Annales Henri Lebesgue
7 (2024) 1175-1243

ANNALES
HENRI LEBESGUE

ANNE-LAURE DALIBARD

CHARLOTTE PERRIN

LOCAL AND GLOBAL WELL-POSEDNESS OF ONE-DIMENSIONAL FREE-CONGESTED EQUATIONS

CARACTÈRE LOCALEMENT ET GLOBALEMENT BIEN POSÉ D'ÉQUATIONS DE CONGESTION UNI-DIMENSIONNELLES

Abstract. — This paper is dedicated to the study of a one-dimensional congestion model, consisting of two different phases. In the congested phase, the pressure is free and the dynamics is incompressible, whereas in the non-congested phase, the fluid obeys a pressureless compressible dynamics.

Keywords: Navier–Stokes equations, free boundary problem, nonlinear stability.
2020 *Mathematics Subject Classification:* 35Q35, 35L67.
DOI: `https://doi.org/10.5802/ahl.218`
(*) This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program Grant agreement No 637653, project BLOC "Mathematical Study of Boundary Layers in Oceanic Motion". This work was supported by the SingFlows and CRISIS projects, grants ANR-18-CE40-0027 and ANR-20-CE40-0020-01 of the French National Research Agency (ANR). A.-L. D. acknowledges the support of the Institut Universitaire de France. This material is based upon work supported by the National Science Foundation under Grant No. DMS-1928930 while the authors participated in a program hosted by the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2021 semester.

We investigate the Cauchy problem for initial data which are small perturbations in the non-congested zone of traveling wave profiles. We prove two different results. First, we show that for arbitrarily large perturbations, the Cauchy problem is locally well-posed in weighted Sobolev spaces. The solution we obtain takes the form $(v_s, u_s)(t, x - \widetilde{x}(t))$, where $x < \widetilde{x}(t)$ is the congested zone and $x > \widetilde{x}(t)$ is the non-congested zone. The set $\{x = \widetilde{x}(t)\}$ is a free boundary, whose evolution is coupled with the one of the solution. Second, we prove that if the initial perturbation is sufficiently small, then the solution is global. This stability result relies on coercivity properties of the linearized operator around a traveling wave, and on the introduction of a new unknown which satisfies better estimates than the original one. In this case, we also prove that traveling waves are asymptotically stable.

Résumé. — Cet article est consacré à l'étude d'un modèle de congestion unidimensionnel, composé de deux phases distinctes. Dans la phase congestionnée, la pression est libre et la dynamique est incompressible, tandis que dans la phase non congestionnée, le fluide obéit à une dynamique compressible avec une pression nulle.

Nous étudions le problème de Cauchy pour des données initiales qui sont de petites perturbations de profils d'ondes progressives dans la zone non congestionnée. Nous prouvons deux résultats différents. Premièrement, nous montrons que pour des perturbations arbitrairement grandes, le problème de Cauchy est localement bien posé dans des espaces de Sobolev à poids. La solution obtenue prend la forme $(v_s, u_s)(t, x - \widetilde{x}(t))$, où $x < \widetilde{x}(t)$ est la zone congestionnée et $x > \widetilde{x}(t)$ est la zone libre. La ligne $\{x = \widetilde{x}(t)\}$ est une frontière libre, dont l'évolution est couplée à celle de la solution. Deuxièmement, nous prouvons que si la perturbation initiale est suffisamment petite, alors la solution est globale. Ce résultat de stabilité repose sur des propriétés de coercivité de l'opérateur linéarisé autour d'une onde progressive, et sur l'introduction d'une nouvelle inconnue qui satisfait de meilleures estimations que l'inconnue initiale. Dans ce cas, nous prouvons également que les ondes progressives sont asymptotiquement stables.

1. Introduction

The purpose of this paper is to construct solutions of the fluid system

$$\begin{cases} \partial_t v - \partial_x u = 0, & (1.1a)\\ \partial_t u + \partial_x p - \mu \partial_x \left(\dfrac{1}{v}\partial_x u\right) = 0, & (1.1b)\\ v \geqslant 1, \quad (v-1)p = 0, \quad p \geqslant 0, & (1.1c)\\ v_{|t=0} = v^0, \quad u_{|t=0} = u^0, & (1.1d) \end{cases}$$

for a large class of initial data (v^0, u^0), with

$$(v, u)(t, x) \underset{x \to \pm\infty}{\longrightarrow} (v_\pm, u_\pm).$$

The variable v represents the specific volume of the fluid, that is the inverse of the density, while u denotes its velocity. Equations (1.1) are actually a reformulation in Lagrangian coordinates of the constrained Navier–Stokes system introduced in [BPZ14]. We further assume that

$$v_- = 1, \quad v_+ > 1, \quad u_- > u_+. \tag{1.2}$$

We do not impose a limit condition on the pressure variable p which is actually linked to (v, u). The pressure is indeed seen as a Lagrange multiplier associated to the constraint $\partial_x u \geqslant 0$ on $\{v = 1\}$. The condition $\partial_x u \geqslant 0$ in $\{v = 1\}$ ensures formally through the mass equation that v remains larger than 1, provided $v_{|t=0} \geqslant 1$

(see also [MRCSV11]). Nevertheless, in the present study where no external force is taken into account and sufficiently regular solutions are considered, we guarantee the incompressibility condition $\partial_x u = 0$ inside the congested region.

Let us recall a few facts about system (1.1), and be more specific about the contents of the present paper. It describes a partially congested system, consisting of two different phases. In the phase $\{v > 1\}$ (non-congested phase where $\rho = 1/v < 1$), the pressure vanishes and the dynamic is compressible. In the phase $\{v = 1\}$ (congested phase), the pressure is activated and the dynamic is incompressible.

From the modeling point of view, the system (1.1) may apply in various contexts. A first example is given by the dynamics of two-phase flows in presence of pure-phase (or saturation) regions as described by Bouchut et al. in [BBCR00]. In this context, the constrained variable is the volume fraction which has to stay between 0 and 1, the extremal values corresponding to the pure-phase states. Another domain of application of Equations (1.1) is the modeling of collective motion (like crowds or vehicular traffic, see for instance [BG17, DHN11, MRCSV11]). There, the maximal density limit (or equivalently the minimal specific volume) corresponds to a microscopic *packing constraint*, constraint which is locally achieved when the agents are in contact. In this framework, models of type (1.1) are called *hard congestion models* (see [Mau12]). Finally, let us also mention the connections between (1.1) and models for wave-structure interactions developed in the very recent years by Lannes [Lan17] and Godlewski et al. [GPSMW18]. A similar constraint to (1.1c) can be indeed formulated to express the two possible states of the flow: *pressurized* in the "interior" domain at the contact with the structure (the height being then constrained by the structure), *free* in the exterior domain.

It can be easily checked that there exist traveling wave solutions $(\bar{u}, \bar{v})(x - st)$ for (1.1). These were constructed in [DP20] and their main features are recalled below in Lemma 2.1. They consist of a congested zone for $x - st < 0$, and of a non-congested zone for $x - st > 0$, in which $\bar{v}$ is the solution of a logistic equation (see Figure 2.1 below). The setting of the current paper is the following: we consider initial data (u^0, v^0) which are smooth perturbations of the traveling wave profiles $(\bar{u}, \bar{v})$ in the non-congested zone $x > 0$ only. In other words, $u^0(x) = \bar{u}(x) = u_-$ and $v^0(x) = v_- = 1$ for $x < 0$. Under compatibility conditions on the initial data, we prove that there exists a local strong solution of (1.1). Furthermore, this solution is global provided the initial perturbation is sufficiently small.

Originally, the study of density constrained fluid systems begins with the proof of the existence of global weak solutions by Lions and Masmoudi [LM99] for the multi-dimensional free-congested Navier–Stokes equations. The result is achieved via a penalty approach: the equations are approximated by a fully compressible Navier–Stokes system in which the maximal density constraint has been relaxed and the (compressible) pressure plays the role of the penalty function. Later, the same type result was obtained in [PZ15] by means of a *soft congestion approximation* which consists of a fully compressible Navier–Stokes system with a singular pressure law blowing up as the density approaches 1. Contrary to the former study, the maximal density constraint is satisfied even at the approximate level which can be useful from the numerical point of view (see for instance [DHN11, PS22]), and

relevant from the physical point of view if one thinks for instance of the influence of repulsive social forces in collective motion. If the constructed weak solutions can theoretically couple the free and congested states, the previous existence results do not give any information about the congested domain and about the time evolution of its boundary. In other words, it was not clear whether free and congested states actually co-exist within a given weak solution.

The existence of more regular solutions to (1.1) for initial mixed compressible-incompressible data is, up to our knowledge, a largely open problem. Comparatively to other compressible-incompressible free boundary problems like the ones studied in [Shi16] or in [CGS16], we have to handle the fact that the interface between the free (compressible) domain and the congested (incompressible) domain is not closed, i.e. matter passes through the boundary and the volume of the congested region evolves with time. Besides, the identification of appropriate transmission conditions across the interface is a non-trivial issue which is for instance raised in [BR17] by Bresch and Renardy. In the hyperbolic framework of wave-structure interactions (WSI), the recent study of Iguchi and Lannes [IL19] provides a one-dimensional existence result in H^m, $m \geqslant 2$ (regularity of the solution in the "exterior" domain, which is called the free domain in our framework). This result has been then extended to the dispersive Boussinesq case in [BLM21, BL22], and to the two-dimensional axisymmetric case in [Boc20]. Finally, the study [MSMTT19] includes viscosity effects, still in an axisymmetric configuration. The congestion problem (1.1) is similar to the viscous WSI problem [MSMTT19] in the sense that it can be formulated as a mixed initial-boundary value problem with a implicit coupling of the (parabolic) PDEs describing the dynamics in the free/exterior zone with a nonlinear ODE. This ODE (see (2.15) below) represents the evolution of the free-congested interface in (1.1), while in [MSMTT19] the ODE models the vertical motion of the structure (the contact line between the fluid and the structure is there constant due to the axisymmetric hypothesis).

As said before, partially congested propagation fronts $(\bar{u}, \bar{v})(x - st)$ for the viscous system (1.1) have already been identified in the previous study [DP20]. These traveling waves are such that $\bar{v}, \bar{u}$ and the (effective) flux $\bar{p} - \mu\partial_x \bar{u}$ are continuous across the free-congested interface. But the core of [DP20] is devoted to the analysis of approximate traveling waves $(v_\varepsilon, u_\varepsilon)_{\varepsilon>0}$ solutions to the *soft congestion approximation* of (1.1). Under some smallness condition (quantified in terms of ε) on the initial perturbation, the profiles $(v_\varepsilon, u_\varepsilon)_{\varepsilon>0}$ are shown to be asymptotically stable. This result is achieved by means of weighted energy estimates. It relies on the use of integrated variables and a reformulation of the system in the variables $(v_\varepsilon, w_\varepsilon)$ where $w_\varepsilon := u_\varepsilon - \mu\partial_x \ln v_\varepsilon$ is the so-called *effective velocity*. Roughly speaking, the use of this new velocity induces regularization effects on the specific volume v_ε, effects previously identified (among others) by Shelukhin [She84], Bresch, Desjardins [BD06], Vasseur and Yao [VY16], Haspot [Has18]. The use of the integrated variables is related to the structure of the dissipation and source terms. As detailed in Section 4.1 below, it enables the derivation of uniform-in-time energy estimates on the solution.

Unfortunately, as $\varepsilon \to 0$ the smallness condition on the initial perturbations degenerates and no stability can be inferred directly for the limit profiles $(\bar{v}, \bar{u})$.

The present study contains three main results related to initial perturbations of the profile $(\bar{v}, \bar{u})$ in the free zone. We demonstrate a local well-posedness result for large data as well as a global result for small initial perturbations. Finally, we prove the asymptotic stability of the profile $(\bar{v}, \bar{u})$. Similarly to the case $\varepsilon > 0$ described above, our analysis is based on energy estimates and the use of the effective velocity to rewrite the equation on the specific volume (1.1a) as a parabolic equation with a nonlinear diffusion. One significant difference between the two studies is that the effective velocity $w = u - \mu\partial_x \ln v$ satisfies in the present case a pure transport equation in the free domain due to the absence of pressure in that region. As a consequence, the two equations in v and w can be decoupled, which simplifies somehow the dynamics and the derivation of estimates on v. One the other hand, and more importantly, the analysis (in particular the global-in-time existence proof) is made here more difficult as a result of the dynamics of the new free boundary, i.e. the interface between the free and congested domains, which is coupled to the dynamics of (u, v) itself through a continuity condition imposed at the interface (see (2.2)-(2.3)). Similarly to the WSI problem tackled by Iguchi and Lannes [IL19], we introduce a new variable (see g_1 defined in the next section) which allows us to have a good control of the motion of the interface.

These results are presented in the next section.

2. Main results and strategy

As explained in the introduction, we will construct solutions in the vicinity of the traveling wave solution $(\bar{u}, \bar{v})(x - st)$. Hence we first recall some features of the profile $(\bar{u}, \bar{v})$:

Lemma 2.1 ([DP20]). — *Assume that $u_- > u_+$, $v_+ > 1, v_- = 1$, and let*

$$s := \frac{u_- - u_+}{v_+ - 1}.$$

Then there exists a unique (up to a shift) traveling wave solution of (1.1). *This traveling wave propagates at speed s and is of the form $(\bar{u}, \bar{v})(x - st)$. Furthermore,*

$$\bar{v}(x) = \begin{cases} 1 & \text{if } x \leqslant 0, \\ \dfrac{v_+}{1 + (v_+ - 1)\exp(-sv_+x/\mu)} & \text{if } x > 0, \end{cases}.$$

$$\bar{u} = u_+ + sv_+ - s\bar{v} = u_- + sv_- - s\bar{v}.$$

In the zone $x < 0$, the pressure is constant and equal to $p_- = s^2(v_+ - 1)$. Eventually, introducing the effective velocity $\bar{w} = \bar{u} - \mu\partial_x \ln \bar{v}$, we have

$$\bar{w}(x) = u_-\mathbf{1}_{x<0} + u_+\mathbf{1}_{x>0}.$$

The profile is represented in Figure 2.1.

Let us now explicit our assumptions on the initial data (u^0, v^0):

(H1) *Partially congested initial data:* $(u^0, v^0) \in (\bar{u}, \bar{v}) + L^1(\mathbb{R})$, and such that $u^0(x) = \bar{u}(x) = u_-$, $v^0(x) = \bar{v}(x) = 1$ if $x < 0$;

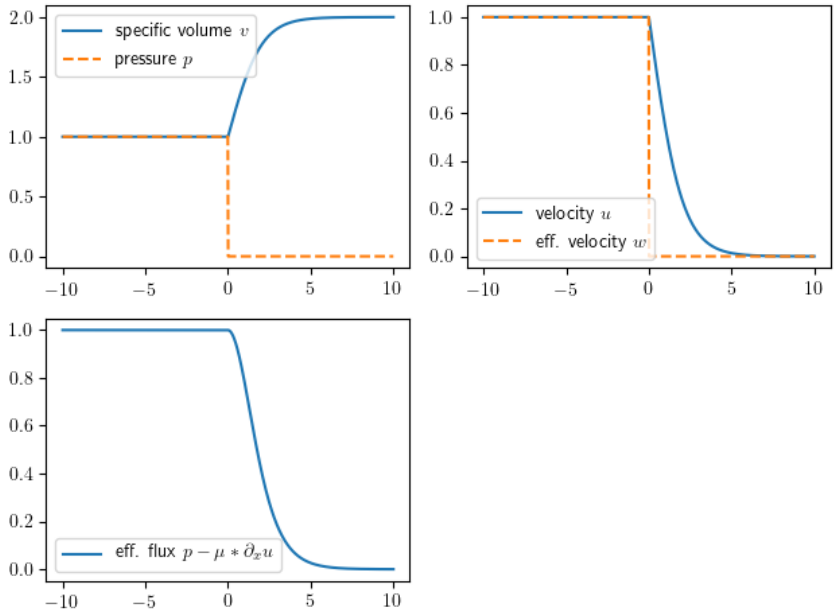

Figure 2.1. Top left: the profiles $\bar{v}$ and $\bar{p}$, top right: the profiles $\bar{u}$ and $\bar{w}$, below: the profile of the effective flux $\bar{p} - \mu\bar{u}'$.

(H2) *Regularity:* $\mathbf{1}_{x>0}(u^0 - \bar{u}, v^0 - \bar{u}) \in H^3(\mathbb{R}_+)$ and $\mathbf{1}_{x>0}\sqrt{x}\partial_x^k w^0 \in L^2(\mathbb{R}_+)$ for $k = 1, 2$, where $w^0 = u^0 - \mu\partial_x \ln v^0$;

(H3) *Compatibility:* $u^0(0^+) = u_-$, $v^0(0^+) = 1$, and

$$\left[-\frac{(\partial_x u^0)^2}{\partial_x v^0} - \mu\partial_x v^0 \partial_x u^0 + \mu\partial_x^2 u^0\right]_{|x=0^+} = 0; \tag{2.1}$$

(H4) *Non-degeneracy:* $\partial_x v^0(0^+) > 0$, $\partial_x u^0(0^+) < 0$ and $v^0(x) > 1$ for $x > 0$;

(H5) *Decay:* $\mathbf{1}_{x>0}V^0 \in L^2(\mathbb{R}_+)$, where $V^0(x) := -\int_x^\infty (v^0 - \bar{v})$, and $\mathbf{1}_{x>0}(1 + \sqrt{x})W^0 \in L^2(\mathbb{R}_+)$, where $W^0 := -\int_x^\infty (w^0 - u_+)$.

Under these assumptions, the solution of (1.1) associated with (u^0, v^0), if it exists, will not be a traveling wave. However, it is reasonable to expect such a solution to be congested in a zone $x < \tilde{x}(t)$, and non-congested in a zone $x > \tilde{x}(t)$, where the free boundary $x = \tilde{x}(t)$ is an unknown of the problem. The dynamics of the interface is actually encoded in the continuity condition that we impose on the specific volume, namely:

$$v(t, \tilde{x}(t)) = 1 \qquad \forall\ t. \tag{2.2}$$

By differentiating the relation with respect to time, we then get:

$$\tilde{x}'(t) = -\frac{\partial_t v_{|x=\tilde{x}^+}}{\partial_x v_{|x=\tilde{x}^+}}. \tag{2.3}$$

The free boundary problem (1.1) differs from "classical" free boundary problems associated with a kinematic condition at the interface. In that latter case, the regularity of $\tilde{x}'$ is the same as the regularity of the solution at the boundary, while

there is here a loss of one derivative for $\widetilde{x}'$ with respect to the solution v. The boundary condition (2.3) is *fully nonlinear* (see the study of Iguchi, Lannes [IL19]). Eventually, using the mass equation, we get the dynamics of $\widetilde{x}$:

$$\widetilde{x}'(t) = -\frac{\partial_x u_{|x=\widetilde{x}^+}}{\partial_x v_{|x=\widetilde{x}^+}}. \tag{2.4}$$

We actually prove the following result:

Theorem 2.2 (Local in time result). — *Let* $(\boldsymbol{u}^0, \boldsymbol{v}^0)$ *satisfying the assumptions* (H1)-(H5).

Then there exists $T > 0$ *and* $\widetilde{x} \in H^2_{\mathrm{loc}}([0,T))$, *with* $\widetilde{x}(0) = 0$, $\widetilde{x}'(0) = -[\frac{\partial_x \boldsymbol{u}^0}{\partial_x \boldsymbol{v}^0}]_{x=0^+}$, *such that* (1.1) *has a unique maximal solution* $(\boldsymbol{u}, \boldsymbol{v})$ *of the form* $(\boldsymbol{u}, \boldsymbol{v})(t,x) = (u_s, v_s)(t, x - \widetilde{x}(t))$ *on the interval* $[0,T)$, *where* $u_s(t,x) = u_-$, $v_s(t,x) = 1$ *and* $p_s(t,x) = -\mu(\partial_x u_s)_{|x=0^+}$ *for* $x < 0$. *Furthermore,*

$$v_s(t,x) > 1 \quad \text{for all } t \in [0,T),\ x > 0, \tag{2.5}$$

and the solution (u_s, v_s) *has the following regularity in the free domain:*

$$v_s - \bar{\boldsymbol{v}},\ u_s - \bar{\boldsymbol{u}} \in L^\infty([0,T); H^3(\mathbb{R}_+)), \tag{2.6}$$

$$\partial_t(v_s - \bar{\boldsymbol{v}}),\ \partial_t(u_s - \bar{\boldsymbol{u}}) \in L^\infty\left([0,T); H^1(\mathbb{R}_+)\right) \cap L^2((0,T); H^2(\mathbb{R}_+)\right). \tag{2.7}$$

Eventually, the pressure in the congested domain satisfies

$$p_s \in H^1(0,T). \tag{2.8}$$

Our second result shows the global existence of the solution provided the initial perturbation is small.

Theorem 2.3 (Global in time result and asymptotic stability). — *Let* $(\boldsymbol{u}^0, \boldsymbol{v}^0)$ *satisfying the assumptions* (H1)-(H5). *Assume moreover that*

$$\mathbf{1}_{\{x>0\}}\sqrt{x}\partial_x^3 \boldsymbol{w}^0 \in L^2(\mathbb{R}), \tag{2.9}$$

and let

$$\begin{aligned}\mathcal{E}_0 := &\left\|\boldsymbol{v}^0 - \bar{\boldsymbol{v}}\right\|^2_{H^3(\mathbb{R}_+)} + \left\|\boldsymbol{u}^0 - \bar{\boldsymbol{u}}\right\|^2_{H^3(\mathbb{R}_+)} + \left\|\boldsymbol{w}^0 - u_+\right\|^2_{L^2(\mathbb{R}_+)} + \left\|V^0\right\|^2_{L^2(\mathbb{R}_+)} \\ &+ \left\|(1+\sqrt{x})W^0\right\|^2_{L^2(\mathbb{R}_+)} + \sum_{k=1}^{3}\left\|(1+\sqrt{x})\partial_x^k \boldsymbol{w}^0\right\|^2_{L^2(\mathbb{R}_+)}.\end{aligned}$$

Then, there exist constants $c_0, \delta_0 > 0$ *depending only on the parameters of the problem* $s, \mu, v_+, u_\pm$, *such that for all* $\delta \in (0, \delta_0)$, *if*

$$\mathcal{E}_0 \leqslant c_0 \delta^2, \tag{2.10}$$

then the solution $(\widetilde{x}, u_s, v_s)$ *is global, and*

$$\|v_s - \bar{\boldsymbol{v}}\|_{L^\infty(\mathbb{R}_+, H^3(\mathbb{R}_+))} + \|u_s - \bar{\boldsymbol{u}}\|_{L^\infty(\mathbb{R}_+, H^3(\mathbb{R}_+))} + \|\widetilde{x}' - s\|_{H^1(\mathbb{R}_+)} \leqslant C\delta.$$

Furthermore

$$|\widetilde{x}'(t) - s| + \sup_{x\in\mathbb{R}}\left|(u_s, v_s, p_s)(t,x) - (\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}}, \bar{\boldsymbol{p}})(x)\right| \longrightarrow 0 \quad \text{as } t \to +\infty, \tag{2.11}$$

where $\bar{\boldsymbol{p}}(x) = p_- \mathbf{1}_{\{x<0\}}$.

The strategy of proof is the following. We work in the shifted variable $x - \widetilde{x}(t)$. Since (u, v) is expected to be constant in $x - \widetilde{x}(t) < 0$, we only consider the system satisfied by (u_s, v_s) in the positive half-line, which reads

$$\begin{cases} \partial_t v_s - \widetilde{x}'(t)\partial_x v_s - \partial_x u_s = 0, \quad t > 0,\ x > 0 & \text{(2.12a)} \\ \partial_t u_s - \widetilde{x}'(t)\partial_x u_s - \mu\partial_x \left(\dfrac{1}{v_s}\partial_x u_s\right) = 0, \quad t > 0,\ x > 0 & \text{(2.12b)} \\ (v_s, u_s)_{|x=0} = (1, u_-), \quad \lim\limits_{x\to\infty}(v_s(t,x), u_s(t,x)) = (v_+, u_+) \quad \forall\ t > 0. & \text{(2.12c)} \end{cases}$$

We have already seen that the dynamics of $\widetilde{x}$ is coupled with the dynamics of (v_s, u_s) through (2.4). In order to construct a solution of (2.12), it will be more convenient to modify the equation on v_s in order to make the regularizing effects of the diffusion more explicit. Indeed, setting $w_s = u_s - \mu\partial_x \ln v_s$, we find that equation (2.12b) can be written as

$$\partial_t v_s - \widetilde{x}'(t)\partial_x v_s - \mu\partial_x^2 \ln v_s = \partial_x w_s, \quad t > 0,\ x > 0.$$

Moreover,

$$\partial_t w_s - \widetilde{x}'(t)\partial_x w_s = 0, \quad t > 0,\ x > 0, \tag{2.13}$$

therefore $w_s(t,x) = \boldsymbol{w}^0(x + \widetilde{x}(t))$ for all $t > 0, x > 0$ provided $\widetilde{x}'(t) > 0$ for all $t > 0$. In particular, letting $x \to 0^+$, we obtain

$$u_- - \mu\partial_x v_{s|x=0^+} = \boldsymbol{w}^0(\widetilde{x}(t)^+). \tag{2.14}$$

Gathering (2.14) and (2.4) leads to

$$\widetilde{x}'(t) = -\mu\frac{\partial_x u_{s|x=0^+}}{u_- - \boldsymbol{w}^0(\widetilde{x}(t)^+)}. \tag{2.15}$$

Since $w_s(t,x) = \boldsymbol{w}^0(x + \widetilde{x}(t))$, the equation on v_s rewrites

$$\begin{aligned} &\partial_t v_s - \widetilde{x}'(t)\partial_x v_s - \mu\partial_{xx}\ln v_s = \partial_x \boldsymbol{w}^0(x + \widetilde{x}(t)), \quad t > 0,\ x > 0, \\ &v_{s|x=0} = 1, \quad \lim_{x\to\infty} v_s(t,x) = v_+, \\ &v_{s|t=0} = v^0. \end{aligned} \tag{2.16}$$

Thus we will build a solution $(\widetilde{x}, v_s, u_s)$ of (2.15)-(2.16)-(2.12c) thanks to the following fixed point argument:

(1) For any given $\widetilde{y} \in H^2_{\mathrm{loc}}(\mathbb{R}_+)$, such that $\widetilde{y}(0) = 0$ and $\widetilde{y}'(0) = -\frac{\partial_x u^0_{|x=0^+}}{\partial_x v^0_{|x=0^+}}$, we consider the solution v of the equation

$$\begin{aligned} &\partial_t v - \widetilde{y}'(t)\partial_x v - \mu\partial_{xx}\ln v = \partial_x \boldsymbol{w}^0(x + \widetilde{y}(t)), \quad t > 0,\ x > 0, \\ &v_{|x=0} = 1, \quad \lim_{x\to\infty} v(t,x) = v_+, \\ &v_{|t=0} = v^0. \end{aligned} \tag{2.17}$$

We prove that under suitable conditions on the initial data, there exists a unique solution $v \in \bar{v} + L^\infty_{\mathrm{loc}}(\mathbb{R}_+, H^1(\mathbb{R}_+))$, and we derive higher regularity estimates.

(2) We then consider the unique solution $u \in \bar{\boldsymbol{u}} + L^\infty_{\mathrm{loc}}(\mathbb{R}_+, H^1(\mathbb{R}_+))$ of

$$\begin{aligned} &\partial_t u - \widetilde{y}'(t)\partial_x u - \mu\partial_x\left(\frac{1}{v}\partial_x u\right) = 0 \quad t>0,\ x>0,\\ &u_{|x=0} = u_-, \quad \lim_{x\to\infty} u(t,x) = u_+,\\ &u_{|t=0} = u^0, \end{aligned} \tag{2.18}$$

where v is the solution of (2.17). Once again, we derive regularity estimates on u.

(3) Eventually, we define

$$\widetilde{z}(t) := -\mu\int_0^t \frac{\partial_x u(\tau,0^+)}{u_- - \boldsymbol{w}^0(\widetilde{y}(\tau)^+)}d\tau,$$

and we consider the application $\mathcal{T} : \widetilde{y} \in H^2_{\mathrm{loc}}(\mathbb{R}_+) \mapsto \widetilde{z} \in H^2_{\mathrm{loc}}(\mathbb{R}_+)$.
We prove that for $T_0 > 0$ small enough the application $\mathcal{T}$ is a contraction on $H^2(0,T_0)$, and therefore has a unique fixed point.

We then need to prove that the solution $(\widetilde{x}, v_s, u_s)$ provided by the fixed point of $\mathcal{T}$ is global when the initial energy is small (see Hypothesis (2.10)). First, we will use space-integrated variables in order to avoid an exponential dependency with respect to time in the energy estimates. Next, we prove in Section 6 that if the initial data is sufficiently small, then $\|\widetilde{x}' - s\|_{H^1}$ remains bounded (and small) on the existence time of the solution. The key ingredient to get this property is the introduction of the new unknown $g_1 = -s(v-\bar{v}) - \mu\partial_x(\frac{v-\bar{v}}{\bar{v}}) = \mathcal{A}(v-\bar{v})$ where $\partial_x\mathcal{A}$ is the linearized operator around $\bar{v}$. We exhibit coercivity properties for the linearized operator, and we prove that g_1 satisfies an equation of the type

$$\partial_t g_1 + \mathcal{A}\partial_x g_1 = \text{quadratic terms}.$$

Whence we deduce good estimates on both g_1 and $\widetilde{x}' - s$.

We finally need to check that the solution $(\widetilde{x}, v_s, u_s)$ of (2.15)-(2.16)-(2.12c) is indeed a solution of the original problem. Since system (2.12) has been modified, this is not completely obvious. In fact, we need to check that the function $w_s = u_s - \mu\ln v_s$ is indeed equal to $\boldsymbol{w}^0(x+\widetilde{x}(t))$. To that end, let us compute the equation satisfied by w_s if v_s is the solution of (2.16) and if u_s is the solution of (2.12c). Combining (2.16) and (2.12c), we have

$$\partial_t w_s - \widetilde{x}'(t)\partial_x w_s - \mu\partial_x\left(\frac{1}{v_s}\partial_x w_s\right) = -\mu\partial_x\left(\frac{1}{v_s}\partial_x \boldsymbol{w}^0(x+\widetilde{x}(t))\right). \tag{2.19}$$

Furthermore, the condition $u_{s|x=0^+} = u_-$ ensures that

$$w_{s|x=0^+} = u_- - \mu\partial_x v_{s|x=0^+},$$

and using the equation (2.16) together with (2.15)

$$\begin{aligned} \partial_x w_{s|x=0^+} &= \partial_x u_{s|x=0^+} - \mu\partial_{xx}\ln v_{s|x=0^+}\\ &= \frac{\boldsymbol{w}^0(\widetilde{x}(t)) - u_-}{\mu}\widetilde{x}'(t) + \widetilde{x}'(t)\partial_x v_{s|x=0^+} + \partial_x \boldsymbol{w}^0(\widetilde{x}(t)). \end{aligned}$$

Taking a linear combination of these two equations leads to

$$\mu\partial_x w_{s|x=0^+} + \widetilde{x}'(t) w_{s|x=0^+} = \widetilde{x}'(t)\boldsymbol{w}^0(\widetilde{x}(t)) + \mu\partial_x \boldsymbol{w}^0(\widetilde{x}(t)). \tag{2.20}$$

It can be easily proved that the unique solution of (2.19)-(2.20) endowed with the initial data $\boldsymbol{w}^0$ is the function $\boldsymbol{w}^0(\cdot+\widetilde{x}(t))$. Thus the function (v_s,u_s) constructed as the solution of (2.16)-(2.12c), where $\widetilde{x}$ is the solution of (2.15), is in fact a solution of (2.12). We extend this solution in $x<0$ by setting $v_s(t,x)=1$, $u_s(t,x)=u_-$, and we set by continuity of the effective flux

$$p_s(t,x)=-\mu\partial_x u_{s|x=0^+}=\widetilde{x}'(t)\left(u_- - \boldsymbol{w}^0(\widetilde{x}(t))\right) \quad \forall\, x<0.$$

Eventually, we come back to the original variables and set

$$(v,u,p)(t,x)=(v_s,u_s,p_s)(t,x-\widetilde{x}(t)).$$

Then it is easily checked that (v,u,p) is a solution of the original system (1.1).

Remark 2.4. — As said in the introduction, System (1.1) is a Lagrangian formulation of the constrained Navier–Stokes equations

$$\begin{cases} \partial_t\rho+\partial_{\mathrm{x}}(\rho u)=0,\\ \partial_t(\rho u)+\partial_{\mathrm{x}}(\rho u^2)+\partial_{\mathrm{x}}p-\mu\partial_{\mathrm{x}}^2u=0,\\ 0\leqslant\rho\leqslant 1,\quad p\geqslant 0,\quad (1-\rho)p=0,\end{cases} \tag{2.21}$$

where x denotes the Lagrangian space variable and $\rho=1/v$ is density of the fluid. Wagner proved in [Wag87] the equivalence (for weak solutions) between the Eulerian and the Lagrangian frameworks for unconstrained equations. Hence, our result provides the existence and the uniqueness of a partially congested solution to the Eulerian equations (2.21), solution which is regular on both sides of the interface. Similarly to the computations presented above, the continuity of the density at the interface encodes the dynamics of the Eulerian interface, denoted $\widetilde{\mathrm{x}}$. We get

$$\widetilde{\mathrm{x}}'(t)=u_-+\frac{\partial_{\mathrm{x}}u_{|\widetilde{\mathrm{x}}(t)+}}{\partial_{\mathrm{x}}\rho_{|\widetilde{\mathrm{x}}(t)+}}.$$

Observe that the difference between the Eulerian dynamics and the Lagrangian one is the shift by the material velocity u_-. This is in accordance with the Lagrangian standpoint.

Let us finally emphasize that the passage to Lagrangian mass coordinates is classical for the study of one-dimensional Navier–Stokes equations. Indeed, the Lagrangian mass coordinates provide regularizing effects on the specific volume (or the density) via the introduction of the effective velocity w.

Remark 2.5 (About the regularity of $\widetilde{x}$). — In the above discussion, we have claimed that we will prove the existence of a fixed point $\widetilde{x}$ in $H^2_{loc}(\mathbb{R}_+)$. Let us discuss why this regularity is required on $\widetilde{x}$. First, we need a control of $\widetilde{x}'$ in $L^\infty(\mathbb{R}_+)$ in order to control the transport equation (2.13) satisfied by w_s. Next, we see in (2.15) that the control in L^∞ of $\widetilde{x}$ requires a bound on $\partial_x u_s$ in $L^\infty(\mathbb{R}_+\times\mathbb{R}_+)$, while this latter bound would a priori rely on a control of $\widetilde{x}''$ in $L^2_{loc}(\mathbb{R}_+)$ (see Proposition A.1 in the Appendix). Therefore the regularity $\widetilde{x}\in H^2_{loc}(\mathbb{R}_+)$ is the minimal regularity which allows us to formally close the fixed point argument.

