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DEFORMING A FINSLER METRIC ON THE TWO-TORUS TO A FLAT FINSLER METRIC WITH CONJUGATE GEODESIC FLOWS

DÉFORMATION D'UNE MÉTRIQUE DE FINSLER SUR LE TORE BIDIMENSIONNEL EN UNE MÉTRIQUE DE FINSLER PLATE AVEC DES FLOTS GÉODÉSIQUES CONJUGUÉS

Abstract. — We show that the space of (reversible) Finsler metrics on the two-torus $\mathbb{T}^2$ whose geodesic flow is conjugate to the geodesic flow of a flat Finsler metric strongly deformation retracts to the space of flat Finsler metrics with respect to the uniform convergence topology. Along the proof, we also show that two Finsler metrics on $\mathbb{T}^2$ without conjugate points, whose Heber foliations are smooth and with the same marked length spectrum, have conjugate geodesic flows.

Keywords: Finsler metrics, dynamical systems, geodesic flow, conjugate flows, conjugate points, integral geometry, Crofton formula, Heber foliation, curve shortening flow.
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Résumé. — Nous montrons que l'espace des métriques de Finsler (réversibles) sur le tore bidimensionnel $\mathbb{T}^2$, dont le flot géodésique est conjugué au flot géodésique d'une métrique de Finsler plate se rétracte par déformation forte sur l'espace des métriques de Finsler plates par rapport à la topologie de la convergence uniforme. Au cours de la preuve, nous montrons également que deux métriques de Finsler sur $\mathbb{T}^2$ sans points conjugués, dont les feuilletages d'Heber sont lisses et ont le même spectre marqué des longueurs, ont des flots géodésiques conjugués.

1. Introduction

The goal of this article is to study the space of Finsler two-tori whose geodesic flow is (smoothly) conjugate to the geodesic flow of a flat Finsler two-torus. (By definition, all Finsler metrics are reversible and quadratically convex; see Definition 2.1). In particular, we will determine the topology of this space up to homotopy with respect to the uniform convergence topology; see Corollary 1.2.

This problem can be seen as the counterpart of a famous question about Zoll metrics (i.e., metrics all of whose geodesics are simple closed curves of the same length) asking whether the space of Zoll metrics, say in the two-sphere, is path-connected; see [Ber03, Question 200]. (Though this question is wide open, it has a positive answer for Zoll Finsler metrics on the projective plane; see [Sab19].) Since the geodesic flow of a Zoll metric on the two-sphere is conjugate to that of the canonical metric, see [Bes78, § 4F] or [ABHS17], this question amounts to asking whether the space of metrics whose geodesic flow is conjugate to that of the round two-sphere is path-connected. Clearly, the two-torus does not admit any Zoll metric, but the latter formulation of the problem makes sense if one replaces the round two-sphere with a flat two-torus.

Complete Finsler manifolds without conjugate points have for characteristic property that any pair of points in their universal cover can be joined by a unique geodesic; see Definition 2.2. There are strong ties between Finsler metrics whose geodesic flow is conjugate to the geodesic flow of a flat Finsler two-torus and metrics without conjugate points. Indeed, every Finsler metric F on the two-torus whose geodesic flow is conjugate to that of a flat Finsler metric $F_\diamond$ has no conjugate points; see [Cro90] or Lemma 11.3. For a Riemannian metric F, this implies that F is flat by Hopf's theorem [Hop48] (see [BI94] for a generalization to any dimension), in which case, the metrics F and $F_\diamond$ coincide. The problem is therefore interesting only for non-Riemannian Finsler metrics. It is an open question whether the geodesic flow of every Finsler two-torus without conjugate points is conjugate to that of some flat Finsler metric; see [BC21, CK94]. The issue is related to the regularity of the so-called Heber foliation (a continuous foliation of $T^*\mathbb{T}^2$ by Lipschitz, Lagrangian, flow-invariant graphs given by the covectors of the same norm generating geodesics with the same asymptotic direction); see Section 2. Actually, it can be proved that two Finsler two-tori without conjugate points having the same marked length spectrum and smooth Heber foliations have conjugate geodesic flows; see Section 11.5. To further illustrate our poor understanding of Finsler metrics without conjugate points, let us also mention that it is still an open question whether a geodesic with irrational direction on a Finsler two-torus without conjugate points is dense; see [BC21].

Before stating our main result, it should be noted that Finsler two-tori without conjugate points are incredibly flexible: given any point on a Finsler surface, there exists a neighborhood of this point which isometrically embeds into a Finsler two-torus without conjugate points; see [Che19]. In particular, modulo isometries and rescaling, Finsler two-tori without conjugate points form a infinite-dimensional space. Similarly, the space of Finsler two-tori with geodesic flow conjugate to that of a flat Finsler metric (modulo isometries) has infinite dimension; see [Sab19, Appendix] for instance.

Our main theorem is the following.

Theorem 1.1. — *Let $M = (\mathbb{T}^2, F)$ be a Finsler two-torus whose geodesic flow is conjugate to the geodesic flow of a flat Finsler two-torus $M_\diamond = (\mathbb{T}^2, F_\diamond)$. Then there exists a canonical deformation $(F_t)_{t \geqslant 0}$ of Finsler metrics on $\mathbb{T}^2$ with $F_0 = F$ such that*

(1) *the geodesic flow of F_t is conjugate to the geodesic flow of $F_\diamond$;*
(2) *the metric F_t converges to $F_\diamond$ for the uniform convergence topology, up to isometry, as t goes to infinity.*

In this theorem, we consider the uniform convergence of metric spaces, where a sequence (d_n) of metrics on a given set X converges to a metric d on X if $d_n \to d$ uniformly on $X \times X$ as n goes to infinity. We refer to the topology it induces on the space of metrics on X as the *uniform convergence topology*.

The following result is a consequence of the main theorem.

Corollary 1.2. — *The space of Finsler metrics on the two-torus, modulo isometries, whose geodesic flow is conjugate to that of a flat Finsler metric strongly deformation retracts to the space of flat Finsler metrics, modulo isometries (which is contractible).*

In addition, the strong deformation retraction is induced by the deformation of the geodesic foliation by the curve shortening flow on the Euclidean plane.

Our results are of the same flavor as those of [Sab19], where it is proved that the space of Zoll Finsler metrics on the projective plane $\mathbb{RP}^2$ whose geodesic length is equal to π strongly deformation retracts to the canonical round metric. In this case too, the deformation retraction is induced by the curve shortening flow on the canonical round projective plane.

In the proof of the main theorem, we will also establish the following result regarding the deformation of the geodesic flows of a two-torus without conjugate points.

Theorem 1.3. — *Let $M = (\mathbb{T}^2, F)$ be a Finsler two-torus without conjugate points. Denote by $\rho : \mathbb{R} \times U_0\mathbb{R}^2 \to U_0\mathbb{R}^2$ the action induced by the geodesic flow on the unit tangent bundle $U\bar{M} \simeq U_0\mathbb{R}^2$ of $\bar{M}$. Then, there exists a deformation*

$$\rho_t : \mathbb{R} \times U_0\mathbb{R}^2 \to U_0\mathbb{R}^2$$

of smooth, free, proper, $\mathbb{Z}^2$-equivariant actions that starts at $\rho_0 = \rho$ and converges to the action $\rho_\infty : \mathbb{R} \times U_0\mathbb{R}^2 \to U_0\mathbb{R}^2$ induced by the geodesic flow on the unit tangent bundle $U_0\mathbb{R}^2$ of the Euclidean plane. Here, the convergence is in the compact-open C^k-topology for any given $k \geqslant 0$.

Furthermore, for every $t \in [0, \infty]$, every ρ_t-orbit projects to an embedding of $\mathbb{R}$ into $\mathbb{R}^2$ under the canonical projection $U_0\mathbb{R}^2 \to \mathbb{R}^2$.

By construction, the deformation $\rho_t : \mathbb{R} \times U_0\mathbb{R}^2 \to U_0\mathbb{R}^2$ in Theorem 1.3 is induced by the deformation of the geodesics of $\bar{M}$ under the Euclidean curve shortening flow. Since the Euclidean curve shortening flow preserves the asymptotic directions of these geodesics, we deduce the following result regarding the deformation of the Heber foliation.

Corollary 1.4. — *Let $M = (\mathbb{T}^2, F)$ be a Finsler two-torus without conjugate points. Then, the Heber foliation of $T^*\bar{M}$ is deformed into the canonical Heber foliation of $T^*\mathbb{R}^2$ (induced by straight lines) under the deformation $\rho_t : \mathbb{R} \times U_0\mathbb{R}^2 \to U_0\mathbb{R}^2$, via the Legendre transform.*

The proof strategy of our main theorem relies on arguments from dynamical systems, geometric flow, and integral geometry, and follows the approach developed in [Sab19] to study the space of Zoll Finsler metrics on the projective plane. We work on the universal cover $\bar{M}$ of Finsler two-tori M without conjugate points, where all geodesics are minimizing (but not closed), and perform a series of $\mathbb{Z}^2$-equivariant constructions. The space of (oriented) geodesics of $\bar{M}$ is a manifold diffeomorphic to $S^1 \times \mathbb{R}$ equipped with a natural smooth measure induced by the canonical symplectic form on the cotangent bundle $T^*\bar{M}$ of $\bar{M}$. Now, the idea is to make use of the one-to-one correspondence between Finsler metrics without conjugate points and their space of geodesics endowed with their natural smooth measure through Crofton's formula. This correspondence, due to Álvarez Paiva and Berck [ÁPB10], allows us to construct Finsler metrics with prescribed geodesics on the plane. We then deform the geodesics of $\bar{M}$ into straight lines while preserving their asymptotic direction and intersection properties through the curve-shortening flow for the Euclidean metric (with an analysis of the underlying parabolic PDE). This deformation of the geodesics of $\bar{M}$ along with the deformation of the natural measure on their moduli space induces a canonical deformation $(\bar{F}_t)$ of the initial Finsler metric $\bar{F}$ on $\bar{M}$. Using a fine analysis of the continuity of the curve-shortening flow at the limit for irrational directions, we show that the deformation process preserves the natural measure on the space of deformed geodesics when the geodesic flow of the Finsler metric F is conjugate to that of a flat Finsler metric $F_\diamond$. This ensures that the Finsler metric deformation $(\bar{F}_t)$ converges to $\bar{F}_\diamond$. By construction, the Finsler metrics $(\bar{F}_t)$ of the deformation have no conjugate points. The final step is to show that these metrics have conjugate geodesic flows modulo the action of $\mathbb{Z}^2$. This is precisely where we need to assume the geodesic flow of our initial metric is conjugate to that of a flat Finsler metric. We first observe that the Finsler metrics $(\bar{F}_t)$ of the deformation have the same length spectrum. Then we show that their Heber foliations are smooth. To conclude, we apply the following dynamical result proved in Section 12.

Theorem 1.5. — *Let $M_1 = (\mathbb{T}^2, F_1)$ and $M_2 = (\mathbb{T}^2, F_2)$ be two Finsler tori without conjugate points whose Heber foliations are smooth. Assume that both metrics have the same marked length spectrum. Then the geodesic flows of M_1 and M_2 are conjugate.*

To implement the proof scheme described above, numerous difficulties need to be faced: lack of regularity, lack of compactness, convergence issues, lack of averaging procedure, non-continuous nature of group actions, etc. Consequently, the proof relies on a wide range of techniques in dynamical systems, integral geometry and partial differential equations.

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2. Asymptotic directions, Busemann functions and the Heber foliation

In this section, we introduce the asymptotic direction of a minimizing geodesic, define Busemann functions and present some general results regarding the Heber foliation of Finsler two-tori without conjugate points.

Definition 2.1. — *A (reversible)* Finsler metric *on a manifold M is a continuous function $F : TM \to [0, \infty)$ on the tangent bundle TM of M satisfying the following properties (here, $F_x := F_{|T_xM}$ for short):*

1. *Smoothness: F is smooth outside the zero section;*
2. *Homogeneity: $F_x(tv) = |t|\, F_x(v)$ for every $v \in T_xM$ and $t \in \mathbb{R}$;*
3. *Quadratic convexity: for every $x \in M$, the square of the function F_x has positive definite second derivatives on $T_xM \setminus \{0\}$, that is, for every $p \in T_xM \setminus \{0\}$, and $u, v \in T_xM$, the symmetric bilinear form*
$$g_p(u,v) = \frac{1}{2}\,\frac{\partial^2}{\partial s \partial t} F_x^2(p + tu + sv)_{|t=s=0}$$
is an inner product.

The F-length of a smooth curve $\gamma : [a,b] \to M$ is defined by
$$\mathrm{length}_F(\gamma) = \int_a^b F(\gamma'(t))\, dt$$
and the distance $d_F(p,q)$ between two points $p, q \in M$ is the infimum F-length of the curves joining p and q.

Consider the Legendre transform
$$\mathcal{L} : TM \to T^*M \tag{2.1}$$
of the Lagrangian $\frac{1}{2}F^2$ between the tangent and cotangent bundles of M; see [Bes78]*. Since F is quadratically convex, the Legendre transform $\mathcal{L}$ is a diffeomorphism between $TM \setminus \{0\}$ and $T^*M \setminus \{0\}$. By the homogeneity of F, it preserves the norm on each fiber of the bundle vectors TM and T^*M. In addition, it induces a diffeomorphism between the unit tangent and cotangent bundles UM and U^*M of M.*

Geometrically, this diffeomorphism is defined as follows: For every vector $v \in U_xM$, the image $\mathcal{L}(v)$ is the unique co-vector of U_x^*M such that $\mathcal{L}(v)(v) = 1$.

The quadratically convex condition (as opposed to a mere convex condition) also allows us to define the geodesic and cogeodesic flows on UM and U^*M; see [Bes78]. *Note that both flows are conjugate by the Legendre transform.*

The tautological one-form *on T^*M is defined by*

$$\alpha_\xi(X) = \xi\left(dp_\xi(X)\right) \tag{2.2}$$

for every $\xi \in T^*M$ and $X \in T_\xi T^*M$, where $p : T^*M \to M$ is the canonical projection. Similarly, the canonical symplectic form *on T^*M is defined as*

$$\omega = d\alpha. \tag{2.3}$$

Note that neither α nor ω depend on the Finsler metric on M. Still, by the Liouville theorem, see [Bes78, Remark 2.12], *both the one-form α and the symplectic form ω are invariant under the cogeodesic flow of M.*

Finsler metrics without conjugate points can be defined as follows.

Definition 2.2. — *A complete Finsler manifold M has no conjugate points if the exponential map of its universal Finsler cover $\bar{M}$ is a diffeomorphism at every point, that is, if $\exp_x : T_x\bar{M} \to \bar{M}$ is a diffeomorphism for every $x \in \bar{M}$. In this case, any pair of points in $\bar{M}$ can be joined by a unique geodesic.*

We will also need the following definition about Finsler geodesics.

Definition 2.3. — *Let M be a complete Finsler manifold with universal Finsler cover $\bar{M}$. A geodesic of $\bar{M}$ is* minimizing *if it minimizes the length between any pair of its points. (Such geodesics are also referred to as A-*geodesics.*)*

Before giving a characterization of Finsler tori without conjugate points in terms of integrable cogeodesic flow, we need to introduce the following definitions.

Definition 2.4. — *A* Lipschitz graph *of T^*M, where M is a manifold, is a Lipschitz section of the canonical projection $T^*M \to M$. In particular, it is a Lipschitz submanifold of T^*M. By Rademacher's theorem, a Lipschitz submanifold of T^*M admits a tangent subspace almost everywhere with respect to the Lebesgue measure on the submanifold.*

A Lipschitz graph of T^*M is Lagrangian *if its (almost everywhere defined) tangent subspaces are Lagrangian with respect to the canonical symplectic form ω on T^*M; see* (2.3).

A subset of T^*M is flow-invariant *if it is invariant under the cogeodesic flow of M, where M is a complete Finsler manifold.*

The following result has been established in [MS11] in the context of Tonelli Hamiltonians and generalized in higher dimension in [AABZ15]. See also [Ban88, BP86, Hed32, Mat91] and [Sch15].

Theorem 2.5. — *Let $M = (\mathbb{T}^2, F)$ be a Finsler two-torus. Then the torus M has no conjugate points if and only if there exists a continuous foliation of T^*M by Lipschitz, Lagrangian, flow-invariant graphs.*

In the rest of this section, we will present a construction of the continuous foliation of T^*M by Lipschitz, Lagrangian, flow-invariant graphs given by Theorem 2.5 when M is a Finsler two-torus without conjugate points. This torus foliation is referred to as the *Heber foliation* of T^*M.

The notion of asymptotic direction of a Finsler geodesic will play a key role regarding geodesic foliations of Finsler two-tori without conjugate points.

Definition 2.6. — *The* asymptotic direction *of a curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ going to infinity (i.e., $\|\gamma(t)\| \to +\infty$ as $t \to +\infty$) is defined as*

$$\theta(\gamma) = \lim_{t \to +\infty} \frac{\gamma(t)}{\|\gamma(t)\|} \in S^1 \tag{2.4}$$

where $\|\cdot\|$ represents the Euclidean norm of $\mathbb{R}^2$ (if the limit exists). The asymptotic direction of a curve in $\mathbb{T}^2$ is defined as the asymptotic direction of any of its lifts.

The asymptotic direction of a curve of $\mathbb{R}^2$ or $\mathbb{T}^2$ is rational *if it is proportional to a vector with rational coefficients and* irrational *otherwise.*

The following result about the existence of asymptotic directions for minimizing geodesics in the universal cover of a Finsler two-torus is due to Hedlund [Hed32] in the Riemannian case; see [Ban88, BP86, Mat91, Sch15] for further generalizations encompassing the Finsler case.

Theorem 2.7. — *Let $M = (\mathbb{T}^2, F)$ be a Finsler two-torus with universal Finsler cover $\bar{M} = (\mathbb{R}^2, \bar{F})$. Then, there exists $w > 0$ such that for every minimizing geodesic $\gamma : \mathbb{R} \to \bar{M}$, the asymptotic direction $\theta(\gamma)$ of γ is well-defined (that is, the limit in* (2.4) *exists) and the geodesic γ lies in a strip bounded by two parallel lines in $\mathbb{R}^2$ of width at most w. Furthermore, $\theta(\bar{\gamma}) = -\theta(\gamma)$, where $\bar{\gamma} : \mathbb{R} \to \bar{M}$ is the minimizing geodesic obtained by reversing the orientation, that is, $\bar{\gamma}(t) = \gamma(-t)$.*

Conversely, for every $\theta_0 \in S^1$, there exists a minimizing geodesic $\gamma : \mathbb{R} \to \bar{M}$ such that $\theta(\gamma) = \theta_0$ and $\theta(\bar{\gamma}) = -\theta_0$. Moreover, for every $x \in \bar{M}$ and every $\theta_0 \in S^1$, there exists a minimizing geodesic ray based at x with asymptotic direction θ_0.

We will also need the following definition of Busemann functions; see [CK95, CS86, Esc77, Heb94] and [Gro99] for further details.

Definition 2.8. — *Let $M = (\mathbb{T}^2, F)$ be a Finsler two-torus without conjugate points. Denote by $\bar{M} = (\mathbb{R}^2, \bar{F})$ the universal Finsler cover of M. The* Busemann function *based at the origin $o \in \mathbb{R}^2$ and pointing in the direction of θ is defined as*

$$B_\theta(x) = \lim_{t \to +\infty} d_{\bar{M}}(x, c_\theta(t)) - t$$

where c_θ is the arc length parametrized geodesic ray arising from o with asymptotic direction θ. This limit is well-defined since the function $t \mapsto d_{\bar{M}}(x, c_\theta(t)) - t$ is monotone nonincreasing and bounded from below by the triangle inequality.

