B(H; H^*)^{(1)}$  such that

$$\begin{aligned} (3.21) \quad U &= \left( \text{Id} - iP(v \cdot \xi)R_0 + iR_0(v \cdot \xi)P + r_4(\xi) \right) \left( \text{Id} + r_5(\xi) \right) \\ &= \text{Id} - iP(v \cdot \xi)R_0 + iR_0(v \cdot \xi)P + r_6(\xi), \end{aligned}$$

where  $r_k(\xi) \in B(H; H^*) \cdot |\xi|^k$  for  $k = 1, \dots, 6$ . Note that there holds as well

$$U^{-1} = \text{Id} - iR_0(v \cdot \xi)P + iP(v \cdot \xi)R_0 + r_7(\xi),$$

with  $r_7(\xi) \in B(H; H^*) \cdot |\xi|^k$ . Notice that in the above decompositions of  $U$  and  $U^{-1}$ , all terms (except Id) are in  $B(H; H^*)$ . We thus compute the second order expansion

$$L = PLP = P(U^{-1}L U)P$$

as follows. Observe that  $PR_0 = R_0P = 0$  so that

$$(3.22) \quad PU^{-1} = P + iP(v \cdot \xi)R_0 + Pr_7(\xi), \quad UP = P + iR_0(v \cdot \xi)P + r_6(\xi)P,$$

and that  $\text{Range}(r_6(\xi)P) \subset D(L)$  so that  $r_8(\xi) := LR_6(\xi)P$  is well-defined and in the end we have  $r_8(\xi) \in B(H; H^*) \cdot |\xi|^2$ . Therefore, using also that  $LP = 0$  while  $LR_0 = \text{Id} - P$ , we deduce that

$$\begin{aligned} L &= \left( P + iP(v \cdot \xi)R_0 + Pr_7(\xi) \right) (L - iv \cdot \xi) \left( P + iR_0(v \cdot \xi)P + r_6(\xi)P \right) \\ &= \left( P + iP(v \cdot \xi)R_0 + Pr_7(\xi) \right) \left( -iP(v \cdot \xi)P + (v \cdot \xi)R_0(v \cdot \xi)P + r_8(\xi) + \tilde{r}_1(\xi) \right) \\ &= -iP(v \cdot \xi)P + P(v \cdot \xi)R_0(v \cdot \xi)P + \tilde{r}_2(\xi), \end{aligned}$$

where  $\tilde{r}_k(\xi) \in B(H; H^*) \cdot |\xi|^k$  for  $k = 1, 2$  because the second order remainder term  $Pr_8(\xi)$  vanishes since  $PL = 0$ . The lemma is proved.

*Remark 3.6.* — Note that in order to compute a second order expansion of the rectified operator, only a first order expansion of the spectral projector is needed.

<sup>(1)</sup>with for instance  $r_4(\xi) = r_1(\xi) + r_2(\xi)$  and  $r_5(\xi) = \sum_{n=1}^{\infty} \left(-\frac{1}{n}\right) (-r_3(\xi))^n$ .

LEMMA 3.7 (*Block matrix representation of the rectified operator*). — For any non-zero  $|\xi| \notin \alpha_0$ , the rectified operator  $L$  writes, along the  $H$ -orthogonal decomposition

$$\text{Ker}(L) = \left\{ u \cdot v\mu ; u \perp \xi \right\} \text{ Span } \left\{ \psi_{\text{Bou}}, \psi_{+\text{wave}}(\tilde{\xi}), \psi_{-\text{wave}}(\tilde{\xi}) \right\}, \quad \tilde{\xi} = \frac{\xi}{|\xi|},$$

as a block matrix:

$$(3.23) \quad L = \begin{pmatrix} \lambda_{\text{inc}}(\xi) \text{Id}_{(d-1) \times (d-1)} & 0_{(d-1) \times 3} \\ 0_{3 \times (d-1)} & M(\xi) \end{pmatrix}, \quad M(\xi) \in M_{3 \times 3}(\mathbb{R})$$

where the “incompressible” eigenvalue  $\lambda_{\text{inc}}(\xi)$  can be expressed for any normalized  $u \perp \xi$  as

$$\lambda_{\text{inc}}(\xi) = \frac{d}{E} \langle L(u \cdot v\mu), u \cdot v\mu \rangle_H.$$

*Proof.* — Fix some non-zero  $|\xi| \notin \alpha_0$  and consider some  $u \perp \xi$ , using the first point of Lemma 3.5, the functions  $L(u \cdot v\mu)$  and  $(\mu, \xi \cdot v\mu, |v|^2 \mu)$  are respectively odd and even in the direction  $u$ , and thus  $H$ -orthogonal. Similarly,  $L(\mu, \xi \cdot v\mu, |v|^2 \mu)$  and  $u \cdot v\mu$  are also  $H$ -orthogonal. Thus, one may write

$$L = \begin{pmatrix} M(\xi) & 0_{(d-1) \times 3} \\ 0_{3 \times (d-1)} & M(\xi) \end{pmatrix},$$

for some  $(d - 1) \times (d - 1)$  matrix  $M(\xi)$ , because  $\{(a + b\xi + c|v|^2)\mu \mid a, b, c \in \mathbb{R}\}$  is spanned by  $\psi_{\text{Bou}}, \psi_{-\text{wave}}(\tilde{\xi})$  and  $\psi_{+\text{wave}}(\tilde{\xi})$ .

In the case  $d > 3$ , we still have to show that  $M(\xi)$  is a multiplication matrix. To do so, consider a pair of  $H$ -orthogonal functions  $\varphi = u \cdot v\mu$  and  $\varphi = u \cdot v\mu$  for some vectors  $u, u \perp \mathbb{R}^d$  such that  $(u, u, \xi)$  is an orthogonal triple. From the first point of Lemma 3.5, the function  $L\varphi$  is odd in the direction  $u$  and even in the direction  $u$ , thus

$$\langle L(u \cdot v\mu), u \cdot v\mu \rangle_H = \langle L(u \cdot v\mu), u \cdot v\mu \rangle_H = 0.$$

Furthermore, choosing an orthogonal matrix mapping  $(\xi, u, u)$  onto  $(\xi, u, u)$ , we have

$$\langle L(u \cdot v\mu), u \cdot v\mu \rangle_H = \langle L(u \cdot v\mu), (u \cdot v\mu) \rangle_H = \langle L(u \cdot \mu), u \cdot v\mu \rangle_H.$$

We conclude by applying these two relations to any orthogonal basis of

$$\left\{ u \cdot v\mu \mid u \perp \xi \right\}.$$

This concludes the proof.

LEMMA 3.8 (*Expansion in matrix form*). — Recall that  $\kappa$  and  $c$  are defined in Theorem 1.8. With the notations of Lemma 3.7, the “incompressible” eigenvalue expands as

$$\lambda_{\text{inc}}(\xi) = -\kappa_{\text{inc}}/|\xi|^2 + O(|\xi|^{-3}).$$

Furthermore, in the basis  $\psi_{\text{Bou}}, \psi_{-\text{wave}}(\tilde{\xi}), \psi_{+\text{wave}}(\tilde{\xi})$ , the matrix  $M(\xi)$  can be written as

$$(3.24) \quad M(\xi) = \frac{i}{|\xi|} \begin{pmatrix} 0 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & -c \end{pmatrix} + \frac{\tilde{Z}}{|\xi|^2} \begin{pmatrix} -\kappa_{\text{Bou}} & & \\ & -\kappa_{\text{wave}} & \\ & & -\kappa_{\text{wave}} \end{pmatrix} + O(|\xi|^{-3}).$$

*Proof.* — Straightforward calculations show that, in the basis  $\psi_{\text{Bou}}, \psi_{-\text{wave}}(\tilde{\xi}), \psi_{+\text{wave}}(\tilde{\xi})$ , the first order coefficient is diagonal:

$$M(\xi) = -iP(v \cdot \xi) + O(|\xi|^{-2}) = \frac{i}{|\xi|} \begin{pmatrix} 0 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & -c \end{pmatrix} + O(|\xi|^{-2}).$$

We then turn to the second order coefficient  $(v \cdot \tilde{\xi})R_0(v \cdot \tilde{\xi})$ .

*Step 1: The diffusion coefficient  $\kappa_{\text{inc}}$ .* — In this step, we will use the following identity:

$$(3.25) \quad \langle L^{-1} \mathbf{A}u \cdot u, \mathbf{A}u \cdot u \rangle_H = \langle L^{-1} \mathbf{A}(u) \cdot (u), \mathbf{A}(u) \cdot (u) \rangle_H,$$

which holds for any  $u, u \in \mathbb{R}^d$  and any orthogonal matrix  $\mathbf{A}$ , and is a consequence of the identities (where we recall  $R_0 = -L^{-1}(\text{Id} - P)$ )

$$R_0 = \mathbf{A}R_0 \mathbf{A}^T, \quad \mathbf{A}(v)u \cdot u = \mathbf{A}(v)(u) \cdot (u).$$

The diffusion coefficient  $\kappa_{\text{inc}}$  is given for any  $\sigma \in S^{d-1}$  orthogonal to  $\tilde{\xi}$  by

$$\kappa_{\text{inc}} = \frac{d-1}{E} \langle R_0(v \cdot \tilde{\xi})(v \cdot \sigma)\mu, v \cdot \tilde{\xi}(v \cdot \sigma)\mu \rangle_H = \langle L^{-1} \mathbf{A}\tilde{\xi} \cdot \sigma, \mathbf{A}\tilde{\xi} \cdot \sigma \rangle_H,$$

which, for any orthogonal pair  $\omega, \sigma \in S^{d-1}$ , rewrites using (3.25) as

$$(3.26) \quad \kappa_{\text{inc}} = -\langle L^{-1} \mathbf{A}\omega \cdot \sigma, \mathbf{A}\omega \cdot \sigma \rangle_H.$$

Choosing in particular  $\omega = \frac{1}{2}(u - u)$  and  $\sigma = \frac{1}{2}(u + u)$ , where  $u, u \in S^{d-1}$  are orthogonal, we have

$$\kappa_{\text{inc}} = -\frac{1}{2} \langle L^{-1}(\mathbf{A}u \cdot u - \mathbf{A}u \cdot u), \mathbf{A}u \cdot u - \mathbf{A}u \cdot u \rangle_H.$$

where we used the fact that  $A$  is symmetric. Consequently, we have by (3.25)

$$\kappa_{\text{inc}} = -\langle L^{-1} \mathbf{A}u \cdot u, \mathbf{A}u \cdot u \rangle_H + \langle L^{-1} \mathbf{A}u \cdot u, \mathbf{A}u \cdot u \rangle_H,$$

and since  $\mathbf{A}$  is trace-free, one can get rid of the term involving  $u$  by averaging over some orthonormal family  $u = u_1, \dots, u_{d-1} \in \mathbb{R}^d$  :

$$\begin{aligned} \kappa_{\text{inc}} &= -\langle L^{-1} \mathbf{A}u \cdot u, \mathbf{A}u \cdot u \rangle_H + \frac{1}{d-1} \langle L^{-1} \mathbf{A}u \cdot u, \sum_{i=1}^{d-1} \mathbf{A}u_i \cdot u_i \rangle_H \\ &= -\langle L^{-1} \mathbf{A}u \cdot u, \mathbf{A}u \cdot u \rangle_H - \frac{1}{d-1} \langle L^{-1} \mathbf{A}u \cdot u, \mathbf{A}u \cdot u \rangle_H, \end{aligned}$$

and therefore

$$(3.27) \quad \kappa_{\text{inc}} = -\frac{d}{d-1} \langle L^{-1} \mathbf{A}u \cdot u, \mathbf{A}u \cdot u \rangle_H.$$

Summing (3.26) and (3.27) over all pairs of vectors in the canonical basis of  $\mathbb{R}^d$ , we rewrite the coefficient  $\kappa_{\text{inc}}$  as a Hilbert–Schmidt norm of matrices:

$$\kappa_{\text{inc}} = -\frac{1}{(d-1)(d+1)} \langle L^{-1} \mathbf{A}, \mathbf{A} \rangle_H.$$

*Step 2: The diffusion coefficient  $\kappa_{\text{Bou}}$ .* — The coefficient  $\kappa_{\text{Bou}}$  writes

$$\begin{aligned} \kappa_{\text{Bou}} &= -\mathbb{R}_0 \int_{\mathbb{S}^d} v \cdot \tilde{\xi} \psi_{\text{Bou}}, v \cdot \tilde{\xi} \psi_{\text{Bou}} \, d\tilde{\xi} \\ &= -\int_{\mathbb{S}^d} L^{-1} \mathbf{B} \cdot \tilde{\xi}, \mathbf{B} \cdot \tilde{\xi} \, d\tilde{\xi} = -\frac{1}{d} \langle L^{-1} \mathbf{B}, \mathbf{B} \rangle_H, \end{aligned}$$

where we used the invariance of  $L^{-1}$  again, as well as the identity  $\mathbf{B}(v) \cdot u = \mathbf{B}(-v) \cdot (-u)$ , allowing to sum over  $\tilde{\xi}$  taken in the canonical basis of  $\mathbb{R}^d$ .

*Step 3: The diffusion coefficient  $\kappa_{\text{wave}}$ .* — The coefficient  $\kappa_{\text{wave}}$  writes

$$\begin{aligned} \kappa_{\text{wave}} &= -\mathbb{R}_0 \int_{\mathbb{S}^d} v \cdot \tilde{\xi} \psi_{\pm\text{wave}}, v \cdot \tilde{\xi} \psi_{\pm\text{wave}} \, d\tilde{\xi} \\ &= -\frac{1}{2} \int_{\mathbb{S}^d} L^{-1} \mathbf{A} \tilde{\xi} \cdot \tilde{\xi}, \mathbf{A} \tilde{\xi} \cdot \tilde{\xi} \, d\tilde{\xi} - \frac{E^2(K-1)}{2} \int_{\mathbb{S}^d} L^{-1} \mathbf{B} \cdot \tilde{\xi}, \mathbf{B} \cdot \tilde{\xi} \, d\tilde{\xi}, \end{aligned}$$

where we have used the fact that  $\mathbf{A} \tilde{\xi} \cdot \tilde{\xi}$ , and thus  $L^{-1} \mathbf{A} \tilde{\xi} \cdot \tilde{\xi}$  again by the invariance of  $L^{-1}$ , is even in the direction  $\tilde{\xi}$ , whereas  $\mathbf{B} \cdot \tilde{\xi}$  is odd in this direction, and therefore these functions are  $H$ -orthogonal. Using the results of the previous steps, we actually have

$$\kappa_{\text{wave}} = \frac{d-1}{2d} \kappa_{\text{inc}} + \frac{E^2(K-1)}{2} \kappa_{\text{Bou}}.$$

The lemma is proved.

**LEMMA 3.9 (Second order diagonalization and decomposition of the rectified operator).** — *With the notations of Lemma 3.7, the rectified operator  $L$  has four distinct eigenvalues which expand as*

$$\begin{aligned} \lambda_{\text{inc}}(\xi) &= -\kappa_{\text{inc}}/|\xi|^2 + O(|\xi|^{-3}), \\ \lambda_{\text{Bou}}(\xi) &= -\kappa_{\text{Bou}}/|\xi|^2 + O(|\xi|^{-3}), \quad \lambda_{\pm\text{wave}}(\xi) = \pm ic/|\xi| - \kappa_{\text{wave}}/|\xi|^2 + O(|\xi|^{-3}). \end{aligned}$$

Furthermore, the “incompressible” eigenvalue is associated with the following spectral projector:

$$P_{\text{inc}}(\xi) [a\mu + u \cdot v\mu + c/v^2\mu] = \int_{\mathbb{S}^d} \tilde{u} \cdot v \mu,$$

and the spectral projectors associated with the “Boussinesq” and “waves” eigenvalues expand in the basis  $\psi_{\text{Bou}}, \psi_{-\text{wave}}, \psi_{+\text{wave}}$  as

$$(3.28) \quad P_{\star}(\xi) = P^{(0)} + |\xi|^{-1} P^{(1)} + O(|\xi|^{-2}), \quad \star = \text{Bou}, \pm\text{wave}$$

where

$$(3.29) \quad P_{\text{Bou}}^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_{-\text{wave}}^{(0)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_{+\text{wave}}^{(0)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and  $P^{(1)}$  is a constant  $3 \times 3$ -matrix for  $\star = \text{Bou}, \pm\text{wave}$ .

*Proof.* — Because the operator  $M(\xi)$  has the asymptotic expansion

$$M(\xi) = |\xi| \left( M^{(0)} \frac{\epsilon \tilde{\xi}}{\xi} + |\xi| M^{(1)} \frac{\epsilon \tilde{\xi}}{\xi} + O(|\xi|^2) \right)$$

where  $M^{(0)} \frac{\epsilon \tilde{\xi}}{\xi}$  has three distinct eigenvalues by Lemma 3.8, we know from [Kat66, Chapter 2, Theorem 5.4] that  $M(\xi)$  has three distinct eigenvalues for  $\xi$  small, and thus admits the following spectral decomposition:

$$M(\xi) = \lambda_{\text{Bou}}(\xi) P_{\text{Bou}}(\xi) + \lambda_{+\text{wave}}(\xi) P_{+\text{wave}}(\xi) + \lambda_{-\text{wave}}(\xi) P_{-\text{wave}}(\xi).$$

The expansions of the eigenvalues is given by [Kat66, Chapter II-(5.12)] applied to (3.24):

$$\begin{aligned} \lambda_{\text{inc}}(\xi) &= -\kappa_{\text{inc}}/|\xi|^2 + O(|\xi|^\beta), & \lambda_{\text{Bou}}(\xi) &= -\kappa_{\text{Bou}}/|\xi|^2 + O(|\xi|^\beta), \\ \lambda_{\pm\text{wave}}(\xi) &= \pm ic/|\xi| - \kappa_{\text{wave}}/|\xi|^2 + O(|\xi|^\beta), \end{aligned}$$

and the expansions of the spectral projectors are given by [Kat66, Chapter II-(5.9)] applied to (3.24) yielding (3.28) and (3.29) and we point out that the first order coefficients  $P^{(1)}$  in matrix form has coefficients independent of  $\tilde{\xi}$  and can be explicitly computed, although their expression will not be needed.

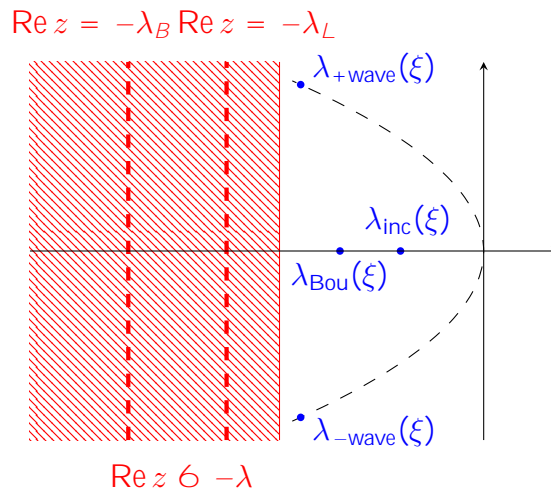


Figure 3.2. Localization of the spectrum of  $L - i(v \cdot \xi)$  for  $|\xi| \leq \alpha_0$ .

LEMMA 3.10 (**Expansion of the spectral projectors**). — For  $\alpha_0 > 0$  small enough, the following spectral decomposition holds for any  $|\xi| \leq \alpha_0$ :

$$\begin{aligned} L P(\xi) = P(\xi) L &= \lambda_{\text{Bou}}(\xi) P_{\text{Bou}}(\xi) + \lambda_{\text{inc}}(\xi) P_{\text{inc}}(\xi) \\ &+ \lambda_{+\text{wave}}(\xi) P_{+\text{wave}}(\xi) + \lambda_{-\text{wave}}(\xi) P_{-\text{wave}}(\xi) \end{aligned}$$

where the projector operators  $P(\xi)$  ( $\star = \text{Bou}, \text{inc}, \pm\text{wave}$ ) expand in  $B(H; H^*)$  as (1.13) with the zeroth order coefficients being defined in Theorem 1.8.

*Proof.* — Recall that  $L$  and  $\tilde{L}$  are related through  $\tilde{L} = (U^{-1}L U)_{|\text{Ker}(L)}$ , thus, using the fact that  $U P = P(\xi)U$ , we deduce that

$$P(\xi)L = L P(\xi) = U L U^{-1}.$$

This lemma is therefore a lifted version of Lemma 3.9, and the corresponding projectors are related through

$$P(\xi) = U P(\xi)U^{-1} = (U P)P(\xi)(P U^{-1}).$$

We then deduce the expansion of each  $P(\xi)$  from those of  $P(\xi)$  in  $B(\text{Ker}(L))$  established in Lemma 3.9 and those of  $U P$  and  $P U^{-1}$  in  $B(H; H^*)$  from (3.22):

$$P(\xi) = \left( P + i\xi \cdot R_0 v P + P r_7(\xi) \right) \left( P^{(0)} \frac{\epsilon_{\tilde{S}}}{\xi} + |\xi| P^{(1)} \frac{\epsilon_{\tilde{S}}}{\xi} + S(\xi) \right) \left( P + i\xi \cdot P v R_0 + r_6(\xi) P \right)$$

where we recall that  $r_j(\xi) \in B(H; H^*) \cdot |\xi|^j$  while  $S(\xi) \in M_{(d+2) \times (d+2)}(\mathbb{R})$  with norm of order  $O(|\xi|^2)$ . We can expand further to deduce

$$P(\xi) =: P^{(0)} \frac{\epsilon_{\tilde{S}}}{\xi} + i\xi \cdot P^{(1)} \frac{\epsilon_{\tilde{S}}}{\xi} + S(\xi),$$

where we have denoted <sup>(2)</sup>

$$P^{(0)} \frac{\epsilon_{\tilde{S}}}{\xi} := P P^{(0)} \frac{\epsilon_{\tilde{S}}}{\xi} P, \quad P^{(1)} \frac{\epsilon_{\tilde{S}}}{\xi} := P^{(0)}(\tilde{\xi}) v R_0 + R_0 v P^{(0)} \frac{\epsilon_{\tilde{S}}}{\xi} - i\tilde{\xi} P P^{(1)} \frac{\epsilon_{\tilde{S}}}{\xi} P.$$

We notice that both  $R_0 v P^{(0)}(\tilde{\xi})$  and  $P P^{(1)}(\tilde{\xi}) P$  vanish on  $\text{Ker}(L)$ :

$$\varphi \in \text{Ker}(L) \quad P^{(1)} \frac{\epsilon_{\tilde{S}}}{\xi} \varphi = P^{(0)} \frac{\epsilon_{\tilde{S}}}{\xi} v R_0 \varphi = P^{(0)} \frac{\epsilon_{\tilde{S}}}{\xi} v \varphi.$$

Therefore, the first order term of the projector associated with the “incompressible” eigenvalue writes explicitly for any  $\varphi \in \text{Ker}(L)$ , and  $\omega \in S^{d-1}$

$$\begin{aligned} P_{\text{inc}}^{(1)}(\omega)\varphi &= \frac{d}{E} \left( \left[ (\text{Id} - \omega \otimes \omega) v_j R_0 \varphi, v \mu_H \right] \cdot v \mu \right)_{j=1}^d \\ &= \frac{d}{E} \left( v_j R_0 \varphi, v \mu_H \cdot \left[ (\text{Id} - \omega \otimes \omega) v \right] \mu \right)_{j=1}^d \\ &= \frac{d}{E} \langle \varphi, L^{-1} \mathbf{A} \rangle_H \left[ (\text{Id} - \omega \otimes \omega) v \right] \mu, \end{aligned}$$

and, in particular,

$$\begin{aligned} \omega \cdot P_{\text{inc}}^{(1)}(\omega)\varphi &= \frac{d}{E} \left[ (\text{Id} - \omega \otimes \omega) v \cdot \omega R_0 \varphi, v \mu_H \right] \cdot v \mu \\ &= \frac{d}{E} \left[ (\text{Id} - \omega \otimes \omega) \langle \varphi, L^{-1} \mathbf{A} \omega \rangle_H \right] \cdot v \mu, \end{aligned}$$

the one associated with the “Boussinesq” eigenvalue writes for any  $\varphi \in \text{Ker}(L)$

$$P_{\text{Bou}}^{(1)}(\omega)\varphi = R_0 \varphi, v \psi_{\text{Bou}} \in H \psi_{\text{Bou}} = \langle \varphi, L^{-1} \mathbf{B} \rangle_H \psi_{\text{Bou}},$$

and the ones associated with the “waves” eigenvalues write, for  $\varphi \in \text{Ker}(L)$ ,

<sup>(2)</sup> Of course,  $PP^{(0)}(\cdot)P$  can be identified with  $P^{(0)}(\cdot)$  but we make here the (slight) distinction between operators defined on the finite dimensional space and the associated matrices.

$$\begin{aligned}
 P_{\pm\text{wave}}^{(1)}(\omega)\varphi &= vR_0\varphi, \psi_{\pm\text{wave}}(\omega) \int_H \psi_{\pm\text{wave}}(\omega) \left( \pm \frac{1}{2} \langle \varphi, L^{-1}A(v)\omega \rangle_H + E \frac{K-1}{2} \langle \varphi, L^{-1}B(v) \rangle_H \right) \psi_{\pm\text{wave}}(\omega).
 \end{aligned}$$

This concludes the proof of Lemma 3.10.

We recall now the following *hypocoercivity* result extracted from [Dua11]:

LEMMA 3.11 (**Hypocoercivity** [Dua11, Lemma 4.1]). — Assume that  $L : D(L) \rightarrow H$  satisfies Assumptions (L1)–(L4). Then, for any  $\xi \in \mathbb{R}^d$ , there exists some  $\xi$ -dependent bilinear symmetric form  $[\cdot, \cdot] : H \times H \rightarrow \mathbb{R}^d$  defined through

$$[f, f] = \frac{\xi}{1 + |\xi|^2} \cdot \left\langle Pf, T_1Pf + T_2Pv(\text{Id} - P)f \right\rangle_H,$$

where  $T_1 \in B(\text{Ker}(L); \mathbb{R}^d)$  and  $T_2 \in B((\text{Ker}(L))^d; \mathbb{R}^d)$ , such that there holds for some  $c > 0$

$$(3.30) \quad [L f, f] \leq -\frac{c|\xi|^2}{1 + |\xi|^2} \|Pf\|_H^2 + \frac{1}{c} \|(\text{Id} - P)f\|_H^2,$$

uniformly in  $\xi \in \mathbb{R}^d$  and  $f \in D(L)$ .

With this at hands, we may turn to the proof of the decay estimates of Theorem 1.8.

LEMMA 3.12 (**Resolvent bounds and decay estimates of the semigroup**). With the notations of Lemma 3.3, let  $0 < \lambda < \lambda_L$ . There exist some constants  $C, \gamma > 0$  such that, for any  $|\xi| > \alpha_0$ , the spectrum is localized as follows:

$$\sigma_H(L) \subset \{z \in \mathbb{C} : \text{Re}(z) \leq -\gamma\},$$

and the resolvent satisfies

$$\sup_{z \in \mathbb{C}, \text{Re}(z) = -\gamma} \|R(z, L)\|_{B(H)} \leq C.$$

Furthermore, for any  $0 < \sigma < \sigma_0 := \min\{\lambda, \gamma\}$ , the decay estimates

$$\sup_{t>0} e^{2\sigma t} \|U(t)(\text{Id} - P(\xi))f\|_H^2 + \int_0^t e^{2\sigma s} \|U(s)(\text{Id} - P(\xi))f\|_H^2 ds \leq C \|(\text{Id} - P(\xi))f\|_H^2$$

and

$$\int_0^t e^{2\sigma s} \|U(s)(\text{Id} - P(\xi))f\|_H^2 ds \leq C \|(\text{Id} - P(\xi))f\|_H^2$$

hold uniformly in  $\xi \in \mathbb{R}^d$  and  $f \in H$ , where  $(U(t))_{t>0}$  denotes the  $C^0$ -semigroup in  $H$  generated by  $(L, D(L))$ .

*Proof.* — In a first step, we prove resolvent bounds using the above hypocoercivity result as well as the uniform decay estimate in  $H$ . In a second step, we prove the  $H^* - S$  integral decay estimate using an energy method, from which we deduce the  $H - H$  one in a third step. Let us fix  $0 < \lambda < \lambda_L$ .

*Step 1: Resolvent bounds and uniform decay estimate.* — Using the above Lemma 3.11, we define, for any  $|\xi| > \alpha_0$  the equivalent inner product on  $H$

$$(\cdot, \cdot)_{H, \xi} := (\cdot, \cdot)_H + \eta \|\cdot\|_{H, \xi}^2$$

with some small  $\eta > 0$ . By combining the control of  $(\text{Id} - \text{P})f$  from **(L3)** and the control of  $\text{P}f$  from (3.30), we have for any  $|\xi| > \alpha_0$

$$\begin{aligned} (L f, f)_{H, \xi} &\leq \left(\frac{\eta}{c} - \lambda_L\right) \|(\text{Id} - \text{P})f\|_{H, \xi}^2 - \frac{c|\xi|^2}{1 + |\xi|^2} \|\text{P}f\|_{H, \xi}^2 \\ &\leq \min\left\{\frac{\lambda_L}{2}, \frac{c\alpha_0^2}{1 + \alpha_0^2}\right\} \|f\|_{H, \xi}^2, \end{aligned}$$

where we chose  $\eta \leq \frac{c}{2}\lambda_L$ . Assuming  $\eta$  small enough so that the norm induced by  $(\cdot, \cdot)_{H, \xi}$  is equivalent to  $\|\cdot\|_{H, \xi}$  uniformly in  $\xi \in \mathbb{R}^d$ , we have for some  $\gamma > 0$

$$(L f, f)_{H, \xi} \leq -\gamma \|f\|_{H, \xi}^2, \quad |\xi| > \alpha_0, f \in D(L).$$

We thus deduce that for  $|\xi| > \alpha_0$

$$U(t) \Big|_{B(H)} \cdot e^{-\gamma t},$$

as well as (up to a reduction of  $\gamma$  for the resolvent bound)

$$\mathfrak{G}(L) \Big|_{B(H)} = \gamma, \quad \sup_{z \in \mathbb{C}^-} \text{Re}(R(z, L) \Big|_{B(H)}) \leq 1.$$

On the other hand, for  $|\xi| \leq \alpha_0$ , the resolvent of  $(\text{Id} - \text{P}(\xi))L$  is given for  $z \in \mathbb{C}^-$  by

$$R(z, L(\text{Id} - \text{P}(\xi))) = R(z, L) - \sum_{\text{inc, Bou, } \pm \text{wave}} (z - \lambda(\xi))^{-1} \text{P}(\xi)$$

which is therefore holomorphic in  $z \in \mathbb{C}^-$  and thus can be explicitly bounded using the bound (3.5) and the maximum principle. We deduce that the semigroup it generates is bounded by the Gearhart–Pruss theorem [EN00, Theorem V.1.10], i.e.

$$U(t) (\text{Id} - \text{P}(\xi)) f \Big|_{B(H)} \cdot e^{-\gamma t} \|f\|_H, \quad |\xi| \leq \alpha_0, f \in H.$$

To sum up, putting together both decay estimates and denoting  $\sigma_0 := \min\{\lambda, \gamma\}$ , there holds

$$\sup_{\mathbb{R}^d} U(t) (\text{Id} - \text{P}(\xi)) f \Big|_{B(H)} \cdot e^{-\sigma_0 t} \|f\|_H, \quad f \in H,$$

where we recall that  $\text{P}(\xi) = 0$  for  $|\xi| > \alpha_0$ . This concludes this step.

*Step 2: Integral decay estimates.* — Let us prove both integral decay estimates.

*Step 2a: The  $H^* - H$ -integral decay estimate.* — Let  $f \in \text{Range}(\text{Id} - \text{P}(\xi)) \cap D(L)$  and denote  $f(t) = U_\xi(t)f$  the unique solution to

$$\begin{aligned} \partial_t f(t) &= L f(t) = L (\text{Id} - \text{P}(\xi)) f(t), \\ f(0) &= f. \end{aligned}$$

Using that  $L$  is self-adjoint in  $H$  and that the multiplication by  $iv \cdot \xi$  is skew-adjoint, we have the energy estimate

$$\frac{d}{dt} \|f(t)\|_H^2 = 2 \text{Re} \langle L f(t), f(t) \rangle_H = 2 \langle L f(t), f(t) \rangle_H.$$



Furthermore, using the dissipativity estimate of  $L$  from **(L3)**, we get

$$\frac{d}{dt} \|f(t)\|_H^2 + 2\lambda_L \|(\text{Id} - P)f(t)\|_{H^*}^2 \leq 0,$$

which we complete using (3.15) and  $P \in \mathcal{B}(H; H^*)$  as

$$(3.31) \quad \frac{d}{dt} \|f(t)\|_H^2 + \lambda_L \|f(t)\|_{H^*}^2 \leq \|Pf(t)\|_{H^*}^2 \leq \|f(t)\|_H^2 \cdot e^{-2\lambda_L t} \|f\|_H^2,$$

where we also used the decay estimate established in the previous step. Multiplying this by  $e^{2\lambda_L t}$  and integrating, one easily deduces

$$\sup_{t>0} e^{2\lambda_L t} \|f(t)\|_H^2 + \lambda_L \int_0^t e^{2\lambda_L s} \|f(s)\|_{H^*}^2 ds \leq \|f\|_H^2.$$

This identity holds for any  $f \in (\text{Id} - P(\xi)) D(L)$  and we conclude the proof using that  $D(L)$  is dense in  $H$ .

*Step 2b: The  $H - H^*$ -integral decay estimate.* — As before, we assume now that  $f \in (\text{Id} - P(\xi)) H$  and set  $f(t) = U(t)f$  for any  $t > 0$ . We use a duality argument together with a density argument and prove that

$$\langle e^{-\lambda_L t} f(t), \phi \rangle_{L_t^2(H)} \leq \|f\|_H \|\phi\|_{L_t^2(H)}.$$

To perform the duality argument, we need to point out that  $L$  is self-adjoint, thus

$$(U(t)(\text{Id} - P(\xi)))^* = U_{-}(t)(\text{Id} - P(-\xi)).$$

Since the step functions span a dense subspace of  $L_t^2(H)$ , it is enough to check that the dual estimate holds for such a following function:

$$\phi(t) = \begin{cases} \phi_0 & t \in [t_1, t_2], \\ 0 & t \notin [t_1, t_2], \end{cases} \quad \|\phi\|_{L_t^2(H)} = \sqrt{t_2 - t_1} \|\phi_0\|_H.$$

For such a step function, the inner product writes explicitly as

$$\langle e^{-\lambda_L t} f(t), \phi \rangle_{L_t^2(H)} = \int_0^{t_2} \langle e^{-\lambda_L t} f(t), \phi(t) \rangle_H dt = \int_{t_1}^{t_2} \langle e^{-\lambda_L t} U(t)(\text{Id} - P(\xi)) f, \phi_0 \rangle_H dt.$$

We then get by duality and using Cauchy-Schwarz's inequality

$$\begin{aligned} \langle e^{-\lambda_L t} f(t), \phi \rangle_{L_t^2(H)} &= \int_{t_1}^{t_2} \langle f, e^{\lambda_L t} U_{-}(t)(\text{Id} - P(-\xi)) \phi_0 \rangle_H dt \\ &\leq \|f\|_H \int_{t_1}^{t_2} e^{\lambda_L t} \|U_{-}(t)(\text{Id} - P(-\xi)) \phi_0\|_H dt \\ &\leq \|f\|_H \sqrt{t_2 - t_1} \int_0^{t_2} e^{\lambda_L t} \|U_{-}(t)(\text{Id} - P(-\xi)) \phi_0\|_H^2 dt < \frac{1}{2} \end{aligned}$$

thus, using the  $H^* - H$  estimate from Step a

$$\langle e^{-\lambda_L t} f(t), \phi \rangle_{L_t^2(H)} \leq \|f\|_H \sqrt{t_2 - t_1} \|\phi_0\|_H \leq \|f\|_H \|\phi\|_{L_t^2(H)}.$$

This concludes the proof of Lemma 3.12.

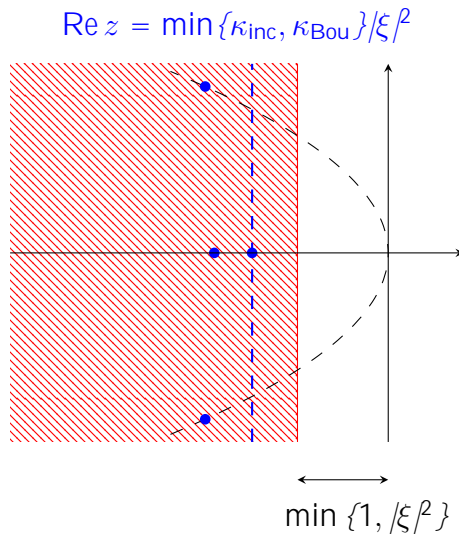


Figure 3.3. Localization of the spectrum of  $L$  provided by the hypocoercivity Lemma 3.11 for  $\xi \in \mathbb{R}^d$ , compared with the localization of the hydrodynamic eigenvalues defined for  $|\xi| \leq \alpha_0$ .