Remark 2.6 (About uniqueness and more general initial perturbations). — Let us emphasize that Theorem 2.8 provides uniqueness for equation (1.1) within the

class of solutions that are congested on a half line $\{x < \widetilde{x}(t)\}$, and free on a half line $\{x > \widetilde{x}(t)\}$. An interesting question, which is out of the scope of the present paper, would be to prove uniqueness within a larger class of functions, e.g. smooth perturbations of translations of the profile $(\bar{u}, \bar{v})$. Note that such functions may have several disconnected congested zones. Hence the study of the associated solution involves several free boundary problems, at each junction between a free and a congested zone. We identify at least two new and different difficulties. First, the congested zones have finite lengths, and therefore the different interfaces are connected to each other. Second, at every junction within the congested zone of the traveling profile, for a smooth perturbation, the first derivative of v is continuous and equal to zero. Therefore, the dynamics given by (2.3) is no longer valid, and the analysis cannot be extended easily.

In a related fashion, let us recall that this study is concerned with perturbations only affecting the "free part" of the traveling wave profile $(\bar{v}, \bar{u}, \bar{p})$. This allows us to concentrate the analysis on the free domain $\{x > \widetilde{x}(t)\}$. Nevertheless, it would be interesting to study more general perturbations of $(\bar{v}, \bar{u}, \bar{p})$, including for instance a free zone $\{v^0 > 1\}$ in $\{x < 0\}$. It would be then necessary to analyze the coupled dynamics between the two disconnected free domains through the interior congested phase. This is actually out of the scope of the present paper and it is left for future work.

The organization of the paper is the following. Given the discussion above, the goal is to construct a fixed point of the application $\mathcal{T}$. Most of the proof is devoted to the derivation of suitable energy estimates on v_s. We first prove the well-posedness of equation (2.17) in Section 3. Higher regularity estimates on v_s and estimates for the solution u_s of (2.18) are then derived in Section 4. We prove in Section 5 that $\mathcal{T}$ is a contraction and get therefore the local-wellposedness result stated in Theorem 2.8. Eventually, for small initial perturbations, we extend in Section 6 the local solutions and show the asymptotic stability of the profiles $(\bar{v}, \bar{u})$ which completes the proof of Theorem 2.3. We have postponed to the Appendix several technical results.

Notation. Throughout the paper, we denote by C a constant depending only on the parameters of the problem, i.e. u_+, u_-, v_+, s, and μ.

3. Well-posedness of equation (2.17)

This section is devoted to the proof of well-posedness of equation (2.17), and to the derivation of some preliminary bounds. In order to alleviate the notation, we drop the indices s. Throughout this section, we assume that $\widetilde{y}$ is a given function in $H^2(0,T)$, such that $\widetilde{y}(0) = 0$, $\widetilde{y}'(t) > 0$ for all $t \in [0,T]$, and satisfying the compatibility condition

$$\widetilde{y}'(0) = -\frac{\partial_x u^0_{|x=0^+}}{\partial_x v^0_{|x=0^+}}. \tag{3.1}$$

We approximate equation (2.17) by considering an equation in a truncated, bounded domain $(0,R)$ with $R > 0$. We derive L^∞, L^1 and energy bounds on the sequence

of approximated solutions, which are all uniform in R. Passing to the limit as the domain fills out the whole half-line, we recover the well-posedness of (2.17).

Therefore, for $R \geqslant 2$, we consider the following equation in $(0,T)\times(0,R)$

$$\begin{cases} \partial_t v^R(t,x) - \widetilde{y}'(t)\partial_x v^R(t,x) - \mu\partial_{xx} a(v^R(t,x)) = (\chi_R \partial_x \boldsymbol{w}^0)\,(x+\widetilde{y}(t)), & \text{(3.2a)}\\ v^R_{|t=0} = \boldsymbol{v}^{0,R}, & \text{(3.2b)}\\ v^R_{|x=0} = 1, \quad v^R_{|x=R} = \bar{\boldsymbol{v}}_{|x=R}, & \text{(3.2c)} \end{cases}$$

where the nonlinearity in the diffusion term has been changed in order to avoid degeneracy. The function $a \in \mathcal{C}^\infty(\mathbb{R})$ is such that $a(x) = \ln(x)$ for $x \in [1/2, \bar{C}]$ where $\bar{C} := 2\sup \boldsymbol{v}^0$. Note that with this choice, $\bar{C} \geqslant v_+$. We also assume that there exists $\nu > 0$ such that $\nu \leqslant a'(x) \leqslant \nu^{-1}$ for all $x \in \mathbb{R}$. The initial datum $\boldsymbol{v}^{0,R}$ is defined by $\boldsymbol{v}^{0,R} = \boldsymbol{v}^0\chi_R + \bar{\boldsymbol{v}}_{|x=R}(1-\chi_R)$, where $\chi_R \in \mathcal{C}^\infty(\mathbb{R})$ satisfies $\chi_R(x) = 1$ if $x \leqslant R-2$ and $\chi_R(x) = 0$ if $x \geqslant R-1$. This change in the initial condition and in the source term ensures that the compatibility conditions at first and second order are satisfied at $t=0$, $x=R$, namely

$$v^R_{|t=0,x=R} = \bar{\boldsymbol{v}}_{|x=R}, \qquad \widetilde{y}'(0)\partial_x v^R_{|t=0,x=R} + \mu\left(\partial_x^2 \ln v^R\right)_{|t=0,x=R} = 0. \tag{3.3}$$

The following result is a straightforward consequence of [LSU68, Theorem 6.2, Chapter 5]:

Lemma 3.1 ([LSU68, Theorem 6.2]). — *There exists $\alpha > 0$ such that system (3.2c) has a unique solution $v^R \in \mathcal{C}([0,T]\times[0,R])$ such that $\partial_t v^R \in \mathcal{C}^\alpha((0,T)\times(0,R))$ and $\partial_{xx} v^R \in \mathcal{C}^\alpha((0,T)\times(0,R))$.*

Proof. — We merely check the assumptions of [LSU68, Theorem 6.2, Chapter 5]. Due to the assumptions on the function a, we only need to verify that the initial data satisfies the compatibility conditions. Note that the function $(t,x) \in (0,T)\times(0,R) \mapsto \boldsymbol{v}^{0,R}(x)$ coincides with v^R on the sets $\{t=0\}$, $\{x=0\}$ and $\{x=R\}$. Furthermore, thanks to (3.1) and to (3.3) we have the compatibility conditions

$$\begin{aligned} \left(\widetilde{y}'(0)\partial_x \boldsymbol{v}^{0,R} + \mu\partial_{xx}\ln \boldsymbol{v}^{0,R}\right)_{|x=0} &= \partial_x \boldsymbol{w}^0_{|x=0},\\ \left(\widetilde{y}'(0)\partial_x \boldsymbol{v}^{0,R} + \mu\partial_{xx}\ln \boldsymbol{v}^{0,R}\right)_{|x=R} &= 0. \end{aligned} \tag{3.4}$$

We infer that the assumptions of [LSU68, Theorem 6.1, Chapter 5] are satisfied, and $\partial_t v^R, \partial_{xx} v^R \in \mathcal{C}^\alpha([0,T]\times[0,R])$ for some $\alpha > 0$. □

3.1. Local-in-time L^∞ estimate

Lemma 3.2. — *Let $T > 0$ be arbitrary. Assume that there exists $M \geqslant 1$ such that*

$$\frac{1}{M} \leqslant \widetilde{y}'(t) \leqslant M \quad \forall\ t \in [0,T], \qquad \inf_{x\in[0,1]} \partial_x \boldsymbol{v}^0(x) \geqslant \frac{1}{M}. \tag{3.5}$$

Then there exists $T_0 \in (0,T]$, depending only on $\|\partial_x \boldsymbol{w}^0\|_{L^\infty(\mathbb{R}_+)}$, $\sup \boldsymbol{v}^0$, M and on the parameters of the system, such that for all $t \in [0,T_0]$,

$$1 < v^R(t,x) \leqslant \bar{C} \quad \forall\, x \in (0,R), \tag{3.6}$$

where $\bar{C} = 2\sup v^0$ *is the constant involved in the definition of* a*. As a consequence,* $a(v^R) = \ln(v^R)$ *almost everywhere.*

Proof. — The proof relies on the maximum principle. We construct by hand a super and a sub-solution, which give the desired estimates.

- *Sub-solution.* We set

$$\underline{\underline{v}}(t,x) = 1 + \underline{\underline{g}}(x)\exp(-At),$$

where $A > 0$ is a parameter that will be chosen later on and where the function $\underline{\underline{g}}$ will be chosen so that $0 \leqslant \underline{\underline{g}} \leqslant (v_+ - 1)/2$. It follows that $a(\underline{\underline{v}}) = \ln(\underline{\underline{v}})$. We will also take T_0 so that $\exp(-AT_0) = 1/2$. With this choice, we have, on the one hand

$$\begin{aligned}(3.7)\quad & \partial_t \underline{\underline{v}} - \widetilde{y}'(t)\partial_x \underline{\underline{v}} - \mu\partial_x^2 a(\underline{\underline{v}}) \\ &= -Ae^{-At}\underline{\underline{g}}(x) - \widetilde{y}'(t)\underline{\underline{g}}'(x)e^{-At} - \mu\partial_x^2 \ln\left(1 + \underline{\underline{g}}(x)e^{-At}\right) \\ &= -Ae^{-At}\underline{\underline{g}}(x) - \widetilde{y}'(t)\underline{\underline{g}}'(x)e^{-At} - \mu\frac{\underline{\underline{g}}''(x)e^{-At}}{1+\underline{\underline{g}}(x)e^{-At}} + \mu\frac{(\underline{\underline{g}}'(x))^2 e^{-2At}}{\left(1+\underline{\underline{g}}(x)e^{-At}\right)^2}.\end{aligned}$$

On the other hand,

$$\left(\chi_R \partial_x \boldsymbol{w}^0\right)(x + \widetilde{y}(t)) \geqslant -\left\|\partial_x \boldsymbol{w}^0\right\|_\infty.$$

We choose the function $\underline{\underline{g}}$ as follows:

- For $x \leqslant x_0 \leqslant 1$, we take $\underline{\underline{g}}(x) = \alpha_1 x + \alpha_2 x^2$, with $\alpha_1 > 0$ small enough and $\alpha_2 > 0$. Then, if x_0 is small enough, $\underline{\underline{g}}(x_0) = \alpha_1 x_0 + \alpha_2 x_0^2 \leqslant 1$, and therefore

$$-Ae^{-At}\underline{\underline{g}}(x) - \widetilde{y}'(t)\underline{\underline{g}}'(x)e^{-At} - \mu\frac{\underline{\underline{g}}''(x)e^{-At}}{1+\underline{\underline{g}}(x)e^{-At}} \leqslant -\frac{2}{5}\mu\alpha_2.$$

  Choose α_2 so that $\mu\alpha_2 \geqslant 5\|\partial_x \boldsymbol{w}^0\|_\infty$, so that the right-hand side is smaller than $\chi_R \partial_x \boldsymbol{w}^0(\cdot + \widetilde{y})$. We further choose $0 < \alpha_1 \leqslant 1/(2M)$. Then for x_0 small enough, $\underline{\underline{v}}(0,x) \leqslant \boldsymbol{v}^0(x)$ for all $x \in [0, x_0]$. Furthermore, the last term in (3.7) can be treated as a perturbation provided α_1 and x_0 are small enough. It can be checked that it is sufficient to take $\alpha_1^2 \leqslant \alpha_2/20$ and $x_0 \leqslant \alpha_1/(2\alpha_2)$.
- For $x \geqslant x_0$, we choose $\underline{\underline{g}}$ to be monotone increasing and such that $1 + \underline{\underline{g}} \leqslant \boldsymbol{v}^0$ a.e. and $\underline{\underline{g}} \leqslant (v_+ - 1)/2$. We then choose A so that

$$\begin{aligned}&\frac{A}{4}\underline{\underline{g}}(x_0) \geqslant \left\|\partial_x \boldsymbol{w}^0\right\|_\infty, \\ &A\underline{\underline{g}}(x_0) \geqslant 4\mu\left(\|\underline{\underline{g}}''\|_\infty + \|\underline{\underline{g}}'\|_\infty^2\right).\end{aligned}$$

  It follows that the right-hand side of (3.7) is lower than $-\|\partial_x \boldsymbol{w}^0\|_\infty$.

Note that by construction, we have $\underline{\underline{v}} \leqslant v^R$ on the sets $\{t = 0\} \cup \{x = 0\} \cup \{x = R\}$ provided R is large enough.

Thus, by the maximum principle, $\underline{\underline{v}} \leqslant v^R$.

- *Super-solution:* We take

$$\overline{\overline{v}}(t,x) := C_1\left(2 - e^{-At}\right),$$

where $C_1 \geqslant \sup v^0$. We keep choosing T_0 so that $\exp(-AT_0) = 1/2$. Then

$$\partial_t \overline{\overline{v}} - \widetilde{y}'(t)\partial_x \overline{\overline{v}} - \mu\partial_x^2 a(\overline{\overline{v}}) = C_1 A e^{-At}.$$

Take A so that $C_1 A/2 \geqslant \|\partial_x w^0\|_\infty$. Then $\overline{\overline{v}}$ is a super-solution of (3.2c), and by the maximum principle, $v_R \leqslant \overline{\overline{v}}$.

□

Remark 3.3. — The use of the maximum principle is actually not necessary for the local existence results that follow. Indeed, since we work in high regularity spaces, we could adapt our fixed point argument by linearizing our different systems around the profile $(\bar{v}, \bar{u})$, treat the non-linear term perturbatively and use the inequality $\|v - \bar{v}\|_{L^\infty(\mathbb{R}_+)} \leqslant \|v - \bar{v}\|^{1/2}_{L^2(\mathbb{R}_+)} \|\partial_x(v - \bar{v})\|^{1/2}_{L^2(\mathbb{R}_+)}$. However, since this would lead to unnecessary technicalities, we have decided to use the maximum principle in this paper.

3.2. L^1 estimate

Anticipating Section 4.1 and the derivation of uniform-in-time estimates on $g = v - \bar{v}$, we aim at proving that $g(t,\cdot) \in L^1(\mathbb{R}_+)$ for all times $t \geqslant 0$. This property will be used in Section 4.1 to justify the passage to the integrated quantity V.

The goal of this Subsection is to prove the following result:

Lemma 3.4. — *Assume that $v^0 - \bar{v} \in L^1(\mathbb{R}_+)$, and that the assumptions of Lemma 3.2 are satisfied. Then there exists a constant C, depending only on T_0, $\sup v^0$, M (see (3.5)) and the parameters of the system, such that the solution v^R of (3.2c) provided by Lemma 3.1 satisfies the following estimate:*

$$\sup_{t\in[0,T_0]} \left\|v^R - \bar{v}\right\|_{L^1(0,R)} \leqslant C\Big[\left\|v^{0,R} - \bar{v}\right\|_{L^1(0,R)} + \left\|\partial_x w^0\right\|_{L^1(\mathbb{R}_+)} + \|\partial_x \bar{v}\|_{L^1(\mathbb{R}_+)}\Big]. \tag{3.8}$$

Proof. — Using the identity $s\partial_x \bar{v} + \mu\partial_x^2 \ln \bar{v} = 0$, we have

$$\begin{aligned}\partial_t\left(v^R - \bar{v}\right) - \widetilde{y}'(t)\partial_x\left(v^R - \bar{v}\right) - \mu\partial_x^2\left(\ln v^R - \ln \bar{v}\right)\\ = \left(\chi_R \partial_x w^0\right)(x + \widetilde{y}(t)) + (\widetilde{y}'(t) - s)\,\partial_x \bar{v}.\end{aligned} \tag{3.9}$$

For $n > 0$, let us introduce $j_n \in \mathcal{C}^2(\mathbb{R})$ defined by

$$j_n(r) = \sqrt{r^2 + \frac{1}{n}} - \sqrt{\frac{1}{n}} \quad \forall\, r \in \mathbb{R}$$

which is a smooth convex approximation of the function $r \mapsto |r|$ as $n \to +\infty$. Hence, $j_n'(r) = r/\sqrt{r^2 + 1/n}$ is an approximation of the sign function, and $j_n(0) = j_n'(0) = 0$.

Multiplying (3.9) by $j_n'\left(\frac{v^R-\bar{v}}{\bar{v}}\right)$, integrating the result between 0 and R and noting that $(v^R-\bar{v})_{|x=0,R}=0$, we get

$$\begin{aligned}
&\int_0^R \bar{v}\ \partial_t j_n\left(\frac{v^R-\bar{v}}{\bar{v}}\right)\ dx - \tilde{y}'(t)\int_0^R \partial_x(v^R-\bar{v})\ j_n'\left(\frac{v^R-\bar{v}}{\bar{v}}\right)\ dx\\
&\quad+\mu\int_0^R j_n''\left(\frac{v^R-\bar{v}}{\bar{v}}\right)\partial_x\ln\left(\frac{v^R}{\bar{v}}\right)\partial_x\left(\frac{v^R-\bar{v}}{\bar{v}}\right)dx\\
&=\int_0^R(\chi_R\partial_x w^0)(x+\tilde{y}(t))\ j_n'\left(\frac{v^R-\bar{v}}{\bar{v}}\right)dx+(\tilde{y}'(t)-s)\int_0^R\partial_x\bar{v}\ j_n'\left(\frac{v^R-\bar{v}}{\bar{v}}\right)dx.
\end{aligned}$$

Since $\frac{v^R-\bar{v}}{\bar{v}}\geqslant\frac{1}{v_+}-1>-1$, we have

$$\begin{aligned}
&\mu\int_0^R j_n''\left(\frac{v^R-\bar{v}}{\bar{v}}\right)\partial_x\ln\left(\frac{v^R}{\bar{v}}\right)\partial_x\left(\frac{v^R-\bar{v}}{\bar{v}}\right)dx\\
&\qquad\qquad=\mu\int_0^R\frac{1}{1+\frac{v^R-\bar{v}}{\bar{v}}}j_n''\left(\frac{v^R-\bar{v}}{\bar{v}}\right)\left(\partial_x\left(\frac{v^R-\bar{v}}{\bar{v}}\right)\right)^2dx\geqslant 0,
\end{aligned}$$

and, thanks to the boundary conditions $(v^R-\bar{v})_{|x=0,R}=0$,

$$\begin{aligned}
&\tilde{y}'(t)\int_0^R\partial_x(v^R-\bar{v})\ j_n'\left(\frac{v^R-\bar{v}}{\bar{v}}\right)\ dx\\
&\quad=\tilde{y}'(t)\int_0^R\bar{v}\ \partial_x\left(\frac{v^R-\bar{v}}{\bar{v}}\right)\ j_n'\left(\frac{v^R-\bar{v}}{\bar{v}}\right)\ dx\\
&\qquad+\tilde{y}'(t)\int_0^R\bar{v}\frac{\partial_x\bar{v}}{\bar{v}^2}(v^R-\bar{v})\ j_n'\left(\frac{v^R-\bar{v}}{\bar{v}}\right)\ dx\\
&\quad=\tilde{y}'(t)\int_0^R\bar{v}\ \partial_x j_n\left(\frac{v^R-\bar{v}}{\bar{v}}\right)\ dx+\tilde{y}'(t)\int_0^R\frac{v^R-\bar{v}}{\bar{v}}\partial_x\bar{v}\ j_n'\left(\frac{v^R-\bar{v}}{\bar{v}}\right)\ dx\\
&\quad=-\tilde{y}'(t)\int_0^R\partial_x\bar{v}\ j_n\left(\frac{v^R-\bar{v}}{\bar{v}}\right)\ dx+\tilde{y}'(t)\int_0^R\frac{v^R-\bar{v}}{\bar{v}}\partial_x\bar{v}\ j_n'\left(\frac{v^R-\bar{v}}{\bar{v}}\right)\ dx.
\end{aligned}$$

Hence, using once again the L^∞ control of v^R from Lemma 3.2, we get

$$\begin{aligned}
&\int_0^R\bar{v}\ j_n\left(\frac{v^R-\bar{v}}{\bar{v}}\right)(t)\ dx-\int_0^R|v^{0,R}-\bar{v}|\ dx\\
&\qquad\leqslant C\|\tilde{y}'\|_{L^1(0,t)}\|\partial_x\bar{v}\|_{L^1(\mathbb{R}_+)}+t\|\partial_x w^0\|_{L^1(\mathbb{R}_+)}+\|\tilde{y}'(\cdot)-s\|_{L^1(0,t)}\|\partial_x\bar{v}\|_{L^1(\mathbb{R}_+)}.
\end{aligned}$$

where we have used the fact that $j_n(r)\leqslant|r|$. Passing to the limit $n\to+\infty$ and using Fatou's lemma, we finally obtain (3.8). □

3.3. Energy estimates and existence of v

The goal of this subsection is to prove the existence and uniqueness of $v\in L^\infty([0,T],H^1_{loc}(\mathbb{R}_+))\cap L^2([0,T],H^2(\mathbb{R}_+))$ solution to (2.17). We proceed in two steps.

First we derive classical a priori estimates on v^R, exponentially growing with time but independent of R. These estimates then ensure the existence of v by passing to the limit in (3.9) as $R \to +\infty$.

Lemma 3.5. — *Assume that $1 \leqslant v^R(t,x) \leqslant \bar{C}$ for all $t \leqslant T_0$, $x \in (0,R)$, where we recall that $\bar{C} = 2\sup v^0$, and that there exists $M \geqslant 1$ such that $M^{-1} \leqslant \widetilde{y}'(t) \leqslant M$ for all $t \in [0,T_0]$.*

There exists a constant C, depending only on s, μ, v_+, M, and $\bar{C}$, such that the solution v^R of (3.2c) satisfies

$$\begin{aligned}(3.10)\quad & \frac{d}{dt}\left\|\left(v^R-\bar{v}\right)(t)\right\|^2_{H^1(0,R)} + \left\|\partial_x\left(v^R-\bar{v}\right)(t)\right\|^2_{L^2(0,R)} + \left\|\partial_t v^R(t)\right\|^2_{L^2(0,R)} \\ & \leqslant C\left(\left\|\left(v^R-\bar{v}\right)(t)\right\|^2_{L^2(0,R)} + \left\|\partial_x w^0(\cdot+\widetilde{y}(t))\right\|^2_{L^2(\mathbb{R}_+)} + |\widetilde{y}'(t)-s|^2\right) \qquad \forall\ t\in(0,T_0),\end{aligned}$$

so that, by a Gronwall inequality,

$$\begin{aligned}(3.11)\quad & \sup_{t\in[0,T]}\left\|\left(v^R-\bar{v}\right)(t)\right\|^2_{H^1(0,R)} \\ & \qquad + \left\|\partial_x\left(v^R-\bar{v}\right)\right\|^2_{L^2((0,T)\times(0,R))} + \left\|\partial_t v^R\right\|^2_{L^2((0,T)\times(0,R))} \\ & \qquad \leqslant C\left(\left\|v^{R,0}-\bar{v}\right\|^2_{H^1(0,R)} + \left\|\sqrt{x}\partial_x w^0\right\|^2_{L^2(\mathbb{R}_+)} + \|\widetilde{y}'-s\|^2_{L^2(0,T)}\right)\exp(CT)\end{aligned}$$

for all $T \leqslant T_0$.

Remark 3.6. — Note that throughout this section and the next ones (i.e. in all energy estimates), the constant C might depend on $\bar{C}$, hence on $\sup v^0$. Hence, all constants will depend on the initial size of the perturbation.

Proof. — This is an immediate consequence of Lemma B.1 in the Appendix. We set $\bar{g} = \bar{v}$, $g = v^R - \bar{v}$, $G = (\chi_R \partial_x w^0)(x+\widetilde{y}(t)) + (\widetilde{y}'(t)-s)\partial_x \bar{v}$. It can be easily checked that the assumptions of Lemma B.1 are satisfied. Furthermore, $\bar{v}$ does not depend on time, and its $W^{2,\infty}$ norm only depends on μ, s and v_+.

There only remains to evaluate $\|G\|_{L^2}$. We use Lemma C.1 in the Appendix and we find

$$\|G\|_{L^2((0,T)\times(0,R))} \leqslant C\left(\|\partial_x\bar{v}\|_{L^2(\mathbb{R}_+)}\|\widetilde{y}'-s\|_{L^2(0,T)} + \left\|\sqrt{x}\partial_x w^0\right\|_{L^2(\mathbb{R}_+)}\right).$$

This concludes the proof of the lemma. □

We are now ready to prove the existence and uniqueness of v.

Lemma 3.7. — *Let $\widetilde{y} \in H^2(0,T)$ such that $\widetilde{y}(0) = 0$ and $\widetilde{y}$ satisfies (3.1). Assume that there exists a constant $M > 1$ such that $\widetilde{y}, v^0$ satisfy (3.5). There exists $T_0 \leqslant T$, depending on M and on the parameters of the problem, such that there exists a unique solution $v \in L^\infty([0,T_0], H^1_{loc}(\mathbb{R}_+)) \cap L^2([0,T_0], H^2(\mathbb{R}_+))$ to*

$$(3.12)\qquad \begin{cases}\partial_t v(t,x) - \widetilde{y}'(t)\partial_x v(t,x) - \mu\partial_{xx}\ln v(t,x) = \partial_x w^0(x+\widetilde{y}(t)),\\ v_{|t=0} = v^0,\\ v_{|x=0} = 1, \quad \lim_{x\to+\infty} v(t,x) = v_+ \qquad \forall\ t\geqslant 0,\end{cases}$$

and such that $\partial_t v \in L^2((0,T_0)\times\mathbb{R}_+)$. *Furthermore, there exists a constant* $C>0$ *depending* $M, v_+, u_\pm, s, \mu$ *such that* v *satisfies the estimates*

$$1 < v \leqslant 2\sup \boldsymbol{v}^0 \quad a.e., \tag{3.13}$$

$$\begin{aligned} E_1(T_0) &:= \|v-\bar{v}\|^2_{L^\infty((0,T_0),H^1)} + \|\partial_x(v-\bar{v})\|^2_{L^2((0,T_0)\times\mathbb{R}_+)} + \|\partial_t v\|^2_{L^2((0,T_0)\times\mathbb{R}_+)} \\ &\leqslant C\left(\left\|\boldsymbol{v}^0-\bar{\boldsymbol{v}}\right\|^2_{H^1(\mathbb{R}_+)} + \left\|\sqrt{x}\partial_x \boldsymbol{w}^0\right\|^2_{L^2(\mathbb{R}_+)} + \|\widetilde{y}'-s\|^2_{L^2(0,T_0)}\right)e^{CT_0}, \end{aligned} \tag{3.14}$$

and

$$\left\|\partial_x^2(v-\bar{v})\right\|_{L^2((0,T_0)\times\mathbb{R}_+)} \leqslant C\left(E_1(T_0) + \inf(1,T_0)E_1(T_0)^2\right).$$

Proof. — We consider the function v^R, which satisfies the estimates of Lemmas 3.2 and 3.5.

Extending v^R to $[0,T_0]\times\mathbb{R}_+$ by taking $v^R(t,x) = \bar{v}(x=R)$ for $x \geqslant R$, the following limits hold:

- Up to a subsequence, $v^R \to v$ in $L^2([0,T_0]\times K)$ for any compact set $K\subset\mathbb{R}_+$;
- $v^R \overset{*}{\rightharpoonup} v \quad w^* - L^\infty([0,T_0]\times\mathbb{R}_+)$
- $v^R - \bar{v} \rightharpoonup v - \bar{v} \quad w - H^1([0,T_0]\times\mathbb{R}_+)$;

Thanks to the strong compactness property, we can pass to the limit in (3.2c) and we infer that v satisfies

$$\partial_t v(t,x) - \widetilde{y}'(t)\partial_x v(t,x) - \mu\partial_{xx}\ln v(t,x) = \partial_x \boldsymbol{w}^0(x+\widetilde{y}(t)).$$

Passing to the limit in the estimates on v^R, we also know that v satisfies the estimates of Lemmas 3.2 and 3.5. This completes the proof of existence. Additionally, using the last item of Lemma B.1, we obtain the estimate on $\partial_x^2(v-\bar{v})$.

Uniqueness is quite classical. For instance, if v_1, v_2 are two solutions, then setting $\widehat{v} = v_1 - v_2$, we find that $\widehat{v}$ satisfies

$$\partial_t\widehat{v} - \widetilde{y}'(t)\partial_x\widehat{v} - \mu\partial_{xx}\ln\left(1+\frac{\widehat{v}}{v_2}\right) = 0.$$

Multiplying the above equation by $\widehat{v}$ and integrating by parts, we find that

$$\frac{d}{dt}\int_0^\infty |\widehat{v}|^2\,dx + \mu\int_0^\infty \frac{(\partial_x\widehat{v})^2}{v_1}\,dx \leqslant C\|\partial_x v_2\|^2_{L^\infty(\mathbb{R}_+)}\int_0^\infty |\widehat{v}|^2 dx.$$

Writing

$$\begin{aligned} \|\partial_x v_2(t)\|_{L^\infty(\mathbb{R}_+)} &\leqslant \|\partial_x(v_2(t)-\bar{v})\|_{L^\infty(\mathbb{R}_+)} + \|\partial_x\bar{v}\|_{L^\infty(\mathbb{R}_+)} \\ &\leqslant \|\partial_x(v_2-\bar{v})\|^{1/2}_{L^\infty([0,T],L^2(\mathbb{R}_+))}\,\|\partial_{xx}(v_2(t)-\bar{v})\|^{1/2}_{L^2(\mathbb{R}_+)} \\ &\quad + \|\partial_x\bar{v}\|_{L^\infty(\mathbb{R}_+)} \end{aligned}$$

and using a Gronwall type argument, we infer that $\widehat{v}\equiv 0$. □

4. Energy estimates on v and u

This section is devoted to the derivation of high regularity estimates on $v-\bar{v}$ and $u-\bar{u}$, where v, u are respectively solutions of (2.17) and (2.18). The existence

and uniqueness of v follows from Lemma 3.7. Preliminary energy estimates on v were derived in the previous section (see (3.14)). However, these estimates are not completely satisfactory, because of the exponential growth in the right-hand side. This exponential growth is not really a problem for the local well-posedness result we will prove in the next section (see Theorem 2.8), but it could prevent us from proving the global stability in Section 6. Fortunately, we are able to prove that energy bounds hold *without* the exponential loss.

The outline of this section is the following:

(1) We first revisit the energy estimates from Lemma 3.5 in order to prove that there is no exponential growth of the energy. To that end, we consider the equation satisfied by the integrated variable $V := -\int_x^{+\infty}(v-\bar{v})$ and derive energy estimates for V.
(2) We then prove higher order regularity estimates for v. More precisely, we prove that $v-\bar{v} \in L^\infty_t(H^3)\cap W^{1,\infty}_t(H^1)\cap H^1_t(H^2)\cap H^2_t(L^2)$.
(3) The existence and uniqueness of u then follows from classical results on linear parabolic equations with smooth coefficients. We also derive energy estimates for $u-\bar{u}$ in $W^{1,\infty}_t(H^1)\cap H^1_t(H^2)\cap H^2_t(L^2)$.

As in the previous section, we consider a given function $\widetilde{y}\in H^2(0,T)$ such that $\widetilde{y}(0)=0$ and satisfying the compatibility condition (3.1). Throughout this section, we will use the fact that the solution v of (2.17) provided by Lemma 3.7 satisfies the L^∞ estimates of Lemma 3.2, namely

$$1 < v \leqslant \bar{C} = 2\sup v^0 \text{ a.e} \tag{4.1}$$

and that there exists a constant $M\geqslant 1$ such that

$$\begin{aligned} &M^{-1}\leqslant \widetilde{y}'(t)\leqslant M \quad \forall\, t\in[0,T],\\ &M^{-1}\leqslant \partial_x v^0_{|x=0}\leqslant M. \end{aligned} \tag{4.2}$$

We shall see that the total initial energy is

$$\begin{aligned} \mathcal{E}_0 := &\left\|v^0-\bar{v}\right\|^2_{H^3(\mathbb{R}_+)} + \left\|u^0-\bar{u}\right\|^2_{H^3(\mathbb{R}_+)} + \left\|V^0\right\|^2_{L^2(\mathbb{R}_+)}\\ &+\left\|\left(1+\sqrt{x}\right)W^0\right\|^2_{L^2(\mathbb{R}_+)} + \left\|\left(1+\sqrt{x}\right)\partial_x w^0\right\|^2_{L^2(\mathbb{R}_+)} + \left\|\left(1+\sqrt{x}\right)\partial_x^2 w^0\right\|^2_{L^2(\mathbb{R}_+)}, \end{aligned} \tag{4.3}$$

where $W^0(x) := -\int_x^\infty (w^0-u_+)$. Note that this total energy differs slightly from the one of Theorem 2.3. Indeed, in order to prove our global existence result, we will need an additional bound on $\|(1+\sqrt{x})\partial_x^3 w^0\|_{L^2(\mathbb{R}_+)}$, which is not required for the local existence. However, for the sake of simplicity, we keep the same notation for the initial energy.

The energy at time T is defined by

$$\mathcal{E}_T := \mathcal{E}_0 + \|\widetilde{y}'-s\|^2_{H^1(0,T)}. \tag{4.4}$$

Let us now state the results we shall prove for u and v:

Proposition 4.1 (Energy estimates for v). — *Assume that* $\widetilde{y}'(0) = -\frac{\partial_x u^0_{|x=0}}{\partial_x v^0_{|x=0}}$ *and that* $\widetilde{y}$ *and* v^0 *satisfy* (4.2). *There exists a universal constant* $p>1$, *a constant* C

depending on $\mu, v_+, s, \bar{C}$ *and* M*, and a time* $T^0 > 0$ *depending only on* $\mathcal{E}_0$ *and on* $\|\widetilde{y}' - s\|_{L^2([0,T])}$*, such that if* $T \leqslant T^0$*, then*

$$\|v - \bar{v}\|^2_{L^\infty([0,T],H^2(\mathbb{R}_+))} + \|\partial_t v\|^2_{L^\infty([0,T],L^2(\mathbb{R}_+))} + \|\partial_t v\|^2_{L^2([0,T],H^1(\mathbb{R}_+))} \leqslant C\mathcal{E}_T,$$

and

$$\|\partial_t\partial_x v\|^2_{L^\infty([0,T],L^2(\mathbb{R}_+))} + \|\partial_x^3(v - \bar{v})\|^2_{L^\infty([0,T],L^2(\mathbb{R}_+))} + \|\partial_t^2 v\|^2_{L^2([0,T]\times\mathbb{R}_+)} + \|\partial_x^2\partial_t v\|^2_{L^2([0,T]\times\mathbb{R}_+)} \leqslant C\mathcal{E}_T(1+\mathcal{E}_T)^p.$$

A similar result holds for u:

Proposition 4.2 (Energy estimates for u). — *Assume that* $\widetilde{y}'(0) = -\frac{\partial_x u^0_{|x=0}}{\partial_x v^0_{|x=0}}$ *and that* $\widetilde{y}$ *satisfies* (4.2)*. Assume furthermore that*

$$\partial_x u^0_{|x=0}\Big(\widetilde{y}'(0) - \mu\left(\partial_x v^0\right)_{|x=0}\Big) + \mu\left(\partial_x^2 u^0\right)_{|x=0} = 0, \tag{4.5}$$

Then there exists a time $T^0 > 0$ *depending only on* $\mathcal{E}_0$ *and on* $\|\widetilde{y}' - s\|_{L^2([0,T])}$*, such that for all* $0 < T < T^0$*, with the same notation as in Proposition 4.1,*

$$\|u - \bar{u}\|^2_{H^1([0,T],H^2(\mathbb{R}_+))} + \|u - \bar{u}\|^2_{W^{1,\infty}([0,T],H^1(\mathbb{R}_+))} + \|u - \bar{u}\|^2_{H^2([0,T],L^2(\mathbb{R}_+))} \leqslant C\mathcal{E}_T(1+\mathcal{E}_T)^p.$$

Remark 4.3. — Recalling the compatibility condition $\widetilde{y}'(0) = -(\partial_x u^0/\partial_x v^0)_{|x=0}$, the compatibility condition (4.5) actually amounts to

$$\left[-\frac{(\partial_x u^0)^2}{\partial_x v^0} - \mu\partial_x v^0\partial_x u^0 + \mu\partial_x^2 u^0\right]_{|x=0} = 0,$$

which is exactly (H3). Notice that this second compatibility condition only involves the initial data u^0, v^0. In other words, no further condition on $\widetilde{y}$ is required.