The collection of Busemann functions B_θ with $\theta \in S^1$ gives rise to the Busemann map

$$\begin{aligned} B : S^1 \times \bar{M} &\to \mathbb{R} \\ (\theta, x) &\mapsto B_\theta(x). \end{aligned}$$

Let us recall some regularity properties of Busemann functions for Finsler tori without conjugate points; see [CK95, CS86, Esc77, Heb94].

Proposition 2.9. — *Let $M = (\mathbb{T}^2, F)$ be a Finsler two-torus without conjugate points. Denote by $\bar{M} = (\mathbb{R}^2, \bar{F})$ its universal Finsler cover. Then*

(1) *The Busemann map $B : S^1 \times \bar{M} \to \mathbb{R}$ is continuous.*
(2) *Every Busemann function $B_\theta : \bar{M} \to \mathbb{R}$ is $C^{1,1}$ (i.e., B_θ is C^1 and its differential dB_θ is locally Lipschitz on M).*
(3) *The differential $dB_\theta : T\bar{M} \to \mathbb{R}$ is $\mathbb{Z}^2$-invariant and has unit norm at every point of $\bar{M}$ (i.e., $\|d_x B_\theta\| = 1$ for every $x \in \bar{M}$).*
(4) *Via the Legendre transform $\mathcal{L} : T\bar{M} \to T^*\bar{M}$, see* (2.1)*, the differential $-dB_\theta$ at $x \in \bar{M}$ corresponds to the unit tangent vector at x generating a geodesic ray with asymptotic direction θ.*

We can now describe the Heber foliation of a Finsler two-torus without conjugate points.

Definition 2.10. — *Let $M = (\mathbb{T}^2, F)$ be a Finsler two-torus without conjugate points. The unit tangent bundle $U\bar{M}$ of the universal cover $\bar{M}$ of M identifies with $\mathbb{R}^2 \times S^1$ as follows*

$$\begin{aligned} U\bar{M} &\to \mathbb{R}^2 \times S^1 \\ (x, v) &\mapsto (x, \theta) \end{aligned} \tag{2.5}$$

where θ is the asymptotic direction of the geodesic γ_v of $\bar{M}$ induced by v. By Proposition 2.9.(4)*, this map is a homeomorphism whose inverse map given by*

$$v = -\mathcal{L}^{-1}(dB_\theta(x)). \tag{2.6}$$

By definition, the Heber homeomorphism *is the inverse map of* (2.5)*.*

By $\mathbb{Z}^2$-invariance of the differential of the Busemann functions, see Proposition 2.9 (3)*, the map*

$$\begin{aligned} \bar{M} &\to T^*\bar{M} \\ x &\mapsto \varrho\, dB_\theta(x) \end{aligned}$$

passes to the quotient under the $\mathbb{Z}^2$-action on $\bar{M}$ and $T^*\bar{M}$, and induces a map

$$M \to T^*M \tag{2.7}$$

for $\theta \in S^1$ and $\varrho \in \mathbb{R}_+$.

The Heber foliation *associated to M is a foliation of T^*M whose leaves are the images of the maps* (2.7)*. Note that the leaves* (2.7) *of the Heber foliation are Lipschitz graphs of T^*M; see Proposition 2.9*(2)*. The Heber foliation is said to be* smooth *if the Heber homeomorphism $\mathbb{R}^2 \times S^1 \to U\bar{M}$ is a smooth diffeomorphism.*

The main properties of the Heber foliation are given by the following proposition; see [AABZ15, Ban88, BP86, Hed32, Mat91, MS11] and [Sch15].

Proposition 2.11. — *Let $M = (\mathbb{T}^2, F)$ be a Finsler two-torus without conjugate points. Then its Heber foliation is a continuous foliation of T^*M by Lipschitz, Lagrangian, flow-invariant graphs. Each of these graphs, except for the zero section,* *can be described as the space of covectors of the same norm generating a geodesic of M with the same asymptotic direction in S^1.*

Moreover, the geodesics of M do not have self-intersection and can be classified according to their asymptotic direction as follows:

(1) *the geodesics of M with a given rational asymptotic direction are closed of the same length and smoothly foliate M;*
(2) *the geodesics of M with a given irrational asymptotic direction are non-closed and foliate M.*

Remark 2.12. — It is an open question whether the Heber foliation associated to a Finsler two-torus M without conjugate points is always smooth. Observe however that the closed geodesics of M homotopic to a simple closed curve define a smooth geodesic foliation of M (and have the same length); see the case (1) of Proposition 2.11 (or [AABZ15, Section 1.2]). The situation is unclear in the case (2): the foliation of M by geodesics with a given irrational asymptotic direction may not be smooth.

3. Standard identification of Finsler two-tori without conjugate points

In this technical section, we present a “standard” identification of Finsler two-tori without conjugate points, where the horizontal and vertical closed curves are geodesics.

Proposition 3.1. — *Every Finsler two-torus $M = (\mathbb{T}^2, F)$ without conjugate points can be identified to $S^1 \times S^1$ so that the horizontal and vertical curves $S^1 \times \{t\}$ and $\{s\} \times S^1$ are geodesics with $s, t \in S^1$.*

In this case, any nonvertical geodesic in the universal cover $\mathbb{R}^2$ of $\mathbb{T}^2$ is the graph of a monotonic smooth function $\mathfrak{u} : \mathbb{R} \to \mathbb{R}$ over its horizontal axis.

Proof. — Two closed geodesics of an orientable closed Finsler surface of minimal length in their homotopy classes have a minimal number of intersection points among homotopic loops; see [FHS82].

Let α and β be two simple closed geodesics of M parameterized proportionally by arclength intersecting once. We know that the closed geodesics homotopic to α have the same length and form a smooth geodesic foliation α_t of M; see Proposition 2.11. Observe that the geodesics α_t transversely intersect the geodesic β at a single point. Thus, we can choose the parameter t in the geodesic foliation (α_t) so that α_t is the unique closed geodesic homotopic to α with $\alpha_t(0) = \beta(t)$. Similarly, there exists a smooth geodesic foliation β_s of M so that β_s is the unique closed geodesic homotopic to β (of the same length) with $\beta_s(0) = \alpha(s)$. Note that both α_t and β_s have minimal length in their homotopy classes and smoothly depend on t and s. It follows from the observation at the beginning of the proof that the closed geodesics α_t and β_s intersect once.

Since (α_t) and (β_s) are two transverse foliation of $\mathbb{T}^2$ whose leaves intersect once, the map

$$\begin{array}{rccc} \phi : & S^1 \times S^1 & \to & \mathbb{T}^2 \\ & (s,t) & \mapsto & \alpha_t \cap \beta_s \end{array}$$

is a bijection. Let us show that ϕ is a diffeomorphism. Since both α_t and β_s smoothly depend on t and s, the map ϕ is smooth. The curves $\phi(\cdot,t)=\alpha_t$ define a geodesic variation. Since the Finsler metric on M has no conjugate points, the Jacobi vector field $\frac{\partial\phi}{\partial t}$ it generates along the closed geodesic α_t does not vanish; see [BCS00, § 5.4]. Thus, $\frac{\partial\phi}{\partial t}$ is a nonvanishing vector field parallel to β'_s. Similarly, $\frac{\partial\phi}{\partial s}$ is a nonvanishing vector field parallel to α'_t.

Since α'_t and β'_s are noncolinear, the same goes for $\frac{\partial\phi}{\partial s}$ and $\frac{\partial\phi}{\partial t}$. It follows that the bijective map $\phi : S^1\times S^1\to\mathbb{T}^2$ is a local diffeomorphism. With this identification, the horizontal and vertical curves $S^1\times\{t\}$ and $\{s\}\times S^1$ coincide with the closed geodesics α_t and β_s.

By the description of geodesics on a two-torus given by Theorem 2.7, every nonvertical geodesic γ on the universal cover of a Finsler two-torus $\mathbb{T}^2$ without conjugate points has a nonvertical direction. Thus, the geodesic γ transversely intersects the vertical geodesic lines of $\mathbb{R}^2$ at least once and so exactly once. Therefore, the geodesic γ is the graph of a smooth function $\mathfrak{u}:\mathbb{R}\to\mathbb{R}$ over the horizontal axis of $\mathbb{R}^2$. Similarly, the geodesic γ intersects every horizontal line exactly once unless it has a horizontal direction in which case it coincides with a horizontal line. It follows that the function $\mathfrak{u}:\mathbb{R}\to\mathbb{R}$ is monotonic. □

4. Convergence of the curve shortening flow

We introduce the curve shortening flow in the Euclidean plane and show the existence of a limit when applied to the graph of a function representing a (minimizing) geodesic on the universal cover of a Finsler two-torus without conjugate points.

Definition 4.1. — *Consider the curve shortening flow for (planar) graphs given by the following quasilinear parabolic equation*

$$\mathfrak{u}_t=\frac{\mathfrak{u}_{xx}}{1+\mathfrak{u}_x^2} \tag{4.1}$$

with a C^∞ initial condition $\mathfrak{u}(.,0)=\mathfrak{u}_0$. Here, the function $\mathfrak{u}:\mathbb{R}\times[0,+\infty)\to\mathbb{R}$ takes (x,t) to $\mathfrak{u}(x,t)$. By [EH91] *(see also* [CZ98]*), for every C^∞ initial condition $\mathfrak{u}_0$, the curve shortening flow* (4.1) *has a unique solution defined for every $t\in[0,+\infty)$. Note that this flow extends to curves of the square flat two-torus $\mathbb{T}^2=\mathbb{R}^2/\mathbb{Z}^2$ whose lifts γ to the universal cover $\mathbb{R}^2$ are smooth graphs.*

The curve shortening flow for curves in $\mathbb{R}^2$ is defined using the following equation

$$\frac{\partial\gamma}{\partial t}=\kappa\,\nu \tag{4.2}$$

where $\gamma(.,t)$ is a family of curves in $\mathbb{R}^2$ with curvature κ and unit normal vector ν.

It is known that for an initial curve $\gamma(.,0)$ which is the graph of a function $\mathfrak{u}_0$, the evolution equations (4.1) *and* (4.2) *are equivalent: the evolution curve $\gamma(.,t)$ is given by the graph of $\mathfrak{u}(.,t)$; see* [EH91] *or* [CZ98] *for instance.*

For a smooth, complete, properly embedded curve $\gamma_0=\gamma(.,0)$ dividing the plane into two regions of infinite area (which is the case when the initial condition γ_0 is the graph of a smooth function $\mathfrak{u}_0$), the curve shortening flow (4.2) has a unique solution; see [CZ98]. As previously, this flow extends to curves of the flat two-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ whose lifts to the universal cover $\mathbb{R}^2$ satisfy the previous existence and uniqueness condition.

The curve shortening flow satisfies the following crucial property; see [CZ98].

Theorem 4.2. — *Two disjoint smooth, complete, properly embedded curves γ_1 and γ_2 dividing the plane into two regions of infinite area remain disjoint and embedded through the curve shortening flow.*

The following result applies to all geodesics in the universal cover of a Finsler two-torus without conjugate points.

Proposition 4.3. — *The asymptotic direction of a minimizing geodesic in the universal cover $\bar{M} = (\mathbb{R}^2, \bar{F})$ of a Finsler two-torus $M = (\mathbb{T}^2, F)$ is preserved under the curve shortening flow.*

Proof. — Every minimizing geodesic γ in $\bar{M} \simeq \mathbb{R}^2$ lies in an open strip S bounded by two straight lines of $\mathbb{R}^2$ with the same asymptotic direction as γ; see Theorem 2.7. Since these two lines are fixed under the curve shortening flow of $\mathbb{R}^2$ and disjoint curves remain disjoint, see Theorem 4.2, the curve γ evolves within the strip S under the curve shortening flow, keeping the same asymptotic direction. □

In general, the curve shortening flow of a smooth function $\mathfrak{u}_0 : \mathbb{R} \to \mathbb{R}$ does not necessarily converge; see [EH89]. Conditions under which such a convergence occurs can be found in [NT07] and [WW11]. In our case, we show the following result.

Theorem 4.4. — *Let $M = (\mathbb{T}^2, F)$ be a Finsler two-torus without conjugate points. Denote by $\bar{M} = (\mathbb{R}^2, \bar{F})$ its universal Finsler cover. Let $\mathfrak{u}_0 : \mathbb{R} \to \mathbb{R}$ be a smooth function whose graph $\mathcal{G}_0$ is a (minimizing) Finsler geodesic of $\bar{M}$. Then, the solution $\mathfrak{u}(., t)$ of the curve shortening flow* (4.1) *with initial condition $\mathfrak{u}_0$ converges to an affine function $\mathfrak{u}_\infty$ (with the same asymptotic direction as $\mathfrak{u}_0$). Here, the convergence is in the C^k-topology for any given $k \geqslant 0$.*

We will consider two cases in the proof of Theorem 4.4: first, the rational case where the asymptotic direction of the minimizing geodesic represented by the graph of $\mathfrak{u}_0$ is rational (proved at the end of this section, even for metrics with conjugate points), then the irrational case (proved in the next section). In the latter case, we will actually need a stronger result, namely Theorem 5.1, giving a uniform convergence of the curve shortening flow for geodesics whose asymptotic direction lies in a small enough neighborhood of a fixed irrational direction.

Remark 4.5. — We emphasize that the geometric nature of the curves and functions considered in Theorem 4.4 is crucial. Indeed, the conclusion fails for general functions as exemplified by the grim reaper curve $y = -\log \cos x$, which is a translating soliton of the curve shortening flow; see [Hal12] for a classification of self-similar solutions to the curve shortening flow in the plane.

We will first prove Theorem 4.4 in the rational case, without assuming that M has no conjugate points (as long as the geodesic represented by $\mathcal{G}_0$ is minimizing).

Let us start with a first lemma about the growth of $\mathfrak{u}_0$, which will also be used in Section 7.

Lemma 4.6. — *The graph $\mathcal{G}_0$ of $\mathfrak{u}_0$ has linear growth. More precisely, there exists $C_M > 0$ (which does not depend on $\mathfrak{u}_0$) such that $|\mathfrak{u}_0'| \leqslant C_M$.*

Proof. — Let $a = \min_{[-1,1]} \mathfrak{u}_0$ and $b = \max_{[-1,1]} \mathfrak{u}_0$. Choose $L > 0$ such that every vertical open interval of Euclidean length L in $\mathbb{R}^2$ has a translate by an element of the lattice $\mathbb{Z}^2$ which intersects the rectangle $[-1,1] \times [a-1, b+1]$ along a segment $\{x\} \times [a-1, b+1]$ with $x \in [-\frac{1}{2}, \frac{1}{2}]$, and in particular, transversally intersects the part of the graph $\mathcal{G}_0$ over $[-\frac{1}{2}, \frac{1}{2}]$. For example, we can take $L = b - a + 4$.

By cocompactness of the Finsler metric on $\mathbb{R}^2$ (due to its $\mathbb{Z}^2$-periodicity), for every $\varepsilon > 0$, there exists $\delta > 0$ such that every pair of δ-close tangent vectors to $\mathbb{R}^2$ generate ε-close length L geodesic rays.

By contradiction, suppose that $|\mathfrak{u}_0'|$ becomes arbitrarily large, that is, some tangent vector to the graph $\mathcal{G}_0$ becomes δ-close to a vertical vector. Since the vertical lines of $\mathbb{R}^2$ are Finsler geodesics, this implies that the geodesic $\mathcal{G}_0$ becomes ε-close to some vertical segment of Euclidean length L. By our choice of L and for ε small enough, this implies that a translate of $\mathcal{G}_0$ by a (nontrivial) element of $\mathbb{Z}^2$ intersects the rectangle $[-1,1] \times [a,b]$ along an arc joining $[-1,1] \times \{a\}$ and $[-1,1] \times \{b\}$ and in particular, transversally intersects $\mathcal{G}_0$. Thus, the geodesic of M corresponding to the projection of $\mathcal{G}_0$ to the torus has a transverse selfintersection. Hence a contradiction with Proposition 2.11. $\square$

The following result showing that the curve shortening flow flattens out the graph of $\mathfrak{u}_0$ is a direct consequence of the upper bound on the curvature obtained in [EH89, Proposition 4.4] combined with Lemma 4.6.

Lemma 4.7. — *The curvature of the graph $\mathcal{G}_t$ of $\mathfrak{u}(.,t)$ converges to zero in the C^k-topology as t goes to infinity (uniformly with respect to $\mathfrak{u}_0$). More precisely, there exists $c_k = c_k(M) > 0$ (which does not depend on $\mathfrak{u}_0$) such that*

$$\|\kappa_t\|^2_{C^k} \leqslant \frac{c_k}{t^{k+1}}$$

for every $t > 0$, where κ_t is the curvature of $\mathcal{G}_t$.

Proof. — A uniform upper bound on the second fundamental form of entire (hypersurface) graphs in $\mathbb{R}^n$ moving by the mean curvature flow can be found in [EH89, Proposition 4.4]. In the case of the graph $\mathcal{G}_t$, this bound can be written

$$\|\kappa_t\|^2_{C^k} \leqslant \frac{c_k}{t^{k+1}}$$

where $c_k = c_k(M)$ is a constant depending only on k and the constant C_M given by Lemma 4.6. $\square$

Our second result is about the width of the graph $\mathcal{G}_t$ defined as follows.

Definition 4.8. — *By Theorem 2.7, the graph $\mathcal{G}_0$ lies in a strip of $\mathbb{R}^2$ bounded by two parallel straight lines and so does the graph $\mathcal{G}_t$. The minimal Euclidean distance between two such lines is called the* width *of $\mathcal{G}_t$.*

By the properties of the curve shortening flow, see Theorem 4.2, the width of $\mathcal{G}_t$ is nonincreasing. Moreover, we have

Lemma 4.9. — *Suppose that the asymptotic direction of the geodesic represented by $\mathcal{G}_0$ is rational. Then, the width of $\mathcal{G}_t$ tends to zero as t goes to infinity.*

Proof. — By contradiction, assume that there exists $w_0 > 0$ such that for every $t \geqslant 0$, the graph $\mathcal{G}_t$ is of width greater than w_0.

Since the asymptotic direction θ_0 of $\mathcal{G}_0$ is rational, then the projection γ_0 of $\mathcal{G}_0$ to the torus $\mathbb{T}^2$ closes up. The cover of $\mathbb{T}^2$ corresponding to the subgroup generated by the homotopy class of γ_0 is a flat cylinder $C = S^1 \times \mathbb{R}$ containing γ_0. The projection γ_t of $\mathcal{G}_t$ to C lies in the minimal annulus $S^1 \times I$ of C containing γ_0. Since the curvature of γ_t uniformly converges to zero as t goes to infinity, see Lemma 4.7, the closed curve γ_t of bounded length becomes arbitrarily C^1-close to a circle of C, for t large enough. Thus, the width of γ_t, and so of $\mathcal{G}_t$, tends to zero as t goes to infinity.

□

Combining the zero convergence results of the curvature of $\mathcal{G}_t$ and of its width when the asymptotic direction of $\mathcal{G}_0$ is rational, see Lemma 4.7 and Lemma 4.9, we immediately deduce Theorem 4.4 in the rational case. To prove the theorem in the remaining irrational case, it is enough to show that the conclusion of Lemma 4.9 still holds when the asymptotic direction of $\mathcal{G}_0$ is irrational. This is done in the next section where a stronger result is proved; see Theorem 5.1.

5. Convergence of the width at irrational directions

We show that the width of a geodesic in the universal cover of a Finsler two-torus without conjugate points converges to zero under the curve shortening flow for geodesics whose asymptotic direction lies in a small enough neighborhood of a fixed irrational direction. We deduce that the straight lines obtained at the limit vary continuously at irrational directions.

