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L^p CARLEMAN ESTIMATES FOR ELLIPTIC BOUNDARY VALUE PROBLEMS AND APPLICATIONS TO THE QUANTIFICATION OF UNIQUE CONTINUATION

ESTIMÉES DE CARLEMAN L^p POUR DES PROBLÈMES AUX LIMITES ELLIPTIQUES ET APPLICATIONS À LA QUANTIFICATION DU PROLONGEMENT UNIQUE

Abstract. — The aim of this work is to prove global L^p Carleman estimates for the Laplace operator in dimension $d \geqslant 3$. Our strategy relies on precise Carleman estimates in strips and a suitable gluing of local and boundary estimates obtained through a change of

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variables. The delicate point and most of the work thus consists in proving Carleman estimates in the strip with a linear weight function for a second-order operator with coefficients depending linearly on the normal variable. This is done by constructing an explicit parametrix for the conjugated operator, which is estimated through the use of Stein–Tomas restriction theorems. As an application, we deduce quantified versions of the unique continuation property for solutions of $\Delta u = Vu + W_1 \cdot \nabla u + \mathrm{div}(W_2 u)$ in terms of the norms of V in $L^{q_0}(\Omega)$, of W_1 in $L^{q_1}(\Omega)$ and of W_2 in $L^{q_2}(\Omega)$ for $q_0 \in (\frac{d}{2}, \infty]$ and q_1 and q_2 satisfying either q_1, $q_2 > \frac{3d-2}{2}$ and $\frac{1}{q_1} + \frac{1}{q_2} < 4(1 - \frac{1}{d})/(3d-2)$, or q_1, $q_2 > \frac{3d}{2}$.

Résumé. — L'objectif de ce travail est de démontrer des estimations de Carleman L^p globales pour l'opérateur Laplacien en dimension $d \geqslant 3$. Notre stratégie repose sur des estimations de Carleman sur des bandes puis un recollement approprié des estimations locales et au bord obtenues grâce à un changement de variables. L'essentiel du travail consiste à prouver des estimations de Carleman dans la bande avec une fonction poids linéaire pour un opérateur du second ordre à coefficients dépendant linéairement de la variable normale. Cela est réalisé par la construction d'une paramétrice explicite pour l'opérateur conjugué, qui est estimée grâce à l'utilisation des théorèmes de restriction de Stein–Tomas. En application, nous déduisons des versions quantifiées de la propriété de prolongement unique pour les solutions de $\Delta u = Vu + W_1 \cdot \nabla u + \mathrm{div}(W_2 u)$ en termes des normes de V dans $L^{q_0}(\Omega)$, de W_1 dans $L^{q_1}(\Omega)$ et de W_2 dans $L^{q_2}(\Omega)$ pour $q_0 \in (\frac{d}{2}, \infty]$ et q_1 et q_2 satisfaisant soit q_1, $q_2 > \frac{3d-2}{2}$ et $\frac{1}{q_1} + \frac{1}{q_2} < 4(1 - \frac{1}{d})/(3d-2)$, soit q_1, $q_2 > \frac{3d}{2}$.

1. Introduction

Main result

The goal of this article is to prove global L^p Carleman estimates for the flat Laplace operator in a smooth bounded domain of $\mathbb{R}^d$ ($d \geqslant 3$) for a general weight function satisfying the sub-ellipticity conditions of Hörmander. As an application, we will show how these can be used to obtain quantitative unique continuation results for solutions of elliptic equation with respect to the norms of the potentials.