We now turn towards the proof of these two propositions, following the outline above. In order to alleviate the notation, we introduce the following quantities:

$$g := v - \bar{v}, \quad h := u - \bar{u}.$$

4.1. Uniform in time estimates for $g = v - \bar{v}$

As announced previously, the first step consists in removing the exponential dependency with respect to time in (3.14). For that purpose, we use the integrated variable

$$V(t,x) := -\int_x^{+\infty}(v - \bar{v})(t,z)\,dz. \tag{4.6}$$

The integrated variable has been previously used in the literature concerned with the stability of viscous shocks, the interested reader is referred for instance to

Goodman [Goo86] or, to the more recent study of Vasseur and Yao [VY16]. Using Lemma 3.4 and passing to the limit as $R \to \infty$, we recall that V is well-defined and uniformly bounded in $L^\infty([0,T]\times\mathbb{R}_+)$. Furthermore, using the identity $s\partial_x\bar{v} + \mu\partial_{xx}\ln\bar{v} = 0$, we find that g is a solution of

$$\partial_t g - \widetilde{y}'\partial_x g - \mu\partial_{xx}\ln\left(1+\frac{g}{\bar{v}}\right) = \partial_x w^0(x+\widetilde{y}(t)) - \frac{\mu}{s}(\widetilde{y}'-s)\partial_{xx}\ln\bar{v}. \tag{4.7}$$

Integrating the above equation with respect to x, we find that V is solution to

$$\begin{aligned}\partial_t V - \widetilde{y}'(t)\partial_x V - \mu\partial_x\ln\left(1+\frac{\partial_x V}{\bar{v}}\right) &\\ &= \Big(w^0(x+\widetilde{y}(t)) - u_+\Big) - \frac{\mu}{s}(\widetilde{y}'-s)\partial_x\ln\frac{\bar{v}}{v_+}\\ &= \partial_x W^0(x+\widetilde{y}(t)) - \frac{\mu}{s}(\widetilde{y}'-s)\partial_x\ln\frac{\bar{v}}{v_+},\end{aligned} \tag{4.8}$$

where $W^0(x) = -\int_x^\infty(w^0 - u_+)$. As detailed below, we expect to control $\partial_x V = v - \bar{v}$ in $L^2((0,T)\times\mathbb{R}_+)$ uniformly with respect to the time T. Replacing this estimate in (3.10), we get the following result.

Lemma 4.4. — *The solution v constructed in Lemma 3.7 satisfies the following estimate*

$$\begin{aligned}E_1(T) &\leqslant C\Bigg[\left\|V^0\right\|^2_{L^2(\mathbb{R}_+)} + \left\|\left(1+\sqrt{x}\right)W^0\right\|^2_{L^2(\mathbb{R}_+)} + \left\|v^0-\bar{v}\right\|^2_{H^1(\mathbb{R}_+)}\\ &\qquad + \left\|\sqrt{x}\partial_x w^0\right\|^2_{L^2(\mathbb{R}_+)} + \|\widetilde{y}'-s\|^2_{L^2(0,T)}\Bigg]\\ &\leqslant C\mathcal{E}_T,\end{aligned} \tag{4.9}$$

where the energy $E_1(T)$ was defined in Lemma 3.7.

Note that the upper bound on $E_1(T)$ only depends on $\mathcal{E}_0$ and on $\|\widetilde{y}'-s\|_{L^2(0,T)}$. Hence, assuming that T is such that $\inf(1,T)\mathcal{E}_T \leqslant 1$, we obtain the first set of inequalities announced in Proposition 4.1.

Proof. — We multiply Eq. (4.8) by V and integrate over $\mathbb{R}_+$:

$$\begin{aligned}\frac{d}{dt}\int_0^{+\infty}\frac{|V|^2}{2}dx + \frac{\widetilde{y}'(t)}{2}(V_{|x=0})^2 + \mu\int_{\mathbb{R}_+}\ln\left(1+\frac{\partial_x V}{\bar{v}}\right)\,\partial_x V\,dx &\\ &= \int_{\mathbb{R}_+}\partial_x W^0(x+\widetilde{y}(t))\,V dx + \frac{\mu}{s}\Big(\widetilde{y}'(t)-s\Big)\int_{\mathbb{R}_+}\partial_x\ln\frac{\bar{v}}{v_+}\,V\,dx\\ &= -\int_{\mathbb{R}_+}W^0(x+\widetilde{y}(t))\,\partial_x V dx - \frac{\mu}{s}\Big(\widetilde{y}'(t)-s\Big)\int_{\mathbb{R}_+}\ln\frac{\bar{v}}{v_+}\,\partial_x V\,dx\\ &\quad - \left[W^0_{|x=\widetilde{y}(t)} + \frac{\mu}{s}\Big(\widetilde{y}'(t)-s\Big)\ln\frac{1}{v_+}\right]\,V_{|x=0}.\end{aligned}$$

Using Lemma 3.2, we recall that

$$\frac{\partial_x V}{\bar{v}} = \frac{v-\bar{v}}{\bar{v}} \in \left[\frac{1}{v_+} - 1, \bar{C} - 1\right].$$

Observing that

$$\mu \int_{\mathbb{R}_+} \ln\left(1 + \frac{\partial_x V}{\bar{v}}\right) \partial_x V dx \geqslant c_0 \int_{\mathbb{R}_+} |\partial_x V|^2 dx$$

for some $c_0 = c_0(\mu, \bar{C}, v_+) > 0$, we have by the Cauchy–Schwarz inequality

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}_+} \frac{|V|^2}{2} dx + \frac{\widetilde{y}'(t)}{2}(V_{|x=0})^2 + c_0 \int_{\mathbb{R}_+} |\partial_x V|^2 dx \\
&\leqslant \frac{c_0}{2}\int_{\mathbb{R}_+} |\partial_x V|^2 dx + \frac{\widetilde{y}'(t)}{4}(V_{|x=0})^2 \\
&\quad + C\Big[\left\| W^0(\cdot + \widetilde{y}(t)) \right\|^2_{L^2(\mathbb{R}_+)} + \left(W^0_{|x=\widetilde{y}(t)}\right)^2 \\
&\quad + |\widetilde{y}'(t) - s|^2 \left(\|\bar{v} - v_+\|^2_{L^2(\mathbb{R}_+)} + |v_+ - 1|^2 \right) \Big]
\end{aligned}$$

for some positive constant $C = C(c_0, \mu, s, M)$. Using Lemma C.1 in the Appendix, we have

$$\begin{aligned}
\left\| W^0(\cdot + \widetilde{y}) \right\|_{L^2((0,T)\times\mathbb{R}_+)} &\leqslant C \left\| \sqrt{x} W^0 \right\|_{L^2(\mathbb{R}_+)}, \\
\left\| W^0(\widetilde{y}(\cdot)) \right\|_{L^2(0,T)} &\leqslant C \left\| W^0 \right\|_{L^2(\mathbb{R}_+)}.
\end{aligned} \tag{4.10}$$

Hence

$$\begin{aligned}
E_0(T) &:= \|V\|^2_{L^\infty([0,T],L^2(\mathbb{R}_+))} + \|v - \bar{v}\|^2_{L^2([0,T]\times\mathbb{R}_+)} + \|V_{|x=0}\|^2_{L^2(0,T)} \\
&\leqslant C\Bigg[\left\| V^0 \right\|^2_{L^2(\mathbb{R}_+)} + \left\| (1+\sqrt{x}) W^0 \right\|^2_{L^2(\mathbb{R}_+)} \\
&\qquad + \|\widetilde{y}'(t) - s\|^2_{L^2(0,T)} \left(\|\bar{v} - v_+\|^2_{L^2(\mathbb{R}_+)} + |v_+ - 1|^2 \right) \Bigg] \\
&\leqslant C\left[\left\| V^0 \right\|^2_{L^2(\mathbb{R}_+)} + \left\| \left(1+\sqrt{x}\right) W^0 \right\|^2_{L^2(\mathbb{R}_+)} + \|\widetilde{y}'(t) - s\|^2_{L^2(0,T)} \right] && (4.11) \\
&\leqslant C\mathcal{E}_T. && (4.12)
\end{aligned}$$

By Lemma 3.7, we have

$$\begin{aligned}
&\frac{d}{dt}\|(v-\bar{v})(t)\|^2_{H^1(\mathbb{R}_+)} + \|\partial_x(v-\bar{v})\|^2_{L^2(\mathbb{R}_+)} + \|\partial_t v\|^2_{L^2(\mathbb{R}_+)} \\
&\quad \leqslant C\left(\|(v-\bar{v})(t)\|^2_{L^2(\mathbb{R}_+)} + \|\partial_x \bar{v}\|^2_{L^2(\mathbb{R}_+)} |\widetilde{y}' - s|^2 + \left\| \partial_x w^0(\cdot + \widetilde{y}(t)) \right\|^2_{L^2(\mathbb{R}_+)} \right)
\end{aligned}$$

so that

$$
\begin{aligned}
E_1(T) = \sup_{t\in[0,T]} \|(v-\bar{v})(t)\|^2_{H^1(\mathbb{R}_+)} &+ \|\partial_x(v-\bar{v})\|^2_{L^2([0,T]\times\mathbb{R}_+)} \\
&+ \|\partial_t v\|^2_{L^2([0,T]\times\mathbb{R}_+)} \\
\leqslant \left\|\boldsymbol{v}^0-\bar{\boldsymbol{v}}\right\|^2_{H^1(\mathbb{R}_+)} &+ C\left(E_0(T) + \|\widetilde{y}'-s\|^2_{L^2(0,T)} + \left\|\sqrt{x}\partial_x \boldsymbol{w}^0\right\|^2_{L^2(\mathbb{R}_+)}\right) \\
\leqslant C\mathcal{E}_T.
\end{aligned}
\tag{4.13}
$$

This concludes the proof of the Lemma 4.4. □

4.2. Higher order regularity estimates for $g = v - \bar{v}$

This section is devoted to the derivation of energy estimates in $L^\infty([0,T], H^3(\mathbb{R}_+))$ for $v-\bar{v}$, in $L^\infty([0,T], H^1(\mathbb{R}_+)) \cap L^2([0,T], H^2(\mathbb{R}_+))$ for $\partial_t v$, and in $L^2([0,T]\times\mathbb{R}_+)$ for $\partial_t^2 v$.

We split the rest of the proof of Proposition 4.1 into two steps corresponding to Lemmas 4.5 and 4.6.

Lemma 4.5 (Higher regularity - I). — *Assume that* $\widetilde{y}'(0) = -\frac{\partial_x \boldsymbol{u}^0_{|x=0}}{\partial_x \boldsymbol{v}^0_{|x=0}}$ *and that* $\inf(1, T^{1/2})E_1(T) \leqslant 1$, *where* $E_1(T)$ *is defined in* (3.14).

The solution v *of* (3.12) *satisfies*

$$
\begin{aligned}
v-\bar{v} &\in L^\infty\left([0,T], H^2(\mathbb{R}_+)\right), \\
\partial_t(v-\bar{v}) &\in L^\infty\left([0,T]; L^2(\mathbb{R}_+)\right) \cap L^2\left([0,T]; H^1(\mathbb{R}_+)\right).
\end{aligned}
$$

Furthermore, the following estimate holds:

$$
\begin{aligned}
E_2(T) := \sup_{t\in[0,T]} &\left[\|\partial_t(v-\bar{v})(t)\|^2_{L^2(\mathbb{R}_+)} + \left\|\partial_x^2(v-\bar{v})(t)\right\|^2_{L^2(\mathbb{R}_+)}\right] \\
&+ \|\partial_t\partial_x(v-\bar{v})\|^2_{L^2([0,T]\times\mathbb{R}_+)} \\
&\leqslant \mathcal{E}_T.
\end{aligned}
\tag{4.14}
$$

Proof. — Let us first derive formally these regularity estimates, and then explain how they can be proved rigorously. We recall that we have set $g := v - \bar{v}$.

Differentiating (4.7) with respect to time leads to

$$
\partial_t\partial_t g - \mu\partial_x\left(\frac{1}{v}\partial_x\partial_t g\right) = \partial_t F - \mu\partial_x\left(\frac{\partial_t v}{v^2}\partial_x g\right)
\tag{4.15}
$$

where we have denoted

$$
F = \partial_x \boldsymbol{w}^0(x+\widetilde{y}(t)) + \widetilde{y}'(t)\partial_x g + (\widetilde{y}'-s)\partial_x\bar{\boldsymbol{v}} - \mu\partial_x\left(\frac{\partial_x\bar{\boldsymbol{v}}}{v\bar{\boldsymbol{v}}}g\right).
\tag{4.16}
$$

Multiplying by $\partial_t g (= \partial_t v)$ and integrating we get

$$\frac{d}{dt}\int_{\mathbb{R}_+}\frac{|\partial_t g|^2}{2}+\mu\int_{\mathbb{R}_+}\frac{|\partial_x\partial_t g|^2}{v}+\mu\underbrace{\left[\frac{1}{v}\partial_x\partial_t g\ \partial_t g\right]_{|x=0}}_{=0}$$
$$=\int_{\mathbb{R}_+}\partial_t F\partial_t g+\mu\int_{\mathbb{R}_+}\frac{\partial_t v}{v^2}\partial_x g\partial_x\partial_t g+\mu\underbrace{\left[\frac{\partial_t v}{v^2}\partial_x g\ \partial_t g\right]_{|x=0}}_{=0}$$

We estimate the two terms of the right-hand side as follows

$$\begin{aligned}\left|\int_{\mathbb{R}_+}\partial_t F\partial_t g\right| &\leqslant \|\partial_t F\|_{L^2(\mathbb{R}_+)}\|\partial_t g\|_{L^2(\mathbb{R}_+)}\\ &\leqslant \frac{\mu}{4}\int_{\mathbb{R}_+}\frac{|\partial_x\partial_t g|^2}{v}+|\widetilde{y}'(t)|^2\left\|\partial_x^2\boldsymbol{w}^0\left(\cdot+\widetilde{y}(t)\right)\right\|^2_{L^2(\mathbb{R}_+)}\\ &\quad+C|\widetilde{y}''(t)|\left(\|\partial_x g\|_{L^2(\mathbb{R}_+)}+\|\partial_x\bar{v}\|_{L^2(\mathbb{R}_+)}\right)\|\partial_t g\|_{L^2(\mathbb{R}_+)}\\ &\quad+C\|\partial_t g\|^2_{L^2(\mathbb{R}_+)}\Big(\|\partial_x g\|_{L^\infty(\mathbb{R}_+)}+|\widetilde{y}'(t)|^2+1\Big),\end{aligned}$$

and

$$\begin{aligned}\left|\mu\int_{\mathbb{R}_+}\frac{\partial_t v}{v^2}\partial_x g\partial_x\partial_t g\right| &\leqslant \frac{\mu}{4}\int_{\mathbb{R}_+}\frac{|\partial_x\partial_t g|^2}{v}+\mu\int_{\mathbb{R}_+}\frac{|\partial_t g|^2}{v^3}|\partial_x g|^2\\ &\leqslant \frac{\mu}{4}\int_{\mathbb{R}_+}\frac{|\partial_x\partial_t g|^2}{v}+C\,\|\partial_x g\|^2_{L^\infty(\mathbb{R}_+)}\,\|\partial_t g\|^2_{L^2(\mathbb{R}_+)}.\end{aligned}$$

Hence, using Young's inequality and recalling that $\widetilde{y}'\leqslant M$,

$$\begin{aligned}&\frac{d}{dt}\|\partial_t g(t)\|^2_{L^2(\mathbb{R}_+)}+\mu\,\|\partial_t\partial_x g\|^2_{L^2(\mathbb{R}_+)}\\ &\qquad\leqslant C\Big[\,|\widetilde{y}'(t)|^2\left\|\partial_x^2\boldsymbol{w}^0\left(\cdot+\widetilde{y}(t)\right)\right\|^2_{L^2(\mathbb{R}_+)}\\ &\qquad\quad+|\widetilde{y}''(t)|\left(\|\partial_x g\|_{L^2(\mathbb{R}_+)}+\|\partial_x\bar{v}\|_{L^2(\mathbb{R}_+)}\right)\|\partial_t g\|_{L^2(\mathbb{R}_+)}\Big]\\ &\qquad\quad+C\left(1+\|\partial_x g\|_{L^2(\mathbb{R}_+)}\left\|\partial_x^2 g\right\|_{L^2(\mathbb{R}_+)}\right)\|\partial_t g\|^2_{L^2(\mathbb{R}_+)}\\ &\qquad\leqslant C\Big[\left\|\partial_x^2\boldsymbol{w}^0\left(\cdot+\widetilde{y}(t)\right)\right\|^2_{L^2(\mathbb{R}_+)}+|\widetilde{y}''(t)|^2+\|\partial_t g\|^2_{L^2(\mathbb{R}_+)}\Big]\\ &\qquad\quad+C\left(\|\partial_x g\|_{L^2(\mathbb{R}_+)}\left\|\partial_x^2 g\right\|_{L^2(\mathbb{R}_+)}+\|\partial_x g\|^2_{L^2(\mathbb{R}_+)}\right)\|\partial_t g\|^2_{L^2}.\end{aligned}$$

Applying Gronwall's inequality together with Lemma C.1, we deduce that

$$\begin{aligned}&\sup_{t\in[0,T]}\|\partial_t g(t)\|^2_{L^2(\mathbb{R}_+)}+\|\partial_t\partial_x g\|^2_{L^2([0,T],L^2(\mathbb{R}_+)}\\ &\quad\leqslant C\Big[\left\|(\partial_t g)_{|t=0}\right\|^2_{L^2(\mathbb{R}_+)}+\left\|\sqrt{x}\partial_x^2\boldsymbol{w}^0\right\|^2_{L^2(\mathbb{R}_+)}+\|\widetilde{y}''\|^2_{L^2(0,T)}+\|\partial_t g\|^2_{L^2([0,T]\times\mathbb{R}_+)}\Big]\\ &\qquad\times\exp\left(C\left[\|\partial_x g\|_{L^2L^2}\left\|\partial_x^2 g\right\|_{L^2L^2}+\|\partial_x g\|^2_{L^2L^2}\right]\right).\end{aligned}$$

We now use the control of $\|\partial_x g\|_{L^2(\mathbb{R}_+)}$ and $\|\partial_x^2 g\|_{L^2(\mathbb{R}_+)}$ provided by Lemma 4.4. Thanks to the assumption $\inf(1, T^{1/2})E_1(T) \leqslant 1$, we have $\|\partial_x^2 g\|_{L^2((0,T)\times\mathbb{R}_+)} \leqslant C{E_1}^{1/2}$. Moreover, we have $\|\partial_x g\|_{L^2((0,T)\times\mathbb{R}_+)} \leqslant C\inf(1, T^{1/2})E_1^{1/2}$ (for small times, we use the $L^\infty(L^2)$ estimate). It follows that, up to a change of constant C,

$$\exp\left(C\left[\|\partial_x g\|_{L^2L^2}\left\|\partial_x^2 g\right\|_{L^2L^2} + \|\partial_x g\|_{L^2L^2}^2\right]\right) \leqslant \exp\left(C\inf\left(1, T^{1/2}\right)E_1\right) \leqslant C.$$

As a consequence, we obtain

$$\begin{aligned}(4.17)\quad &\sup_{t\in[0,T]} \|\partial g(t)\|_{L^2(\mathbb{R}_+)}^2 + \|\partial_t\partial_x g\|_{L^2([0,T],L^2(\mathbb{R}_+))}^2 \\ &\qquad \leqslant C\Big[\left\|(\partial_t g)_{|t=0}\right\|_{L^2(\mathbb{R}_+)}^2 + \left\|\sqrt{x}\partial_x^2 w^0\right\|_{L^2(\mathbb{R}_+)}^2 + \|\widetilde{y}''\|_{L^2(0,T)}^2 + E_1\Big].\end{aligned}$$

Next, using (4.16), we have

$$\begin{aligned}(4.18)\quad \left\|(\partial_t g)_{|t=0}\right\|_{L^2(\mathbb{R}_+)}^2 &\leqslant \left\|F_{|t=0}\right\|_{L^2(\mathbb{R}_+)}^2 + C\left\|g_{|t=0}\right\|_{H^2(\mathbb{R}_+)}^2 \\ &\leqslant \left\|\partial_x w^0\right\|_{L^2(\mathbb{R}_+)}^2 + C\left\|v^0 - \bar{v}\right\|_{H^2(\mathbb{R}_+)}^2 + C\left|\widetilde{y}'(0) - s\right|^2.\end{aligned}$$

Gathering (4.17) and (4.18), we obtain the inequality announced in the statement of the lemma.

To complete the proof, it remains to estimate $\partial_x^2 g$ in $L^\infty L^2$. For that purpose, we write

$$\begin{aligned}\mu^2\int_{\mathbb{R}_+}\left|\partial_x\left(\frac{\partial_x g}{v}\right)\right|^2(t,x)\,dx &= \int_{\mathbb{R}_+}|\partial_t g - F|^2(t,x)\,dx \\ &\leqslant 2\|\partial_t g(t)\|_{L^2(\mathbb{R}_+)}^2 + 2\|F(t)\|_{L^2(\mathbb{R}_+)}^2\end{aligned}$$

for any $t \in [0,T]$. Hence, considering the sup in time and combining with (4.17) and (4.13), we get,

$$\begin{aligned}&\sup_{t\in[0,T]}\mu^2\left\|\partial_x\left(\frac{\partial_x g}{v}(t)\right)\right\|_{L^2(\mathbb{R}_+)}^2 \\ &\quad\leqslant C\Big[\left\|v^0 - \bar{v}\right\|_{H^2(\mathbb{R}_+)}^2 + \left\|\left(1+\sqrt{x}\right)\partial_x w^0\right\|_{L^2(\mathbb{R}_+)}^2 + \|\sqrt{x}\partial_x^2 w^0\|_{L^2(\mathbb{R}_+)}^2 \\ &\qquad\qquad + \|\widetilde{y}''\|_{L^2(0,T)}^2 + \|\widetilde{y}' - s\|_{L^\infty(0,T)}^2 + E_0(T)\Big].\end{aligned}$$

Using the compatibility condition $\widetilde{y}'(0) = -(\partial_x u^0/\partial_x v^0)_{|x=0}$, we obtain

$$\begin{aligned}(4.19)\quad &\|\widetilde{y}' - s\|_{L^\infty([0,T])} \\ &\quad\leqslant |\widetilde{y}'(0) - s| + C\|\widetilde{y}' - s\|_{L^2(0,T)}^{1/2}\|\widetilde{y}''\|_{L^2(0,T)}^{1/2} \\ &\quad\leqslant C\left(\left\|u^0 - \bar{u}\right\|_{H^2(\mathbb{R}_+)} + \left\|v^0 - \bar{v}\right\|_{H^2(\mathbb{R}_+)} + \|\widetilde{y}' - s\|_{L^2(0,T)}^{1/2}\|\widetilde{y}''\|_{L^2(0,T)}^{1/2}\right).\end{aligned}$$

Using the estimate on $E_0(T)$ leads to the desired inequality.

Let us now say a few words about a more rigorous derivation of these estimates. Once again, we consider the approximation v^R solution to (3.2c). We can perform

the previous estimates for v^R, and we obtain uniform bounds in R. Passing to the limit $R \to +\infty$, we get inequality (4.14). $\square$

Lemma 4.6 (Higher regularity - II). — *Assume that* $\widetilde{y}'(0) = -\frac{\partial_x u^0_{|x=0}}{\partial_x v^0_{|x=0}}$.

The solution v *of* (3.12) *satisfies* $v - \bar{v} \in L^\infty([0,T], H^3(\mathbb{R}_+))$, *and* $\partial_t(v-\bar{v}) \in L^\infty([0,T]; H^1(\mathbb{R}_+)) \cap L^2([0,T]; H^2(\mathbb{R}_+))$.

Furthermore, the following estimate holds: assume that $\inf(1, T^{1/2})\mathcal{E}_T \leqslant 1$, *where* $\mathcal{E}_T$ *is defined in* (4.4). *Then, for some* $p > 1$,

$$\begin{aligned} E_3(T) := \sup_{t\in[0,T]} &\left[\|\partial_x\partial_t(v-\bar{v})(t)\|^2_{L^2(\mathbb{R}_+)} + \left\|\partial_x^3(v-\bar{v})(t)\right\|^2_{L^2(\mathbb{R}_+)} \right] \\ &+ \left\|\partial_t^2(v-\bar{v})\right\|^2_{L^2([0,T]\times\mathbb{R}_+)} + \left\|\partial_x^2\partial_t(v-\bar{v})\right\|^2_{L^2([0,T]\times\mathbb{R}_+)} \\ &\leqslant \mathcal{E}_T(1+\mathcal{E}_T)^p. \end{aligned} \tag{4.20}$$

Proof. — Let us now consider (4.15) as a linear parabolic equation on $\partial_t(v - \bar{v}) = \partial_t g$, with homogeneous boundary conditions, endowed with the initial data $\widetilde{y}'(0)\partial_x v^0 + \mu\partial_{xx}\ln v^0 + \partial_x w^0 = \widetilde{y}'(0)\partial_x v^0 + \partial_x u^0$. Thanks to the compatibility condition (3.1), the initial condition belongs to $H^1_0(\mathbb{R}_+)$, and we can apply estimate (A.4) in Proposition A.1 in the Appendix. Indeed, setting $a = 1/v$, we have $\partial_t a = -\frac{\partial_t v}{v^2}$, $\partial_x a = -\frac{\partial_x v}{v^2}$. Using Lemma 4.5, we infer that $\|\partial_t a\|_{L^\infty(L^2)} \leqslant E_2^{1/2}$ and

$$\|\partial_x a\|_{L^\infty([0,T];L^4(\mathbb{R}_+))} \leqslant C\left(1 + E_2^{1/2}\right).$$

Moreover, setting

$$f = \partial_t F - \mu\partial_x\left(\frac{\partial_t v}{v^2}\partial_x\left(v-\bar{v}\right)\right),$$

we have, as shown in the course of the proof of Lemma 4.5,

$$\begin{aligned} \|f\|_{L^2([0,T]\times\mathbb{R}_+)} &\leqslant C\|\partial_t F\|_{L^2((0,T)\times\mathbb{R}_+)} + C\left\|\partial_x(v-\bar{v})\right\|_{L^\infty}\left\|\partial_x\partial_t v\right\|_{L^2L^2} \\ &\quad + C\|\partial_t v\|_{L^2L^\infty}\left\|\partial_x^2\left(v-\bar{v}\right)\right\|_{L^\infty L^2} \\ &\quad + C\left\|\partial_x v\right\|_{L^\infty}\left\|\partial_t v\right\|_{L^2L^2}\left\|\partial_x\left(v-\bar{v}\right)\right\|_{L^\infty} \\ &\leqslant C\mathcal{E}_T^{1/2}(1+\mathcal{E}_T). \end{aligned}$$

Hence, according to Proposition A.1

$$\begin{aligned} \sup_{t\in[0,T]}\|\partial_x\partial_t g\|^2_{L^2(\mathbb{R}_+)} + \left\|\partial_t^2 g\right\|^2_{L^2([0,T]\times\mathbb{R}_+)} + \left\|\partial_x^2\partial_t g\right\|^2_{L^2([0,T]\times\mathbb{R}_+)} \\ \leqslant C\left\|\partial_x\partial_t g_{|t=0}\right\|^2_{L^2(\mathbb{R}_+)} + C\mathcal{E}_T(1+\mathcal{E}_T)^2. \end{aligned}$$

We have

$$[\partial_t\partial_x g]_{|t=0} - \mu\left[\partial_x^2\left(\frac{1}{v}\partial_x g\right)\right]_{t=0} = [\partial_x F]_{|t=0}$$

where

$$\partial_x F = \partial_x^2 w^0(\cdot + \widetilde{y}(t)) + \widetilde{y}'(t)\partial_x^2 g + (\widetilde{y}' - s)\partial_x^2\bar{v} - \mu\partial_x^2\left(\frac{\partial_x\bar{v}}{v\bar{v}}g\right),$$

so that

$$\left\|(\partial_{t,x}g)_{|t=0}\right\|_{L^2(\mathbb{R}_+)} \leqslant C\Big(\left\|g_{|t=0}\right\|_{H^3(\mathbb{R}_+)} \left(1+\left\|g_{|t=0}\right\|^2_{H^2(\mathbb{R}_+)}\right) + \left\|\partial_x^2 \boldsymbol{w}^0\right\|_{L^2(\mathbb{R}_+)} + |\widetilde{y}'(0)-s| \left\|\partial_x^2 \bar{\boldsymbol{v}}\right\|_{L^2(\mathbb{R}_+)}\Big).$$

We then estimate $\widetilde{y}'(0)-s$ as in (4.19), so that

$$\left\|(\partial_{t,x}g)_{|t=0}\right\|_{L^2(\mathbb{R}_+)} \leqslant C\mathcal{E}_0^{1/2}(1+\mathcal{E}_0).$$

To conclude, coming back to

$$\mu\partial_x^2\left(\frac{1}{v}\partial_x g\right) = \partial_t(\partial_x g) - \partial_x F,$$

we have

$$\sup_{t\in[0,T]} \mu^2 \int_{\mathbb{R}_+} \left|\partial_x^2\left(\frac{1}{v}\partial_x g\right)(t)\right|^2 \leqslant 2\|\partial_t\partial_x g\|^2_{L^\infty([0,T],\,L^2(\mathbb{R}_+))} + 2\|\partial_x F\|^2_{L^\infty([0,T],\,L^2(\mathbb{R}_+))},$$

with

$$\|\partial_x F\|_{L^\infty([0,T],\,L^2(\mathbb{R}_+))} \leqslant C\Bigg[\left\|\partial_x^2 \boldsymbol{w}^0\right\|_{L^2(\mathbb{R}_+)} + \|g\|_{L^\infty([0,T],H^2(\mathbb{R}_+))} + \|\widetilde{y}'-s\|_{L^\infty(0,T)} + \|g\|_{L^\infty((0,T),H^2)}\left(1+\|g\|_{L^\infty((0,T),\,H^2)}\right)^2\Bigg].$$

Finally, using the previous estimates as well as the inequality (4.19), we obtain, for some $p \geqslant 1$,

$$\sup_{t\in[0,T]} \left\|\partial_x^3 g(t)\right\|^2_{L^2(\mathbb{R}_+)} \leqslant C\mathcal{E}_T(1+\mathcal{E}_T)^p. \qquad \square$$

Gathering the estimates of Lemmas 4.4, 4.5 and 4.6, we obtain Proposition 4.1.

4.3. Estimates on $h = u - \bar{\boldsymbol{u}}$

This subsection is devoted to the study of (2.18) satisfied by u, which we rewrite in terms of $h := u - \bar{\boldsymbol{u}}$:

$$\begin{aligned}
(4.21)\quad & \partial_t h - \widetilde{y}'(t)\partial_x h - \mu\partial_x\left(\frac{1}{v}\partial_x h\right) \\
&\qquad = (\widetilde{y}'-s)\partial_x\bar{\boldsymbol{u}} + \mu\partial_x\left(\left(\frac{1}{v}-\frac{1}{\bar{\boldsymbol{v}}}\right)\partial_x\bar{\boldsymbol{u}}\right), \quad t>0,\ x>0, \\
& h_{|x=0} = 0, \quad \lim_{x\to\infty} h(t,x) = 0, \\
& h_{|t=0} = \boldsymbol{u}^0 - \bar{\boldsymbol{u}},
\end{aligned}$$

where v is the solution of (2.17) constructed in the previous subsections. The goal is to prove Proposition 4.2.

Throughout this subsection, we use the notations introduced in (4.3), (4.4), and we assume that the time T is such that $\inf(1,T)E_1(T) \leqslant 1$, where $E_1(T)$ was introduced in Lemma 4.4. We also assume that assumption (4.2) is satisfied.

Our first result is the classical energy estimate for (4.21):

Lemma 4.7 (Existence and energy estimates). — *There exists a unique* $u \in \bar{u} + L^\infty([0,T],\, H^1(\mathbb{R}_+))$ *solution to* (4.21) *which is such that:*

$$E_4(T) := \sup_{t\in[0,T]} \|(u-\bar{u})(t)\|^2_{H^1(\mathbb{R}_+)} + \|\partial_x(u-\bar{u})\|^2_{L^2([0,T];H^1(\mathbb{R}_+))} + \|\partial_t(u-\bar{u})\|^2_{L^2([0,T]\times\mathbb{R}_+)} \tag{4.22}$$

$$\leqslant \mathcal{E}_T(1+\mathcal{E}_T)^p, \tag{4.23}$$

for some $p \geqslant 1$.

Proof. — The existence and uniqueness of u in

$$\bar{u} + L^\infty\left([0,T], L^2(\mathbb{R}_+)\right) \cap L^2([0,T], H^1(\mathbb{R}_+)),$$

follows from classical variational arguments. The energy estimates in $L^\infty([0,T], H^1(\mathbb{R}_+)) \cap L^2([0,T], H^2(\mathbb{R}_+))$ are a consequence of Proposition A.1 in the Appendix, setting $b(t,x) = -\widetilde{y}'(t)$, $c \equiv 0$, $a = 1/v$, and

$$f = (\widetilde{y}' - s)\partial_x\bar{u} + \mu\partial_x\left(\left(\frac{1}{v} - \frac{1}{\bar{v}}\right)\partial_x\bar{u}\right).$$

We obtain in particular, using the lower bound $a \geqslant \bar{C}^{-1}$ and setting $h = u - \bar{u}$,

$$\|h\|_{L^\infty([0,T],L^2(\mathbb{R}_+))} + \|\partial_x h\|_{L^2((0,T)\times\mathbb{R}_+)} \tag{4.24}$$
$$\leqslant C\left(\left\|u^0 - \bar{u}\right\|_{L^2(\mathbb{R}_+)} + \|\widetilde{y}' - s\|_{L^2(0,T)}\,\|\bar{u} - u_+\|_{L^2(\mathbb{R}_+)} + \|v - \bar{v}\|_{L^2([0,T],L^2(\mathbb{R}_+))}\right),$$

and the right-hand side is bounded by $(\mathcal{E}_T)^{1/2}$.