Theorem 5.1. — *Let $M = (\mathbb{T}^2, F)$ be a Finsler two-torus without conjugate points. Denote by $\bar{M} = (\mathbb{R}^2, \bar{F})$ its universal Finsler cover. Let θ be an irrational direction. For every $\varepsilon > 0$, there exists $t_0 > 0$ such that for every geodesic γ of $\bar{M}$ with asymptotic direction close enough to θ and every $t \geqslant t_0$, the width of γ_t satisfies* $\mathrm{width}(\gamma_t) < \varepsilon$.

Remark 5.2. — The point of Theorem 5.1 is that t_0 does not depend on γ as long as its asymptotic direction remains close enough to the irrational direction θ. Of course, Theorem 5.1 also applies when the asymptotic direction of the geodesic γ is irrational, which gives an analogue of Lemma 4.9 in the irrational case and allows us to derive Theorem 4.4 in this case too. We will need this stronger version of Theorem 4.4 to establish the continuity of the Euclidean curve shortening flow at the limit for Finsler geodesics with irrational asymptotic directions.

We decompose the proof of the proposition into several lemmas.

Definition 5.3. — *The* vertical distance *between two subsets $A, B \subset \mathbb{R}^2$ is defined as*

$$\sup_V \inf\{d(x,y) \mid x \in A \cap V, y \in B \cap V\}$$

where V runs over all vertical lines of $\mathbb{R}^2$ intersecting A and B, and d represents the Euclidean distance in $\mathbb{R}^2$.

The following lemma is a quantitative version of the fact that an irrational geodesic in the square flat torus is dense.

Lemma 5.4. — *Let θ be an irrational direction. For every $\varepsilon \in (0,1)$, there exists $L = L(\theta, \varepsilon) > 0$ such that every segment c of $\mathbb{R}^2$ of direction close enough to θ which projects onto an interval of the x-axis of length at least L passes below a point x_+ of $\mathbb{Z}^2$ and above a point x_- of $\mathbb{Z}^2$, both at vertical distance less than ε from the segment c.*

Proof. — Let $\varepsilon \in (0,1)$. Fix a point $\bar{x} \in \mathbb{Z}^2$. Let $\bar{x}_+$ and $\bar{x}_-$ be the points of $\mathbb{R}^2$ at vertical distance ε from $\bar{x}$, with $\bar{x}_+$ above $\bar{x}$ and $\bar{x}_-$ below x. Let $\bar{m}_+$ and $\bar{m}_-$ be the midpoints of $[\bar{x}, \bar{x}_+]$ and $[\bar{x}, \bar{x}_-]$. Denote by $\pi(\bar{m}_+)$ and $\pi(\bar{m}_-)$ the projections of $\bar{m}_+$ and $\bar{m}_-$ to $\mathbb{T}^2$, where $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ is the covering projection. Suppose that the segment c is directed along the irrational direction θ. If the segment c is long enough (depending on θ and ε), its projection to $\mathbb{T}^2$ passes at (vertical) distance less than $\frac{\varepsilon}{2}$ from any point, and in particular from $\pi(\bar{m}_+)$ and $\pi(\bar{m}_-)$. Thus, the segment c passes at vertical distance less than $\frac{\varepsilon}{2}$ from some $\mathbb{Z}^2$-translates m_+ and m_- of $\bar{m}_+$ and $\bar{m}_-$ (i.e., some lifts of $\pi(\bar{m}_+)$ and $\pi(\bar{m}_-)$). Define x_+ and x_- as the corresponding $\mathbb{Z}^2$-translates of $\bar{x}_+$ and $\bar{x}_-$. By construction, the segment c passes below x_+ and above x_- at vertical distance less than ε from them. The same holds for a segment c of direction close enough to θ. ☐

Denote by $(\vec{\imath}, \vec{\jmath})$ the canonical basis of $\mathbb{R}^2$. The following lemma is a quantitative version of the fact that the curve shortening flow straighten curves.

Lemma 5.5. — *Assume that the horizontal and vertical curves of M are geodesic. For every $\varepsilon \in (0,1)$ and every $L > 0$, there exists $t_0 = t_0(\varepsilon, L) > 0$ such that for every $t \geqslant t_0$ and every geodesic γ of $\bar{M}$ whose asymptotic direction forms an angle lying between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$ with the horizontal direction, the following holds: every arc of γ_t over an interval I of length L of the x-axis is at vertical distance at most ε from a segment of $\mathbb{R}^2$ over the same interval I.*

Proof. — By Theorem 2.7, the geodesic γ lies in a strip S of width w with the same asymptotic direction, where w does not depend on γ (only on the Finsler metric on M). Since the vertical curves of $\bar{M}$ are geodesic, the tangent vector γ' is uniformly bounded away from a vertical direction, that is, $|\langle \gamma', \vec{\jmath} \rangle| \leqslant C \|\gamma'\|$ for some constant $C \in (0,1)$ not depending on the geodesic γ. Otherwise, the geodesic γ would be close to a vertical geodesic and would leave the strip S whose vertical width is bounded.

By Lemma 4.7, there exists $c_0 > 0$ not depending on γ such that the curvature $\kappa(\gamma_t)$ of γ_t satisfies

$$|\kappa(\gamma_t)| \leqslant \frac{c_0}{\sqrt{t}}.$$

Thus, for t large enough, the curve γ_t is almost straight. Hence the desired result. ☐

We can now proceed to the proof of Theorem 5.1.

Proof of Theorem 5.1. — By Proposition 3.1, we can assume that the horizontal and vertical curves of $\mathbb{T}^2$ are geodesics. Switching the roles of the x- and y-axis, and the orientation of γ if necessary, we can further assume that the angle between the asymptotic direction of γ and the horizontal vector $\vec{\imath}$ lies between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$. Recall that the curve γ_t lies in a (closed) Euclidean strip $S_t = S(\gamma_t)$ of $\mathbb{R}^2$ bounded by two straight lines $\Delta_+ = \Delta_+(\gamma_t)$ and $\Delta_- = \Delta_-(\gamma_t)$ with the same asymptotic direction as γ, with Δ_+ above Δ_- in $\mathbb{R}^2$; see Theorem 2.7.

Let us show that if the asymptotic direction of γ is close enough to θ (more precisely, if the asymptotic direction of γ is in the neighborhood of θ given by Lemma 5.4), then the width of S_t is less than 14ε for $t \geqslant t_0$. Take a point $p_+ = p_+(\gamma_t)$ of γ_t at vertical distance at most ε from Δ_+ (below Δ_+) and denote by $\bar{p}_+$ its projection to the x-axis. Let I be the interval of the x-axis centered at $\bar{p}_+$ of length L, where L is given in Lemma 5.4. Consider the arc α_+ of γ_t over I. By Lemma 5.5, this arc α_+ is at vertical distance at most ε from a segment c of $\mathbb{R}^2$. Thus, the segment c lies below the line $\Delta_+ + \varepsilon\vec{\jmath}$ above Δ_+ at vertical distance ε from Δ_+ and passes through a point at vertical distance at most ε from p_+. Therefore, the segment c lies at vertical distance at most 6ε from $\Delta_+ + \varepsilon\vec{\jmath}$. We deduce that the arc α_+ of γ_t lies below Δ_+ at vertical distance at most 6ε from Δ_+. See Figure 5.1.

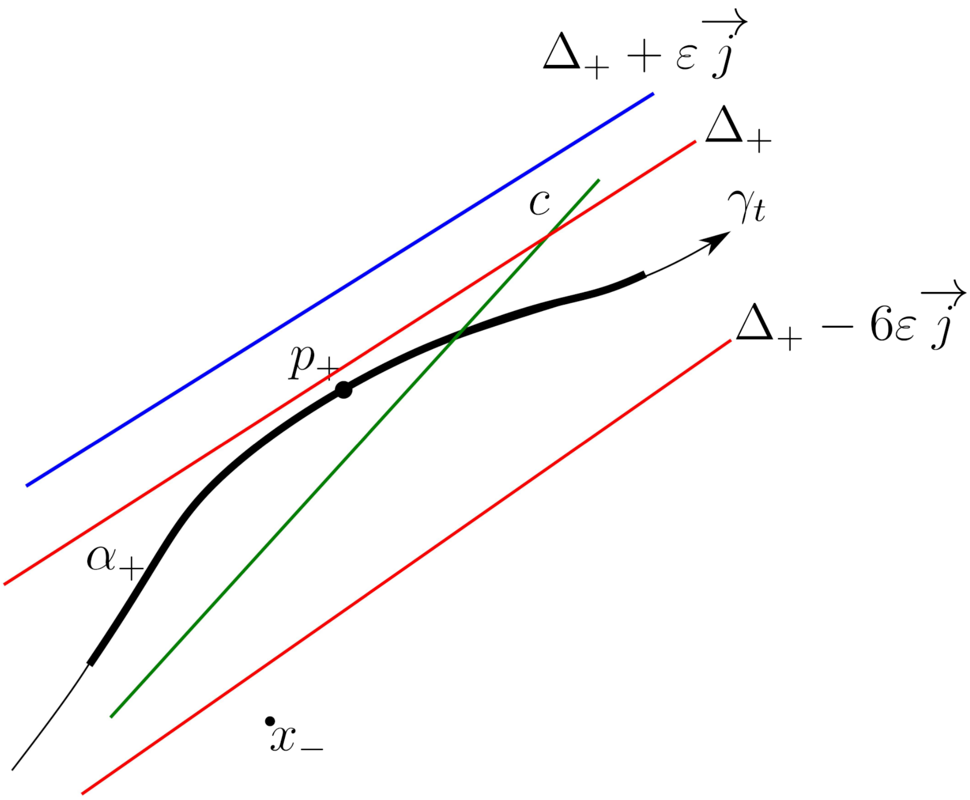

Figure 5.1. Relative positions of the curves α_+, c and Δ_+

Apply Lemma 5.4 to the segment lying in $\Delta_+ - 6\varepsilon\vec{\jmath}$ above the interval I of length L, assuming the asymptotic direction of Δ_+ is close enough to the irrational direction θ. This yields a point x_- of $\mathbb{Z}^2$ below $\Delta_+ - 6\varepsilon\vec{\jmath}$ at vertical distance at most ε from it and so at vertical distance at most 7ε from Δ_+. It follows that the geodesic arc α_+ of γ_t passes above the point x_- of $\mathbb{Z}^2$ at vertical distance at most 7ε from Δ_+. Similarly, the curve γ_t passes below a point x_+ of $\mathbb{Z}^2$ at vertical distance at most 7ε from Δ_-.

Let T be the translation of $\mathbb{R}^2$ by the integral vector $x_- - x_+ \in \mathbb{Z}^2$. Note that T is an isometry both for the Euclidean metric and the Finsler metric. By construction, the point x_- lies below γ_t and above $T(\gamma_t)$, and is at vertical distance at most 7ε from Δ_+ and $T(\Delta_-)$. The geodesics γ and $T(\gamma)$ have the same asymptotic direction and do not intersect. The same holds for their images γ_t and $T(\gamma_t)$ under the curve shortening flow; see Section 4. In addition, $T(\gamma_t)$ is proved to be below γ_t and $T(\Delta_-)$ is below Δ_-, hence $T(\Delta_-)$ bounds γ_t from below. It follows that the curve γ_t lies in the strip of width at most 14ε bounded by Δ_+ on top and by $T(\Delta_-)$ at the bottom. Thus, $\mathrm{width}(\gamma_t) \leqslant 14\varepsilon$. □

By Theorem 4.4 and Proposition 4.3, a geodesic in the universal cover of a Finsler two-torus converges to an (oriented) straight line of $\mathbb{R}^2$ with the same asymptotic direction. An oriented straight line Δ of $\mathbb{R}^2$ is determined by its asymptotic direction $\theta(\Delta)$ and its signed distance $p(\Delta)$ to the origin in $\mathbb{R}^2$; see Section 9 for a more detailed description. The following result which is a consequence of the previous proposition, is about the regularity of the signed distance at the limit when the geodesic varies.

Proposition 5.6. — *Let $M = (\mathbb{T}^2, F)$ be a Finsler two-torus without conjugate points. Denote by $\bar{M} = (\mathbb{R}^2, \bar{F})$ its universal Finsler cover. Then the function*

$$\begin{array}{rccc} p_\infty : & U\bar{M} & \to & \mathbb{R} \\ & v & \mapsto & p\left(\gamma_v^\infty\right) \end{array}$$

is continuous at vectors v generating a geodesic γ_v with irrational asymptotic direction.

Proof. — Fix a vector v_0 generating a geodesic γ_{v_0} with irrational asymptotic direction. Let $\varepsilon > 0$. By Theorem 5.1, we can choose $t_0 > 0$ such that for every geodesic γ_v of $\bar{M}$ with $v \in U\bar{M}$ close enough to v_0, the curve $\gamma_v^{t_0}$ lies in a (closed) Euclidean strip of width less than ε. Since its limit γ_v^∞ lies in the same strip, the Hausdorff distance between $\gamma_v^{t_0}$ and γ_v^∞ in $\mathbb{R}^2$ is less than ε.

For $v \in U\bar{M}$ close enough to v_0, the curve $\gamma_v^{t_0}$ is at Hausdorff distance less than ε from $\gamma_{v_0}^{t_0}$ on any given compact subset of $\mathbb{R}^2$. It follows that for $v \in U\bar{M}$ close enough to v_0, the Hausdorff distance between the limit straight lines γ_v^∞ and $\gamma_{v_0}^\infty$ on any given compact subset of $\mathbb{R}^2$ is less than 3ε. Thus,

$$\left|p\left(\gamma_v^\infty\right) - p\left(\gamma_{v_0}^\infty\right)\right| \leqslant \left|p\left(\gamma_v^\infty\right) - p\left(\gamma_v^{t_0}\right)\right| + \left|p\left(\gamma_v^{t_0}\right) - p\left(\gamma_{v_0}^{t_0}\right)\right| + \left|p\left(\gamma_{v_0}^{t_0}\right) - p\left(\gamma_{v_0}^\infty\right)\right| < 3\varepsilon.$$

Hence, the function $p_\infty : U\bar{M} \to \mathbb{R}$ is continuous at v_0. □

It is unclear whether the function $p_\infty : U\bar{M} \to \mathbb{R}$ is continuous at vectors generating geodesics pointing in rational directions. The following example illustrates possible issues. Note however that these issues may not occur for geodesics in the universal cover of a Finsler torus without conjugate points.

Example 5.7. — We can construct a smooth family of graphs asymptotic to the lines $y = mx$ converging to the horizontal line $y = 1$ in the smooth topology on compact sets, as m goes to zero. Such a family converges to a non-continuous family of lines under the curve shortening flow, namely the family formed of the lines $y = mx$ for $m > 0$ and $y = 1$ for $m = 0$. See Figure 5.2.

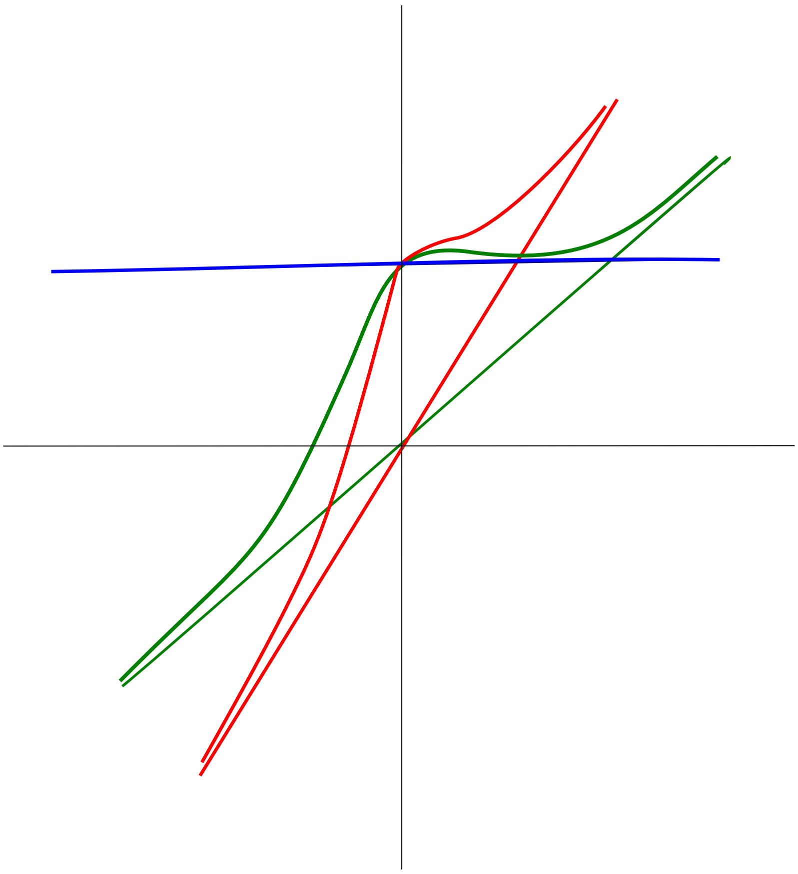

Figure 5.2. Non-continuity of the limit under the curve shortening flow.

6. Analytic expression of the limit under the curve shortening flow

In this section, we derive the following analytic expression of the limit under the curve shortening flow of a function whose graph represents a (minimizing) geodesic in the universal cover of a Finsler two-torus without conjugate points; see Theorem 4.4.

Proposition 6.1. — *Let $M = (\mathbb{T}^2, F)$ be a Finsler two-torus without conjugate points. Denote by $\bar{M} = (\mathbb{R}^2, \bar{F})$ its universal Finsler cover. Let $\mathfrak{u} : \mathbb{R} \to \mathbb{R}$ be a* *smooth function whose graph $\mathcal{G}$ in $\mathbb{R}^2$ is a (minimizing) Finsler geodesic of $\bar{M}$ with the same asymptotic direction as the line $y = ax$. Then, the limit $\mathfrak{u}_\infty$ of $\mathfrak{u}$ under the curve shortening flow* (4.1) *satisfies*

$$\mathfrak{u}_\infty(0) = \lim_{R \to +\infty} \frac{1}{2R} \int_{-R}^{R} \mathfrak{u}(x) - ax \, dx$$

and, more generally,

$$\mathfrak{u}_\infty(x_0) = \lim_{R \to +\infty} \frac{1}{2R} \int_{-R}^{R} \mathfrak{u}(x + x_0) - ax \, dx \tag{6.1}$$

for every $x_0 \in \mathbb{R}$. In particular, the limits exist.

Actually, the convergence in (6.1) *is uniform, that is,*

$$\lim_{R \to +\infty} \sup_{x_0 \in \mathbb{R}} \frac{1}{2R} \left| \int_{-R}^{R} \mathfrak{u}(x + x_0) - ax - \mathfrak{u}_\infty(x_0) \, dx \right| = 0.$$

Proof. — Assume that $a = 0$. By Theorem 4.4, the solution of the curve shortening flow (4.1) with initial condition $\mathfrak{u}$ converges to a constant function $\mathfrak{u}_\infty = \mathfrak{u}_\infty(0)$. Now, recall that the solution of the curve shortening flow equation (4.1) converges uniformly to the solution of the heat equation with the same initial condition; see [NT07, Theorem 1.4]. Thus, the solution of the heat equation with initial condition $\mathfrak{u}$ uniformly converges to the same constant function $\mathfrak{u}_\infty = \mathfrak{u}_\infty(0)$. Since the initial condition $\mathfrak{u}$ is bounded, see Lemma 4.9, it follows from the expression of the limit of the solution of the heat equation, see [RÈ66, RÈ67], that

$$\mathfrak{u}_\infty(0) = \lim_{R \to +\infty} \frac{1}{2R} \int_{-R}^{R} \mathfrak{u}(x) \, dx.$$

See also [NT07, § 3]. By a change of variable, we obtain (6.1). Actually, since the solution of the heat equation with the initial condition $\mathfrak{u}$ uniformly converges, the convergence in (6.1) is uniform.