### 3.4. Proof of Theorem 1.19

We present here the full proof of the “enlarged” version of Theorem 1.8 as provided in Theorem 1.19. We present in a first step how to extend the resolvent bounds, in a second step how to extend the decay estimate. In a third step, we extend the projector bounds from Lemma 3.4 as this is enough to deduce the same bounds on the expansion of  $U$  and in turn of  $\mathcal{P}(\xi)$ .

*Proof.* —

*Step 1: Resolvent bounds.* — Using the factorization formulae

$$\begin{aligned}
 (3.32) \quad R(z, L) &= R(z, B^{(0)}) \mathcal{S} + R(z, L) A^{(0)} R(z, B^{(0)}) \\
 &= R(z, B^{(0)}) \mathcal{S} + R(z, B^{(0)}) A^{(0)} R(z, L)
 \end{aligned}$$

and the fact that the function  $z \mapsto R(z, B^{(0)}) \mathcal{S} \in B(H) \rightarrow B(X)$  is holomorphic, as well as  $H \subset X$  and  $A^{(0)} \in B(X; H)$ , we deduce that, for any  $z \in \mathbb{C} \setminus \mathfrak{S}(L)$

$$R(z, L) \in B(X) \iff R(z, L) \in B(H),$$

or in other words, the spectrum of  $L$  in  $\mathbb{C} \setminus \mathfrak{S}(L)$  does not depend on the space  $H$  or  $X$ :

$$\mathfrak{S}_H(L) \cap \mathbb{C} \setminus \mathfrak{S}(L) = \mathfrak{S}_X(L) \cap \mathbb{C} \setminus \mathfrak{S}(L).$$

More precisely, since  $R(z, B^{(0)}) \mathcal{S} \in B(H) \rightarrow B(X)$  uniformly in  $\xi \in \mathbb{R}^d$  and  $z \in \mathbb{C} \setminus \mathfrak{S}(L)$ , there holds for any  $z \in \mathbb{C} \setminus \mathfrak{S}(L)$

$$1 + R(z, L) \in B(H) \iff 1 + R(z, L) \in B(X) \iff 1 + R(z, L) \in B(H).$$

This implies that the bounds in  $B(H)$  on the resolvent from Theorem 1.8 also hold in  $B(X)$ . Actually, also the bounds (3.5) in Lemma 3.3 can be refined for  $|\xi| \leq \alpha_0$  small enough as

$$(3.33) \sup_{|z|=r} R(z, L)_{B(X; X^*)} + \sup_{|z|=r} R(z, L)_{B(X; X)} + \sup_z R(z, L)_{B(X)} \leq C_1,$$

where  $r$  is small enough and  $\cdot := \cdot_{\{|z| > r\}}$ . Indeed, starting from the dissipativity estimate involving  $X^*$  and  $A^{(0)}$  in  $B(X)$  from **(LE)**:

$$\begin{aligned} \operatorname{Re} (L - z)f, f_X &= \operatorname{Re} B^{(0)}f, f_X + \operatorname{Re} A^{(0)}f, f_X - z f^2_X \\ &\leq -\lambda_B f^2_{X^*} + \left( \|A^{(0)}\|_{B(X)} - z \right) f^2_X \end{aligned}$$

which gives for some  $z_0 > \|A^{(0)}\|_{B(X)}$

$$R(z_0, L)_{B(X; X^*)} \leq \lambda_B^{-1}.$$

Performing the same computations with the decomposition

$$L = B^{(0)} + iw \cdot \xi + A^{(0)},$$

where  $B^{(0)}$  satisfies the same dissipativity as  $B^{(0)}$ , we get  $R(z_0, L)_{B(X; X^*)} \leq \lambda_B^{-1}$  and thus by the adjoint identity (B.2)

$$R(z_0, L)_{B(X; X)} \leq \lambda_B^{-1}.$$

Using the resolvent identity as in the proof of Lemma 3.3, we deduce (3.33).

*Step 2: Decay estimates.* — We first prove the uniform estimates, and then the integral ones.

*Step 2a: The uniform decay estimate.* — To improve the decay estimate of (1.20a) to (1.26a), we apply the Duhamel formula to the decomposition  $L = B^{(0)} + A^{(0)}$

$$U(t) = V^{(0)}(t) + \int_0^t U(t - \tau)A^{(0)}V^{(0)}(\tau)d\tau,$$

where  $(V^{(0)}(t))_{t>0}$  denotes the  $C^0$ -semigroup in  $X$  generated by  $(B^{(0)}, D(L))$ . After composing with  $\operatorname{Id} - P(\xi)$  from the left, we get

$$(\operatorname{Id} - P(\xi))U(t) = (\operatorname{Id} - P(\xi))V^{(0)}(t) + \int_0^t (\operatorname{Id} - P(\xi))U(t - \tau)A^{(0)}V^{(0)}(\tau)d\tau.$$

Since  $\|P(\xi)\|_{B(X)} \leq 1$  from (3.33), and using  $A^{(0)} \in B(X; H)$ , we have

$$\begin{aligned} \|(\operatorname{Id} - P(\xi))U(t)\|_{B(X)} &\leq \|V^{(0)}(t)\|_{B(X)} \\ &\quad + \int_0^t \|(\operatorname{Id} - P(\xi))U(t - \tau)\|_{B(H)} \|V^{(0)}(\tau)\|_{B(X)} d\tau, \end{aligned}$$

and using the decay estimate of  $(\operatorname{Id} - P(\xi))U(t)$  in  $B(H)$  from Theorem 1.8, as well as the dissipativity hypothesis for  $B^{(0)}$  from **(LE)**, we then get (recall that  $\sigma_0 \leq \lambda$ )

$$\|(\operatorname{Id} - P(\xi))U(t)\|_{B(X)} \leq e^{-\lambda t} + \int_0^t e^{-\sigma_0(t-\tau)} e^{-\lambda \tau} d\tau \leq e^{-\sigma_0 t}.$$

This proves the uniform in time decay.

*Step 2b: The integral  $X^* - X$  and  $X - X$  decay estimates.* — From then on, the proof of the integral decay estimate follows the same strategy as the one adopted for the proof of Lemma 3.12, starting from the decomposition  $L = B^{(0)} + A^{(0)}$  and, resuming the computations of Lemma 3.12. Typically, estimate (3.31) can be adapted to give now

$$\frac{d}{dt} \|f(t)\|_X^2 + 2\lambda_B \|f(t)\|_{X^*}^2 \leq \|A^{(0)}\|_{B(X)} \|f(t)\|_X^2 + \|A^{(0)}\|_{B(X)}^2 e^{-2\lambda_B t} \|f\|_X^2.$$

After integration, one obtains easily (1.26a) as in Lemma 3.12, as well as the corresponding estimate for  $(U(t)(\text{Id} - P(\xi)))$ , from which we deduce (1.26b) by a similar duality argument.

*Step 3: Expansion of the projectors.* — To establish the uniform bounds in  $B(X; H^*)$  on the expansion of the spectral projectors

$$P(\xi) = P^{(0)}(\xi) + i\xi \cdot P^{(1)}(\xi) + S(\xi),$$

we use a similar bootstrap strategy as in the proofs of Lemma 3.4 and Theorem 3.14. More precisely, in each step, we will prove uniform bounds in  $B(X; X)$  and then combine with those in  $B(H; H^*)$  from Lemma 3.4 to conclude. Similarly, we will need the following regularization properties for  $P(\xi)$ :

$$(3.34) \quad P(\xi)_{B(X; X_2)} + P(\xi)_{B(X; H^*)} \leq 1,$$

which, as in the original proof, comes from combining the identity  $P(\xi)^2 = P(\xi)$  with the resolvent bound (3.33) (for the  $X - X$  bound), with Lemma 3.4 (for the  $H - H^*$  bound), and with the regularization hypothesis (LEd) for  $A^{(0)}$  (for the  $X - X_2$  and  $X - H$  bounds) applied to the representation

$$P(\xi) = \frac{1}{2i\pi} \oint_{|z|=r} R(z, B^{(0)}) \tilde{S} A^{(0)} \tilde{S}^2 R(z, L) dz.$$

Similarly, we will need the regularization properties

$$(3.35) \quad P(\xi)_{B(X; X_j)} \leq 1, \quad j = 1, 2,$$

which comes from the regularization hypothesis (LEd) for  $A^{(0)}$  applied to the representation

$$P(\xi) = \frac{1}{2i\pi} \oint_{|z|=r} R(z, B^{(0)}) \tilde{S} A^{(0)} \tilde{S}^{-j} (R(z, L)) dz.$$

Finally, we will need the resolvent bounds

$$(3.36) \quad R(z, L)_{B(X_j)} \leq 1 + \frac{1}{|z|}, \quad j = 0, 1, 2, \quad z \in \mathbb{C} \setminus \{0\},$$

which also come from a factorization strategy as in the original proof.

*Step 3a: First order expansion of  $P(\xi)$ .* — We use the bootstrap formula (B.6):

$$P^{(1)}(\xi) = P(\xi)P^{(1)}(\xi) + P^{(1)}(\xi)P,$$

and the adjoint identity  $T_{B(X; X)} = T_{B(X; X^*)}$  from (B.2) to establish the bound

$$\begin{aligned} \|\mathbf{P}^{(1)}(\xi)\|_{B(X; X)} &\leq \|\mathbf{P}^{(1)}(\xi) \mathbf{P}(\xi)\|_{B(X; X^*)} + \|\mathbf{P}^{(1)}(\xi) \mathbf{P}\|_{B(X; X)} \\ &\leq \|\mathbf{P}^{(1)}(\xi)\|_{B(X_1; X^*)} \|\mathbf{P}(\xi)\|_{B(X; X_1)} + \|\mathbf{P}^{(1)}(\xi)\|_{B(X_1; X)} \|\mathbf{P}\|_{B(X; X_1)} \\ &\quad + \|\mathbf{P}^{(1)}(\xi)\|_{B(X_1; X^*)} + \|\mathbf{P}^{(1)}(\xi)\|_{B(X_1; X)} \end{aligned}$$

where we used the regularization property (3.34) for  $\mathbf{P} = \mathbf{P}(0)$  and (3.35) for  $\mathbf{P}(\xi)$  in the last estimate. We now turn to the first term  $\|\mathbf{P}^{(1)}(\xi)\|_{B(X_1; X^*)}$ :

$$\begin{aligned} \|\mathbf{P}^{(1)}(\xi)\|_{B(X_1; X^*)} &\leq \frac{1}{2\pi} \oint_{|z|=r} \left\| \left( R(z, L) v R(z, L) \right) \right\|_{B(X_1; X^*)} d|z| \\ &\leq \frac{1}{2\pi} \oint_{|z|=r} \|R(z, L) v R(z, L)\|_{B(X_1; X^*)} d|z| \\ &\quad + \oint_{|z|=r} \|R(z, L)\|_{B(X; X^*)} \|R(z, L)\|_{B(X_1)} d|z|, \end{aligned}$$

where we used the fact that the adjoint of the multiplication by  $v$  is in  $B(X_1; X)$  according to (LEb). We can rewrite this estimate without the adjoints and estimate it using the resolvent bounds (3.33) and (3.36):

$$\|\mathbf{P}^{(1)}(\xi)\|_{B(X_1; X^*)} \leq \oint_{|z|=r} \|R(z, L)\|_{B(X; X)} \|R(z, L)\|_{B(X_1)} d|z| \leq 1.$$

The second term  $\|\mathbf{P}^{(1)}(\xi)\|_{B(X_1; X)} \leq 1$  is estimated in the same way, thus we obtain

$$\|\mathbf{P}^{(1)}(\xi)\|_{B(X; X)} \leq 1.$$

Integrating the formula (3.32) and combining with the  $H - H^*$  resolvent bound (3.5), one proves the estimate

$$\|\mathbf{P}(\xi)\|_{B(X; H^*)} \leq 1,$$

which then allows to perform another simpler (duality-free) bootstrap argument by combining the above estimates with the bounds of Lemma 3.4:

$$\begin{aligned} \|\mathbf{P}^{(1)}(\xi)\|_{B(X; H^*)} &\leq \|\mathbf{P}(\xi) \mathbf{P}^{(1)}(\xi)\|_{B(X; H^*)} + \|\mathbf{P}^{(1)}(\xi) \mathbf{P}\|_{B(X; H^*)} \\ &\leq \|\mathbf{P}(\xi)\|_{B(X; H^*)} \|\mathbf{P}^{(1)}(\xi)\|_{B(X; X)} + \|\mathbf{P}^{(1)}(\xi)\|_{B(X; H^*)} \|\mathbf{P}\|_{B(X; X)} \\ &\leq 1. \end{aligned}$$

This concludes this step.

*Step 3b: Second order expansion.* — We use this time the bootstrap formula (B.8) and the first order estimates, together with the duality identity (B.2)

$$\begin{aligned} \|\mathbf{P}^{(2)}(\xi)\|_{B(X; X)} &\leq 1 + \|\mathbf{P} \mathbf{P}^{(2)}(\xi)\|_{B(X; X)} + \|\mathbf{P}^{(2)}(\xi) \mathbf{P}(\xi)\|_{B(X; X)} \\ &\leq 1 + \|\mathbf{P}^{(2)}(\xi) \mathbf{P}\|_{B(X; X^*)} + \|\mathbf{P}^{(2)}(\xi) \mathbf{P}(\xi)\|_{B(X; X)}. \end{aligned}$$

Using the regularization estimate (3.34) on  $\mathbf{P}$ , we obtain

$$\|\mathbf{P}^{(2)}(\xi)\|_{B(X; X)} \leq 1 + \|\mathbf{P}^{(2)}(\xi)\|_{B(X_2; X^*)} \|\mathbf{P}\|_{B(X; X_2)} + \|\mathbf{P}^{(2)}(\xi)\|_{B(X_2; X)} \|\mathbf{P}\|_{B(X; X_2)}$$

$$1 + \|\mathbf{P}^{(2)}(\xi)\|_{B(X_2; X^*)} + \|\mathbf{P}^{(2)}(\xi)\|_{B(X_2; X)}.$$

We conclude as in the previous step that  $\|\mathbf{P}^{(2)}(\xi)\|_{B(X; X)} \leq 1$ , and then perform a second bootstrap to deduce  $\|\mathbf{P}^{(2)}(\xi)\|_{B(X; H^*)} \leq 1$  from the estimates of Lemma 3.4. This concludes the proof.

*Remark 3.13.* — Notice that, since

$$[(\text{Id} - \mathbf{P}(\xi))U(t)]_{B(X)} = (\text{Id} - \mathbf{P}(\xi))U(t)_{B(X)}$$

the decay estimates (1.26) extends easily to the adjoint  $U(t)(\text{Id} - \mathbf{P}(\xi))$ .

### 3.5. Regularized version of the spectral result

We present here yet another improved version of Theorem 1.8, taking now advantage of possible alternative splittings of the linearized operator  $L$ . In order to prove a “regularized” version of our main result, we will need the following extra assumption.

**LR** Besides Assumptions **(L1)**–**(L4)**, assume that the operator can be decomposed in a way  $L = B^{(1)} + A^{(1)}$  compatible with a hierarchy of **Banach** spaces  $(W_j)_{j=-\ell}^2$ , where  $\ell > 0$ , such that

(a) the spaces  $W_j$  embed into one another and the regular space embeds into the original space:

$$W_2 \subset W_1 \subset W = W_0 \subset W_{-1} \subset \dots \subset W_{-\ell}, \quad W \subset H,$$

(b) the multiplication by  $v$  is bounded from  $W_j$  to  $W_{j-1}$  for some  $j$ :

$$vf_{W_j} \in f_{W_{j+1}}, \quad j = 0, 1,$$

(c) the operator  $A^{(1)}$  is bounded from  $W_j$  to  $W_{j+1}$  and from  $H$  to  $W_{-\ell}$ :

$$A^{(1)} \in \mathcal{B}(W_j; W_{j+1}) \cup \mathcal{B}(H; W_{-\ell}), \quad j = 1, \dots, -\ell,$$

(d) the part  $B^{(1)}$  is dissipative on  $Y = W_{-\ell}, \dots, W_2, H$  in the sense that

$$\sup_{z \in \mathbb{R}^d} \left\| R(z, B^{(1)}) \right\|_{B(Y)} \leq |\text{Re } z + \lambda_B|^{-1},$$

uniformly in  $z \in \mathbb{R}^d$ .

Under this new set of Assumptions, we derive the following version of Theorem 1.8:

**THEOREM 3.14 (Regularized result).** — *If Assumptions **(LR)** are in force, then the spectral projectors from Theorem 1.8 are regularizing in the sense that in the decomposition (1.13)*

$$\mathbf{P}(\xi) = \mathbf{P}^{(0)} \tilde{\xi} + i\xi \cdot \mathbf{P}^{(1)} \tilde{\xi} + \mathbf{S}(\xi),$$

*each term belongs to  $\mathcal{B}(H; W)$  uniformly in  $|\xi| \leq \alpha_0$ , and  $\mathbf{S}(\xi)_{B(H; W)} \leq |\xi|^2$ .*

*Remark 3.15.* — Once again, we illustrate this set of assumptions in the case of the Boltzmann equation for hard spheres. In this context the hierarchy of spaces can be taken to be

$$W_j = L^\infty \int \mu^{-1/2} v^{j+} dv, \quad j = 2, \dots, -\ell,$$

for some integer  $\ell > \frac{d}{2}$ , and the splitting is also Grad’s splitting (see Remark 1.4).

*Proof of Theorem 3.14.* — Let us prove that the coefficients of the expansion

$$P(\xi) = P + \xi \cdot P^{(1)} + \xi \otimes \xi : P^{(2)}(\xi)$$

belong to  $B(H; W)$  uniformly in  $\xi$  small enough. As pointed out in the proof of Theorem 1.19, this will be enough to deduce it also holds for  $P(\xi)$ .

*Step 1: Estimate for the resolvent in the regular space  $W$ .* — Starting from the factorization formula

$$(3.37) \quad R(z, L) = \sum_{n=0}^{-1} R(z, B^{(1)}) \check{A}^{(1)} \check{S}_n R(z, B^{(1)}) + R(z, B^{(1)}) \check{A}^{(1)} \check{S} R(z, L),$$

one gets from (3.5), the embedding  $W \subset H$ , as well as the bounds (LRd) on  $R(z, B^{(1)})$  and the regularization hypothesis (LRC) for  $A^{(1)}$  that, for any  $0 < \lambda < \lambda_L$ , there are some  $\alpha_0, r > 0$  small enough such that

$$(3.38) \quad \sup_z R(z, L)_{B(W)} \leq C, \quad |\xi| \leq \alpha_0$$

with, as before,  $\xi = \xi_{\{|z| > r\}}$ .

*Step 2: Behavior of the spectral projector as  $\xi \rightarrow 0$ .* — We use a similar bootstrap strategy. It is simpler because no duality argument is involved, in exchange we replace the use of adjoint operators by estimates in  $W_{-1}$  and  $W_{-2}$ .

The first step is to extend the bound (3.38) from  $W$  to  $W_j$  for  $j = -2, \dots, 2$  using similar factorization arguments as in the previous proofs.

The second step is then to deduce, using the bounds (LRC)–(LRd) and the representation formula

$$P(\xi) = \frac{1}{2i\pi} \int_{|z|=r} R(z, B^{(1)}) \check{A}^{(1)} \check{S}_2 R(z, L) dz$$

the regularization bounds

$$P(\xi)_{B(W; W_2)} + P(\xi)_{B(W_{-2}; W)} \leq 1, \quad |\xi| \leq \alpha_0.$$

From then, we follow a simplified version of the bootstrap procedures used in the proofs of Lemma 3.4 and Theorem 1.19. This concludes this proof.

#### 4. Properties of the linearized semigroup in the physical space

If one assumes only (L1)–(L4), we denote in this section  $X = H$ . Under the extra assumption (LE),  $X$  is the space from (LE).

In this section, we exploit the spectral description of the previous Section to study the main properties of the semigroup  $(U(t))_{t>0}$ . We adopt the notations and definitions introduced in Section 2. We only recall that

$$U(t) = U_{\text{kin}}(t) + U_{\text{hydro}}(t), \quad U_{\text{hydro}}(t) = U_{\text{NS}}(t) + U_{\text{wave}}(t)$$

where the various semigroups are defined in Definitions 2.7 and 2.11. As explained in the Introduction and in Section 2, it is important for the definition of the various terms  ${}_{\text{hydro}}[f, g]$  and  ${}_{\text{kin}}[f, g]$  to study suitable bounds on the semigroups  $U_{\text{kin}}(t)$  and  $U_{\text{hydro}}(t)$  as well as their convolution with suitable time-dependent functions.

We begin with the following uniform estimates on  $U_{\text{hydro}}(\cdot)$

**LEMMA 4.1 (Bounds for the hydrodynamic semigroup).** — For  $\star = \text{NS, wave, disp}$ , the hydrodynamic semigroups  $U(\cdot)$  are bounded from  $X$  to  $H$ :

$$(4.1) \quad U(\cdot)g \in H \cdot \left\{ g \in X + g \in \dot{H}_x^{-\alpha}(X) \right\}.$$

Furthermore, if  $\varphi \in L^2([0, T]; X)$  where  $T \in (0, \infty]$  is such that  $\text{P}\varphi(t) = 0$ , then for  $\star = \text{NS, wave}$  and thus  $\star = \text{hydro}$

$$(4.2a) \quad \frac{1}{\varepsilon} U(\cdot) \varphi \in H \cdot \left\{ w, (t) \int_0^t \varphi(\tau) \frac{2}{X} \dot{H}_x^{-\alpha}(X_v) d\tau \right\}^{< \frac{1}{2}},$$

and

$$(4.2b) \quad \frac{1}{\varepsilon} U(\cdot) \varphi \in H \cdot \left\{ w, (t) \int_0^t \varphi(\tau) \frac{2}{X^{1+\alpha}} d\tau \right\}^{< \frac{1+\alpha}{2}},$$

**Remark 4.2.** — The above estimates still hold true for  $\star = \text{hydro}$  since

$$U_{\text{hydro}}(t) = U_{\text{wave}}(t) + U_{\text{NS}}(t).$$

Notice also that (4.2b) shows that, with respect to (4.2a), no use of Sobolev space of negative order is required under the stronger integrability assumption  $\varphi \in L^{\frac{2}{1+\alpha}}([0, T]; X)$ .

*Proof.* — Let us fix  $\star = \text{Bou, inc, } \pm\text{wave}$ . First of all, notice that for  $\varepsilon/|\xi| \ll \alpha_0$ , where we recall from (1.9) that  $\alpha_0$  can be taken small enough

$$\text{Re}(\varepsilon^{-2}\lambda(\varepsilon\xi)) = -\kappa |\xi|^2 + O(\varepsilon/|\xi|^\beta) \ll -\frac{\kappa}{2} |\xi|^2,$$

and thus, the following estimate holds:

$$\exp(\varepsilon^{-2}t\lambda(\varepsilon\xi)) \ll \exp\left(-t\kappa \frac{|\xi|^2}{2}\right).$$

Let us prove in the first step (4.1) and in the second step (4.2).



*Step 1: Proof of (4.1).* — Using the Fourier representation from Definition 2.7 of  $U f$ , together with the boundedness  $\mathbf{P}(\varepsilon\xi)_{B(X;H^*)}$  of Theorems 1.8 and 1.19, we easily have for  $d > 2$

$$U(t)g \Big|_{H^*}^2 = \int_{\mathbb{R}^d} F_x[U(t)g](\xi) \Big|_{H^*}^2 \xi^{2s} d\xi \\ = \int_{\mathbb{R}^d} e^{-t|\xi|^2} \widehat{g}(\xi) \Big|_{X}^2 \xi^{2s} d\xi \cdot g \Big|_{X}^2,$$

as well as

$$\int_0^T \left\| |\xi|^{1-} U(t)g \right\|_{H^*}^2 dt = \int_0^T dt \int_{\mathbb{R}^d} |\xi|^{2-2} F_x[U(t)g](\xi) \Big|_{H^*}^2 \xi^{2s} d\xi \\ = \int_0^T \int_{\mathbb{R}^d} |\xi|^{2-2} e^{-t|\xi|^2} \widehat{g}(\xi) \Big|_{X}^2 \xi^{2s} d\xi dt \\ = \int_{\mathbb{R}^d} \widehat{g}(\xi) \Big|_{X}^2 \xi^{2s} \int_0^T |\xi|^{2-2} e^{-t|\xi|^2} dt d\xi.$$

Consequently,

$$(4.3) \quad \int_0^T \left\| |\xi|^{1-} U(t)g \right\|_{H^*}^2 dt \leq \int_{\mathbb{R}^d} \widehat{g}(\xi) \Big|_{X}^2 |\xi|^{-2} \xi^{2s} \int_0^T |\xi|^2 e^{-t|\xi|^2} dt d\xi.$$

Using that

$$(4.4) \quad |\xi|^{-2} \xi^{2s} \leq \xi^{2s} \mathbf{1}_{|\xi|>1} + |\xi|^{-2} \mathbf{1}_{|\xi|\leq 1} \quad \text{and} \quad \int_0^T |\xi|^2 e^{-t|\xi|^2} dt \leq 1$$

we deduce that

$$\int_0^T \left\| |\xi|^{1-} U(t)g \right\|_{H^*}^2 dt \leq g \Big|_{X}^2 + g \Big|_{H_x^{-\alpha}(X_v)}^2.$$

This concludes this step thanks to (2.4).

*Step 2: Proof of (4.2).* — Recall from the expansion (1.13) of  $\mathbf{P}$  that

$$(4.5) \quad \mathbf{P}(\varepsilon\xi) = \mathbf{P}^{(0)} \Big|_{\widetilde{S}}^{\varepsilon\xi} + i\varepsilon\xi \cdot \mathbf{P}^{(1)}(\widetilde{\xi}) + S(\varepsilon\xi) \\ =: \mathbf{P}^{(0)} \Big|_{\widetilde{S}}^{\varepsilon\xi} + \varepsilon\xi \cdot \overline{\mathbf{P}}^{(1)}(\varepsilon\xi),$$

where the remainder  $\overline{\mathbf{P}}^{(1)}(\varepsilon\xi)$  satisfies

$$\sup_{|\xi| \leq \varepsilon} \overline{\mathbf{P}}^{(1)}(\varepsilon\xi)_{B(X;H^*)} \leq 1$$

by virtue of Theorems 1.8 and 1.19 whereas  $\mathbf{P}^{(0)}(\widetilde{\xi})$  is an  $H$ -orthogonal projection on a subspace of  $\text{Ker}(L)$ . In particular, we have

$$(4.6) \quad \mathbf{P}(\varepsilon\xi)\widehat{\varphi}(t, \xi) = \varepsilon\xi \cdot \overline{\mathbf{P}}^{(1)}(\varepsilon\xi)\widehat{\varphi}(t, \xi)$$

and thus there holds, for any  $t > 0, \tau > 0$

$$\left\| \exp(\varepsilon^{-2} t \lambda(\varepsilon \xi)) P(\varepsilon \xi) \widehat{\varphi}(\tau, \xi) \right\|_{H^s} \leq \varepsilon |\xi| e^{-t \varepsilon^2 \frac{|\xi|^2}{2}} \left\| P^{(1)}(\varepsilon \xi) \widehat{\varphi}(\tau, \xi) \right\|_{H^s} \\ \leq \varepsilon |\xi| e^{-t \varepsilon^2 \frac{|\xi|^2}{2}} \widehat{\varphi}(\tau, \xi) \chi.$$

Therefore,

$$(4.7) \quad \frac{1}{\varepsilon^2} \|U(\cdot) \varphi(t)\|_{H^s}^2 = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} \xi^{2s} \left\| \int_0^t F_x[U(t-\tau)\varphi(\tau)](\xi) d\tau \right\|_{H^s}^2 d\xi \\ \leq \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} \xi^{2s} \int_0^t \varepsilon |\xi| e^{-(t-\tau) \varepsilon^2 \frac{|\xi|^2}{2}} \widehat{\varphi}(\tau, \xi) \chi d\tau d\xi.$$

Using Cauchy–Schwarz’s inequality to estimate the integral over  $[0, t]$ , we have

$$\frac{1}{\varepsilon^2} \|U(\cdot) \varphi(t)\|_{H^s}^2 \leq \int_{\mathbb{R}^d} \xi^{2s} \int_0^t |\xi| e^{-(t-\tau) \varepsilon^2 \frac{|\xi|^2}{2}} d\tau \int_0^t \widehat{\varphi}(\tau, \xi) \chi d\tau d\xi \\ \leq \int_{\mathbb{R}^d} \xi^{2s} d\xi \int_0^t \widehat{\varphi}(\tau, \xi) \chi d\tau \int_0^t \varphi(\tau) \chi d\tau.$$

In the same way,

$$(4.8) \quad \frac{1}{\varepsilon^2} \int_0^T \| |x|^{\beta-1} U(\cdot) \varphi(t) \|_{H^s}^2 dt \\ \leq \frac{1}{\varepsilon^2} \int_0^T dt \int_{\mathbb{R}^d} \xi^{2s} |\xi|^{\beta-2} \int_0^t \varepsilon |\xi| e^{-(t-\tau) \varepsilon^2 \frac{|\xi|^2}{2}} \widehat{\varphi}(\tau, \xi) \chi d\tau d\xi \\ \leq \int_{\mathbb{R}^d} \xi^{2s} |\xi|^{\beta-2} d\xi \int_0^T \int_0^t |\xi|^\beta e^{-(t-\tau) \varepsilon^2 \frac{|\xi|^2}{2}} \widehat{\varphi}(\tau, \xi) \chi d\tau dt.$$

Recalling (4.4) and using Young’s convolution inequality in the form  $L^1([0, T]) \cdot L^2([0, T]) \subset L^2([0, T])$  we deduce that

$$\frac{1}{\varepsilon^2} \int_0^T \| |x|^{\beta-1} U(t) \varphi \|_{H^s}^2 dt \leq \int_{\mathbb{R}^d} \xi^{2s} |\xi|^{\beta-2} \int_0^T \widehat{\varphi}(t, \xi) \chi dt d\xi \\ \leq \int_0^T \varphi(t) \chi dt + \int_0^T \varphi(t) \chi_{H_x^{-\alpha}(X_v)} dt$$

which easily prove (4.2a). To prove (4.2b), we rewrite (4.8) as

$$\frac{1}{\varepsilon^2} \int_0^T \| |x|^{\beta-1} U(\cdot) \varphi(t) \|_{H^s}^2 dt \\ \leq \int_{\mathbb{R}^d} \xi^{2s} d\xi \int_0^T \int_0^t |\xi|^\beta e^{-(t-\tau) \varepsilon^2 \frac{|\xi|^2}{2}} \widehat{\varphi}(\tau, \xi) \chi d\tau dt.$$

Using now Young’s convolution inequality in the form  $L^{\frac{2}{2-\alpha}}([0, T]) \cdot L^{\frac{2}{1+\alpha}}([0, T]) \subset L^2([0, T])$  we deduce that

$$\frac{1}{\varepsilon^2} \int_0^T \left\| \int_{\mathbb{R}^d} \xi^{2s} \int_0^T \widehat{\varphi}(t, \xi) \frac{1}{X^{\frac{2}{1+\alpha}}} dt \right\|_{H^s}^2 dt \leq \int_{\mathbb{R}^d} \xi^{2s} \int_0^T \widehat{\varphi}(t, \xi) \frac{1}{X^{\frac{2}{1+\alpha}}} dt \int_0^T \varphi(t) \frac{1}{X^{\frac{2}{1+\alpha}}} dt$$

where we used Minkowski’s integral inequality for the last estimate. Since the estimates established are uniform in  $T$ , this concludes the proof of Lemma 4.1.

We now make precise the asymptotic equivalence between the semigroup

$$(U_{\text{wave}}(t))_{t>0}$$

and its leading order  $(U_{\text{disp}}(t))_{t>0}$ .

LEMMA 4.3 (*Asymptotic equivalence of the oscillating semigroups*). — Given  $s > 0$  and some regularity parameter  $r \in (s, s + 1]$ , it holds

$$(4.9) \quad U_{\text{wave}}(\cdot)f - U_{\text{disp}}(\cdot)f \in \mathcal{H}^s \cdot \varepsilon^{r-s} \mathcal{H}^{r, r} + \mathcal{H}^{-\alpha}(X_v),$$

for any  $f \in \mathcal{H}^r(X_v)$  while there holds

$$(4.10) \quad \lim_0 U_{\text{wave}}(\cdot)f - U_{\text{disp}}(\cdot)f \in \mathcal{H} = 0$$

for any  $f \in \mathcal{H}^r(X_v)$ , i.e. whenever  $r = s$ .

Proof. — We start by expanding the symbol of  $U_{\pm\text{wave}}(t)$  using the decomposition of  $P_{\pm\text{wave}}(\varepsilon\xi)$  from Step 2 of the proof of Lemma 4.1, we obtain

$$\begin{aligned} & \exp(\varepsilon^{-2}t\lambda_{\pm\text{wave}}(\varepsilon\xi)) P_{\pm\text{wave}}(\varepsilon\xi) \\ &= \exp(\varepsilon^{-2}t\lambda_{\pm\text{wave}}(\varepsilon\xi)) \left[ P_{\pm\text{wave}}^{(0)}(\xi) + i\varepsilon\xi \cdot \overline{P}_{\pm\text{wave}}^{(1)}(\varepsilon\xi) \right] \\ &= \exp(\pm ic\varepsilon^{-1}t/|\xi| - t\kappa_{\text{wave}}/|\xi|^2) P^{(0)}(\xi) \\ &+ \left[ \exp(\varepsilon^{-2}t\lambda(\varepsilon\xi)) - \exp(\pm ic\varepsilon^{-1}t/|\xi| - t\kappa_{\text{wave}}/|\xi|^2) \right] P_{\pm\text{wave}}^{(0)}(\xi) \\ &+ \varepsilon \exp(\varepsilon^{-2}t\lambda(\varepsilon\xi)) \xi \cdot \overline{P}_{\pm\text{wave}}^{(1)}(\varepsilon\xi). \end{aligned}$$

Thus, the symbol of the difference  $U_{\text{wave}}(t) - U_{\text{disp}}(t)$  writes, for  $\varepsilon/|\xi| \ll \alpha_0$ , as the sum of the two terms (corresponding to  $\star = \pm\text{wave}$ ):

$$\begin{aligned} & \left[ \exp(\varepsilon^{-2}t\lambda(\varepsilon\xi)) - \exp(\pm ic\varepsilon^{-1}t/|\xi| - t\kappa_{\text{wave}}/|\xi|^2) \right] P^{(0)}(\xi) \\ &+ \varepsilon \exp(\varepsilon^{-2}t\lambda(\varepsilon\xi)) \xi \cdot \overline{P}^{(1)}(\varepsilon\xi). \end{aligned}$$

On the one hand, when  $\varepsilon/|\xi| > \alpha_0$ , since  $\overline{P}^{(1)}(\varepsilon\xi)$  is supported in  $\{\varepsilon/|\xi| \ll \alpha_0\}$ , the symbol reduces to

$$- \exp(ic\varepsilon^{-1}t/|\xi| - t\kappa_{\text{wave}}/|\xi|^2) P_{\text{wave}}^{(0)}(\xi) - \exp(-ic\varepsilon^{-1}t/|\xi| - t\kappa_{\text{wave}}/|\xi|^2) P_{-\text{wave}}^{(0)}(\xi).$$

On the other hand, when  $\varepsilon/\xi \leq \alpha_0$ , we estimate the difference of exponentials using the inequality  $|1 - e^a| \leq ae^{|a|}$  as well as the expansion (1.9) of  $\lambda_{\pm \text{wave}}(\xi)$ :

$$\begin{aligned} & \left| \exp(\varepsilon^{-2}t\lambda_{\pm}(\varepsilon\xi)) - \exp(\pm ic\varepsilon^{-1}/\xi - t\kappa_{\text{wave}}/\xi^2) \right| \\ &= \left| \exp(\pm ic\varepsilon^{-1}/\xi - t\kappa_{\text{wave}}/\xi^2) \right| \left| \exp(O(t\varepsilon/\xi^3)) - 1 \right| \\ & \leq (\varepsilon/\xi) (t/\xi^2) \exp(-t\kappa_{\text{wave}}/\xi^2) \exp(O(t\varepsilon/\xi^3)), \end{aligned}$$

thus, using  $re^{-r} \leq e^{-\frac{1}{2}r}$  and assuming  $\alpha_0$  small enough so that  $O(\varepsilon/\xi^3) \leq \frac{1}{4}\kappa_{\text{wave}}/\xi^2$ , we obtain

$$\begin{aligned} & \left| \exp(\varepsilon^{-2}t\lambda_{\pm}(\varepsilon\xi)) - \exp(\pm ic\varepsilon^{-1}/\xi - t\kappa_{\text{wave}}/\xi^2) \right| \\ & \leq \varepsilon/\xi \exp\left(-\frac{t}{2}\kappa_{\text{wave}}/\xi^2\right) \exp(O(t\varepsilon/\xi^3)) \\ & \leq \varepsilon/\xi \exp\left(-\frac{t}{4}\kappa_{\text{wave}}/\xi^2\right). \end{aligned}$$

Putting together the previous estimates, we then bound the operator norm in  $B(X; H^s)$  of the symbol of the difference  $U_{\text{wave}}(t) - U_{\text{disp}}(t)$ . It is controlled by

$$\mathbf{1}_{\varepsilon/\xi \leq \alpha_0} \exp\left(-\frac{t}{4}\kappa_{\text{wave}}/\xi^2\right) + \mathbf{1}_{\varepsilon/\xi > \alpha_0} \exp\left(-\frac{t}{4}\kappa_{\text{wave}}/\xi^2\right) (\varepsilon/\xi)^{r-s} \exp\left(-\frac{t}{4}\kappa_{\text{wave}}/\xi^2\right),$$

where we used the comparison  $u\mathbf{1}_{u \leq \alpha_0} + \mathbf{1}_{u > \alpha_0} \leq u^{r-s}$  for any  $u > 0$  since  $r-s \in [0, 1]$ . As in the proof of Lemma 4.1, such an estimate on the symbol of  $U_{\text{wave}}(t) - U_{\text{disp}}(t)$  yields the controls (4.9), from which we deduce (4.10) by density.