Furthermore, note that with the notations of Lemma 4.4

$$\|\partial_t a\|_{L^\infty([0,T],L^2(\mathbb{R}_+))} \leqslant \|\partial_t v\|_{L^\infty([0,T],L^2(\mathbb{R}_+))} \leqslant E_1^{1/2},$$
$$\|\partial_x a\|_{L^\infty([0,T],L^4(\mathbb{R}_+))} \leqslant C\Big(1 + \|\partial_x(v-\bar{v})\|_{L^\infty([0,T],H^1(\mathbb{R}_+))}\Big) \leqslant C\Big(1 + E_1^{1/2}\Big).$$

The estimate announced in the Proposition follows easily from Proposition A.1. □

Remark 4.8. — Note that, compared to Proposition A.1 and Lemma 3.5 concerning the $L^\infty L^2$ estimate on $v - \bar{v}$, we do not have the exponential dependency with respect to T in (4.24). This results from the divergence structure of the diffusion term ($\partial_x(\frac{\partial_x h}{v})$ here, compared to $\partial_x^2 \ln(\frac{v}{\bar{v}})$ in Lemma 3.5), but also from the structure of the source term f. Indeed $f = \partial_x\mathcal{F}$ with $\mathcal{F} = (\widetilde{y}'(t) - s)(\bar{u} - u_+) + (\frac{1}{v} - \frac{1}{\bar{v}})\partial_x\bar{u}$, therefore

$$\left|\int_0^T \int_{\mathbb{R}_+} f\ h\ dxdt\right| \leqslant C_\eta\|\mathcal{F}\|^2_{L^2([0,T]\times\mathbb{R}_+)} + \eta\|\partial_x h\|^2_{L^2([0,T]\times\mathbb{R}_+)},$$

where the second term is absorbed in the left-hand side of the energy inequality. Hence, compared to Proposition A.1, we can close the energy estimate without need of $\|u - \bar{u}\|_{L^2([0,T]\times\mathbb{R}_+)}$.

Lemma 4.9 (Improved regularity). — *Under the compatibility condition* (4.5), *namely*

$$\partial_x u^0_{|x=0}\Big(\widetilde{y}'(0) - \mu\left(\partial_x v^0\right)_{|x=0}\Big) + \mu\left(\partial_x^2 u^0\right)_{|x=0} = 0,$$

the solution u to (4.21) *is such that:*

$$\begin{aligned} E_5(T) &:= \|\partial_{t,x}(u-\bar{u})\|^2_{L^\infty([0,T];L^2(\mathbb{R}_+))} + \left\|\partial_t^2(u-\bar{u})\right\|^2_{L^2([0,T]\times\mathbb{R}_+)} \\ &\quad + \left\|\partial_t\partial_x^2(u-\bar{u})\right\|^2_{L^2([0,T]\times\mathbb{R}_+)} \\ &\leqslant \mathcal{E}_T(1+\mathcal{E}_T)^p, \end{aligned}$$

for some integer $p \geqslant 1$.

Proof. — We begin with the control of $\partial_t h = \partial_t(u-\bar{u})$ in $L^\infty([0,T],\, L^2(\mathbb{R}_+)) \cap L^2([0,T],\, H^1(\mathbb{R}_+))$ using estimate (A.5) from Proposition A.1 and Lemma 4.7:

$$\begin{aligned} &\|\partial_t h\|^2_{L^\infty([0,T],\,L^2(\mathbb{R}_+))} + \|\partial_{t,x} h\|^2_{L^2((0,T)\times\mathbb{R}_+)} \\ &\leqslant C\left(\left\|u^0-\bar{u}\right\|^2_{H^2(\mathbb{R}_+)}(1+E_1) + \left(1+\|\widetilde{y}''\|^2_{L^2(0,T)} + E_2\right)E_4\right) \\ &\quad + C\,\|\widetilde{y}''\|^2_{L^2(0,T)} + C\,\|\partial_t g\|^2_{L^2([0,T],\,H^1(\mathbb{R}_+))}\left(1+\|\partial_x g\|^2_{L^\infty([0,T],\,L^2(\mathbb{R}_+))}\right) \\ &\leqslant \mathcal{E}_T\,(1+\mathcal{E}_T)^p\,. \end{aligned} \tag{4.25}$$

Now, we see $\partial_t h$ as the solution to the problem

$$\partial_t(\partial_t h) - \widetilde{y}'(t)\partial_{t,x} h - \mu\partial_x\left(\frac{1}{v}\partial_{t,x} h\right) = H \tag{4.26}$$

with

$$H := \widetilde{y}''(t)\partial_x h + \widetilde{y}''(t)\partial_x\bar{u} - \mu\partial_x\left(\frac{\partial_t v}{v^2}\partial_x h\right) - \mu\partial_x\left(\frac{\partial_t v}{v^2}\partial_x\bar{u}\right),$$

endowed with the initial condition

$$\partial_t h_{|t=0} = \widetilde{y}'(0)\partial_x u^0 + \mu\partial_x\left(\frac{1}{v^0}\partial_x u^0\right).$$

Under the compatibility condition (4.5), we ensure that $\partial_t h_{|t=0,x=0} = 0$. We can now apply Proposition A.1 and we get thanks to (A.4)

$$\begin{aligned} &\|\partial_{t,x} h\|^2_{L^\infty([0,T],\,L^2(\mathbb{R}_+))} + \left\|\partial_t^2 h\right\|^2_{L^2([0,T]\times\mathbb{R}_+)} + \left\|\partial_t\partial_x^2 h\right\|^2_{L^2([0,T]\times\mathbb{R}_+)} \\ &\leqslant C\Bigg[\left\|\partial_t\partial_x h_{|t=0}\right\|^2_{L^2(\mathbb{R}_+)} + \|\widetilde{y}''\|^2_{L^2(0,T)}\,\|\partial_x h\|^2_{L^\infty([0,T],\,L^2(\mathbb{R}_+))} + \|\widetilde{y}''\|^2_{L^2(0,T)} \\ &\quad + \|\partial_t v\|^2_{L^\infty([0,T],\,H^1(\mathbb{R}_+))}\left(1+\|\partial_x(v-\bar{v})\|^2_{L^\infty([0,T];H^1(\mathbb{R}_+))}\right)\|\partial_x h\|^2_{L^2([0,T],\,H^1(\mathbb{R}_+))} \\ &\quad + \|\partial_t v\|^2_{L^2([0,T],\,H^1(\mathbb{R}_+))}\left(1+\|v-\bar{v}\|^2_{L^\infty([0,T],\,H^1(\mathbb{R}_+))}\right) \\ &\quad + \Big(\|\partial_t v\|_{L^\infty([0,T],\,L^2(\mathbb{R}_+))} + \|\partial_t v\|^2_{L^\infty([0,T],\,L^2(\mathbb{R}_+))} + \|\partial_x v\|^2_{L^\infty([0,T],\,L^4(\mathbb{R}_+))} \end{aligned}$$

$$+ \|\partial_x v\|^4_{L^\infty([0,T],\, L^4(\mathbb{R}_+))} + \|\widetilde{y}'\|^2_{L^\infty(0,T)} \Big) \|\partial_{t,x} h\|^2_{L^2([0,T]\times\mathbb{R}_+)} \Bigg].$$

The right-hand side is controlled as follows. Proposition 4.1 allows us to upper-bound v and its derivatives. We also use (4.25) to estimate $\|\partial_{t,x} h\|_{L^2L^2}$ and Lemma 4.7 to estimate h in $L^\infty(H^1) \cap L^2(H^2)$. Next, using (4.21), we observe that

$$\begin{aligned}\partial_t \partial_x h_{|t=0} &= \widetilde{y}'(0)\partial_x^2(u^0 - \bar{u}) + (\widetilde{y}'(0) - s)\partial_x^2 \bar{u} + \mu \partial_x^2 \left(\frac{1}{v^0} \partial_x (u^0 - \bar{u}) \right) \\ &\quad + \mu \partial_x^2 \left(\left(\frac{1}{v^0} - \frac{1}{\bar{v}} \right) \partial_x \bar{u} \right)\end{aligned}$$

and therefore, using the compatibility condition $\widetilde{y}'(0) = -\partial_x u^0_{x=0} / \partial_x v^0_{|x=0}$,

$$\begin{aligned}&\left\| \partial_t \partial_x h_{|t=0} \right\|_{L^2(\mathbb{R}_+)} \\ &\leqslant C \left(\left\| \partial_x \left(u^0 - \bar{u} \right) \right\|_{H^2(\mathbb{R}_+)} + \left\| v^0 - \bar{v} \right\|_{H^2(\mathbb{R}_+)} \right) \left(1 + |\widetilde{y}'(0) - s| + \left\| v^0 - \bar{v} \right\|_{H^2(\mathbb{R}_+)} \right) \\ &\qquad + C\, |\widetilde{y}'(0) - s| \\ &\leqslant C \mathcal{E}_0^{1/2} \left(1 + \mathcal{E}_0^{1/2} \right).\end{aligned}$$

Gathering all the estimates, we find that

$$\|\partial_{t,x} h\|^2_{L^\infty L^2} + \left\| \partial_t^2 h \right\|^2_{L^2L^2} + \left\| \partial_t \partial_x^2 h \right\|^2_{L^2L^2} \leqslant \mathcal{E}_T \left(1 + \mathcal{E}_T \right)^p$$

for some large and computable constant $p > 1$. □

5. Existence and uniqueness of local solutions

The purpose of this section is to prove Theorem 2.8, following the strategy outlined in Section 2. We shall construct the solution $\widetilde{x}$ of (2.15) as the fixed point of a nonlinear application $\mathcal{T}$, whose definition we now recall: $\mathcal{T} : \widetilde{y} \in H^2(0,T) \mapsto \widetilde{z} \in H^2(0,T)$, where

$$\widetilde{z}'(t) = -\mu \frac{\partial_x u(t, 0^+)}{u_- - w^0(\widetilde{y}(t)^+)}, \quad \widetilde{z}(0) = 0$$

and u is the unique solution of (2.18).

As a consequence, recalling that $s = -\mu \partial_x \bar{u}_{|x=0} / (u_- - u_+)$,

$$\begin{aligned}\widetilde{z}' - s &= -\mu \frac{(\partial_x u - \partial_x \bar{u})_{|x=0^+}(t)}{u_- - w^0\left(\widetilde{y}(t)\right)} - \mu \partial_x \bar{u}_{|x=0^+} \frac{w^0\left(\widetilde{y}(t)^+\right) - u_+}{\left(u_- - w^0(\widetilde{y}(t))\right)\left(u_- - u_+\right)} \\ &= -\mu \frac{(\partial_x u - \partial_x \bar{u})_{|x=0^+}(t)}{u_- - w^0\left(\widetilde{y}(t)^+\right)} + s \frac{w^0\left(\widetilde{y}(t)^+\right) - u_+}{u_- - w^0\left(\widetilde{y}(t)^+\right)}.\end{aligned} \tag{5.1}$$

We will first prove that for any initial data satisfying (H1)-(H5), the application $\mathcal{T}$ has an invariant set provided T is chosen sufficiently small. We will then prove that $\mathcal{T}$ is a contraction on this invariant set. In order not to burden the notation, within this section, we will systematically drop the exponent $+$ in the traces, i.e. we will write $\partial_x u_{|x=0}$, $w^0(\widetilde{y}(t))$ instead of $\partial_x u_{|x=0^+}$, $w^0(\widetilde{y}(t)^+)$.

Let us define

$$\mathcal{I}_M := \Bigg\{ \widetilde{y} \in H^2([0,T]),\ \widetilde{y}(0) = 0,\ \widetilde{y}'(0) = -\frac{\partial_x u^0_{|x=0}}{\partial_x v^0_{|x=0}}$$
$$\text{and } M^{-1} \leqslant \widetilde{y}'(t) \leqslant M \ \ \forall\, t \in [0,T],\ \|\widetilde{y}' - s\|_{H^1([0,T])} \leqslant M \Bigg\}.$$

The set $\mathcal{I}_M$ will be our invariant set provided M is chosen sufficiently large, and T sufficiently small. Without loss of generality, we will assume that $M \geqslant 1$.

Following the previous section (see (4.3) and (4.4)), we set

$$\mathcal{E}_0 := \left\| v^0 - \bar{v} \right\|^2_{H^3(\mathbb{R}_+)} + \left\| u^0 - \bar{u} \right\|^2_{H^3(\mathbb{R}_+)} + \left\| V^0 \right\|^2_{L^2(\mathbb{R}_+)}$$
$$+ \left\| (1+\sqrt{x}) W^0 \right\|^2_{L^2(\mathbb{R}_+)} + \left\| \left(1+\sqrt{x}\right) \partial_x w^0 \right\|^2_{L^2(\mathbb{R}_+)} + \left\| \left(1+\sqrt{x}\right) \partial_x^2 w^0 \right\|^2_{L^2(\mathbb{R}_+)},$$
$$\mathcal{E}_T := \mathcal{E}_0 + \|\widetilde{y}' - s\|^2_{H^1([0,T])}.$$

Theorem 2.8 is an immediate consequence of the following result:

Proposition 5.1. — *Let $M > 1$ be such that*

$$\sqrt{\frac{2}{M}} \leqslant \partial_x v^0_{|x=0} \leqslant \sqrt{\frac{M}{2}}, \quad \mathcal{E}_0 \leqslant M^2, \quad -\sqrt{\frac{M}{2}} \leqslant \partial_x u^0_{|x=0} \leqslant -\sqrt{\frac{2}{M}}. \tag{5.2}$$

Then there exists $T > 0$ depending on M and the parameters of the system, such that $\mathcal{T}$ has a unique fixed point on $\mathcal{I}_M$.

5.1. $\mathcal{T}$ has an invariant set

First, notice that by definition, $\widetilde{z}(0) = 0$ and

$$\widetilde{z}'(0) = -\mu \frac{\partial_x u^0_{|x=0}}{u_- - w^0(0)}.$$

We recall that $w^0 = u^0 - \mu \partial_x \ln v^0$, so that $u_- - w^0(0) = \mu \partial_x v^0_{|x=0}$. It follows that

$$\widetilde{z}'(0) = -\frac{\partial_x u^0_{|x=0}}{\partial_x v^0_{|x=0}}.$$

We assume that $M \geqslant 1$ is chosen so that (5.2) is satisfied. Notice that these conditions ensure that

$$\frac{2}{M} \leqslant \widetilde{y}'(0) = \widetilde{z}'(0) = -\frac{\partial_x u^0_{|x=0}}{\partial_x v^0_{|x=0}} \leqslant \frac{M}{2}.$$

Since

$$\widetilde{z}'(0) - \sqrt{t}\, \|\widetilde{z}''\|_{L^2(0,T)} \leqslant \widetilde{z}'(t) \leqslant \widetilde{z}'(0) + \sqrt{t}\, \|\widetilde{z}''\|_{L^2(0,T)},$$

it suffices to prove that $\|\widetilde{z}' - s\|_{H^1(0,T)} \leqslant M$ for T sufficiently small (depending on M), and to further choose T so that $\sqrt{T} M \leqslant (2M)^{-1}$, i.e. $T \leqslant (4M^4)^{-1}$.

Now, let us consider the solutions v, u of (2.17), (2.18) respectively. Using the notation of Lemma 4.4, we have

$$E_1 \leqslant C\left(\mathcal{E}_0 + \|\widetilde{y}' - s\|^2_{L^2(0,T)}\right) \leqslant C_M.$$

(We recall that the constant C in the above inequality may depend on M.) Now, let us choose T_M so that $\inf(1, T_M)C_M \leqslant 1$. Then the solutions v, u of (2.17), (2.18) satisfy the estimates of Propositions 4.1 and 4.2.

Let us now turn towards the bound on $\|\widetilde{y}' - s\|_{H^1(0,T)}$. Note that $\mathcal{E}_T \leqslant 2M^2$ according to our choice of M. We also note that

$$u_- - w^0(0) = \mu\partial_x v^0_{|x=0} \geqslant \mu\sqrt{\frac{2}{M}}.$$

Since $\|\widetilde{y}'\|_{L^\infty(0,T)} \leqslant M$ and $\|\partial_x w^0\|_\infty \leqslant \mathcal{E}_0^{1/2} \leqslant M$, we obtain, for $t \in [0,T]$,

$$u_- - w^0(\widetilde{y}(t)) \geqslant \mu\sqrt{\frac{2}{M}} - \|\widetilde{y}'\|_\infty t\|\partial_x w^0\|_\infty \geqslant \frac{\mu}{\sqrt{M}} \tag{5.3}$$

provided $T \leqslant CM^{-5/2}$. We infer, using (5.1) and Proposition 4.2,

$$\begin{aligned}\|\widetilde{z}' - s\|_{L^2(0,T)} \leqslant & C_M\left(\|\partial_x(u - \bar{u})\|_{L^2((0,T),L^\infty)} + T^{1/2}\left\|w^0 - u_+\right\|_\infty\right)\\ \leqslant & C_M\mathcal{E}_T^{1/2}(1+\mathcal{E}_T)^{p/2}T^{1/2} \leqslant C_M T^{1/2}.\end{aligned}$$

Thus, for T sufficiently small, the right-hand side is smaller than $M/2$.

We now consider $\widetilde{z}''$. We recall that we set $h = u - \bar{u}$. Using that

$$\begin{aligned}\widetilde{z}''(t) &= -\mu\frac{(\partial_t\partial_x u)_{|x=0}}{u_- - w^0(\widetilde{y}(t))} - \mu\frac{(\partial_x u)_{|x=0}}{(u_- - w^0(\widetilde{y}(t)))^2}\widetilde{y}'(t)\partial_x w^0(\widetilde{y}(t))\\ &= -\mu\frac{\partial_t\partial_x h_{|x=0}}{u_- - w^0(\widetilde{y}(t))} - \mu\frac{\partial_x h_{|x=0}}{(u_- - w^0(\widetilde{y}(t)))^2}\widetilde{y}'(t)\partial_x w^0(\widetilde{y}(t)) \qquad (5.4)\\ &\quad + s\widetilde{y}'(t)\partial_x w^0(\widetilde{y}(t))\frac{u_- - u_+}{(u_- - w^0(\widetilde{y}(t)))^2}, \qquad (5.5)\end{aligned}$$

and recalling (5.3), it follows that

$$\begin{aligned}\|\widetilde{z}''\|_{L^2(0,T)} \leqslant & C\sqrt{M}\,\|\partial_t\partial_x h\|_{L^2((0,T),L^\infty(\mathbb{R}_+))} + CM^3\|\partial_x h\|_{L^2((0,T),L^\infty(\mathbb{R}_+))}\\ & + CM^3T^{1/2}.\end{aligned}$$

We now use Proposition 4.2: there exists a constant $p \geqslant 1$ such that

$$\|\partial_t\partial_x h\|^2_{L^\infty((0,T),L^2)} + \left\|\partial_t\partial_x^2 h\right\|^2_{L^2((0,T)\times\mathbb{R}_+)} \leqslant C_M M^{2p+2}.$$

Hence

$$\begin{aligned}\|\partial_t\partial_x h\|_{L^2((0,T),L^\infty(\mathbb{R}_+))} &\leqslant C\,\|\partial_t\partial_x h\|^{1/2}_{L^2((0,T)\times\mathbb{R}_+)}\left\|\partial_t\partial_x^2 h\right\|^{1/2}_{L^2((0,T)\times\mathbb{R}_+)}\\ &\leqslant C_M T^{1/4}M^{p+1}.\end{aligned}$$

Similarly,

$$\begin{aligned}\|\partial_x h\|_{L^2((0,T),L^\infty(\mathbb{R}_+))} &\leqslant T^{1/4}\,\|\partial_x h\|^{1/2}_{L^\infty([0,T];L^2(\mathbb{R}_+))}\left\|\partial_x^2 h\right\|^{1/2}_{L^2([0,T]\times\mathbb{R}_+)}\\ &\leqslant C_M T^{1/4} M^p.\end{aligned}$$

Hence, choosing T sufficiently small (depending on M), we obtain

$$\|\widetilde{z}''\|_{L^2(0,T)} \leqslant \frac{M}{2}.$$

We deduce that $\mathcal{I}_M$ is stable by $\mathcal{T}$.

5.2. $\mathcal{T}$ is a contraction on $\mathcal{I}_M$

Let $\widetilde{y}_1, \widetilde{y}_2 \in \mathcal{I}_M$. We consider the associated solutions v_i, u_i of (2.17), (2.18), and we set $\widetilde{z}_i = \mathcal{T}(\widetilde{y}_i) \in \mathcal{I}_M$. We define $D_v = v_1 - v_2$, $D_u = u_1 - u_2$.

Let us now evaluate $\widetilde{z}_1' - \widetilde{z}_2'$ and $\widetilde{z}_1'' - \widetilde{z}_2''$. First, we have

$$\widetilde{z}_1'(t) - \widetilde{z}_2'(t) = -\mu\frac{\partial_x D_{u|x=0}}{u_- - \boldsymbol{w}^0(\widetilde{y}_1)} - \mu\partial_x u_{2|x=0}\frac{\boldsymbol{w}^0(\widetilde{y}_1) - \boldsymbol{w}^0(\widetilde{y}_2)}{(u_- - \boldsymbol{w}^0(\widetilde{y}_1))(u_- - \boldsymbol{w}^0(\widetilde{y}_2))},$$

so that, using Lemma C.2 and (5.3)

$$\|\widetilde{z}_1' - \widetilde{z}_2'\|_{L^2(0,T)} \leqslant C\left\|\partial_x D_{u|x=0}\right\|_{L^2(0,T)} + C\left\|\partial_x u_{2|x=0}\right\|_{L^\infty(0,T)}\|\widetilde{y}_1' - \widetilde{y}_2'\|_{L^2(0,T)}. \tag{5.6}$$

The constant C depends on parameters of the problem, on Sobolev norms of $\boldsymbol{w}^0$, and on M. In a similar way, using identity (5.4), we find that

$$\begin{aligned}\|\widetilde{z}_1'' - \widetilde{z}_2''\|_{L^2(0,T)} &\leqslant C\left(\left\|\partial_t\partial_x D_{u|x=0}\right\|_{L^2(0,T)} + \left\|\partial_x D_{u|x=0}\right\|_{L^2(0,T)}\right)\\ &\quad + C\left(\left\|\partial_x u_{2|x=0}\right\|_{L^\infty(0,T)} + \left\|\partial_t\partial_x u_{2|x=0}\right\|_{L^2(0,T)}\right)\|\widetilde{y}_1' - \widetilde{y}_2'\|_{L^2(0,T)}.\end{aligned} \tag{5.7}$$

There remains to evaluate the traces of $\partial_t\partial_x D_u$ and $\partial_x D_u$ at $x = 0$ in terms of $\widetilde{y}_1 - \widetilde{y}_2$. In order to do so, we follow the order of the energy estimates in the previous sections and start by evaluating D_v. Note that D_v is a solution of

$$\begin{aligned}\partial_t D_v - \widetilde{y}_1'\partial_x D_v - \mu\partial_{xx}\ln\left(1 + \frac{D_v}{v_2}\right) &= (\widetilde{y}_1' - \widetilde{y}_2')\,\partial_x v_2,\\ D_{v|x=0} = 0, \quad \lim_{x\to\infty} D_v(t,x) &= 0,\\ D_{v|t=0} &= 0.\end{aligned}$$

We apply Lemma B.1 in the Appendix to the function D_v, with $\bar{g} = v_2$, $G = (\widetilde{y}_1' - \widetilde{y}_2')\partial_x v_2$. Since $T \leqslant T_M$, $\exp((1 + \|\partial_x v_2\|^2_{L^\infty((0,T)\times\mathbb{R}_+)})T_M) \leqslant C_M$, it follows that

$$\begin{aligned}&\|D_v\|_{L^\infty((0,T),H^1(\mathbb{R}_+))} + \|\partial_t D_v\|_{L^2((0,T)\times\mathbb{R}_+)} + \|\partial_x D_v\|_{L^2((0,T)\times\mathbb{R}_+)}\\ &\qquad\leqslant C\,\|\widetilde{y}_1' - \widetilde{y}_2'\|_{L^2(0,T)}\left(\|\partial_x(v_2 - \bar{v})\|_{L^\infty((0,T),L^2(\mathbb{R}_+))} + \|\partial_x\bar{v}\|_{L^2(\mathbb{R}_+)}\right)\\ &\qquad\qquad\times\exp\left(\|\partial_x v_2\|_{L^\infty((0,T)\times\mathbb{R}_+)}T\right)\\ &\qquad\leqslant C_M\,\|\widetilde{y}_1' - \widetilde{y}_2'\|_{L^2(0,T)},\end{aligned}$$

and

$$\begin{aligned}\left\|\partial_x^2\left(\frac{D_v}{v_2}\right)\right\|_{L^2((0,T)\times\mathbb{R}_+)} &\leqslant C_M\left(\|\widetilde{y}_1'-\widetilde{y}_2'\|_{L^2(0,T)}+T^{1/2}\|\widetilde{y}_1'-\widetilde{y}_2'\|^2_{L^2(0,T)}\right)\\ &\leqslant C_M\|\widetilde{y}_1'-\widetilde{y}_2'\|_{L^2(0,T)}.\end{aligned}$$

Following the estimates of Section 4.2 (see also Proposition A.1 in the Appendix), we infer that

$$\begin{aligned}\|\partial_t D_v\|^2_{L^\infty((0,T),H^1(\mathbb{R}_+))}+\|\partial_x\partial_t D_v\|^2_{L^2((0,T),H^1(\mathbb{R}_+))}+\left\|\partial_x^2 D_v\right\|^2_{L^\infty((0,T),H^1(\mathbb{R}_+))}\\ \leqslant C_M\left(\|\widetilde{y}_1'-\widetilde{y}_2'\|^2_{L^2(0,T)}+\|\widetilde{y}_1''-\widetilde{y}_2''\|^2_{L^2(0,T)}\right).\end{aligned}$$

We now turn towards the estimates on D_u, which satisfies the equation

$$\partial_t D_u-\widetilde{y}_1'\partial_x D_u-\mu\partial_x\left(\frac{1}{v_1}\partial_x D_u\right)=S$$

with

$$S:=(\widetilde{y}_1'-\widetilde{y}_2')\,\partial_x u_2+\mu\partial_x\left(\left(\frac{1}{v_1}-\frac{1}{v_2}\right)\partial_x u_2\right).$$

In order to complete the energy estimates on D_u, we need to evaluate S and $\partial_t S$ in $L^2([0,T]\times\mathbb{R}_+)$. We have

$$\begin{aligned}\|S\|_{L^2([0,T]\times\mathbb{R}_+)} &\leqslant \|\partial_x u_2\|_{L^\infty([0,T],L^2(\mathbb{R}_+))}\|\widetilde{y}_1'-\widetilde{y}_2'\|_{L^2(0,T)}\\ &\quad+C\Big(\|D_v\|_{L^\infty([0,T]\times\mathbb{R}_+)}\left\|\partial_x^2 u_2\right\|_{L^2([0,T]\times\mathbb{R}_+)}\\ &\quad+\|\partial_x D_v\|_{L^\infty([0,T],L^2(\mathbb{R}_+))}\|\partial_x u_2\|_{L^2([0,T],L^\infty(\mathbb{R}_+))}\\ &\quad+\|D_v\|_{L^\infty([0,T]\times\mathbb{R}_+)}\|\partial_x(v_1+v_2)\|_{L^\infty([0,T]\times\mathbb{R}_+)}\|\partial_x u_2\|_{L^2([0,T],L^\infty(\mathbb{R}_+))}\Big)\\ &\leqslant C_M\|\widetilde{y}_1'-\widetilde{y}_2'\|_{L^2(0,T)}\end{aligned}$$

and similarly,

$$\|\partial_t S\|_{L^2([0,T]\times\mathbb{R}_+)}\leqslant C_M\|\widetilde{y}_1'-\widetilde{y}_2'\|_{H^1(0,T)}.$$

Using once again Proposition A.1 in the Appendix, we obtain

$$\begin{aligned}&\|D_u\|_{L^\infty((0,T),H^1(\mathbb{R}_+))}+\|\partial_x D_u\|_{L^2((0,T),H^1(\mathbb{R}_+))}+\|\partial_t D_u\|_{L^2((0,T)\times\mathbb{R}_+)}\\ &\leqslant C_M\|\widetilde{y}_1'-\widetilde{y}_2'\|_{L^2(0,T)}\end{aligned}$$

and

$$\|\partial_t D_u\|_{L^\infty((0,T),H^1(\mathbb{R}_+))}+\|\partial_x\partial_t D_u\|_{L^2((0,T),H^1(\mathbb{R}_+))}\leqslant C_M\|\widetilde{y}_1'-\widetilde{y}_2'\|_{H^1(0,T)}.$$

We are now ready to prove the contraction property. We focus on the estimate of $\widetilde{z}_1''-\widetilde{z}_2''$, since

$$\|\widetilde{z}_1'-\widetilde{z}_2'\|_{L^2(0,T)}\leqslant\frac{T}{\sqrt{2}}\|\widetilde{z}_1''-\widetilde{z}_2''\|_{L^2(0,T)}.$$

Using the estimates on D_u, we obtain

$$\begin{aligned}
\left\|(\partial_t\partial_x D_u)_{|x=0}\right\|_{L^2(0,T)} &+ \left\|\partial_x D_{u|x=0}\right\|_{L^2(0,T)} \\
&\leqslant C\left\|\partial_t\partial_x D_u\right\|^{1/2}_{L^2((0,T)\times\mathbb{R}_+)}\left\|\partial_t\partial_x^2 D_u\right\|^{1/2}_{L^2((0,T)\times\mathbb{R}_+)} \\
&\quad + C\left\|\partial_x D_u\right\|^{1/2}_{L^2((0,T)\times\mathbb{R}_+)}\left\|\partial_x^2 D_u\right\|^{1/2}_{L^2((0,T)\times\mathbb{R}_+)} \\
&\leqslant CT^{1/4}\Big(\left\|\partial_t\partial_x D_u\right\|^{1/2}_{L^\infty((0,T),L^2(\mathbb{R}_+))}\left\|\partial_t\partial_x^2 D_u\right\|^{1/2}_{L^2((0,T)\times\mathbb{R}_+)} \\
&\qquad\qquad + \left\|\partial_x D_u\right\|^{1/2}_{L^\infty((0,T),L^2(\mathbb{R}_+))}\left\|\partial_x^2 D_u\right\|^{1/2}_{L^2((0,T)\times\mathbb{R}_+)}\Big) \\
&\leqslant C_M T^{1/4}\left\|\widetilde{y}_1' - \widetilde{y}_2'\right\|_{H^1(0,T)}.
\end{aligned}$$

Furthermore,

$$\left\|\partial_x u_{2|x=0}\right\|_{L^\infty(0,T)} + \left\|\partial_t\partial_x u_{2|x=0}\right\|_{L^2(0,T)} \leqslant \left|\partial_x \bar{u}_{|x=0^+}\right| + C\mathcal{E}_T(1+\mathcal{E}_T)^p \leqslant C_M.$$

Coming back to (5.7), it follows that

$$\begin{aligned}
\left\|\widetilde{z}_1'' - \widetilde{z}_2''\right\|_{L^2(0,T)} &\leqslant C_M T^{1/4}\left\|\widetilde{y}_1' - \widetilde{y}_2'\right\|_{H^1(0,T)} + C_M\left\|\widetilde{y}_1' - \widetilde{y}_2'\right\|_{L^2(0,T)} \\
&\leqslant C_M T^{1/4}\left\|\widetilde{y}_1' - \widetilde{y}_2'\right\|_{H^1(0,T)} + C_M T\left\|\widetilde{y}_1'' - \widetilde{y}_2''\right\|_{L^2(0,T)} \\
&\leqslant C_M T^{1/4}\left\|\widetilde{y}_1' - \widetilde{y}_2'\right\|_{H^1(0,T)}.
\end{aligned}$$

Hence, for T sufficiently small (depending on M), $\mathcal{T}$ is a contraction on $\mathcal{I}_M$.

5.3. Conclusion

We deduce from the previous subsections that there exists $T_M > 0$ such that for all $T \leqslant T_M$, $\mathcal{T}$ has a unique fixed point in $\mathcal{I}_M$. Let us consider the solution (v_s, u_s) associated with this fixed point. According to the argument at the end of section 2, $(\widetilde{x}, v_s, u_s)$ is a solution of (2.16). Furthermore, v_s and u_s satisfy the properties listed in Propositions 4.1 and 4.2 respectively, and v_s also satisfies the L^∞ estimates of Lemma 3.2, provided T_M is small enough.

Thus $(\widetilde{x}, v_s, u_s)$ satisfy all the properties listed in Theorem 2.8. Eventually, the pressure in the congested domain is given by

$$p_s(t,x) = p_s(t) = \widetilde{x}'(t)\left(u_- - w^0(\widetilde{x}(t))\right).$$

Since $\widetilde{x} \in H^2(0,T)$, $p_s \in H^1(0,T)$. This completes the proof of Theorem 2.8. □

6. Global solutions: existence for small data and stability

In this section, we start from a strong solution (u_s, v_s) provided by Theorem 2.8 (see the previous section). We prove that if $\mathcal{E}_0$ is small enough, the existence time of the solution is infinite. Furthermore, the traveling wave is asymptotically stable.

Once again, we drop the indices s throughout the section in order to alleviate the notation.