In the general case (i.e., when a is not necessarily zero), we apply the same argument to $\mathfrak{v}(x) = \mathfrak{u}(x) - ax$. ☐

Remark 6.2. — By Theorem 4.4, the limit function $\mathfrak{u}_\infty$ is an affine function of the form $\mathfrak{u}_\infty(x) = ax + b$. It follows from Proposition 6.1 that

$$a = \mathfrak{u}_\infty(1) - \mathfrak{u}_\infty(0) = \lim_{R \to +\infty} \frac{1}{2R} \int_{-R}^{R} \mathfrak{u}(x+1) - \mathfrak{u}(x) \, dx \tag{6.2}$$

and

$$b = \mathfrak{u}_\infty(0) = \lim_{R \to +\infty} \frac{1}{2R} \int_{-R}^{R} \mathfrak{u}(x) - ax \, dx. \tag{6.3}$$

Thus, the limit affine function $\mathfrak{u}_\infty : \mathbb{R} \to \mathbb{R}$ is given by the limits of some linear integrals of $\mathfrak{u}$.

Remark 6.3. — As for Theorem 4.4, the conclusion of Proposition 6.1 still holds in the rational case when M has conjugate points (as long as the geodesic represented by $\mathcal{G}$ is minimizing).

7. Curve shortening flow and unit bundle diffeomorphism

We show that the deformation of the geodesic foliation of a Finsler two-torus without conjugate points under the curve shortening flow induces a family of diffeomorphisms on the unit tangent bundle of the torus. This gives rise to a deformation of the Heber foliation on the cotangent bundle.

Let $M = (\mathbb{T}^2, F)$ be a Finsler two-torus without conjugate points, where $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Denote by $\bar{M} = (\mathbb{R}^2, \bar{F})$ the universal Finsler cover of M. The isometric action of $\mathbb{Z}^2$ by deck transformations on $\bar{M}$ induces a natural action on $U\bar{M}$ and $T\bar{M}$.

Definition 7.1. — *Consider the map*

$$\overline{\Psi}_t : U\bar{M} \to T\mathbb{R}^2 \setminus \{0\}$$

defined as

$$\overline{\Psi}_t(v) = (\gamma_v^t)'(0)$$

for every $v \in U\bar{M}$ and $t \in [0, \infty)$, where (γ_v^t) is the Euclidean curve shortening flow of the Finsler geodesic γ_v induced by v; see Definition 4.1. Note that $\overline{\Psi}_0$ is the identity map on $U\bar{M}$. Since $\mathbb{Z}^2$ acts by isometries both on $\bar{M}$ and $\mathbb{R}^2$, the map $\overline{\Psi}_t$ is $\mathbb{Z}^2$-equivariant.

Consider also the map

$$\Psi_t : U\bar{M} \to U_0\mathbb{R}^2$$

to the unit tangent bundle $U_0\mathbb{R}^2$ of the Euclidean plane $\mathbb{R}^2$ defined as

$$\Psi_t(v) = \pi\left[\overline{\Psi}_t(v)\right] = \pi\left[(\gamma_v^t)'(0)\right]$$

where $\pi : T\bar{M} \setminus \{0\} \to U_0\mathbb{R}^2$ is the radial projection onto the unit circle of each tangent plane of the Euclidean plane $\mathbb{R}^2 = \bar{M}$. The map Ψ_t is also $\mathbb{Z}^2$-equivariant and its quotient map is a π_1-isomorphism since it is the radial projection of the identity map for $t = 0$.

The following result is derived from an analysis of the (parabolic) partial differential equation satisfied by the curve shortening flow. This is the analogue of [Sab19, Proposition 4.4] for the family of diffeomorphisms of the unit tangent bundle of the round projective plane obtained by applying the curve shortening flow to the geodesics of a Zoll Finsler metric.

Theorem 7.2. — *Let $M = (\mathbb{T}^2, F)$ be a Finsler torus without conjugate points. For every $t \in [0, +\infty)$, the map $\Psi_t : U\bar{M} \to U_0\mathbb{R}^2$ is a diffeomorphism.*

Its quotient map, still denoted by $\Psi_t : UM \to U_0\mathbb{T}^2$, is also a diffeomorphism.

Proof. — First, let us show that the map $\overline{\Psi}_t : U\bar{M} \to T\mathbb{R}^2$ is an immersion.

For $t = 0$, this is true since, by construction, for every $v \in U\bar{M}$,

$$\overline{\Psi}_0(v) = \gamma_v'(0) = v.$$

Assume $t > 0$. Let $v_0 \in U\bar{M}$ and $\zeta \in T_{v_0}U\bar{M}$. Let $v = v(\lambda)$ be a smooth vector variation in $U\bar{M}$ with $v(0) = v_0$ and $v'(0) = \zeta$. We want to show that for every $t > 0$, the differential

$$d\overline{\Psi}_t(v_0) : T_{v_0}U\bar{M} \to T_{\overline{\Psi}_t(v_0)}T\mathbb{R}^2$$

of $\overline{\Psi}_t$ at v_0 is injective. That is, if the derivative $d\overline{\Psi}_t(v_0)(\zeta)$ of $\overline{\Psi}_t(v(\lambda))$ vanishes at $\lambda = 0$, then the vector $\zeta = v'(0)$ is zero.

By Proposition 3.1, we can assume that the horizontal and vertical lines of $\bar{M}$ are geodesics. Switching the horizontal and vertical lines if necessary, we can assume that the vector v_0 is nonvertical and therefore each geodesic $\gamma_\lambda = \gamma_{v(\lambda)}$ of $\bar{M}$ is represented in $\bar{M} = \mathbb{R}^2$ as the graph of some smooth function $\mathfrak{u}_0(.,\lambda)$ for λ close enough to zero. The map $(x, y) \mapsto (x, y - \mathfrak{u}_0(x, 0))$ is a diffeomorphism of $\mathbb{R}^2$ taking vertical lines to vertical lines and the geodesic γ_0 to the x-axis. In this new coordinate system, every curve γ_λ^t with λ close enough to zero obtained by applying the curve shortening flow to γ_λ, see (4.2), is represented as the graph

$$\{(x, \mathfrak{u}(x, t, \lambda)) \mid x \in \mathbb{R}\}$$

of a smooth function $\mathfrak{u}(., t, \lambda)$ with $\mathfrak{u}(., 0, 0) = 0$. More specifically,

$$\gamma_\lambda^t(s) = (x(s, t, \lambda), \mathfrak{u}(x(s, t, \lambda), t, \lambda))$$

where s is the arclength-parameter of γ_λ^t for the Finsler metric $\bar{F}$ (with an orientation compatible with x). Note that the partial derivative x_s does not vanish (and is positive). Observe also that $\mathfrak{u}_x(., 0, 0) = \mathfrak{u}_{xx}(., 0, 0) = 0$.

In the new coordinate system, the equation of the curve shortening flow is no longer given by (4.1). It satisfies a new equation which has been derived in [Ang90, Eq. (3.2)] and [Gag90, Appendix]. More precisely, the function

$$\mathfrak{u} : \mathbb{R} \times [0, \infty) \times (-\varepsilon, \varepsilon) \to \mathbb{R}$$

satisfies the following (parabolic) partial differential equation of the curve shortening flow

$$\mathfrak{u}_t = \mathcal{F}(x, \mathfrak{u}, \mathfrak{u}_x, \mathfrak{u}_{xx}) \tag{7.1}$$

where $\mathcal{F}$ is a smooth function defined on $\mathbb{R}^4$ with

$$\mathcal{F}_q(x, u, p, q) > 0$$

which can be expressed in terms of the coefficients of the Euclidean metric in the new coordinate system (where γ_0 coincides with the x-axis). Here, the subscripts refer to the partial differentiations. Note that we do not need an explicit form of $\mathcal{F}$.

The tangent vector $\overline{\Psi}_t(v(\lambda)) = (\gamma_\lambda^t)'(0) \in U_0\mathbb{R}^2$ is given by

$$\overline{\Psi}_t(v(\lambda)) = (x, \mathfrak{u}, x_s, x_s\, \mathfrak{u}_x) \tag{7.2}$$

where $x_* = x_*(0, t, \lambda)$ and $\mathfrak{u}_* = \mathfrak{u}_*(x(0, t, \lambda), t, \lambda)$. Taking the derivative of (7.2) with respect to λ, we obtain for $\lambda = 0$

$$d\overline{\Psi}_t(v_0)(\zeta) = (x_\lambda, x_\lambda\, \mathfrak{u}_x + \mathfrak{u}_\lambda, x_{s\lambda}, x_{s\lambda}\, \mathfrak{u}_x + x_s\, x_\lambda\, \mathfrak{u}_{xx} + x_s\, \mathfrak{u}_{x\lambda}) \tag{7.3}$$

where $x_* = x_*(0, t, 0)$ and $\mathfrak{u}_* = \mathfrak{u}_*(x(0, t, 0), t, 0)$.

Suppose that $d\overline{\Psi}_\tau(v_0)(\zeta) = 0$. Recall that x_s does not vanish. In this case, the functions x_λ, $\mathfrak{u}_\lambda$, $x_{s\lambda}$ and $\mathfrak{u}_{x\lambda}$ vanish at $s = 0$, $t = \tau$, $\lambda = 0$. In particular, the function $\mathfrak{v} = \mathfrak{u}_\lambda$ has a multiple zero at $(x(0, \tau, 0), \tau, 0)$, i.e., both $\mathfrak{v}$ and $\mathfrak{v}_x$ vanish at this point.

Using the fact that $\mathfrak{v}$ has a multiple zero and that the Finsler metric has no conjugate points, our goal is to show that $\zeta = 0$.

The following result shows that $\mathfrak{v}$ vanishes when $\lambda = 0$.

Lemma 7.3. — *For every $x \in \mathbb{R}$ and $t \geqslant 0$, we have*

$$\mathfrak{v}(x,t,0) = 0.$$

Proof. — Let us derive the evolution equation of $\mathfrak{v}$. Differentiating the relation (7.1) with respect to λ leads to the following parabolic partial differential equation

$$\mathfrak{v}_t = a(x,t,\lambda)\,\mathfrak{v}_{xx} + b(x,t,\lambda)\,\mathfrak{v}_x + c(x,t,\lambda)\,\mathfrak{v} \tag{7.4}$$

where $a = \mathcal{F}_q(x,\mathfrak{u},\mathfrak{u}_x,\mathfrak{u}_{xx})$, $b = \mathcal{F}_p(x,\mathfrak{u},\mathfrak{u}_x,\mathfrak{u}_{xx})$ and $c = \mathcal{F}_u(x,\mathfrak{u},\mathfrak{u}_x,\mathfrak{u}_{xx})$.

By Lemma 4.7 (still for $\lambda = 0$), the function $\mathfrak{u}$ has linear growth with respect to $x \in \mathbb{R}$ and all the partial derivatives of $\mathfrak{u}$ (and so a, a^{-1}, b and all their partial derivatives) are uniformly bounded with respect to $x \in \mathbb{R}$ and $t \geqslant 0$.

It follows from [Ang88, Theorem B] that the function $\mathfrak{v}(.,t,0)$ has at least two zeros for $t < \tau$ (the number of zeros does not increase under the flow and $\mathfrak{v}(.,\tau,0)$ has a double zero). In particular, $\mathfrak{v}(.,0,0)$ vanishes at least twice.

Consider the geodesic variation

$$\gamma_\lambda(s) = (x(s,0,\lambda), \mathfrak{u}(x(s,0,\lambda),0,\lambda)).$$

Taking the derivative with respect to λ and using that $\mathfrak{u}_x(.,0,0) = 0$, we obtain for $\lambda = 0$ the Jacobi field

$$J = (x, \mathfrak{u}, x_\lambda, \mathfrak{v})$$

where $x_* = x_*(s,0,0)$, $\mathfrak{u} = \mathfrak{u}(x(s,0,0),0,0)$ and $\mathfrak{v} = \mathfrak{u}_\lambda = \mathfrak{u}_\lambda(x(s,0,0),0,0)$. Since $s \mapsto x(s,0,0)$ is a diffeomorphism and $\mathfrak{v}(.,0,0)$ vanishes twice, the Jacobi field J is parallel to the horizontal axis γ_0, at two points s_1 and s_2. That is, $J(s_i) = \alpha_i\,\gamma_0'(s_i)$ with $\alpha_i \in \mathbb{R}$, for $i = 1,2$.

Consider the decomposition

$$J = \left(\frac{s_2 - s}{s_2 - s_1}\alpha_1 + \frac{s_1 - s}{s_1 - s_2}\alpha_2\right)\gamma' + J_\perp.$$

By construction, $J_\perp$ is a Jacobi field along γ_0 which vanishes twice, namely at s_1 and s_2. Since the Finsler metric has no conjugate points, the Jacobi field $J_\perp$ is trivial. Thus, J is parallel to γ_0' and $\mathfrak{v}(.,0,0)$ is constant equal to zero. Now, since $\mathfrak{v}$ satisfies the parabolic partial differential equation (7.4), we deduce from the initial condition $\mathfrak{v}(.,0,0) = 0$ that $\mathfrak{v}(.,.,0)$ is zero. □

We can now derive the desired result.

Lemma 7.4. — *We have $\zeta = 0$.*

Proof. — By Lemma 7.3, the function $\mathfrak{v}(.,.,0)$ and its derivative $\mathfrak{v}_x(.,.,0)$ are zero. In particular, $\mathfrak{u}_\lambda(.,0,0)$ and $\mathfrak{u}_{x\lambda}(.,0,0)$ are zero. Besides, we also have $\mathfrak{u}_x(.,0,0) = \mathfrak{u}_{xx}(.,0,0) = 0$. It follows from the relation $d\overline{\Psi}_0(v_0)(\zeta) = \zeta$ and the expression (7.3) that

$$\zeta = d\overline{\Psi}_0(v_0)(\zeta) = (x_\lambda, 0, x_{s\lambda}, 0)$$

where $x_* = x_*(0,0,0)$. By our choice of coordinate, the geodesic γ_0 agrees with the x-axis and its tangent vector v_0 is horizontal. Now, since $\zeta \in T_{v_0}U\mathbb{R}^2$ is tangent at v_0 to the unit tangent bundle of $\mathbb{R}^2$ with the Finsler metric, the vector $(x_{s\lambda}, 0)$ of $\mathbb{R}^2$ formed of the last two coordinates of ζ in $\mathbb{R}^4$ is not colinear to v_0 unless it is trivial. Since both vectors are horizontal, it follows that $x_s(0,0,0) = 0$.

To show that x_λ also vanishes at $(0,0,0)$, we consider a smooth variation of *horizontal* vectors $v(\lambda)$ with $v(0)=v_0$ and $v'(0)=\zeta$ (recall that both v_0 and ζ are horizontal). Since $v(\lambda)$ is tangent to the geodesic γ_0, the induced geodesic variation γ_λ is given by a change of variable

$$\gamma_\lambda(s)=\gamma_0(s+\bar{s}(\lambda))$$

where $\bar{s}:\mathbb{R}\to\mathbb{R}$ is a smooth function with $\bar{s}(0)=0$ so that

$$\gamma_\lambda'(0)=\gamma_0'(\bar{s}(\lambda))=v(\lambda). \tag{7.5}$$

Following the curve shortening flow, this implies that

$$\gamma_\lambda^t(s)=\gamma_0^t(s+\bar{s}(\lambda))=\gamma_0(s+\bar{s}(\lambda)).$$

In particular,

$$\overline{\Psi}_t(v(\lambda))=(\gamma_\lambda^t)'(0)=\gamma_0'(\bar{s}(\lambda)).$$

Taking the derivative of this expression with respect to λ, we obtain for $\lambda=0$

$$d\overline{\Psi}_t(v_0)(\zeta)=\bar{s}'(0)\,\frac{\partial}{\partial s}(\gamma_0')_{|s=0}.$$

Observe that the vector $\frac{\partial}{\partial s}(\gamma_0')_{|s=0}$ of $T_{\overline{\Psi}_t(v_0)}T\mathbb{R}^2$ is nonzero since the vector $\gamma_0'(0)$ formed of its first two coordinates is nonzero.

Now, by assumption, $d\overline{\Psi}_\tau(v_0)(\zeta)=0$. This implies that $\bar{s}'(0)=0$. Differentiating the equation (7.5) with respect to λ and plugging in $\lambda=0$, we derive that $\zeta=v'(0)$ is zero. □

It follows from Lemma 7.4 that the map $\overline{\Psi}_t: U\bar{M}\to T\mathbb{R}^2\setminus\{0\}$ is an immersion.

Let us show that this immersion is transverse to the rays $\mathbb{R}_+^* u=\{su\mid s>0\}$, where the vector u runs over $U\mathbb{R}^2$. By contradiction, assume that there exist $v\in U\bar{M}$ and a nonzero vector $\zeta\in T_vU\bar{M}$ such that the unit tangent vector u pointing to $\overline{\Psi}_\tau(v)$ and the vector $d\overline{\Psi}_\tau(v)(\zeta)$ are colinear in the tangent space $T_{\overline{\Psi}_\tau(v)}T\mathbb{R}^2$ to $T\mathbb{R}^2\setminus\{0\}$ at $\overline{\Psi}_\tau(v)$. In the previous coordinate system, see (7.2) and (7.3), the unit tangent vector u is proportional to

$$u\sim(0,0,1,\mathfrak{u}_x)$$

where $\mathfrak{u}_x=\mathfrak{u}_x(x(0,\tau,0),\tau,0)$, while

$$d\overline{\Psi}_\tau(v)(\zeta)=(x_\lambda, x_\lambda\,\mathfrak{u}_x+\mathfrak{u}_\lambda, x_{s\lambda}, x_{s\lambda}\,\mathfrak{u}_x+x_s\,x_\lambda\,\mathfrak{u}_{xx}+x_s\,\mathfrak{u}_{x\lambda})$$

where $x_*=x_*(0,\tau,0)$ and $\mathfrak{u}_*=\mathfrak{u}_*(x(0,\tau,0),\tau,0)$. Since these two vectors are colinear, the functions x_λ, $\mathfrak{u}_\lambda$ and the determinant

$$\begin{vmatrix} 1 & x_{s\lambda} \\ \mathfrak{u}_x & x_{s\lambda}\,\mathfrak{u}_x+x_s\,x_\lambda\,\mathfrak{u}_{xx}+x_s\,\mathfrak{u}_{x\lambda}\end{vmatrix}=x_s\,(x_\lambda\,\mathfrak{u}_{xx}+\mathfrak{u}_{x\lambda})$$

vanish at $(s,t,\lambda)=(0,\tau,0)$. So does the function $\mathfrak{u}_{x\lambda}$ (recall that x_s does not vanish). Thus, both $\mathfrak{v}$ and $\mathfrak{v}_x$ vanish at $(0,\tau,0)$, where $\mathfrak{v}=\mathfrak{u}_\lambda$. That is, the function $\mathfrak{v}$ has a multiple zero at $(0,\tau,0)$. By Lemmas 7.3 and 7.4, this implies that $\zeta=0$, which is absurd. Therefore, the map $\overline{\Psi}_t: U\bar{M}\to T\mathbb{R}^2\setminus\{0\}$ is transverse to the rays of $T\mathbb{R}^2\setminus\{0\}$.