A similar result holds for the difference between  $U_{\text{NS}}(t) - U_{\text{NS}}(t)$ .

**LEMMA 4.4 (Asymptotic equivalence of the Navier–Stokes semigroup).** *Given  $s > 0$  and consider some regularity parameter  $r \in (s, s + 1]$ , the part  $U_{\text{NS}}(\cdot)$  of the hydrodynamic semigroup is such that*

$$(4.11) \quad U_{\text{NS}}(\cdot)f - U_{\text{NS}}(\cdot)f \Big|_{H^s} \leq \varepsilon^{r-s} \left( \|f\|_{X^{r,s}} + \|f\|_{\dot{H}_x^{-\alpha}(X_v)} \right),$$

for any  $f \in X^{r,s} \cap \dot{H}_x^{-\alpha}(X_v)$ , while, for  $f \in X \cap \dot{H}_x^{-\alpha}(X_v)$  (i.e.  $r = s$ ), there holds

$$(4.12) \quad \lim_0 U_{\text{NS}}(\cdot)f - U_{\text{NS}}(\cdot)f \Big|_H = 0.$$

Furthermore, if  $\varphi \in L^2([0, T]; X)$  where  $T \in (0, \infty]$  is such that  $P\varphi(t) = 0$ , then

$$(4.13) \quad \frac{1}{\varepsilon} \|U_{\text{NS}}(\cdot)\varphi - \varepsilon \cdot V_{\text{NS}}(\cdot)\varphi\|_{H^s} \leq \varepsilon^{r-s} \sup_{0 \leq t < T} \left\{ w_{X^{r,s}}(t) \int_0^t \|\varphi(\tau)\|_{X^{r,s} \cap \dot{H}_x^{-\alpha}(X_v)}^2 d\tau \right\}^{\frac{1}{2}}.$$

*Proof.* — Let us fix  $\star = \text{Bou, inc}$ . As in the previous proof, we start by expanding the symbol of  $U_{\text{NS}}(t)$  so as to compare it with those of  $U_{\text{NS}}(t)$  and  $V_{\text{NS}}(t)$ . We first prove (4.11) and (4.12), and then (4.13).

*Step 1: Proof of (4.11) and (4.12).* — For  $\varepsilon/\xi \leq \alpha_0$ , using the decomposition (4.5), there holds

$$\begin{aligned} \exp(\varepsilon^{-2}t\lambda(\varepsilon\xi)) P(\varepsilon\xi) &= \exp(-t\kappa/\xi^2) P^{(0)} \xi^{\sim S} + \varepsilon \exp(\varepsilon^{-2}t\lambda(\varepsilon\xi)) \xi \cdot \bar{P}^{(1)}(\varepsilon\xi) \\ &\quad + \left[ \exp(\varepsilon^{-2}t\lambda(\varepsilon\xi)) - \exp(-t\kappa/\xi^2) \right] P^{(0)} \xi^{\sim S} \end{aligned}$$

whereas, for  $\varepsilon/\xi > \xi_0$ , since  $P(\varepsilon\xi)$  vanishes, the symbol of the difference  $U_{\text{NS}}(t) - U_{\text{NS}}(t)$  reduces to that of  $-U_{\text{NS}}(t)$  given by

$$-\exp(-t\kappa_{\text{Bou}}/\xi^2) P_{\text{Bou}}^{(0)} \xi^{\sim S} - \exp(-t\kappa_{\text{inc}}/\xi^2) P_{\text{inc}}^{(0)} \xi^{\sim S}.$$

To sum up, the symbol of the difference  $U_{\text{NS}}(t) - U_{\text{NS}}(t)$  writes as the sum over  $\star = \text{Bou, inc}$  of the symbols

$$\begin{aligned} &\mathbf{1}_{\varepsilon/\xi \leq \alpha_0} \left( \varepsilon \exp(\varepsilon^{-2}t\lambda(\varepsilon\xi)) \xi \cdot \bar{P}^{(1)}(\varepsilon\xi) \right. \\ &\left. + \left[ \exp(\varepsilon^{-2}t\lambda(\varepsilon\xi)) - \exp(-t\kappa/\xi^2) \right] P^{(0)} \xi^{\sim S} \right) - \mathbf{1}_{\varepsilon/\xi > \alpha_0} \exp(-t\kappa/\xi^2) P^{(0)} \xi^{\sim S}, \end{aligned}$$

and its operator norm in  $B(X; H^\bullet)$  is controlled as in the proof of Lemma 4.1 by

$$\varepsilon/\xi \exp(-t\kappa \frac{|\xi|^2}{4}).$$

We then deduce (4.11) as well as (4.12) by density as in the proof of Lemma 4.3.

*Step 2: Proof of (4.13).* — In the case  $P_\varphi(t) = 0$ , the projector  $P^{(0)}_\varphi(t)$  vanishes, and we use the second order expansion of  $P(\varepsilon\xi)$  provided in (1.13) in Theorem 1.8:

$$P(\varepsilon\xi) = i\varepsilon\xi \cdot P^{(1)} \xi^{\sim S} + S(\varepsilon\xi),$$

where we recall that  $S(\varepsilon\xi) \in B(X; H^\bullet) \cdot \varepsilon^2/|\xi|^2$  uniformly in  $\varepsilon/\xi \leq \alpha_0$ . Similarly, the symbol of  $U_{\text{NS}}(t) - \varepsilon \cdot V_{\text{NS}}(t)$  restricted to  $\text{Ker}(P)$  then writes

$$\begin{aligned} &\mathbf{1}_{\varepsilon/\xi \leq \alpha_0} \left( \varepsilon^2 \exp(\varepsilon^{-2}t\lambda(\varepsilon\xi)) |\xi|^2 S(\varepsilon\xi) \right. \\ &\quad \left. + i\varepsilon \left[ \exp(\varepsilon^{-2}t\lambda(\varepsilon\xi)) - \exp(-t\kappa/\xi^2) \right] \xi \cdot P^{(1)} \xi^{\sim S} \right) \\ &\quad - \mathbf{1}_{\varepsilon/\xi > \alpha_0} i\varepsilon \exp(-t\kappa/\xi^2) \xi \cdot P^{(1)} \xi^{\sim S}, \end{aligned}$$

which is similarly controlled by

$$\varepsilon/\xi / (\varepsilon/\xi)^{r-s} \exp(-t\kappa \frac{|\xi|^2}{4}).$$

These representations allow to proceed as in the proofs of Lemmas 4.1 and 4.3 to get the desired conclusion.

We present now a dispersive estimate for the semigroup  $(U_{\text{disp}}(t))_{t>0}$  which is deduced from a general result about the decay rate for solutions to the wave equation.

LEMMA 4.5 (*Dispersive estimate*). — The part  $U_{\text{disp}}$  of the hydrodynamic semigroup satisfies the dispersive estimate

$$(4.14) \quad \left\| U_{\text{disp}}(t)g \right\|_{W_x^s, (H_v)} \leq \left( \frac{\varepsilon}{t} \right)^{\frac{d-1}{2}} g \Big|_{\dot{B}_{1,1}^{\frac{d+1}{2}+s}(X_v)}.$$

*Proof.* — In virtue of the macroscopic representation of  $U_{\text{disp}}$  from Proposition 2.10 and the continuity of the heat semigroup on  $L^1$ , we can deduce (4.14) directly from Lemma B.4.

LEMMA 4.6 (*Vanishing estimate for the convoluted oscillating semigroup*). — Suppose  $\varphi \in L^1([0, T]; X)$  is such that  $\| |x|^{\beta-} \varphi \|_{L^2([0, T]; X)}$  and  $\mathbf{P}\varphi(t) = 0$  for any  $t > 0$  together with

$$\partial_t \varphi \in L^2 \left( L^{\frac{2}{1+\alpha}}([0, T]; X^{s-1}) \right) \cap L^{\frac{4}{3}} \left( L^{\frac{4}{3+2\alpha}}([0, T]; \dot{H}_x^{-\frac{1}{2}}(X_v)) \right).$$

Then, there holds

$$\begin{aligned} \frac{1}{\varepsilon^2} U_{\text{wave}}(\cdot) \varphi_H &\leq \varphi(0) \Big|_{\dot{H}_x^{-\alpha}(X_v)} + \varphi \Big|_{L^1([0, T]; X)} + \| |x|^{\beta-} \varphi \|_{L^2([0, T]; X)} \\ &+ \partial_t \varphi \Big|_{L^2 \left( L^{\frac{2}{1+\alpha}}([0, T]; X^{s-1}) \right)} \\ &+ \partial_t \varphi \Big|_{L^{\frac{4}{3}} \left( L^{\frac{4}{3+2\alpha}}([0, T]; \dot{H}_x^{-\frac{1}{2}}(X_v)) \right)}. \end{aligned}$$

Remark 4.7. — Note that if  $T < \frac{1}{\varepsilon}$ , we have  $L^{\frac{4}{3}} \left( L^{\frac{4}{3+2\alpha}} \right) = L^{\frac{4}{3}}$  and  $L^{\frac{2}{1+\alpha}} \left( L^2 \right) = L^2$ .

*Proof.* — In the first step, we establish a preparatory estimate for any  $\xi \in \mathbb{R}^d$  satisfying  $\varepsilon/|\xi| \leq \alpha_0$ , which we will use in the following step to prove the lemma. Since  $w \leq 1$ , we neglect it for the estimates on  $\partial_t \varphi$ .

*Step 1: Preparatory estimate.* — Recall that  $U_{\text{wave}}(\cdot) \varphi(t) \Big|_{H^s}$  is given by (4.7) which allows us to work, as in the proof of Lemma 4.3, on the two parts of the symbol of  $U_{\text{wave}}(t)$ . Recalling (4.6), for any fixed  $t > 0$  and any  $\tau \in [0, t]$  one has

$$\begin{aligned} \exp(\varepsilon^{-2}\tau\lambda_{\pm\text{wave}}(\varepsilon\xi)) \mathbf{P}_{\pm\text{wave}}(\varepsilon\xi) \widehat{\varphi}(t - \tau, \xi) \\ = \varepsilon \exp(\varepsilon^{-2}\tau\lambda_{\pm\text{wave}}(\varepsilon\xi)) \xi \cdot \overline{\mathbf{P}}_{\pm\text{wave}}^{(1)}(\varepsilon\xi) \widehat{\varphi}(t - \tau, \xi) \\ = \varepsilon \exp(\varepsilon^{-2}\tau\lambda_{\pm\text{wave}}(\varepsilon\xi)) \xi \cdot \phi^{\pm}(\tau, \xi) \end{aligned}$$

where we denoted  $\phi^{\pm}(\tau, \xi) := \overline{\mathbf{P}}_{\pm\text{wave}}^{(1)}(\varepsilon\xi) \widehat{\varphi}(t - \tau, \xi)$ . We now integrate with respect to  $\tau \in [0, t]$  using integration by parts:

$$\begin{aligned} \int_0^t \exp(\varepsilon^{-2}\tau\lambda_{\pm\text{wave}}(\varepsilon\xi)) \mathbf{P}_{\pm\text{wave}}(\varepsilon\xi) \widehat{\varphi}(t - \tau, \xi) d\tau \\ = \varepsilon \xi \cdot \int_0^t \exp(\varepsilon^{-2}\tau\lambda_{\pm\text{wave}}(\varepsilon\xi)) \phi^{\pm}(\tau, \xi) d\tau \\ = - \frac{\varepsilon^3 \xi}{\lambda_{\pm\text{wave}}(\varepsilon\xi)} \cdot \int_0^t \exp(\varepsilon^{-2}\tau\lambda_{\pm\text{wave}}(\varepsilon\xi)) \partial_{\tau} \phi^{\pm}(\tau, \xi) d\tau \\ + \frac{\varepsilon^3 \xi}{\lambda_{\pm\text{wave}}(\varepsilon\xi)} \cdot [\phi^{\pm}(t, \xi) \exp(\varepsilon^{-2}t\lambda_{\pm\text{wave}}(\varepsilon\xi)) - \phi^{\pm}(0, \xi)]. \end{aligned}$$

As in Lemma 4.3, we can choose  $\alpha_0$  small enough so that

$$|\lambda_{\pm\text{wave}}(\varepsilon\xi)| \leq \varepsilon/|\xi|, \quad \text{Re}(\varepsilon^{-2}\lambda_{\pm\text{wave}}(\varepsilon\xi)) \leq -\frac{1}{2}\kappa_{\text{wave}}/|\xi|^2,$$

uniformly in  $|\xi| \leq \alpha_0$ , and one notices

$$\|\phi^\pm(t, \xi)\|_{H^\bullet} = \|\mathbf{P}_{\pm\text{wave}}^{(1)}(\varepsilon\xi)\widehat{\varphi}(0, \xi)\|_{H^\bullet} \leq \widehat{\varphi}(0, \xi) \leq \dots,$$

while, in the same way,

$$\|\phi^\pm(0, \xi)\|_{H^\bullet} \leq \widehat{\varphi}(t, \xi) \leq \dots, \quad \|\partial \phi^\pm(\tau, \xi)\|_{H^\bullet} \leq \partial \widehat{\varphi}(t - \tau, \xi) \leq \dots.$$

Those considerations lead to

$$\begin{aligned} & \frac{1}{\varepsilon^2} \left\| \int_0^t \exp(\varepsilon^{-2}\tau\lambda_{\pm\text{wave}}(\varepsilon\xi)) \mathbf{P}_{\pm\text{wave}}(\varepsilon\xi)\widehat{\varphi}(t - \tau, \xi) d\tau \right\|_{H^\bullet} \\ & \leq \int_0^t \exp\left(-\frac{\tau}{2}\kappa_{\text{wave}}/|\xi|^2\right) \|\partial \widehat{\varphi}(t - \tau, \xi)\|_{H^\bullet} d\tau + \|\widehat{\varphi}(t, \xi)\|_{H^\bullet} \\ & \leq \|\widehat{\varphi}(0, \xi)\|_{H^\bullet} \exp\left(-\frac{t}{2}\kappa_{\text{wave}}/|\xi|^2\right) \leq \dots \end{aligned}$$

In other words, we have shown that

$$(4.15) \quad \frac{1}{\varepsilon^2} \left\| F_x[U_{\text{wave}}(\cdot) \varphi](t, \xi) \right\|_{H^\bullet} \leq \int_0^t \exp\left(-\frac{\tau}{2}\kappa_{\text{wave}}/|\xi|^2\right) \|\partial \widehat{\varphi}(t - \tau, \xi)\|_{H^\bullet} d\tau + \|\widehat{\varphi}(t, \xi)\|_{H^\bullet} + \|\widehat{\varphi}(0, \xi)\|_{H^\bullet} \exp\left(-\frac{t}{2}\kappa_{\text{wave}}/|\xi|^2\right) \leq \dots$$

**Step 2: Completion of the proof.** — We first deduce from (4.15) that

$$\frac{1}{\varepsilon^2} \|U_{\text{wave}}(\cdot) \varphi(t)\|_{H^\bullet} \leq \sup_{0 \leq \tau \leq t} \|\varphi(\tau)\|_{H^\bullet} + \left\| \xi^s \int_0^t e^{-\frac{\tau}{2}\kappa_{\text{wave}}/|\xi|^2} \|\partial \widehat{\varphi}(t - \tau, \xi)\|_{H^\bullet} d\tau \right\|_{L^2_\xi}.$$

For notations simplicity, we call  $J = J(t, \varphi)$  the above  $L^2$ -norm and split it according to  $|\xi| \leq 1$  or  $|\xi| > 1$ , i.e.  $J = J_1 + J_2$  where

$$J_1^2 = \int_{|\xi| \leq 1} \xi^{2s} \int_0^t e^{-\frac{\tau}{2}\kappa_{\text{wave}}/|\xi|^2} \|\partial \widehat{\varphi}(t - \tau, \xi)\|_{H^\bullet}^2 d\tau d\xi \leq \dots$$

and

$$\begin{aligned} J_2^2 &= \int_{|\xi| > 1} \xi^{2s} \int_0^t e^{-\frac{\tau}{2}\kappa_{\text{wave}}/|\xi|^2} \|\partial \widehat{\varphi}(t - \tau, \xi)\|_{H^\bullet}^2 d\tau d\xi \leq \dots \\ &\leq \int_{|\xi| > 1} \xi^{2s-2} \int_0^t |\xi| e^{-\frac{\tau}{2}\kappa_{\text{wave}}/|\xi|^2} \|\partial \widehat{\varphi}(t - \tau, \xi)\|_{H^\bullet}^2 d\tau d\xi \leq \dots \end{aligned}$$

On the one hand, using Cauchy–Schwarz inequality and the second estimate in (4.4), one has

$$J_2^2 \leq \int_{\mathbb{R}^d} \xi^{2s-2} d\xi \int_0^t \|\partial \widehat{\varphi}(\tau, \xi)\|_{H^\bullet}^2 d\tau = \|\partial_t \varphi\|_{L^2_t H^{s-1}(X_v)}^2.$$

On the other hand, invoking Hölder's inequality (with exponents  $p = 4, q = \frac{4}{3}$ ) to estimate the time integral, we deduce that

$$J_1^2 \cdot \int_{|\xi| \leq 1} \int_0^t e^{-\text{wave} \frac{|\xi|^2}{2} \tau} d\tau \int_0^t \partial \widehat{\varphi}(t - \tau, \xi) \frac{4}{3} d\tau d\xi < \frac{3}{2}$$

$$\cdot \int_{\mathbb{R}^d} |\xi|^{-1} \int_0^t \partial \widehat{\varphi}(\tau, \xi) \frac{4}{3} d\tau d\xi < \frac{3}{2}$$

which, thanks to Minkowski's integral inequality, yields

$$J_1^{\frac{4}{3}} \cdot \int_0^t \int_{\mathbb{R}^d} |\xi|^{-1} \partial \widehat{\varphi}(\tau, \xi) \frac{2}{3} d\xi d\tau = \int_0^t \partial_t \varphi(t) \frac{4}{3} \dot{H}_x^{-\frac{1}{2}}(X_v) d\tau.$$

Therefore,

$$J \cdot \partial_t \varphi_{L_t^2 H_x^{s-1}(X_v)} + \partial_t \varphi_{L_t^{\frac{4}{3}} \dot{H}_x^{-\frac{1}{2}} X_v}$$

i.e.

$$\frac{1}{\varepsilon^2} U_{\text{wave}}(\cdot) \varphi(t)_{H^s} \cdot \sup_{0 \leq t} \varphi(\tau)_X + \partial_t \varphi_{L_t^{\frac{4}{3}} \dot{H}_x^{-\frac{1}{2}} X_v} + \partial_t \varphi_{L_t^2 H_x^{s-1} X_v}.$$

Furthermore, coming back to (4.15),

$$\frac{1}{\varepsilon^4} \int_0^T |\xi|^{\beta-2} \left\| F_x [U_{\text{wave}}(\cdot) \varphi](t, \xi) \right\|_{H^s}^2 dt$$

$$\cdot \int_0^T \int_0^t |\xi|^{-1} e^{-\frac{\tau}{2} \text{wave} |\xi|^2} \partial \widehat{\varphi}(t - \tau, \xi) d\tau d\xi < 2$$

$$+ \int_0^T |\xi|^{\beta-2} \left\| \widehat{\varphi}(t, \xi) \right\|_X^2 dt + \widehat{\varphi}(0, \xi) \frac{2}{3} \int_0^T |\xi|^{\beta-2} e^{-t \text{wave} |\xi|^2} dt \quad \text{CE}$$

Using now Young's convolution inequality in the form  $L^{\frac{2p}{3p-2}}[0, T] \cdot L^p([0, T]) \subset L^2([0, T])$  ( $p \in [1, 2]$ ) in the first time integral, we deduce that

$$\frac{1}{\varepsilon^4} \int_0^T |\xi|^{\beta-2} \left\| F_x [U_{\text{wave}}(\cdot) \varphi](t, \xi) \right\|_{H^s}^2 dt$$

$$\cdot \int_0^T |\xi|^{\beta-2} \left\| \widehat{\varphi}(t, \xi) \right\|_X^2 dt + |\xi|^{-2} \widehat{\varphi}(0, \xi) \frac{2}{3}$$

$$+ \int_0^T \left( |\xi|^{-(2+\frac{2}{p})} \partial \widehat{\varphi}(\tau, \xi) \right)^p d\tau \quad \text{CE}_{\frac{2}{p}}.$$

We integrate this inequality against  $|\xi|^{-2s}$  with the choice  $p = \frac{2}{1+}$   $[\frac{4}{3}, 2]$  on the region  $|\xi| > 1$  and with  $p = \frac{4}{3+2}$   $[1, \frac{4}{3}]$  on the region  $|\xi| \leq 1$ , to obtain, after a simple use of Minkowski's integral inequality,



$$\frac{1}{\varepsilon^2} \left( \int_0^T \left\| \langle x \rangle^{\beta-1} U_{\text{wave}}(\cdot) \varphi(t) \right\|_{H^s}^2 dt \right)^{\frac{1}{2}} \cdot \left\| \langle x \rangle^{\beta-1} \varphi \right\|_{L^2([0,T]; X)} + \varphi(0)_{\dot{H}_x^{-\alpha}(X_v)} + \partial_t \varphi_{L_t^{\frac{4}{3+2\alpha}} \dot{H}_x^{-\frac{1}{2}} X_v} + \partial_t \varphi_{L_t^{\frac{2}{1+\alpha}} H_x^{s-1} X_v}.$$

Since the estimates established are uniform in  $T$  and  $w, \delta < 1$ , this concludes the proof of Lemma 4.6.

The decay and regularization estimates for  $(U_{\text{kin}}(t))_{t>0}$  are given by scaling the estimates from Theorem 1.8, or under the enlargement assumptions (LE), Theorem 1.19.

LEMMA 4.8 (**Decay and regularization of the kinetic semigroup**). — For any fixed decay rate  $\sigma \in (0, \sigma_0)$ , the kinetic part  $(U_{\text{kin}}(t))_{t>0}$  of the semigroup satisfies the decay and regularization estimates

$$\sup_{t>0} e^{2\sigma t/2} U_{\text{kin}}(t) f_{X^2} + \frac{1}{\varepsilon^2} \int_0^t e^{2\sigma \tau/2} U_{\text{kin}}(\tau) f_{X^2} \cdot dt \leq f_{X^2}$$

as well as

$$\frac{1}{\varepsilon^2} \int_0^t e^{2\sigma \tau/2} U_{\text{kin}}(\tau) f_{X^2} dt \leq f_{X^2},$$

with exactly the same estimate satisfied by the adjoint  $((U_{\text{kin}}(t))^*)_{t>0}$ .

As for the hydrodynamic semigroup  $U_{\text{hydro}}(\cdot)$ , we establish now suitable convolution estimates:

LEMMA 4.9 (**Decay and regularization of the convoluted kinetic semigroup**). — Consider  $T \in (0, \infty]$ . For any  $\varphi \in L^2([0, T]; H)$ , there holds uniformly in  $\varepsilon$

$$(4.16a) \quad \frac{1}{\varepsilon} U_{\text{kin}}(\cdot) \varphi_{F(T, \cdot, \cdot)} \leq \sup_{0 \leq t < T} \left\{ w_{\cdot}(t) \int_0^t \varphi(\tau)_{H^2}^2 d\tau \right\}^{\frac{1}{2}}.$$

Furthermore, consider  $\sigma \in [0, \sigma_0)$ , there holds uniformly in  $\varepsilon$

$$(4.16b) \quad \frac{1}{\varepsilon} U_{\text{kin}}(\cdot) \varphi_{X(\cdot, \cdot)} \leq \int_0^T e^{2\sigma \tau/2} \varphi(\tau)_{X^2}^2 dt \quad \mathbb{E}_{\frac{1}{2}}$$

for any  $\varphi$  for which the right-hand-side is finite.

*Proof.* — Denote by  $(Y, Y, Y^*, Y)$  either  $(H, H, H^*, H)$  or  $(X, X, X^*, X)$ . In a first step, we use a duality argument to prove the  $Y - Y$ -integral decay:

$$\frac{1}{\varepsilon^2} \int_0^T e^{2\sigma \tau/2} U_{\text{kin}}(\cdot) \varphi(t)_{Y^*} \cdot dt \leq \varepsilon^2 \int_0^T \varphi(t)_{Y^*}^2 dt.$$

and we deduce from it the  $Y - Y$ -uniform decay together with the stronger  $Y^* - Y$ -integral decay using an energy method in a second step:

$$\sup_{0 \leq t < T} e^{2\sigma t/2} U_{\text{kin}}(\cdot) \varphi(t)_{Y^*} + \frac{1}{\varepsilon^2} \int_0^T e^{2\sigma \tau/2} U_{\text{kin}}(\cdot) \varphi(\tau)_{Y^*} \cdot \leq \varepsilon^2 \int_0^T \varphi(\tau)_{Y^*}^2 dt.$$

Note that this proves (4.16b), and it is enough to prove (4.16a) as it follows from the particular case  $\sigma = 0$  and  $Y = H$ .

*Step 1: Integral decay in  $Y - Y^*$ .* — We will prove the following estimate uniformly in  $T \in (0, \infty]$  and  $\phi \in L^2([0, T]; Y)$ :

$$\|e^{t/2} (U_{\text{kin}}(\cdot) \varphi), \phi\|_{L^2([0, T]; Y)} \leq \varepsilon^2 \|\phi\|_{L^2([0, T]; Y)} \int_0^T e^{2t/2} \|\varphi(\tau)\|_{Y^*}^2 d\tau \leq \varepsilon^2 \|\phi\|_{L^2([0, T]; Y)}^2 \mathfrak{E}_{\frac{1}{2}},$$

and as in the proof of Lemma 3.12, it is enough to check that it holds for  $\phi$  of the form

$$\phi(t) = \begin{cases} \phi_0 & Y, \quad t \in [t_1, t_2] \\ 0, & t \notin [t_1, t_2] \end{cases}, \quad \|\phi\|_{L^2([0, T]; Y)} = \sqrt{t_2 - t_1} \|\phi_0\|_{Y^*}.$$

By duality, we have

$$\begin{aligned} & \|e^{t/2} (U_{\text{kin}}(\cdot) \varphi), \phi\|_{L^2([0, T]; Y)} \\ &= \int_{t_1}^{t_2} \int_0^{t-\tau} e^{t/2} U_{\text{kin}}(t - \tau) \varphi(\tau), \phi_0\|_{Y^*} d\tau dt \\ &= \int_{t_1}^{t_2} \int_0^{t-\tau} e^{t/2} \|\varphi(\tau)\|_{Y^*} \|e^{(t-\tau)/2} U_{\text{kin}}(t - \tau) \phi_0\|_{Y^*} d\tau dt \\ &\leq \int_{t_1}^{t_2} \int_0^t \|e^{t/2} \|\varphi(\tau)\|_{Y^*} \|e^{(t-\tau)/2} U_{\text{kin}}(t - \tau) \phi_0\|_{Y^*} d\tau dt, \end{aligned}$$

and so, using first Cauchy–Schwarz’s inequality and then Young’s convolution inequality in the form  $L^2([0, T]) \times L^1([0, T]) \hookrightarrow L^2([0, T])$ , there holds

$$\begin{aligned} & \|e^{t/2} (U_{\text{kin}}(\cdot) \varphi), \phi\|_{L^2([0, T]; Y)} \\ &\leq \sqrt{t_2 - t_1} \int_0^T \int_0^t \|e^{t/2} \|\varphi(\tau)\|_{Y^*} \|e^{(t-\tau)/2} U_{\text{kin}}(t - \tau) \phi_0\|_{Y^*} d\tau dt \leq 2 \mathfrak{E}_{\frac{1}{2}} \\ &\leq \sqrt{t_2 - t_1} \int_0^T \|e^{t/2} \|\varphi(t)\|_{Y^*}^2 dt \leq \mathfrak{E}_{\frac{1}{2}} \int_0^T \|e^{t/2} U_{\text{kin}}(t) \phi_0\|_{Y^*}^2 dt. \end{aligned}$$

Furthermore, using Cauchy–Schwarz’s inequality, we deduce for some  $\sigma \in (\sigma, \sigma_0)$

$$\|e^{t/2} (U_{\text{kin}}(\cdot) \varphi), \phi\|_{L^2([0, T]; Y)} \leq \varepsilon \sqrt{t_2 - t_1} \int_0^T \|e^{t/2} \|\varphi(t)\|_{Y^*}^2 dt \leq \mathfrak{E}_{\frac{1}{2}} \int_0^T \|e^{t/2} U_{\text{kin}}(t) \phi_0\|_{Y^*}^2 dt.$$

from which, using the  $Y^* - Y$ -integral estimate for  $((U_{\text{kin}}(t)))_{t>0}$  obtained in Lemma 4.8, we obtain

$$\|e^{t/2} (U_{\text{kin}}(\cdot) \varphi), \phi\|_{L^2([0, T]; Y)} \leq \varepsilon^2 \sqrt{t_2 - t_1} \int_0^T \|e^{t/2} \|\varphi(t)\|_{Y^*}^2 dt \leq \mathfrak{E}_{\frac{1}{2}} \|\phi_0\|_{Y^*}.$$

This concludes this step.

*Step 2: Regularized uniform and integral decay.* — Denote  $u(t) := U_{\text{kin}} \varphi(t) = U - P_{\text{kin}}\varphi(t)$ , it satisfies the evolution equation

$$\partial_t u = \frac{1}{\varepsilon^2} (L - \varepsilon v \cdot \nabla_x) u + P_{\text{kin}}\varphi, \quad u(0) = 0.$$

Note that, considering the decomposition  $L = (L - P) + P$  in the case  $Y = H$  (from Assumption **(L1)**–**(L4)**), or  $L = B + A$  in the case  $Y = X$  (from Assumption **(LE)**), the following degenerate dissipativity estimate holds for some  $\lambda > 0$ :

$$\text{Re } Lf, f_Y + \lambda \|f_Y\|^2 \leq \|f_Y\|^2.$$

We now write an energy estimate, using the skew-adjointness of  $v \cdot \nabla_x$ :

$$\frac{1}{2} \frac{d}{dt} \|u_Y\|^2 + \frac{\lambda}{\varepsilon^2} \|u_Y\|^2 \leq \|u_Y\|^2 + |P_{\text{kin}}\varphi, u_Y|.$$

Recall that  $P_{\text{kin}} = \text{Id} - P_{\text{hydro}}$ , where we know from the spectral analysis performed in Theorems 1.8 or 1.19 that  $P_{\text{hydro}} : B(Y; Y^*) \rightarrow B(Y)$  uniformly in  $\varepsilon$  from the embedding  $Y^* \hookrightarrow Y \hookrightarrow Y$ . Thus, we have  $P_{\text{kin}} : B(Y) \rightarrow B(Y)$  uniformly in  $\varepsilon$ , from which we deduce

$$\frac{1}{2} \frac{d}{dt} \|u_Y\|^2 + \frac{\lambda}{\varepsilon^2} \|u_Y\|^2 \leq \|u_Y + P_{\text{kin}}\varphi_Y\|^2 + \|\varphi_Y\| \|u_Y\|.$$

Therefore, multiplying by  $e^{2t/\varepsilon^2}$ , we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} e^{2t/\varepsilon^2} \|u_Y\|^2 + \frac{\lambda}{\varepsilon^2} e^{2t/\varepsilon^2} \|u_Y\|^2 \\ \leq \frac{1}{\varepsilon^2} e^{2t/\varepsilon^2} \|u_Y + \varepsilon e^{t/\varepsilon^2} \varphi_Y\|^2 + \frac{1}{\varepsilon} e^{t/\varepsilon^2} \|\varphi_Y\| \|u_Y\|, \end{aligned}$$

or more simply by Young’s inequality

$$\frac{1}{2} \frac{d}{dt} e^{2t/\varepsilon^2} \|u_Y\|^2 + \frac{\lambda}{2\varepsilon^2} e^{2t/\varepsilon^2} \|u_Y\|^2 \leq \frac{1}{\varepsilon^2} e^{2t/\varepsilon^2} \|u_Y + \varepsilon^2 e^{t/\varepsilon^2} \varphi_Y\|^2.$$

Integrating in time, we finally deduce from the previous step

$$\|u_{X(\cdot, \cdot)}\|^2 \leq \varepsilon^2 \int_0^T e^{2t/\varepsilon^2} \|\varphi(t)\|_Y^2 dt.$$

This concludes the proof of Lemma 4.9.

### 5. Bilinear theory

We come now to the main nonlinear estimates involving the various sti terms

$$[f, g] = \frac{1}{\varepsilon^2} U \cdot Q^{\text{sym}}(f, g).$$

We will exploit the decomposition of  $U(t)$  given in (2.12) and the associated non-linear decomposition

$$[f, g](t) = \text{hydro}[f, g](t) + \text{kin}[f, g](t),$$

with

$$[f, g](t) := \mathbb{P} \quad [f, g](t) = \frac{1}{\varepsilon} \int_0^t U(t - \tau) Q^{\text{sym}}(f(\tau), g(\tau)) d\tau.$$

We first need the following spatially inhomogeneous nonlinear estimates of  $Q$ .

**LEMMA 5.1 (Nonlinear Sobolev estimates for  $Q$ ).** — Denote  $Y = H$  under assumption **(B3)**, or  $Y = X$  under assumption **(BE)**. Consider  $s > \frac{d}{2}$  and recall that  $\alpha \in (0, \frac{1}{2})$  if  $d = 2$ , or  $\alpha = 0$  if  $d > 3$ . There holds

$$(5.1a) \quad \begin{aligned} & Q(f, g)_{\dot{H}_x^{-\alpha}(Y_v)} + Q(f, g)_{Y^*, s} \\ & \leq \|f\|_{Y^*, s} \| |x|^{\wedge -} g \|_{Y^*, s - (1-\alpha)} + \| |x|^{\wedge -} f \|_{Y^*, s - (1-\alpha)} \|g\|_{Y^*, s}, \end{aligned}$$

$$(5.1b) \quad \begin{aligned} & Q(f, g)_{\dot{H}_x^{-\alpha}(Y_v)} + Q(f, g)_{Y^*, s} \\ & \leq \|f\|_{Y^*, s} \| |x|^{\wedge -} g \|_{Y^*, s - (1-\alpha)} + \|f\|_{Y^*, s} \| |x|^{\wedge -} g \|_{Y^{s - (1-\alpha)}}. \end{aligned}$$

Furthermore, we have the following control when  $g \in W_x^{s, \cdot}(Y_v^*)$  for  $s > s^*$ :

$$(5.2) \quad Q(f, g)_{Y^*, s} \leq \|f\|_{Y^*, s} \|g\|_{W_x^{s, \cdot}(Y_v^*)} + \|f\|_{Y^*, s} \|g\|_{W_x^{s, \cdot}(Y_v)}.$$

*Remark 5.2.* — Because the estimates provided in the Lemma are involving fractional Sobolev spaces in the variable  $x$ , and due to the locality in  $x$  of the bilinear operator  $Q$ , the proof requires some estimates reminiscent to paradi erential calculus. This is not the case when dealing with mere  $H_x^k(X_v)$  spaces with  $k \in \mathbb{N}$  as in Lemma 7.2 where  $L^p$ -estimates and Sobolev embeddings will allow to recover the needed estimates.