Let us now introduce our setting. We assume that $\mathcal{E}_0 \leqslant c_0\delta^2$, where δ is a small constant, $\mathcal{E}_0$ is the initial energy, defined in (4.3) and c_0 is a constant depending only on the parameters of the problem, to be defined later on. At this stage, we merely choose c_0 small enough so that $\|v_0 - \bar{v}\|_{L^\infty(\mathbb{R}_+)} \leqslant \delta/2$. We denote by T^* the maximal existence time of the solution. For $T \in (0, T^*)$ small enough, using a continuity argument together with the dominated convergence theorem, $\|\widetilde{x}' - s\|_{H^1([0,T])} + \|v - \bar{v}\|_{L^\infty([0,T]\times\mathbb{R}_+)} \leqslant \delta$. We set

$$\overline{T} := \sup\left\{T \in (0, T^*),\ \|\widetilde{x}' - s\|_{H^1([0,T])} \leqslant \delta\right\}.$$

The global existence of the solution relies on the following bootstrap result:

Proposition 6.1. — *There exist constants $c_0, \delta_0 > 0$, depending only on the parameters s, μ, v_+, such that the following result holds.*

For all $\delta \in (0, \delta_0)$, if $\mathcal{E}_0 \leqslant c_0\delta^2$, and $\|(1+\sqrt{x})\partial_x^k w^0\|_{L^2(\mathbb{R}_+)} \leqslant c_0\delta^{3/2}$ for $k = 1, 2, 3$, then

$$\forall\, T \in [0, \overline{T}], \quad \|\widetilde{x}' - s\|_{H^1([0,T])} \leqslant \frac{\delta}{2}.$$

The proof of Proposition 6.1 is the main purpose of this section. Before describing the strategy of the proof, let us deduce the global existence result of Theorem 2.3 from Proposition 6.1. First, it is clear from Proposition 6.1 and from a simple continuity argument that $\overline{T} = T^*$. Second, notice that for all $T < \overline{T}$, $\mathcal{E}_T = \mathcal{E}_0 + \|\widetilde{x}' - s\|^2_{H^1(0,T)} \leqslant 2\delta^2 \leqslant 1$ provided $\delta_0 \leqslant 1/\sqrt{2}$ and $c_0 \leqslant 1$. Hence the total energy remains bounded on $(0, T^*)$. Classically, this implies that the solution is global. Let us explain why in the present context.

First, note that for all $t \in [0, \overline{T})$,

$$\frac{s}{2} \leqslant s - \|\widetilde{x}' - s\|_{L^\infty} \leqslant \widetilde{x}'(t) \leqslant s + \|\widetilde{x}' - s\|_{L^\infty} \leqslant \frac{3s}{2} \tag{6.1}$$

provided δ is sufficiently small.

As a consequence, as long as $1 \leqslant v \leqslant 2(v_+ + 1)$, the estimates of Propositions 4.1 and 4.2 hold, with a constant C depending only on the parameters of the problem $s, \mu, v_+, u_\pm$ (note that we chose here $\bar{C} = 2(v_+ + 1)$). Hence, let us introduce

$$\overline{T}' := \sup\left\{T \in (0, \overline{T}), \quad 1 \leqslant v \leqslant 2(v_+ + 1)\right\}.$$

By continuity, if δ is small enough, we have $\overline{T}' > 0$. Furthermore, there exists a constant C depending only on the parameters of the problem such that

$$\|\partial_x(v - \bar{v})\|_{L^\infty((0,\overline{T}')\times\mathbb{R}_+)} \leqslant C\delta.$$

Consequently, for all $t \in [0, \overline{T}']$,

$$\inf_{x\in[0,1]} \partial_x v(t,x) \geqslant \inf_{x\in[0,1]} \partial_x \bar{v}(x) - \|\partial_x(v - \bar{v})\|_\infty \geqslant \frac{1}{2}\inf_{x\in[0,1]} \partial_x \bar{v}(x) > 0$$

provided δ is small enough. It follows that $\partial_x v_{|t=\overline{T}'}$ satisfies the assumption (3.5) from Lemma 3.2.

Now, let us prove that $\overline{T}' = \overline{T}$ provided δ is small enough. We argue by contradiction and assume that $\overline{T}' < \overline{T}$. We then consider the Cauchy problem at $t = \overline{T}'$. As mentioned above, (3.5) is satisfied, with a constant M depending only on the parameters of the problem. Furthermore,

$$\sup_x v\left(\overline{T}', x\right) \leqslant \sup \bar{v} + \left\|v(\overline{T}') - \bar{v}\right\|_{L^\infty(\mathbb{R}_+)} \leqslant v_+ + C\mathcal{E}_{\overline{T}'} \leqslant v_+ + \frac{1}{2}$$

provided δ is small enough. As a consequence, according to Lemma 3.2, there exists a time $\tau > 0$, depending only on the parameters of the problem, such that

$$1 < v(t,x) \leqslant 2\left(v_+ + \frac{1}{2}\right) = 2v_+ + 1$$

for all $t \in [\overline{T}', \inf(\overline{T}, \overline{T}' + \tau)]$. This contradicts the definition of $\overline{T}'$, and we infer that $\overline{T}' = \overline{T}$.

Thus the constants in Propositions 4.1 and 4.2 and in all Lemmas of section 4 depend only on $s, \mu, v_+, u_\pm$ for all estimates bearing on the interval $[0, \overline{T}]$. At last, let us emphasize that since $\mathcal{E}_T \leqslant 1$ for all $T \leqslant \overline{T}$, the condition $\inf(1, T^{1/2})E_1(T) \leqslant 1$ of Lemmas 4.5, 4.6 etc. is always satisfied.

It then follows from Proposition 5.1 that the time T_M on which the fixed point argument is valid depends only on the parameters of the problem $s, \mu, v_+, u_\pm$. By a classical induction argument, we may solve the Cauchy problem on $[nT_M, (n+1)T_M]$ for all $n \geqslant 0$, and we deduce that $T^* = +\infty$.

Let us now explain our strategy of proof of Proposition 6.1. It relies on two sets of estimates:

- The first set of estimates was obtained in Propositions 4.1 and 4.2. It ensures that if δ is small enough, as long as $T \leqslant \overline{T}$, setting $g := v - \bar{v}$, $h := u - \bar{u}$,

$$\begin{aligned} \|g\|_{L^\infty([0,T],H^3(\mathbb{R}_+))} + \|\partial_t g\|_{L^\infty([0,T],H^1(\mathbb{R}_+))} + \|\partial_t g\|_{L^2((0,T),H^2(\mathbb{R}_+))} \\ \leqslant C\left(\mathcal{E}_0^{1/2} + \|\widetilde{x}' - s\|_{H^1(0,T)}\right), \end{aligned} \tag{6.2}$$

$$\begin{aligned} \|h\|_{L^\infty([0,T],H^1(\mathbb{R}_+))} + \|\partial_t h\|_{L^2([0,T],H^2(\mathbb{R}_+))} + \left\|\partial_t^2 h\right\|_{L^2((0,T)\times\mathbb{R}_+)} \\ \leqslant C(\mathcal{E}_0^{1/2} + \|\widetilde{x}' - s\|_{H^1(0,T)}), \end{aligned} \tag{6.3}$$

  where the constant C depends only on the parameters of the problem, namely s, μ, v_+ and u_+, as explained above (recall that $\mathcal{E}_T \leqslant 1$ for all $T < \overline{T}$).

  From there, using the equations (4.7) and (4.21) on g and h respectively, we also deduce that for all $T \leqslant \overline{T}$,

$$\begin{aligned} \|g\|_{L^2((0,T),H^4(\mathbb{R}_+))} + \|h\|_{L^2((0,T),H^4(\mathbb{R}_+))} \\ \leqslant C\left(\mathcal{E}_0^{1/2} + \left\|(1+\sqrt{x})\partial_x^3 w^0\right\|_{L^2(\mathbb{R}_+)} + \|\widetilde{x}' - s\|_{H^1(0,T)}\right). \end{aligned} \tag{6.4}$$

- The second set of estimates relies on a coercivity inequality for the linearized operator around $\bar{v}$, which will be the main focus of this section. One crucial observation lies in the fact that $\partial_x \bar{v}$ belongs to the kernel of this linearized

operator[(1)]. This allows us to define a new unknown

$$g_1 := -s(v - \bar{v}) - \mu\partial_x \left(\frac{v - \bar{v}}{\bar{v}} \right), \tag{6.5}$$

which will satisfy better estimates than $g = v - \bar{v}$ (cf Remark 6.2).

In order to keep the presentation as simple as possible, we will start with the case when $w^0 \equiv u_+$, which contains the main ideas of the proof. In subsection 6.2, we will address the case of non-constant w^0, and we will point out the main differences with the simplified case. We conclude this section with a proof of the long time stability in subsection 6.3.

6.1. Case $w^0 \equiv u_+$

In this section, we assume that w^0 is constant and equal to u_+. The equation satisfied by v on $[0, T^*)$ is therefore

$$\begin{aligned} \partial_t v - \widetilde{x}'\partial_x v - \mu\partial_x^2 \ln v &= 0, \\ v_{|x=0} = 1, \quad \lim_{x\to\infty} v(t,x) &= v_+, \\ v_{|t=0} &= v^0. \end{aligned}$$

In the rest of this section, we introduce the linearized operator around $\bar{v}$, namely $\partial_x\mathcal{A}$, where

$$\mathcal{A} := -s\mathrm{Id} - \mu\partial_x \left(\frac{\cdot}{\bar{v}} \right).$$

We also set $\beta(t) = \widetilde{x}'(t) - s$ and $g = v - \bar{v}$, as in the previous sections. The equation on g can be written as

$$\begin{aligned} \partial_t g + \partial_x\mathcal{A}g = \beta\partial_x\bar{v} + \beta\partial_x g + \mu\partial_x^2 \left(\ln\left(1 + \frac{g}{\bar{v}}\right) - \frac{g}{\bar{v}} \right), \\ g_{|t=0} = v^0 - \bar{v}, \\ g_{|x=0} = 0. \end{aligned} \tag{6.6}$$

Let us comment a little on the structure of this equation. We will prove that the operators $\partial_x\mathcal{A}$ and $\mathcal{A}\partial_x$ enjoy nice coercivity properties (see Lemma 6.3 below). The second and third terms in the right-hand side of (6.6) are quadratic and will be treated perturbatively, using the first set of estimates on g, namely (6.2) and (6.4). Eventually, $\partial_x\bar{v} \in \ker\mathcal{A}$. Hence the first term in the right-hand side disappears when $\mathcal{A}$ is applied to the equation. As a consequence, $g_1 = \mathcal{A}g$ satisfies the equation

$$\begin{aligned} \partial_t g_1 + \mathcal{A}\partial_x g_1 = \beta\mathcal{A}\partial_x g + \mu\mathcal{A}\partial_x^2 \left(\ln\left(1 + \frac{g}{\bar{v}}\right) - \frac{g}{\bar{v}} \right), \\ g_{1|t=0} = \mathcal{A}\left(v^0 - \bar{v}\right). \end{aligned} \tag{6.7}$$

(1) Note that this is a classical property of nonlinear equations with constant coefficients, linearized around a given stationary solution. It is related to the (formal) space invariance of the equation.

From definition (6.5) and estimates (6.2), (6.4), we know that

$$\begin{aligned} &g_1 \in L^\infty\left((0,\overline{T}), H^2(\mathbb{R}_+)\right), \quad g_1 \in L^2\left((0,\overline{T}), H^3(\mathbb{R}_+)\right) \\ \text{and that} \quad &\partial_t g_1 \in L^\infty\left((0,\overline{T}), L^2(\mathbb{R}_+)\right) \cap L^2\left((0,\overline{T}), H^1(\mathbb{R}_+)\right). \end{aligned} \tag{6.8}$$

Using the identity $u - \mu\partial_x \ln v = u_+$, we also infer that $h - \mu\partial_x \ln(1 + g/\bar{v}) = 0$. Using (6.3) and (6.4), we deduce that

$$\begin{aligned} &g_1 \in L^2\left((0,\overline{T}), H^4(\mathbb{R}_+)\right), \\ &\partial_t g_1 \in L^2\left((0,\overline{T}), H^2(\mathbb{R}_+)\right) \quad \text{and} \quad \partial_t^2 g_1 \in L^2\left((0,\overline{T}) \times \mathbb{R}_+\right). \end{aligned} \tag{6.9}$$

Remark 6.2. — In Equation (6.7), all the terms in the right-hand side are quadratic (recall that $\|g\| \lesssim \mathcal{E}_0^{1/2} + \|\beta\|$), so that g_1 can be expected to satisfy better estimates than g.

Before addressing the estimates on $g_1 = \mathcal{A}g$, let us derive some information on the traces of $\partial_x^k g_1$ at $x = 0$ for $k = 0, 1, 2$. Since $\boldsymbol{w}^0 = u_+$, we know that

$$u(t,x) - \mu\partial_x \ln v(t,x) = u_+ \quad \forall\, t > 0,\ x > 0.$$

As a consequence,

$$\partial_x v_{|x=0} = \frac{u_- - u_+}{\mu} = \partial_x \bar{v}_{|x=0},$$

and thus $g_{1|x=0} = \partial_x g_{|x=0} = 0$. Taking the trace of (6.6) at $x = 0$ (which is legitimate since all terms belong to $L^2((0,\overline{T}), H^2(\mathbb{R}_+))$, we find that

$$\partial_x g_{1|x=0} = \beta\partial_x \bar{v}_{|x=0} = \beta\frac{s(v_+ - 1)}{\mu}. \tag{6.10}$$

We now take the trace of (6.7) at $x = 0$, noticing that all terms in (6.7) belong to $L^2((0,T), H^1(\mathbb{R}_+))$. Note that

$$\mathcal{A}\partial_x g = \partial_x g_1 + [\mathcal{A}, \partial_x]g = \partial_x g_1 - \mu\partial_x\left(\frac{\partial_x \bar{v}}{\bar{v}^2} g\right). \tag{6.11}$$

Furthermore, $\ln(1+X) - X$ is quadratic close to $X = 0$, and we recall that $(g)_{|x=0} = (\partial_x g)_{|x=0} = 0$. It follows that

$$\left(\mathcal{A}\partial_x^2\left(\ln\left(1 + \frac{g}{\bar{v}}\right) - \frac{g}{\bar{v}}\right)\right)_{|x=0} = 0.$$

We infer from (6.7) and (6.11) that

$$(\mathcal{A}\partial_x g_1)_{|x=0} = \beta\left(\mathcal{A}\partial_x g\right)_{|x=0} = \beta\partial_x g_{1|x=0} = \beta^2\frac{s(v_+ - 1)}{\mu},$$

and thus

$$\mu\partial_x\left(\frac{\partial_x g_1}{\bar{v}}\right)_{|x=0} = -s\partial_x g_{1|x=0} - \beta\partial_x g_{1|x=0} \tag{6.12}$$

$$= -\beta\frac{s^2(v_+ - 1)}{\mu} - \beta^2\frac{s(v_+ - 1)}{\mu}. \tag{6.13}$$

Let us now state a lemma giving some coercivity properties on $\mathcal{A}$:

Lemma 6.3 (Coercivity properties of $\partial_x\mathcal{A}$ and $\mathcal{A}\partial_x$). —

- *Estimate for $\mathcal{A}\partial_x$: let $\varphi \in H^2(\mathbb{R}_+)$. Then*

$$\int_0^\infty (\mathcal{A}\partial_x\varphi)\,\varphi = \mu\int_0^\infty \frac{(\partial_x\varphi)^2}{\bar{v}} + \frac{s}{2}(\varphi(0))^2 + \mu\varphi'(0)\varphi(0)$$
$$\geqslant \frac{\mu}{v_+}\int_0^\infty (\partial_x\varphi)^2 + \frac{s}{2}(\varphi(0))^2 + \mu\varphi'(0)\varphi(0).$$

- *Estimate for $\partial_x\mathcal{A}$: let $\varphi \in H^2(\mathbb{R}_+)$. Then, for any weight function $\rho \in \mathcal{C}_b^2(\mathbb{R}_+)$ such that $\rho > 0$ a.e.*

$$\int_0^\infty (\partial_x\mathcal{A}\varphi)\frac{\varphi}{\bar{v}}\rho \geqslant \mu\int_0^\infty \left(\partial_x\frac{\varphi}{\bar{v}}\right)^2\rho - C\|\rho\|_{W^{2,\infty}}\int_0^\infty \varphi^2$$
$$+ \varphi^2(0)\left(\frac{s}{2}\rho(0) - \frac{\mu}{2}\rho'(0)\right) + \mu\partial_x\left(\frac{\varphi}{\bar{v}}\right)_{|x=0}\varphi(0)\rho(0).$$

The proof of Lemma 6.3 is straightforward and provided in Appendix D for the reader's convenience.

We are now ready to tackle the estimates on g_1. In the following, we denote by C any constant depending only on the parameters of the problem (i.e. $v_+, u_\pm, s, \mu$).

$\triangleright L^\infty(L^2)\cap L^2(H^1)$ *estimate:*

We multiply equation (6.7) by g_1. Using the first point in Lemma 6.3 together with the observation (6.11), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^\infty g_1^2 + \frac{\mu}{v_+}\int_0^\infty (\partial_x g_1)^2 \leqslant \left|\mu\beta\int_0^\infty \partial_x\left(\frac{\partial_x\bar{v}}{\bar{v}^2}g\right)g_1\right| + \mu\left|\int_0^\infty \mathcal{A}\partial_x^2\left(\ln\left(1+\frac{g}{\bar{v}}\right) - \frac{g}{\bar{v}}\right)g_1\right|. \tag{6.14}$$

We now evaluate separately the two terms in the right-hand side.

- Integrating by parts the first term, we obtain

$$\left|\int_0^\infty \partial_x\left(\frac{\partial_x\bar{v}}{\bar{v}^2}g\right)g_1\right| \leqslant C\|\partial_x g_1\|_{L^2(\mathbb{R}_+)}\|g\|_{L^2(\mathbb{R}_+)}.$$

  Recalling that

$$\|g\|^2_{L^\infty([0,\overline{T}],L^2(\mathbb{R}_+))} \leqslant \mathcal{E}_0 + C\|\beta\|^2_{L^2(0,\overline{T})},$$

  we obtain

$$\left|\mu\beta\int_0^\infty \partial_x\left(\frac{\partial_x\bar{v}}{\bar{v}^2}g\right)g_1\right| \leqslant \frac{\mu}{4v_+}\|\partial_x g_1\|^2_{L^2(\mathbb{R}_+)} + C\left(\mathcal{E}_0 + \|\beta\|^2_{L^2(0,\overline{T})}\right)|\beta|^2.$$

- Using the definition of the differential operator $\mathcal{A}$ and integrating by parts, we have

$$\left|\int_0^\infty \mathcal{A}\partial_x^2\left(\ln\left(1+\frac{g}{\bar{v}}\right) - \frac{g}{\bar{v}}\right)g_1\right|$$
$$\leqslant C\left|\int_0^\infty \partial_x\left(\ln\left(1+\frac{g}{\bar{v}}\right) - \frac{g}{\bar{v}}\right)\partial_x g_1\right| + C\left|\int_0^\infty \partial_x^2\left(\ln\left(1+\frac{g}{\bar{v}}\right) - \frac{g}{\bar{v}}\right)\partial_x g_1\right|.$$

We now consider the nonlinear term. We have

$$\partial_x\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right)=-\partial_x\left(\frac{g}{\bar{v}}\right)\frac{\frac{g}{\bar{v}}}{1+\frac{g}{\bar{v}}}. \tag{6.15}$$

Consequently, recalling that $\bar{v}+g=v\geqslant 1$ and using the fact that $H^1(\mathbb{R}_+)$ is an algebra, we have

$$\begin{aligned}\left|\int_0^\infty \mathcal{A}\partial_x^2\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right)g_1\right| &\leqslant C\|\partial_x g_1\|_{L^2(\mathbb{R}_+)}\left\|\partial_x\left(\frac{g}{\bar{v}}\right)\frac{g}{v}\right\|_{H^1(\mathbb{R}_+)}\\ &\leqslant C\|\partial_x g_1\|_{L^2(\mathbb{R}_+)}\left\|\partial_x\left(\frac{g}{\bar{v}}\right)\right\|_{H^1(\mathbb{R}_+)}\|g\|_{H^1(\mathbb{R}_+)}\\ &\leqslant C\|\partial_x g_1\|_{L^2(\mathbb{R}_+)}\|g\|_{L^\infty\left([0,\overline{T}],H^2(\mathbb{R}_+)\right)}\|g\|_{H^1(\mathbb{R}_+)}\\ &\leqslant C\left(\mathcal{E}_0^{1/2}+\|\beta\|_{H^1\left(0,\overline{T}\right)}\right)\|g\|_{H^1(\mathbb{R}_+)}\|\partial_x g_1\|_{L^2(\mathbb{R}_+)}.\end{aligned}$$

We obtain eventually

$$\begin{aligned}&\left|\mu\int_0^\infty \mathcal{A}\partial_x^2\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right)g_1\right|\\ &\qquad\leqslant \frac{\mu}{4v_+}\|\partial_x g_1\|^2_{L^2(\mathbb{R}_+)}+C\left(\mathcal{E}_0+\|\beta\|^2_{H^1\left(0,\overline{T}\right)}\right)\|g\|^2_{H^1(\mathbb{R}_+)}.\end{aligned}$$

Gathering all the terms, we are led to

$$\frac{d}{dt}\int_0^\infty g_1^2+\frac{\mu}{v_+}\int_0^\infty(\partial_x g_1)^2\leqslant C\left(\mathcal{E}_0+\|\beta\|^2_{H^1\left(0,\overline{T}\right)}\right)\left(|\beta|^2+\|g\|^2_{H^1(\mathbb{R}_+)}\right).$$

Integrating in time and using the bound

$$\|g\|_{L^2\left([0,\overline{T}],H^2(\mathbb{R}_+)\right)}\leqslant C\left(\mathcal{E}_0^{1/2}+\|\beta\|_{H^1\left(0,\overline{T}\right)}\right),$$

we get

$$\|g_1\|^2_{L^\infty\left([0,\overline{T}],L^2(\mathbb{R}_+)\right)}+\frac{\mu}{v_+}\|\partial_x g_1\|^2_{L^2\left([0,\overline{T}],L^2(\mathbb{R}_+)\right)}\leqslant \mathcal{E}_0+C\left(\mathcal{E}_0+\|\beta\|^2_{H^1\left(0,\overline{T}\right)}\right)^2. \tag{6.16}$$

$\triangleright L^\infty(H^1)\cap L^2(H^2)$ *estimate:* Differentiating (6.7) with respect to x, we obtain, using (6.11)

$$\partial_t\partial_x g_1+\partial_x\mathcal{A}\partial_x g_1=\beta\partial_x^2 g_1-\mu\beta\partial_x^2\left(\frac{\partial_x\bar{v}}{\bar{v}^2}g\right)+\mu\partial_x\mathcal{A}\partial_x^2\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right). \tag{6.17}$$

We also recall (6.10) and (6.12):

$$\partial_x g_{1|x=0}=\beta\frac{s(v_+-1)}{\mu},\quad \mu\partial_x\left(\frac{\partial_x g_1}{\bar{v}}\right)_{|x=0}=-s\partial_x g_{1|x=0}-\beta\partial_x g_{1|x=0}.$$

We multiply (6.17) by $\partial_x g_1\rho/\bar{v}$, where $\rho\in\mathcal{C}_b^2(\mathbb{R}_+)$ is a weight function that will be chosen momentarily, and such that $\rho\geqslant 1$. Using the second part of Lemma 6.3 and replacing $\mu\partial_x(\frac{\partial_x g_1}{\bar{v}})_{|x=0}$, we have

$$\int_0^\infty (\partial_x \mathcal{A}\partial_x g_1)\ \partial_x g_1\ \frac{\rho}{\bar{v}} \geqslant \mu \int_0^\infty \left(\partial_x \left(\frac{\partial_x g_1}{\bar{v}}\right)\right)^2 \rho - C\|\rho\|_{W^{2,\infty}} \int_0^\infty (\partial_x g_1)^2 + \left(\partial_x g_{1|x=0}\right)^2 \left(-\frac{s}{2}\rho(0) - \frac{\mu}{2}\rho'(0) - \beta\rho(0)\right).$$

We now choose the function ρ so that the last term is positive at main order. More precisely, we take

$$\rho(0) = 2, \quad \rho'(0) = -\frac{4s}{\mu}, \quad 1 \leqslant \rho(x) \leqslant 2 \quad \forall\, x \in \mathbb{R}_+.$$

Then

$$\begin{aligned}(6.18)\quad & \int_0^\infty (\partial_x \mathcal{A}\partial_x g_1)\ \partial_x g_1\ \frac{\rho}{\bar{v}} \\ & \geqslant \mu \int_0^\infty \left(\partial_x \frac{\partial_x g_1}{\bar{v}}\right)^2 + \beta^2 \frac{s^3(v_+ - 1)^2}{\mu^2} - C\|\rho\|_{W^{2,\infty}} \int_0^\infty (\partial_x g_1)^2 - C|\beta|^3.\end{aligned}$$

Note that the term $\|\partial_x g_1\|_{L^2(\mathbb{R}_+)}$ is controlled thanks to (6.16).

We now address the terms in the right-hand side of (6.17).

- The first term is easily bounded thanks to the L^2L^2 bound (6.16) on $\partial_x g_1$. We have
$$\int_0^\infty \partial_x^2 g_1 \partial_x g_1 \frac{\rho}{\bar{v}} = -\frac{1}{2}\int_0^\infty (\partial_x g_1)^2 \partial_x \left(\frac{\rho}{\bar{v}}\right) - \left(\partial_x g_{1|x=0}\right)^2,$$
so that
$$\left|\beta \int_0^\infty \partial_x^2 g_1 \partial_x g_1 \frac{\rho}{\bar{v}}\right| \leqslant C|\beta|\|\partial_x g_1\|^2_{L^2(\mathbb{R}_+)} + C|\beta|^3.$$
- For the second term in the right-hand side, we use the bounds on g. We have
$$\left|\mu\beta \int_0^\infty \partial_x^2 \left(\frac{\partial_x \bar{v}}{\bar{v}^2} g\right) \partial_x g_1 \frac{\rho}{\bar{v}}\right| \leqslant C|\beta|\|\partial_x g_1\|_{L^2(\mathbb{R}_+)}\|g\|_{H^2(\mathbb{R}_+)}.$$
- Let us now address the nonlinear term, which is quadratic in g. Integrating by parts and recalling that $\mathcal{A}\partial_x^2(\ln(1 + g/\bar{v}) - g/\bar{v})$ vanishes at $x = 0$, we have
$$\begin{aligned}\int_0^\infty \partial_x \mathcal{A}\partial_x^2 \left(\ln\left(1 + \frac{g}{\bar{v}}\right) - \frac{g}{\bar{v}}\right) \partial_x g_1 \frac{\rho}{\bar{v}} & \\ = -\int_0^\infty \mathcal{A}\partial_x^2 \left(\ln\left(1 + \frac{g}{\bar{v}}\right) - \frac{g}{\bar{v}}\right) \left(\partial_x \frac{\partial_x g_1}{\bar{v}}\right) \rho & \\ - \int_0^\infty \mathcal{A}\partial_x^2 \left(\ln\left(1 + \frac{g}{\bar{v}}\right) - \frac{g}{\bar{v}}\right) \left(\frac{\partial_x g_1}{\bar{v}}\right) \partial_x \rho. &\end{aligned}$$
Furthermore, by definition of the operator $\mathcal{A}$,
$$\left\|\mathcal{A}\partial_x^2 \left(\ln\left(1 + \frac{g}{\bar{v}}\right) - \frac{g}{\bar{v}}\right)\right\|_{L^2(\mathbb{R}_+)} \leqslant \left\|\partial_x \left(\ln\left(1 + \frac{g}{\bar{v}}\right) - \frac{g}{\bar{v}}\right)\right\|_{H^2(\mathbb{R}_+)}.$$

Remembering (6.15) and using the fact that $H^2(\mathbb{R}_+)$ is an algebra, we obtain

$$\begin{aligned}\left\|\mathcal{A}\partial_x^2\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right)\right\|_{L^2(\mathbb{R}_+)} &\leqslant \left\|\partial_x\left(\frac{g}{\bar{v}}\right)\frac{g}{v}\right\|_{H^2(\mathbb{R}_+)}\\ &\leqslant C\|\partial_x g\|_{H^2(\mathbb{R}_+)}\|g\|_{H^2(\mathbb{R}_+)},\end{aligned}$$

where we have used the fact that $\|g\|_{L^\infty([0,\overline{T}],W^{1,\infty}(\mathbb{R}_+))}$ remains small.

As a consequence,

$$\begin{aligned}(6.19)\quad &\left|\mu\int_0^\infty \partial_x\mathcal{A}\partial_x^2\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right)\partial_x g_1\frac{\rho}{\bar{v}}\right|\\ &\leqslant C\left(\left\|\partial_x\frac{\partial_x g_1}{\bar{v}}\right\|_{L^2(\mathbb{R}_+)}+\|\partial_x g_1\|_{L^2(\mathbb{R}_+)}\right)\times\|g\|_{L^\infty\left([0,\overline{T}],H^2(\mathbb{R}_+)\right)}\|\partial_x g\|_{H^2(\mathbb{R}_+)}.\end{aligned}$$

Gathering all the terms and using several times the Cauchy–Schwarz inequality, we obtain, with $c=s^3(v_+-1)^2/\mu^2$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^\infty(\partial_x g_1)^2\frac{\rho}{\bar{v}}+\frac{\mu}{2}\int_0^\infty\left(\partial_x\frac{\partial_x g_1}{\bar{v}}\right)^2+c\beta^2\\ &\leqslant C\left(1+\|\beta\|_{L^\infty(0,\overline{T})}\right)\int_0^\infty(\partial_x g_1)^2+C|\beta|^3+C\left(|\beta|^2+\|\partial_x g\|_{H^2}\right)\|g\|^2_{L^\infty\left([0,\overline{T}],H^2(\mathbb{R}_+)\right)}.\end{aligned}$$

Recall that $\|\beta\|_{L^\infty(0,\overline{T})}\lesssim\|\beta\|_{H^1(0,\overline{T})}\leqslant 1$ for δ small enough. Integrating with respect to time and using (6.16), we get

$$\begin{aligned}(6.20)\quad &\|\partial_x g_1\|^2_{L^\infty\left([0,\overline{T}],L^2(\mathbb{R}_+)\right)}+\left\|\partial_x\frac{\partial_x g_1}{\bar{v}}\right\|^2_{L^2\left([0,\overline{T}]\times\mathbb{R}_+\right)}+\|\beta\|^2_{L^2\left([0,\overline{T}]\right)}\\ &\qquad\leqslant C\mathcal{E}_0+C\left(\mathcal{E}_0+\|\beta\|^2_{H^1\left(0,\overline{T}\right)}\right)^2+C\left(\mathcal{E}_0^{1/2}+\|\beta\|_{H^1\left(0,\overline{T}\right)}\right)\|\beta\|^2_{L^2\left(0,\overline{T}\right)}.\end{aligned}$$

Note additionally that the last term in the right hand side can be absorbed in the left-hand side provided $\mathcal{E}_0^{1/2}+\|\beta\|_{H^1(0,\overline{T})}$ is small enough.

$\triangleright W^{1,\infty}(L^2)\cap H^1(H^1)$ *estimate:* Let us now consider the time derivative of g_1 satisfying

$$(6.21)\quad\begin{cases}\partial_t(\partial_t g_1)+\mathcal{A}\partial_x\partial_t g_1=\beta\mathcal{A}\partial_x\partial_t g+\mu\mathcal{A}\partial_x^2\partial_t\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right)+\beta'\mathcal{A}\partial_x g,\\ (\partial_t g_1)_{|x=0}=0.\end{cases}$$

Recall that we have by (6.11):

$$\mathcal{A}\partial_x\partial_t g=\partial_x\partial_t g_1-\mu\partial_x\partial_t\left(\frac{\partial_x\bar{v}}{\bar{v}^2}g\right).$$

Multiplying Equation (6.21) by $\partial_t g_1$ and using once again Lemma 6.3, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}_+}|\partial_t g_1|^2+\frac{\mu}{v_+}\int_{\mathbb{R}_+}|\partial_x\partial_t g_1|^2 \\ \leqslant \mu|\beta|\left|\int_{\mathbb{R}_+}\partial_x\partial_t\left(\frac{\partial_x\bar{v}}{\bar{v}^2}g\right)\partial_t g_1\right|+\mu\left|\int_{\mathbb{R}_+}\mathcal{A}\partial_x^2\partial_t\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right)\partial_t g_1\right| \\ +|\beta'|\left|\int_{\mathbb{R}_+}\mathcal{A}\partial_x g\partial_t g_1\right|.\end{aligned}$$

Integrating by parts and using the Cauchy–Schwarz inequality together with (6.2), we have for the different terms of the right-hand side:

- First

$$\mu|\beta|\left|\int_{\mathbb{R}_+}\partial_x\partial_t\left(\frac{\partial_x\bar{v}}{\bar{v}^2}g\right)\partial_t g_1\right|\leqslant\frac{\mu}{4v_+}\|\partial_t\partial_x g_1\|^2_{L^2(\mathbb{R}_+)}+C\|\partial_t g\|^2_{L^\infty\left([0,\overline{T}],L^2(\mathbb{R}_+)\right)}|\beta|^2;$$

- Next, for the nonlinear term, we use an analogue of (6.15), which yields

$$\begin{aligned}\mu\left|\int_{\mathbb{R}_+}\mathcal{A}\partial_x^2\partial_t\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right)\partial_t g_1\right| \\ \leqslant C\|\partial_t\partial_x g_1\|_{L^2(\mathbb{R}_+)}\left\|\partial_t\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right)\right\|_{H^2(\mathbb{R}_+)} \\ \leqslant C\|\partial_t\partial_x g_1\|_{L^2(\mathbb{R}_+)}\left\|\partial_t\left(\frac{g}{\bar{v}}\right)\frac{g}{v}\right\|_{H^2(\mathbb{R}_+)} \\ \leqslant C\|\partial_t\partial_x g_1\|_{L^2(\mathbb{R}_+)}\|\partial_t g\|_{H^2(\mathbb{R}_+)}\|g\|_{H^2(\mathbb{R}_+)} \\ \leqslant\frac{\mu}{4v_+}\|\partial_t\partial_x g_1\|^2_{L^2(\mathbb{R}_+)}+C\|\partial_t g\|^2_{H^2(\mathbb{R}_+)}\|g\|^2_{L^\infty\left([0,\overline{T}],H^2(\mathbb{R}_+)\right)}.\end{aligned}$$

- Finally, recalling (6.11)

$$\begin{aligned}|\beta'|\left|\int_{\mathbb{R}_+}\mathcal{A}\partial_x g\partial_t g_1\right| \\ \leqslant|\beta'|\left|\int_{\mathbb{R}_+}\left[g_1-\mu\frac{\partial_x\bar{v}}{\bar{v}^2}g\right]\partial_x\partial_t g_1\right| \\ \leqslant\frac{\mu}{4v_+}\|\partial_t\partial_x g_1\|^2_{L^2(\mathbb{R}_+)}+C|\beta'|^2\Big(\|g_1\|^2_{L^\infty\left([0,\overline{T}],L^2(\mathbb{R}_+)\right)}+\|g\|^2_{L^\infty\left([0,\overline{T}],L^2(\mathbb{R}_+)\right)}\Big).\end{aligned}$$

Eventually, we obtain after time integration of all previous terms and the use of the estimates (6.16) on g_1, (6.2) on g

$$\|\partial_t g_1\|^2_{L^\infty\left([0,\overline{T}],L^2(\mathbb{R}_+)\right)}+\|\partial_t\partial_x g_1\|^2_{L^2\left((0,\overline{T})\times\mathbb{R}_+\right)}\leqslant C\Big[\mathcal{E}_0+\Big(\mathcal{E}_0+\|\beta\|^2_{H^1\left(0,\overline{T}\right)}\Big)^2\Big]. \tag{6.22}$$

From there, using equation (6.7), we infer that

$$\left\|\partial_x^2 g_1\right\|^2_{L^\infty\left([0,\overline{T}],L^2(\mathbb{R}_+)\right)}\leqslant C\Big[\mathcal{E}_0+\Big(\mathcal{E}_0+\|\beta\|^2_{H^1\left(0,\overline{T}\right)}\Big)^2\Big].$$

$\triangleright W^{1,\infty}(H^1)\cap H^1(H^2)$ *estimate:* We differentiate Eq (6.17) with respect to time:

$$\begin{aligned}(6.23)\quad &\partial_t(\partial_t\partial_x g_1)+\partial_x\mathcal{A}\partial_t\partial_x g_1\\ &=\beta\partial_x^2\partial_t g_1+\beta'\partial_x^2 g_1-\mu\beta'\partial_x^2\left(\frac{\partial_x\bar{v}}{\bar{v}^2}g\right)-\mu\beta\partial_x^2\left(\frac{\partial_x\bar{v}}{\bar{v}^2}\partial_t g\right)+\mu\partial_x\mathcal{A}\partial_x^2\partial_t\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right).\end{aligned}$$

This equation is endowed with the following boundary condition, which is to be understood in a weak sense

$$\begin{aligned}\mu\partial_x\left(\frac{\partial_t\partial_x g_1}{\bar{v}}\right)_{|x=0}&=-\beta'(\partial_x g_1)_{|x=0}-(\beta+s)\left(\partial_t\partial_x g_1\right)_{|x=0}\\ &=-\beta'(2\beta+s)s\frac{(v_+-1)}{\mu}.\end{aligned}\tag{6.24}$$

In other words, for any test function $\psi\in L^2((0,T),H^1(\mathbb{R}_+))$, for almost every $t\in[0,T]$ (recall the regularities (6.8)-(6.9) on g_1),

$$\begin{aligned}(6.25)\quad &\langle\partial_t(\partial_t\partial_x g_1)(t),\psi(t)\rangle_{H^{-1},H^1}-\int_0^\infty\mathcal{A}(\partial_t\partial_x g_1)\,\partial_x\psi\\ &\quad+\left(s\partial_t\partial_x g_{1|x=0}-\beta'(2\beta+s)s\frac{(v_+-1)}{\mu}\right)\psi_{|x=0}\\ &=-\int_0^\infty\mu\mathcal{A}\partial_x^2\partial_t\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right)\partial_x\psi(t,x)\\ &\quad+\int_0^\infty\left[\beta\partial_x^2\partial_t g_1+\beta'\partial_x^2 g_1-\mu\beta'\partial_x^2\left(\frac{\partial_x\bar{v}}{\bar{v}^2}g\right)-\mu\beta\partial_x^2\left(\frac{\partial_x\bar{v}}{\bar{v}^2}\partial_t g\right)\right]\psi.\end{aligned}$$

In particular, we have used the fact that

$$\mu\partial_x^3\partial_t\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right)_{|x=0}=0$$

in a weak sense. The weak formulation (6.25) can be obtained by writing the weak formulation of (6.17) associated with $\partial_x g_1$, taking a test function of the form $\partial_t\psi$ with $\psi\in\mathcal{C}^1([0,\overline{T}),H^1(\mathbb{R}_+))$, and integrating by parts in time.