To be more precise, our main result is the following one:

Theorem 1.1. — *Let $d \geqslant 3$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of class C^3, and ω be a non-empty open subset of Ω with $\overline{\omega} \subset \Omega$. Let $\varphi \in C^3(\overline{\Omega})$ be such that*

$$\forall\, x \in \partial\Omega,\ \varphi(x) = 0 \text{ and } \partial_n \varphi(x) < 0, \tag{1.1}$$

and there exists $\alpha, \beta > 0$ for which

$$\inf_{\overline{\Omega} \setminus \omega} |\nabla \varphi| > \alpha, \tag{1.2}$$

and

$$\begin{aligned} &\forall\, x \in \overline{\Omega} \setminus \omega, \forall\, \xi \in \mathbb{R}^d \quad \text{with} \quad |\nabla\varphi(x)| = |\xi| \quad \text{and} \quad \nabla\varphi(x) \cdot \xi = 0, \\ &\qquad (\mathrm{Hess}\,\varphi(x))\nabla\varphi(x) \cdot \nabla\varphi(x) + (\mathrm{Hess}\,\varphi(x))\xi \cdot \xi \geqslant \beta |\nabla\varphi(x)|^2, \end{aligned} \tag{1.3}$$

where $\mathrm{Hess}\,\varphi$ denotes the Hessian matrix of φ. Let ω_1 be an open subset of Ω so that $\overline{\omega} \subset \omega_1$ and $\overline{\omega}_1 \subset \Omega$, and η be a smooth radial non-negative cut-off function (in $\mathscr{C}^\infty_c(\mathbb{R}^d)$) vanishing outside the ball of radius 1 and equal to one in the ball of radius $\frac{1}{2}$.

Then there exist $C > 0$ and $\tau_0 \geqslant 1$ (depending only on α, β, $\|\varphi\|_{C^3(\overline{\Omega})}$, η and the geometric configuration Ω, ω, and ω_1) such that for all $u \in H^1(\Omega)$ solution of

$$(1.4) \qquad \begin{cases} -\Delta u = f_2 + f_{2'_*} + \operatorname{div} F & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

with

$$f_2 \in L^2(\Omega), \quad f_{2'_*} \in L^{\frac{2d}{d+2}}(\Omega), \quad F \in L^2(\Omega;\mathbb{C}^d), \quad \text{and} \quad g \in H^{\frac{1}{2}}(\partial\Omega),$$

we have, for all $\tau \geqslant \tau_0$,

$$(1.5) \quad \tau^{\frac{3}{2}} \left\| e^{\tau\varphi} u \right\|_{L^2(\Omega)} + \tau^{\frac{1}{2}} \left\| e^{\tau\varphi} \nabla u \right\|_{L^2(\Omega)} \leqslant C \Bigg(\left\| e^{\tau\varphi} f_2 \right\|_{L^2(\Omega)} + \tau^{\frac{3}{4}-\frac{1}{2d}} \left\| e^{\tau\varphi} f_{2'_*} \right\|_{L^{\frac{2d}{d+2}}_{\eta,\tau}(\Omega)} + \tau \left\| e^{\tau\varphi} F \right\|_{L^2(\Omega)} + \tau^{\frac{3}{4}} \|g\|_{H^{\frac{1}{2}}(\partial\Omega)} + \tau^{\frac{3}{2}} \left\| e^{\tau\varphi} u \right\|_{L^2(\omega_1)} + \tau^{\frac{3}{4}} \left\| e^{\tau\varphi} u \right\|_{L^{\frac{2d}{d-2}}_{\eta,\tau}(\omega_1)} \Bigg),$$

and

$$(1.6) \quad \tau^{\frac{3}{4}+\frac{1}{2d}} \left\| e^{\tau\varphi} u \right\|_{L^{\frac{2d}{d-2}}_{\eta,\tau}(\Omega)} \leqslant C \Bigg(\left\| e^{\tau\varphi} f_2 \right\|_{L^2(\Omega)} + \tau^{\frac{3}{4}+\frac{1}{2d}} \left\| e^{\tau\varphi} f_{2'_*} \right\|_{L^{\frac{2d}{d+2}}_{\eta,\tau}(\Omega)} + \tau \left\| e^{\tau\varphi} F \right\|_{L^2(\Omega)} + \tau^{\frac{3}{4}+\frac{1}{2d}} \|g\|_{H^{\frac{1}{2}}(\partial\Omega)} + \tau^{\frac{3}{2}} \left\| e^{\tau\varphi} u \right\|_{L^2(\omega_1)} + \tau^{\frac{3}{4}+\frac{1}{2d}} \left\| e^{\tau\varphi} u \right\|_{L^{\frac{2d}{d-2}}_{\eta,\tau}(\omega_1)} \Bigg).$$