This implies that the map $\Psi_t: U\bar{M}\to U_0\mathbb{R}^2$ defined from $\overline{\Psi}_t$ by taking the radial projection $\pi: T\mathbb{R}^2\setminus\{0\}\to U_0\mathbb{R}^2$ is a local diffeomorphism. Since the map Ψ_t is

$\mathbb{Z}^2$-equivariant and the action of $\mathbb{Z}^2$ on $U\bar{M}$ is cocompact, it follows that the map Ψ_t is a proper local diffeomorphism. Therefore, it is a covering map. Now, the quotient map of Ψ_t is a π_1-isomorphism as observed at the end of Definition 7.1. Hence, the covering $\Psi_t : U\bar{M} \to U_0\mathbb{R}^2$ is a diffeomorphism and so is its quotient map. □

Observation 7.5. — *Using the natural identifications $U_0\mathbb{T}^2 \simeq UM$ (by radial projection on each tangent plane) and $U^*M \simeq UM$ (by the Legendre transform), Theorem 7.2 yields a family of diffeomorphisms $U^*M \to U^*M$ which, by homogeneity, extend to diffeomorphisms $T^*M \to T^*M$. We still denote by $\Psi_t : T^*M \to T^*M$ this family of diffeomorphisms. Since the asymptotic directions of the geodesics of M are preserved under the curve shortening flow, see Proposition 4.3, each diffeomorphism $\Psi_t : T^*M \to T^*M$ sends the Heber foliation of T^*M to a continuous foliation of T^*M by Lipschitz Lagrangian graphs. Furthermore, if the Heber foliation is smooth, so is the image foliation.*

8. Geodesic flow deformation

Using the curve shortening flow and the family of diffeomorphisms of the previous section, we construct a deformation of the geodesic flow of a Finsler two-torus without conjugate points to the geodesic flow of the square flat two-torus.

Let $M = (\mathbb{T}^2, F)$ be a Finsler two-torus. Denote by $\bar{M} = (\mathbb{R}^2, \bar{F})$ the universal Finsler cover of M. Identify $U\bar{M} \simeq U_0\mathbb{R}^2$ by radial projection on each tangent plane. With this identification, the geodesic flow of the (quadratically convex) Finsler metric $\bar{F}$ on $\bar{M}$ induces a smooth free proper $\mathbb{Z}^2$-equivariant $\mathbb{R}$-action on $U_0\mathbb{R}^2$ by conjugation. This action is denoted by

$$\rho : \mathbb{R} \times U_0\mathbb{R}^2 \to U_0\mathbb{R}^2$$

and is defined as

$$\rho(s, v) = \gamma_v'(s)$$

for every $s \in \mathbb{R}$ and $v \in U_0\mathbb{R}^2$, where γ_v is the arclength parametrized $\bar{F}$-geodesic induced by v. In this expression, the vector $v \in U_0\mathbb{R}^2$ is identified with a vector of $U\bar{M}$ and the vector $\gamma_v'(s) \in U\bar{M}$ is identified with a vector of $U_0\mathbb{R}^2$ by radial projection. By construction, the orbits of the action ρ of $\mathbb{R}$ on $U_0\mathbb{R}^2$ project down to geodesics of $\bar{M}$.

The following result yields a deformation of the action

$$\rho : \mathbb{R} \times U_0\mathbb{R}^2 \to U_0\mathbb{R}^2$$

induced by the geodesic flow of $\bar{M}$ into the corresponding action induced by the geodesic flow of the Euclidean plane $\mathbb{R}^2$.

Theorem 8.1. — *Let $M = (\mathbb{T}^2, F)$ be a Finsler two-torus without conjugate points and $\bar{M} = (\mathbb{R}^2, \bar{F})$ be its universal Finsler cover. Then there exists a smooth free proper $\mathbb{Z}^2$-equivariant $\mathbb{R}$-action*

$$\rho_t : \mathbb{R} \times U_0\mathbb{R}^2 \to U_0\mathbb{R}^2$$

induced by the curve shortening flow, which starts at $\rho_0 = \rho$, varies smoothly with respect to $t \in [0, \infty)$, $s \in \mathbb{R}$ and $v \in U_0\mathbb{R}^2$, and converges to the action $\rho_\infty : \mathbb{R} \times U_0\mathbb{R}^2 \to U_0\mathbb{R}^2$ of the geodesic flow of the Euclidean plane $\mathbb{R}^2$. Here, the convergence is in the compact-open C^k-topology for any given $k \geqslant 0$.

Furthermore, for every $t \in [0, \infty]$, every ρ_t-orbit projects to an embedding of $\mathbb{R}$ into $\mathbb{R}^2$ under the canonical projection $U_0\mathbb{R}^2 \to \mathbb{R}^2$.

Proof. — First, let us modify the ρ-action without changing its orbits through a Euclidean arclength reparametrization. For every Finsler geodesic $\gamma : \mathbb{R} \to \bar{M}$, let $\widehat{\gamma}$ be the orientation-preserving arclength reparametrization of γ with the same initial point, i.e., $\widehat{\gamma}(0) = \gamma(0)$, with respect to the Euclidean metric on $\mathbb{R}^2$. Denote by $\sigma : \mathbb{R} \to \mathbb{R}$ the corresponding change of parameter with

$$\widehat{\gamma}(s) = \gamma(\sigma(s)).$$

For every $t \in [0, 1]$, define $\gamma_t : \mathbb{R} \to \mathbb{R}^2$ as

$$\gamma_t(s) = \gamma(t\,\sigma(s) + (1-t)\,s)$$

for every $s \in \mathbb{R}$. Clearly, the curve γ_t is a proper, regular, orientation-preserving reparametrization of γ with the same initial point as γ. By construction, the isotopy (γ_t) connects $\gamma_0 = \gamma$ to $\gamma_1 = \widehat{\gamma}$. Note that the isotopy (γ_t) smoothly depends on γ (where the curve space on the plane is endowed with the metrizable compact-open C^k-topology).

This reparametrization allows us to define an $\mathbb{R}$-action deformation of ρ

$$\rho_t : \mathbb{R} \times U_0\mathbb{R}^2 \to U_0\mathbb{R}^2$$

with $\rho_0 = \rho$ such that

$$\rho_t(s, v) = (\gamma_v)_t'(s)$$

for every $s \in \mathbb{R}$ and $v \in U_0\mathbb{R}^2$. Note that the ρ_t-orbits remain the same for every $t \in [0, 1]$ and that

$$\rho_1(s, v) = (\widehat{\gamma_v})'(s).$$

Next, we extend the deformation (ρ_t) of the geodesic flow to $t \geqslant 1$ using the isotopy of diffeomorphisms $\Psi_t : U\bar{M} \to U_0\mathbb{R}^2$ given by Theorem 7.2. For every $v \in U_0\mathbb{R}^2$, consider the unique curve γ_u^t tangent to v at $s = 0$ and pointing in the same direction as v. That is, $u = \Psi_t^{-1}(v)$ with the identification $U\bar{M} \simeq U_0\mathbb{R}^2$. Reparametrize this curve proportionally to its Euclidean arclength into $\widehat{\gamma_u^t}$ preserving both its initial point and its orientation. Define the $\mathbb{R}$-action

$$\rho_t : \mathbb{R} \times U_0\mathbb{R}^2 \to U_0\mathbb{R}^2$$

such that $\rho_{t+1}(s, v)$ is the vector of $U_0\mathbb{R}^2$ tangent to $\widehat{\gamma_u^t}$ at the point of parameter s for every $t \geqslant 0$. That is,

$$\rho_{t+1}(s, v) = \left(\widehat{\gamma^t_{\Psi_t^{-1}(v)}}\right)'(s)$$

for every $t \geqslant 0$, $s \in \mathbb{R}$ and $v \in U_0\mathbb{R}^2$. Since $\Psi_t : U_0\mathbb{R}^2 \simeq U\bar{M} \to U_0\mathbb{R}^2$ is a diffeomorphism, see Theorem 7.2, the map $\rho_t(s, .)$ is also a diffeomorphism of $U_0\mathbb{R}^2$. Clearly, the $\mathbb{R}$-action ρ_t on $U_0\mathbb{R}^2$ is smooth, free and proper. Since $\mathbb{Z}^2$ acts by isometries both on $\bar{M}$ and $\mathbb{R}^2$, the $\mathbb{R}$-action $\rho_t : \mathbb{R} \times U_0\mathbb{R}^2 \to U_0\mathbb{R}^2$ is $\mathbb{Z}^2$-equivariant. It also satisfies the symmetry property

$$\rho_t(s, -v) = -\rho_t(-s, v)$$

for every $t \in [0, \infty)$, $s \in \mathbb{R}$ and $v \in U_0\mathbb{R}^2$. It follows from Lemma 4.7 that for every $\varepsilon > 0$ and every $t \geqslant 0$ large enough, the C^k-norm of the curvature of the curves $\widehat{\gamma_u^t}$ in the Euclidean plane $\mathbb{R}^2$ is at most ε for every $u \in U_0\mathbb{R}^2$. Hence, these curves are uniformly close to segments in any given compact set of $\mathbb{R}^2$ as t goes to infinity. By construction, this implies that the action ρ_t is C^k-close to the action ρ_∞ induced by the geodesic flow of the Euclidean plane $\mathbb{R}^2$ on any compact set for t large enough. Moreover, every ρ_t-orbit is transverse to the fibers of $U_0\mathbb{R}^2 \to \mathbb{R}^2$ and projects to an embedding of $\mathbb{R}$ into $\mathbb{R}^2$ by the canonical projection $U_0\mathbb{R}^2 \to \mathbb{R}^2$. ☐

Remark 8.2. — The Heber foliation of $T^*\bar{M}$ is deformed into the canonical Heber foliation of $T^*\mathbb{R}^2$ (i.e., the foliation induced by straight lines via the Legendre transform) under the curve shortening flow of the Euclidean plane $\mathbb{R}^2$.

9. Crofton's formula for Finsler metrics without conjugate points

We review a general Crofton formula for Finsler metrics without conjugate points on surfaces.

Let $M = (\mathbb{T}^2, F)$ be a Finsler two-torus without conjugate points. Denote by $\bar{M} = (\mathbb{R}^2, \bar{F})$ the universal Finsler cover of M, where the Finsler metric $\bar{F}$ is the lift of F. We will identify $T\bar{M}$ with $T^*\bar{M}$, and $U\bar{M}$ with $U^*\bar{M}$ via the Legendre transform; see (2.1). Using these identifications, the action $\rho_{\bar{F}}$ of $\mathbb{R}$ on $T\bar{M}$ (resp. $U\bar{M}$) given by the geodesic flow of $\bar{F}$ induces an action on $T^*\bar{M}$ (resp. $U^*\bar{M}$) by conjugation by the Legendre transform, namely the cogeodesic flow of $\bar{F}$. Both $\mathbb{R}$-actions will be denoted by $\rho_{\bar{F}}$. Note that the $\mathbb{R}$-orbits of $\rho_{\bar{F}}$ on $U^*\bar{M}$ are transverse to the contact structure given by the kernel of the tautological one-form α defined in (2.2).

Recall that the quotient manifold theorem, see [Lee13, Theorem 7.10], asserts that if G is a Lie group acting smoothly, freely and properly(1) on a smooth manifold N, then the quotient space N/G is a topological manifold with a unique smooth structure such that the quotient map $N \to N/G$ is a smooth submersion (actually, a locally trivial fibration by Ehresmann's fibration theorem [Ehr50]). This result applies to the $\mathbb{R}$-action $\rho_{\bar{F}}$ of the cogeodesic flow of $\bar{F}$ on $U^*\bar{M}$. Indeed, since $\bar{M}$ is simply connected and the Finsler metric $\bar{F}$ on $\bar{M}$ has no conjugate points, the cogeodesic flow action on $U^*\bar{M}$ is free and proper. Denote by

$$\Gamma_{\bar{F}} = U^*\bar{M}/\rho_{\bar{F}}$$

the quotient manifold and by

$$q_{\bar{F}} : U^*\bar{M} \to \Gamma_{\bar{F}}$$

(1) A Lie group G acts properly on a manifold N if for every compact set $K \subset N$, the subset $\{g \in G |\ g.K \cap K \neq \emptyset\}$ is compact.

the quotient fibration. The quotient manifold $\Gamma_{\bar{F}}$ represents the space of unparametrized oriented geodesics of the Finsler metric $\bar{F}$ without conjugate points. It identifies with $S^1 \times \mathbb{R}$, that is,

$$\Gamma_{\bar{F}} \simeq S^1 \times \mathbb{R}.$$

Indeed, recall that every geodesic of $\bar{M}$ has an asymptotic direction and that the geodesics of $\bar{M}$ with a given asymptotic direction foliate $\bar{M}$; see Section 2. Conversely, every direction is the asymptotic direction of a geodesic of $\bar{M}$; see Section 2. Thus, every oriented unparametrized geodesic of $\bar{M}$ is determined by its asymptotic direction and its signed Euclidean distance to the origin of $\mathbb{R}^2$ where the sign is positive if the orientation of the geodesic matches the orientation of the Euclidean circle centered at the origin tangent to it, and negative otherwise. This yields the desired one-to-one correspondence between $\Gamma_{\bar{F}}$ and $S^1 \times \mathbb{R}$.

The isometric action of $\mathbb{Z}^2$ on $\bar{M}$ by deck transformations induces a natural action on $\Gamma_{\bar{F}}$. Note that the action of $\mathbb{Z}^2$ on $\Gamma_{\bar{F}}$ is neither free, nor continuous.

By construction, the fibration $q_{\bar{F}} : U^*\bar{M} \to \Gamma_{\bar{F}}$ is $\mathbb{Z}^2$-equivariant. It takes a unit cotangent vector $\xi \in U^*\bar{M}$ to the unparametrized oriented $\bar{F}$-geodesic of $\bar{M}$ it generates. Thus, for every $\gamma \in \Gamma_{\bar{F}}$, the projection $p(q_{\bar{F}}^{-1}(\gamma))$, where $p : T^*\bar{M} \to \bar{M}$ is the canonical projection, represents the unparametrized geodesic of $\bar{F}$ on $\bar{M}$ given by γ. We will sometimes identify γ and $p(q_{\bar{F}}^{-1}(\gamma))$.

Consider the double fibration

$$\begin{tikzcd} & U^*\bar{M} \arrow[r, "i"] \arrow[dl, "p"'] \arrow[dr, "q_{\bar{F}}"] & T^*\bar{M} \\ \bar{M} & & \Gamma_{\bar{F}} \end{tikzcd}$$

where $i : U^*\bar{M} \hookrightarrow T^*\bar{M}$ is the canonical injection. Note that the product map $p \times q_{\bar{F}} : U^*\bar{M} \to \bar{M} \times \Gamma_{\bar{F}}$ is an embedding. Since the canonical symplectic form ω is invariant under the cogeodesic flow, see (2.3), then there exists a unique $\mathbb{Z}^2$-invariant symplectic area form $\Omega_{\bar{F}}$ on $\Gamma_{\bar{F}}$ such that

$$q_{\bar{F}}^* \Omega_{\bar{F}} = i^*\omega. \tag{9.1}$$

We will need the following result about the Crofton formula on Finsler surfaces established in [ÁPB10, Theorem 5.2].

Theorem 9.1. — *The length of every smooth curve c on $\bar{M}$ with the Finsler metric $\bar{F}$ satisfies the following equation*

$$\mathrm{length}_{\bar{F}}(c) = \frac{1}{4} \int_{\gamma \in \Gamma_{\bar{F}}} \#(\gamma \cap c)\, |\Omega_{\bar{F}}| \tag{9.2}$$

where $|\Omega_{\bar{F}}|$ is the smooth positive $\mathbb{Z}^2$-invariant area density on $\Gamma_{\bar{F}}$ induced by $\Omega_{\bar{F}}$.

Remark 9.2. — The Crofton formula (9.2) shows that the Finsler metric $\bar{F}$ is uniquely determined by the fibration $q_{\bar{F}}$ (and the symplectic area form $\Omega_{\bar{F}}$ on $\Gamma_{\bar{F}}$ derived from $q_{\bar{F}}$).

Remark 9.3. — We will denote by $\lambda_{\bar{F}}$ the smooth positive $\mathbb{Z}^2$-invariant measure on $\Gamma_{\bar{F}}$ corresponding to the area density $|\Omega_{\bar{F}}|$.

We derive the following corollary.

Corollary 9.4. — *Let c be a closed geodesic of M and $\langle c\rangle$ be the subgroup of $\pi_1(\mathbb{T}^2)=\mathbb{Z}^2$ generated by the homotopy class of c. Then*

$$\mathrm{length}_F(c)=\frac{1}{4}\,\lambda_{\bar F}(\Gamma_{\bar F}/\langle c\rangle).$$

In this relation, we still denote by $\lambda_{\bar F}$ the push-forward under the quotient map $\Gamma_{\bar F}\to\Gamma_{\bar F}/\langle c\rangle$ of the restriction of $\lambda_{\bar F}$ to a Borel fundamental domain.

Proof. — Let $\bar c$ be a lift of c in $\bar M$ (of the same length). Every generic geodesic $\gamma\in\Gamma_{\bar F}$ intersects the geodesic arc $\bar c$ at most once. Moreover, for every generic geodesic $\gamma\in\Gamma_{\bar F}$, there is a unique $\langle c\rangle$-translate of γ intersecting the geodesic arc $\bar c$. Thus, by Theorem 9.1, we obtain

$$\mathrm{length}_F(c)=\frac{1}{4}\int_{\gamma\in\Gamma_{\bar F}}\#(\gamma\cap\bar c)\,d\lambda_{\bar F}=\frac{1}{4}\,\lambda_{\bar F}(\Gamma_{\bar F}/\langle c\rangle).$$

□

This leads us to the following definition.

Definition 9.5. — *A smooth positive $\mathbb{Z}^2$-invariant measure λ on $\Gamma=\Gamma_{\bar F}$ satisfies the F*-closing condition *if*

$$\lambda(\Gamma/\alpha)=\lambda_{\bar F}\left(\Gamma_{\bar F}/\alpha\right)$$

for every $\alpha\in\mathbb{Z}^2$.

Remark 9.6. — The closing condition is stable under convex combinations: if we have two measures λ_1 and λ_2 on Γ satisfying the F-closing condition, then any convex combination of λ_1 and λ_2 will also satisfy the F-closing condition.

10. Constructing Finsler metrics with prescribed geodesics

In this section, we go over the geometric construction of Finsler metrics with prescribed geodesics on a surface given by Álvarez Paiva and Berck in [ÁPB10] and adapt this construction to our situation.

Let $\mathbb{R}^2$ be the Euclidean plane with the natural action of $\mathbb{Z}^2$ by translations. Consider the double fibration of the bundle $S^*\mathbb{R}^2$ of cooriented contact elements on $\mathbb{R}^2$

$$\begin{tikzcd} & S^*\mathbb{R}^2\simeq U_0^*\mathbb{R}^2 \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{R}^2 & & \Gamma \end{tikzcd} \tag{10.1}$$

where $p:S^*\mathbb{R}^2\to\mathbb{R}^2$ is the canonical projection and $q:S^*\mathbb{R}^2\to\Gamma$ is a $\mathbb{Z}^2$-equivariant fibration onto an oriented surface Γ endowed with a $\mathbb{Z}^2$-action (in our case, $\Gamma=S^1\times\mathbb{R}$). Identify $S^*\mathbb{R}^2$ with the unit cotangent bundle $U_0^*\mathbb{R}^2$ of the Euclidean metric on $\mathbb{R}^2$ using the canonical identification. We will assume the following:

(1) The fibration $q : S^*\mathbb{R}^2 \to \Gamma$ is Legendrian (i.e., its fibers are Legendrian curves γ of $S^*\mathbb{R}^2 \simeq U_0^*\mathbb{R}^2$ with respect to the tautological contact structure α, see (2.2), that is, $\gamma^*\alpha = 0$).
(2) The product map $p \times q : S^*\mathbb{R}^2 \to \mathbb{R}^2 \times \Gamma$ is an embedding.