Similarly to the $L^\infty(H^1)$ estimate we take $\psi=\partial_x(\partial_t g_1)\rho/\bar{v}$ in the weak formulation above, with the same weight function ρ satisfying $\rho(0)=2$, $\rho'(0)=-\frac{4s}{\mu}$, $1\leqslant\rho(x)\leqslant 2\ \ \forall\, x\in\mathbb{R}_+$, and we evaluate each term separately. At the boundary $x=0$, we have

$$(\partial_t\partial_x g_1)_{|x=0}=\frac{s(v_+-1)}{\mu}\beta',\tag{6.26}$$

and the equality holds in $L^2(0,T)$.

• For the transport-diffusion term we have, replacing ψ by $\partial_x\partial_t g_1\rho/\bar{v}$ in the left-hand side of (6.25) and using Lemma 6.3,

$$\begin{aligned}
-\int_0^\infty \mathcal{A}(\partial_t\partial_x g_1)\,\partial_x\left(\partial_x\partial_t g_1\frac{\rho}{\bar{v}}\right)& \\
&+\left(s\partial_t\partial_x g_{1|x=0}-\beta'(2\beta+s)s\frac{(v_+-1)}{\mu}\right)\left(\partial_x\partial_t g_1\frac{\rho}{\bar{v}}\right)_{|x=0}\\
&\geqslant \mu\int_{\mathbb{R}_+}\left(\partial_x\left(\frac{\partial_x\partial_t g_1}{\bar{v}}\right)\right)^2\rho-C\|\rho\|_{W^{2,\infty}}\int_{\mathbb{R}_+}\left(\partial_x\partial_t g_1\right)^2\\
&\quad+\left(\partial_t\partial_x g_{1|x=0}\right)^2\left(\frac{s}{2}\rho(0)-\frac{\mu}{2}\rho'(0)\right)\\
&\quad-\beta'(2\beta+s)s\frac{(v_+-1)}{\mu}\left(\partial_t\partial_x g_1\right)_{|x=0}\rho(0),
\end{aligned}$$

so that, with the boundary condition (6.26) and with a computation identical to (6.18), the right-hand side of the above inequality is equal to

$$\begin{aligned}
\mu\int_{\mathbb{R}_+}\left(\partial_x\left(\frac{\partial_x\partial_t g_1}{\bar{v}}\right)\right)^2\rho-C\|\rho\|_{W^{2,\infty}}\int_{\mathbb{R}_+}\left(\partial_x\partial_t g_1\right)^2&\\
+2\frac{s^3(v_+-1)^2}{\mu^2}|\beta'|^2-4\frac{s^2(v_+-1)^2}{\mu^2}\beta|\beta'|^2.&
\end{aligned}$$

Similarly to the $L^\infty(H^1)$ estimate, the last term will be absorbed in the penultimate one when $\|\beta\|_{L^\infty}\leqslant\delta$ is small enough, while the second integral is controlled thanks to the previous $H^1(H^1)$-estimate.

• In the right-hand side of (6.25) with $\psi=\partial_t\partial_x g_1\rho/\bar{v}$, we have to control the following integrals

$$\begin{aligned}
I_1&=\beta\int_{\mathbb{R}_+}\partial_x^2\partial_t g_1\,\partial_t\partial_x g_1\frac{\rho}{\bar{v}}\\
I_2&=\beta'\int_{\mathbb{R}_+}\partial_x^2 g_1\partial_t\partial_x g_1\frac{\rho}{\bar{v}}\\
I_3&=\mu\beta'\int_{\mathbb{R}_+}\partial_x^2\left(\frac{\partial_x\bar{v}}{\bar{v}^2}g\right)\partial_t\partial_x g_1\frac{\rho}{\bar{v}}\\
I_4&=\mu\beta\int_{\mathbb{R}_+}\partial_x^2\left(\frac{\partial_x\bar{v}}{\bar{v}^2}\partial_t g\right)\partial_t\partial_x g_1\frac{\rho}{\bar{v}}\\
I_5&=\mu\int_{\mathbb{R}_+}\mathcal{A}\partial_x^2\partial_t\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right)\partial_x\left(\partial_t\partial_x g_1\frac{\rho}{\bar{v}}\right).
\end{aligned}$$

(1) The first integral is controlled via an integration by parts:

$$\begin{aligned}
|I_1|&\leqslant\frac{|\beta|}{2}\int_{\mathbb{R}_+}(\partial_t\partial_x g_1)^2\partial_x\left(\frac{\rho}{\bar{v}}\right)+\beta\left(\partial_t\partial_x g_1\right)^2_{|x=0}\\
&\leqslant C|\beta|\|\partial_t\partial_x g_1\|^2_{L^2(\mathbb{R}_+)}+C|\beta'|^2\beta.
\end{aligned}$$

(2) For I_2, I_3, I_4, the Cauchy-Schwarz inequality yields

$$\begin{aligned}|I_2| + |I_3| + |I_4| \leqslant \|\partial_x\partial_t g_1\|^2_{L^2(\mathbb{R}_+)} &\\ &+ C|\beta'|^2\Big(\left\|\partial_x^2 g_1\right\|^2_{L^\infty\left([0,\overline{T}],L^2(\mathbb{R}_+)\right)} + \|g\|^2_{L^\infty\left([0,\overline{T}],H^2(\mathbb{R}_+)\right)}\Big)\\ &+ \|\beta\|^2_{L^\infty\left(0,\overline{T}\right)}\|\partial_t g(t)\|^2_{H^2(\mathbb{R}_+)}\,.\end{aligned}$$

(3) For the nonlinear term I_5, we have similarly to (6.19):

$$\begin{aligned}&\left|\mu\int_0^\infty \mathcal{A}\partial_x^2\partial_t\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right)\partial_x\left(\partial_t\partial_x g_1\frac{\rho}{\bar{v}}\right)\right|\\ &\quad\leqslant C\left(\left\|\partial_x\frac{\partial_x\partial_t g_1}{\bar{v}}\right\|_{L^2(\mathbb{R}_+)} + \|\partial_t\partial_x g_1\|_{L^2(\mathbb{R}_+)}\right)\left\|\frac{\partial_t g}{\bar{v}}\right\|_{H^3(\mathbb{R}_+)}\left\|\frac{g}{v}\right\|_{H^3(\mathbb{R}_+)}\\ &\quad\leqslant C\left(\left\|\partial_x\frac{\partial_x\partial_t g_1}{\bar{v}}\right\|_{L^2(\mathbb{R}_+)} + \|\partial_t\partial_x g_1\|_{L^2(\mathbb{R}_+)}\right)\|\partial_t g\|_{H^3(\mathbb{R}_+)}\|g\|_{L^\infty\left([0,\overline{T}],H^3(\mathbb{R}_+)\right)}.\end{aligned}$$

To control $\|\partial_t\partial_x g\|_{H^2(\mathbb{R}_+)}$, we write in the case $\boldsymbol{w}^0 \equiv u_+$:

$$\mu\partial_t\partial_x^3\ln\left(1+\frac{g}{\bar{v}}\right) = \mu\partial_t\partial_x^3\ln\frac{v}{\bar{v}} = \partial_t\partial_x^2\left(u-\bar{u}\right),$$

so that $\|\partial_t\partial_x g\|_{H^2(\mathbb{R}_+)} \leqslant C(\|\partial_t h\|_{H^2(\mathbb{R}_+)} + \|\partial_t g\|_{H^2(\mathbb{R}_+)})$, where we recall that $h = u - \bar{u}$.

Gathering all terms and integrating in time, we get, using (6.2), (6.3) and (6.16)

$$\begin{aligned}\|\partial_t\partial_x g_1\|^2_{L^\infty\left([0,\overline{T}],L^2(\mathbb{R}_+)\right)} + \left\|\partial_x\frac{\partial_x\partial_t g_1}{\bar{v}}\right\|^2_{L^2\left([0,\overline{T}]\times\mathbb{R}_+\right)} + \|\beta'\|^2_{L^2\left([0,\overline{T}]\right)}&\\ \leqslant C\mathcal{E}_0 + C\left(\mathcal{E}_0 + \|\beta\|^2_{H^1\left(0,\overline{T}\right)}\right)^2 + C\left(\mathcal{E}_0^{1/2} + \|\beta\|_{H^1\left(0,\overline{T}\right)}\right)\|\beta'\|^2_{L^2\left(0,\overline{T}\right)}.&\end{aligned} \tag{6.27}$$

Observing that the last term in the right-hand side can be absorbed in the left-hand side provided $\mathcal{E}_0^{1/2} + \|\beta\|_{H^1(0,\overline{T})}$ is small enough, we obtain eventually

$$\begin{aligned}\|\partial_t\partial_x g_1\|^2_{L^\infty\left([0,\overline{T}],L^2\right)} + \left\|\partial_x\left(\frac{\partial_t\partial_x g_1}{\bar{v}}\right)\right\|^2_{L^2\left([0,\overline{T}]\times\mathbb{R}_+\right)} + \|\beta'\|^2_{L^2\left([0,\overline{T}]\right)}&\\ \leqslant C\left[\mathcal{E}_0 + \left(\mathcal{E}_0 + \|\beta\|^2_{H^1(0,\overline{T})}\right)^2\right].&\end{aligned} \tag{6.28}$$

▷ *Conclusion:* Combining (6.20) and (6.28), and recalling that $\|\beta\|_{H^1([0,\overline{T}]} \leqslant \delta$ be definition of $\overline{T}$, we deduce that for $\mathcal{E}_0 \leqslant c_0\delta^2$ with $c_0 > 0$ small enough (depending only on the parameters $\mu, v_+, u_\pm$),

$$\|\beta\|^2_{H^1\left([0,\overline{T}]\right)} \leqslant C\left(\mathcal{E}_0 + \left(\mathcal{E}_0 + \|\beta\|^2_{H^1\left(0,\overline{T}\right)}\right)^2\right) \leqslant Cc_0\delta^2 + C\delta^4 \leqslant \frac{\delta^2}{4}.$$

This completes the proof of Proposition 6.1 in the case $\boldsymbol{w}^0 \equiv u_+$. Note that in this case $\partial_x^k \boldsymbol{w}^0 = 0$ for $k = 1, 2, 3$. Hence the additional conditions on $\boldsymbol{w}^0$ are automatically satisfied.

6.2. Case when w^0 is arbitrary

In the case when $\boldsymbol{w}^0$ is not constant, the estimates of the previous section involve new contributions coming both from the equations, which contain additional terms linked to $\partial_x \boldsymbol{w}^0$, and from the boundary $x = 0$, since $(\partial_x g)_{|x=0}$ (and thus $(g_1)_{|x=0}$) is not equal to 0 anymore. More precisely,

$$\begin{aligned} \partial_t g_1 + \mathcal{A}\partial_x g_1 &= \beta \mathcal{A}\partial_x g + \mu \mathcal{A}\partial_x^2 \left(\ln\left(1 + \frac{g}{\bar{v}}\right) - \frac{g}{\bar{v}} \right) + \mathcal{A}\partial_x \boldsymbol{w}^0(x + \widetilde{x}(t)), \\ g_{1|t=0} &= \mathcal{A}\left(v^0 - \bar{v}\right), \end{aligned} \tag{6.29}$$

with at the boundary $x = 0$

$$(g_1)_{|x=0} = -s\underbrace{g_{|x=0}}_{=0} - \mu \left(\partial_x \left(\frac{g}{\bar{v}} \right) \right)_{|x=0} = -\mu(\partial_x(v - \bar{v}))_{|x=0} = \boldsymbol{w}^0(\widetilde{x}) - u_+ \tag{6.30}$$

since $u(t,x) - \mu\partial_x \ln v(t,x) = \boldsymbol{w}^0(x + \widetilde{x}(t))$ for all t, x.

Nevertheless, we will show that these new contributions are all controlled provided $\|\boldsymbol{w}^0 - u_+\|_{L^\infty(\mathbb{R}_+)}$ and $\|(1+\sqrt{x})\partial_x^k \boldsymbol{w}^0\|_{L^2(\mathbb{R}_+)}$, $k = 1, 2, 3$, are small. We recall that using the definition of g_1 together with the estimates (6.3) and with the equation on u, we have the regularities (6.8)-(6.9):

$$g_1 \in L^2\left((0,\overline{T}), H^3(\mathbb{R}_+)\right), \quad \partial_t g_1 \in L^2\left((0,\overline{T}), H^2(\mathbb{R}_+)\right), \quad \partial_t^2 g_1 \in L^2\left((0,\overline{T}) \times \mathbb{R}_+\right).$$

As a consequence,

$$\partial_x^k g_{1|x=0},\ \partial_t \partial_x^l g_{1|x=0} \in L^2(0,\overline{T})$$

for $0 \leqslant k \leqslant 3$ and $0 \leqslant l \leqslant 1$.

6.2.1. Estimates on the traces

Let us detail the different traces associated to g_1 that will be involved in the estimates and how they are controlled in $L^2([0,\overline{T}])$.

Lemma 6.4 (Trace estimates). — *Assume that* $\|\boldsymbol{w}^0 - u_+\|_{W^{1,\infty}(\mathbb{R}_+)} \leqslant 1$. *Then*

(1) *Estimates on* $g_{1|x=0}$*:*

$$\|(g_1)_{|x=0}\|^2_{L^2(0,\overline{T})} + \|(\partial_t g_1)_{|x=0}\|^2_{L^2(0,\overline{T})} \leqslant C\left\|\boldsymbol{w}^0 - u_+\right\|^2_{H^1(\mathbb{R}_+)} \leqslant C\mathcal{E}_0; \tag{6.31}$$

(2) *Estimates on* $\partial_x g_{1|x=0}$*:*

$$\partial_x g_{1|x=0} = \beta \frac{s(v_+ - 1)}{\mu} + R_1, \tag{6.32}$$

where

$$\begin{aligned}\|R_1\|_{H^1(0,\overline{T})} &\leqslant C\|\beta\|_{H^1(0,\overline{T})}\left\|\boldsymbol{w}^0-u_+\right\|_{W^{1,\infty}(\mathbb{R}_+)}\\ &\quad + C\left(\left\|\boldsymbol{w}^0-u_+\right\|^2_{W^{1,4}(\mathbb{R}_+)}+\left\|\partial_x\boldsymbol{w}^0\right\|_{H^1(\mathbb{R}_+)}\right)\\ &\leqslant C\mathcal{E}_0^{1/2}\left(1+\|\beta\|_{H^1([0,\overline{T}])}\right)\leqslant C\mathcal{E}_0^{1/2};\end{aligned}$$

(3) *Estimates on the traces of the nonlinear term:*

$$\partial_x^k\left(\ln\left(1+\frac{g}{\overline{v}}\right)-\frac{g}{\overline{v}}\right)_{|x=0}=T_k,\quad k=1,2,3,$$

with

$$\begin{aligned}&T_1=\left[\partial_x\left(\frac{g}{\overline{v}}\right)\frac{g}{\overline{v}}\right]_{|x=0}=0,\\ &\|T_2\|_{H^1(0,\overline{T})}\leqslant C\left\|\boldsymbol{w}^0-u_+\right\|^2_{W^{1,4}(\mathbb{R}_+)}\leqslant C\mathcal{E}_0,\\ &\|T_3\|_{H^1(0,\overline{T})}\leqslant C\left(\|\beta\|_{H^1(0,\overline{T})}\left\|\boldsymbol{w}^0-u_+\right\|_{W^{1,\infty}(\mathbb{R}_+)}+\left\|\boldsymbol{w}^0-u_+\right\|^2_{W^{2,4}(\mathbb{R}_+)}\right)\\ &\qquad\leqslant C\left(\mathcal{E}_0+\mathcal{E}_0^{1/2}\|\beta\|_{H^1([0,\overline{T}])}\right);\end{aligned}\tag{6.33}$$

(4) *Estimates on* $\partial_x^2 g_{1|x=0}$*:*

$$\mu\left[\partial_x\left(\frac{\partial_x g_1}{\overline{v}}\right)\right]_{|x=0}=-(s+\beta)\left(\partial_x g_1\right)_{|x=0}-R_2,\tag{6.34}$$

where

$$\begin{aligned}\|R_2\|_{H^1(0,\overline{T})} &\leqslant C\|\beta\|_{H^1(0,\overline{T})}\left\|\boldsymbol{w}^0-u_+\right\|_{W^{1,\infty}(\mathbb{R}_+)}\\ &\quad + C\left(\left\|\boldsymbol{w}^0-u_+\right\|^2_{W^{2,4}(\mathbb{R}_+)}+\left\|\partial_x\boldsymbol{w}^0\right\|_{H^2(\mathbb{R}_+)}\right)\\ &\leqslant C\mathcal{E}_0^{1/2}.\end{aligned}\tag{6.35}$$

Proof. — Throughout the proof, we recall that for all $t\in[0,\overline{T}]$, $\widetilde{x}'$ satisfies (6.1), so that

$$\frac{st}{2}\leqslant\widetilde{x}(t)\leqslant\frac{3st}{2}.$$

(1) First, the estimate on $g_{1|x=0}$ follows directly from (6.30), writing

$$\begin{aligned}\|g_{1|x=0}\|^2_{L^2([0,\overline{T}])}&=\int_0^{\overline{T}}\frac{1}{\widetilde{x}'(t)}\left|\boldsymbol{w}^0(\widetilde{x}(t))-u_+\right|^2\widetilde{x}'(t)\,dt\\ &\leqslant\frac{2}{s}\int_0^{3\overline{T}/2}\left|\boldsymbol{w}^0(x)-u_+\right|^2\,dx\leqslant C\left\|\boldsymbol{w}^0-u_+\right\|^2_{L^2(\mathbb{R}_+)}.\end{aligned}$$

For the time derivative, we write

$$\partial_t g_{1|x=0}=\widetilde{x}'\partial_x\boldsymbol{w}^0(\widetilde{x})$$

and integrate in time.

(2) Estimates on $\partial_x g_{1|x=0}$: First, we have by (6.30)

$$\begin{aligned}\partial_x^2\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right)_{|x=0} &= -\left(\partial_x^2\left(\frac{g}{\bar{v}}\right)\frac{g/\bar{v}}{1+g/\bar{v}}\right)_{|x=0}\\ &\quad+\left(\partial_x\left(\frac{g}{\bar{v}}\right)\right)^2_{|x=0}\left(\frac{1}{(1+g/\bar{v})^2}\right)_{|x=0}\\ &= \frac{1}{\mu^2}\left(\boldsymbol{w}^0(\widetilde{x})-u_+\right)^2 =: T_2\end{aligned} \tag{6.36}$$

Note that the estimate of (6.33) with $k=2$ follows.

Next, taking the trace of the equation satisfied by g (6.6) (with the additional $\partial_x \boldsymbol{w}^0$ in the right-hand side), we infer that

$$\begin{aligned}(\partial_x g_1)_{|x=0} &= \beta(\partial_x v)_{|x=0}+\mu\partial_x^2\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right)_{|x=0}+\partial_x\boldsymbol{w}^0(\widetilde{x})\\ &= \frac{u_- - \boldsymbol{w}^0(\widetilde{x})}{\mu}\beta+\mu T_2+\partial_x\boldsymbol{w}^0(\widetilde{x})\\ &= \beta\frac{s(v_+-1)}{\mu}+R_1,\end{aligned} \tag{6.37}$$

with

$$R_1 := \beta\frac{u_+-\boldsymbol{w}^0(\widetilde{x})}{\mu}+\mu T_2+\partial_x\boldsymbol{w}^0(\widetilde{x}).$$

We then bound R_1 in $H^1(0,T)$. We have

$$\begin{aligned}\|R_1\|_{H^1(0,\overline{T})} \leqslant C\Big(&\|\beta\|_{H^1(0,\overline{T})}\left\|\boldsymbol{w}^0-u_+\right\|_{L^\infty(\mathbb{R}_+)}+\|\beta\|_{L^2(0,\overline{T})}\left\|\partial_x\boldsymbol{w}^0\right\|_{L^\infty(\mathbb{R}_+)}\\ &+\|T_2\|_{H^1(0,\overline{T})}+\left\|\partial_x\boldsymbol{w}^0\right\|_{H^1(\mathbb{R}_+)}\Big).\end{aligned}$$

Recalling that $\|\beta\|_{H^1(0,\overline{T})}\leqslant\delta\leqslant 1$, we obtain the desired estimate.

(3) Estimate of T_3: we have

$$\begin{aligned}&\partial_x^3\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right)\\ &\quad= -\partial_x^3\left(\frac{g}{\bar{v}}\right)\frac{\frac{g}{\bar{v}}}{1+\frac{g}{\bar{v}}}-\frac{3}{(1+\frac{g}{\bar{v}})^2}\partial_x\left(\frac{g}{\bar{v}}\right)\partial_x^2\left(\frac{g}{\bar{v}}\right)+\frac{2}{(1+\frac{g}{\bar{v}})^3}\left(\partial_x\left(\frac{g}{\bar{v}}\right)\right)^3.\end{aligned}$$

Recalling the definition of $\mathcal{A}$, we have

$$\begin{aligned}\mu\partial_x^2\left(\frac{g}{\bar{v}}\right)_{|x=0} &= -s\partial_x g_{|x=0}-\partial_x g_{1|x=0}\\ &= \frac{s}{\mu}(\boldsymbol{w}^0(\widetilde{x})-u_+)-\beta\frac{s(v_+-1)}{\mu}-R_1.\end{aligned}$$

It follows that

$$\begin{aligned}
T_3 &= -3\left[\partial_x^2\left(\frac{g}{\bar{v}}\right)\right]_{|x=0}\left[\partial_x\left(\frac{g}{\bar{v}}\right)\right]_{|x=0} + 2\left[\partial_x\left(\frac{g}{\bar{v}}\right)\right]^3_{|x=0} \\
&= -3\beta\frac{s(v_+-1)}{\mu^3}\left(\boldsymbol{w}^0(\widetilde{x})-u_+\right) + 3(\beta+s)\frac{\left(\boldsymbol{w}^0(\widetilde{x})-u_+\right)^2}{\mu^3} \\
&\quad -\frac{3}{\mu^2}\partial_x\boldsymbol{w}^0(\widetilde{x})\left(\boldsymbol{w}^0(\widetilde{x})-u_+\right) + \frac{\left(\boldsymbol{w}^0(\widetilde{x})-u_+\right)^3}{\mu^3}.
\end{aligned}$$

Note that $(\boldsymbol{w}^0(\widetilde{x})-u_+)^3$ can be considered as negligible compared to $(\boldsymbol{w}^0(\widetilde{x})-u_+)^2$ since we assume that $\|\boldsymbol{w}^0-u_+\|_{W^{1,\infty}(\mathbb{R}_+)} \leqslant 1$. Hence, taking the H^1 norm of the right-hand side, we obtain

$$\|T_3\|_{H^1(0,\overline{T})} \leqslant C\left(\|\beta\|_{H^1(0,\overline{T})}\left\|\boldsymbol{w}^0-u_+\right\|_{W^{1,\infty}(\mathbb{R}_+)} + \left\|\boldsymbol{w}^0-u_+\right\|^2_{W^{2,4}(\mathbb{R}_+)}\right).$$

(4) Traces of $\partial_x^2 g_1$:
We have

$$\mu\left(\partial_x\frac{\partial_x g_1}{\bar{v}}\right)_{|x=0} = -s(\partial_x g_1)_{|x=0} - (\mathcal{A}\partial_x g_1)_{|x=0}$$

where, thanks to (6.29)-(6.30) and (6.11) (recalling also that $\partial_x\bar{v}_{|x=0} = \mu^{-1}s(v_+-1)$):

$$\begin{aligned}
(\mathcal{A}\partial_x g_1)_{|x=0} &= \beta\,(\mathcal{A}\partial_x g)_{|x=0} + \mu\left[\mathcal{A}\partial_x^2\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right)\right]_{|x=0} \\
&\quad + \left(\mathcal{A}\partial_x\boldsymbol{w}^0(x+\widetilde{x})\right)_{|x=0} - (\partial_t g_1)_{|x=0} \\
&= \beta\,(\partial_x g_1)_{|x=0} - \mu\beta\left[\partial_x\left(\frac{\partial_x\bar{v}}{\bar{v}^2}g\right)\right]_{|x=0} + \mu\left[\mathcal{A}\partial_x^2\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right)\right]_{|x=0} \\
&\quad + s(v_+-2)\partial_x\boldsymbol{w}^0(\widetilde{x}) - \mu\partial_x^2\boldsymbol{w}^0(\widetilde{x}) - \widetilde{x}'(t)\partial_x\boldsymbol{w}^0(\widetilde{x}).
\end{aligned}$$

Hence, we deduce that

$$\mu\left(\partial_x\frac{\partial_x g_1}{\bar{v}}\right)_{|x=0} = -(s+\beta)\,(\partial_x g_1)_{|x=0} - R_2, \tag{6.38}$$

with, compared to (6.12), a remainder:

$$\begin{aligned}
R_2 = -\mu\beta\left[\partial_x\left(\frac{\partial_x\bar{v}}{\bar{v}^2}g\right)\right]_{|x=0} &+ \mu\left[\mathcal{A}\partial_x^2\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right)\right]_{|x=0} \\
&+ s(v_+-2)\partial_x\boldsymbol{w}^0(\widetilde{x}) - \mu\partial_x^2\boldsymbol{w}^0(\widetilde{x}) - \widetilde{x}'(t)\partial_x\boldsymbol{w}^0(\widetilde{x}).
\end{aligned}$$

Let us now estimate R_2. We have by definition of $\mathcal{A}$

$$\begin{aligned}
&\mu\left[\mathcal{A}\partial_x^2\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right)\right]_{|x=0}\\
&=-s\mu\left[\partial_x^2\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right)\right]_{|x=0}-\mu^2\left[\partial_x\left(\frac{1}{\bar{v}}\partial_x^2\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right)\right)\right]_{|x=0}\\
&=\left(\mu\left(\partial_x\bar{v}\right)_{|0}-s\right)\mu\left[\partial_x^2\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right)\right]_{|x=0}-\mu^2\left[\partial_x^3\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right)\right]_{|x=0}\\
&=(v_+-2)s\mu T_2-\mu^2T_3.
\end{aligned}$$

Replacing in the expression of R_2 and using (6.30), we obtain

$$\begin{aligned}
R_2=\frac{s(v_+-1)}{\mu}\beta(\boldsymbol{w}^0(\widetilde{x})-u_+)+(v_+-2)s\mu T_2-\mu^2T_3&\\
+s(v_+-2)\partial_x\boldsymbol{w}^0(\widetilde{x})-\mu\partial_x^2\boldsymbol{w}^0(\widetilde{x})-\widetilde{x}'(t)\partial_x\boldsymbol{w}^0(\widetilde{x})&.
\end{aligned}$$

Thus

$$\begin{aligned}
\|R_2\|_{H^1\left(0,\overline{T}\right)}&\leqslant C\left(\|\beta\|_{H^1\left(0,\overline{T}\right)}\left\|\boldsymbol{w}^0-u_+\right\|_{W^{1,\infty}(\mathbb{R}_+)}+\|T_2\|_{H^1\left(0,\overline{T}\right)}\right.\\
&\qquad\left.+\|T_3\|_{H^1(0,\overline{T})}+\left\|\partial_x\boldsymbol{w}^0\right\|_{H^2(\mathbb{R}_+)}\right)\\
&\leqslant C\|\beta\|_{H^1\left(0,\overline{T}\right)}\left\|\boldsymbol{w}^0-u_+\right\|_{W^{1,\infty}(\mathbb{R}_+)}\\
&\quad+C\left(\left\|\boldsymbol{w}^0-u_+\right\|^2_{W^{2,4}(\mathbb{R}_+)}+\left\|\partial_x\boldsymbol{w}^0\right\|_{H^2(\mathbb{R}_+)}\right).
\end{aligned}$$

This completes the proof of Lemma 6.4. □

6.2.2. Estimates on g_1

We now turn towards the energy estimates for g_1. As announced before, we focus on the additional terms coming either from the boundary terms or from the non-zero source term involving $\partial_x\boldsymbol{w}^0$.

$\triangleright L^\infty(L^2)\cap L^2(H^1)$ *estimate:* Multiplying (6.29) by g_1 and proceeding as in the previous subsection, we have now

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^\infty g_1^2+\frac{\mu}{v_+}\int_0^\infty(\partial_xg_1)^2&\leqslant C\left(\mathcal{E}_0+\|\beta\|^2_{H^1\left(0,\overline{T}\right)}\right)\left(|\beta|^2+\|g(t)\|^2_{H^2(\mathbb{R}_+)}\right)\\
&\quad+\left|\int_{\mathbb{R}_+}\mathcal{A}\partial_x\boldsymbol{w}^0(x+\widetilde{x})g_1\right|+\frac{s+\beta}{2}(g_1)^2_{|x=0}+\mu\left|(g_1)_{|x=0}(\partial_xg_1)_{|x=0}\right|\\
&\quad+\mu^2\left|\left(\partial_x^2\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right)\right)_{|x=0}(g_1)_{|x=0}\right|\\
&\quad+s\mu\underbrace{\left|\partial_x\left(\ln\left(1+\frac{g}{\bar{v}}\right)-\frac{g}{\bar{v}}\right)_{|x=0}\right|}_{=0}|(\partial_xg_1)_{|x=0}|
\end{aligned}\tag{6.39}$$

where the two last lines contain the additional contributions compared to the case $\boldsymbol{w}^0 \equiv u_+$ (cf. (6.14)). We treat the integral involving $\partial_x \boldsymbol{w}^0$ as follows:

$$\int_{\mathbb{R}_+} \mathcal{A}\partial_x \boldsymbol{w}^0(x+\widetilde{x})g_1 = s\int_{\mathbb{R}_+} \left(\boldsymbol{w}^0(x+\widetilde{x}) - u_+\right)\partial_x g_1 + \mu\int_{\mathbb{R}_+} \frac{\partial_x \boldsymbol{w}^0}{\bar{\boldsymbol{v}}}\partial_x g_1 \\ + s\left(\boldsymbol{w}^0(\widetilde{x}) - u_+\right)(g_1)_{|x=0} + \mu\partial_x \boldsymbol{w}^0(\widetilde{x})(g_1)_{|x=0}$$

so that

$$\left|\int_{\mathbb{R}_+} \mathcal{A}\partial_x \boldsymbol{w}^0(x+\widetilde{x})g_1\right| \\ \leqslant \frac{\mu}{8v_+}\int_{\mathbb{R}_+}(\partial_x g_1)^2 + s\left|\boldsymbol{w}^0(\widetilde{x}) - u_+\right|\left|(g_1)_{|x=0}\right| + \mu\left|\partial_x \boldsymbol{w}^0(\widetilde{x})\right|\left|(g_1)_{|x=0}\right| \\ + C\Big[\left\|\boldsymbol{w}^0(\cdot+\widetilde{x}) - u_+\right\|^2_{L^2(\mathbb{R}_+)} + \left\|\partial_x \boldsymbol{w}^0(\cdot+\widetilde{x})\right\|^2_{L^2(\mathbb{R}_+)}\Big],$$

where the first integral can be absorbed in the left-hand side of (6.39).