*Proof.* — As just said, due to the locality in  $x$  of the bilinear operator  $Q$ , one can adapt classical results from paradi erential calculus, replacing the multiplication  $(u, v) \mapsto uv$  (resp. the modulus  $|\cdot|$ ) by the collision operator  $(u, v) \mapsto Q(u, v)$  (resp. the  $Y$ -norm). In particular, we redefine the homogeneous paraproduct and remainder (see Appendix B.1) as

$$\dot{T}uv = \sum_j Q(\dot{S}_{j-1}u, \dot{S}_jv), \quad \dot{R}(u, v) = \sum_{|j-k| \leq 1} Q(\dot{S}_k u, \dot{S}_jv),$$

which satisfy, under the assumption **(B3)** or **(BE)**, the estimates

$$\begin{aligned} & \|Q(\dot{S}_{j-1}u, \dot{S}_jv)\|_{L_x^p(Y_v)} \leq \| \dot{S}_{j-1}u \|_{Y_v^*} \| \dot{S}_jv \|_{Y_v} \|_{L_x^p} + \| \dot{S}_{j-1}u \|_{Y_v} \| \dot{S}_jv \|_{Y_v^*} \|_{L_x^p}, \\ & \|Q(\dot{S}_k u, \dot{S}_jv)\|_{L_x^p(Y_v)} \leq \| \dot{S}_k u \|_{Y_v^*} \| \dot{S}_jv \|_{Y_v} \|_{L_x^p} + \| \dot{S}_k u \|_{Y_v} \| \dot{S}_jv \|_{Y_v^*} \|_{L_x^p}. \end{aligned}$$

One then checks that (5.2) is the  $Q$ -version of Proposition B.3. Furthermore, denoting for compactness  $s_1 = 1 - \alpha \in (0, \frac{d}{2})$ , one gets from the  $Q$ -version of Proposition B.1 with  $s_1 = \frac{d}{2} - 1 \in [0, \frac{d}{2})$  and  $s_2 = s$ , so that  $s_1 + s_2 - \frac{d}{2} = -\alpha$ :

$$\begin{aligned} & Q(f, g)_{\dot{H}_x^{-\alpha}(Y_v)} \leq \|f\|_{\dot{H}_x^{\frac{d}{2}-1}(Y_v)} \| |x|^{\wedge -} g \|_{L_x^2(Y_v^*)} + \| |x|^{\wedge -} f \|_{L_x^2(Y_v^*)} \|g\|_{\dot{H}_x^{\frac{d}{2}-1}(Y_v)} \\ & \leq \|f\|_{H_x^s(Y_v)} \| |x|^{\wedge -} g \|_{L_x^2(Y_v^*)} + \| |x|^{\wedge -} f \|_{L_x^2(Y_v^*)} \|g\|_{H_x^s(Y_v)}. \end{aligned}$$

We now turn to the estimate in  $Y^{s-} = H_x^s(Y_v)$ . Using **(B3)** or **(BE)**, we have

$$\begin{aligned} Q(f, g)_{H_x^s(Y_v)} &= \left\| \xi^{-s} \int_{\mathbb{R}^d} Q \widehat{f}(\xi - \zeta) \widehat{g}(\zeta) \widehat{S} \, d\zeta \right\|_{L_\xi^2(Y_v)} \\ &\leq \left\| \xi^{-s} \int_{\mathbb{R}^d} \|\widehat{f}(\xi - \zeta)\|_{Y_v} \|\widehat{g}(\zeta)\|_{Y_v} \, d\zeta \right\|_{L_\xi^2} \\ &\quad + \left\| \xi^{-s} \int_{\mathbb{R}^d} \|\widehat{f}(\xi - \zeta)\|_{Y_v} \|\widehat{g}(\zeta)\|_{Y_v} \, d\zeta \right\|_{L_\xi^2} =: I_1 + I_2. \end{aligned}$$

We split the frequency weight as

$$\xi^{-s} = 1 + |\xi|^s \cdot (1 + |\xi - z|^s + |\zeta|^s) \cdot |\xi - z|^{-s-} \cdot \xi^{-z^{-s-}} + z^s,$$

since  $s - > 0$ , which allows to control the term  $I_1$  as follows:

$$\begin{aligned} I_1 &\leq \left\| \int_{\mathbb{R}^d} \left[ |\xi - z|^{-s-} \|\widehat{f}(\xi - z)\|_{Y_v} \right] \widehat{g}(z) \, dz \right\|_{L_\xi^2} \\ &\quad + \left\| \int_{\mathbb{R}^d} \|\widehat{f}(\xi - z)\|_{Y_v} \left[ |\zeta|^s \widehat{g}(z) \right] \, dz \right\|_{L_\xi^2}. \end{aligned}$$

Using Young’s convolution inequality  $L^2 \cdot L^1 \subset L^2$  we deduce that

$$\begin{aligned} I_1 &\leq \left\| |\xi|^{-s-} \|\widehat{f}\|_{Y_v} \right\|_{L_\xi^2} \|\widehat{g}(\xi)\|_{Y_v} + \left\| \|\widehat{f}(\xi)\|_{Y_v} \right\|_{L_\xi^1} \|\xi^{-s} \widehat{g}\|_{Y_v} \\ &\leq \|\widehat{f}\|_{H_x^{s-}(Y_v)} \|\widehat{g}\|_{L_\xi^1(Y_v)} + \|\widehat{f}\|_{L_\xi^1(Y_v)} \|g\|_{H_x^s(Y_v)}. \end{aligned}$$

Using the fact that  $\xi^{-s-} \in L^2$  (resp.  $|\xi|^{-s-} \in L^2$ ), a simple use of Cauchy-Schwarz’s inequality allows to estimate  $L^1$ -norms with weighted  $L^2$ -norms resulting in

$$I_1 \leq \|\widehat{f}\|_{H_x^{s-}(Y_v)} \|g\|_{H_x^s(Y_v)}.$$

We prove in the exact same way that

$$I_2 \leq \|f\|_{H_x^s(Y_v)} \|\widehat{g}\|_{H_x^{s-}(Y_v)},$$

thus (5.1a) is proved, and the proof of (5.1b) is similar. The proofs of (7.4) and (7.5) are also similar. This concludes the proof of Lemma 5.1.

### 5.1. Bilinear and linear hydrodynamic estimates

We have all in hands to estimate the bilinear “hydrodynamic” operator  $U_{\text{hydro}}(f, g) = {}^1U_{\text{hydro}}(\cdot) \circ Q^{\text{sym}}(f, g)$ . **The results of this section hold under any assumption **(B3)**, **(BE)** or **(BED)**.** They are based upon the above properties of  $Q$  as well as the results of Section 4 on the various semigroups involved:

PROPOSITION 5.3 (**General bilinear hydrodynamic estimates**). — The bilinear operator  $\text{hydro}$  satisfies the following continuity estimates in  $H$  when at least one argument is in  $H$  :

$$(5.3a) \quad \left\| \text{hydro}[f, g] \right\|_H \cdot w, (T)^{-1} f_H g_H,$$

$$(5.3b) \quad \left\| \text{hydro}[f, g] \right\|_H \cdot \varepsilon w, (T)^{-1} f_H g_F,$$

$$(5.3c) \quad \left\| \text{hydro}[f, g] \right\|_H \cdot \varepsilon f_H g_X,$$

as well as the following ones when at least one argument is in  $F$  :

$$(5.4a) \quad \left\| \text{hydro}[f, g] \right\|_H \cdot \varepsilon w, (T)^{-1} f_F g_F,$$

$$(5.4b) \quad \left\| \text{hydro}[f, g] \right\|_H \cdot \varepsilon f_F g_X,$$

and the following one when both arguments are in  $X$  :

$$(5.5) \quad \left\| \text{hydro}[f, g] \right\|_H \cdot \varepsilon f_X g_X.$$

Furthermore, it is strongly continuous at  $t = 0$ :

$$\lim_t \left\| \text{hydro}(f, g)(t) \right\|_H = 0$$

in all cases considered above.

*Proof.* — We recall the definition of  $\text{hydro}$  and the orthogonality property of  $Q$ :

$$\text{hydro}[f, g](t) = \frac{1}{\varepsilon} \int_0^t U_{\text{hydro}}(t - \tau) Q(f(\tau), g(\tau)) d\tau, \quad PQ = 0,$$

thus, denoting for compactness  $w = w_{\cdot, \cdot}$ , the convolution estimate (4.2) gives

$$(5.6) \quad \left\| \text{hydro}[f, g] \right\|_{H^s}^2 \cdot \sup_{0 \leq t < T} w(t)^2 \int_0^t Q(f(\tau), g(\tau))_{Y, s}^2 \dot{H}_x^{-\alpha}(Y_v) d\tau,$$

where  $(Y, Y) = (H, H), (X, X)$  or  $(X_{-1}, X_{-1})$ , and we recall that  $w$  is non-increasing and bounded from above and below:

$$(5.7) \quad 0 \leq t_1 \leq t_2 < T, \quad 0 < w(T) \leq w(t_2) \leq w(t_1) \leq 1.$$

The continuity at  $t = 0$  will be an easy consequence of the estimate (5.6).

*Step 1: Proof of (5.3) for  $f \in H$ .* — When  $g \in H$ , we combine (5.6) with the bilinear estimate (5.1a) for  $Q$ , to deduce the following, where  $\alpha := 1 - \alpha$

$$\begin{aligned} \left\| \text{hydro}[f, g] \right\|_H^2 \cdot \sup_{0 \leq t < T} w(t)^2 \int_0^t & \left\{ f(\tau)_{H^s}^2 / |x| g(\tau)_{H^{s-\alpha}}^2 \right. \\ & \left. + |x| f(\tau)_{H^{s-\alpha}}^2 g(\tau)_{H^s}^2 \right\} d\tau \\ & \cdot \sup_{0 \leq t < T} \int_0^t \left[ w^2(\tau) f(\tau)_{H^s}^2 / |x| g(\tau)_{H^{s-\alpha}}^2 \right. \\ & \left. + |x| f(\tau)_{H^{s-\alpha}}^2 w(\tau)^2 g(\tau)_{H^s}^2 \right] d\tau \end{aligned}$$

where we used (5.7) in the second inequality. Recalling that

$$h_{H^s}^2 := \sup_{0 \leq t < T} \int_{\mathbb{S}} w(t)^2 h(t)_{H^s}^2 + w(t)^2 \int_0^t \left\| |x|^{1-\cdot} h(\tau) \right\|_{H^s}^2 d\tau,$$

and using  $H^s \subset H$  and (5.7), we deduce

$$\begin{aligned} \left\| \text{hydro}(f, g) \right\|_{H^s}^2 &\leq f_{H^s}^2 \int_0^T \left\| |x| g(t) \right\|_{H^s}^2 dt + g_{H^s}^2 \int_0^T \left\| |x| f(t) \right\|_{H^s}^2 dt \\ &\leq w(T)^{-2} f_{H^s}^2 g_{H^s}^2 \end{aligned}$$

which is exactly (5.3a). When  $g \in F$ , using furthermore  $H^s \subset H$ , we similarly have (5.3b):

$$\begin{aligned} \left\| \text{hydro}[f, g] \right\|_{H^s}^2 &\leq \varepsilon^2 \int_0^T (w(\tau) f(\tau)_{H^s})^2 \frac{1}{\varepsilon} g(\tau)_{H^s}^2 d\tau \\ &\leq \varepsilon^2 f_{H^s}^2 \int_0^T \frac{1}{\varepsilon} g(\tau)_{H^s}^2 d\tau \leq \varepsilon^2 w(T)^{-2} f_{H^s}^2 g_{F^s}^2, \end{aligned}$$

which gives now (5.3b). In the same way, using also  $H^s \subset X^s$ , we have (5.3c).

*Step 2: Proof of (5.4) and (5.5).* — When  $f \in F$  and  $g \in X^s$ , we combine (5.6) with the bilinear estimate (5.1a) for  $Q$ . Using that  $H^s \subset X^s$  and the property (5.7) of  $w$ , we have:

$$\begin{aligned} \left\| \text{hydro}(f, g) \right\|_{H^s}^2 &\leq \varepsilon^2 \sup_{0 \leq t < T} \left\{ w(t)^2 \int_0^t f(\tau)_{H^s}^2 \frac{1}{\varepsilon} g(\tau)_{X^s}^2 \right. \\ &\quad \left. + \frac{1}{\varepsilon} f(\tau)_{H^s}^2 g(\tau)_{X^s}^2 d\tau \right\} \\ &\leq \varepsilon^2 \int_0^T w(\tau)^2 f(\tau)_{H^s}^2 \frac{1}{\varepsilon} g(\tau)_{X^s}^2 dt \\ &\quad + \varepsilon^2 \sup_{0 \leq t < T} \left\{ w(t)^2 \int_0^t \frac{1}{\varepsilon} f(\tau)_{H^s}^2 g(\tau)_{X^s}^2 d\tau \right\} \\ &\leq \varepsilon^2 f_{F^s}^2 g_{X^s}^2. \end{aligned}$$

This proves (5.4b) The proofs of (5.4a) and (5.5) are similar and omitted.

**PROPOSITION 5.4 (Special bilinear hydrodynamic estimates).** — When  $f \in H$  and  $\phi$  is the parameter defining the  $H$ -norm

$$(5.8) \quad \left\| \text{hydro}[f, \phi] \right\|_{H^s} \leq \eta f_{H^s}.$$

Furthermore, when  $g_{\text{disp}} = U_{\text{disp}}(\cdot)g$  where  $g = P_{\text{wave}}g \in H^s \dot{H}_X^{-s}(H_V)$ , there holds

$$(5.9) \quad \left\| \text{hydro}[f, g_{\text{disp}}] \right\|_{H^s} \leq \beta_{\text{disp}}(g, \varepsilon) f_{H^s}, \quad \lim_0 \beta_{\text{disp}}(g, \varepsilon) = 0,$$

and in the case  $d > 3$ , the rate of convergence is explicit assuming  $g \in \dot{B}_{1,1}^{s+(d+1)/2}(H_\nu)$   $H^s$  for some  $s > s$ :

$$\beta_{\text{disp}}(g, \varepsilon) \leq \bar{\varepsilon} \left( \|g\|_{H^s} + \|g\|_{\dot{B}_{1,1}^{s+(d+1)/2}(H_\nu)} \right).$$

*Proof.* — We start by proving (5.8) and then prove (5.9). We use again the short-hand notation  $w = w$ , and recall that, besides (5.6), the convolution estimate (4.2) also leads to

$$(5.10) \quad \left\| \text{hydro}[f, g] \right\|_{H^s}^2 \leq \sup_{0 \leq t < T} w(t)^2 \int_0^t Q(f(\tau), g(\tau)) \frac{2}{1+\alpha} d\tau \ll \dots$$

*Step 1: Proof of (5.8).* — We combine the convolution estimate (5.6) with the nonlinear bound (5.1b), and use  $H^{\bullet-\alpha} \subset H$  to obtain (where we denote for compactness  $\alpha = 1 - \alpha$ )

$$\left\| \text{hydro}[f, \phi] \right\|_{H^s}^2 \leq \sup_{0 \leq t < T} w(t)^2 \int_0^t \left[ w(\tau) f(\tau) \right]_{H^{\bullet,s}}^2 \left[ w(\tau)^{-1} |x| \phi(\tau) \right]_{H^{\bullet,s-\alpha}}^2 d\tau$$

$$\leq \|f\|_{H^s}^2 \sup_{0 \leq t < T} \left\{ w(t)^2 \int_0^t \left[ w(\tau)^{-1} |x| \phi(\tau) \right]_{H^{\bullet,s-\alpha}}^2 d\tau \right\}$$

and then, using (2.3), we finally get  $\text{hydro}(f, \phi) \in H^s$  with  $\| \cdot \|_{H^s} \leq \eta^2 \|f\|_{H^s}$ , which is exactly (5.8).

*Step 2: Proof of (5.9).* — As in the previous step, but using the nonlinear bound (5.1b) for  $Q$ , one has

$$\left\| \text{hydro}[f, g_{\text{disp}}] \right\|_{H^s}^2 \leq \|f\|_{H^s}^2 \int_0^T \left\| |x| g_{\text{disp}}(t) \right\|_{H^{\bullet,s-\alpha}}^2 dt,$$

which, according to (4.3) and (4.4), satisfies for some universal  $\kappa > 0$

$$(5.11) \quad \left\| \text{hydro}[f, g_{\text{disp}}] \right\|_{H^s}^2 \leq \|f\|_{H^s}^2 \varepsilon \sup_{\text{supp } \hat{g} \subset H_\nu} \int_0^T |\xi|^\rho e^{-t|\xi|^\rho} dt.$$

Since the supremum term is bounded uniformly in  $T \in (0, \infty)$  and  $g \in H^s \cap \dot{H}_x^{-\alpha}(H_\nu)$ , it is enough to prove (5.9) in the case where  $g \in H^s \cap \dot{B}_{1,1}^{s+(d+1)/2}(H_\nu)$  for some  $s > s$  and  $\xi \in \text{supp } \hat{g}(\xi) \subset H_\nu$  is supported away from 0, as it will allow to conclude by a density argument. We assume therefore there is some  $\nu > 0$  such that

$$|\xi| \geq \nu, \quad \hat{g}(\xi) = 0,$$

and we point out that when  $T < \infty$ , using  $e^{-t|\xi|^\rho} \leq t^{-1} R^{-1}$ , for any  $R > 0$

$$\sup_{|\xi| \geq \nu} \int_R^T |\xi|^\rho e^{-t|\xi|^\rho} dt \leq \int_R^T \frac{dt}{t} = \log(T) - \log(R),$$

and when  $T = \infty$

$$\sup_{|\xi| \geq \nu} \int_R^\infty |\xi|^\rho e^{-t|\xi|^\rho} dt = \sup_{|\xi| \geq \nu} \int_{R|\xi|^\rho}^\infty e^{-t} dt \leq \exp(-R\nu^\rho),$$



so that, in both cases, we have

$$C(R, T) := \sup_{|I| \geq R} \int_R^T |\xi|^2 e^{-t|\xi|^2} dt \stackrel{R \ll T}{\sim} 0.$$

We split for some  $0 < R < T$  the nonlinear term:

$$Q(g_{\text{disp}}, f) = \mathbf{1}_{0 \leq t \leq R} Q(g_{\text{disp}}, f) + \mathbf{1}_{R \leq t < T} Q(g_{\text{disp}}, f) =: \varphi^-(t) + \varphi^+(t),$$

so that, using an estimate analogous to (5.11) we have

$$\frac{1}{\varepsilon} \left\| U_{\text{hydro}}(\cdot) \varphi^+ \right\|_{H^s} \leq C \left( \|f\|_{H^s}^2 + \|g\|_{\dot{H}_x^{-\alpha}(H_v)}^2 \right) \stackrel{S_2}{\leq} C(R, T).$$

Furthermore, combining this time (5.10) with the boundedness estimate (4.1) (resp. the dispersive estimate (4.14))  $U_{\text{disp}}(\cdot)$ , together with the corresponding nonlinear bound (5.1) (resp. (5.2)) for  $Q$ , we have for  $\varphi^-(t)$

$$(5.12) \quad \frac{1}{\varepsilon} \left\| U_{\text{hydro}}(\cdot) \varphi^- \right\|_{H^s} \leq C \left( \|f\|_{H^s}^2 + \|g\|_{\dot{H}_x^{-\alpha}(H_v)}^2 \right) \times \int_0^R 1 - \left(\frac{\varepsilon}{t}\right)^{\frac{d-1}{1+\alpha}} dt \stackrel{E_{1+}}{\leq} C(R, T).$$

Put together, the two previous controls yield uniformly in  $R \in (0, T)$

$$\left\| U_{\text{hydro}}[f, g_{\text{disp}}] \right\|_{H^s}^2 \leq C \left( \|f\|_{H^s}^2 + \|g\|_{\dot{H}_x^{-\alpha}(H_v)}^2 \right) \int_0^R 1 - \left(\frac{\varepsilon}{t}\right)^{\frac{d-1}{1+\alpha}} dt + \frac{1}{\varepsilon} \left( \|f\|_{H^s}^2 + \|g\|_{\dot{H}_x^{-\alpha}(H_v)}^2 \right) C(R, T).$$

Letting  $\varepsilon \rightarrow 0$  and then  $R \rightarrow T$ , one deduces (5.9) for  $g \in \dot{B}_{1,1}^{s+(d+1)/2}(H_v)$  with  $s > s_c$  and whose Fourier transform is supported away from 0. We conclude to the general case of  $g \in \dot{H}_x^{-s}$  by density thanks to (5.11). Note that in dimension  $d > 3$ , there holds  $\alpha = 0$ , thus one has

$$\int_0^T 1 - \left(\frac{\varepsilon}{t}\right)^{d-1} dt = \int_0^T dt + \varepsilon^{d-1} \int_0^T t^{1-d} dt \sim \varepsilon.$$

This concludes the proof of Proposition 5.4.

### 5.2. Bilinear kinetic and mixed estimates

The results of this section hold assuming **(B3)**. We point out that only one estimate, namely (5.15a), holds assuming **(B3)** or **(BE)** but not **BED**, which is why it has to be treated using the alternative strategy of Section 7. We have the analogue of Proposition 5.3 for the kinetic bilinear operator  $\text{kin}(f, g) = \mathbb{1} U_{\text{kin}} Q^{\text{sym}}(f, g)$ .

**PROPOSITION 5.5 (General bilinear kinetic and mixed estimates).** — *The bilinear operator  $\text{kin}$  satisfies the following continuity estimates in the mixed space  $F$  when at least one argument is in  $F$ :*

$$(5.13) \quad \text{kin}[f, g]_F \leq \varepsilon \omega_\varepsilon(T)^{-1} \|f\|_F \min \{ \|g\|_F, \|g\|_{H^s} \},$$

and the following one when  $f, g \in H$  (note the absence of a factor  $\varepsilon$ ):

$$(5.14) \quad \text{kin}[f, g]_{F \cdot w, (T)^{-1} f \in H, g \in H}.$$

Furthermore, considering  $X = H$  in the definition of  $\times$  under Assumption **(B3)**, or considering  $X$  to be the space from **(BE)** under this assumption, there holds in the kinetic space  $\times$  when at least one argument is in  $\times$

$$(5.15a) \quad \text{kin}[f, g]_{\times} \cdot \varepsilon f_{\times} g_{\times},$$

$$(5.15b) \quad \text{kin}[f, g]_{\times} \cdot \varepsilon w, (T)^{-1} f_{\times} \min \{ g_{F}, g_{H} \}.$$

Finally, it is strongly continuous at  $t = 0$  in the sense that in the corresponding cases

$$\lim_{t \rightarrow 0} \text{kin}[f, g](t)_{\times} = 0, \quad \lim_{t \rightarrow 0} \text{kin}[f, g](t)_{H} = 0.$$

*Proof.* — Recall the definition of  $\text{kin}$ :

$$\text{kin}[f, g](t) = \frac{1}{\varepsilon} \int_0^t U_{\text{kin}}(t - \tau) Q(f(\tau), g(\tau)) d\tau,$$

thus denoting for compactness  $w, (t) = w(t)$ , the convolution estimates (4.16b) and (4.16a) give respectively

$$(5.16) \quad \text{kin}[f, g]_{\times}^2 \cdot \int_0^T e^{2-t/2} Q(f(t), g(t))_{\times}^2 dt,$$

and

$$(5.17) \quad \text{kin}[f, g]_{F \cdot w}^2 \cdot \sup_{0 < t < T} w(t)^2 \int_0^T Q(f(\tau), g(\tau))_{H}^2 dt \ll$$

We also recall the bound (5.7) for  $w$ . The continuity at  $t = 0$  will be immediate from the estimates below by letting  $T \rightarrow 0$ . For the reader convenience, we also recall the definitions of  $\cdot_{\times}$  and  $\cdot_{F}$ :

$$f_{\times}^2 := \sup_{0 < t < T} e^{2-t/2} f(t)_{\times}^2 + \frac{1}{\varepsilon^2} \int_0^T e^{2-t/2} f(t)_{\times}^2 dt,$$

$$g_{F \cdot w}^2 := \sup_{0 < t < T} w(t)^2 g(t)_{H}^2 + \frac{w(t)^2}{\varepsilon^2} \int_0^t g(\tau)_{H}^2 d\tau.$$

*Step 1: Proof of (5.15) for  $f \in \times$ .* — On the one hand, if  $g \in \times$ , combining the estimate (5.16) with the bilinear estimate (5.1a) for  $Q$ , one has:

$$\begin{aligned} \text{kin}[f, g]_{\times}^2 &\cdot \varepsilon^2 \int_0^T \frac{1}{\varepsilon} e^{-t/2} f(t)_{\times} \cdot g(t)_{\times}^2 \\ &+ f(t)_{\times} \cdot \frac{1}{\varepsilon} e^{-t/2} g(t)_{\times}^2 dt \\ &\cdot \varepsilon^2 f_{\times}^2 g_{\times}^2, \end{aligned}$$

which is (5.15a). Similarly, when  $g \in F$ , we have

$$\text{kin}[f, g]_{\times}^2 \cdot \varepsilon^2 \int_0^T \frac{1}{\varepsilon} e^{-t/2} f(t)_{\times} \cdot g(t)_{H}^2$$

$$\begin{aligned} & + \epsilon e^{-t/\epsilon^2} \|f(t)\|_{X^*}^2 \frac{1}{\epsilon} \|g(t)\|_{H^*}^2 \ll \|w(T)\|_{H^*}^2 \\ & \cdot \epsilon^2 w(T)^{-2} \|f\|_{X^*}^2 \|g\|_{H^*}^2, \end{aligned}$$

where we used (5.7). This proves (5.15b) for  $g \in H^*$ . On the other hand, if  $g \in H$ , using furthermore  $X^* \hookrightarrow X$  and  $H^* \hookrightarrow X^*$ , we have

$$\| \text{kin}[f, g] \|_{X^*}^2 \leq \epsilon^2 \int_0^T \frac{1}{\epsilon} e^{-t/\epsilon^2} \|f(t)\|_{X^*}^2 \|g(t)\|_{H^*}^2 dt \leq \epsilon^2 w(T)^{-2} \|f\|_{X^*}^2 \|g\|_{H^*}^2$$

which gives (5.15b).

*Step 2: Proof of (5.13) for  $f \in F$ .* — In the case  $f, g \in F$ , combining the estimate (5.17) with the bilinear estimate (5.1a), and using the bound (5.7) for  $w$ , we have

$$\| \text{kin}[f, g] \|_{F^*}^2 \leq \epsilon^2 \int_0^T \frac{1}{\epsilon} \|f(t)\|_{H^*}^2 \left[ \|w(t)g(t)\|_{H^*}^2 + \|w(t)f(t)\|_{H^*}^2 \right] \frac{1}{\epsilon} \|g(t)\|_{H^*}^2 dt$$

which readily gives (5.13) for  $f, g \in F$ . In the case  $f \in F$  and  $g \in H$ , using furthermore  $H^* \hookrightarrow H$ , we have

$$\| \text{kin}[f, g] \|_{F^*}^2 \leq \epsilon^2 \int_0^T \frac{1}{\epsilon} \|f(t)\|_{H^*}^2 \|w(t)g(t)\|_{H^*}^2 dt \leq \epsilon^2 w(T)^{-2} \|f\|_{F^*}^2 \|g\|_{H^*}^2.$$

This shows (5.13). Similarly, using the nonlinear estimate (5.1b) for  $Q$ , denoting for compactness  $\alpha = 1 - \alpha$ , we have

$$\| \text{kin}[f, g] \|_{F^*}^2 \leq \int_0^T \|w(t)f(t)\|_{H^*}^2 \|g(t)\|_{H^*,s}^2 dt \leq w(T)^{-2} \|f\|_{H^*}^2 \|g\|_{H^*}^2.$$

This proves (5.14) and concludes the proof of Proposition 5.5.

This next proposition is proved as Proposition 5.4 and its proof is omitted.

**PROPOSITION 5.6 (Special bilinear mixed estimates).** — *When  $f \in H$  and  $\phi$  is the parameter defining the  $H$ -norm*

$$(5.18) \quad \| \text{kin}[f, \phi] \|_{F^*} \leq \eta \|f\|_{H^*},$$

*furthermore, when  $g_{\text{disp}} = U_{\text{disp}}(\cdot)g$  where  $g = P_{\text{disp}}g \in H^s \dot{H}_x^- (H_v)$ , there holds*

$$(5.19) \quad \left\| \text{kin}[f, g_{\text{disp}}] \right\|_{F^*} \leq \beta_{\text{disp}}(g, \epsilon) \|f\|_{H^*}, \quad \lim_0 \beta_{\text{disp}}(g, \epsilon) = 0,$$

*and in the case  $d > 3$ , the rate of convergence is explicit assuming  $g \in \dot{B}_{1,1}^{s+(d+1)/2} (H_v)$   $H^{s^*}$  for some  $s > s$ :*

$$\beta_{\text{disp}}(g, \epsilon) \leq \bar{\epsilon} \left( \|g\|_{H^s} + \|g\|_{\dot{B}_{1,1}^{s+(d+1)/2} (H_v)} \right).$$

## 6. Proof of Theorems 1.12 and 1.20

In this section, we denote  $X = H$  under the sole assumptions **(L1)**–**(L4)** and **(B1)**–**(B3)**, and  $X$  is the space from assumptions **(LE)** and **(BE)** under these extra assumptions.

In this section, we construct a solution of the perturbed equation and then show it must be unique. We follow the approach described in Section 2. We refer more specifically to Section 2.6 that we briefly resume here. Recall that we look for a solution of the form

$$(6.1) \quad \begin{aligned} f(t) &= f_{\text{kin}}(t) + f_{\text{mix}}(t) + f_{\text{hydro}}(t) \\ &= f_{\text{kin}}(t) + f_{\text{mix}}(t) + f_{\text{disp}}(t) + f_{\text{NS}}(t) + g(t) \end{aligned}$$

where  $f_{\text{NS}}(\cdot)$  as well as  $f_{\text{in}}$  (and thus  $f_{\text{disp}}(t) = U_{\text{disp}}(t)f_{\text{in}}$ ) are functions to be considered as fixed parameters since they depend only on the initial datum  $f_{\text{in}}$  (and  $\varepsilon$ ). We point out that by Lemma 4.8, Lemma 4.1 and Lemma C.5 respectively

$$U_{\text{kin}}(\cdot)f_{\text{in}} \times \cdot 1, \quad \|f_{\text{disp}}\|_H \cdot 1, \quad f_{\text{NS}} \in H \cdot 1,$$

without those quantities being necessarily small. The smallness necessary to our fixed-point argument is coming from Proposition 5.4 which yields

$$\| \text{hydro}(f_{\text{NS}}, \cdot) \|_{B(H)} + \| \text{kin}(f_{\text{NS}}, \cdot) \|_{B(H;F)} \cdot \eta,$$

and from Proposition 5.6 which yields

$$\lim_0 \left( \| \text{hydro}(f_{\text{disp}}, \cdot) \|_{B(H)} + \| \text{kin}(f_{\text{disp}}, \cdot) \|_{B(H;F)} \right) = 0.$$

From now on, we work with the space  $F$  and  $H$  associated to the solution  $f_{\text{NS}}$  which corresponds, in Eq. (2.2), to the choice of the weight function

$$w_{f_{\text{NS}}}(t) = \exp \left( \frac{1}{2\eta^2} \int_0^t \| |x|^{\lambda-1} f_{\text{NS}}(\tau) \|_{H^{*,s}}^2 d\tau \right) \quad t > 0,$$

with  $\eta > 0$  still to be chosen.

We recall that we showed in Section 2.6 that solving equation

$$f(t) = U(t)f_{\text{in}} + [f, f](t)$$

can be reformulated, under the above *ansatz*, into the system of coupled nonlinear equations

$$(6.2) \quad \begin{cases} f_{\text{kin}}(t) &= U_{\text{kin}}(t)f_{\text{in}} + \text{kin}[f_{\text{kin}}, f_{\text{kin}}](t) + 2 \text{kin}[f_{\text{kin}}, f_{\text{hydro}} + f_{\text{mix}}](t), \\ f_{\text{mix}}(t) &= \text{kin}[(f_{\text{NS}} + f_{\text{disp}}) + f_{\text{mix}} + g; (f_{\text{NS}} + f_{\text{disp}}) + f_{\text{mix}} + g](t), \\ g(t) &= [f_{\text{kin}}, f_{\text{mix}}](t)g(t) + \text{hydro}[g, g](t) + S(t), \end{cases}$$

where the source term  $S(t)$  is defined through (2.15). We construct a solution  $(f_{\text{kin}}, f_{\text{mix}}, g)$  of this system in the space  $X \times F \times H$  and more specifically, in a product of the following balls for some small radii  $c_2, c_3 > 0$ :



$$\left\| \text{hydro} [h, f_{\text{NS}} + f_{\text{disp}} + g_{\text{mix}} + g_{\text{kin}}] \right\|_H \cdot (\eta + \beta_{\text{disp}}(\mathbf{P}_{\text{disp}} f_{\text{in}}, \varepsilon) + \varepsilon w_{f_{\text{NS}}}, (T)^{-1} g_{\text{mix}} + \varepsilon g_{\text{kin}}) h_H,$$

as well as the following stability estimate:

$$\begin{aligned} & \left\| \text{hydro} [h, f_{\text{NS}} + f_{\text{disp}} + g_{\text{mix}} + g_{\text{kin}}] - \text{hydro} [h, f_{\text{NS}} + f_{\text{disp}} + g_{\text{mix}} + g_{\text{kin}}] \right\|_H \\ & \cdot h - h_H (\eta + \beta_g(\varepsilon) + \varepsilon w_{f_{\text{NS}}}, (T)^{-1} g_{\text{mix}} + \varepsilon g_{\text{kin}}) \\ & + h_H (\varepsilon w_{f_{\text{NS}}}, (T)^{-1} g_{\text{mix}} - g_{\text{mix}} + \varepsilon g_{\text{kin}} - g_{\text{kin}}). \end{aligned}$$

*Proof.* — The two estimates are direct consequences of Propositions 5.3 and 5.4 since

$$\begin{aligned} & \text{hydro} [h, f_{\text{NS}} + f_{\text{disp}} + g_{\text{mix}} + g_{\text{kin}}] - \text{hydro} [h, f_{\text{NS}} + f_{\text{disp}} + g_{\text{mix}} + g_{\text{kin}}] \\ & = \text{hydro} [h - h, f_{\text{NS}} + f_{\text{disp}} + g_{\text{mix}} + g_{\text{kin}}] \\ & \quad + 2 \text{hydro} [h, g_{\text{mix}} - g_{\text{mix}}] + 2 \text{hydro} [h, g_{\text{kin}} - g_{\text{kin}}]. \end{aligned}$$

This proves the result.

The first part  $S_1$  of the source term  $S$  which depends only on the initial data  $f_{\text{in}}$  and the Navier-Stokes solution  $f_{\text{NS}}$  (but not on the partial solutions  $f_{\text{kin}}$ ,  $f_{\text{mix}}$  or  $g$ ) is estimated in this next lemma.

**LEMMA 6.2 (Estimate of the first source term  $S_1$ ).** — Consider some  $f_{\text{in}} \in X$ . The source term  $S_1$  satisfies

$$S_1 \in H \hookrightarrow \beta_{\text{NS}}(f_{\text{NS}}, f_{\text{in}}, \varepsilon), \quad \lim_0 \beta_{\text{NS}}(f_{\text{NS}}, f_{\text{in}}, \varepsilon) = 0.$$

If we assume additionally that the initial data  $f_{\text{in}}$  lies in  $X^{s+\delta} \cap \dot{H}_x^{-\alpha}(X_v)$  for some  $\delta \in (0, 1]$ , then the rate of convergence can be made explicit as

$$\begin{aligned} \beta_{\text{NS}}(f_{\text{NS}}, f_{\text{in}}, \varepsilon) \leq & \varepsilon \left( 1 + \|f_{\text{in}}\|_{X^{s+\delta}} + \|f_{\text{in}}\|_{\dot{H}_x^{-\alpha}(X_v)} \right. \\ & \left. + \|f_{\text{NS}}\|_{L^2([0, T]; H^{s+\delta})} + \|x f_{\text{NS}}\|_{L^2([0, T]; H^{s+\delta})} \right)^3. \end{aligned}$$

*Proof.* — Recalling that  $U_{\text{hydro}}(t) = U_{\text{NS}}(t) + U_{\text{wave}}(t)$ , we write the source term  $S_1(t)$  as

$$\begin{aligned} S_1(t) = & (U_{\text{wave}}(t) f_{\text{in}} - U_{\text{disp}}(t) f_{\text{in}}) + (U_{\text{NS}}(t) f_{\text{in}} - U_{\text{NS}}(t) f_{\text{in}}) \\ & + \text{wave} [f_{\text{NS}}, f_{\text{NS}}](t) + (\text{NS} [f_{\text{NS}}, f_{\text{NS}}](t) - \text{NS} [f_{\text{NS}}, f_{\text{NS}}](t)). \end{aligned}$$

Using Lemmas 4.3 and 4.4, we have for a smooth initial data  $f_{\text{in}}$

$$\begin{aligned} & \left\| U_{\text{hydro}}(\cdot) f_{\text{in}} - U_{\text{NS}}(\cdot) f_{\text{in}} - U_{\text{disp}}(\cdot) f_{\text{in}} \right\|_H \\ & \leq \left\| U_{\text{wave}}(\cdot) f_{\text{in}} - U_{\text{disp}}(\cdot) f_{\text{in}} \right\|_H + \left\| U_{\text{NS}}(\cdot) f_{\text{in}} - U_{\text{NS}}(\cdot) f_{\text{in}} \right\|_H \\ & \leq \varepsilon \left( \|f_{\text{in}}\|_{X^{s+\delta}} + \|f_{\text{in}}\|_{\dot{H}_x^{-\alpha}(X_v)} \right), \end{aligned}$$

and in general, by a limiting argument

$$\lim_0 \left\| U_{\text{hydro}}(\cdot) f_{\text{in}} - U_{\text{NS}}(\cdot) f_{\text{in}} - U_{\text{disp}}(\cdot) f_{\text{in}} \right\|_H = 0.$$

Furthermore, using the estimate of Lemma 4.6 with  $\varphi = Q(f_{\text{NS}}, f_{\text{NS}})$ , and where we point out that, for  $d = 2$ , we have  $0 < \alpha < \frac{1}{2}$  and thus

$$\frac{4}{3 + 2\alpha} < 1, \frac{4}{3} < 2, \frac{2}{1 + \alpha} < \frac{4}{3},$$

whereas, for  $d > 3$ , we have  $\alpha = 0$  and thus

$$\frac{4}{3 + 2\alpha} = \frac{4}{3}, \quad \frac{2}{1 + \alpha} = 2,$$

we estimate  $\varphi(0) \dot{H}_x^{-\alpha}(H_v)$  thanks to (5.1a), and the other ones using Lemmas C.4–C.5 to deduce that

$$\begin{aligned} & \text{wave} [f_{\text{NS}}, f_{\text{NS}}]_H \\ & \leq \varepsilon \left( 1 + \|f_{\text{NS}}(0)\|_{\dot{H}_x^{-\alpha}(H_v)} + \|f_{\text{NS}}\|_{L^2([0, T]; H^s)} + \|x f_{\text{NS}}\|_{L^2([0, T]; H^s)} \right)^3. \end{aligned}$$

Finally, one proves as for (5.3) using this time (4.13), that for any  $\delta \in [0, 1]$

$$\|NS [f_{\text{NS}}, f_{\text{NS}}] - NS [f_{\text{NS}}, f_{\text{NS}}]\|_{H^{s+\delta}} \leq \varepsilon \|f_{\text{NS}}\|_{H^{s+\delta}}^2,$$

which, on the one hand, implies by Lemma C.5

$$\begin{aligned} & \|NS [f_{\text{NS}}, f_{\text{NS}}] - NS [f_{\text{NS}}, f_{\text{NS}}]\|_{H^s} \\ & \leq \varepsilon \left( \|f_{\text{NS}}(0)\|_{\dot{H}_x^{-\alpha}(H_v)} + \|f_{\text{NS}}\|_{L^2([0, T]; H^{s+\delta})} + \|x f_{\text{NS}}\|_{L^2([0, T]; H^{s+\delta})} \right)^2, \end{aligned}$$

and on the other hand, since  $f_{\text{NS}}$  can be approximated by elements of  $H^{s+1}$  by Lemma C.5, we have in general

$$\lim_0 \|NS [f_{\text{NS}}, f_{\text{NS}}] - NS [f_{\text{NS}}, f_{\text{NS}}]\|_{H^s} = 0.$$

This concludes the proof of Lemma 6.2.