Here, the norms $\|\cdot\|_{L^p_{\eta,\tau}(\Omega)}$ are defined for $p \in [1,\infty]$ for $f \in L^p(\Omega)$ by the formula

$$(1.7) \qquad \|f\|^2_{L^p_{\eta,\tau}(\Omega)} = \tau^{\frac{d}{3}} \int_{x_0 \in \Omega} \left\| \eta\left(\tau^{\frac{1}{3}}(\cdot - x_0)\right) f(\cdot) \right\|^2_{L^p(\Omega)} dx_0.$$

Remark 1.2. — The notations 2_* and $2'_*$ stem from the Sobolev's embedding $H^1(\Omega) \subset L^{2_*}(\Omega)$, with $2_* = \frac{2d}{d-2}$ and $L^{2'_*}(\Omega) \subset H^{-1}(\Omega)$, with $2'_* = \frac{2d}{d+2}$.

Before going further, let us remark that the existence of a function satisfying the conditions (1.1)–(1.2)–(1.3) for any arbitrary geometric setting is due to Fursikov and Imanuvilov [FI96, Lemma 1.1] (see also [LRLR22, Proposition 3.31]). The conditions (1.2)–(1.3) are the *sub-ellipticity conditions* of the weight function φ with respect to the Laplace operator, which are known to be necessary and sufficient conditions to get a local L^2 Carleman estimate (i.e. (1.5) for compactly supported functions u, and with $f_{2'_*} = 0$) for the Laplace operator with the same powers of the Carleman parameter τ, see [Hör94, Chapter XXVIII] and, for instance [LRLR22, Definition 3.2, Section 3.6 and Section 4.1.2] for a more recent perspective.

The Carleman estimate (1.5) coincides with the one in [IP03] except for the terms involving the norm

$$L^{\frac{2d}{d+2}}_{\eta,\tau}(\Omega).$$

This term and the estimate (1.6) on

$$u \text{ in } L^{\frac{2d}{d-2}}_{\eta,\tau}(\Omega)$$

are the main novelties of our result and allow us to quantify efficiently unique continuation properties for solutions of elliptic equations with respect to the norms of potentials in $L^p(\Omega)$.

Note that estimate (1.5) implies an estimate on $\tau^{\frac{1}{2}}\|e^{\tau\varphi}u\|_{H^1(\Omega)}$ from the right hand side of (1.5), thus on $\tau^{\frac{1}{2}}\|e^{\tau\varphi}u\|_{L^{\frac{2d}{d-2}}(\Omega)}$. Therefore, estimate (1.5) does not allow to recover estimate (1.6) directly from classical Sobolev's embeddings.

Local L^p Carleman estimates (i.e Carleman estimates for compactly supported functions) have been derived in many situations, but usually to focus on questions related to unique continuation. We should in particular quote the breakthrough article [JK85] obtained for a radial weight $\log(|x|)$, which rather corresponds to a limiting Carleman weight in the sense that the second condition (1.3) is satisfied with $\beta = 0$ (see also the previous results [ABG81, Hör83]). Later, several works have been devoted to get local Carleman estimates with some specific strictly convex weights, see e.g. [BKRS88, Sog89, Sog90], which have later been revisited and improved in the works [KT01, DSF05]. We also point out the more recent works [Dav20, DZ19] for local Carleman estimates with some specific strictly convex weight. Here, we emphasize that we will consider general weight functions satisfying the sub-ellipticity conditions (1.2)–(1.3), similarly as in [DSF05], which considers the more general case of second order real principal type operators of order 2. In fact, in our context, the article [DSF05] proves that, for all $x_0 \in \mathbb{R}^d$, if φ is subelliptic at x_0 (that is $|\nabla\varphi(x_0)| \neq 0$ and condition (1.3) at $x = x_0$), there exists a neighborhood K of x_0 such that the local Carleman estimate $\|e^{\tau\varphi}u\|_{L^{\frac{2d}{d-2}}} \leqslant C\|e^{\tau\varphi}\Delta u\|_{L^{\frac{2d}{d+2}}}$ holds for all u compactly supported in K. The estimates (1.5)–(1.6) thus extend the result in [DSF05] by providing a global Carleman estimate, allowing source terms in $H^{-1}(\Omega)$ and boundary conditions in $H^{\frac{1}{2}}(\partial\Omega)$, and estimating u in the $H^1(\Omega)$-norm as well.