We then come back to (6.39) and integrate in time. The traces of g_1, $\partial_x g_1$ and of the nonlinear term are estimated thanks to Lemma 6.4. Note in particular that by the Cauchy-Schwarz inequality

$$\int_0^T \left|(g_1)_{|x=0}(\partial_x g_1)_{|x=0}\right| \leqslant C\left(\mathcal{E}_0 + \mathcal{E}_0^{1/2}\|\beta\|_{H^1(0,\overline{T})}\right).$$

Gathering all terms, we deduce that

$$\|g_1\|^2_{L^\infty\left([0,\overline{T}],L^2(\mathbb{R}_+)\right)} + \frac{\mu}{v_+}\|\partial_x g_1\|^2_{L^2\left([0,\overline{T}],L^2(\mathbb{R}_+)\right)} \\ \leqslant C\left(\mathcal{E}_0 + \mathcal{E}_0^{1/2}\|\beta\|_{H^1\left(0,\overline{T}\right)} + \left(\mathcal{E}_0 + \|\beta\|^2_{H^1\left(0,\overline{T}\right)}\right)^2\right). \tag{6.40}$$

$\triangleright L^\infty(H^1)\cap L^2(H^2)$ *estimate:* Differentiating with respect to x Eq. (6.29), we obtain (6.17) with the additional contribution $\partial_x\mathcal{A}\partial_x \boldsymbol{w}^0$ in the right-hand side. Following the same steps as in the constant case $\boldsymbol{w}^0 \equiv u_+$, we use Lemma 6.3 to handle the diffusion term:

$$\int_0^\infty (\partial_x\mathcal{A}\partial_x g_1)\ \partial_x g_1\ \frac{\rho}{\bar{\boldsymbol{v}}} \geqslant \mu\int_0^\infty \left(\partial_x\left(\frac{\partial_x g_1}{\bar{\boldsymbol{v}}}\right)\right)^2\rho - C\|\rho\|_{W^{2,\infty}}\int_0^\infty(\partial_x g_1)^2 \\ + \left(\partial_x g_{1|x=0}\right)^2\left(\frac{s}{2}\rho(0) - \frac{\mu}{2}\rho'(0)\right) \\ + \mu\left(\partial_x\frac{\partial_x g_1}{\bar{\boldsymbol{v}}}\right)_{|0}(\partial_x g_1)_{|x=0}\rho(0)$$

Choosing once again $\rho(0) = 2$, $\rho'(0) = -4s/\mu$ and using (6.32), we have

$$\left(\partial_x g_{1|x=0}\right)^2\left(\frac{s}{2}\rho(0) - \frac{\mu}{2}\rho'(0)\right) + \mu\left(\partial_x\frac{\partial_x g_1}{\bar{\boldsymbol{v}}}\right)_{|x=0}(\partial_x g_1)_{|x=0}\,\rho(0) \\ \geqslant \left(\frac{s(v_+-1)}{\mu}\right)^2(s+\beta)\beta^2 - C\left(R_1^2 + |R_1|\,|\beta| + |R_2|\,|\beta| + |R_2|\,|R_1|\right).$$

We then control the time integral of the second term in the right-hand side thanks to Lemma 6.4.

Following the same steps as in the constant case $\boldsymbol{w}^0 \equiv u_+$, we can check that we have the following other additional contributions:

- the additional integral involving $\partial_x \mathcal{A} \partial_x \boldsymbol{w}^0$ is treated by integration by parts with boundary terms of the type $\partial_x^k \boldsymbol{w}^0_{|x=0} (\partial_x g_1)_{|x=0}$, $k = 1, 2$. The integral is easily controlled by use of the Cauchy-Schwarz inequality;
- from the nonlinear term, we get the following boundary term:

$$\left[\mathcal{A}\partial_x^2 \left(\ln\left(1+\frac{g}{\overline{v}}\right) - \frac{g}{\overline{v}}\right)\right]_{|x=0} (\partial_x g_1)_{|x=0}\,,$$

which we also treat using Lemma 6.4, noticing that

$$\begin{aligned}\left[\mathcal{A}\partial_x^2 \left(\ln\left(1+\frac{g}{\overline{v}}\right) - \frac{g}{\overline{v}}\right)\right]_{|x=0} &= -sT_2 - \mu\partial_x \left(\frac{1}{\overline{v}}\partial_x^2 \left(\ln\left(1+\frac{g}{\overline{v}}\right) - \frac{g}{\overline{v}}\right)\right)_{|x=0} \\ &= s(v_+ - 2)T_2 - \mu T_3.\end{aligned} \tag{6.41}$$

We obtain

$$\begin{aligned}\|\partial_x g_1\|^2_{L^\infty\left([0,\overline{T}],L^2(\mathbb{R}_+)\right)} + \left\|\partial_x \frac{\partial_x g_1}{\overline{v}}\right\|^2_{L^2\left([0,\overline{T}]\times\mathbb{R}_+\right)} + \|\beta\|^2_{L^2\left([0,\overline{T}]\right)} \\ \leqslant C\left(\mathcal{E}_0 + \mathcal{E}_0^{1/2}\|\beta\|_{H^1\left(0,\overline{T}\right)} + \left(\mathcal{E}_0 + \|\beta\|^2_{H^1\left(0,\overline{T}\right)}\right)^2\right).\end{aligned}$$

$\triangleright W^{1,\infty}(L^2)\cap H^1(H^1)$ *estimate:* Differentiating with respect to time Equation (6.29), we get Equation (6.21) with an additional term in the right-hand side, namely $\widetilde{x}'(t)\mathcal{A}\partial_x^2 \boldsymbol{w}^0(x+\widetilde{x}(t))$. Multiplying the equation by $\partial_t g_1$ and integrating in space, we have, similarly to (6.39)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^\infty (\partial_t g_1)^2 + \frac{\mu}{v_+}\int_0^\infty (\partial_t\partial_x g_1)^2 \\ \leqslant C\left(\mathcal{E}_0 + \|\beta\|^2_{H^1\left(0,\overline{T}\right)}\right)\left(|\beta'|^2 + |\beta|^2 + \|g\|^2_{H^2(\mathbb{R}_+)} + \|\partial_t g\|^2_{H^2(\mathbb{R}_+)}\right) \\ + \left|\int_{\mathbb{R}_+} \widetilde{x}'(t)\int_{\mathbb{R}_+} \mathcal{A}\partial_x^2 \boldsymbol{w}^0(x+\widetilde{x}(t))\partial_t g_1\right| \\ + \frac{s+\beta}{2}(\partial_t g_1)^2_{|x=0} + \mu\left|(\partial_t g_1)_{|x=0}(\partial_t\partial_x g_1)_{|x=0}\right| + \mu^2\left|\partial_t T_2\,(\partial_t g_1)_{|x=0}\right| \\ + |\beta'|\,\left|g_{1|x=0}\right|\,\left|\partial_t g_{1|x=0}\right|.\end{aligned} \tag{6.42}$$

We first consider the integral term involving $\mathcal{A}\partial_x^2 \boldsymbol{w}^0(x+\widetilde{x}(t))$:

$$\begin{aligned}\widetilde{x}'(t)\int_{\mathbb{R}_+} &\mathcal{A}\partial_x^2 \boldsymbol{w}^0\left(x+\widetilde{x}(t)\right)\partial_t g_1\\ &= -\widetilde{x}'(t)\int_{\mathbb{R}_+}\left(\mathcal{A}\partial_x \boldsymbol{w}^0\left(x+\widetilde{x}(t)\right)-\mu\left(\frac{\partial_x\bar{\boldsymbol{v}}}{\bar{\boldsymbol{v}}^2}\partial_x \boldsymbol{w}^0\left(x+\widetilde{x}(t)\right)\right)\right)\partial_x\partial_t g_1\\ &\quad -\widetilde{x}'(t)\left[\mathcal{A}\partial_x \boldsymbol{w}^0\left(x+\widetilde{x}(t)\right)-\mu\left(\frac{\partial_x\bar{\boldsymbol{v}}}{\bar{\boldsymbol{v}}^2}\partial_x \boldsymbol{w}^0\left(x+\widetilde{x}(t)\right)\right)\right]_{|x=0}\left(\partial_t g_1\right)_{|x=0}.\end{aligned}$$

The integral is then controlled by a Cauchy–Schwarz inequality, using the trace estimates from Lemma 6.4 together with Lemma C.1

$$\begin{aligned}\int_0^{\overline{T}}\int_{\mathbb{R}_+}&\left|\widetilde{x}'(t)\left(\mathcal{A}\partial_x \boldsymbol{w}^0\left(x+\widetilde{x}(t)\right)-\mu\left(\frac{\partial_x\bar{\boldsymbol{v}}}{\bar{\boldsymbol{v}}^2}\partial_x \boldsymbol{w}^0\left(x+\widetilde{x}(t)\right)\right)\right)\partial_x\partial_t g_1\right|\\ &\leqslant \frac{\mu}{4v_+}\left\|\partial_t\partial_x g_1\right\|^2_{L^2\left((0,\overline{T})\times\mathbb{R}_+\right)}+C\left(\left\|\sqrt{x}\partial_x \boldsymbol{w}^0\right\|^2_{L^2(\mathbb{R}_+)}+\left\|\sqrt{x}\partial_x^2 \boldsymbol{w}^0\right\|^2_{L^2(\mathbb{R}_+)}\right)\\ &\qquad + C\left\|\partial_x \boldsymbol{w}^0\right\|_{H^1(\mathbb{R}_+)}\left\|\left(1+\sqrt{x}\right)\partial_x \boldsymbol{w}^0\right\|_{L^2(\mathbb{R}_+)}.\end{aligned}$$

To conclude as in the constant case, we have to control the different boundary terms in (6.42). Using Lemma 6.4, their $L^2(0,\overline{T})$ norm is bounded by

$$\begin{aligned}C\Bigg(&\left\|g_{1|x=0}\right\|^2_{H^1\left(0,\overline{T}\right)}+\left\|\partial_t T_2\right\|^2_{L^2\left(0,\overline{T}\right)}+\left\|\partial_t g_{1|x=0}\right\|_{L^2\left(0,\overline{T}\right)}\left\|\partial_t\partial_x g_{1|x=0}\right\|_{L^2\left(0,\overline{T}\right)}\\ &\qquad+\left\|\beta'\right\|_{L^2\left(0,\overline{T}\right)}\left\|g_{1|x=0}\right\|^2_{H^1\left(0,\overline{T}\right)}\Bigg)\\ &\leqslant C\left(\mathcal{E}_0+\mathcal{E}_0^{1/2}\left\|\beta\right\|_{H^1\left(0,\overline{T}\right)}\right).\end{aligned}$$

Note that we used here the smallness of $\mathcal{E}_0$. We obtain eventually

$$\begin{aligned}\left\|\partial_t g_1\right\|^2_{L^\infty\left([0,\overline{T}],L^2(\mathbb{R}_+)\right)}&+\left\|\partial_x\partial_t g_1\right\|^2_{L^2\left([0,\overline{T}]\times\mathbb{R}_+\right)}\\ &\leqslant C\left(\mathcal{E}_0+\left(\mathcal{E}_0+\left\|\beta\right\|^2_{H^1\left(0,\overline{T}\right)}\right)^2+\mathcal{E}_0^{1/2}\left\|\beta\right\|_{H^1\left(0,\overline{T}\right)}\right).\end{aligned}$$

$\triangleright W^{1,\infty}(H^1)\cap H^1(H^2)$ *estimate:* We differentiate the equation (6.17) satisfied by $\partial_x g_1$ with respect to time:

$$\begin{aligned}\partial_t(\partial_t\partial_x g_1)&+\partial_x\mathcal{A}\partial_t\partial_x g_1\\ &=\beta\partial_x^2\partial_t g_1+\beta'\partial_x^2 g_1+\mu\beta'\partial_x^2\left(\frac{\partial_x\bar{\boldsymbol{v}}}{\bar{\boldsymbol{v}}^2}g\right)+\mu\beta\partial_x^2\left(\frac{\partial_x\bar{\boldsymbol{v}}}{\bar{\boldsymbol{v}}^2}\partial_t g\right)\\ &\quad+\mu\partial_x\mathcal{A}\partial_x^2\partial_t\left(\ln\left(1+\frac{g}{\bar{\boldsymbol{v}}}\right)-\frac{g}{\bar{\boldsymbol{v}}}\right)+\widetilde{x}'(t)\partial_x\mathcal{A}\partial_x^2 \boldsymbol{w}^0(x+\widetilde{x}(t))\end{aligned}\tag{6.43}$$

As in the constant case $\boldsymbol{w}^0\equiv u_+$, we aim at obtaining a control of $\|\partial_t\partial_x g_1\|_{L^\infty(L^2)}+\|\beta'\|_{L^2}$. For that purpose, we want to use $(\partial_t\partial_x g_1)\rho/\bar{\boldsymbol{v}}$ as a test function in the weak formulation associated with (6.43). This weak formulation is similar to (6.25), with extra source and boundary terms. Using (6.41) and Lemma 6.4, we find that for

any test function $\psi \in L^\infty((0,\overline{T}), L^2(\mathbb{R}_+)) \cap L^2((0,\overline{T}), H^1(\mathbb{R}_+))$, for almost every $t \in (0,\overline{T})$,

$$\begin{aligned}
\left\langle \partial_t^2 \partial_x g_1(t), \psi(t) \right\rangle_{H^{-1},H^1} &- \int_0^\infty \mathcal{A}(\partial_t \partial_x g_1)\, \partial_x \psi \\
&- \left(\beta \partial_t \partial_x g_{1|x=0} + \beta' \partial_x g_{1|x=0} + \partial_t R_2\right) \psi_{|x=0} \\
= &- \int_0^\infty \mu \mathcal{A} \partial_x^2 \partial_t \left(\ln\left(1 + \frac{g}{\bar{v}}\right) - \frac{g}{\bar{v}}\right) \partial_x \psi(t,x)\, dx \\
&+ \int_0^\infty \left[\beta \partial_x^2 \partial_t g_1 + \beta' \partial_x^2 g_1 + \mu \beta' \partial_x^2 \left(\frac{\partial_x \bar{v}}{\bar{v}^2} g\right) + \mu \beta \partial_x^2 \left(\frac{\partial_x \bar{v}}{\bar{v}^2} \partial_t g\right)\right] \psi \\
&+ \tilde{x}'(t) \int_0^\infty \partial_x \mathcal{A} \partial_x^2 w^0(x + \tilde{x}(t)) \psi \\
&- \left[s\mu(v_+ - 2)\partial_t T_2 - \mu \partial_t T_3\right] \psi_{|x=0}.
\end{aligned}$$

We now take $\psi = (\partial_t \partial_x g_1)\rho/\bar{v}$, with $\rho \in \mathcal{C}_b^2(\mathbb{R}_+)$ such that $\rho(0) = 2$, $\rho'(0) = -4s/\mu$ as before.

The diffusion term is treated as previously:

$$\begin{aligned}
&- \int_0^\infty \mathcal{A}(\partial_t \partial_x g_1)\, \partial_x \left(\partial_x \partial_t g_1 \frac{\rho}{\bar{v}}\right) \\
&\qquad \geqslant \mu \int_{\mathbb{R}_+} \left(\partial_x \left(\frac{\partial_x \partial_t g_1}{\bar{v}}\right)\right)^2 \rho - C\|\rho\|_{W^{2,\infty}} \int_{\mathbb{R}_+} \left(\partial_x \partial_t g_1\right)^2 + 3s \left(\partial_t \partial_x g_{1|x=0}\right)^2.
\end{aligned}$$

Recalling (6.32), we have

$$(\partial_x \partial_t g_1)_{|x=0} = \beta' \frac{s(v_+ - 1)}{\mu} + \partial_t R_1. \tag{6.44}$$

Gathering all the boundary terms of the left-hand side of the weak formulation, we obtain, using once again Lemma 6.4

$$\begin{aligned}
&- 2\left(\beta \partial_t \partial_x g_{1|x=0} + \beta' \partial_x g_{1|x=0} + \partial_t R_2\right) (\partial_t \partial_x g_1)_{|x=0} + 3s \left(\partial_t \partial_x g_{1|x=0}\right)^2 \\
&\qquad \geqslant C^{-1} \left(\partial_t \partial_x g_{1|x=0}\right)^2 - C\left((\beta')^2 \left(\partial_x g_{1|x=0}\right)^2 + (\partial_t R_2)^2\right) \\
&\qquad \geqslant C^{-1}(\beta')^2 - C\left((\partial_t R_1)^2 + |\beta'|^2 \left(\|\beta\|^2_{H^1(0,\overline{T})} + \|R_1\|^2_{H^1(0,\overline{T})}\right) + (\partial_t R_2)^2\right),
\end{aligned}$$

for some constant $C \geqslant 1$ depending only on the parameters of the problem. Note that

$$\begin{aligned}
\int_0^{\overline{T}} &\left((\partial_t R_1)^2 + |\beta'|^2 \left(\|\beta\|^2_{H^1(0,\overline{T})} + \|R_1\|^2_{H^1(0,\overline{T})}\right) + (\partial_t R_2)^2\right) \\
&\qquad \leqslant C\left(\mathcal{E}_0 + \left(\mathcal{E}_0 + \|\beta\|^2_{H^1(0,\overline{T})}\right)^2\right).
\end{aligned}$$

The reader may check that the other additional boundary terms can be controlled as well. Eventually, we are led to

$$\begin{aligned}\|\partial_t\partial_x g_1\|^2_{L^\infty\left([0,\overline{T}],L^2(\mathbb{R}_+)\right)} + \int_0^{\overline{T}}\int_0^\infty \left(\partial_x\left(\frac{\partial_t\partial_x g_1}{\bar{v}}\right)\right)^2 + \|\beta'\|^2_{L^2\left(0,\overline{T}\right)} \\ \leqslant C\left(\mathcal{E}_0 + \left(\mathcal{E}_0 + \|\beta\|^2_{H^1\left(0,\overline{T}\right)}\right)^2\right) \\ + C\,\|\partial_t\partial_x g_1\|^2_{L^2\left((0,\overline{T})\times\mathbb{R}_+\right)} \\ \leqslant C\left(\mathcal{E}_0 + \left(\mathcal{E}_0 + \|\beta\|^2_{H^1\left(0,\overline{T}\right)}\right)^2 + \mathcal{E}_0^{1/2}\|\beta\|_{H^1\left(0,\overline{T}\right)}\right).\end{aligned}$$

Conclusion. We infer that

$$\|\beta\|^2_{H^1\left(0,\overline{T}\right)} \leqslant C\left(\mathcal{E}_0 + \left(\mathcal{E}_0 + \|\beta\|^2_{H^1\left(0,\overline{T}\right)}\right)^2 + \mathcal{E}_0^{1/2}\|\beta\|_{H^1\left(0,\overline{T}\right)}\right).$$

The Cauchy–Schwarz inequality then entails

$$\|\beta\|^2_{H^1\left(0,\overline{T}\right)} \leqslant C\left(\mathcal{E}_0 + \left(\mathcal{E}_0 + \|\beta\|^2_{H^1\left(0,\overline{T}\right)}\right)^2\right).$$

Assume that

$$\mathcal{E}_0 \leqslant c_0\delta^2.$$

Then

$$\|\beta\|^2_{H^1\left(0,\overline{T}\right)} \leqslant Cc_0\delta^2 + C\delta^4.$$

Once again, choosing the constant c_0 small enough, we obtain the desired result. This completes the proof of Proposition 6.1. □

6.3. Long time behavior

Corollary 6.5. — *Assume that the hypotheses of Theorem 2.3 are satisfied. Let $(\widetilde{x}, v_s, u_s)$ be the unique global smooth solution to* (2.15)-(2.16)-(2.12c) *constructed in Proposition 5.1. Then*

$$|\widetilde{x}'(t) - s| + \sup_{x\in\mathbb{R}_+}\Big|(v_s,u_s)(t,x) - (\bar{v},\bar{u})(x)\Big| \longrightarrow 0 \quad \text{as } t\to+\infty. \tag{6.45}$$

Moreover, setting $p_s(t,x) = p_s(t) = -\mu[\partial_x u_s]_{|x=0}$, we also ensure that

$$\Big|(p_s - p_-)(t)\Big| \longrightarrow 0 \quad \text{as } t\to+\infty,$$

where $p_- = -\mu[\partial_x\bar{u}]_{|x=0} = s^2(v_+ - 1)$ has been defined in Lemma 2.1.

Proof. — The result easily follows from the controls of the solution $(\widetilde{x}, v_s, u_s)$ and its time derivative. For instance, $v_s - \bar{v}$ is controlled in $L^2\cap L^\infty(\mathbb{R}_+,\, H^1(\mathbb{R}_+))$ and $\partial_t v_s \in L^2(\mathbb{R}_+,\, H^1(\mathbb{R}_+))$. Therefore,

$$\|(v_s - \bar{v})(t)\|_{H^1(\mathbb{R}_+)} \longrightarrow 0 \quad \text{as } t\to+\infty,$$

and

$$\sup_{x\in\mathbb{R}_+}\left|(v_s-\bar{v})(t,x)\right| \leqslant C\,\|(v_s-\bar{v})(t)\|^{1/2}_{L^2(\mathbb{R}_+)}\,\|\partial_x(v_s-\bar{v})(t)\|^{1/2}_{L^2(\mathbb{R}_+)} \longrightarrow 0 \quad \text{as } t\to+\infty.$$

The long time behavior of $u_s-\bar{u}$ and $\widetilde{x}'-s$ can be derived with the same arguments. Finally, the long time behavior of p_s is obtained through the control of $\partial_x(u_s-\bar{u})$ in $L^2\cap L^\infty(\mathbb{R}_+;H^1(\mathbb{R}_+))$ and of $\partial_t\partial_x(u_s-\bar{u})\in L^2(\mathbb{R}_+;H^1(\mathbb{R}_+))$. □

Appendix A. Linear parabolic equations with Dirichlet boundary conditions

In this section, we consider equations of the type

$$\begin{aligned}\partial_t u+\partial_x(b(t,x)u)+c(t,x)u-\partial_x(a(t,x)\partial_x u)=f(t,x),\quad t>0,\ x>0\\ u_{|x=0}=0,\quad u_{|t=0}=u^0.\end{aligned}\tag{A.1}$$

We denote $\Omega_T:=(0,T)\times\mathbb{R}_+$. We will always assume that there exists $\alpha>0$ such that

$$a,b,c\in L^\infty([0,T]\times\mathbb{R}_+),\qquad \inf_{[0,T]\times\mathbb{R}_+} a\geqslant\alpha>0.\tag{A.2}$$

Note that under such assumptions, it is classical that for any $f\in L^2([0,T],H^{-1}(\mathbb{R}_+))$, for any $u^0\in L^2(\mathbb{R}_+)$, equation (A.1) has a unique weak solution

$$u\in L^\infty\left([0,T],L^2(\mathbb{R}_+)\right)\cap L^2\left([0,T],H^1_0(\mathbb{R}_+)\right).$$

We will now derive regularity estimates for u. These estimates are quite standard, and can be found in numerous textbooks on partial differential equations. Since they are used repeatedly in this paper, we gathered them in the following Proposition, for which we provide a proof for the reader's convenience.

Proposition A.1. — *Let $T>0$. Assume that*

$$\begin{aligned}&c\in W^{1,\infty}\left([0,T],L^\infty(\mathbb{R}_+)\right),\quad b\in W^{1,\infty}(\Omega_T),\\ &a\in H^1\left([0,T],\,L^2(\mathbb{R}_+)\right)\cap L^2\left([0,T],\,H^2(\mathbb{R}_+)\right).\end{aligned}$$

Let $u^0\in H^1(\mathbb{R}_+)$ such that $u^0(0)=0$, and let $f\in L^2(\Omega_T)$.
Let $u\in L^\infty([0,T],L^2(\mathbb{R}_+))\cap L^2([0,T],H^1_0(\mathbb{R}_+))$ be the unique weak solution of (A.1).

Then $u\in L^\infty([0,T],\,H^1(\mathbb{R}_+))\cap H^1([0,T],\,L^2(\mathbb{R}_+))\cap L^2([0,T],\,H^2(\mathbb{R}_+))$ and we have the classical energy estimate

$$A_0:=\|u\|^2_{L^\infty([0,T],L^2)}+\|\partial_x u\|^2_{L^2(\Omega_T)}\leqslant C\left(\|f\|^2_{L^2(\Omega_T)}+\|u^0\|^2_{L^2(\mathbb{R}_+)}\right)e^{CT}\tag{A.3}$$

for some constant $C = C(\alpha, \|b\|_{L^\infty(\Omega_T)}, \|c\|_{L^\infty(\Omega_T)}$, *as well as the higher regularity estimates*

$$\text{(A.4)}\quad A_1 := \|\partial_x u\|^2_{L^\infty L^2} + \|\partial_t u\|^2_{L^2L^2} + \left\|\partial_x^2 u\right\|^2_{L^2L^2}$$
$$\leqslant C\Bigg[\|a\|_{L^\infty}\left\|\partial_x u^0\right\|^2_{L^2} + \Big(\|\partial_x b\|^2_{L^2L^\infty} + \|c\|^2_{L^2L^\infty}\Big)\|u\|^2_{L^\infty L^2} + \|f\|^2_{L^2L^2}$$
$$+\Big(\|\partial_t a\|_{L^\infty L^2} + \|\partial_t a\|^2_{L^\infty L^2} + \|\partial_x a\|^2_{L^\infty L^4} + \|\partial_x a\|^4_{L^\infty L^4} + \|b\|^2_{L^\infty}\Big)\|\partial_x u\|^2_{L^2L^2}\Bigg]$$

and

$$\text{(A.5)}\quad \|\partial_t u\|^2_{L^\infty([0,T],L^2)} + \|\partial_x\partial_t u\|^2_{L^2L^2}$$
$$\leqslant C\Big[\left\|u^0\right\|^2_{H^2(\mathbb{R}_+)}\Big(1 + \|b\|^2_{L^\infty_t(W^{1,\infty})} + \|c\|^2_{L^\infty(\Omega_T)} + \|a\|^2_{L^\infty_t(W^{1,\infty})}\Big) + A_0\|\partial_t b\|^2_{L^2L^\infty}$$
$$+ A_1\Big(\|\partial_t c\|^2_{L^2(\Omega_T)} + \|c\|_{L^\infty} + \|\partial_t a\|_{L^\infty L^2} + \|b\|_{L^\infty(\Omega_T)} + 1\Big) + \|\partial_t f\|^2_{L^2(\Omega_T)}\Big].$$

Proof. — The weak solution u of (A.1) can be defined as the limit as $R \to \infty$ of the solution u_R of the linear equation in a bounded domain

$$\text{(A.6)}\quad \partial_t u_R + \partial_x(b(t,x)u_R) + c(t,x)u_R - \partial_x(a(t,x)\partial_x u_R)$$
$$= f(t,x), \quad t > 0,\ x \in (0,R)$$
$$u_{R|x=0} = u_{R|x=R} = 0, \quad u_{|t=0} = u^0_R,$$

where $u^0_R = u^0\chi_R$, where $\chi_R(x) = 1$ if $x \leqslant R-2$ and $\chi_R(x) = 0$ if $x \geqslant R-1$, with $\|\chi_R\|_{W^{2,\infty}} \leqslant C$. In turn, the function u_R is defined as the limit of the Galerkin approximation

$$u_{R,n}(t,x) = \sum_{k=1}^n d_k^{R,n}(t)w_{k,R}(x),$$

where $w_{k,R}$ are eigenfunctions of the Laplacian with Dirichlet boundary conditions in $(0,R)$, normalized in L^2, namely $w_{k,R}(x) = (2/R)^{1/2}\sin(k\pi x/R)$. More precisely, for all $k \in \{1,\cdots,n\}$, for a.e. $t \in [0,T]$

$$\text{(A.7)}\quad \int_0^R \partial_t u_{R,n}(t)w_{k,R} - \int_0^R b(t)u_{R,n}(t)\partial_x w_{k,R} + \int_0^R c(t)u_{R,n}(t)w_{k,R}$$
$$+ \int_0^R a(t)\partial_x u_{R,n}(t)\partial_x w_{k,R} = \int_0^R f(t)w_{k,R}.$$

The initial data $u_{R,n|t=0}$ is the orthogonal projection onto $\mathrm{Span}(w_{1,R},\cdots,w_{n,R})$ in $L^2(0,R)$ of u^0_R.

Energy estimate (A.3). The classical energy estimate is obtained by multiplying (A.7) by $d_k^{R,n}(t)$ and summing for $k \in \{1, \cdots, n\}$. Using Cauchy-Schwarz inequalities, we infer that

$$\text{(A.8)} \quad \frac{d}{dt} \|u_{R,n}(t)\|^2_{L^2(0,R)} + \alpha \|\partial_x u_{R,n}(t)\|^2_{L^2(0,R)} \leqslant \|f(t)\|^2_{L^2(0,R)} + \left(1 + \frac{\|b\|^2_{L^\infty(\Omega_T)}}{\alpha} + \|c\|_{L^\infty(\Omega_T)}\right) \|u_{R,n}(t)\|^2_{L^2(0,R)}.$$

Integrating in time and recalling that $\|u_{R,n}(t=0)\|_{L^2(0,R)} \leqslant \|u_R^0\|_{L^2([0,R])} \leqslant \|u^0\|_{L^2(\mathbb{R}_+)}$, we deduce that $u_{R,n}$ satisfies (A.3) for all R, n. Passing to the limit as $n \to \infty$, we deduce that u_R also satisfies (A.3).

Estimate (A.4). We first multiply (A.7) by $(d_k^{R,n})'(t)$ and sum for $k \in \{1, \cdots, n\}$. After integrations by parts in space and time, we get

$$\begin{aligned}
&\int_0^R \frac{a(t)}{2} |\partial_x u_{R,n}(t)|^2 + \int_0^t \int_0^R |\partial_t u_{R,n}|^2 \\
&\quad = \int_0^R \frac{a(0)}{2} |\partial_x u_{R,n}(0)|^2 - \int_0^t \int_0^R \partial_x (b u_{R,n}) \partial_t u_{R,n} - \int_0^t \int_0^R c u_{R,n} \partial_t u_{R,n} \\
&\qquad + \int_0^t \int_0^R f\ \partial_t u_{R,n} + \int_0^t \int_0^R \frac{\partial_t a}{2} |\partial_x u_{R,n}|^2
\end{aligned}$$

and therefore

$$\begin{aligned}
\text{(A.9)} \quad &\int_0^R \frac{a(t)}{2} |\partial_x u_{R,n}(t)|^2 + \frac{1}{2} \int_0^t \int_0^R |\partial_t u_{R,n}|^2 \\
&\leqslant \int_0^R \frac{a(0)}{2} |\partial_x u_{R,n}(0)|^2 + \int_0^t \int_0^R \frac{|\partial_t a|}{2} |\partial_x u_{R,n}|^2 \\
&+ C\Big[\|b\|^2_{L^\infty(\Omega_T)} \|\partial_x u_{R,n}\|^2_{L^2(\Omega_T)} + \left(\|\partial_x b\|^2_{L^2L^\infty} + \|c\|^2_{L^2L^\infty}\right) \|u_{R,n}\|^2_{L^\infty L^2} + \|f\|^2_{L^2(\Omega_T)}\Big]
\end{aligned}$$

The crucial point here is to notice that thanks to the compatibility condition $u_R^0(0) = u_R^0(R) = 0$, we have

$$\text{(A.10)} \qquad \|\partial_x u_{R,n}(t=0)\|_{L^2(0,R)} \leqslant \left\|\partial_x u_R^0\right\|_{L^2(0,R)} \leqslant C \left\|u^0\right\|_{H^1(\mathbb{R}_+)}.$$

Indeed,

$$\partial_x u_{R,n}(t=0) = \sum_{k=0}^n d_k^{R,n}(0) \partial_x w_{k,R}.$$

Computing $d_k^{R,n}(0)$ and using the compatibility conditions on u_R^0, we infer that

$$d_k^{R,n}(0) = \sqrt{\frac{2}{R}} \int_0^R u_R^0(x) \sin\left(\frac{k\pi x}{R}\right) dx = \sqrt{\frac{2}{R}} \frac{R}{k\pi} \int_0^R \partial_x u_R^0(x) \cos\left(\frac{k\pi x}{R}\right) dx.$$

Setting $v_{k,R} := \sqrt{\frac{2}{R}} \cos(k\pi x/R)$ for $k \geqslant 1$, we notice that $(v_{k,R})_{k\geqslant 1}$ is an orthonormal family in $L^2(0,R)$, and that

$$\partial_x u_{R,n}(t=0) = \sum_{k=0}^n \left(\partial_x u_R^0, v_{k,R}\right) v_{k,R},$$

where $(\cdot,\cdot)$ is the usual L^2 scalar product. Inequality (A.10) then follows from the Bessel inequality.