By [ÁPB10, Theorem 3.3], for every area form Ω on Γ, there exists a unique Finsler metric $\bar{F}$ on $\mathbb{R}^2$ satisfying the Crofton formula

$$\mathrm{length}_{\bar{F}}(c) = \frac{1}{4}\int_{\gamma \in \Gamma} \#(\gamma \cap c)\,|\Omega|$$

for any piecewise smooth curve c on $\mathbb{R}^2$. In this formula, we identify $\gamma \in \Gamma$ with the curve $p(q^{-1}(\gamma))$ of $\mathbb{R}^2$.

The Finsler metric $\bar{F}$ is given by the Gelfand transform

$$\bar{F} := p_*\left(q^*|\Omega|\right) : T\mathbb{R}^2 \to \mathbb{R} \tag{10.2}$$

of the area density $|\Omega|$; see [ÁPB10, Theorem 2.2], which is defined as follows. For every $v \in T_x\mathbb{R}^2$, take the area density on the fiber $p^{-1}(x) = S_x^*\mathbb{R}^2$ by contracting $q^*|\Omega|$ at each point $\xi \in S_x^*\mathbb{R}^2$ with every vector $\widehat{v} \in T_\xi S^*\mathbb{R}^2$ such that $dp(\widehat{v}) = v$. By definition, $\bar{F}(x; v)$ is the integral of this area density over $S_x^*\mathbb{R}^2$.

More precisely, for every $x \in \mathbb{R}^2$, there exists a unique non-vanishing one-form β_x on $S_x^*\mathbb{R}^2$ such that

$$\bar{F}(x, v) = \int_{\xi \in S_x^*\mathbb{R}^2} |\xi(v)|\,\beta_x$$

for every $v \in T_x\mathbb{R}^2$, where the one-form β_x defined by

$$(q^*\Omega)_{(x,\xi)} = p^*\xi \wedge \beta_{(x,\xi)}$$

depends smoothly on x; see [ÁPB10, Lemma 2.3].

Alternatively, the Finsler metric $\bar{F}$ can be defined from a smooth positive measure λ on Γ by the same formula

$$\mathrm{length}_{\bar{F}}(c) = \frac{1}{4}\int_{\gamma \in \Gamma} \#(\gamma \cap c)\,d\lambda$$

using the correspondence between smooth positive measures λ on Γ and area densities $|\Omega|$ of area forms Ω on Γ.

From now on, we will assume that the smooth positive measure λ on Γ is $\mathbb{Z}^2$-invariant. In this case, the Finsler metric $\bar{F}$ on $\mathbb{R}^2$ passes to the quotient and induces a Finsler metric F on $\mathbb{T}^2$. We will sometimes denote this metric by F_λ to emphasize that the construction is induced by the measure λ on Γ (and the double fibration (10.1)). Similarly, we denote by

$$M_\lambda = \left(\mathbb{T}^2, F_\lambda\right) \tag{10.3}$$

the two-torus $\mathbb{T}^2$ with the Finsler metric F_λ.

In order to apply the previous construction to the fibration induced by a smooth free proper $\mathbb{Z}^2$-equivariant $\mathbb{R}$-action

$$\rho : \mathbb{R} \times U_0\mathbb{R}^2 \to U_0\mathbb{R}^2$$

(whose orbits are transverse to the contact structure $\ker \alpha$ induced by the tautological one-form α), we first modify the action to ensure that the corresponding fibration is Legendrian. To this end, consider the map

$$\mathcal{T} : U_0\mathbb{R}^2 \to U_0\mathbb{R}^2$$

sending every vector $v \in U_{0,x}\mathbb{R}^2$ to the unique vector $w \in U_{0,x}\mathbb{R}^2$ such that $\mathcal{L}(w)(v) = 0$ with (v, w) positively oriented. Here, $\mathcal{L}$ is the Legendre transform of the Euclidean metric and the letter $\mathcal{T}$ stands for turn. The map $\mathcal{T} : U_0\mathbb{R}^2 \to U_0\mathbb{R}^2$ and its restriction $\mathcal{T}_x : U_{0,x}\mathbb{R}^2 \to U_{0,x}\mathbb{R}^2$ are diffeomorphisms. Thus, the $\mathbb{R}$-action ρ on $U_0\mathbb{R}^2$ induces a smooth free proper $\mathbb{Z}^2$-equivariant $\mathbb{R}$-action $\bar{\rho}$ on $U_0\mathbb{R}^2$ by conjugation by $\mathcal{T}$ defined as

$$\bar{\rho}(s, u) = \mathcal{T}^{-1}\Big(\rho(s, \mathcal{T}(u))\Big)$$

for every $s \in \mathbb{R}$ and $u \in U_0\mathbb{R}^2$. This action induces a smooth free proper $\mathbb{Z}^2$-equivariant $\mathbb{R}$-action on $U_0^*\mathbb{R}^2 \simeq U_0\mathbb{R}^2$ by conjugation by the Legendre transform, which is still denoted by $\bar{\rho}$.

Consider the $\mathbb{Z}^2$-equivariant fibration

$$q_{\bar{\rho}} : U_0^*\mathbb{R}^2 \to \Gamma_{\bar{\rho}}$$

induced by the smooth free proper $\mathbb{Z}^2$-equivariant $\mathbb{R}$-action $\bar{\rho}$ on $U_0^*\mathbb{R}^2$, where $\Gamma_{\bar{\rho}} = U_0^*\mathbb{R}^2/\bar{\rho}$. Since the actions ρ and $\bar{\rho}$ are conjugate, there is a natural diffeomorphism $\Gamma_{\bar{\rho}} \simeq \Gamma_\rho$.

We shall need the following results about the fibration $q_{\bar{\rho}}$ corresponding to the assumptions (1) and (2).

Lemma 10.1. — *The fibers of $q_{\bar{\rho}} : U_0^*\mathbb{R}^2 \to \Gamma_{\bar{\rho}}$ are Legendrian with respect to the contact structure induced by α on $U_0^*\mathbb{R}^2$.*

Proof. — For every $\gamma \in \Gamma_{\bar{\rho}}$, let $\xi \in q_{\bar{\rho}}^{-1}(\gamma)$ and $X \in T_\xi q_{\bar{\rho}}^{-1}(\gamma)$. Here, we identify $\gamma \in \Gamma_{\bar{\rho}}$ with the curve $p(q_{\bar{\rho}}^{-1}(\gamma))$ of $\mathbb{R}^2$. The vector $dp_\xi(X)$ is tangent to γ at $p(\xi)$. By the definition of $q_{\bar{\rho}}$, the vectors tangent to γ at $p(\xi)$ lie in the kernel of ξ. Hence, $\xi(dp_\xi(X)) = 0$. That is, $\alpha_\xi(X) = 0$ as desired. □

Lemma 10.2. — *The product map $\phi_{\bar{\rho}} = p \times q_{\bar{\rho}} : U_0^*\mathbb{R}^2 \to \mathbb{R}^2 \times \Gamma_{\bar{\rho}}$ is an embedding.*

Proof. — Let us start by proving that $\phi_{\bar{\rho}}$ is an immersion. Consider $\xi \in U_0^*\mathbb{R}^2$ and $X \in T_\xi q_{\bar{\rho}}^{-1}(\gamma)$ such that $d\phi_{\bar{\rho}}(\xi)(X) = 0$. Thus, the vector X lies in the kernel of the differential of the fibration $q_{\bar{\rho}}$ at ξ. It follows that X is tangent to the $\bar{\rho}$-orbit of $U_0^*\mathbb{R}^2$ at ξ. Since the restriction of p to each $\bar{\rho}$-orbit of $U_0^*\mathbb{R}^2$ is an embedding into $\mathbb{R}^2$, the relation $dp_\xi(X) = 0$ implies that $X = 0$. Thus, $d\phi_{\bar{\rho}}$ is injective and $\phi_{\bar{\rho}}$ is an immersion.

Let us prove now that $\phi_{\bar{\rho}}$ is injective. Let $\xi_1, \xi_2 \in U_0^*\mathbb{R}^2$ such that $\phi_{\bar{\rho}}(\xi_1) = \phi_{\bar{\rho}}(\xi_2)$. That is, $p(\xi_1) = p(\xi_2)$ and $q_{\bar{\rho}}(\xi_1) = q_{\bar{\rho}}(\xi_2)$. Thus, the covectors ξ_1 and ξ_2 are based at the same point x of $\mathbb{R}^2$. Now, the projections of the $\bar{\rho}$-orbits to $\mathbb{R}^2$ are embeddings of $\mathbb{R}$ into $\mathbb{R}^2$. Therefore, for every $\gamma \in \Gamma_{\bar{\rho}}$ such that $x \in p(q_{\bar{\rho}}^{-1}(\gamma))$, there exists a unique $\xi \in U_{0,x}^*\mathbb{R}^2$ such that $q_{\bar{\rho}}(\xi) = \gamma$. Since $q_{\bar{\rho}}(\xi_1) = q_{\bar{\rho}}(\xi_2)$, this implies that $\xi_1 = \xi_2$.

Since the map $\phi_{\bar{\rho}}$ is clearly proper, it is an embedding. □

Now, we can apply the construction given by (10.3) to the double fibration

$$\begin{tikzcd} & U_0^*\mathbb{R}^2 \arrow[dl, "p"'] \arrow[dr, "q_{\bar{\rho}}"] & \\ \mathbb{R}^2 & & \Gamma_{\bar{\rho}} \simeq \Gamma_\rho \end{tikzcd}$$

induced by the smooth free proper $\mathbb{Z}^2$-equivariant $\mathbb{R}$-action

$$\rho : \mathbb{R} \times U_0\mathbb{R}^2 \to U_0\mathbb{R}^2.$$

Thus, to any smooth positive $\mathbb{Z}^2$-invariant measure λ on Γ_ρ corresponds a Finsler metric F_λ on $\mathbb{T}^2$. Moreover, the geodesics of the lift $\bar{F}_\lambda$ of F_λ on $\mathbb{R}^2$ coincide with the projection of the curves corresponding to the fibers of $q_{\bar{\rho}}$ or q_ρ. More precisely, the geodesics of $\bar{F}_\lambda$ coincide with the curves $p(q_\rho^{-1}(\gamma))$, where γ runs over Γ_ρ. We will sometimes identify γ with $p(q_\rho^{-1}(\gamma))$. Under this identification, the space of geodesics of $\bar{F}_\lambda$ agrees with Γ_ρ.

Observation 10.3. — *In practice, we will apply this construction to a double fibration where the curves $p(q_\rho^{-1}(\gamma))$ intersect each other at most once (for instance, when Γ is the space of geodesics of the universal cover $\bar{M}$ of a Finsler two-torus M without conjugate points or the space of their images under the Euclidean curve shortening flow). In this case, the resulting Finsler metric F_λ has no conjugate points since the geodesics on its universal cover intersect each other at most once.*

11. Conjugate geodesic flows and the curve shortening flow

We show that, at the limit, the curve shortening flow preserves the natural measure on the space of geodesics of a Finsler metric on the two-torus with geodesic flow conjugate to the geodesic flow of a flat Finsler metric.

Let $M_\diamond = (\mathbb{T}^2, F_\diamond)$ be a flat Finsler two-torus and $M = (\mathbb{T}^2, F)$ be a Finsler two-torus. Suppose that the geodesic flow of M is conjugate to the geodesic flow of $M_\diamond$. This means that there exists a smooth diffeomorphism

$$h : UM_\diamond \to UM$$

which intertwines the geodesic flows of $M_\diamond$ and M, that is,

$$\varphi_t \circ h = h \circ \varphi_t^\diamond$$

where $\varphi_t^\diamond : UM_\diamond \to UM_\diamond$ and $\varphi_t : UM \to UM$ are the geodesic flows of $M_\diamond$ and M on their unit tangent bundles.

We will need the following result proved in [Cro90, § IV].

Lemma 11.1. — *The map on geodesics induced by the conjugacy $h : UM_\diamond \to UM$ induces an isomorphism $h_* : \pi_1(M_\diamond) \to \pi_1(M)$.*

Proof. — First, note that $UM_\diamond$ is homeomorphic to $S^1 \times S^1 \times S^1$ and that $\pi_1(UM_\diamond)$ is isomorphic to $\mathbb{Z}^3$ with generators a_1, a_2, a_3. One may assume that a_1 and a_2 come from tangent vector fields to closed geodesics on $M_\diamond$, while a_3 comes from the tangent fiber. In particular, there is a natural identification between the lattice $\mathbb{Z}^2$ spanned by a_1 and a_2, and $\pi_1(M_\diamond)$. This identification is given by lifting a closed geodesic to its tangent vector field in $UM_\diamond$. Consider the homomorphism $(p\circ h)_*$: $\operatorname{span}\{a_1,a_2\}\simeq\pi_1(M_\diamond)\to\pi_1(M)$, where $p:UM\to M$ is the canonical projection. This homomorphism is surjective since each element of $\pi_1(M)$ can be represented by a closed geodesic γ of M and the inverse image of γ by h is a geodesic $\gamma_\diamond$ of $M_\diamond$, hence it lies in the span of a_1 and a_2. Now, every surjective homomorphism $\mathbb{Z}^2\to\mathbb{Z}^2$ is an isomorphism. Hence the result. ☐

Observation 11.2. — *The isomorphism $h_*:\pi_1(M_\diamond)\to\pi_1(M)$, where $\pi_1(M_\diamond)=\pi_1(M)=\mathbb{Z}^2$, extends to an automorphism $A\in\mathrm{GL}_2(\mathbb{Z})$ of $\mathbb{T}^2$. Its inverse A^{-1} : $\mathbb{T}^2\to\mathbb{T}^2$ is an isometry between $(A^{-1})^*F_\diamond$ and $F_\diamond$ which induces a diffeomorphism $h_{A^{-1}}:U_{(A^{-1})^*F_\diamond}\mathbb{T}^2\to U_{F_\diamond}\mathbb{T}^2$ between the unit tangent bundles of $(A^{-1})^*F_\diamond$ and $F_\diamond$. Replacing $F_\diamond$ with $(A^{-1})^*F_\diamond$, and h with $h\circ h_{A^{-1}}$, we can assume that the map on geodesics induced by the conjugacy $h:UM_\diamond\to UM$ agrees with the identity map between $\pi_1(M_\diamond)$ and $\pi_1(M)$.*

The following result can be extracted from [Cro90, § IV].

Lemma 11.3. — *A Finsler two-torus M whose geodesic flow is conjugate to that of a flat Finsler torus $M_\diamond$ has no conjugate points.*

Proof. — We claim that every closed geodesic γ in M is the shortest in its homotopy class. To see this, let τ be a closed geodesic homotopic to γ. By Lemma 11.1, the corresponding geodesics $\gamma_\diamond$ and $\tau_\diamond$ in $M_\diamond$ are homotopic and hence have the same length (since $M_\diamond$ is a flat Finsler torus). Thus, the geodesics γ and τ of M have the same length. Since this applies as well to all iterates of γ, we see that the lift $\bar{\gamma}$ of γ to the universal cover $\bar{M}$ of M is minimizing and hence has no conjugate points. Now, observe that since $M_\diamond$ is a flat Finsler torus, the closed geodesics of $M_\diamond$ induce a dense subset in $UM_\diamond$. The same holds on M through the conjugacy $h:UM_\diamond\to UM$. We deduce that M has no conjugate points. ☐

Remark 11.4. — It is unknown whether a Finsler torus without conjugate points has a geodesic flow conjugate to the geodesic flow of a flat Finsler torus.

We will also need to introduce the equivariant diffeomorphism between spaces of geodesics induced by the geodesic flow conjugacy.

Definition 11.5. — *The conjugacy $h:UM_\diamond\to UM$ between the geodesic flows of $M_\diamond$ and M lifts to a $\mathbb{Z}^2$-equivariant conjugacy $\bar{h}:U\bar{M}_\diamond\to U\bar{M}$ between the lifted geodesic flows on the universal covers $\bar{M}_\diamond$ and $\bar{M}$. The conjugacy $\bar{h}:U\bar{M}_\diamond\to U\bar{M}$ induces a $\mathbb{Z}^2$-equivariant smooth diffeomorphism*

$$\mathfrak{h}:\Gamma_\diamond\to\Gamma$$

between the spaces of unparametrized oriented geodesics of $\bar{M}_\diamond$ and $\bar{M}$.

Now, assume that the conjugacy $h:UM_\diamond\to UM$ induces the identity map between $\pi_1(M_\diamond)$ and $\pi_1(M)$; see Observation 11.2. In this case, the map $\mathfrak{h}:\Gamma_\diamond\to\Gamma$ takes an oriented straight line to a geodesic line of the same asymptotic direction.

Recall that $\Gamma_\diamond$ identifies with the space of oriented straight lines of $\mathbb{R}^2$ parametrized by $S^1 \times \mathbb{R}$, that is, $\Gamma_\diamond \simeq S^1 \times \mathbb{R}$. Here, an oriented straight line in $\mathbb{R}^2$ is determined by its asymptotic direction and its signed Euclidean distance to the origin of $\mathbb{R}^2$; see Section 9.

Consider the deformation $\rho_t : \mathbb{R} \times U_0^*\mathbb{R}^2 \to U_0^*\mathbb{R}^2$ of the cogeodesic flow $\rho_{\bar{F}}$ on $U^*\bar{M} \simeq U_0\mathbb{R}^2$; see Section 8. Denote by

$$\Gamma_t = U^*\mathbb{R}^2/\rho_t$$

the quotient manifold and by

$$q_t : U_0^*\mathbb{R}^2 \to \Gamma_t$$

the quotient fibration; see Section 9. Since the $\mathbb{Z}^2$-translations are Euclidean isometries, the curve shortening flow induces a family of $\mathbb{Z}^2$-equivariant diffeomorphisms

$$f_t : \Gamma \to \Gamma_t \tag{11.1}$$

which, at the limit, gives rise to a $\mathbb{Z}^2$-equivariant map

$$f_\infty : \Gamma \to \Gamma_\diamond;$$

see Theorems 7.2 and 8.1. By construction, the map $f_\infty : \Gamma \to \Gamma_\diamond$ takes a geodesic in the universal cover $\bar{M}$ of M to a straight line in the plane obtained at the limit by applying the curve shortening flow; see Theorem 4.4. This map is measurable as a limit of continuous functions, it is continuous at irrational directions, see Propositions 5.6 and 11.6 below, but it is unclear whether it is continuous on Γ; see Example 5.7. In particular, it is unclear whether the function $p_\infty : S^1 \to \mathbb{R}$ in the following proposition is continuous at every direction $\theta \in S^1$.

The map $f_\infty : \Gamma \to \Gamma_\diamond$ is bijective and admits a simple expression after reparametrization by the map $\mathfrak{h} : \Gamma_\diamond \to \Gamma$ induced by the conjugacy.

Proposition 11.6. — *There exists a function $p_\infty : S^1 \to \mathbb{R}$ continuous at every irrational direction such that*

$$(f_\infty \circ \mathfrak{h})(\theta, p) = (\theta, p + p_\infty(\theta))$$

for every irrational direction $\theta \in S^1$ and every $p \in \mathbb{R}$.

Proof. — Let $\Delta = (\theta, p) \in \Gamma_\diamond$ be a straight line in $\mathbb{R}^2$. Its image $f_\infty \circ \mathfrak{h}(\Delta)$ is the limit of the curve $\mathfrak{h}(\Delta)$ under the curve shortening flow. Since the map $\mathfrak{h} : \Gamma_\diamond \to \Gamma$ and the curve shortening flow preserve the asymptotic direction of a curve, the image $f_\infty \circ \mathfrak{h}(\Delta)$ can be represented by $(\theta, p') \in \Gamma_\diamond$. For $\Delta_0 = (\theta, 0)$, we have $f_\infty \circ \mathfrak{h}(\Delta_0) = (\theta, p_\infty(\theta))$, where the function $p_\infty : S^1 \to \mathbb{R}$ is continuous at every irrational direction; see Proposition 5.6.