**PROPOSITION 6.3 (Estimate for the source term  $S$ ).** — Consider some  $f_{\text{in}} \in X$  and denote  $f_{\text{disp}} = U_{\text{disp}}(\cdot) f_{\text{in}}$ , the source term  $S$  satisfies in  $H^s = H^s(T, f_{\text{NS}}, \eta)$

$$\begin{aligned} S [g_{\text{kin}}, g_{\text{mix}}]_H & \leq \beta_{\text{disp}}(\mathbf{P}_{\text{disp}} f_{\text{in}}, \varepsilon) (\|f_{\text{NS}}\|_H + \|g\|_H) + \beta_{\text{NS}}(f_{\text{NS}}, f_{\text{in}}, \varepsilon) \\ & \quad + \varepsilon w_{f_{\text{NS}}}(T)^{-1} (\|g_{\text{kin}}\|_X + \|g_{\text{mix}}\|_F) \\ & \quad \times (\|g_{\text{kin}}\|_X + \|g_{\text{mix}}\|_F + \|f_{\text{NS}}\|_H + \|\mathbf{P}_{\text{disp}} f_{\text{in}}\|_H + \|\mathbf{P}_{\text{disp}} f_{\text{in}}\|_{\dot{H}_x^{-\alpha}(H_v)}), \end{aligned} \tag{5}$$

where there holds

$$\lim_0 \beta_{\text{disp}}(\mathbf{P}_{\text{disp}} f_{\text{in}}, \varepsilon) = \lim_0 \beta_{\text{NS}}(f_{\text{NS}}, f_{\text{in}}, \varepsilon) = 0.$$

The rate of convergence of the term  $\beta_{\text{NS}}(f_{\text{NS}}, f_{\text{in}}, \varepsilon)$  can be made explicit if the initial data  $f_{\text{in}}$  lies in  $X^{s+\delta} \dot{H}_x^{-\alpha}(X_v)$  for some  $\delta \in (0, 1]$ :

$$\beta_{NS}(f_{NS}, f_{in}, \varepsilon)$$

$$\cdot \varepsilon \left( 1 + \|f_{in}\|_{X^{s+\delta}} + \|f_{in}\|_{\dot{H}_x^{-\alpha}(X_v)} + \|f_{NS}\|_{L^2([0,T]; H^{s+\delta})} + \|f_{NS}\|_{L^2([0,T]; H^{s+\delta})} \right)^3.$$

and if  $d > 3$ , the rate of convergence of  $\beta_{disp}(f_{in}, \varepsilon)$  is explicit if  $\|P_{disp}f_{in}\|_{\dot{B}_{1,1}^{s+(d+1)/2}(H_v)} \leq H^s$  for some  $s > s_0$ :

$$\beta_{disp}(f_{in}, \varepsilon) \leq \bar{\varepsilon} \left( \|P_{disp}f_{in}\|_{H^s} + \|P_{disp}f_{in}\|_{\dot{B}_{1,1}^{s+(d+1)/2}(H_v)} \right).$$

Furthermore, the source term  $S$  satisfies the stability estimate

$$\begin{aligned} & S_3[g_{kin}, g_{mix}] - S_3[g_{kin}, g_{mix}]_H \\ & \leq \varepsilon w_{f_{NS}}(T)^{-1} (\|g_{kin} - g_{kin}\|_X + \|g_{mix} - g_{mix}\|_F) \\ & \times (\|g_{kin}\|_X + \|g_{kin}\|_X + \|g_{mix}\|_F + \|g_{mix}\|_F + \|f_{NS}\|_H + \|P_{disp}f_{in}\|_H + \|P_{disp}f_{in}\|_{\dot{H}_x^{-\alpha}(H_v)}). \end{aligned}$$

*Remark 6.4.* — Note that the terms  $\beta_g$  and  $\beta_{NS}$  depend respectively on  $g$ , which stands for  $P_{disp}f_{in}$ , and  $f_{NS}$  which are to be considered as fixed data of the problem, thus the lack of uniform estimate for their convergence is not an issue for the iterative scheme.

*Proof.* — Recalling the definition of  $S_2$ :

$$S_2 = \text{hydro} [g_{disp}, 2f_{NS} + f_{disp}]$$

we easily have thanks to Eq. (5.9) in Proposition 5.4 and Lemma 4.1

$$S_2 \leq \beta_{disp}(P_{disp}f_{in}, \varepsilon) \|2f_{NS} + f_{disp}\|_H \leq \beta_{disp}(P_{disp}f_{in}, \varepsilon) (\|f_{NS}\|_H + \|g\|_H).$$

Furthermore, recalling the definition of  $S_3$ :

$$S_3[g_{kin}, g_{mix}] = \text{hydro} [g_{kin} + g_{mix}, g_{kin} + g_{mix} + f_{NS} + f_{disp}],$$

we easily have thanks to the various estimates of Proposition 5.3 and Lemma 4.1 together with the bilinearity of  $\text{hydro}$

$$\begin{aligned} & S_3[g_{kin}, g_{mix}]_H \leq \varepsilon w_{f_{NS}}(T)^{-1} (\|g_{kin}\|_X + \|g_{mix}\|_F) \\ & \times (\|g_{kin}\|_X + \|g_{mix}\|_F + \|f_{NS}\|_H + \|P_{disp}f_{in}\|_H + \|P_{disp}f_{in}\|_{\dot{H}_x^{-\alpha}(H_v)}). \end{aligned}$$

The stability estimate comes from the identity

$$\begin{aligned} & S_3[g_{kin}, g_{mix}] - S_3[g_{kin}, g_{mix}] \\ & = \text{hydro} [g_{kin} - g_{kin} + g_{mix} - g_{mix}, g_{kin} + g_{kin} + g_{mix} + g_{mix}] \\ & \quad + \text{hydro} [g_{kin} - g_{kin} + g_{mix} - g_{mix}, f_{NS} + f_{disp}] \end{aligned}$$

which we control using the same estimates. This concludes the proof.



### 6.2. The mapping is a contraction

In what follows, we will simplify some estimates by using the fact that

$$c_2, c_3, \eta, \varepsilon \leq 1, \quad 1 \leq w_{f_{NS}}, (T)^{-1}$$

and that  $\beta(\varepsilon) = \beta_{disp}(P_{disp}f_{in}, \varepsilon) + \beta_{NS}(f_{NS}, f_{in}, \varepsilon)$  (see Proposition 6.3) can be assumed to vanish at a slower rate than  $\varepsilon$ :

$$\varepsilon \leq \beta(\varepsilon).$$

To prove the existence and uniqueness of a fixed point for  $\Xi$ , we need to check that  $\Xi$  is a contraction on  $B$ . We begin by showing that  $B$  is stable under the action of  $\Xi$  under suitable smallness assumption on  $\varepsilon, c_3, \eta, c_2$ :

LEMMA 6.5. — For a suitable choice of

$$\varepsilon \leq c_3 \quad \eta \leq c_2 \leq 1$$

the mapping  $\Xi$  is well-defined on  $B$  and  $\Xi(B) \subset B$ .

*Proof.* — Let us check that the first component  $\Xi_1$  is well defined and takes values in  $B_1$ . We assume  $(\phi, \varphi, \psi) \in B$

$$\begin{aligned} \Xi_1[\phi, \varphi, \psi] - U_{kin}(\cdot)f_{in} &\leq C_{kin}[\phi, \phi]_X + 2 \left\| \text{kin}[\phi, f_{NS} + f_{disp}] \right\|_X \\ &\quad + 2 C_{kin}[\phi, \psi]_X + 2 C_{kin}[\phi, \varphi]_X. \end{aligned}$$

Using (5.15), we have the estimates

$$\begin{aligned} C_{kin}[\phi, \phi]_X &\leq \varepsilon \phi^2_X \leq \varepsilon, \\ 2 C_{kin}[\phi, \varphi]_X &\leq \varepsilon w_{f_{NS}}, (T)^{-1} \phi \times \psi_F \leq \varepsilon w_{f_{NS}}, (T)^{-1}, \end{aligned}$$

as well as

$$\begin{aligned} 2 \left\| \text{kin}[\phi, f_{NS} + f_{disp}] \right\|_X + 2 C_{kin}[\phi, \psi]_X &\leq \varepsilon \left\| f_{NS} + f_{disp} \right\|_H + \psi_H \\ &\leq \varepsilon w_{f_{NS}}, (T)^{-1} \phi \times \left\| f_{NS} + f_{disp} \right\|_H + \psi_H \\ &\leq \varepsilon w_{f_{NS}}, (T)^{-1}. \end{aligned}$$

Consequently

$$\Xi_1[\phi, \varphi, \psi] - U_{kin}(\cdot)f_{in} \leq \varepsilon w_{f_{NS}}, (T)^{-1},$$

thus, considering  $\varepsilon \leq \eta$ , we conclude that  $\Xi_1[\phi, \varphi, \psi] \in B_1$ .

The second component  $\Xi_2$  is also well-defined. We have

$$\begin{aligned} \left\| \Xi_2[\phi, \varphi, \psi] \right\|_F &\leq C_{kin}[\varphi, \varphi]_F + C_{kin}[f_{NS}, f_{NS}]_F + \left\| \text{kin}[f_{disp}, f_{disp}] \right\|_F + C_{kin}[\psi, \psi]_F \\ &\quad + 2 C_{kin}[\varphi, f_{NS}]_F + 2 \left\| \text{kin}[\varphi, f_{disp}] \right\|_F + 2 C_{kin}[\varphi, \psi]_F \\ &\quad + 2 \left\| \text{kin}[f_{NS}, f_{disp}] \right\|_F + 2 C_{kin}[f_{NS}, \psi]_F + 2 \left\| \text{kin}[f_{disp}, \psi] \right\|_F. \end{aligned}$$

Using (5.13), (5.18), (5.19) and (5.14) respectively, we have

$$\begin{aligned}
& \text{kin} [\varphi, \varphi]_F + \text{kin} [f_{\text{NS}}, f_{\text{NS}}]_F + \left\| \text{kin} [f_{\text{disp}}, f_{\text{disp}}] \right\|_F + \text{kin} [\psi, \psi]_F \\
& \cdot \varepsilon w_{f_{\text{NS}}}, (T)^{-1} \varphi^2_F + \eta f_{\text{NS}}_H + \beta(\varepsilon) f_{\text{disp}}_H + w_{f_{\text{NS}}}, (T)^{-1} \psi^2_H \\
& \cdot \varepsilon c_2^2 w_{f_{\text{NS}}}, (T)^{-1} + \eta + \beta(\varepsilon) + c_3^2 w_{f_{\text{NS}}}, (T)^{-1} \\
& \cdot (\beta(\varepsilon) + c_3) w_{f_{\text{NS}}}, (T)^{-1} + \eta.
\end{aligned}$$

Furthermore, using (5.13), we have

$$\begin{aligned}
2 \text{kin} [\varphi, f_{\text{NS}}]_F + 2 \left\| \text{kin} [\varphi, f_{\text{disp}}] \right\|_F + 2 \text{kin} [\varphi, \psi]_F & \leq \\
& \cdot \varepsilon w_{f_{\text{NS}}}, (T)^{-1} \varphi_F f_{\text{NS}}_H + \left\| f_{\text{disp}} \right\|_H + \psi_H \\
& \cdot \varepsilon c_2(1 + c_3) w_{f_{\text{NS}}}, (T)^{-1} \cdot \varepsilon w_{f_{\text{NS}}}, (T)^{-1},
\end{aligned}$$

whereas, (5.18) and (5.19) give respectively

$$\begin{aligned}
2 \left\| \text{kin} [f_{\text{NS}}, f_{\text{disp}}] \right\|_F + 2 \text{kin} [f_{\text{NS}}, \psi]_F & \leq \eta \left\| f_{\text{disp}} \right\|_H + \psi_H \\
& \cdot \eta(1 + c_3) \cdot \eta,
\end{aligned}$$

and

$$2 \left\| \text{kin} [f_{\text{disp}}, \psi] \right\|_F \leq \beta(\varepsilon) \psi_H \leq \beta(\varepsilon) c_3 \leq \beta(\varepsilon).$$

Gathering these estimates yield

$$2 [\phi, \varphi, \psi]_X \leq (\beta(\varepsilon) + c_3) w_{f_{\text{NS}}}, (T)^{-1} + \eta.$$

We deduce for  $\max\{\varepsilon, c_3\} \leq \eta \leq c_2$  that  $\mathfrak{B}_2$  takes value in  $B_2$ .

Finally, the third component  $\mathfrak{B}_3$  is well defined. We have

$$[\phi, \varphi, \psi]_H \leq [\phi, \varphi] \psi_H + \left\| \text{hydro}[\psi, \psi] \right\|_H + S[\phi, \varphi]_H.$$

By Proposition 6.1

$$[\phi, \varphi] \psi_H \leq (\eta + \beta(\varepsilon) + \varepsilon c_2 w_{f_{\text{NS}}}, (T)^{-1} + \varepsilon) c_3 \leq \beta(\varepsilon) w_{f_{\text{NS}}}, (T)^{-1} + \eta c_3,$$

while, from (5.3) and Proposition 6.3

$$\left\| \text{hydro}[\psi, \psi] \right\|_H \leq c_3^2 w_{f_{\text{NS}}}, (T)^{-1}, \quad S[\phi, \varphi]_H \leq \beta(\varepsilon).$$

All these estimates gathered together give

$$[\phi, \varphi, \psi]_H \leq \beta(\varepsilon) w_{f_{\text{NS}}}, (T)^{-1} + (\eta + c_3 w_{f_{\text{NS}}}, (T)^{-1}) c_3$$

thus  $\mathfrak{B}_3$  takes value in  $B_3$  by taking  $\varepsilon \leq c_3 \leq \eta \leq 1$ . This completes the proof of Lemma 6.5.

We show now that, up to reducing further the parameters  $\varepsilon, c_2, c_3, \eta$ , the mapping  $\Xi$  is a contraction on  $\mathbf{B}$ .

**PROPOSITION 6.6.** — *Under the smallness assumption*

$$\max\{\varepsilon, c_3\} \leq \eta \leq 1,$$

*the mapping  $\Xi : \mathbf{B} \times \mathbf{B} \times X \times F \times H$  is a contraction.*

*Proof.* — Let us fix  $(\phi, \varphi, \psi) \in \mathbf{B}$ ,  $(\phi, \varphi, \psi) \in \mathbf{B}$ . We prove that each component of  $\Xi$  is contractive. We have

$$\begin{aligned} & \|\phi_1[\phi, \varphi, \psi] - \phi_1[\phi, \varphi, \psi]\|_X \\ & \leq \|\text{kin}[\phi - \phi, \phi + \phi]\|_X + 2 \|\text{kin}[\phi - \phi, f_{NS} + f_{disp}]\|_X \\ & \quad + 2 \|\text{kin}[\phi - \phi, \psi]\|_X + 2 \|\text{kin}[\phi, \psi - \psi]\|_X \\ & \quad + 2 \|\text{kin}[\phi - \phi, \varphi]\|_X + 2 \|\text{kin}[\phi, \varphi - \varphi]\|_X \end{aligned}$$

As in the previous proof, using (5.15) we have

$$\begin{aligned} & \|\text{kin}[\phi - \phi, \phi + \phi]\|_X + 2 \|\text{kin}[\phi - \phi, f_{NS} + f_{disp}]\|_X \\ & \leq \varepsilon (1 + w_{f_{NS}}(T)^{-1}) \|\phi - \phi\|_X \\ & \leq \varepsilon w_{f_{NS}}(T)^{-1} \|\phi - \phi\|_X, \end{aligned}$$

and

$$\begin{aligned} & 2 \|\text{kin}[\phi - \phi, \psi]\|_X + 2 \|\text{kin}[\phi, \psi - \psi]\|_X \\ & \leq \varepsilon c_3 w_{f_{NS}}(T)^{-1} \|\phi - \phi\|_X + \varepsilon w_{f_{NS}}(T)^{-1} \|\psi - \psi\|_H \\ & \leq \varepsilon w_{f_{NS}}(T)^{-1} \|(\phi, \psi) - (\phi, \psi)\|_{X \times H}, \end{aligned}$$

as well as

$$\begin{aligned} & 2 \|\text{kin}[\phi - \phi, \varphi]\|_X + 2 \|\text{kin}[\phi, \varphi - \varphi]\|_X \\ & \leq \varepsilon c_2 w_{f_{NS}}(T)^{-1} \|\phi - \phi\|_X + \varepsilon w_{f_{NS}}(T)^{-1} \|\varphi - \varphi\|_F \\ & \leq \varepsilon w_{f_{NS}}(T)^{-1} \|(\phi, \varphi) - (\phi, \varphi)\|_{X \times F}. \end{aligned}$$

This shows that

$$\|\phi_1[\phi, \varphi, \psi] - \phi_1[\phi, \varphi, \psi]\|_X \leq \varepsilon w_{f_{NS}}(T)^{-1} \|(\phi, \varphi, \psi) - (\phi, \varphi, \psi)\|_{X \times F \times H}.$$

Thus, taking  $\varepsilon = \eta$ , the first component  $\phi_1$  is indeed a contraction. We argue in the same way for the second component. It holds

$$\begin{aligned} & \|\phi_2[\phi, \varphi, \psi] - \phi_2[\phi, \varphi, \psi]\|_F \\ & \leq \|\text{kin}[\varphi - \varphi, \varphi + \varphi]\|_F \\ & \quad + 2 \|\text{kin}[\varphi - \varphi, f_{NS} + f_{disp}]\|_F \\ & \quad + 2 \|\text{kin}[\varphi - \varphi, \psi]\|_F + \|\text{kin}[\varphi, \psi - \psi]\|_F \\ & \quad + 2 \|\text{kin}[f_{NS}, \psi - \psi]\|_F + 2 \|\text{kin}[f_{disp}, \psi - \psi]\|_F. \end{aligned}$$

As in the previous proof, resorting to (5.13), one deduces that

$$\|\text{kin}[\varphi - \varphi, \varphi + \varphi]\|_F \leq \varepsilon c_2 w_{f_{NS}}(T)^{-1} \|\varphi - \varphi\|_F \leq \varepsilon w_{f_{NS}}(T)^{-1} \|\varphi - \varphi\|_F$$

and

$$2 \|\text{kin}[\varphi - \varphi, f_{NS} + f_{disp}]\|_F \leq \varepsilon w_{f_{NS}}(T)^{-1} \|\varphi - \varphi\|_F,$$

whereas

$$2 \|\text{kin}[\varphi - \varphi, \psi]\|_F + \|\text{kin}[\varphi, \psi - \psi]\|_F$$

$$\begin{aligned} & \cdot \varepsilon c_3 w_{f_{NS}, (T)^{-1}} \varphi - \varphi_F + \varepsilon c_2 w_{f_{NS}, (T)^{-1}} \psi - \psi_H \\ & \cdot \varepsilon w_{f_{NS}, (T)^{-1}} (\varphi, \psi) - (\varphi, \psi)_{F \times H} . \end{aligned}$$

Finally, using (5.18) and (5.19), one has as previously

$$2 \quad \text{kin} [f_{NS}, \psi - \psi]_F + 2 \left\| \text{kin} [f_{disp}, \psi - \psi] \right\|_F \cdot (\eta + \beta(\varepsilon)) \psi - \psi_H$$

which, together with the previous estimates, yields

$$\begin{aligned} & 2 [\phi, \varphi, \psi] - 2 [\phi, \varphi, \psi]_F \\ & \cdot [\varepsilon w_{f_{NS}, (T)^{-1}} + \eta + \beta(\varepsilon)] (\phi, \varphi, \psi) - (\phi, \varphi, \psi)_{X \times F \times H} , \end{aligned}$$

thus, taking  $\varepsilon = \eta$ , the second component  $\text{kin}$  is also a contraction. As far as the third component is concerned, one has

$$\begin{aligned} & 3 [\phi, \varphi, \psi] - 3 [\phi, \varphi, \psi]_H \\ & 6 \quad [\phi, \varphi] \psi - [\phi, \varphi] \psi_H + \left\| \text{hydro} [\psi - \psi, \psi + \psi] \right\|_H \\ & \quad + S [\phi - \phi, \varphi]_H + S [\phi, \varphi - \varphi]_H . \end{aligned}$$

Now, using Proposition 6.1,

$$\begin{aligned} & [\phi, \varphi] \psi - [\phi, \varphi] \psi_H \\ & \cdot (\eta + \beta(\varepsilon) + \varepsilon c_2 w_{f_{NS}, (T)^{-1}} + \varepsilon) \psi - \psi_H \\ & \quad + c_3 \varepsilon w_{f_{NS}, (T)^{-1}} \varphi - \varphi_F + c_3 \varepsilon \phi - \phi_X \\ & \cdot (\eta + \beta(\varepsilon) + \varepsilon c_2 w_{f_{NS}, (T)^{-1}} + \varepsilon) (\phi, \varphi, \psi) - (\phi, \varphi, \psi)_{X \times F \times H} \\ & \cdot (\eta + \beta(\varepsilon) w_{f_{NS}, (T)^{-1}}) (\phi, \varphi, \psi) - (\phi, \varphi, \psi)_{X \times F \times H} , \end{aligned}$$

while (5.3a) yields

$$\left\| \text{hydro} [\psi - \psi, \psi + \psi] \right\|_H \cdot w_{f_{NS}, (T)^{-1}} c_3 \psi - \psi_H .$$

Finally, Proposition 6.3 easily gives

$$\begin{aligned} & S [\phi, \varphi] - S [\phi, \varphi]_H \leq S_3 [\phi - \phi, \varphi]_H + S_3 [\phi, \varphi - \varphi]_H \\ & \cdot \varepsilon \phi - \phi_X + \varphi - \varphi_F . \end{aligned}$$

All these estimates yield

$$\begin{aligned} & 3 [\phi, \varphi, \psi] - 3 [\phi, \varphi, \psi]_H \\ & \cdot [\varepsilon + \eta + (\beta(\varepsilon) + c_3) w_{f_{NS}, (T)^{-1}}] (\phi, \varphi, \psi) - (\phi, \varphi, \psi)_{X \times F \times H} , \end{aligned}$$

thus, taking  $\max\{\varepsilon, c_3\} = \eta = 1$ , the third component  $\text{kin}$  is indeed a contraction.

**6.3. Proof of Theorem 1.12: existence and convergence**

We have established that  $\Xi$  is a well-defined contraction on  $\mathbf{B} = B_1 \times B_2 \times B_3$  under the smallness assumption

$$\varepsilon \leq c_3 \eta \leq c_2^{-1},$$

thus it admits a unique fixed point denoted  $(f_{\text{kin}}, f_{\text{mix}}, g)$ . The part  $g$  satisfies (for some sufficiently small  $c > 0$ )

$$\|g\|_H \leq c \|g\|_H + \|S\|_H,$$

and therefore, considering  $c$  small enough, we have:

$$\|g\|_H \leq \|S\|_H \leq \beta(\varepsilon).$$

The part  $f_{\text{mix}}$  satisfies the equation

$$\begin{aligned} f_{\text{mix}} - \text{kin}[f_{\text{NS}}, f_{\text{NS}}] &= \text{kin}[f_{\text{mix}}, f_{\text{mix}}] + \text{kin}[g, g] + \text{kin}[f_{\text{disp}}, f_{\text{disp}}] \\ &+ 2 \text{kin}[f_{\text{mix}}, f_{\text{NS}}] + 2 \text{kin}[f_{\text{mix}}, f_{\text{disp}}] + 2 \text{kin}[f_{\text{mix}}, g] \\ &+ 2 \text{kin}[f_{\text{NS}}, f_{\text{disp}}] + 2 \text{kin}[f_{\text{NS}}, g] + 2 \text{kin}[f_{\text{disp}}, g], \end{aligned}$$

therefore, from the computations of Section 6.2, we have

$$\|f_{\text{mix}} - \text{kin}[f_{\text{NS}}, f_{\text{NS}}]\|_F \leq \|g\|_H + \beta(\varepsilon) \leq \beta(\varepsilon).$$

Furthermore, by a duality argument similar to the one from the proof of (4.16b)

$$\begin{aligned} \int_{t_1}^{t_2} \langle \text{kin}[f_{\text{NS}}, f_{\text{NS}}](t), \phi_0 \rangle_H dt &= \frac{1}{\varepsilon} \int_{t_1}^{t_2} \int_0^t \langle Q(f_{\text{NS}}(\tau), f_{\text{NS}}(\tau)), U_{\text{kin}}(t - \tau) \phi_0 \rangle_H d\tau dt \\ &\leq \frac{1}{\varepsilon} \|f_{\text{NS}}\|_{L^2([0, T]; H^*)} (t_2 - t_1) \int_0^{t_2} \|U_{\text{kin}}(t) \phi_0\|_H dt \\ &\leq (t_2 - t_1) \int_0^{t_2} e^{2t} \|U_{\text{kin}}(t) \phi_0\|_H^2 dt < \frac{1}{2} \\ &\leq \varepsilon (t_2 - t_1) \|\phi_0\|_H, \end{aligned}$$

where we used that  $\|f_{\text{NS}}\|_{L^2([0, T]; H^*)} \leq 1$ . Thus, we deduce

$$\|\text{kin}[f_{\text{NS}}, f_{\text{NS}}]\|_{L^2([0, T]; H)} \leq \varepsilon,$$

from which we conclude that  $\|f_{\text{mix}} + g\|_{L^2([0, T]; H)} \leq \beta(\varepsilon)$ . We conclude to Theorem 1.12 by letting

$$f_{\text{err}} := g + f_{\text{mix}}.$$

This concludes the proof of Theorem 1.12.

**6.4. Proof of Theorem 1.12: uniqueness**

Consider another solution associated with the same initial data  $f_{in}$ :

$$\bar{f} \in L^\infty([0, T]; X) \cap L^2_{loc}([0, T]; X^*)$$

satisfying for some universal small  $c > 0$  the bound (note that the same bound holds for  $f$  since  $\|f\|_{L^2_t(X)} \leq 1$ )

$$\|\bar{f}\|_{L^\infty([0, T]; X)} \leq \frac{c}{\varepsilon}.$$

Define the difference of solutions

$$h = f - \bar{f}$$

and observe it satisfies the equation

$$\partial_t h = \frac{1}{\varepsilon^2} (L - \varepsilon v \cdot \nabla_x) h + \frac{1}{\varepsilon} Q(h, f + \bar{f}), \quad h(0) = 0.$$

We write an energy estimate for  $h$  (see *Step 2* of the proof of Lemma 4.9 for the dissipative part):

$$\frac{1}{2} \frac{d}{dt} \|h\|_{X^*}^2 + \frac{\lambda}{\varepsilon^2} \|h\|_{X^*}^2 \leq \frac{1}{\varepsilon^2} \|h\|_{X^*}^2 + \frac{1}{\varepsilon} \|h\|_{X^*}^2 \cdot \|f + \bar{f}\|_X + \frac{1}{\varepsilon} \|h\|_{X^*} \|h\|_{X^*} \cdot \|f + \bar{f}\|_{X^*},$$

which gives after integrating on  $[0, t]$  in the space  $X^*$  ( $\sigma = 0, t, \varepsilon$ ):

$$\|h\|_{X^*}^2 \leq \frac{t}{\varepsilon^2} \|h\|_{X^*}^2 + c \|h\|_{X^*}^2 + \|h\|_{X^*}^2 \int_0^t \|f(\tau) + \bar{f}(\tau)\|_{X^*}^2 d\tau \leq c^{1/2}.$$

Thus, since  $c$  is supposed to be small, taking  $t$  close enough to 0 yields (for instance)

$$\|h\|_{X^*} \leq \frac{1}{2} \|h\|_{X^*},$$

which in turn implies  $h(\tau) = 0$ , or equivalently  $f(\tau) = \bar{f}(\tau)$  for any  $\tau \in [0, t]$ . Repeating this argument yields the uniqueness of the solution.

**7. Proof of Theorem 1.22**

We prove here Theorem 1.22 under the assumption **(BED)**. Note that the following strategy can also be seen as an alternative proof under assumption **(BE)**.

### 7.1. Modification of the strategy

Under the assumption **(BED)**, the arguments of  $Q(f, g)$  in  $X$  no longer play symmetric roles, so **we no longer consider  $Q$  in its symmetrized form**. This does not induce any change for the parts  $f_{\text{hydro}}$  and  $f_{\text{mix}}$  of the solution since they are constructed using assumption **(B3)** and not **(BED)**, for which both arguments play symmetric roles. The only modification is therefore the need to adjust the iterative scheme constructing  $f_{\text{kin}}$  as well as its space-velocity functional space so as to take into account the assumptions **(BED)** following the strategy adopted in [CG24, CM17, CTW16, GMM17, HTT20]. We detail below this new strategy.

- We no longer consider the equation on  $f_{\text{kin}}$  in integral form

$$f_{\text{kin}}(t) = U_{\text{kin}}(t)f_{\text{in}} + \mathbf{P}_{\text{kin}} [f_{\text{kin}}, f_{\text{kin}}](t) + 2 \mathbf{P}_{\text{kin}} [f_{\text{kin}}, f_{\text{hydro}} + f_{\text{mix}}](t),$$

but we study the evolution of  $f_{\text{kin}}$  in its differential form

$$(7.1) \quad \partial_t f_{\text{kin}} = \frac{1}{\varepsilon^2} (L - \varepsilon v \cdot \nabla_x) f_{\text{kin}} + \frac{1}{\varepsilon} \mathbf{P}_{\text{kin}} Q(f_{\text{kin}}, f_{\text{kin}}) + \frac{2}{\varepsilon} \mathbf{P}_{\text{kin}} Q^{\text{sym}}(f_{\text{kin}}, f_{\text{hydro}} + f_{\text{mix}}).$$

This equation can be studied through a suitable energy method so as to be able to use the “closing estimate” of **(BED)** (which does not translate in integral form).

- Since the roles played by both arguments of  $Q(f, g)$  in  $X$  under the assumption **(BED)** are different, we do not construct  $f_{\text{kin}}$  using Banach’s theorem, which, as far as (7.1) is concerned, would correspond to the convergence an iterative scheme of the form

$$\partial_t f_{\text{kin},N} = \frac{1}{\varepsilon^2} (L - \varepsilon v \cdot \nabla_x) f_{\text{kin},N} + \frac{1}{\varepsilon} \mathbf{P}_{\text{kin}} Q(f_{\text{kin},N-1}, f_{\text{kin},N-1}) + \frac{2}{\varepsilon} \mathbf{P}_{\text{kin}} Q^{\text{sym}}(f_{\text{kin},N-1}, f_{\text{hydro},N-1} + f_{\text{mix},N-1})$$

but using a variation of such a scheme which allows to use the “closing estimate” of **(BED)**. Namely, we prove the stability of the scheme (7.2) hereafter.

- We define a new hierarchy of spaces  $(\mathcal{X}_j)_{j=-2-s}^1$  of the form

$$\mathcal{X}_j = L^2_X(\mathbf{X}_j) \cap \dot{H}^s_X(\mathbf{X}_{j-s})$$

which allows to prove spatially inhomogeneous counterparts of the estimates of **(BED)**. Notice here that we assume our “regularity parameter”  $s$  to be integer  $s \geq N$  and it is now assumed an additional role in the hierarchy of spaces  $\mathcal{X}_{-2-s}, \dots, \mathcal{X}_1$ .

- The operator  $L - \varepsilon v \cdot \nabla_x$  is not dissipative for the inner product of  $\mathbf{X}$ , but it is *hypo-dissipative* on  $\text{Range}(\mathbf{P}_{\text{kin}})$ , so, we introduce an equivalent inner product of the form

$$(f, g)_{\mathcal{X}_j} := \delta (f, g)_{\mathcal{X}_j} + \frac{1}{\varepsilon^2} \int_0^t \langle U_{\text{kin}}(t) f, U_{\text{kin}}(t) g \rangle_{\mathcal{X}_{j-1}} dt,$$

for which  $L - \varepsilon v \cdot \nabla_x$  is dissipative and  $Q$  satisfies the same estimates as **(BED)**.

To summarize, our approach will be aimed at constructing a solution  $(f_{\text{kin}}, f_{\text{mix}}, g)$  as the limit of a sequence of approximate solutions  $\{(f_{\text{kin},N}, f_{\text{mix},N}, g_N)\}_{N>0}$ , where the first component  $f_{\text{kin},N}$  is constructed inductively by solving the following differential equation:

$$(7.2) \quad \begin{cases} \partial_t f_{\text{kin},N} &= \frac{1}{\varepsilon^2} (L - \varepsilon v \cdot \nabla_x) f_{\text{kin},N} + \frac{1}{\varepsilon} \mathbf{P}_{\text{kin}} Q(f_{\text{kin},N-1}, f_{\text{kin},N}) \\ &+ \frac{2}{\varepsilon} \mathbf{P}_{\text{kin}} Q^{\text{sym}}(f_{\text{kin},N}, f_{\text{hydro},N-1} + f_{\text{mix},N-1}) \\ f_{\text{kin},N}(0) &= \mathbf{P}_{\text{kin}} f_{\text{in}}, \\ f_{\text{kin},0} &= 0, \end{cases}$$

where we naturally denoted  $f_{\text{hydro},N} = f_{\text{NS}} + f_{\text{disp}} + g_N$ , and the other parts are still constructed as in Section 6:

$$\begin{cases} g_N &= [f_{\text{kin},N-1}, f_{\text{mix},N-1}] g_{N-1} + \text{hydro} [g_{N-1}, g_{N-1}] \\ &+ S [f_{\text{kin},N-1}, f_{\text{mix},N-1}], \\ f_{\text{mix},N} &= \text{kin} [f_{\text{mix},N-1} + f_{\text{kin},N-1}, f_{\text{mix},N-1} + f_{\text{kin},N-1}], \\ g_0 &= 0, \quad f_{\text{mix},0} = 0. \end{cases}$$

Let us define the new functional space we will use in this section.

**DEFINITION 7.1.** — *Suppose  $s \in \mathbb{N}$  and satisfies  $s > 3$  if  $d = 2$  or  $s > \frac{d}{2} + 1$  if  $d > 3$ , we define*

$$f \in \mathcal{X}_j^s := f \in L_x^2(\mathcal{X}_j) + \sum_{x'}^s f \in L_x^2(\mathcal{X}_{j-s}),$$

and note that, since  $\mathcal{X}_k \subset \mathcal{X}_j$  as soon as  $j \subset k$ , the following equivalence of norms holds:

$$f \in \mathcal{X}_j^s \iff \sum_{k=0}^s f \in H_x^k(\mathcal{X}_{j-k}) \iff \sum_{k=0}^s f \in \mathcal{X}_j^k.$$

In particular, this hierarchy of spaces is decreasing in both indexes:

$$(7.3) \quad s_1 \subset s_2 \text{ and } j_1 \subset j_2 \implies \mathcal{X}_{j_2}^{s_2} \subset \mathcal{X}_{j_1}^{s_1}.$$

**LEMMA 7.2.** — *The bilinear operator  $Q$  satisfies in  $\mathcal{X}_j$  the estimates*

$$(7.4) \quad Q(f, g) \in \mathcal{X}_j \cdot f \in \mathcal{X}_j, g \in \mathcal{X}_{j+1}^{\bullet} + f \in \mathcal{X}_j^{\bullet}, g \in \mathcal{X}_{j+1},$$

$$(7.5) \quad Q(f, g), g \in \mathcal{X}_j \cdot f \in \mathcal{X}_j, g \in \mathcal{X}_j^{2\bullet} + f \in \mathcal{X}_j^{\bullet}, g \in \mathcal{X}_j, g \in \mathcal{X}_j^{\bullet}.$$

*Proof.* — The general non-closed control (7.4) is easily obtained from its spatially homogeneous counterpart of **(BED)** so we only prove the closed control (7.5).