Next, we multiply (A.7) by $\lambda_k^R d_k^{R,n}(t)$ where λ_k^R is the kth eigenvalue of the Laplacian. Summing for $k \in \{1, \cdots, n\}$, we get after integrations by parts:

$$\begin{aligned}-\int_0^t\int_0^R \partial_x(a\partial_x u_{R,n})\partial_x^2 u_{R,n} = &-\int_0^t\int_0^R \partial_x(bu_{R,n})\partial_x^2 u_{R,n} - \int_0^t\int_0^R cu_{R,n}\partial_x^2 u_{R,n}\\ &+\int_0^t\int_0^R f\ \partial_x^2 u_{R,n} - \int_0^t\int_0^R \partial_t u_{R,n}\partial_x^2 u_{R,n}.\end{aligned}$$

Using the Cauchy-Schwarz inequality, we deduce that

$$\begin{aligned}\text{(A.11)}\quad &\int_0^t\int_0^R a\left|\partial_x^2 u_{R,n}\right|^2\\ &\leqslant C\Big[\left\|\partial_t u_{R,n}\right\|^2_{L^2(\Omega_T)} + \int_0^t\int_0^R\left|\frac{\partial_x a}{\sqrt{a}}\right|^2 |\partial_x u_{R,n}|^2 + \|b\|^2_{L^\infty(\Omega_T)}\left\|\partial_x u_{R,n}\right\|^2_{L^2(\Omega_T)}\\ &\quad + \left(\|\partial_x b\|^2_{L^2L^\infty} + \|c\|^2_{L^2L^\infty}\right)\|u_{R,n}\|^2_{L^\infty L^2} + \|f\|^2_{L^2(\Omega_T)}\Big].\end{aligned}$$

Combining (A.9) and (A.11), we obtain

$$\begin{aligned}&\int_0^R \frac{a(t)}{2}\left|\partial_x u_{R,n}(t)\right|^2 + \frac{1}{4}\int_0^t\int_0^R\left|\partial_t u_{R,n}\right|^2 + \frac{1}{4C}\int_0^t\int_0^R a\left|\partial_x^2 u_{R,n}\right|^2\\ &\leqslant \int_0^R \frac{a(0)}{2}\left|\partial_x u_{R,n}(0)\right|^2 + \int_0^t\int_0^R\frac{|\partial_t a|}{2}\left|\partial_x u_{R,n}\right|^2 + \frac{1}{4C}\int_0^t\int_0^R\left|\frac{\partial_x a}{\sqrt{a}}\right|^2\left|\partial_x u_{R,n}\right|^2\\ &+ C\Big[\|b\|^2_{L^\infty(\Omega_T)}\left\|\partial_x u_{R,n}\right\|^2_{L^2(\Omega_T)} + \left(\|\partial_x b\|^2_{L^2L^\infty} + \|c\|^2_{L^2L^\infty}\right)\|u_{R,n}\|^2_{L^\infty L^2} + \|f\|^2_{L^2(\Omega_T)}\Big].\end{aligned}$$

Moreover,

$$\begin{aligned}\int_0^R |\partial_t a|\ \left|\partial_x u_{R,n}\right|^2 &\leqslant \|\partial_t a\|_{L^2(0,R)}\left\|\partial_x u_{R,n}\right\|_{L^2(0,R)}\left\|\partial_x u_{R,n}\right\|_{L^\infty(0,R)}\\ &\leqslant C\,\|\partial_t a\|_{L^2(0,R)}\left\|\partial_x u_{R,n}\right\|_{L^2(0,R)}\left\|\partial_x u_{R,n}\right\|_{H^1(0,R)}\end{aligned}$$

so that

$$\begin{aligned}&\int_0^t\int_0^R |\partial_t a|\ \left|\partial_x u_{R,n}\right|^2\\ &\qquad\leqslant \eta\left\|\partial_x^2 u_{R,n}\right\|^2_{L^2(\Omega_T)} + C_\eta\left(\|\partial_t a\|_{L^\infty L^2} + \|\partial_t a\|^2_{L^\infty L^2}\right)\left\|\partial_x u_{R,n}\right\|^2_{L^2(\Omega_T)}.\end{aligned}$$

In the same manner,

$$\begin{aligned}&\int_0^t\int_0^R\left|\frac{\partial_x a}{\sqrt{a}}\right|^2\left|\partial_x u_{R,n}\right|^2\\ &\qquad\leqslant \eta\left\|\partial_x^2 u_{R,n}\right\|^2_{L^2(0,R)} + C_\eta\left(\|\partial_x a\|^2_{L^\infty L^4} + \|\partial_x a\|^4_{L^\infty L^4}\right)\left\|\partial_x u_{R,n}\right\|^2_{L^2(\Omega_T)}.\end{aligned}$$

As a consequence, choosing η small enough, and setting $J(\xi) = \xi + \xi^2$ for $\xi \in \mathbb{R}$,

$$\begin{aligned}
&\alpha \int_0^R |\partial_x u_{R,n}(t)|^2 + \int_0^t \int_0^R |\partial_t u_{R,n}|^2 + \alpha \int_0^t \int_0^R \left|\partial_x^2 u_{R,n}\right|^2 \\
&\leqslant C \Bigg[\|a\|_{L^\infty} \|\partial_x u_{R,n}(0)\|^2_{L^2(0,R)} + \left(J\left(\|\partial_t a\|_{L^\infty L^2}\right) + J\left(\|\partial_x a\|^2_{L^\infty L^4}\right)\right) \|\partial_x u_{R,n}\|^2_{L^2(\Omega_T)} \\
&\quad + \|b\|^2_{L^\infty(\Omega_T)} \|\partial_x u_{R,n}\|^2_{L^2(\Omega_T)} + \left(\|\partial_x b\|^2_{L^2L^\infty} + \|c\|^2_{L^2L^\infty}\right) \|u_{R,n}\|^2_{L^\infty L^2} + \|f\|^2_{L^2(\Omega_T)} \Bigg]
\end{aligned}$$

and eventually, passing to the limit $n \to +\infty$

$$\begin{aligned}
&\|\partial_x u_R\|^2_{L^\infty((0,T),L^2((0,R))} + \|\partial_t u_R\|^2_{L^2((0,T)\times(0,R))} + \left\|\partial_x^2 u_R\right\|^2_{L^2((0,T)\times(0,R))} \\
&\quad \leqslant C \Bigg[\|a\|_{L^\infty} \left\|\partial_x u_R^0\right\|^2_{L^2} + \left(J\left(\|\partial_t a\|_{L^\infty L^2}\right) + J\left(\|\partial_x a\|^2_{L^\infty L^4}\right)\right) \|\partial_x u_R\|^2_{L^2L^2} \\
&\qquad + \|b\|^2_{L^\infty} \|\partial_x u_R\|^2_{L^2L^2} + \left(\|\partial_x b\|^2_{L^2L^\infty} + \|c\|^2_{L^2L^\infty}\right) \|u_R\|^2_{L^\infty L^2} + \|f\|^2_{L^2L^2} \Bigg].
\end{aligned}$$

Estimate (A.5). In a last step, we differentiate (A.7) with respect to t. We find that $u'_{R,n} := \partial_t u_{R,n}$ also satisfies a weak formulation similar to (A.7), namely

$$\begin{aligned}
\text{(A.12)} \quad &\int_0^R \partial_t u'_{R,n}(t) w_{k,R} - \int_0^R b(t) u'_{R,n}(t) \partial_x w_{k,R} + \int_0^R c(t) u'_{R,n}(t) w_{k,R} + \\
&\qquad + \int_0^R a(t) \partial_x u'_{R,n}(t) \partial_x w_{k,R} = \int_0^R \widetilde{f}_{R,n}(t) w_{k,R} + \int_0^R g_{R,n}(t) \partial_x w_{k,R}
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{f}_{R,n} &:= \partial_t f + \partial_t c u_{R,n}, \\
g_{R,n} &:= -\partial_t b u_{R,n} + \partial_t a \partial_x u_{R,n}
\end{aligned}$$

and with the initial data

$$u'_{R,n}(0) = \mathbb{P}_n \left[-\partial_x \left(b(t=0) u_R^0 \right) - c(t=0) u_R^0 + \partial_x \left(a(t=0) \partial_x u_R^0 \right) \right],$$

where $\mathbb{P}_n$ is the orthogonal projection in $L^2(0,R)$ onto $\mathrm{Span}(w_{1,R}, \cdots, w_{n,R})$. Using (A.4), we obtain

$$\begin{aligned}
\left\|\widetilde{f}_{R,n}\right\|_{L^2((0,T)\times(0,R))} &\leqslant C \left(\|\partial_t f\|_{L^2(\Omega_T)} + \|u_{R,n}\|_{L^\infty([0,T],H^1(0,R))} \|\partial_t c\|_{L^2(\Omega_T)} \right), \\
\|g_{R,n}\|_{L^2((0,T)\times(0,R))} &\leqslant C \Big[\|u_{R,n}\|_{L^\infty([0,T],L^2(0,R))} \|\partial_t b\|_{L^2([0,T];L^\infty(\mathbb{R}_+))} \\
&\qquad + \|\partial_t a\|_{L^\infty([0,T],L^2(\mathbb{R}_+))} \|\partial_x u_{R,n}\|_{L^2([0,T],H^1)} \Big],
\end{aligned}$$

so that

$$\begin{aligned}
&\left\|u'_{R,n}\right\|^2_{L^\infty L^2} + \left\|\partial_x u'_{R,n}\right\|^2_{L^2L^2} \\
&\quad \leqslant C\Big[\left\|u^0\right\|^2_{H^2(\mathbb{R}_+)}\left(1+\|b\|^2_{L^\infty_t(W^{1,\infty})}+\|c\|^2_\infty+\|a\|^2_{L^\infty_t(W^{1,\infty})}\right)+A_0\left\|\partial_t b\right\|^2_{L^2L^\infty} \\
&\qquad\qquad + A_1\Big(\|\partial_t c\|^2_{L^2(\Omega_T)}+\|c\|_{L^\infty}+\|\partial_t a\|_{L^\infty L^2}+\|b\|_\infty+1\Big)+\|\partial_t f\|^2_{L^2(\Omega_T)}\Big]
\end{aligned}$$

Passing to the limit as $n\to\infty$, we deduce that u_R satisfies (A.5).

We then extend u_R to $(0,T)\times(0,+\infty)$ in such a way that the extension satisfies the same bounds as u_R. The family thus obtained is compact in $L^2_{\mathrm{loc}}([0,T]\times\mathbb{R}_+)$, and we can extract a subsequence converging weakly in $w^*-L^\infty([0,T],H^2(\mathbb{R}_+)$, and whose time-derivative converges weakly in $w^*-L^\infty([0,T],L^2(\mathbb{R}_+)$ and in $w-L^2([0,T],H^1(\mathbb{R}_+))$. Passing to the limit in (A.6), we find that the limit is a solution of (A.1), satisfying the estimates (A.3), (A.4) and (A.5). □

Appendix B. L^2 estimates for some nonlinear parabolic equations

In this section, we consider diffusion equations of the form

$$\begin{aligned}
&\partial_t g-\widetilde{y}'(t)\partial_x g-\mu\partial_{xx}\ln\left(1+\frac{g}{\bar g}\right)=G,\quad t\in(0,T),\ x\in(0,R),\\
&\qquad\qquad g_{|t=0}=g_0\in H^1_0((0,R)),\\
&\qquad\qquad g_{|x=0}=0,\\
&g_{|x=R}=0\quad\text{if } R<\infty,\quad \lim_{x\to\infty}g(t,x)=0\quad\text{if } R=+\infty.
\end{aligned}\tag{B.1}$$

We prove the following

Lemma B.1. — *Let $G\in L^2((0,T)\times(0,R))$, $g_0\in H^1_0((0,R))$. Assume that $\bar g\in W^{1,\infty}\cap H^1((0,T)\times(0,R))$ and that $\bar g_{|x=0}=1$. Assume furthermore that $M^{-1}\leqslant\widetilde{y}'(t)\leqslant M$ for a.e. $t\in(0,T)$, for some constant $M\geqslant 1$.*

Let $g\in L^\infty([0,T],H^1(0,R))$ be a solution of (B.1) such that $\partial_t g\in L^2((0,T)\times(0,R))$. Assume furthermore that $1\leqslant g+\bar g\leqslant\bar C$, $1\leqslant\bar g\leqslant\bar C$ a.e. for some constant $\bar C>1$.

Then $g/\bar g\in L^\infty([0,T],H^1(0,R))\cap L^2((0,T),H^2((0,R)))$, and there exists a constant C depending only on μ, M and $\|\bar g\|_{W^{1,\infty}([0,T]\times[0,R])}$ such that

$$\begin{aligned}
&\|g\|_{L^\infty([0,T],H^1(0,R))}+\|\partial_t g\|_{L^2((0,T)\times(0,R))}+\|\partial_x g\|_{L^2((0,T)\times(0,R))}\\
&\qquad\qquad\leqslant C\left(\|g_0\|_{H^1}+\|G\|_{L^2((0,T)\times(0,R))}+\|g\|_{L^2((0,T)\times(0,R))}\right),
\end{aligned}$$

and, if $R=+\infty$,

$$\begin{aligned}
\left\|\partial_x^2\left(\frac{g}{\bar g}\right)\right\|_{L^2((0,T)\times\mathbb{R}_+)}&\leqslant C\left(\|g_0\|_{H^1}+\|G\|_{L^2((0,T)\times\mathbb{R}_+)}+\|g\|_{L^2((0,T)\times\mathbb{R}_+)}\right)\\
&\quad+C\inf\left(1,T^{\frac12}\right)\left(\|g_0\|_{H^1}+\|G\|_{L^2((0,T)\times\mathbb{R}_+)}+\|g\|_{L^2((0,T)\times\mathbb{R}_+)}\right)^2.
\end{aligned}$$

Additionally, for all $R > 0$, g satisfies the exponential growth estimate

$$\begin{aligned}&\|g\|_{L^\infty([0,T],H^1(0,R))} + \|\partial_t g\|_{L^2((0,T)\times(0,R))} + \|\partial_x g\|_{L^2((0,T)\times(0,R))}\\ &\leqslant C\left(\|g_0\|_{H^1} + \|G\|_{L^2((0,T)\times(0,R))}\right)\exp((1+\|\partial_x\bar{g}\|_\infty^2)T).\end{aligned}$$

Proof. — The proof is quite classical. The only subtlety lies in the derivation of the $L^\infty(H^1)$ estimate. We only sketch the main steps in order to highlight the dependency on $\bar{g}$. Note that our purpose here is merely to derive energy estimates once the regularity of the solution is known, rather than proving extra regularity for the solution.

• We start with the $L^\infty(L^2)$ estimate. Multiplying (B.1) by g and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^R |g|^2 + \mu\int_0^R \frac{\partial_x(\frac{g}{\bar{g}})}{1+\frac{g}{\bar{g}}}\partial_x g = \int_0^R Gg.$$

Recalling that $\bar{g} \geqslant 1,\ g+\bar{g} \geqslant 1$, we have

$$\int_0^R \frac{\partial_x(\frac{g}{\bar{g}})}{1+\frac{g}{\bar{g}}}\partial_x g \geqslant \frac{1}{2}\int_0^R \frac{(\partial_x g)^2}{g+\bar{g}} - \frac{1}{2}\|\partial_x\bar{g}\|_\infty^2\int_0^R g^2,$$

and therefore

$$\begin{aligned}\text{(B.2)}\quad &\|g\|_{L^\infty([0,T],L^2(0,R))} + \|\partial_x g\|_{L^2((0,T)\times(0,R))}\\ &\leqslant C\left(\|g_0\|_{L^2(0,R)} + \|G\|_{L^2((0,T)\times(0,R))} + (1+\|\partial_x\bar{g}\|_\infty)\,\|g\|_{L^2((0,T)\times(0,R))}\right).\end{aligned}$$

Note that we also obtain the exponential estimate

$$\begin{aligned}&\|g\|_{L^\infty([0,T],L^2(0,R))} + \|\partial_x g\|_{L^2((0,T)\times(0,R))}\\ &\qquad\leqslant C\left(\|g_0\|_{L^2(0,R)} + \|G\|_{L^2((0,T)\times(0,R))}\right)\exp\left(\left(1+\|\partial_x\bar{g}\|_\infty^2\right)T\right).\end{aligned}$$

• Let us now tackle the $L^\infty(H^1)$ estimate. We multiply (B.1) by $\partial_t\ln(1+\frac{g}{\bar{g}})$ and examine each term separately. We have

$$\begin{aligned}&\int_0^R \partial_t g\partial_t\ln\left(1+\frac{g}{\bar{g}}\right)\\ &\qquad= \int_0^R \frac{\left(\partial_t\frac{g}{\bar{g}}\right)^2}{1+\frac{g}{\bar{g}}}\bar{g} - \int_0^R \frac{\partial_t\bar{g}}{\bar{g}}g\frac{\partial_t\frac{g}{\bar{g}}}{1+\frac{g}{\bar{g}}} \geqslant \frac{1}{2}\int_0^R \frac{\left(\partial_t\frac{g}{\bar{g}}\right)^2}{g+\bar{g}}\bar{g}^2 - \frac{1}{2}\|\partial_t\bar{g}\|_\infty^2\int_0^R g^2.\end{aligned}$$

As for the diffusion term, we recognize a time derivative

$$\begin{aligned}-\mu\int_0^R \partial_{xx}\ln\left(1+\frac{g}{\bar{g}}\right)\partial_t\ln\left(1+\frac{g}{\bar{g}}\right) &= \mu\int_0^R \partial_x\ln\left(1+\frac{g}{\bar{g}}\right)\partial_t\partial_x\ln\left(1+\frac{g}{\bar{g}}\right)\\ &= \frac{\mu}{2}\frac{d}{dt}\int_0^R\left(\partial_x\ln\left(1+\frac{g}{\bar{g}}\right)\right)^2.\end{aligned}$$

Transferring the transport term into the right-hand side and using a Cauchy-Schwarz inequality, we obtain

$$\left|\int_0^R (G-\widetilde{y}'(t)\partial_x g)\,\partial_t \ln\left(1+\frac{g}{\bar g}\right)\right| \leqslant \frac{1}{4}\int_0^R \frac{\left(\partial_t \frac{g}{\bar g}\right)^2}{g+\bar g}\bar g^2 + C\int_0^R \left(G^2+(\partial_x g)^2\right).$$

Gathering all the terms and recalling that $\bar g \geqslant 1$, $g+\bar g \leqslant \bar C$, we obtain

$$\begin{aligned}\left\|\partial_t \frac{g}{\bar g}\right\|_{L^2((0,T)\times(0,R))} &+ \left\|\partial_x \frac{g}{\bar g}\right\|_{L^\infty((0,T),L^2(0,R))}\\ &\leqslant C\Bigg(\left\|\partial_x \frac{g_0}{\bar g(t=0)}\right\|_{L^2(0,R)} + \|G\|_{L^2((0,T)\times(0,R))} + \|\partial_x g\|_{L^2((0,T)\times(0,R))}\\ &\quad + \|\partial_t \bar g\|_{L^\infty}\,\|g\|_{L^2((0,T)\times(0,R))}\Bigg).\end{aligned}$$

Recalling (B.2), we obtain the estimate announced in the Lemma for $\|\partial_x g\|_{L^\infty(H^1)}$ and $\|\partial_t g\|_{L^2(L^2)}$.

• Let us now derive the estimate on the second derivative in the case $R=+\infty$. Let us set $f=\partial_x \ln(1+\frac{g}{\bar g})$. According to the previous estimates, we know that $f\in L^\infty((0,T),L^2(\mathbb{R}_+)$. Furthermore, using equation (B.1), we have

$$\mu\partial_x f = \partial_t g - \widetilde{y}'(t)\partial_x g - G \in L^2((0,T)\times\mathbb{R}_+).$$

Hence, using the previous estimates,

$$\begin{aligned}\|\partial_x f\|_{L^2((0,T)\times\mathbb{R}_+)} &\leqslant C\left(\|\partial_t g\|_{L^2((0,T)\times\mathbb{R}_+)} + \|\partial_x g\|_{L^2((0,T)\times\mathbb{R}_+)} + \|G\|_{L^2((0,T)\times\mathbb{R}_+)}\right)\\ &\leqslant C\left(\|g_0\|_{H^1(\mathbb{R}_+)} + \|g\|_{L^2((0,T)\times\mathbb{R}_+)} + \|G\|_{L^2((0,T)\times\mathbb{R}_+)}\right).\end{aligned}$$

In particular, the Gagliardo–Nirenberg–Sobolev inequality entails that

$$\begin{aligned}\|f\|_{L^4((0,T)\times\mathbb{R}_+)} &\leqslant C\left(\int_0^T \|f(t)\|^3_{L^2(\mathbb{R}_+)}\,\|\partial_x f(t)\|_{L^2(\mathbb{R}_+)}\right)^{1/4}\\ &\leqslant C\|f\|^{1/2}_{L^\infty((0,T),L^2(\mathbb{R}_+))}\|f\|^{1/4}_{L^2((0,T)\times\mathbb{R}_+)}\,\|\partial_x f\|^{1/4}_{L^2((0,T)\times\mathbb{R}_+)}\\ &\leqslant C\left(\|g_0\|_{H^1(\mathbb{R}_+)} + \|g\|_{L^2((0,T)\times\mathbb{R}_+)} + \|G\|_{L^2((0,T)\times\mathbb{R}_+)}\right).\end{aligned}$$

Now

$$\partial_x f = \partial_{xx}\ln\left(1+\frac{g}{\bar g}\right) = \partial_x\left(\frac{\partial_x \frac{g}{\bar g}}{1+\frac{g}{\bar g}}\right) = \frac{\partial_{xx}\frac{g}{\bar g}}{1+\frac{g}{\bar g}} - f^2.$$

Thus

$$\begin{aligned}&\|\partial_{xx} g\|_{L^2((0,T)\times\mathbb{R}_+)}\\ &\qquad\leqslant C\left(\|\partial_x f\|_{L^2((0,T)\times\mathbb{R}_+)} + \|f\|^2_{L^4((0,T)\times\mathbb{R}_+)}\right)\\ &\qquad\leqslant C\left(\|g_0\|_{H^1(\mathbb{R}_+)} + \|g\|_{L^2((0,T)\times\mathbb{R}_+)} + \|G\|_{L^2((0,T)\times\mathbb{R}_+)}\right)\\ &\qquad+ C\inf\left(1,T^{1/2}\right)\left(\|g_0\|_{H^1(\mathbb{R}_+)} + \|g\|_{L^2((0,T)\times\mathbb{R}_+)} + \|G\|_{L^2((0,T)\times\mathbb{R}_+)}\right)^2.\end{aligned}$$

□

Appendix C. Two technical Lemmas

Lemma C.1. — *Let $F \in L^2(\mathbb{R}_+)$ such that $\sqrt{x}F \in L^2(\mathbb{R}_+)$, and let $M \geqslant 1$. Let $\widetilde{y} \in L^\infty([0,T])$ such that $\widetilde{y}(t) \geqslant t/M$ a.e. Then for all $T > 0$, $R > 0$*

$$\int_0^T \int_0^R F^2\left(x + \widetilde{y}(t)\right) dx\, dt \leqslant M \int_0^\infty z F^2(z)\, dz.$$

Proof. — We write

$$\int_0^T \int_0^R F^2(x + \widetilde{y}(t)) dx\, dt = \int_0^T \int_{\widetilde{y}(t)}^{R+\widetilde{y}(t)} F^2(z)\, dz\, dt \leqslant \int_0^T \int_0^\infty \mathbf{1}_{z > \widetilde{y}(t)} F^2(z)\, dz\, dt.$$

Using the assumption on $\widetilde{y}$, we deduce

$$\begin{aligned}\int_0^T \int_0^R F^2\left(x + \widetilde{y}(t)\right) dx\, dt &\leqslant \int_0^T \int_0^\infty \mathbf{1}_{z\,>\,t/M} F^2(z)\, dz\, dt \\ &\leqslant M \int_0^\infty z F^2(z)\, dz. \qquad \square\end{aligned}$$

Lemma C.2. — *Let $\widetilde{y}_1, \widetilde{y}_2 \in L^\infty_{\mathrm{loc}}([0,T])$ and such that $\widetilde{y}_i' \in L^2 \cap L^\infty(0,T)$. Assume that $\widetilde{y}_1(0) = \widetilde{y}_2(0) = 0$ and that $M^{-1} \leqslant \widetilde{y}_i' \leqslant M$ for some constant $M \geqslant 1$ and for $i = 1, 2$. Let $w^0 \in W^{1,\infty}(\mathbb{R}_+)$ such that $\sqrt{x}\partial_x w^0, \sqrt{x}\partial_x^2 w^0 \in L^2(\mathbb{R}_+)$.*

Then there exists a constant C depending only on M such that

$$\begin{aligned}&\left\|w^0(\widetilde{y}_1) - w^0(\widetilde{y}_2)\right\|_{L^\infty(0,T)} \\ &\quad \leqslant C\left(\left\|\sqrt{x}\partial_x w^0\right\|_{L^2(\mathbb{R}_+)} + \left\|\sqrt{x}\partial_x^2 w^0\right\|_{L^2(\mathbb{R}_+)} + \left\|\partial_x w^0\right\|_{L^\infty(\mathbb{R}_+)}\right) \|\widetilde{y}_1' - \widetilde{y}_2'\|_{L^2(0,T)}\end{aligned}$$

and

$$\left\|w^0(\widetilde{y}_1) - w^0(\widetilde{y}_2)\right\|_{L^2(0,T)} \leqslant C\, \|\widetilde{y}_1' - \widetilde{y}_2'\|_{L^2(0,T)} \left\|\sqrt{z}\partial_x w^0(z)\right\|_{L^2(\mathbb{R}_+)}.$$

Proof. — First, using the Sobolev embedding $H^1(0,T) \subset L^\infty(0,T)$, we have

$$\begin{aligned}\|w^0(\widetilde{y}_1) - w^0(\widetilde{y}_2)\|_{L^\infty(0,T)} \leqslant C\Big(&\|w^0(\widetilde{y}_1) - w^0(\widetilde{y}_2)\|_{L^2(0,T)} \\ &+ \|\partial_t(w^0(\widetilde{y}_1) - w^0(\widetilde{y}_2))\|_{L^2(0,T)}\Big).\end{aligned}$$

Let us start with the first term in the right-hand side. Using a Taylor formula,

$$w^0(\widetilde{y}_1(t)) - w^0(\widetilde{y}_2(t)) = (\widetilde{y}_1(t) - \widetilde{y}_2(t)) \int_0^1 \partial_x w^0\left(\tau \widetilde{y}_1(t) + (1-\tau)\widetilde{y}_2(t)\right)\, d\tau.$$

For a.e. $t \in [0,T]$, the Cauchy-Schwarz inequality implies

$$|\widetilde{y}_1(t) - \widetilde{y}_2(t)| = \left|\int_0^t (\widetilde{y}_1' - \widetilde{y}_2')\right| \leqslant t^{1/2}\, \|\widetilde{y}_1' - \widetilde{y}_2'\|_{L^2(0,T)}.$$

Hence, setting $z = \tau\widetilde{y}_1(t) + (1-\tau)\widetilde{y}_2(t)$ and observing that $z/M \leqslant t \leqslant Mz$,

$$\begin{aligned}\left\|w^0(\widetilde{y}_1) - w^0(\widetilde{y}_2)\right\|^2_{L^2(0,T)} &\leqslant \|\widetilde{y}_1' - \widetilde{y}_2'\|^2_{L^2(0,T)} \\ &\quad \times \int_0^T \int_0^1 t\left(\partial_x w^0\left(\tau\widetilde{y}_1(t) + (1-\tau)\widetilde{y}_2(t)\right)\right)^2 d\tau\, dt \\ &\leqslant M^2 \|\widetilde{y}_1' - \widetilde{y}_2'\|^2_{L^2(0,T)} \int_0^{MT}\int_0^1 z\left(\partial_x w^0(z)\right)^2 d\tau\, dz \\ &\leqslant M^2 \|\widetilde{y}_1' - \widetilde{y}_2'\|^2_{L^2(0,T)} \left\|\sqrt{z}\partial_x w^0(z)\right\|^2_{L^2(\mathbb{R}_+)}.\end{aligned}$$

The second term is treated in a similar fashion. □

Appendix D. Proof of Lemma 6.3

We recall that

$$\mathcal{A} = -s\mathrm{Id} - \mu\partial_x\left(\frac{\cdot}{\bar{v}}\right).$$

Let $\varphi \in H^2(\mathbb{R}_+)$ be arbitrary.

- Integrating by parts and recalling that $\bar{v}(0) = 1$,

$$\begin{aligned}\int_0^\infty (\mathcal{A}\partial_x\varphi)\varphi &= -\int_0^\infty s\partial_x\varphi\,\varphi - \mu\int_0^\infty \partial_x\left(\frac{\partial_x\varphi}{\bar{v}}\right)\varphi \\ &= \frac{s}{2}(\varphi(0))^2 + \mu\int_0^\infty \frac{(\partial_x\varphi)^2}{\bar{v}} + \mu\varphi(0)\varphi'(0).\end{aligned}$$

- In a similar fashion, for any $\rho \in \mathcal{C}^2_b(\mathbb{R}_+)$,

$$\begin{aligned}\int_0^\infty (\partial_x\mathcal{A}\varphi)\frac{\varphi}{\bar{v}}\rho &= -s\int_0^\infty \partial_x\varphi\,\frac{\varphi}{\bar{v}}\rho - \mu\int_0^\infty \left(\partial_x^2\left(\frac{\varphi}{\bar{v}}\right)\right)\left(\frac{\varphi}{\bar{v}}\right)\rho \\ &= \frac{s}{2}\varphi(0)^2\rho(0) + \frac{s}{2}\int_0^\infty \varphi^2\partial_x\left(\frac{\rho}{\bar{v}}\right) + \mu\int_0^\infty\left(\partial_x\left(\frac{\varphi}{\bar{v}}\right)\right)^2\rho \\ &\quad + \mu\int_0^\infty\left(\partial_x\left(\frac{\varphi}{\bar{v}}\right)\right)\left(\frac{\varphi}{\bar{v}}\right)\partial_x\rho + \mu\partial_x\left(\frac{\varphi}{\bar{v}}\right)_{|x=0}\varphi(0)\rho(0) \\ &= \mu\int_0^\infty\left(\partial_x\left(\frac{\varphi}{\bar{v}}\right)\right)^2\rho + \frac{s}{2}\int_0^\infty \varphi^2\partial_x\left(\frac{\rho}{\bar{v}}\right) \\ &\quad - \frac{\mu}{2}\int_0^\infty\left(\frac{\varphi}{\bar{v}}\right)^2\rho'' \\ &\quad + \frac{s}{2}\varphi(0)^2\rho(0) - \frac{\mu}{2}\varphi(0)^2\rho'(0) + \mu\partial_x\left(\frac{\varphi}{\bar{v}}\right)_{|x=0}\varphi(0)\rho(0).\end{aligned}$$

The result follows.

BIBLIOGRAPHY

[BBCR00] François Bouchut, Yann Brenier, Julien Cortes, and Jean-François Ripoll, *A hierarchy of models for two-phase flows*, J. Nonlinear Sci. **10** (2000), no. 6, 639–660. ↑1177

[BD06] Didier Bresch and Benoît Desjardins, *On the construction of approximate solutions for the 2D viscous shallow water model and for compressible Navier–Stokes models*, J. Math. Pures Appl. **86** (2006), no. 4, 362–368. ↑1178

[BG17] Florent Berthelin and Paola Goatin, *Particle approximation of a constrained model for traffic flow*, NoDEA, Nonlinear Differ. Equ. Appl. **24** (2017), no. 5, article no. 55. ↑1177

[BL22] Geoffrey Beck and David Lannes, *Freely floating objects on a fluid governed by the Boussinesq equations*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire **39** (2022), no. 3, 575–646. ↑1178

[BLM21] Didier Bresch, David Lannes, and Guy Métivier, *Waves interacting with a partially immersed obstacle in the Boussinesq regime*, Anal. PDE **14** (2021), no. 4, 1085–1124. ↑1178

[Boc20] Edoardo Bocchi, *Floating structures in shallow water: local well-posedness in the axisymmetric case*, SIAM J. Math. Anal. **52** (2020), no. 1, 306–339. ↑1178

[BPZ14] Didier Bresch, Charlotte Perrin, and Ewelina Zatorska, *Singular limit of a Navier–Stokes system leading to a free/congested zones two-phase model*, C. R. Math. **352** (2014), no. 9, 685–690. ↑1176

[BR17] Didier Bresch and Micheal Renardy, *Development of congestion in compressible flow with singular pressure*, Asymptotic Anal. **103** (2017), no. 1-2, 95–101. ↑1178

[CGS16] Rinaldo Colombo, Graziano Guerra, and Veronika Schleper, *The Compressible to Incompressible Limit of One Dimensional Euler Equations: The Non Smooth Case*, Arch. Ration. Mech. Anal. **219** (2016), no. 2, 701–718. ↑1178

[DHN11] Pierre Degond, Jiale Hua, and Laurent Navoret, *Numerical simulations of the Euler system with congestion constraint*, J. Comput. Phys. **230** (2011), no. 22, 8057–8088. ↑1177

[DP20] Anne-Laure Dalibard and Charlotte Perrin, *Existence and stability of partially congested propagation fronts in a one-dimensional Navier–Stokes model*, Commun. Math. Sci. **18** (2020), no. 7, 1775–1813. ↑1177, 1178, 1179

[Goo86] Jonathan Goodman, *Nonlinear asymptotic stability of viscous shock profiles for conservation laws*, Arch. Ration. Mech. Anal. **95** (1986), 325–344. ↑1194

[GPSMW18] Edwige Godlewski, Martin Parisot, Jacques Sainte-Marie, and Fabien Wahl, *Congested shallow water model: roof modeling in free surface flow*, ESAIM, Math. Model. Numer. Anal. **52** (2018), no. 5, 1679–1707. ↑1177

[Has18] Boris Haspot, *Vortex solutions for the compressible Navier–Stokes equations with general viscosity coefficients in 1D: regularizing effects or not on the density*, preprint HAL hal-01716150, `https://hal.science/hal-01716150/`, 2018. ↑1178

[IL19] Tatsuo Iguchi and David Lannes, *Hyperbolic free boundary problems and applications to wave-structure interactions*, Indiana Univ. Math. J. **70** (2019), 353–464. ↑1178, 1179, 1181

[Lan17] David Lannes, *On the dynamics of floating structures*, Ann. PDE **3** (2017), no. 1, article no. 11. ↑1177

[LM99] Pierre-Louis Lions and Nader Masmoudi, *On a free boundary barotropic model*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire **16** (1999), no. 3, 373–410. ↑1177

[LSU68] Olga A. Ladyženskaja, Vsevolod A. Solonnikov, and Nina N. Ural'tseva, *Linear and quasilinear equations of parabolic type*, Translations of Mathematical Monographs, vol. 23, American Mathematical Society, 1968, Translated from the Russian by S. Smith. ↑1186

[Mau12] Bertrand Maury, *Prise en compte de la congestion dans les modèles de mouvements de foules*, Actes des colloques "EDP-Normandie", Caen 2010 – Rouen 2011 (Christian Dogbe et al., eds.), Normandie-Mathématiques, Fédération Normandie-Mathématiques, 2012, pp. 7–20. ↑1177

[MRCSV11] Bertrand Maury, Aude Roudneff-Chupin, Filippo Santambrogio, and Juliette Venel, *Handling congestion in crowd motion modeling*, Netw. Heterog. Media **6** (2011), no. 3, 485–519. ↑1177

[MSMTT19] Debayan Maity, Jorge San Martín, Takéo Takahashi, and Marius Tucsnak, *Analysis of a simplified model of rigid structure floating in a viscous fluid*, J. Nonlinear Sci. **29** (2019), no. 5, 1975–2020. ↑1178

[PS22] Charlotte Perrin and Khaled Saleh, *Numerical Staggered Schemes for the Free-Congested Navier–Stokes Equations*, SIAM J. Numer. Anal. **60** (2022), no. 4, 1824–1852. ↑1177

[PZ15] Charlotte Perrin and Ewelina Zatorska, *Free/congested two-phase model from weak solutions to multi-dimensional compressible Navier–Stokes equations*, Commun. Partial Differ. Equations **40** (2015), no. 8, 1558–1589. ↑1177

[She84] Vladimir V. Shelukhin, *On the structure of generalized solutions of the one-dimensional equations of a polytropic viscous gas*, J. Appl. Math. Stochastic Anal. **48** (1984), no. 6, 665–672. ↑1178

[Shi16] Yoshihiro Shibata, *On the $\mathcal{R}$-Boundedness for the Two Phase Problem with Phase Transition: Compressible-Incompressible Model Problem*, Funkc. Ekvacioj, Ser. Int. **59** (2016), no. 2, 243–287. ↑1178

[VY16] Alexis F. Vasseur and Lei Yao, *Nonlinear stability of viscous shock wave to one-dimensional compressible isentropic Navier–Stokes equations with density dependent viscous coefficient*, Commun. Math. Sci. **14** (2016), no. 8, 2215–2228. ↑1178, 1194

[Wag87] David H Wagner, *Equivalence of the Euler and Lagrangian equations of gas dynamics for weak solutions*, J. Differ. Equations **68** (1987), no. 1, 118–136. ↑1184

Manuscript received on 8th November 2021,
revised on 24th July 2023,
accepted on 28th November 2023.

Recommended by Editors San Vu Ngoc and N. Seguin.

eISSN: 2644-9463
This journal is a member of Centre Mersenne.

Anne-Laure DALIBARD
Sorbonne Université,
Université Paris-Diderot SPC,

CNRS, Laboratoire Jacques-Louis Lions,
LJLL, F-75005 Paris
& École Normale Supérieure,
Université PSL,
Département de Mathématiques
et applications,
F-75005, Paris (France)
anne-laure.dalibard@sorbonne-universite.fr

Charlotte PERRIN
Aix Marseille Univ,
CNRS, I2M, Marseille (France)
charlotte.perrin@univ-amu.fr