The signed Euclidean distance $d_\pm(\gamma \cdot \Delta, \Delta)$ between the parallel oriented lines $\gamma \cdot \Delta$ and γ depends only on γ and θ, but not on p. Thus,

$$d_\pm\Big(\gamma \cdot (f_\infty \circ \mathfrak{h}(\Delta)), f_\infty \circ \mathfrak{h}(\Delta)\Big) = d_\pm(\gamma \cdot \Delta, \Delta).$$

By $\mathbb{Z}^2$-equivariance of $f_\infty \circ \mathfrak{h}$, we obtain

$$f_\infty \circ \mathfrak{h}(\gamma \cdot \Delta_0) = \gamma \cdot (f_\infty \circ \mathfrak{h}(\Delta_0)) = \Big(\theta, p_\infty(\theta) + d_\pm(\gamma \cdot \Delta_0, \Delta_0)\Big).$$

Observe that

$$\gamma \cdot \Delta_0 = \Big(\theta, d_\pm(\gamma \cdot \Delta_0, \Delta_0)\Big)$$

under the identification $\Gamma_\diamond \simeq S^1 \times \mathbb{R}$.

Now, if θ is irrational, the subset

$$\Big\{\gamma \cdot \Delta_0 = (\theta, d_\pm(\gamma \cdot \Delta_0, \Delta_0)) \,\Big|\, \gamma \in \mathbb{Z}^2\Big\}$$

is dense in $\{(\theta, p) \in \Gamma_\diamond \mid p \in \mathbb{R}\}$. By continuity of $f_\infty \circ \mathfrak{h}$ at irrational directions, it follows that

$$f_\infty \circ \mathfrak{h}(\theta, p) = (\theta, p + p_\infty(\theta))$$

for every $(\theta, p) \in \Gamma_\diamond \simeq S^1 \times \mathbb{R}$ of irrational direction. □

We deduce the following measure invariance property.

Proposition 11.7. — *The $\mathbb{Z}^2$-equivariant map $f_\infty \circ \mathfrak{h} : \Gamma_\diamond \to \Gamma_\diamond$ preserves the measure $\lambda_\diamond$, that is,*

$$(f_\infty \circ \mathfrak{h})_* \lambda_\diamond = \lambda_\diamond.$$

Proof. — It is enough to show that the measures $\lambda_\diamond$ and $(f_\infty \circ \mathfrak{h})_*\lambda_\diamond$ agree on the rectangles $[\theta_1, \theta_2] \times [p_1, p_2]$ of $\Gamma_\diamond \simeq S^1 \times \mathbb{R}$. Since the flat Finsler metric $F_\diamond$ is invariant by the translations of $\mathbb{R}^2$, the measure $\lambda_\diamond$ associated to $F_\diamond$ is invariant by the action on $\Gamma_\diamond$ induced by these translations. It follows that the smooth measure $\lambda_\diamond$ on $\Gamma_\diamond \simeq S^1 \times \mathbb{R}$ can be written as

$$\lambda_\diamond = K(\theta)\,|d\theta \wedge dp|$$

where $K : S^1 \to \mathbb{R}$ is a smooth function which does not depend on p. By Proposition 11.6, the map $f_\infty \circ \mathfrak{h} : \Gamma_\diamond \to \Gamma_\diamond$ agrees with the map $(\theta, p) \mapsto (\theta, p + p_\infty(\theta))$ almost everywhere on $\Gamma_\diamond$, where $p_\infty : S^1 \to \mathbb{R}$ is continuous at every irrational direction. Thus,

$$\begin{aligned}
[(f_\infty \circ \mathfrak{h})_*\lambda_\diamond]\,([\theta_1, \theta_2] \times [p_1, p_2]) &= \int_{\theta_1}^{\theta_2} \int_{p_1 - p_\infty(\theta)}^{p_2 - p_\infty(\theta)} K(\theta)\, dp\, d\theta \\
&= \int_{\theta_1}^{\theta_2} \int_{p_1}^{p_2} K(\theta)\, dp\, d\theta \\
&= \lambda_\diamond\,([\theta_1, \theta_2] \times [p_1, p_2])\,.
\end{aligned}$$

Hence the desired result. □

12. Smooth Heber foliations and conjugate geodesic flows

We show that two Finsler two-tori without conjugate points having the same marked length spectrum and smooth Heber foliations have the same dynamics, namely their geodesic flows are conjugate. Note that this result has already been stated (without proof) in the introduction of [CK95], at least for Riemannian metrics.

We introduce the definition of the stable norm in the special case of a Finsler two-torus and refer to [Gro99] for a general discussion.

Definition 12.1. — *Let $M = (\mathbb{T}^2, F)$ be a Finsler two-torus. Denote by $\bar{M} = (\mathbb{R}^2, \bar{F})$ its universal Finsler cover. The* stable norm *of M is a norm on $H_1(\mathbb{T}^2;\mathbb{R})$ defined as follows. For every $\gamma \in H_1(\mathbb{T}^2;\mathbb{Z})$,*

$$|\gamma|_{\mathrm{st}} = \lim_{k \to +\infty} \frac{d_{\bar{M}}\left(o, \gamma^k \cdot o\right)}{k}$$

where $o \in \bar{M}$ is a fixed origin. Note that the limit exists and does not depend on the origin o. The function $|.|_{\mathrm{st}}$ defined by this function extends to a vector-space norm on $H_1(\mathbb{T}^2;\mathbb{R})$, the so-called stable norm of M, still denoted by $|.|_{\mathrm{st}}$. Alternatively, the stable norm of an integral homology class can be defined as the minimal length of a cycle representing this integral homology class; see [Gro99, Lemma 4.32].

Let $M = (\mathbb{T}^2, F)$ be a Finsler two-torus without conjugate points. We will endow $H_1(\mathbb{T}^2;\mathbb{R})$ with the stable norm $|.|_{\mathrm{st}}$ of M. Denote by $\mathcal{B} \subseteq H_1(\mathbb{T}^2;\mathbb{R})$ the unit ball of the stable norm of M. The space S^1 of asymptotic directions of $\bar{M}$ identifies with the boundary $\partial\mathcal{B}$ of $\mathcal{B}$. Moreover, we have the following regularity result obtained in [MS11, Lemma 5] using Aubry-Mather theory and weak KAM theory.

Lemma 12.2. — *The stable norm $|.|_{\mathrm{st}}$ of a Finsler two-torus M without conjugate points is C^1, except possibly at 0. In particular, the boundary $\partial\mathcal{B}$ of $\mathcal{B}$ admits a unique tangent line at every point.*

Proof. — Since M has no conjugate points, the cotangent space T^*M admits a Heber foliation; see Theorem 2.5. By [MS11, Lemma 5], the existence of a Heber foliation implies that Mather's average action $\beta_F : H_1(\mathbb{T}^2;\mathbb{R}) \to \mathbb{R}$ is C^1. We do not need the general definition of β_F, simply that in our case $\beta_F = \frac{1}{2}|.|_{\mathrm{st}}^2$; see [BM08, Proposition 3.3]. This implies that the stable norm $|.|_{\mathrm{st}}$ of M is C^1, except possibly at 0. ☐

For the rest of this section, we will assume that the Heber foliation is smooth; see Definition 2.10. Thus, the unit cotangent bundle $U^*\bar{M}$ of $\bar{M}$ identifies with $\mathbb{R}^2 \times S^1$ through the smooth diffeomorphism

$$\begin{aligned} U^*\bar{M} &\to \mathbb{R}^2 \times S^1 \\ (x,\xi) &\mapsto (x,\theta) \end{aligned} \tag{12.1}$$

where θ is the asymptotic direction of the geodesic induced by ξ; see (2.5). The inverse map of this diffeomorphism is given by $\xi = -dB_\theta(x)$; see (2.6).

Consider the $\mathbb{Z}^2$-invariant symplectic area form Ω on the space Γ of unparametrized oriented geodesics on $\bar{M}$; see (9.1). Define $\Gamma \to S^1$ as the quotient of $U^*\bar{M} \to \mathbb{R}^2 \times S^1 \to S^1$ under the cogeodesic flow, where the first map $U^*\bar{M} \to \mathbb{R}^2 \times S^1$ is the diffeomorphism (12.1). Let η be the pullback of the canonical one-form of S^1 under the smooth map $\Gamma \to S^1$ taking a geodesic of $\bar{M}$ to its asymptotic direction θ. Observe that the one-form η does not vanish on Γ and is $\mathbb{Z}^2$-invariant (by $\mathbb{Z}^2$-invariance of $\Gamma \to S^1$). Thus, there exists a $\mathbb{Z}^2$-invariant differential one-form ζ on Γ such that

$$\Omega = \eta \wedge \zeta. \tag{12.2}$$

The one-form ζ is not uniquely defined. Indeed, every other $\mathbb{Z}^2$-invariant one-form ζ' on Γ with $\Omega = \eta \wedge \zeta'$ satisfies

$$\zeta' = \zeta + h\,\eta$$

where $h : \Gamma \to \mathbb{R}$ is a smooth $\mathbb{Z}^2$-invariant function on Γ. Still, we can think of ζ as "the quotient of the two-form Ω by the one-form η". We will write Ω/η for the class of $\mathbb{Z}^2$-invariant differential one-forms ζ on Γ with $\Omega = \eta \wedge \zeta$. For a curve c of Γ formed of geodesics of $\bar{M}$ with the same asymptotic direction, we will write

$$\int_c \Omega/\eta := \int_c \zeta \tag{12.3}$$

where ζ is a representative of Ω/η. Observe that the one-form η vanishes along c. Thus, even though ζ is not uniquely defined, the value of the integral (12.3) does not depend on the one-form ζ representing Ω/η, which justifies the notation.

The unit cotangent bundle $U^*\bar{M}$ also identifies with $\Gamma \times \mathbb{R} = S^1 \times \mathbb{R} \times \mathbb{R}$. Loosely speaking, a unit covector $(x, \xi) \in U^*\bar{M}$ is represented by the geodesic in Γ (of asymptotic direction θ) it induces and a parameter s measuring the signed distance to the origin of the geodesic (defined as the unique point of the geodesic where the Busemann function B_θ vanishes); see Figure 12.1.

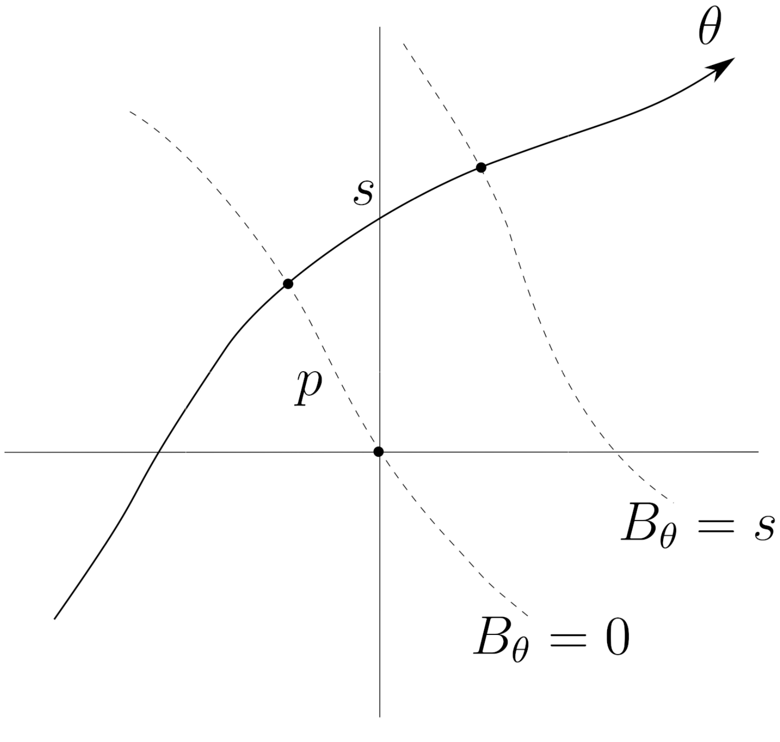

Figure 12.1. (θ, p, s) coordinates

More precisely, this identification is given by

$$\begin{aligned} U^*\bar{M} &\to S^1 \times \mathbb{R} \times \mathbb{R} \\ (x, \xi) &\mapsto (\theta, p, s) \end{aligned} \tag{12.4}$$

In this expression, θ is the asymptotic direction of the geodesic induced by ξ, $s = B_\theta(x)$ and

$$p = \int_c \Omega/\eta$$

where c is the path of Γ formed of the geodesics of $\bar{M}$ (with the same asymptotic direction θ) lying between the geodesic of asymptotic direction θ passing through the origin $o \in \mathbb{R}^2$ and the geodesic of asymptotic direction θ passing through x. Since the Heber foliation is assumed to be smooth, the map (12.4) is also a smooth diffeomorphism.

Under this identification, the symplectic area form Ω on Γ can be written as

$$\Omega = \eta \wedge dp$$

(but we won't need this) and the action of the cogeodesic flow of $\bar{M}$ on $U^*\bar{M}$ is given by

$$\varphi_t(\theta, p, s) = (\theta, p, s + t). \tag{12.5}$$

Finally, the action of $\mathbb{Z}^2$ on $U^*\bar{M}$ takes the following form. There exist two additive functions $\vec{p}_\theta : \mathbb{Z}^2 \to \mathbb{R}$ and $\vec{s}_\theta : \mathbb{Z}^2 \to \mathbb{R}$ such that for every $\gamma \in \mathbb{Z}^2$, we have

$$\gamma \cdot (\theta, p, s) = (\theta, p + \vec{p}_\theta(\gamma), s + \vec{s}_\theta(\gamma))\,. \tag{12.6}$$

The functions $\vec{p}_\theta : \mathbb{Z}^2 \to \mathbb{R}$ and $\vec{s}_\theta : \mathbb{Z}^2 \to \mathbb{R}$ can be defined as

$$\vec{p}_\theta(\gamma) = \int_{(\theta,x)}^{(\theta,\gamma\cdot x)} \Omega/\eta \tag{12.7}$$

$$\vec{s}_\theta(\gamma) = B_\theta(\gamma \cdot x) - B_\theta(x) \tag{12.8}$$

where the integral is along the path of Γ formed of the geodesics of $\bar{M}$ (with the same asymptotic direction θ) lying between the geodesic of asymptotic direction θ passing through x and the geodesic of asymptotic direction θ passing through $\gamma \cdot x$, where x is any point of $\bar{M}$.

It remains to show the following.

Lemma 12.3. —

1. *The functions $\vec{p}_\theta$ and $\vec{s}_\theta$ given by the formulas* (12.7) *and* (12.8) *do not depend on $x \in \bar{M}$.*
2. *The functions $\vec{p}_\theta$ and $\vec{s}_\theta$ are additive.*
3. *The relation* (12.6) *for the action of $\gamma \in \mathbb{Z}^2$ is satisfied.*

Proof. — It follows from the $\mathbb{Z}^2$-invariance of the one-form ζ representing Ω/η that the formula (12.7) defining $\vec{p}_\theta$ does not depend on $x \in \bar{M}$. Indeed, for every $x, x' \in \bar{M}$, we have

$$\begin{aligned}
\vec{p}_\theta(\gamma) &= \int_{(\theta,x)}^{(\theta,x')} \Omega/\eta + \int_{(\theta,x')}^{(\theta,\gamma\cdot x')} \Omega/\eta + \int_{(\theta,\gamma\cdot x')}^{(\theta,\gamma\cdot x)} \Omega/\eta \\
&= \int_{(\theta,x)}^{(\theta,x')} \Omega/\eta + \int_{(\theta,x')}^{(\theta,\gamma\cdot x')} \Omega/\eta + \int_{(\theta,x')}^{(\theta,x)} \Omega/\eta \\
&= \int_{(\theta,x')}^{(\theta,\gamma\cdot x')} \Omega/\eta.
\end{aligned}$$

Similarly, it directly follows from [CS86, Corollary 2.6], using the fact that the (minimal) closed geodesics in every free homotopy class of M cover the torus, that for every rational direction $\theta \in S^1$ and every $\gamma \in \mathbb{Z}^2$, the difference $B_\theta(\gamma \cdot x) - B_\theta(x)$ does not depend on $x \in \bar{M}$. By a continuity argument, see Proposition 2.9, the same holds true for every $\theta \in S^1$ and $\gamma \in \mathbb{Z}^2$.

Since the functions $\vec{p}_\theta$ and $\vec{s}_\theta$ do not depend on $x \in \bar{M}$, this ensures that both functions are additive and that the relation (12.6) for the action of $\gamma \in \mathbb{Z}^2$ is satisfied. $\square$

The additive functions $\vec{p}_\theta : H_1(\mathbb{T}^2;\mathbb{Z}) \to \mathbb{R}$ and $\vec{s}_\theta : H_1(\mathbb{T}^2;\mathbb{Z}) \to \mathbb{R}$ extend to linear forms $\vec{p}_\theta : H_1(\mathbb{T}^2;\mathbb{R}) \to \mathbb{R}$ and $\vec{s}_\theta : H_1(\mathbb{T}^2;\mathbb{R}) \to \mathbb{R}$.

Lemma 12.4. — *Let $\theta \in \partial\mathcal{B} \simeq S^1$.*

(1) *The linear form $\vec{p}_\theta : H_1(\mathbb{T}^2;\mathbb{R}) \to \mathbb{R}$ satisfies*

$$\ker \vec{p}_\theta = \mathbb{R}\,\theta.$$

(2) *The linear form $\vec{s}_\theta : H_1(\mathbb{T}^2;\mathbb{R}) \to \mathbb{R}$ is of norm 1 (with respect to the stable norm of M) and satisfies $\vec{s}_\theta(-\theta) = 1$. In particular, the kernel of $\vec{s}_\theta$ is parallel to the tangent line to $\partial\mathcal{B}$ at $-\theta$.*

Proof. —

(1) Let $\theta \in \partial\mathcal{B}$ be a rational direction and $\gamma \in H_1(\mathbb{T}^2;\mathbb{Z})$ be an integral vector pointing in the direction of θ. The geodesic (θ, x) of $\bar{M}$ of asymptotic direction θ passing through x coincides with its image $(\theta, \gamma \cdot x)$ under the translation of γ. Thus, $\vec{p}_\theta(\gamma) = 0$. Since $\theta = \frac{\gamma}{|\gamma|_{\mathrm{st}}}$, it follows that $\vec{p}_\theta(\theta) = 0$. The same relation also holds true if θ is an irrational direction by a continuity argument. Hence, $\ker \vec{p}_\theta = \mathbb{R}\,\theta$.

(2) As previously, let $\theta \in \partial\mathcal{B}$ be a rational direction and $\gamma \in H_1(\mathbb{T}^2;\mathbb{Z})$ be an integral vector pointing in the direction of θ. By Proposition 2.11.(1), the displacement function $d_\gamma : \bar{M} \to \mathbb{R}$ of $\gamma \in H_1(\mathbb{T}^2;\mathbb{Z})$, defined as $d_\gamma(x) = d_{\bar{M}}(x, \gamma \cdot x)$, is constant equal to $|\gamma|_{\mathrm{st}}$. Since the Busemann function $B_\theta : \bar{M} \to \mathbb{R}$ is 1-Lipschitz, we obtain

$$|\vec{s}_\theta(\gamma)| = |B_\theta(\gamma \cdot x) - B_\theta(x)| \leqslant d_{\bar{M}}(x, \gamma \cdot x) = |\gamma|_{\mathrm{st}}.$$

Thus, the linear form $\vec{s}_\theta : H_1(M;\mathbb{R}) \to \mathbb{R}$ is of norm at most 1.