The inner product writes according to Leibniz's formula and the locality of  $Q$  as

$$\begin{aligned} Q(f, g), h \in \mathcal{X}_j &= Q(f, g), h \in L_x^2(\mathcal{X}_j) + \sum_{|l|=s} \langle \partial_x Q(f, g), \partial_x h \rangle_{L_x^2(\mathcal{X}_{j-s})} \\ &= Q(f, g), h \in L_x^2(\mathcal{X}_j) + \sum_{|+| - |l| = s} \langle Q(\partial_x f, \partial_x^- g), \partial_x h \rangle_{L_x^2(\mathcal{X}_{j-s})}. \end{aligned}$$



We first look at the terms which can be controlled using the closed estimate (1.28) of **(BED)**. Using Hölder's inequality in  $L_x \times L_x^2 \times L_x^2$  (or some appropriate permutation) together with the embeddings  $H_x^s \subset L_x$  and  $\mathbf{X}_{j-s} \subset \mathbf{X}_{-1-s}$  and therefore  $H_x^s(\mathbf{X}_{j-s}) \subset L_x(\mathbf{X}_{-1-s})$ , we immediately have the estimate

$$\begin{aligned} Q(f, g), g \in L_x^2(\mathbf{X}_j) & \cdot \sum_{\{a,b,c\}=\{j,j,-1-s\}} \int_{\mathbb{R}^d} f(x_a) g(x_b) g(x_c) dx + \int_{\mathbb{R}^d} f(x_a) g(x_b) g(x_c) dx \\ & \cdot f(x_j) g(x_j)^2 + f(x_j) g(x_j) g(x_j) \end{aligned}$$

as well as the terms associated with  $|\beta| = s$  and  $|\gamma| = 0$ :

$$\langle Q(f, \partial_x g), \partial_x g \rangle_{L_x^2(\mathbf{X}_{j-s})} \cdot f(x_j) g(x_j)^2 + f(x_j) g(x_j) g(x_j).$$

We are thus left with the terms associated with  $|\beta| = s$  and  $|\gamma| > 1$  which have to be controlled starting from the non-closed estimate (1.27) of **(BED)**:

$$\begin{aligned} (7.6) \quad \langle Q(\partial_x f, \partial_x^- g), \partial_x g \rangle_{L_x^2(\mathbf{X}_{j-s})} & \cdot \int_{\mathbb{R}^d} \partial_x f(x) \mathbf{X}_{j-s} \|\partial_x^- g(x)\|_{\mathbf{X}_{j-(s-1)}} \|\partial_x g(x)\|_{\mathbf{X}_{j-s}} dx \\ & + \int_{\mathbb{R}^d} \partial_x f(x) \mathbf{X}_{j-s} \|\partial_x^- g(x)\|_{\mathbf{X}_{j-(s-1)}} \|\partial_x g(x)\|_{\mathbf{X}_{j-s}} dx, \end{aligned}$$

and we will show in each case that

$$(7.7) \quad \begin{aligned} & \left\| \partial_x f(x) \mathbf{X}_{j-s} \|\partial_x^- g(x)\|_{\mathbf{X}_{j-(s-1)}} \right\|_{L_x^2} \cdot f(x_j) g(x_j) \\ & \left\| \partial_x f(x) \mathbf{X}_{j-s} \|\partial_x^- g(x)\|_{\mathbf{X}_{j-(s-1)}} \right\|_{L_x^2} \cdot f(x_j) g(x_j). \end{aligned}$$

*Step 1: The case  $d = 2$ .* — When  $|\gamma| = s$ , we have  $|\beta - \gamma| = 0$ , thus, using the fact that  $H_x^{s-1} \subset L_x$  since  $s > 3$ , followed by (7.3)

$$\|\partial_x^- g\|_{L_x(\mathbf{X}_{j-(s-1)})} \cdot g \in H_x^{s-1}(\mathbf{X}_{j-(s-1)}) \cdot g(x_j),$$

thus, (7.7) holds. When  $|\gamma| = s - 1$ , using the injection  $H_x^1 \subset L_x^4$ , we have

$$\partial_x f \in L_x^4(\mathbf{X}_{j-s}) \cdot f \in H_x^s(\mathbf{X}_{j-s}) \cdot f(x_j),$$

similarly, since  $|\beta - \gamma| = 1 \leq s - 2$ , we also have

$$\|\partial_x^- g\|_{L_x^4(\mathbf{X}_{j-(s-1)})} \cdot f \in H_x^{s-1}(\mathbf{X}_{j-(s-1)}) \cdot f(x_j),$$

thus, (7.7) holds. When  $|\gamma| \leq s - 2$ , we have

$$\partial_x f \in L_x(\mathbf{X}_{j-s}) \cdot f \in H_x^s(\mathbf{X}_{j-s}), \cdot f(x_j)$$

and similarly, since  $|\beta - \gamma| \leq s - 1$  (recall that  $|\beta| = s$  and  $|\gamma| > 1$ )

$$\|\partial_x^- g\|_{L_x^2(\mathbf{X}_{j-(s-1)})} \cdot f \in H_x^{s-1}(\mathbf{X}_{j-(s-1)}) \cdot f(x_j),$$

thus (7.7) holds. This concludes this step.

*Step 2: The case  $d = 3$ .* — First, a simple use of Cauchy–Schwarz inequality yields

$$\| \partial_x f \|_{L^2_x(\mathcal{X}_{j-s})} \| \partial_x^- g \|_{L^2_x(\mathcal{X}_{j-(s-1)})} \leq \| \partial_x f \|_{L^2_x(\mathcal{X}_{j-s})} \| \partial_x^- g \|_{L^2_x(\mathcal{X}_{j-(s-1)})}$$

and, since  $|\beta - \gamma| + |\gamma| \leq s$ , we deduce that

$$(7.8) \quad \| \partial_x f \|_{L^2_x(\mathcal{X}_{j-s})} \| \partial_x^- g \|_{L^2_x(\mathcal{X}_{j-(s-1)})} \leq \| f \|_{\mathcal{X}_j} \| g \|_{\mathcal{X}_j}.$$

Now, using Sobolev embeddings, we have

$$\| \partial_x f \|_{L^p_x(\mathcal{X}_{j-s})} \leq \| f \|_{H^s_x(\mathcal{X}_{j-s})} \leq \| f \|_{\mathcal{X}_j}, \quad \frac{1}{p} = \frac{1}{2} - \frac{s - |\gamma|}{d}$$

as well as

$$\| \partial_x^- g \|_{L^q_x(\mathcal{X}_{j-(s-1)})} \leq \| g \|_{H^{s-1}_x(\mathcal{X}_{j-(s-1)})} \leq \| g \|_{\mathcal{X}_j}, \quad \frac{1}{q} = \frac{1}{2} - \frac{(s-1) - |\beta - \gamma|}{d}.$$

Since  $|\beta| = s$  and  $s > \frac{d}{2} + 1$ , Hölder’s inequality implies

$$\| \partial_x f \|_{L^p_x(\mathcal{X}_{j-s})} \| \partial_x^- g \|_{L^q_x(\mathcal{X}_{j-(s-1)})} \leq \| f \|_{\mathcal{X}_j} \| g \|_{\mathcal{X}_j}, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} = 1 - \frac{s-1}{d} \leq \frac{1}{2}.$$

Using (7.8) and simple interpolation, we deduce that

$$\| \partial_x f \|_{L^2_x(\mathcal{X}_{j-s})} \| \partial_x^- g \|_{L^2_x(\mathcal{X}_{j-(s-1)})} \leq \| f \|_{\mathcal{X}_j} \| g \|_{\mathcal{X}_j},$$

this proves the first estimate in (7.7) and the second one is done in the same way. Combining then (7.7) with (7.6) and Cauchy–Schwarz inequality yields

$$\left\langle Q(\partial_x f, \partial_x^- g), \partial_x g \right\rangle_{L^2_x(\mathcal{X}_{j-s})} \leq \| f \|_{\mathcal{X}_j} \| g \|_{\mathcal{X}_j}^2 + \| f \|_{\mathcal{X}_j} \| g \|_{\mathcal{X}_j} \| g \|_{\mathcal{X}_j}.$$

This concludes this step and the proof of Lemma 7.2.

As stated above, we will study the equation (7.2) using an equivalent inner product. The next proposition defines it and presents its properties.

**PROPOSITION 7.3 (Kinetic dissipative inner product).** — Let  $j = -1, 0$  and  $\sigma \in (0, \sigma_0)$ . For some  $\delta > 0$  small enough, the inner product defined for any  $f, g \in \text{Range}(P_{\text{kin}}) \cap \mathcal{X}_j$  as

$$f, g \mathcal{X}_j := \delta f, g \mathcal{X}_j + \frac{1}{\varepsilon^2} \int_0^1 e^{2t/\varepsilon} \langle U_{\text{kin}}(t)f, U_{\text{kin}}(t)g \rangle_{\mathcal{X}_{j-1}} dt,$$

induces a norm equivalent to that of  $\mathcal{X}_j$  (uniformly in  $\varepsilon$ ), i.e. there exists  $C > 0$  independent of  $\varepsilon$  such that

$$(7.9) \quad \frac{1}{C} \| f \|_{\mathcal{X}_j} \leq \| f \|_{\mathcal{X}_j} \leq C \| f \|_{\mathcal{X}_j}.$$

Moreover, there is  $\mu > 0$  such that

$$\text{Re} \langle (L - \varepsilon v \cdot \nabla_x) f, f \rangle_{\mathcal{X}_j} \leq -\sigma \| f \|_{\mathcal{X}_j}^2 - \mu \| f \|_{\mathcal{X}_j}^2,$$

and the nonlinear estimates for  $P_{\text{kin}}Q$  are the same as those from (7.4) and (7.5):

$$P_{\text{kin}}Q(f, g), h_{\mathbf{x}_j} \leq h_{\mathbf{x}_j} \left( f_{\mathbf{x}_j} g_{\mathbf{x}_{j+1}} + f_{\mathbf{x}_j} g_{\mathbf{x}_{j+1}} \right),$$

and

$$P_{\text{kin}}Q(f, g), g_{\mathbf{x}_j} \leq f_{\mathbf{x}_j} g_{\mathbf{x}_j}^2 + g_{\mathbf{x}_j} g_{\mathbf{x}_j} f_{\mathbf{x}_j}.$$

*Proof.* —

*Step 1: Proof of the equivalence of norms.* — The norm  $\|\cdot\|_{\mathbf{x}_j}$  writes

$$\|f\|_{\mathbf{x}_j}^2 = \delta \|f\|_{\mathbf{x}_j}^2 + \frac{1}{\varepsilon^2} \int_0^t e^{2t'/2} U_{\text{kin}}(t) f_{\mathbf{x}_{j-1}}^2 dt,$$

and thus, using  $\mathbf{x}_j \subset \mathbf{x}_{j-1}$  and the decay estimate for the semigroup  $U_{\text{kin}}(t)$  from Lemma 4.8, we have for some  $C > 0$

$$\delta \|f\|_{\mathbf{x}_j}^2 \leq \|f\|_{\mathbf{x}_j}^2 \leq (\delta + C) \|f\|_{\mathbf{x}_j}^2.$$

This proves (7.9).

*Step 2: Proof of the dissipative estimate.* — Using the decomposition  $L = B + A$  coming from (LE) in the space  $\mathbf{X}_j$  (recall that, from (BED), the dissipativity estimate of (LE) is valid in  $\mathbf{X}_j$ ) together with the estimate for  $A$  from (BED), we have for some  $K > 0$

$$\begin{aligned} \text{Re} (L - \varepsilon v \cdot x) f, f_{\mathbf{x}_j} &= \delta \text{Re} (L - \varepsilon v \cdot x) f, f_{\mathbf{x}_j} \\ &+ \text{Re} \int_0^t e^{2t'/2} U_{\text{kin}}(t) \frac{1}{\varepsilon^2} (L - \varepsilon v \cdot x) f_{\mathbf{x}_{j-1}}, U_{\text{kin}}(t) f_{\mathbf{x}_{j-1}} dt \\ &\leq -\delta \lambda_B \|f\|_{\mathbf{x}_j}^2 + \delta K \|f\|_{\mathbf{x}_{j-1}}^2 \\ &- \frac{\sigma}{\varepsilon^2} \int_0^t e^{2t'/2} U_{\text{kin}}(t) f_{\mathbf{x}_{j-1}}^2 dt + \frac{1}{2} \int_0^t \frac{d}{dt} e^{2t'/2} U_{\text{kin}}(t) f_{\mathbf{x}_{j-1}}^2 dt, \end{aligned}$$

and thus, since  $\lim_t U(t) f_{\mathbf{x}_{j-1}} = 0$  and  $\|\cdot\|_{\mathbf{x}_j} > \|\cdot\|_{\mathbf{x}_j}$ , we have

$$\begin{aligned} \text{Re} (L - \varepsilon v \cdot x) f, f_{\mathbf{x}_j} &\leq -\delta \lambda_B \|f\|_{\mathbf{x}_j}^2 + \delta K \|f\|_{\mathbf{x}_{j-1}}^2 \\ &- \frac{\sigma}{\varepsilon^2} \int_0^t e^{2t'/2} U_{\text{kin}}(t) f_{\mathbf{x}_{j-1}}^2 dt - \frac{1}{2} \|f\|_{\mathbf{x}_{j-1}}^2 \\ &\leq -\sigma \delta \|f\|_{\mathbf{x}_j}^2 + \frac{1}{\varepsilon^2} \int_0^t e^{2t'/2} U_{\text{kin}}(t) f_{\mathbf{x}_{j-1}}^2 dt \\ &- \delta(\lambda_B - \sigma) \|f\|_{\mathbf{x}_j}^2 - \frac{1}{2} \delta K \|f\|_{H_x^s(\mathbf{x}_{j-1})}^2. \end{aligned}$$

We finally deduce, considering  $\delta \leq (2K)^{-1}$  and letting  $\mu = \delta(\lambda_B - \sigma) > 0$

$$\text{Re} (L - \varepsilon v \cdot x) f, f_{\mathbf{x}_j} \leq -\sigma \|f\|_{\mathbf{x}_j}^2 - \mu \|f\|_{\mathbf{x}_j}^2.$$

This concludes this step.

*Step 3: Proof of the nonlinear estimates.* — Using the definition  $P_{\text{kin}} = \text{Id} - P_{\text{hydro}}$ , we have

$$P_{\text{kin}}Q(f, g), h_{\mathbf{x}_j} = Q(f, g), h_{\mathbf{x}_j} + R(f, g, h),$$

where

$$R(f, g, h) := -\langle P_{\text{hydro}}Q(f, g), h \rangle_{\mathbf{x}_j} + \frac{1}{\varepsilon^2} \int_0^T e^{2t/\varepsilon^2} U_{\text{kin}}(t)Q(f, g), U_{\text{kin}}(t)h_{\mathbf{x}_{j-1}} dt.$$

On the one hand, the boundedness  $P_{\text{hydro}} : B(\mathbf{x}_{j-1}; H) \rightarrow B(\mathbf{x}_{j-1}; \mathbf{x}_j)$  implies

$$\begin{aligned} & \left| \langle P_{\text{hydro}}Q(f, g), h \rangle_{\mathbf{x}_j} \right| \leq \|h_{\mathbf{x}_j}\| \|P_{\text{hydro}}Q(f, g)\|_{\mathbf{x}_j} \\ & \leq \|h_{\mathbf{x}_j}\| \|Q(f, g)\|_{\mathbf{x}_{j-1}}, \end{aligned}$$

and on the other hand, using Cauchy-Schwarz inequality and the integral estimates of Lemma 4.8:

$$\begin{aligned} & \frac{1}{\varepsilon^2} \int_0^T e^{2t/\varepsilon^2} U_{\text{kin}}(t)Q(f, g), U_{\text{kin}}(t)h_{\mathbf{x}_{j-1}} dt \\ & \leq \frac{1}{\varepsilon^2} \int_0^T e^{t/\varepsilon^2} \|U_{\text{kin}}(t)Q(f, g)\|_{\mathbf{x}_{j-1}}^2 dt + \int_0^T e^{t/\varepsilon^2} \|U_{\text{kin}}(t)h_{\mathbf{x}_{j-1}}\|_{\mathbf{x}_{j-1}}^2 dt \\ & \leq \|Q(f, g)\|_{\mathbf{x}_{j-1}} \|h_{\mathbf{x}_{j-1}}\|_{\mathbf{x}_{j-1}} + \|Q(f, g)\|_{\mathbf{x}_{j-1}} \|h_{\mathbf{x}_{j-1}}\|_{\mathbf{x}_{j-1}}. \end{aligned}$$

Combining these estimates and using the injection  $\mathbf{x}_j \hookrightarrow \mathbf{x}_{j-1}$ , we deduce that

$$R(f, g, h) \leq \|h_{\mathbf{x}_j}\| \|Q(f, g)\|_{\mathbf{x}_{j-1}}.$$

Recalling that  $\mathbf{x}_{j-1}$  is defined so that the general non-closed estimate of  $Q$  in  $\mathbf{x}_{j-1}$  only involves the norms of the space  $\mathbf{X}_j \hookrightarrow \mathbf{X}_{j-1}$  and  $\mathbf{X}_j^* \hookrightarrow \mathbf{X}_{j-1}^*$  (but not  $\mathbf{X}_{j+1}$  nor  $\mathbf{X}_{j+1}^*$ ), that is to say,

$$Q(f, g)_{\mathbf{x}_{j-1}} \leq \|f_{\mathbf{x}_j^*}\| \|g_{\mathbf{x}_j}\| + \|f_{\mathbf{x}_j}\| \|g_{\mathbf{x}_j^*}\|,$$

we finally end up with

$$R(f, g, h) \leq \|h_{\mathbf{x}_j}\| \|f_{\mathbf{x}_j^*}\| \|g_{\mathbf{x}_j}\| + \|f_{\mathbf{x}_j}\| \|g_{\mathbf{x}_j^*}\| \|h_{\mathbf{x}_j}\|.$$

This allows to conclude this step and the proof of Proposition 7.3.

### 7.2. Stability estimates

We study here the scheme (7.2) in the kinetic-type time-position-velocity space  $X_j$  which corresponds to the space  $X$  introduced in Definition 2.1 with  $E$  replaced with  $\mathbf{X}_j$ , i.e  $X_j$  is characterized by the norm:

$$(7.10) \quad \|f\|_{X_j}^2 := \sup_{0 \leq t < T} \|f(t)\|_{\mathbf{X}_j}^2 + \frac{1}{\varepsilon^2} \int_0^T e^{2t/\varepsilon^2} \|f(t)\|_{\mathbf{X}_j^*}^2 dt.$$

We have the following estimates, valid for any  $N > 0$ :

LEMMA 7.4. — *With the notations of Section 6, one can choose*

$$\varepsilon \leq c_3 \eta \leq c_2 \varepsilon, \quad C_1 \leq 1$$

such that, for any  $N > 0$  and  $j = -1, 0$ :

$$(7.11) \quad \|f_{\text{kin},N}\|_{X_j} \leq C_1 P_{\text{kin}} f_{\text{in}} \chi_j, \quad \|f_{\text{mix},N}\|_F \leq c_2, \quad g_{N,H} \leq c_3.$$

*Proof.* — The proof of the lemma is made by induction over  $N$ . Thanks to (7.9), it will turn useful to estimate the various norms rather with  $\|\cdot\|_{X_j}$ .

The estimates (7.11) are satisfied for  $N = 0$ . Let us assume they hold at rank  $N - 1$  for some  $N > 1$  and deduce them at rank  $N$ .

We recall that  $f_{\text{kin},N}$  is a solution to (7.2) where, thanks to Proposition 7.3, it holds

$$\frac{1}{\varepsilon^2} (L - \varepsilon v \cdot \nabla_x) f_{\text{kin},N}, f_{\text{kin},N} \chi_j, \leq -\frac{\mu}{\varepsilon^2} \|f_{\text{kin},N}\|_{X_j}^2 - \frac{\sigma}{\varepsilon^2} \|f_{\text{kin},N}\|_{X_j}^2,$$

as well as

$$P_{\text{kin}} Q(f_{\text{kin},N-1}, f_{\text{kin},N}), f_{\text{kin},N} \chi_j, \leq \|f_{\text{kin},N}\|_{X_j} \|f_{\text{kin},N}\|_{X_j} \|f_{\text{kin},N-1}\|_{X_{j-1}} + \|f_{\text{kin},N}\|_{X_j} \|f_{\text{kin},N-1}\|_{X_j}.$$

Still using Proposition 7.3, the coupling term is estimated thanks to the non-closed estimate combined with the injections  $H^s \hookrightarrow X_{j+1}$  and  $H^s \hookrightarrow X_{j+1}$ , and the closed estimate respectively:

$$P_{\text{kin}} Q(f_{\text{kin},N}, f_{\text{hydro},N-1} + f_{\text{mix},N-1}), f_{\text{kin},N} \chi_j, \leq \|f_{\text{kin},N}\|_{X_j}^2 \left( \|f_{\text{hydro},N-1}\|_H + \|f_{\text{mix},N-1}\|_H \right) + \|f_{\text{kin},N}\|_{X_j} \|f_{\text{kin},N}\|_{X_j} \left( \|f_{\text{hydro},N-1}\|_H + \|f_{\text{mix},N-1}\|_H \right),$$

and

$$P_{\text{kin}} Q(f_{\text{hydro},N-1} + f_{\text{mix},N-1}, f_{\text{kin},N}), f_{\text{kin},N} \chi_j, \leq \|f_{\text{kin},N}\|_{X_j}^2 \left( \|f_{\text{hydro},N-1}\|_H + \|f_{\text{mix},N-1}\|_H \right) + \|f_{\text{kin},N}\|_{X_j} \|f_{\text{kin},N}\|_{X_j} \left( \|f_{\text{hydro},N-1}\|_H + \|f_{\text{mix},N-1}\|_H \right).$$

Note that the second part of both previous estimates coincide. Put together, and multiplying by  $e^{2t/\varepsilon^2}$ , we have the energy estimate

$$(7.12) \quad \frac{1}{2} \frac{d}{dt} \left( e^{2t/\varepsilon^2} \|f_{\text{kin},N}\|_{X_j}^2 \right) + \frac{\mu}{\varepsilon^2} e^{2t/\varepsilon^2} \|f_{\text{kin},N}\|_{X_j}^2 \leq I_1(t) + I_2(t) + I_3(t) + I_4(t)$$

where we introduced

$$I_1 = \frac{1}{\varepsilon} e^{2t/\varepsilon^2} \|f_{\text{kin},N}\|_{X_j} \|f_{\text{kin},N}\|_{X_j} \|f_{\text{kin},N-1}\|_{X_j} + \|f_{\text{kin},N}\|_{X_j} \|f_{\text{kin},N-1}\|_{X_j}$$

$$\begin{aligned}
I_2 &= \frac{1}{\varepsilon} e^{2t/\varepsilon} \|f_{\text{kin},N}\|_{\mathbf{x}_j} \|f_{\text{kin},N}\|_{\mathbf{x}_j} \left( \|f_{\text{hydro},N-1}\|_{H^1} + \|f_{\text{mix},N-1}\|_{H^1} \right) \\
I_3 &= \frac{1}{\varepsilon} e^{2t/\varepsilon} \|f_{\text{kin},N}\|_{\mathbf{x}_j}^2 \left( \|f_{\text{hydro},N-1}\|_{H^1} + \|f_{\text{mix},N-1}\|_{H^1} \right) \\
I_4 &= \frac{1}{\varepsilon} e^{2t/\varepsilon} \|f_{\text{kin},N}\|_{\mathbf{x}_j} \|f_{\text{kin},N}\|_{\mathbf{x}_j} \left( \|f_{\text{hydro},N-1}\|_{H^1} + \|f_{\text{mix},N-1}\|_{H^1} \right) .
\end{aligned}$$

One easily sees that

$$\int_0^T I_1(t) dt \leq \varepsilon \|f_{\text{kin},N}\|_{X_j}^2 \|f_{\text{kin},N-1}\|_{X_j}$$

where we recall the definition (7.10) and the fact that  $\cdot \mathbf{x}_j \cdot \cdot \mathbf{x}_j$ . We write the time integral of the second term as follows

$$\begin{aligned}
\int_0^T I_2(t) dt &= \varepsilon \int_0^T \frac{1}{\varepsilon} e^{2t/\varepsilon} \|f_{\text{kin},N}\|_{\mathbf{x}_j} \left( \|f_{\text{hydro},N-1}\|_{H^1} + \|f_{\text{mix},N-1}\|_{H^1} \right) dt \\
&\quad + \varepsilon \int_0^T \frac{1}{\varepsilon} e^{2t/\varepsilon} \|f_{\text{kin},N}\|_{\mathbf{x}_j} \left( \|f_{\text{hydro},N-1}\|_{H^1} + \|f_{\text{mix},N-1}\|_{H^1} \right) \left[ e^{2t/\varepsilon} \|f_{\text{kin},N}\|_{\mathbf{x}_j} \right] dt
\end{aligned}$$

where, in each integral, the first two terms in brackets belong to  $L^2(0, T)$  whereas the third one belongs to  $L^1(0, T)$ . Then, it is easy to deduce that

$$\int_0^T I_2(t) dt \leq \varepsilon w_{\text{fNS}}(T)^{-1} \|f_{\text{kin},N}\|_{X_j}^2 \left( \|f_{\text{hydro},N-1}\|_{H^1} + \|f_{\text{mix},N-1}\|_{H^1} \right)$$

where we used again that  $\cdot \mathbf{x}_j \cdot \cdot \mathbf{x}_j$ . One also sees that

$$\begin{aligned}
\int_0^T I_3(t) dt &\leq \varepsilon \int_0^T \frac{1}{\varepsilon} e^{2t/\varepsilon} \|f_{\text{kin},N}\|_{\mathbf{x}_j}^2 \left( \|f_{\text{hydro},N-1}\|_{H^1} + \|f_{\text{mix},N-1}\|_{H^1} \right) dt \\
&\leq \varepsilon w_{\text{fNS}}(T)^{-1} \|f_{\text{kin},N}\|_{X_j}^2 \left( \|f_{\text{hydro},N-1}\|_{H^1} + \|f_{\text{mix},N-1}\|_{H^1} \right) .
\end{aligned}$$

Finally, the term involving  $I_4$  is dealt with as the one involving  $I_2$  writing

$$\begin{aligned}
\int_0^T I_4(t) dt &= \varepsilon \int_0^T \frac{1}{\varepsilon} e^{2t/\varepsilon} \|f_{\text{kin},N}\|_{\mathbf{x}_j} \left( \|f_{\text{hydro},N-1}\|_{H^1} + \|f_{\text{mix},N-1}\|_{H^1} \right) dt \\
&\quad + \varepsilon \int_0^T \frac{1}{\varepsilon} e^{2t/\varepsilon} \|f_{\text{kin},N}\|_{\mathbf{x}_j} \left( \|f_{\text{hydro},N-1}\|_{H^1} + \|f_{\text{mix},N-1}\|_{H^1} \right) \left[ e^{2t/\varepsilon} \|f_{\text{kin},N}\|_{\mathbf{x}_j} \right] dt
\end{aligned}$$

where for both integrals, the first two terms in brackets belong to  $L^2(0, T)$  whereas the third one belongs to  $L^1(0, T)$ . This gives easily

$$\int_0^T I_4(t) dt \leq \varepsilon w_{\text{fNS}}(T)^{-1} \|f_{\text{kin},N}\|_{X_j}^2 \left( \|f_{\text{hydro},N-1}\|_{H^1} + \|f_{\text{mix},N-1}\|_{H^1} \right) .$$

Coming back to (7.12), we finally deduce the recursive estimate

$$\begin{aligned}
\|f_{\text{kin},N}\|_{X_j}^2 &\leq \|f_{\text{kin},N}\|_{X_j}^2 \|f_{\text{kin},N-1}\|_{X_j} + P_{\text{kin}} f_{\text{in}}^2_{H_x^s(\mathbf{x}_j)} \\
&\quad + \varepsilon w_{\text{fNS}}(T)^{-1} \|f_{\text{kin},N}\|_{X_j}^2 \left( \|f_{\text{hydro},N-1}\|_{H^1} + \|f_{\text{mix},N-1}\|_{H^1} \right) ,
\end{aligned}$$

and using the inductive hypothesis (7.11)

$$\|f_{\text{kin},N}\|_{X_j}^2 \leq \varepsilon \left[ C_1 + w_{\tilde{r}_{\text{NS}}}(T)^{-1} (1 + c_2 + c_3) \right] \|f_{\text{kin},N}\|_{X_j}^2 + \mathbf{P}_{\text{kin}} f_{\text{in}} \Big|_{X_j},$$

so that, for  $\varepsilon$  small enough, there holds

$$\|f_{\text{kin},N}\|_{X_j} \leq \mathbf{P}_{\text{kin}} f_{\text{in}} \Big|_{X_j},$$

which allows to deduce the stability estimate at rank  $N$  for  $f_{\text{kin},N}$ .

The proof of the stability estimates for  $f_{\text{mix},N}$  and  $g_N$  is the same as in Section 6. This concludes the proof of Lemma 7.4.

### 7.3. Convergence of the scheme

The convergence will be proved in the larger space  $X_{-1}$  using, of course, the stability estimate in  $X_{-1}$ , but also those in  $X_0$  because of the non-closed estimates. To do so, we denote the difference and sum of successive approximate solutions as

$$\begin{aligned} d_{\text{kin},N} &:= f_{\text{kin},N} - f_{\text{kin},N-1}, & d_{\text{mix},N} &:= f_{\text{mix},N} - f_{\text{mix},N-1} \\ d_{\text{hydro},N} &:= f_{\text{hydro},N} - f_{\text{hydro},N-1} = g_N - g_{N-1}. \end{aligned}$$

One has the following recursive estimate.

PROPOSITION 7.5. — *With the notations of Lemma 7.4, up to reducing again*

$$\varepsilon \leq c_3 \quad \eta \leq c_2 \leq 1,$$

*the following estimate*

$$\begin{aligned} &\|d_{\text{kin},N}\|_{X_{-1}} + \|d_{\text{hydro},N}\|_H + \|d_{\text{mix},N}\|_F \\ &\leq \frac{1}{2} \left( \|d_{\text{kin},N-1}\|_{X_{-1}} + \|d_{\text{hydro},N-1}\|_H + \|d_{\text{mix},N-1}\|_F \right), \end{aligned}$$

*holds for any  $N > 0$ .*

*Proof.* — As for the proof of Lemma 7.4, the difficulty lies in estimating  $d_{\text{kin},N}$  which solves

$$\begin{cases} \partial_t d_{\text{kin},N} = \frac{1}{\varepsilon^2} (L - \varepsilon v \cdot \nabla_x) d_{\text{kin},N} \\ \quad + \frac{1}{\varepsilon} \mathbf{P}_{\text{kin}} Q(f_{\text{kin},N-1}, d_{\text{kin},N}) + \frac{1}{\varepsilon} \mathbf{P}_{\text{kin}} Q(d_{\text{kin},N-1}, f_{\text{kin},N-1}) \\ \quad + \frac{2}{\varepsilon} \mathbf{P}_{\text{kin}} Q^{\text{sym}}(f_{\text{hydro},N-1} + f_{\text{mix},N-1}, d_{\text{kin},N}) \\ \quad + \frac{2}{\varepsilon} \mathbf{P}_{\text{kin}} Q^{\text{sym}}(f_{\text{kin},N-1}, d_{\text{hydro},N-1} + d_{\text{mix},N-1}), \\ d_{\text{kin},N}(0) = 0. \end{cases}$$

We use as previously the equivalent norms  $\|\cdot\|_{\mathcal{X}_{-1}}$ , which allows the use of dissipativity:

$$\frac{1}{\varepsilon^2} (L - \varepsilon v \cdot \nabla_x) d_{\text{kin},N}, d_{\text{kin},N} \Big|_{\mathcal{X}_{-1}} \leq -\frac{\sigma}{\varepsilon^2} \|d_{\text{kin},N}\|_{\mathcal{X}_{-1}}^2 - \frac{\mu}{\varepsilon^2} \|d_{\text{kin},N}\|_{\mathcal{X}_{-1}^\bullet}^2,$$

as well as the closed estimate:

$$P_{\text{kin}} Q(f_{\text{kin},N-1}, d_{\text{kin},N}), d_{\text{kin},N} \mathbf{x}_{-1},$$

$$\cdot \|d_{\text{kin},N}\|_{\mathbf{x}_{-1}^\bullet}^2 \|f_{\text{kin},N-1}\|_{\mathbf{x}_{-1}} + \|d_{\text{kin},N}\|_{\mathbf{x}_{-1}^\bullet} \|d_{\text{kin},N}\|_{\mathbf{x}_{-1}} \|f_{\text{kin},N-1}\|_{\mathbf{x}_{-1}^\bullet},$$

and the non-closed estimate involving the  $\mathbf{x}_0$  and  $\mathbf{x}_0^\bullet$ -norms of  $f_{\text{kin},N-1}$ :

$$P_{\text{kin}} Q(d_{\text{kin},N-1}, f_{\text{kin},N-1}), d_{\text{kin},N} \mathbf{x}_{-1},$$

$$\cdot \|d_{\text{kin},N}\|_{\mathbf{x}_{-1}^\bullet} \|d_{\text{kin},N-1}\|_{\mathbf{x}_{-1}} \|f_{\text{kin},N-1}\|_{\mathbf{x}_0^\bullet}$$

$$+ \|d_{\text{kin},N}\|_{\mathbf{x}_{-1}^\bullet} \|d_{\text{kin},N-1}\|_{\mathbf{x}_{-1}^\bullet} \|f_{\text{kin},N-1}\|_{\mathbf{x}_0^\bullet}.$$

The coupling term involving  $d_{\text{kin},N}$  is estimated using both the closed and non-closed estimates, together with the injections  $H^c \mathbf{x}_0$  and  $H^\bullet \mathbf{x}_0^\bullet$ :

$$P_{\text{kin}} Q^{\text{sym}}(f_{\text{hydro},N-1} + f_{\text{mix},N-1}, d_{\text{kin},N}), d_{\text{kin},N} \mathbf{x}_{-1},$$

$$\cdot \|d_{\text{kin},N}\|_{\mathbf{x}_{-1}^\bullet}^2 \left( \|f_{\text{hydro},N-1}\|_{H^c} + \|f_{\text{mix},N-1}\|_{H^c} \right)$$

$$+ \|d_{\text{kin},N}\|_{\mathbf{x}_{-1}^\bullet} \|d_{\text{kin},N}\|_{\mathbf{x}_{-1}^\bullet} \left( \|f_{\text{hydro},N-1}\|_{H^\bullet} + \|f_{\text{mix},N-1}\|_{H^\bullet} \right),$$

and the coupling term involving  $f_{\text{kin},N-1}$  using the non-closed estimate, together with the injections  $H^c \mathbf{x}_0 \subset \mathbf{x}_{-1}$  and  $H^\bullet \mathbf{x}_0^\bullet \subset \mathbf{x}_{-1}^\bullet$ :

$$P_{\text{kin}} Q^{\text{sym}}(d_{\text{hydro},N-1} + d_{\text{mix},N-1}, f_{\text{kin},N-1}), d_{\text{kin},N} \mathbf{x}_{-1},$$

$$\cdot \|d_{\text{kin},N}\|_{\mathbf{x}_{-1}^\bullet} \|f_{\text{kin},N-1}\|_{\mathbf{x}_0^\bullet} \left( \|d_{\text{hydro},N-1}\|_{H^c} + \|d_{\text{mix},N-1}\|_{H^c} \right)$$

$$+ \|d_{\text{kin},N}\|_{\mathbf{x}_{-1}^\bullet} \|f_{\text{kin},N-1}\|_{\mathbf{x}_0^\bullet} \left( \|d_{\text{hydro},N-1}\|_{H^\bullet} + \|d_{\text{mix},N-1}\|_{H^\bullet} \right).$$

Multiplying these estimates by  $e^{2t}$ , we get the energy estimate

$$\frac{1}{2} \frac{d}{dt} \left( e^{2t} \|d_{\text{kin},N}\|_{\mathbf{x}_{-1}}^2 \right) + \frac{\mu}{\varepsilon^2} e^{2t} \|d_{\text{kin},N}\|_{\mathbf{x}_{-1}^\bullet}^2$$

$$\cdot \frac{1}{\varepsilon} e^{2t} \|d_{\text{kin},N}\|_{\mathbf{x}_{-1}^\bullet}^2 \|f_{\text{kin},N-1}\|_{\mathbf{x}_{-1}}$$

$$+ \frac{1}{\varepsilon} e^{2t} \|d_{\text{kin},N}\|_{\mathbf{x}_{-1}^\bullet} \left( \|d_{\text{kin},N}\|_{\mathbf{x}_0} \|f_{\text{kin},N-1}\|_{\mathbf{x}_{-1}} \right.$$

$$\left. + \|d_{\text{kin},N-1}\|_{\mathbf{x}_{-1}} \|f_{\text{kin},N-1}\|_{\mathbf{x}_0^\bullet} + \|d_{\text{kin},N-1}\|_{\mathbf{x}_{-1}^\bullet} \|f_{\text{kin},N-1}\|_{\mathbf{x}_1^\bullet} \right)$$

$$+ \frac{1}{\varepsilon} e^{2t} \|d_{\text{kin},N}\|_{\mathbf{x}_{-1}^\bullet}^2 \left( \|f_{\text{hydro},N-1}\|_{H^c} + \|f_{\text{mix},N-1}\|_{H^c} \right)$$

$$+ \frac{1}{\varepsilon} e^{2t} \|d_{\text{kin},N}\|_{\mathbf{x}_{-1}^\bullet} \|d_{\text{kin},N}\|_{\mathbf{x}_{-1}^\bullet} \left( \|f_{\text{hydro},N-1}\|_{H^\bullet} + \|f_{\text{mix},N-1}\|_{H^\bullet} \right)$$

$$+ \frac{1}{\varepsilon} e^{2t} \|d_{\text{kin},N}\|_{\mathbf{x}_{-1}^\bullet} \|f_{\text{kin},N-1}\|_{\mathbf{x}_0^\bullet} \left( \|d_{\text{hydro},N-1}\|_{H^c} + \|d_{\text{mix},N-1}\|_{H^c} \right)$$



$$+ \frac{1}{\varepsilon} e^{2t/\tau^2} \|d_{\text{kin},N}\|_{X_{-1}} \|f_{\text{kin},N-1}\|_{X_0} \|d_{\text{hydro},N-1}\|_H + \|d_{\text{mix},N-1}\|_H .$$

Arguing as in the proof of Lemma 7.4, we then obtain

$$\begin{aligned} & \|d_{\text{kin},N}\|_{X_{-1}}^2 \\ & \cdot \varepsilon \|d_{\text{kin},N}\|_{X_{-1}}^2 \|f_{\text{kin},N-1}\|_{X_{-1}} + \varepsilon \|d_{\text{kin},N}\|_{X_{-1}} \|d_{\text{kin},N-1}\|_{X_{-1}} \|f_{\text{kin},N-1}\|_{X_0} \\ & + \varepsilon w_{f_{\text{NS}}}(T)^{-1} \|d_{\text{kin},N}\|_{X_{-1}}^2 \|f_{\text{hydro},N-1}\|_H + \|f_{\text{mix},N-1}\|_F \\ & + \varepsilon w_{f_{\text{NS}}}(T)^{-1} \|d_{\text{kin},N}\|_{X_{-1}} \|f_{\text{kin},N-1}\|_{X_0} \|d_{\text{hydro},N-1}\|_H + \|d_{\text{mix},N-1}\|_F , \end{aligned}$$

and thus using the stability estimates (7.11)

$$\|d_{\text{kin},N}\|_{X_{-1}} \cdot \varepsilon w_{f_{\text{NS}}}(T)^{-1} \left( \|d_{\text{kin},N-1}\|_{X_{-1}} + \|d_{\text{hydro},N-1}\|_H + \|d_{\text{mix},N-1}\|_F \right)$$

i.e., and assuming  $\varepsilon$  small enough,

$$\|d_{\text{kin},N}\|_{X_{-1}} \leq \frac{1}{4} \left( \|d_{\text{kin},N-1}\|_{X_{-1}} + \|d_{\text{hydro},N-1}\|_H + \|d_{\text{mix},N-1}\|_F \right) .$$

Arguing in the very same way as in Section 6.2, one also shows

$$\|d_{\text{hydro},N}\|_H + \|d_{\text{mix},N}\|_F \leq \frac{1}{4} \left( \|d_{\text{kin},N-1}\|_{X_{-1}} + \|d_{\text{hydro},N-1}\|_H + \|d_{\text{mix},N-1}\|_F \right) .$$

Summing up the last two estimates yields the result.