Furthermore, since γ points in the direction of θ, we have

$$\vec{s}_\theta(\gamma) = B_\theta(\gamma \cdot x) - B_\theta(x) = -d_{\bar{M}}(x, \gamma \cdot x) = -|\gamma|_{\mathrm{st}}.$$

Since $\theta = \frac{\gamma}{|\gamma|_{\mathrm{st}}}$, we obtain $\vec{s}_\theta(\theta) = -1$. The same relation also holds true if θ is an irrational direction by a continuity argument.

Therefore, $\|\vec{s}_\theta\| = 1$ and $\vec{s}_\theta(-\theta) = 1$ for every $\theta \in \partial\mathcal{B}$. $\square$

This construction allows us to prove the following result.

Theorem 12.5. — *Let $M_1 = (\mathbb{T}^2, F_1)$ and $M_2 = (\mathbb{T}^2, F_2)$ be two Finsler two-torus without conjugate points whose Heber foliation are smooth. Suppose that the two metrics have the same marked length spectrum. Then the geodesic flows of M_1 and M_2 are conjugate.*

Proof. — Denote by $\vec{p}_{\theta,i} : H_1(\mathbb{T}^2;\mathbb{R}) \to \mathbb{R}$ and $\vec{s}_{\theta,i} : H_1(\mathbb{T}^2;\mathbb{R}) \to \mathbb{R}$ the linear forms corresponding to the metric F_i defined in (12.7) and (12.8). By Lemma 12.4 (1), the linear forms $\vec{p}_{\theta,1}$ and $\vec{p}_{\theta,2}$ have the same kernel and are therefore proportional. That is,

$$\vec{p}_{\theta,2} = \kappa(\theta)\, \vec{p}_{\theta,1}$$

for some smooth function $\kappa : S^1 \to \mathbb{R}$. Since the metrics F_1 and F_2 have the same length spectrum, they also have the same stable norm. It follows from Lemma 12.4 (2) that the linear forms $\vec{s}_{\theta,1}$ and $\vec{s}_{\theta,2}$ coincide for every direction $\theta \in S^1$.

Using the identification (12.4), consider the map

$$\begin{aligned} U^*\bar{M}_1 &\to U^*\bar{M}_2 \\ (\theta, p, s) &\mapsto (\theta, \kappa(\theta)\, p, s)\,. \end{aligned}$$

By the action of the cogeodesic flow in this coordinate system, see (12.5), this map $U^*\bar{M}_1 \to U^*\bar{M}_2$ conjugates the cogeodesic flows on $\bar{M}_1$ and $\bar{M}_2$. Furthermore, the map $U^*\bar{M}_1 \to U^*\bar{M}_2$ passes to the quotient under the $\mathbb{Z}^2$-action on $\bar{M}_1$ and $\bar{M}_2$; see (12.5). Thus, the quotient map $U^*M_1 \to U^*M_2$ conjugates the cogeodesic flows of M_1 and M_2, and therefore the geodesic flows of M_1 and M_2 through their Legendre transforms, as required. □

13. Deforming Finsler two-tori

Given a Finsler metric on the two-torus whose geodesic flow is conjugate to that of a flat Finsler metric, we show how to canonically connect the two metrics through Finsler metrics with the same dynamics. For this, we make use of the deformation of the geodesic foliation given by the Euclidean curve shortening flow, see Section 4, the invariance of the natural measure on the space of geodesics at the limit under this flow, see Section 11, the construction of Finsler metrics with prescribed geodesics given by the Crofton formula, see Section 10, and the relationship between conjugate geodesic flows and smooth Heber foliations; see Section 12.

Let us start with the following observation.

Proposition 13.1. — *Let $M_1 = (\mathbb{T}^2, F_1)$ and $M_2 = (\mathbb{T}^2, F_2)$ be two Finsler two-torus without conjugate points whose geodesic flows are conjugate. If the Heber foliation of M_1 is smooth then the Heber foliation of M_2 is also smooth.*

Proof. — Since the geodesic flow of M_1 and M_2 are conjugate, there exists a smooth $\mathbb{Z}^2$-equivariant diffeomorphism $\bar{h} : U\bar{M}_1 \to U\bar{M}_2$ which intertwines the geodesic flows of $\bar{M}_1$ and $\bar{M}_2$. We can assume that the homomorphism $h_* : \pi_1(M_1) \to \pi_1(M_2)$ induced by the quotient map is the identity; see Observation 11.2. This implies that a unit vector generating a geodesic of $\bar{M}_1$ with asymptotic direction θ is sent under $\bar{h} : U\bar{M}_1 \to U\bar{M}_2$ to a unit vector generating a geodesic of $\bar{M}_2$ with the same asymptotic direction θ. Thus, we have the following diagram

$$\begin{tikzcd} (x_1, v_1) \in U\bar{M}_1 \arrow[r, "\bar{h}"] \arrow[d, "h_1"] & (x_2, v_2) \in U\bar{M}_2 \arrow[d, "h_2"] \\ (x_1, \theta) \in \mathbb{R}^2 \times S^1 & (x_2, \theta) \in \mathbb{R}^2 \times S^1 \end{tikzcd}$$

where the two vertical maps h_1 and h_2 are the inverse Heber homeomorphism of M_1 and M_2; see Definition 2.10. By assumption, the first vertical map h_1 is a diffeomorphism.

To prove that the Heber foliation of M_2 is smooth, we need to show that the second vertical map $h_2 : U\bar{M}_2 \to \mathbb{R}^2 \times S^1$ is also a diffeomorphism. First, observe that the composition $U\bar{M}_2 \to U\bar{M}_1 \to \mathbb{R}^2 \times S^1 \to S^1$ taking (x_2, v_2) to θ is smooth. This ensures that the vertical map h_2 is smooth. Now, we only need to show that its inverse h_2^{-1} is also smooth. Fix $(x_2, \theta) \in \mathbb{R}^2 \times S^1$. By Proposition 3.1, we can assume that the horizontal and vertical curves of $\bar{M}_1$ are geodesics. Without loss of generality, we can also assume that the direction θ is not horizontal, otherwise we switch the horizontal and vertical axis of $\bar{M}_1$ and $\bar{M}_2$. Thus, the horizontal axis of $\bar{M}_1$ transversely intersects every curve of the geodesic foliation with asymptotic direction θ exactly once. As x runs over the horizontal axis of $\bar{M}_1$, the image under the conjugacy of the geodesic γ_v generated by the vector $h_1^{-1}(x, \theta) = (x, v) \in U\bar{M}_1$ runs over the geodesic foliation of $\bar{M}_2$ with asymptotic direction θ (recall that the conjugacy preserves the asymptotic direction). More precisely, there is a unique vector $(x, v) \in U\bar{M}_1$ and a unique real $t \in \mathbb{R}$ such that the basepoint of the vector $h(\gamma_v'(t)) \in U\bar{M}_2$ agrees with x_2. The vector v and the real t vary smoothly with (x_2, θ) and the same goes for $(x_1, v_1) = \gamma_v'(t) \in U\bar{M}_1$ and $(x_2, v_2) = \bar{h}(x_1, v_1) \in U\bar{M}_2$. As previously noticed, we have $h_2(x_2, v_2) = (x_2, \theta)$ or equivalently $(x_2, v_2) = h_2^{-1}(x_2, \theta)$. It follows that the inverse of the second vertical map $h_2 : U\bar{M}_2 \to \mathbb{R}^2 \times S^1$ in the diagram is also smooth. Hence, the map $h_2 : U\bar{M}_2 \to \mathbb{R}^2 \times S^1$ is a diffeomorphism. $\square$

We can now prove our main theorem.

Theorem 13.2. — *Let $M = (\mathbb{T}^2, F)$ be a Finsler two-torus whose geodesic flow is conjugate to the geodesic flow of a flat Finsler two-torus $M_\diamond = (\mathbb{T}^2, F_\diamond)$. Then there exists a canonical deformation $(F_t)_{t \geqslant 0}$ of Finsler metrics on $\mathbb{T}^2$ with $F_0 = F$ such that*

(1) *the geodesic flow of F_t is conjugate to the geodesic flow of $F_\diamond$;*
(2) *the metric F_t converges to $F_\diamond$ for the uniform convergence topology, up to isometry, as t goes to infinity.*

Proof. — First, observe that since the geodesic flow of M is conjugate to that of the flat Finsler torus $M_\diamond$, the metric F has no conjugate points by Lemma 11.3 and its Heber foliation is smooth by Proposition 13.1. Denote by $h : UM_\diamond \to UM$ the conjugacy between the geodesic flows of $M_\diamond$ and M. By Observation 11.2, we can assume that the conjugacy $h : UM_\diamond \to UM$ induces the identity map between $\pi_1(M_\diamond)$ and $\pi_1(M)$. In particular, this implies that the metrics $F_\diamond$ and F have the same marked length spectrum. Let $\mathfrak{h} : \Gamma_\diamond \to \Gamma$ be the $\mathbb{Z}^2$-equivariant smooth diffeomorphism between the spaces of unparametrized oriented geodesics of $\bar{M}_\diamond$ and $\bar{M}$ induced by $h : UM_\diamond \to UM$; see Definition 11.5. Denote by $\lambda_\diamond$ and λ the smooth positive $\mathbb{Z}^2$-invariant measures on $\Gamma_\diamond$ and Γ induced by $F_\diamond$ and F; see Remark 9.3. The measures

$$\lambda_t = (1-t)\,\lambda + t\,\mathfrak{h}_*\lambda_\diamond$$

with $0 \leqslant t \leqslant 1$ connecting λ to the push-forward $\mathfrak{h}_*\lambda_\diamond$ of $\lambda_\diamond$ by the $\mathbb{Z}^2$-equivariant smooth diffeomorphism $\mathfrak{h} : \Gamma_\diamond \to \Gamma$ are smooth positive $\mathbb{Z}^2$-invariant measures on Γ.

The family (λ_t) of measures on Γ induces a family (F_t) of Finsler metrics on $\mathbb{T}^2$ joining $F_0 = F$ to $F_1 = F_{\mathfrak{h}_*\lambda_\diamond}$ with the same geodesic space Γ; see the end of Section 10. Since the metrics $F_\diamond$ and F have the same marked length spectrum, the measures $\mathfrak{h}_*\lambda_\diamond$ and λ on Γ satisfy the same F-closing condition; see Definition 9.5 and Corollary 9.4. By stability of the F-closing condition under convex combinations, see Remark 9.6, the measures λ_t also satisfy the F-closing condition. It follows that the Finsler metrics F_t induced by the measures λ_t have the same marked length spectrum with the same geodesic space Γ as F; see the end of Section 10 and Corollary 9.4. By construction, these metrics have no conjugate points, see Observation 10.3, and their Heber foliations are smooth (since the metrics F_t and F have the same geodesics and the Heber foliation of F is smooth). Applying Theorem 12.5, we obtain that the Finsler metrics F_t have conjugate geodesic flows.

Now, we need to deform the metric $F_{\mathfrak{h}_*\lambda_\diamond}$ into $F_\diamond$ through a family of Finsler metrics on $\mathbb{T}^2$ with conjugate geodesic flows. Instead of deforming the measure λ induced by F on the geodesic space Γ as previously, we will deform the geodesics. Consider the family of $\mathbb{Z}^2$-equivariant maps $f_t : \Gamma \to \Gamma_t$ induced by the curve shortening flow; see (11.1). The $\mathbb{Z}^2$-equivariant maps $f_t \circ \mathfrak{h} : \Gamma_\diamond \to \Gamma_t$ are smooth for $t \in [0, \infty)$ and measurable for $t = \infty$; see Proposition 11.6. Thus, the push-forward measures $\lambda_t = (f_t \circ \mathfrak{h})_*\lambda_\diamond$ on Γ_t are smooth, positive and $\mathbb{Z}^2$-invariant for every $t \in [0, \infty]$, even for $t = \infty$ where $(f_\infty \circ \mathfrak{h})_*\lambda_\diamond = \lambda_\diamond$ on $\Gamma_\infty = \Gamma_\diamond$ by Proposition 11.7. The family (λ_t) of measures on Γ_t induces a family (F_t) of Finsler metrics on $\mathbb{T}^2$ starting at $F_{\mathfrak{h}_*\lambda_0}$ and ending at $F_\diamond$; see the end of Section 10. Indeed, by the Crofton formula, see Theorem 9.1, the Finsler metric corresponding to the measure $\lambda_t = (f_t \circ \mathfrak{h})_*\lambda_\diamond$ on Γ_t when $t = \infty$ is the flat Finsler metric $F_\diamond$. Now, by $\mathbb{Z}^2$-invariance of the maps $f_t : \Gamma \to \Gamma_t$ for $t \in [0, \infty]$, we have

$$\lambda_t(\Gamma_t/\langle\alpha\rangle) = \mathfrak{h}_*\lambda_\diamond(\Gamma/\langle\alpha\rangle) = \lambda(\Gamma/\langle\alpha\rangle)$$

for every $\alpha \in \mathbb{Z}^2$ (recall that the measures $\mathfrak{h}_*\lambda_\diamond$ and λ on Γ satisfy the same F-closing condition). It follows that the Finsler metrics F_t with $t \in [0, \infty]$ have the same marked length spectrum by Corollary 9.4. Furthermore, since their geodesic space is given by the deformation Γ_t of Γ under the curve shortening flow, the metrics F_t have no conjugate points, see Observation 10.3, and their Heber foliation is smooth (even for $t = \infty$); see Observation 7.5. Applying Theorem 12.5 as previously, we obtain that the Finsler metrics F_t have conjugate geodesic flows for $t \in [0, \infty]$.

In order to conclude the proof of Theorem 13.2, we need to show the point (2).

Fix a compact convex subset K of $\mathbb{R}^2$. Let $\Lambda_K = [0, \infty] \times K \times K$ be the compact set where every point $(t, x, y) \in [0, \infty] \times K \times K$ identifies with the geodesic arc $c_t(x, y)$ of $\bar{M}_t$ joining x to y (with $\bar{M}_\infty = \mathbb{R}^2$). Note that $c_t(x, y)$ lies in a curve of Γ_t.

Consider the map $\varphi : \Lambda_K \times \Gamma_\diamond \to \mathbb{R}_+$ defined as

$$\varphi(t, x, y, \gamma) = \frac{1}{4}\,\#\Big(f_t \circ \mathfrak{h}(\gamma) \cap c_t(x, y)\Big)$$

for every $(t, x, y) \in \Lambda_K$ and $\gamma \in \Gamma_\diamond$. Observe that $f_t \circ \mathfrak{h}(\gamma)$ and $c_t(x, y)$ intersect at most once as long as γ is different from the line whose image $f_t \circ \mathfrak{h}(\gamma)$ under the curve shortening flow passes through x and y. Thus, the map $\varphi : \Lambda_K \times \Gamma_\diamond \to \mathbb{R}_+$ is bounded by $\frac{1}{4}$ almost everywhere.

Let w be the maximal width of a geodesic of $\bar{M}$; see Theorem 2.7. The space $\Gamma_\diamond(K_w)$ of lines in $\Gamma_\diamond$ whose image under $\mathfrak{h} : \Gamma_\diamond \to \Gamma$ intersects the closed Euclidean $(w+1)$-neighborhood K_w of $K \subseteq \mathbb{R}^2$ is compact. By the definition of $\Gamma_\diamond(K_w)$, the image under $\mathfrak{h} : \Gamma_\diamond \to \Gamma$ of a line $\gamma \in \Gamma_\diamond \setminus \Gamma_\diamond(K_w)$ is a geodesic of $\bar{M}$ at Euclidean distance greater than $w+1$ from K. It follows from the definition of w that this image lies in a closed strip disjoint from K and so are its images $f_t \circ \mathfrak{h}(\gamma)$ and $f_\infty \circ \mathfrak{h}(\gamma)$ under the curve shortening flow. By Crofton's formula, see Theorem 9.1, we obtain

$$d_{\bar{F}_t}(x,y) = \int_{\gamma \in \Gamma_\diamond(K_w)} \varphi(t,x,y,\gamma)\, d\lambda_\diamond \tag{13.1}$$

for every $(t,x,y) \in \Lambda_K$.

Let $x_0, y_0 \in K$. By Theorem 4.4, for almost every $\gamma \in \Gamma_\diamond(K_w)$ (actually, as long as the line $f_\infty \circ \mathfrak{h}(\gamma)$ does not pass through x_0 or y_0), the map $\varphi(.,\gamma) : \Lambda_K \to \mathbb{R}_+$ is continuous at (∞, x_0, y_0). Indeed, for (x,y) close to (x_0,y_0) and for t large enough, the arc $c_t(x,y)$ is C^k-close to the segment $c_\infty(x_0,y_0) = [x_0,y_0]$. Similarly, for t large enough, the curve $f_t \circ \mathfrak{h}(\gamma)$ is close to the line $f_\infty \circ \mathfrak{h}(\gamma)$. Thus, if the line $f_\infty \circ \mathfrak{h}(\gamma)$ transversely intersects the interior of the segment $[x_0,y_0]$, then the curve $f_t \circ \mathfrak{h}(\gamma)$ intersects the arc $c_t(x,y)$ at a unique point. Similarly, if the line $f_\infty \circ \mathfrak{h}(\gamma)$ does not intersect the segment $[x_0,y_0]$, then the curve $f_t \circ \mathfrak{h}(\gamma)$ does not intersect the arc $c_t(x,y)$ either. Therefore, the map $\varphi(\cdot,\gamma) : \Lambda_K \to \mathbb{R}_+$ is constant (equal to 0 or 1) in a neighborhood of each point (∞, x_0, y_0), where x_0 and y_0 do not belong to the line $f_\infty \circ \mathfrak{h}(\gamma)$. It follows from Lebesgue's dominated convergence theorem that the integral (13.1) converges to

$$\begin{aligned}\frac{1}{4}\int_{\gamma \in \Gamma_\diamond(K_w)} \#\Big(f_\infty \circ \mathfrak{h}(\gamma) \cap [x_0,y_0]\Big)\, d\lambda_\diamond &= \frac{1}{4}\int_{\gamma \in \Gamma_\diamond} \#(\gamma \cap [x_0,y_0])\, d[(f_\infty \circ \mathfrak{h})_* \lambda_\diamond] \\ &= d_{\bar{F}_\diamond}(x_0,y_0)\end{aligned}$$

as $(t,x,y) \to (\infty, x_0, y_0)$, where the last inequality comes from the measure invariance property of Proposition 11.7 and Crofton's formula. Hence, the distance $d_{\bar{F}_t}$ uniformly converges to $d_{\bar{F}_\infty}$ on any compact set as t goes to infinity.

To show that the metric F_t converges to $F_\diamond$ for the uniform convergence topology, it is enough to prove that the diameter of M_t is uniformly bounded. To this end, fix two (transverse) geodesic foliation $\mathcal{F}'_\diamond$ and $\mathcal{F}''_\diamond$ of $M_\diamond$ by simple closed geodesics, see Proposition 2.11.(1), and denote by ℓ' and ℓ'' their lengths. The corresponding geodesic foliations $\mathcal{F}'_t$ and $\mathcal{F}''_t$ of M_t have the same length (recall that M_t has no conjugate points with the same marked length spectrum as $M_\diamond$). Observe that any pair of points on M_t can be joined by a path formed of an arc lying a geodesic of $\mathcal{F}'_t$ followed by an arc lying a geodesic of $\mathcal{F}''_t$. This implies that the diameter of M_t is uniformly bounded by $\ell' + \ell''$. □

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