The above result allows to prove in a standard way the convergence of the approximate solutions  $\{(f_{\text{kin},N}, f_{\text{mix},N}, g_N)\}_N$  to some limit  $(f_{\text{kin}}, f_{\text{mix}}, g)$  solving the system

$$\begin{cases} \partial_t f_{\text{kin}} &= \frac{1}{\varepsilon^2} (L - \varepsilon v \cdot \nabla_x) f_{\text{kin}} + \frac{1}{\varepsilon} P_{\text{kin}} Q(f_{\text{kin}}, f_{\text{kin}}) \\ &+ \frac{2}{\varepsilon} P_{\text{kin}} Q^{\text{sym}}(f_{\text{kin}}, f_{\text{hydro}} + f_{\text{mix}}) , \\ f_{\text{kin}}(0) &= P_{\text{kin}} f_{\text{in}} , \\ f_{\text{mix}} &= P_{\text{kin}} [f_{\text{mix}} + f_{\text{hydro}}, f_{\text{mix}} + f_{\text{hydro}}] , \\ g &= [f_{\text{kin}}, f_{\text{mix}}] g + P_{\text{hydro}}(g, g) + S . \end{cases}$$

A similar argument as the one from Section 6.4 performed in  $X_{-1}$  yields the uniqueness, which implies the uniqueness in  $X_0 = X$  and achieves the proof of Theorem 1.12 under Assumptions **BED**.

### Appendix A. About Assumptions L1–L4 and B1-B3

The various assumptions **(L1)–(L4)** and **(B1)–(B2)**, as well as the “enlargement ones” **(LE)**, **(BE)** and **(BED)** were identified as being the properties shared by the Boltzmann and Landau equations in a close to equilibrium setting. We specify in this section that the aforementioned assumptions are proven in the literature, with precise references.

**A.1. The case of the classical Boltzmann equation**

Let us recall that, in the case of the classical Boltzmann equations, the distribution  $\mu$  can be taken assumed to be the centered Maxwellian:

$$\mu(v) := (2\pi)^{-d/2} \exp \left( -\frac{|v|^2}{2} \right), \quad E = d, \quad K = 1 + \frac{2}{d}.$$

The linear operator  $L$  is the linearized nonlinear operator  $Q$  around  $\mu$ :

$$Lf := Q(\mu, f) + Q(f, \mu)$$

where  $Q$  is defined as

$$Q(f, f) := \int_{\mathbb{R}_v^3 \times \mathbb{S}_\sigma^{d-1}} |v - v'| b(\cos \theta) (f(v)f(v') - f(v)f(v')) d\sigma dv,$$

for some parameter  $\gamma \in (-3, 1)$  and some positive function  $b$  smooth on  $(0, 1]$ , and the pre-collisional velocities  $v, v'$  as well as the deviation angle are defined as

$$v = \frac{v + v'}{2} + \sigma \frac{|v - v'|}{2}, \quad v' = \frac{v + v'}{2} - \sigma \frac{|v - v'|}{2}, \quad \cos \theta := \sigma \cdot \frac{v - v'}{|v - v'|}$$

and illustrated in Figure A.1.

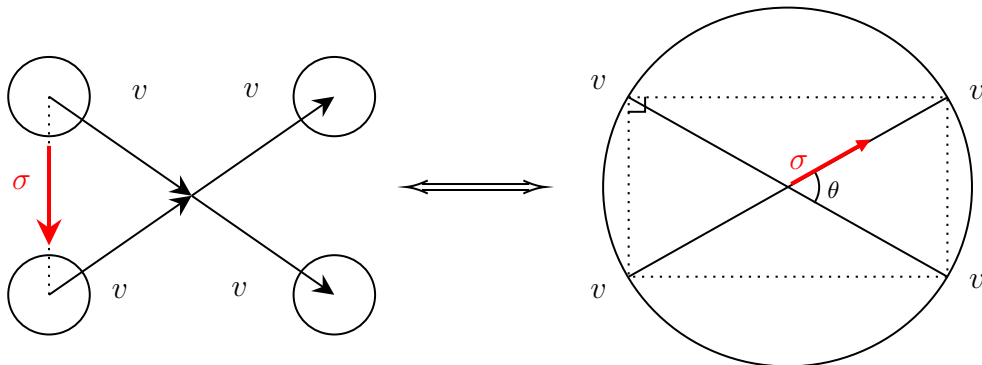


Figure A.1. On the left, the representation of a binary collision. On the right, an illustration of the so-called collisional geometry, representing the distribution of the pre-collisional velocities  $(v, v')$  and the post-collisional ones  $(v, v')$  on the circle centered about (the conserved mean velocity)  $\frac{v+v'}{2}$  of radius (the conserved relative velocity)  $\frac{|v - v'|}{2}$  in the plane  $\text{Span}(v - v', \sigma)$ . This representation allows to visualize the deviation angle  $\theta$ .

A.1.1. The cut-off case

One talks about an *angular cut-off assumption* when the function  $b$  is assumed to be well-behaved enough close to  $\theta = 0$  to satisfy some integrability property. Concerning our basic assumptions **(L1)**–**(L4)** and **(B1)**–**(B2)**, one considers hard or Maxwell potentials under a mild cut-off assumption:

$$\gamma > 0, \quad \sin(\theta)b(\cos \theta) \in L^1([-1, 1]).$$

The hierarchy of spaces  $H_j$  and the dissipation space  $H^\bullet$  are defined as

$$H_j := L^2(\mu^{-1} v^{2j} dv), \quad H^\bullet := L^2 \int \mu^{-1} v^2 dv,$$

and the dissipation estimate is given by the interpolation of the following two estimates coming respectively from Hilbert–Grad’s splitting ([Gra63, Hil12]) and the spectral gap estimate [BM05, Theorem 1.1]:

$$Lg, g_H \leq -c g_{H^\bullet}^2 + C g_H^2, \quad Lg, g_H \leq -c g_H^2.$$

The splitting we consider is

$$Bg := Lg - M\mathbf{1}_{|v| \leq M} g, \quad Ag := M\mathbf{1}_{|v| \leq M} g,$$

for a large enough  $M > 0$ , and it satisfies **L4** thanks to the following weighted estimates coming from [Guo06, Proof of Lemma 3.3 for  $\beta = 0$ ]:

$$Lg, g_{H_j} \leq -c \|v^2 g\|_{H_j}^2 + C \|v^2 f\|_{H_0}^2,$$

for some  $c, C > 0$ . The nonlinear estimate **B3** is given by [Guo04, Lemma 4.1].

The “enlarged” assumptions **(LE)** and **(BE)** are satisfied for some restricted versions of the cut-off model. The linear assumptions **(LE)** are proved under the strong cut-off assumption:

$$b = b(\cos \theta) \in C^1([-1, 1])$$

in the larger energy space  $X$  and dissipation space  $X^\bullet$  defined as

$$X := L^2(v^k dv), \quad X^\bullet := L^2 \int v^{k+2} dv, \quad k > 2.$$

The splitting considered is of the form

$$L = (L - A) + A =: B + A, \\ Ag(v) := \int_{\mathbb{R}^d} a(v, v) g(v) dv, \quad a \in C_c(\mathbb{R}_v^d \times \mathbb{R}_v^d),$$

and the dissipation estimate is given in [BMAM19, Lemma 3.4], whereas the regularization estimate follows from the form of  $A$ . The nonlinear estimate **BE** is given by [BMAM19, Lemma 4.4].

### A.1.2. The non cut-off case

One talks about a *non cut-off case* when the function  $b$  has a non-integrable singularity close to  $\theta = 0$ , and more precisely when it behaves as follows:

$$\sin(\theta)^{d-2} b(\cos \theta) \sim \theta^{-2s}, \quad s \in (0, 1).$$

In this situation, the angular singularity improves the dissipation in the sense that  $L$  presents a spectral gap even for some negative values of  $\gamma$ , at least in the suitable space  $H$ . The basics assumptions **(L1)**–**(L4)** are satisfied for Maxwell, hard and moderately soft potentials:

$$0 < s < 1, \quad \gamma + 2s > 0.$$

The energy space is defined as in the cuto case, whereas this time, the dissipation space  $H^\bullet$  is defined as (see [AHL19, AMU<sup>+</sup>11], we also mention [GS11])

$$\|g\|_{H^\bullet}^2 = \left\| |v|^{-2+s} g \right\|_H^2 + \int_{\mathbb{R}_{v,v}^6 \times \mathbb{S}_\sigma^{d-1}} |v|^{-\mu} (g(v) - g(v)) \, d\sigma dv$$

$$\left\| |v|^{-2+s} g \right\|_H^2 + \left\| |v|^{-2}/v \, |v|^s g \right\|_H^2,$$

where  $|v|^{-2+s}$  is defined as a pseudo-differential operator. The dissipativity estimate **L4** is given by [AMU<sup>+</sup>11, Lemma 2.6], and the nonlinear one (**B3**) is given by [GS11, Theorem 2.1]. The intermediate spaces  $H_j$  are defined as in the cuto case, but this time the associated splitting is

$$Bg := Lg - M\chi \frac{|v|^\kappa}{M} g, \quad Ag := M\chi \frac{|v|^\kappa}{M} g,$$

where  $\chi$  is some smooth bump function and  $M$  is large enough. This splitting satisfies (**L4**) thanks to the following weighted estimates from [GS11, Lemmas 2.4-2.5]:

$$Lg, g_{H_j} \leq -c \|g\|_{H_j}^2 + C \mathbf{1}_{|v| \leq c} \|f\|_{H_j}^2.$$

The enlargement assumptions are known to hold only for hard and Maxwell potentials ( $\gamma > 0$ ) and the enlarged spaces are defined as

$$X_j := L^2(|v|^{-2k+2j} dv), \quad k > 3 + \gamma/2 + 2s,$$

and the dissipation space  $X^\bullet$  is defined in a similar fashion as in the Gaussian case. The decomposition from (**LE**) is defined as in the Gaussian case and the dissipativity estimates follow from the weighted estimates

$$Lg, g_{X_j} \leq -c \|g\|_{X_j}^2 + C \|f\|_{L^2}^2, \quad Lg, g_{X^\bullet} \leq -c \|g\|_{X^\bullet}^2 + C \|f\|_{L^2}^2,$$

which are proved in [HTT20, Lemma 4.2]. The nonlinear estimates are proved in [CDL22, Lemma 2.12] (see also [AMSY21, HTT20]).

### A.2. The case of the classical Landau equation

The Landau equation corresponds in some sense to the non-cuto Boltzmann equation for  $s = 1$ . This time, the nonlinear operator  $Q$  is defined as

$$Q(f, f) = \int_{\mathbb{R}^3} |v - v'|^{-2} (v - v') \left( f(v) |v' f(v) - |v f(v) f(v') \right) dv,$$

where  $(z) = \text{Id} - |z|^{-2} z \otimes z$  is the orthogonal projection onto  $z^\perp$ . The basic assumptions (**L1**)–(**L4**) and (**B1**)–(**B2**) are satisfied for hard, Maxwell and moderately soft potentials ( $\gamma + 2 > 0$ ) and the energy space remains the same as for the Boltzmann equation, but the dissipation space is defined as

$$\|g\|_{H^\bullet}^2 = \left\| |v|^{-2+1} g \right\|_H^2 + \left\| |v|^{-2} (v \otimes v) g \right\|_H^2$$

$$\left\| |v|^{-2+1} g \right\|_H^2 + \left\| |v|^{-2} |v g \right\|_H^2 + \left\| |v|^{-2+1} (v \otimes v) g \right\|_H^2.$$

The dissipativity estimate **(L4)** is given by [Guo02, (24)], and the nonlinear one **(B3)** is given by [Rac21, Lemma 2.2]. The intermediate spaces  $H_j$  and splitting are the same as for the non-cutoff Boltzmann equation, which satisfies **(L4)** using this time the weighted estimates which can be proved as [Guo02, Lemma 6]:

$$\langle Lg, g \rangle_{H_j} \leq -c \|g\|_{H_j}^2 + C \mu \|f\|_{H_j}^2.$$

The enlarged assumptions **(LE)** and **(BED)** hold for Maxwell and hard potentials ( $\gamma > 0$ ) and the enlarged spaces are defined as

$$X_j = L^2(v^{2k+2j} dv), \quad k > \gamma + 17/2,$$

and the dissipation space is defined as

$$\|g\|_{X^*}^2 = \|v^{-\gamma+1} g\|_{X^*}^2 + \|v^{-\gamma} \nu g\|_{X^*}^2 + \|v^{-\gamma+1} (v) \nu g\|_{X^*}^2.$$

The decomposition from **(LE)** is defined as in the gaussian case and the dissipativity is proved in [CTW16, (2.22)-(2.23)]. The nonlinear bounds **(BED)** are proved in [CTW16, Lemma 3.5].

### A.3. Towards more general collision operators

The method tailored in the present contribution should be robust enough to deal with more general models. As in [BGL91], one can tackle the derivation of the Navier–Stokes–Fourier system from a generic collisional kinetic equation conserving mass, momentum and energy, and dissipating entropy of the form

$$(A.1) \quad (\partial_t F + v \cdot \nabla_x) F(t, x, v) = C[F(t, x, \cdot)](v).$$

Neglecting for a while any functional analytic issues, let us explain formally how our framework would adapt to such model.

#### General collisional equation

The macroscopic conservation property reads

$$\int_{\mathbb{R}^d} C[F](v) \frac{1}{|v|^\beta} dv = 0$$

while we assume there exists some convex  $\psi : [0, \infty) \rightarrow \mathbb{R}^+$  such that the following dissipation property (H-theorem) holds

$$D[F] = \int_{\mathbb{R}^d} \psi(F(v)) C[F](v) dv \leq 0$$

together with the equivalence for some universal profile  $\mathcal{M} : \mathbb{R}_m^{d+2} \times \mathbb{R}_v^d \rightarrow [0, \infty)$

$$C[F] = 0 \iff D[F] = 0 \iff F(v) = \mathcal{M}(\mathbf{m}; v)$$

where the macroscopic quantities  $\mathbf{m} = (R_F, U_F, T_F) \in \mathbb{R}^{d+2}$  are defined as

$$R_F = \int_{\mathbb{R}^d} F(v) dv, \quad U_F = \frac{1}{R_F} \int_{\mathbb{R}^d} v F(v) dv, \quad T_F = \frac{1}{dR_F} \int_{\mathbb{R}^d} |v - U_F|^2 F(v) dv.$$

Let us also assume that  $\mathcal{M}$  is related to  $\mathcal{C}$  through

$$(A.2) \quad (\mathcal{M}(\mathbf{m}; v)) = a_{\mathbf{m}} + b_{\mathbf{m}} \cdot v + c_{\mathbf{m}}/|v|^2 \quad \text{for some } (a_{\mathbf{m}}, c_{\mathbf{m}}) \in \mathbb{R}^2, b_{\mathbf{m}} \in \mathbb{R}^d.$$

and that, denoting  $\mathbf{m} = (1, 0, 1)$  and  $\mathcal{M}(v) = \mathcal{M}(\mathbf{m}; v)$ , the macroscopic fluctuations can be factored as

$$\mathcal{M}(\mathbf{m} + \varepsilon(\rho, u, \theta); v) = \mathcal{M}(v) + \varepsilon \left( \rho + u \cdot v + \frac{\theta}{2} (|v|^2 - E) \right) \mu(v) + o(\varepsilon),$$

where we denoted

$$\mu(v) := \frac{1}{(\mathcal{M}(v))}.$$

### The hydrodynamic limit problem

With the above premises, the scaling leading to (1.1) is then

$$F(t, x, v) = \mathcal{M}(v) + \varepsilon f(\varepsilon^2 t, \varepsilon x, v)$$

which, plugged in (A.1), yields

$$\partial_t f + \frac{1}{\varepsilon} v \cdot \nabla_x f = \frac{1}{\varepsilon^2} Lf + \frac{1}{\varepsilon} Q(f, f) + \frac{1}{\varepsilon} R[\varepsilon f]$$

where the operators  $L$ ,  $Q$  and  $R$  are related to  $\mathcal{C}$  through its Taylor expansion about  $\mathcal{M}$ :

$$\mathcal{C}[\mathcal{M} + \varepsilon g] = \varepsilon Lg + \varepsilon^2 Q(g, g) + \varepsilon^2 R[\varepsilon g] \quad \text{where } \lim_0 R[\varepsilon g] = 0.$$

The assumptions **(LE)** and **(BE)** are partially satisfied at a formal level. Denote the spaces

$$N := \text{Span}(\mu, v_1 \mu, \dots, v_d \mu, |v|^2 \mu), \quad H := L^2(\mu(v)^{-1} dv)$$

where  $\mu(v) := \frac{1}{(\mathcal{M}(v))}$ .

The macroscopic conservation property implies the orthogonality properties for  $L$  and  $Q$ :

$$(A.3) \quad Lf, \varphi_H = Q(f, g), \varphi_H = 0, \quad \varphi \in N,$$

and the H-theorem implies, at the quadratic order, the dissipativity of  $L$  for the inner product of  $H$ :

$$\lim_0 \frac{1}{\varepsilon^2} D[\mathcal{M} + \varepsilon g] = Lf, f_H \leq 0$$

thanks to (A.2)–(A.3). Finally, the assumption made on the form of the Maxwellian fluctuations implies the inclusion

$$N \subset \text{Ker}(L)$$

and the reverse inclusion holds as soon as the dissipativity inequality is strict for  $f \in H \setminus N$ .

To handle the presence of the remainder  $R$  taking into account how  $C$  fails to be quadratic, note that it satisfies the macroscopic conservation property

$$R[g], \varphi_H = 0, \quad \varphi = N.$$

As a consequence, we expect it can be handled in a similar way as  $Q$ , thus it is enough to prove uniform bounds  $R[\varepsilon f] = o(1)$  in  $H_x^s(H_v)$ .

### Examples

The above general set of assumptions holds for the classical Boltzmann or Landau equation, as well as the BGK or non-linear Fokker–Planck models with

$$\mathcal{M}(R, U, T; v) = \frac{R}{(2\pi T)^{d/2}} \exp\left(-\frac{|v - U|^2}{2T}\right) \quad \text{and} \quad (r) = \log(r).$$

The above formalism allows to also encompass other models of physical interest. For instance, the BGK operator, defined as

$$C[F](v) = \mathcal{M}(R_F, U_F, T_F; v) - F(v),$$

or the nonlinear Fokker–Planck operator (see [Vil02, Section 1.6]) as defined as

$$C[F](v) = \nu \cdot (T_F \nu + (v - U_F)F).$$

Another set of examples are given by the quantum Boltzmann or Landau equations for which

$$a_m \exp\left(-\frac{|v - b_m|^2}{c_m}\right) = \frac{\mathcal{M}(m, v)}{F[\mathcal{M}(m; v)]} \quad \text{and} \quad (r) = \log\left(\frac{r}{F[r]}\right).$$

In the case of Fermi–Dirac ( $\delta > 0$ ) or Bose–Einstein ( $\delta < 0$ ) statistics (see [Dol94, ABL21]), the function  $F$  is given by

$$F[r] = 1 - \delta r,$$

and in the case of Haldane statistics (see [AN15]),  $F$  is given by

$$F[r] = (1 - \alpha r) (1 + (1 - \alpha)r)^{1-\alpha} \quad \text{for some } \alpha \in (0, 1).$$

The quantum Boltzmann operator is then given by

$$C[F](v) = \int_{\mathbb{R}^d \times S^{d-1}} B(|v - v'|, \sigma) \left\{ f(v) f(v') F[f(v)] F[f(v')] - f(v) f(v') F[f(v)] F[f(v')] \right\} dv' d\sigma,$$

whereas the quantum Landau operator is given by

$$C[F](v) = \nu \int_{\mathbb{R}^d} |v - v'|^{+2} \left\{ f(v) F[f(v')] - \nu f(v) - f(v) F[f(v)] - \nu f(v) \right\} dv'.$$

At the linear level, the quantum Boltzmann and Landau equations fit within our framework. The various assumptions **(L1)**–**(L4)** have been shown to hold for the Boltzmann–Bose–Einstein equation ( $\delta < 0$ ) in [YZ24, Zho22] in the case of very soft potentials ( $\gamma + 2s < 0$ ) for which there does not hold  $\nu \cdot H^* > \nu \cdot H$ , but is more intricate than the one for which **(L4)** would be satisfied ( $\gamma + 2s > 0$ ).

For Fermi-Dirac statistics, a spectral gap estimate can be found in [JXZ22] in the case  $\gamma = b(\cos \theta) = 1$ . Concerning the Landau equation in the case  $\delta > 0$ , the linear assumptions **(L1)**–**(L4)** as well as **(LE)** have been partially checked in [ABL21]. We refer the reader to the work in preparation [GL] for more details on the quantum Boltzmann equation.

## Appendix B. Technical toolbox

### B.1. Littlewood–Paley theory

For some appropriate  $\varphi \in C^\infty(\mathbb{R}^d)$  supported in an annulus centered about 0 and  $\chi \in C^\infty(\mathbb{R}^d)$  supported in a ball centered about 0 such that

$$0 \leq \varphi, \chi \leq 1, \quad \chi(\xi) + \sum_{j=0}^{\infty} \varphi(2^{-j}\xi) = \sum_{j=-\infty}^{\infty} \varphi(2^{-j}\xi) = 1,$$

one defines the *homogeneous Littlewood–Paley projectors* for any  $j \in \mathbb{Z}$  (see [BCD11, Section 2.2]):

$$\dot{S}_j u := F^{-1} [\varphi(2^{-j}\xi) \widehat{u}(\xi)], \quad \dot{S}_j u := F^{-1} [\chi(2^{-j}\xi) \widehat{u}(\xi)],$$

as well as *Bony’s homogeneous decomposition* (see [BCD11, Section 2.6.1]):

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u, v),$$

where the homogeneous paraproduct  $\dot{T}_f g$  and the homogeneous remainder  $\dot{R}(f, g)$  are defined as

$$\dot{T}_f g := \sum_{j=-\infty}^{\infty} \dot{S}_{j-1} f \dot{S}_j g, \quad \dot{R}(f, g) := \sum_{|j-k| \leq 1} \dot{S}_j f \dot{S}_k g.$$

This decomposition allows to prove the following product rule, which follows from the combination of [BCD11, Corollary 2.55] and the embedding  $\dot{B}_{2,1}^s \hookrightarrow \dot{B}_{2,2}^s = \dot{H}^s$ .

**PROPOSITION B.1.** — *For any  $s_1, s_2 \in (-\frac{d}{2}, \frac{d}{2})$  such that  $s_1 + s_2 > 0$ , there holds*

$$uv \in \dot{H}^{-\frac{d}{2} + s_1 + s_2}, \quad u \in \dot{H}^{s_1}, v \in \dot{H}^{s_2}.$$

One also defines the *inhomogeneous Littlewood–Paley projectors* for  $j > -1$  (see [BCD11, Section 2.2]):

$$S_j u := \begin{cases} F^{-1} [\chi(\xi) \widehat{u}(\xi)], & j = -1, \\ \dot{S}_j u, & j > 0, \end{cases} \quad S_j := \sum_{k=-1}^{j-1} \dot{S}_k,$$

as well as Bony’s inhomogeneous decomposition (see [BCD11, Section 2.8.1]):

$$uv = T_u v + T_v u + R(u, v),$$

where the inhomogeneous paraproduct  $T_f g$  and remainder  $R(f, g)$  are defined as

$$T_f g := \sum_{j=-1}^{\infty} S_{j-1} f \dot{S}_j g, \quad R(f, g) := \sum_{|j-k| \leq 1} \dot{S}_j f \dot{S}_k g.$$



This decomposition allows to prove the following product rule, which follows from the combination of [BCD11, Corollary 2.86] and the embedding  $H^\sigma = B_{2,2}^{\sigma} \hookrightarrow L^\infty$  whenever  $\sigma > d/2$ .

**PROPOSITION B.2.** — *For any  $s > 0$  and  $\sigma > d/2$ , there holds*

$$uv \in H^s \quad . \quad u \in H^s \quad v \in H^\sigma \quad + \quad u \in H^\sigma \quad v \in H^s.$$

We also present the following Sobolev–Hölder product rule.

**PROPOSITION B.3.** — *For any  $s > s' > 0$ , there holds*

$$uv \in H^s \quad . \quad u \in H^s \quad v \in W^{s', \infty}.$$

*Proof.* — When  $s \leq N$ , the estimate follows from the combination of Leibniz’s formula and Hölder’s inequality in  $L^2 \times L^\infty$ : for any  $\sigma \leq s$

$$(uv) \in L^2 \quad . \quad \sum_{r=0}^s \left\| \partial^r u \quad \partial^{s-r} v \right\|_{L^2} \quad . \quad \sum_{r=0}^s \left\| \partial^r u \right\|_{L^2} \left\| \partial^{s-r} v \right\|_{L^\infty} \quad . \quad u \in H^\sigma \quad v \in W^{\sigma, \infty},$$

and thus  $uv \in H^s \quad . \quad u \in H^s \quad v \in W^{s, \infty}$ .

When  $s > N$ , then  $W^{s, \infty} = B^s$ , and we rely on Bony’s inhomogeneous decomposition. On the one hand, [BCD11, Theorem 2.82] and [BCD11, Theorem 2.85] yield

$$T_\nu u \in H^s \quad + \quad R(u, v) \in H^s \quad . \quad u \in H^s \quad v \in W^{s, \infty},$$

so we are left with estimating  $T_\nu u$ . Following the proof of [BCD11, Theorem 2.82] (or more precisely [BCD11, Theorem 2.82]), there holds

$$\begin{aligned} T_\nu u \in H^s \quad . \quad & \sum_{j=-1}^s 2^{2js} \left\| S_{j-1} u \quad \partial^j v \right\|_{L^2}^2 \\ & . \quad \sum_{j=-1}^s 2^{-2j(s-s')} \left\| \partial^j v \right\|_{L^\infty}^2 \sum_{j=-1}^s \left\| S_{j-1} u \right\|_{L^2}^2 \\ & . \quad u \in W^{s, \infty} \quad , \quad u \in L^2, \end{aligned}$$

where we used the definition of the norm  $B^s = W^{s, \infty}$  and the fact that  $\left\| S_{j-1} u \right\|_{L^2} \leq C \left\| u \right\|_{L^2}$ .

### B.2. About the wave equation

On the basis of the above Littlewood–Paley decomposition, we also establish the following decay in time of the wave semigroup. Such a result is very likely to be part of the folklore knowledge for dispersive equations and we give a full proof here:

**LEMMA B.4.** — *Given  $u \in \dot{B}_{1,1}^{\frac{d+1}{2}}(\mathbb{R}^d)$  one has*

$$\left\| e^{it|D_x|} u \right\|_{L^\infty} \leq C t^{-\frac{d-1}{2}} \left\| u \right\|_{\dot{B}_{1,1}^{\frac{d+1}{2}}}, \quad t > 0$$

whereas, for  $s > 0$  and  $u \in \dot{B}_{1,1}^{\frac{d+1}{2}+s}$ ,

$$\left\| e^{it|D_x|} u \right\|_{W^{s, \infty}} \leq C t^{-\frac{d-1}{2}} \left\| u \right\|_{\dot{B}_{1,1}^{\frac{d+1}{2}+s}} \quad t > 0.$$

*Proof.* — For any  $j \in \mathbb{Z}$ , we denote  $u_j(x) = u(2^j x)$  and notice

$$e^{it|D_x|} j u = e^{i2^j t|D_x|} {}_0u_{-j} \quad , \quad {}_0u_{-j} = (j u)_{-j} .$$

Using the scaling properties of the  $L^1$  and  $L^\infty$ -norms, we deduce

$$\|e^{it|D_x|} j u\|_{L^1} = \|e^{i2^j t|D_x|} {}_0u_{-j}\|_{L^1} \quad , \quad \|{}_0u_{-j}\|_{L^1_x} = 2^{jd} \|j u\|_{L^1_x} ,$$

and thus, using the dispersive estimate for functions whose frequencies are localized in an annulus (see [BCD11, Proposition 8.15])

$$\begin{aligned} \|e^{it|D_x|} u\|_{L_x} &\leq \sum_{j \in \mathbb{Z}} \|e^{it|D_x|} j u\|_{L_x} \leq \sum_{j \in \mathbb{Z}} (2^j t)^{-\frac{d-1}{2}} \|{}_0u_{-j}\|_{L^1_x} \\ &\leq t^{-\frac{d-1}{2}} \sum_{j \in \mathbb{Z}} (2^j)^{\frac{d+1}{2}} \|j u\|_{L^1_x} = t^{-\frac{d-1}{2}} \|u\|_{\dot{B}_{1,1}^{(d+1)/2}} . \end{aligned}$$

Furthermore, there holds

$$\begin{aligned} \|e^{it|D_x|} u\|_{W_x^s} &\leq \|e^{it|D_x|} u\|_{L_x} + \sup_{j > 0} 2^{js} \|e^{it|D_x|} j u\|_{L_x} \\ &\leq t^{-\frac{d-1}{2}} \|u\|_{\dot{B}_{1,1}^{(d+1)/2}} + \sup_{j > 0} 2^{j(s+\frac{d+1}{2})} \|j u\|_{L^1_x} \\ &\leq t^{-\frac{d-1}{2}} \|u\|_{\dot{B}_{1,1}^{(d+1)/2+s}} . \end{aligned}$$

This concludes the proof.

### B.3. Duality

Consider some surjective isometry  $f : Y^* \rightarrow Y$ , that is to say

$$f^* f^* = f^* .$$

Its adjoint  $f^* : Y \rightarrow Y^*$  for the inner product of  $Y$  is then an isometry as well:

$$f^* f^* = \sup_{g : Y^* = 1} f^* , \quad g^* f^* = \sup_{g : Y = 1} f^* , \quad g^* f^* = f^* f^* ,$$

and it extends naturally to a surjective isometry since  $Y^*$  was defined as the completion of  $Y$ . This allows to write the  $B(Y^* ; Y^*)$ -norm in term of the  $B(Y)$ -norm as well as the isometries  $f^*$  and  $f^*$  :

$$T_{Y^* ; Y^*} = T_{Y ; Y} ,$$

and thus deduce the identity

$$(B.1) \quad T_{Y^* ; Y^*} = T_{Y^* ; Y^*} ,$$

from the classical one  $S_{Y^* ; Y^*} = S_{Y^* ; Y^*}$  and  $(f^*)^* = f^*$ . Similarly, we have

$$(B.2) \quad T_{Y ; Y} = T_{Y ; Y} ,$$

### B.4. Bootstrap formula for projectors

We present some formulas relating the remainder of Taylor expansions for projectors with the lower order terms and remainders. This will allow to prove inductively regularizing properties on each term of said expansion.

LEMMA B.5. — Consider a projector  $P(r) \in \mathcal{B}(E)$  depending on a parameter  $r \in [0, 1]$  and its Taylor expansion at order  $N > 0$ :

$$P(r) = \sum_{n=0}^{N-1} r^n P^{(n)} + r^N P^{(N)}(r),$$

whose (constant) coefficients belong to  $\mathcal{B}(E)$  and satisfy the identities

$$(B.3) \quad \sum_{n=0}^M P^{(n)} P^{(M-n)} = P^{(M)}, \quad 0 \leq M \leq N - 1,$$

then the remainder satisfies a similar one:

$$(B.4) \quad P^{(N)}(r) = P(r)P^{(N)}(r) + \sum_{n=1}^N P^{(n)}(r)P^{(N-n)}$$

$$(B.5) \quad = P^{(N)}(r)P(r) + \sum_{n=1}^N P^{(N-n)}P^{(n)}(r).$$

*Proof.* — We only take care of (B.4). Note that when  $N = 0$ , this reduces to  $P(r) = P(r)P(r)$ , which is true since  $P(r)$  is a projector. We prove the case  $N > 1$  by induction. We start from the induction hypothesis at order  $N$ :

$$P^{(N)}(r) = P(r)P^{(N)}(r) + \sum_{n=1}^N P^{(n)}(r)P^{(N-n)},$$

and inject the expansions  $P^{(N)}(r) = P^{(N)} + rP^{(N+1)}(r)$  and  $P^{(n)}(r) = P^{(n)} + rP^{(n+1)}(r)$ :

$$\begin{aligned} P^{(N)}(r) &= P(r) \left( P^{(N)} + rP^{(N+1)}(r) \right) + \sum_{n=1}^N \left( P^{(n)} + rP^{(n+1)}(r) \right) P^{(N-n)} \\ &= P(r)P^{(N)} + \sum_{n=1}^N P^{(n)}P^{(N-n)} + r \left( P(r)P^{(N+1)}(r) + \sum_{n=1}^N P^{(n+1)}(r)P^{(N-n)} \right). \end{aligned}$$

Next, we expand  $P(r) = P^{(0)} + rP^{(1)}(r)$  in the first line:

$$\begin{aligned} P^{(N)}(r) &= \left( P^{(0)} + rP^{(1)}(r) \right) \left( P^{(N)} + rP^{(N+1)}(r) \right) + \sum_{n=1}^N \left( P^{(n)} + rP^{(n+1)}(r) \right) P^{(N-n)} \\ &= \sum_{n=0}^N P^{(n)}P^{(N-n)} + r \left( P(r)P^{(N+1)}(r) + \sum_{n=1}^N P^{(n+1)}(r)P^{(N-n)} \right). \end{aligned}$$

Since we assumed (B.3), we replace the first term and thus have

$$P^{(N)}(r) = P^{(N)} + r P(r)P^{(N+1)}(r) + \sum_{n=0}^N P^{(n+1)}(r)P^{(N-n)},$$

from which we conclude using  $P^{(N)}(r) = P^{(N)} + rP^{(N+1)}(r)$ .

In the case of the spectral projector from Lemma 3.4, we use the following corollary.

**COROLLARY B.6.** — *The following identities hold for  $N = 1$ :*

$$(B.6) \quad P^{(1)}(\xi) = P(\xi)P^{(1)}(\xi) + P^{(1)}(\xi)P,$$

$$(B.7) \quad = P^{(1)}(\xi)P(\xi) + PP^{(1)}(\xi),$$

and, assuming  $P^{(1)} = PP^{(1)} + P^{(1)}P$ , for  $N = 2$ :

$$(B.8) \quad P^{(2)}(\xi) = P(\xi)P^{(2)}(\xi) + P^{(1)}(\xi) P^{(1)} + P^{(2)}(\xi)P,$$

$$(B.9) \quad = P^{(2)}(\xi)P(\xi) + P^{(1)} P^{(1)}(\xi) + PP^{(2)}(\xi).$$

## Appendix C. Properties of the Navier–Stokes equations

We recall here some classical results on the Navier–Stokes equations and refer to [LR16] and references therein.

**THEOREM C.1 (Cauchy theory for Navier–Stokes).** — *Let  $s > \frac{d}{2} - 1$  and consider a triple of initial conditions  $(\varrho_{\text{in}}, u_{\text{in}}, \theta_{\text{in}}) \in H_x^s$  satisfying*

$$x(\varrho_{\text{in}} + \theta_{\text{in}}) = 0, \quad x \cdot u_{\text{in}} = 0.$$

*There exists a unique maximal lifespan  $T \in (0, \infty]$  such that, for any  $T < T^*$ , the initial data  $(\varrho_{\text{in}}, u_{\text{in}}, \theta_{\text{in}})$  generates a unique solution*

$$(\varrho, u, \theta) \in C([0, T]; H_x^s) \times L^2([0, T]; H_x^{s+1})$$

*to the incompressible Navier–Stokes–Fourier system*

$$(C.1) \quad \begin{aligned} \partial_t u + u \cdot \nabla_x u &= \kappa_{\text{inc}} \nabla_x u - \nabla_x p, & \nabla_x \cdot u &= 0, \\ \partial_t \theta + u \cdot \nabla_x \theta &= \kappa_{\text{Bou}} \nabla_x \theta, & \nabla_x(\varrho + \theta) &= 0, \end{aligned}$$

*and it satisfies for some universal constant  $C > 0$*

$$\begin{aligned} (\varrho, u, \theta) &\in L^2([0, T]; H^s) + \nabla_x(\varrho, u, \theta) \in L^2([0, T]; H^s) \\ &\leq C (\varrho_{\text{in}}, u_{\text{in}}, \theta_{\text{in}})_{H_x^s} \exp \left( C \|u_{\text{in}}\|_{L^2([0, T]; H_x^{\frac{d}{2}-1})} \right). \end{aligned}$$

*If the solution is global (i.e.  $T = \infty$ ), the solution vanishes for large times:*

$$\lim_t (\varrho(t), u(t), \theta(t))_{H^s} = 0,$$

*this is the case if  $d = 2$ , or if  $d > 3$  and  $\|u_{\text{in}}\|_{H_x^{\frac{d}{2}-1}}$  is small.*

Note that, on the one hand,  $\varrho(t), \theta(t) \in L^2_x$  thus the Boussinesq condition  $\operatorname{div}_x(\varrho + \theta) = 0$  is equivalent to  $\varrho + \theta = 0$ , and on the other hand, since  $u$  is incompressible, the pressure (which is to be interpreted as a Lagrange multiplier) can be eliminated using Leray’s projector  $P$  on incompressible fields:

$$(C.2) \quad \begin{cases} \partial_t u + P(\operatorname{div}_x u) = \kappa_{\text{inc}} \operatorname{div}_x u, \\ \partial_t \theta = \kappa_{\text{Bou}} \operatorname{div}_x \theta + u \cdot \operatorname{div}_x \theta, \\ \varrho = -\theta, \end{cases}$$

or, equivalently,

$$(C.3) \quad \begin{cases} \partial_t u + P[\operatorname{div}_x (u \otimes u)] = \kappa_{\text{inc}} \operatorname{div}_x u, \\ \partial_t \theta = \kappa_{\text{Bou}} \operatorname{div}_x \theta + \operatorname{div}_x (u \theta), \\ \varrho = -\theta. \end{cases}$$

The next two results detail in what sense the Navier–Stokes–Fourier system is equivalent to (2.9), proving Proposition 2.5.

LEMMA C.2. — *The following identities hold.*

(1) *For the Burnett function  $\mathbf{A}$ , one has*

$$\langle Q^{\text{sym}}(v_i \mu, v_j \mu), L^{-1} \mathbf{A} \rangle_H = \frac{\vartheta_1}{2} (E_{ij} + E_{j,i} - \frac{2}{d} \delta_{ij} \operatorname{Id}),$$

where  $(E_{ij})_{i,j=1}^d$  is the canonical basis of  $\mathcal{M}_{d \times d}$ , and the coefficient  $\vartheta_1$  is defined as

$$\vartheta_1 := -d \frac{E}{E} \langle Q^{\text{sym}}(v_1 \mu, v_1 \mu), L^{-1} (\operatorname{Id} - P) v_2^2 \mu \rangle_H.$$

Moreover, there holds for  $\varphi = \mu, (|v|^2 - E)\mu$

$$\langle Q^{\text{sym}}(\varphi, v \mu), L^{-1} \mathbf{A} \rangle_H = \langle Q^{\text{sym}}(\varphi, \varphi), L^{-1} \mathbf{A} \rangle_H = 0.$$

(2) *Regarding the Burnett function  $\mathbf{B}$ , one has*

$$\begin{aligned} \langle Q^{\text{sym}}(v_i \mu, \mu), L^{-1} \mathbf{B} \rangle_H &= \vartheta_2 \mathbf{e}_i, \\ \langle Q^{\text{sym}}(v_i \mu, (|v|^2 - E)\mu), L^{-1} \mathbf{B} \rangle_H &= \vartheta_3 \mathbf{e}_i, \end{aligned}$$

where  $(\mathbf{e}_i)_{i=1}^d$  is the canonical basis of  $\mathbb{R}^d$  and the coefficients  $\vartheta_i$  are defined as

$$\begin{cases} \vartheta_2 := -\frac{1}{E \sqrt{K(K-1)}} \langle Q^{\text{sym}}(v_1 \mu, \mu), L^{-1} (\operatorname{Id} - P) v_1 / |v|^2 \mu \rangle_H, \\ \vartheta_3 := -\frac{1}{E \sqrt{K(K-1)}} \langle Q^{\text{sym}}(v_1 \mu, (|v|^2 - E)\mu), L^{-1} (\operatorname{Id} - P) v_1 / |v|^2 \mu \rangle_H. \end{cases}$$

Furthermore, for  $\varphi, \psi = \mu, v \mu, (|v|^2 - E)\mu$ , one has

$$\langle Q^{\text{sym}}(\varphi, \psi), L^{-1} \mathbf{B} \rangle_H = 0 \quad \text{and} \quad \langle Q^{\text{sym}}(v_i \mu, v_j \mu), L^{-1} \mathbf{B} \rangle_H = 0.$$

*Proof.* — We recall the notation from Section 3

$$R_0 := L^{-1}(\text{Id} - P) \quad \text{as well as the identity} \quad L^{-1}\mathbf{A} = \frac{E}{E'} R_0[v \quad v\mu].$$

We also recall that both  $L$  and  $Q^{\text{sym}}$  commute with orthogonal matrices, and in particular preserve the evenness/oddity.

In this proof, we only prove the first identity which is the most intricate, that is we compute for all  $i, j, k, \ell$

$$Q^{\text{sym}}(v_i\mu, v_j\mu), R_0[v_kv\mu].$$

The other identities are proved in a similar yet simpler manner.

*Step 1: The case  $i = j$ .* — If  $\{i, j\} = \{k, \ell\}$ , then  $Q^{\text{sym}}(v_i\mu, v_j\mu)$  is odd in the variables  $v_i$  and  $v_j$ , however  $R_0[v_kv\mu]$  is even in at least one of these variables, thus

$$\langle Q^{\text{sym}}(v_i\mu, v_j\mu), R_0[v_kv\mu] \rangle = 0.$$

If  $\{i, j\} = \{k, \ell\}$ , we use the isometric change of variables  $(v_i, v_j) \xrightarrow{\epsilon} \frac{v_1+v_2}{2}, \frac{v_1-v_2}{2} \xrightarrow{\delta}$ , which is compatible with the invariance of  $L$  and  $Q^{\text{sym}}$ :

$$\begin{aligned} & \langle Q^{\text{sym}}(v_i\mu, v_j\mu), R_0[v_iv_j\mu] \rangle \\ &= \frac{1}{4} \langle Q^{\text{sym}}((v_1+v_2)\mu, (v_1-v_2)\mu), R_0[(v_1-v_2)(v_1+v_2)\mu] \rangle \\ &= \frac{1}{4} \langle Q^{\text{sym}}(v_1\mu, v_1\mu) - Q^{\text{sym}}(v_2\mu, v_2\mu), R_0[v_1^2\mu] - R_0[v_2^2\mu] \rangle \\ &= \frac{1}{2} (\langle Q^{\text{sym}}(v_1\mu, v_1\mu), R_0[v_1^2\mu] \rangle - \langle Q^{\text{sym}}(v_1\mu, v_1\mu), R_0[v_2^2\mu] \rangle) \end{aligned}$$

where we used the change of variables  $(v_1, v_2) \xrightarrow{\delta} (v_2, v_1)$  in the last identity. Using that

$$R_0[v_1^2\mu] + \sum_{j=2}^d R_0[v_j^2\mu] = R_0[|v|^2\mu] = 0$$

together with the change of variables  $v_j \xrightarrow{\delta} v_2$ , we can rewrite the previous identity as

$$\begin{aligned} & \langle Q^{\text{sym}}(v_i\mu, v_j\mu), R_0[v_iv_j\mu] \rangle \\ &= \frac{1}{2} \left( - \sum_{j=2}^d \langle Q^{\text{sym}}(v_1\mu, v_1\mu), R_0[v_j^2\mu] \rangle - \langle Q^{\text{sym}}(v_1\mu, v_1\mu), R_0[v_2^2\mu] \rangle \right) \\ &= -\frac{d}{2} \langle Q^{\text{sym}}(v_1\mu, v_1\mu), R_0[v_2^2\mu] \rangle. \end{aligned}$$

To sum up, if  $i = j$ , we have  $Q^{\text{sym}}(v_i\mu, v_j\mu), L^{-1}\mathbf{A} = \frac{\vartheta_1}{2}(E_{ij} + E_{j,i})$ .

*Step 2: The case  $i \neq j$ .* — If  $k = \ell$ , then  $R_0[v_kv\mu]$  is odd in both  $v_k$  and  $v_j$ , whereas  $Q^{\text{sym}}(v_i\mu, v_i\mu)$  is even in all directions, thus

$$\langle Q^{\text{sym}}(v_i\mu, v_i\mu), R_0[v_kv\mu] \rangle = 0.$$

When  $k = \ell$ , arguing as in *Step 1*, we have

$$\langle Q^{\text{sym}}(v_i\mu, v_i\mu), \mathbf{R}_0[v_k^2\mu] \rangle = \begin{cases} -(d-1) \langle Q^{\text{sym}}(v_1\mu, v_1\mu), \mathbf{R}_0[v_1^2\mu] \rangle, & k = i, \\ \langle Q^{\text{sym}}(v_1\mu, v_1\mu), \mathbf{R}_0[v_2^2\mu] \rangle, & k = i. \end{cases}$$

To sum up, when  $i = j$ , we have  $Q^{\text{sym}}(v_i\mu, v_i\mu), L^{-1}\mathbf{A} = \vartheta_1(E_{i,i} - \frac{1}{d}\text{Id})$  and this proves the result.

*Step 3: Comments on the other coefficients.* — Regarding the Burnett function  $\mathbf{A}$ , one proves similarly for  $\varphi = \mu, (|v|^2 - E)\mu$

$$\langle Q^{\text{sym}}(\varphi, \varphi), L^{-1}\mathbf{A} \rangle_H = \frac{d}{E} \langle Q^{\text{sym}}(\varphi, \varphi), L^{-1}(\text{Id} - \mathbf{P})v_1^2\mu \rangle_H \text{Id},$$

and observing that the  $Q(\varphi, \varphi)$  is radial and that  $|v|^2\mu = \sum_{i=1}^d v_i^2\mu \in \text{Ker}(L)$ , we deduce using the change of variable  $v_1 \rightarrow v_i$  that these coefficients vanish.

Regarding the Burnett function  $\mathbf{B}$ , one proves similarly for  $\varphi = \mu, (|v|^2 - E)\mu$

$$\langle Q^{\text{sym}}(v_i\mu, \varphi), L^{-1}\mathbf{B} \rangle_H = -\frac{1}{E\sqrt{K(K-1)}} \langle Q^{\text{sym}}(v_1\mu, \varphi), L^{-1}(\text{Id} - \mathbf{P})v_1|v|^2\mu \rangle_H,$$

and for  $\varphi, \psi = \mu, (|v|^2 - E)\mu$ , we have that

$$\langle Q^{\text{sym}}(\varphi, \psi), L^{-1}\mathbf{B} \rangle_H = -\frac{1}{E\sqrt{K(K-1)}} \langle Q^{\text{sym}}(\varphi, \psi), L^{-1}(\text{Id} - \mathbf{P})v_1|v|^2\mu \rangle_H = 0,$$

where we used that  $Q(\varphi, \psi)$  is radial and  $L^{-1}(\text{Id} - \mathbf{P})v_1|v|^2\mu$  is odd in  $v_1$ . Finally, there holds

$$Q^{\text{sym}}(v_i\mu, v_j\mu), L^{-1}\mathbf{B} = 0$$

because  $Q^{\text{sym}}(v_i\mu, v_j\mu)$  is odd in both  $v_i$  and  $v_j$  if  $i = j$ , or even in  $v_i = v_j$  otherwise, and  $L^{-1}\mathbf{B}$  is odd. This concludes the proof.

*Remark C.3.* — Note that in the case of the classical Boltzmann and Landau equations, the operator  $L$  is related to  $Q$  through a linearization procedure, and one can show the identity

$$f \in \text{Ker}(L), \quad Q(f, f) = -\frac{1}{2}L(f^2\mu^{-1}),$$

which implies that  $\vartheta_2 = 0$  and the coefficients  $\vartheta_1$  and  $\vartheta_3$  can be computed explicitly.

Thanks to the above result, we are in position to prove the Proposition 2.5.

*Proof of Proposition 2.5.* — We recall the integral formulation of the incompressible Navier–Stokes system:

$$\begin{cases} u(t) = e^{\text{inc}t} \mathbf{x} \mathbf{P} u_{\text{in}} - \vartheta_{\text{inc}} \mathbf{P} \int_0^t e^{(t-\tau)\text{inc} \mathbf{x}} \mathbf{x} \cdot (u \otimes u)(\tau) d\tau, \\ \theta(t) = e^{t \text{Bou} \mathbf{x}} \theta_{\text{in}} - \vartheta_{\text{Bou}} \int_0^t e^{(t-\tau)\text{Bou} \mathbf{x}} \mathbf{x} \cdot (u\theta)(\tau) d\tau, \\ \varrho = -\theta, \end{cases}$$

where we recall that  $\mathbf{P}$  is Leray’s projector on incompressible fields, and we point out the equivalence between  $\mathbf{x}(\varrho + \theta) = 0$  and  $\varrho + \theta = 0$  since  $\varrho(t), \theta(t) \in L^2_x$ .

Regarding the kinetic integral equation, we recall the definitions of  $U_{NS}$  and  $V_{NS}$ :

$$U_{NS}(t) = e^{t \text{ inc}} \cdot x P_{\text{inc}}^{(0)} + e^{t \text{ Bou}} \cdot x P_{\text{Bou}}^{(0)}, \quad V_{NS}(t) = e^{t \text{ inc}} \cdot x P_{\text{inc}}^{(1)} + e^{t \text{ Bou}} \cdot x P_{\text{Bou}}^{(1)},$$

and point out the equivalence coming from Proposition 2.10:

$$(C.4) \quad \text{Id} - P_{\text{inc}}^{(0)} \cdot f = \text{Id} - P_{\text{Bou}}^{(0)} \cdot f = 0 \quad x(\varrho + \theta) = 0 \quad \text{and} \quad x \cdot u = 0,$$

and since **(B1)** assumes  $Q(f, f) \in \text{Ker}(L)$ , we have

$$P_{\text{inc}}^{(0)} P_{\text{inc}}^{(1)} Q(f, f) = P_{\text{inc}}^{(1)} Q(f, f), \quad P_{\text{Bou}}^{(0)} P_{\text{Bou}}^{(1)} Q(f, f) = P_{\text{Bou}}^{(1)} Q(f, f),$$

thus we assume (C.4) from now on.

Since  $U_{NS}$  and  $V_{NS}$  both take values in macroscopic distributions, it is enough to consider their macroscopic components.

*Step 1: Description of  $P_{\text{inc}}^{(1)} Q(f, f)$  and  $P_{\text{Bou}}^{(1)} Q(f, f)$ .* — Plugging the expression (2.7) of  $f$  into the nonlinearity  $Q(f, f)$ , we have

$$\begin{aligned} Q(f, f) &= \varrho^2 Q(\mu, \mu) + Q(u \cdot v\mu, u \cdot v\mu) + \frac{\theta^2}{E^2(K-1)^2} Q((/v^\ell - E)\mu, (/v^\ell - E)\mu) \\ &\quad + 2\varrho u \cdot Q(\mu, v\mu) + \frac{2\varrho\theta}{E(K-1)} Q(\mu, (/v^\ell - E)\mu) \\ &\quad + \frac{2\theta u}{E(K-1)} \cdot Q(v\mu, (/v^\ell - E)\mu). \end{aligned}$$

On the one hand, Lemma C.2 yields

$$\langle Q(f, f), L^{-1} \mathbf{A} \rangle_H = \frac{\vartheta_1}{2} \cdot \left\langle u \cdot \left( u - \frac{2}{d} /u^\ell \text{Id} \right) \right\rangle,$$

and, since for any  $g : \mathbb{R}^d \rightarrow \mathbb{R}$ , there holds  $x \cdot (g\text{Id}) = xg$  and thus  $P(x \cdot (g\text{Id})) = 0$ , we deduce from Proposition 2.10 that

$$\begin{aligned} \left\langle u \cdot x \cdot P_{\text{inc}}^{(1)} Q(f, f) \right\rangle &= \frac{d^{\frac{3}{2}}}{E} P \left\{ x \cdot \langle Q(f, f), L^{-1} \mathbf{A} \rangle_H \right\} \\ &= \frac{d^{\frac{3}{2}}}{E} \frac{\vartheta_1}{2} P \left\{ x \cdot (u \cdot u) \right\}. \end{aligned}$$

On the other hand, Lemma C.2 and (C.4) yield

$$\langle Q(f, f), L^{-1} \mathbf{B} \rangle_H = 2\vartheta_2 u\varrho + \frac{2\vartheta_3}{E(K-1)} u\theta = \left( 2\vartheta_2 + \frac{2\vartheta_3}{E(K-1)} \right) \cdot u\theta,$$

and thus, according to Proposition 2.10,

$$\left\langle \theta \cdot x \cdot P_{\text{Bou}}^{(1)} Q(f, f) \right\rangle = \frac{1}{K\sqrt{K(K-1)}} \left( 2\vartheta_2 + \frac{2\vartheta_3}{E(K-1)} \right) \cdot x \cdot (u\theta).$$

*Step 2: The integral formulation in macroscopic variables.* — We are only left with checking that the  $u$ -part of the kinetic integral system satisfies the Navier–Stokes equations, and that the  $\theta$ -part satisfies the Fourier equation.

Indeed, by Proposition 2.10 and the previous step, there holds



$$\begin{aligned}
 u(t) = u[f(t)] &= u[U_{\text{NS}}(t)f_{\text{in}}] + u \int_0^t V_{\text{NS}}(t - \tau) Q(f(\tau), f(\tau)) \\
 &= e^{t \text{ inc } x} u_{\text{in}} - \vartheta_{\text{inc}} \text{P} \int_0^t e^{(t - \tau) \text{ inc } x} x \cdot (u - u)(\tau) d\tau,
 \end{aligned}$$

as well as

$$\begin{aligned}
 \theta(t) = \theta[f(t)] &= \theta[U_{\text{NS}}(t)f_{\text{in}}] + \theta \int_0^t V_{\text{NS}}(t - \tau) Q(f(\tau), f(\tau)) \\
 &= e^{t \text{ inc } x} \theta_{\text{in}} - \vartheta_{\text{Bou}} \int_0^t e^{(t - \tau) \text{ Bou } x} x \cdot (u\theta)(\tau) d\tau.
 \end{aligned}$$

This concludes the proof of Proposition 2.5.

The next two lemmas provide estimates related to the kinetic version of the Navier–Stokes–Fourier solution, and the first one is essentially a quantitative version of those proved in [GT20].

**LEMMA C.4 (Estimates for Navier–Stokes).** — Suppose  $s > \frac{d}{2}$  and denote the bilinear term  $\varphi = Q(f, f)$  where  $f$  is defined as

$$f(t, x, v) = \varrho(t, x)\mu(v) + u(t, x) \cdot v\mu(v) + \frac{1}{E(K - 1)}\theta(t, x) (|v|^2 - E)\mu(v),$$

and the coefficients  $(\varrho, u, \theta)$  are a solution to the Navier–Stokes–Fourier equations given by Theorem C.1. Then  $\varphi$  satisfies

$$\begin{aligned}
 &\|\varphi\|_{L^p([0, T]; H_x^s(H_v))} + \left\| |x|^{1-\alpha} \varphi \right\|_{L^2([0, T]; H_x^s(H_v))} \\
 &\leq \left( 1 + (\varrho_{\text{in}}, u_{\text{in}}, \theta_{\text{in}})_{\dot{H}^{-\alpha}} + (\varrho, u, \theta)_{L([0, T]; H_x^s)} + \|x(\varrho, u, \theta)\|_{L^2([0, T]; H_x^s)} \right)^2,
 \end{aligned}$$

and for any  $p \in (1, 2]$ , where  $p = 1$  is allowed for  $d > 3$ , its derivative  $\partial_t \varphi$  satisfies

$$\begin{aligned}
 &\|\partial_t \varphi\|_{L^p([0, T]; H_x^{s-1}(H_v))} + \|\partial_t \varphi\|_{L^p([0, T]; \dot{H}_x^{-\frac{1}{2}}(H_v))} \\
 &\leq \left( 1 + (\varrho, u, \theta)_{L([0, T]; H_x^s)} + \|x(\varrho, u, \theta)\|_{L^2([0, T]; H_x^s)} \right)^3.
 \end{aligned}$$

*Proof.* — The function  $\varphi(t, x) \in H$  writes for some  $\varphi_j \in H$  as (note that  $\varrho = -\theta$ )

$$\varphi(t, x) = u(t, x) \cdot u(t, x) : \varphi_1 + u(t, x)\varrho(t, x) \cdot \varphi_2 + \varrho^2(t, x)\varphi_3,$$

and its derivative writes

$$\partial_t \varphi = 2(\partial_t u) \cdot u : \varphi_1 + [(\partial_t u) \varrho + u(\partial_t \varrho)] \cdot \varphi_2 + 2\varrho(\partial_t \varrho) \varphi_3$$

We only prove that the term  $\varrho(\partial_t u)$  satisfies the estimates of the lemma, the other ones being treated the same way. Since  $(\varrho, u, \theta)$  is a solution of the Navier–Stokes–Fourier system, using the formulation (C.2), the term writes (omitting constants)

$$\varrho(\partial_t u) = \varrho \left( \text{P} (u \cdot x u) \right) + \varrho \cdot x u.$$

We will require the product rules recalled in Appendix B.1:

$$(C.5) \quad s_1, s_2 \in \left(-\frac{d}{2}, \frac{d}{2}\right), \quad s_1 + s_2 > 0, \quad gh \in \dot{H}^{s_1+s_2-\frac{d}{2}} \cdot g \in \dot{H}^{s_1} \quad h \in \dot{H}^{s_2},$$

$$(C.6) \quad s_1 > 0, \quad s_2 > \frac{d}{2}, \quad gh \in H_1^s \cdot g \in \dot{H}^{s_1} \quad h \in \dot{H}^{s_2} + g \in \dot{H}^{s_2} \quad h \in \dot{H}^{s_1},$$

and this last one which can be proved as (5.1a):

$$(C.7) \quad s > \frac{d}{2}, \quad r \in \left(0, \frac{d}{2}\right), \quad gh \in H^s \cdot |x|^r g \in H_x^{s-r} \quad h \in H^s.$$

*Step 1: The estimates for  $\varrho(P(u \cdot xu))$ .* — Using the algebra structure of  $H^s$  and  $P \in B(H_x^s)$ :

$$\left\| \varrho(P(u \cdot xu)) \right\|_{H_x^{s-1}} \leq \left\| \varrho(P(u \cdot xu)) \right\|_{H_x^s} \cdot \varrho \in H_x^s \quad u \in H_x^s \quad xu \in H_x^s \quad L_t^2.$$

Applying the product rule (C.5) a first time with the parameters  $(s_1, s_2) = (\frac{1}{2}, \frac{d-1}{2})$  and using the boundedness  $P \in B(\dot{H}^{(d-1)/2})$

$$\left\| \varrho(P(u \cdot xu)) \right\|_{\dot{H}_x^{-\frac{1}{2}}} \leq \varrho \in L_x^2 \quad P(u \cdot xu) \in \dot{H}_x^{(d-1)/2} \cdot \varrho \in L_x^2 \quad u \cdot xu \in \dot{H}_x^{(d-1)/2},$$

and then using the product rule (C.5) a second time with the parameters  $(s_1, s_2) = (\nu, d - \frac{1}{2} - \nu)$  for some  $\nu \in (\frac{d-1}{2}, \frac{d}{2})$  so that  $s_2 \in (\frac{d-1}{2}, \frac{d}{2})$

$$\left\| \varrho(P(u \cdot xu)) \right\|_{\dot{H}_x^{-\frac{1}{2}}} \leq \varrho \in L_x^2 \quad u \in \dot{H}_x^\nu \quad xu \in \dot{H}_x^{d-\nu-\frac{1}{2}}.$$

When  $d > 3$ , we have  $\nu \in (1, s)$  and  $d - \nu - \frac{1}{2} < s$ , and thus

$$\varrho u \cdot xu \in \dot{H}_x^{-\frac{1}{2}} \cdot \varrho \in H_x^s \quad xu \in H_x^{s-1} \quad xu \in H_x^s \quad L_t^1 \quad L_t^2,$$

and when  $d = 2$ , since  $\nu \in (\frac{1}{2}, 1)$ , we have by interpolation

$$\varrho u \cdot xu \in \dot{H}_x^{-\frac{1}{2}} \cdot \varrho \in H_x^s \quad u \in H_x^{1-} \quad xu \in H_x^{1+} \quad L_t^2 \quad L_t^{\frac{2}{1+\nu}},$$

thus, taking  $\nu$  arbitrarily small to 1 yields the result.

*Step 2: The estimates for  $\varrho \cdot xu$ .* — We rewrite this term as

$$\varrho \cdot xu = x \cdot (\varrho \cdot xu) - x\varrho \cdot xu,$$

from which we deduce

$$\varrho \cdot xu \in H^{s-1} \cdot \varrho \cdot xu \in H_x^s + x\varrho \cdot xu \in H^{s-1}.$$

Using for the first term the product rule (C.7) with  $\nu \in (0, 1)$  when  $d = 2$  or  $\nu = 1$  when  $d > 3$ , and (C.6) for the second term, we have

$$\begin{aligned} \varrho \cdot xu \in H^{s-1} \cdot |x| \varrho \in H_x^{s-\nu} \quad xu \in H_x^s + x\varrho \in H_x^{s-1} \quad xu \in H_x^s \\ + x\varrho \in H_x^s \quad xu \in H_x^{s-1} \quad L_t^2 \quad L_t^{\frac{2}{1+\nu}}. \end{aligned}$$

Furthermore, using the product rule (C.5) with the parameters  $(s_1, s_2) = (\nu, \frac{d+1}{2} - \nu)$  for some  $\nu \in (\frac{1}{2}, \frac{d}{2})$ , and with  $(s_1, s_2) = (\frac{d-1}{2}, 0)$ , we have

$$\varrho \cdot xu \in \dot{H}_x^{-\frac{1}{2}} \cdot \varrho \cdot xu \in \dot{H}_x^{\frac{1}{2}} + x\varrho \cdot xu \in \dot{H}_x^{-\frac{1}{2}}$$

$$\begin{aligned} & \cdot \varrho \dot{H}_x^\nu \quad \times u \dot{H}_x^{(d+1)/2-\nu} + \quad \times \varrho \dot{H}_x^{(d-1)/2} \quad \times u \dot{L}_x^2 \\ & \cdot \quad \times u \dot{H}_x^s \quad \varrho \dot{H}_x^\nu + \quad \times \varrho \dot{H}_x^{s-1} \quad \cdot \end{aligned}$$

In the case  $d > 3$ , we deduce taking  $\nu = 1$

$$\varrho \times u \dot{H}_x^{-\frac{1}{2}} \cdot \quad \times u \dot{H}_x^s \quad \times \varrho \dot{H}_x^{s-1} \quad L_t^1 \quad L_t^2,$$

and when  $d = 2$ , by interpolation,

$$\varrho \times u \dot{H}_x^{-\frac{1}{2}} \cdot \quad \times u \dot{H}_x^s \quad \times \varrho \dot{H}_x^{s-1} \quad \varrho \dot{H}_x^{1-\frac{1}{1+\nu}} \quad L_t^{\frac{2}{1+\nu}} \quad L_t^2,$$

from which we conclude the result by taking  $\nu$  arbitrarily close to 1. This concludes the proof of the estimates for  $\partial_t \varphi$ .

For the estimates of  $\varphi$ , one proves similarly

$$\varrho u \dot{H}_x^s \cdot \quad \varrho \dot{H}_x^s \quad u \dot{H}_x^s$$

and

$$\left\| \int x^{l-} (\varrho u) \right\|_{\dot{H}_x^s} \cdot \left( \left\| \int x^{l-} \varrho \right\|_{\dot{H}_x^s} + \quad \times \varrho \dot{H}_x^s \right) \quad u \dot{H}_x^s + \quad \varrho \dot{H}_x^s \quad \times u \dot{H}_x^s$$

which allows to conclude using Lemma C.5. This concludes the proof of Lemma C.4.

**LEMMA C.5 (The Navier–Stokes–Fourier solution and the space  $H$ ).** *The Navier–Stokes solution in its kinetic form belongs to the space  $H^s$  (where the parameter  $\alpha$  defines this space):*

$$f \dot{H} \cdot \quad (\varrho_{\text{in}}, u_{\text{in}}, \theta_{\text{in}}) \dot{H}_x^{-\alpha} + \quad (\varrho, u, \theta) \dot{L} \quad ([0, T]; \dot{H}_x^s) + \quad \times (\varrho, u, \theta) \dot{L}^2 \quad ([0, T]; \dot{H}_x^s) \cdot$$

Furthermore, it can be approximated by a smoother sequence; there exists  $(f)_{(0,1]} \dot{H}^{s+1}$  such that

$$\lim_0 f - f \dot{H}^s = 0.$$

*Proof.* — The proof is given in two steps.

*Step 1: Bound in  $H$ .* — We only need to consider the case  $d = 2$  and only prove the estimate for  $u$ . We start by applying Duhamel’s principle to (C.3):

$$u(t) = e^{t \text{ inc } \times} u_{\text{in}} - \int_0^t e^{(t-\tau) \times} \varphi(\tau) d\tau, \quad \varphi := P \left[ \quad \times \cdot (u \quad u) \right],$$

from which we obtain

$$\begin{aligned} \int_0^T \left\| \int x^{l-} u(t) \right\|_{\dot{L}_x^2}^2 dt & \cdot \int_0^T \left\| \int x^{l-} e^{t \text{ inc } \times} u_{\text{in}} \right\|_{\dot{L}_x^2}^2 dt \\ & + \int_0^T \left\| \int x^{l-} \int_0^t e^{(t-\tau) \times} \varphi(\tau) d\tau \right\|_{\dot{L}_x^2}^2 dt, \end{aligned}$$

or, equivalently, in Fourier variables:

$$\int_0^T \left\| |x|^{1-\alpha} u(t) \right\|_{L_x^2}^2 dt + \int_{\mathbb{R}^d} |\xi|^{-2} |\widehat{u}_{\text{in}}(\xi)|^2 \int_0^T |\xi|^{2\alpha} e^{-2|\xi|t} dt d\xi + \int_{\mathbb{R}^d} |\xi|^{-2} \int_0^T \int_0^t |\xi|^{2\alpha} e^{-(t-\tau)|\xi|} \widehat{\varphi}(\tau, \xi) d\tau dt d\xi.$$

Using Young’s convolution inequality in the form  $L^2([0, T]) \cdot L^1([0, T]) \subset L^2([0, T])$  followed by Minkowski’s integral inequality for the second term, we thus get

$$\int_0^T \left\| |x|^{1-\alpha} u(t) \right\|_{L_x^2}^2 dt + \int_{\mathbb{R}^d} |\xi|^{-2} |\widehat{u}_{\text{in}}(\xi)|^2 d\xi + \int_{\mathbb{R}^d} |\xi|^{-2} \int_0^T |\widehat{\varphi}(t, \xi)| dt d\xi \leq C \left( \|u_{\text{in}}\|_{\dot{H}_x^{-\alpha}}^2 + \int_0^T \|\varphi(t)\|_{\dot{H}_x^{-\alpha}}^2 dt \right).$$

Since  $\varphi \in B(\dot{H}_x^{-\alpha})$ , we get using the product rule (C.5) (using that  $\frac{d}{2} = 1$ ) and then by interpolation

$$\|\varphi\|_{\dot{H}_x^{-\alpha}} \leq C \|u\|_{\dot{H}_x^{1-\alpha}} \|u\|_{\dot{H}_x^{1-\frac{\alpha}{2}}} \leq C \|xu\|_{L_x^2} \left\| |x|^{1-\alpha} u \right\|_{L_x^2}$$

from which we conclude using Cauchy–Schwarz

$$\int_0^T \left\| |x|^{1-\alpha} u(t) \right\|_{L_x^2}^2 dt \leq C \left( \|u_{\text{in}}\|_{\dot{H}_x^{-\alpha}}^2 + \int_0^T \left\| |x|^{1-\alpha} u(t) \right\|_{L_x^2}^2 dt + \int_0^T \|xu(t)\|_{L_x^2}^2 dt \right),$$

and thus, by Young’s inequality

$$\int_0^T \left\| |x|^{1-\alpha} u(t) \right\|_{L_x^2}^2 dt \leq C \left( \|u_{\text{in}}\|_{\dot{H}_x^{-\alpha}}^2 + \int_0^T \|xu(t)\|_{L_x^2}^2 dt \right).$$

This concludes this step.

**Step 2: Approximation by functions in  $H^{s+1}$ .** — Since the solution is instantly regularized in the sense that

$$(\varrho, xu, \theta) \in L^2([0, T]; H_x^s),$$

and in virtue of the control from Theorem C.1, there holds for any  $\delta \in (0, T)$

$$\|(\varrho, u, \theta)\|_{L([0, T]; H^{s+1})} + \|x(\varrho, u, \theta)\|_{L^2([0, T]; H^{s+1})} \leq C \left( \|\varrho(\delta), u(\delta), \theta(\delta)\|_{H_x^{s+1}} \exp(C \|xu\|_{L^2([0, T]; H_x^{\frac{d}{2}-1})} \right),$$

and thus from Step 1, we have that  $f(\delta + \cdot) \in H^{s+1}$ . From this observation, considering (by the continuity of  $f$  and the density of  $H_x^{s+1}$ ) for any  $\varepsilon > 0$  some  $\delta > 0$  and  $f_{\text{in}} \in H_x^{s+1}(H_v)$  such that

$$\sup_{0 \leq t \leq \varepsilon} \|f(t) - f_{\text{in}}\|_{H_x^s(H_v)} \leq \varepsilon, \quad \int_0^\varepsilon \left\| |x|^{1-\alpha} f(t) \right\|_{H_x^s(H_v)}^2 dt \leq \varepsilon,$$

and up to a reduction of  $\delta$

$$\| |x|^{\beta} f_{\text{in}} \|_{H_x^s(H_v)} \leq \frac{\varepsilon}{\delta}.$$

We can therefore define

$$f(t) = \begin{cases} f_{\text{in}}, & t \in [0, \delta), \\ f(t), & t \in [\delta, T), \end{cases}$$

so as to have  $f \in H^{s+1}$  and

$$\begin{aligned} \| f - f_{\text{in}} \|_{H^s}^2 &\leq \sup_{0 \leq t \leq \varepsilon} \| f(t) - f_{\text{in}} \|_{H_x^s(H_v)}^2 \\ &\quad + \int_0^\varepsilon \| |x|^{\beta} f(t) - |x|^{\beta} f_{\text{in}} \|_{H_x^s(H_v)}^2 dt \leq \varepsilon^2, \end{aligned}$$

which concludes the proof of Lemma C.5.

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