Finally, let us also emphasize that Theorem 1.1 presents global L^p Carleman estimates, in the sense that the Carleman estimates (1.5)–(1.6) hold for functions u having possibly non-zero trace on the boundary. To our knowledge, this is new, as all the L^p Carleman estimates with $p \neq 2$ that we have encountered in the literature hold for compactly supported functions.

Properties of the $L^p_{\eta,\tau}(\Omega)$ norms

To understand the norms $L^p_{\eta,\tau}(\Omega)$, let us first remark that for $p = 2$, and $f \in L^2_{\eta,\tau}(\Omega)$, by Fubini's theorem, we have, for τ large enough,

$$\|f\|^2_{L^2_{\eta,\tau}(\Omega)} \leqslant \|\eta\|^2_{L^2(\mathbb{R}^d)}\|f\|^2_{L^2(\Omega)},$$
$$\|f\|^2_{L^2_{\eta,\tau}(\Omega)} \geqslant \|f\|^2_{L^2(\Omega)} \left(\inf_{x_0 \in \Omega} \tau^{\frac{d}{3}} \int_{x \in \Omega} \left|\eta\left(\tau^{\frac{1}{3}}(x - x_0)\right)\right|^2 dx\right) \geqslant c_*\|f\|^2_{L^2(\Omega)}.$$

Here, $c_* > 0$ is independent of $\tau \geqslant \tau_0$ if τ_0 is chosen so that $\tau_0 \geqslant \varepsilon_0^{-3}$, where $\varepsilon_0 > 0$ is such that for all $\varepsilon \in (0, \varepsilon_0]$ and $x \in \Omega$, there exists a ball of radius $\varepsilon/8$ contained in $B(x, \varepsilon/2) \cap \Omega$ (it is not difficult to check that such an $\varepsilon > 0$ exists by compactness and smoothness of the boundary $\partial\Omega$). The norms $L^2_{\eta,\tau}(\Omega)$ are thus equivalent to the usual $L^2(\Omega)$ norm uniformly with respect to the parameter τ.

For other values of $p \in [1, \infty)$, these norms are less easy to describe, as they somehow encode some mean information on the L^p-norms localized in balls of radius $\tau^{-\frac{1}{3}}$, as one can see by writing them under the form

$$\|f\|_{L^p_{\eta,\tau}(\Omega)} = \tau^{\frac{d}{6}} \left\| \left\| \eta\left(\tau^{\frac{1}{3}}(x-x_0)\right) f(x) \right\|_{L^p_x(\Omega)} \right\|_{L^2_{x_0}(\Omega)}.$$

In fact, for $p \in (1,\infty)$, again by Fubini's theorem, there exists $C>0$, such that for $\tau \geqslant \tau_0$ and $f \in L^p(\Omega)$,

$$\begin{aligned}(1.8)\quad \frac{\tau^{\frac{d}{3}}}{C} \int_{x\in\Omega}\int_{x_0\in\Omega} \left|\eta\left(\tau^{\frac{1}{3}}(x-x_0)\right) f(x)\right|^p dx\,dx_0 &\leqslant \|f\|^p_{L^p(\Omega)} \\ &\leqslant C\tau^{\frac{d}{3}} \int_{x\in\Omega}\int_{x_0\in\Omega} \left|\eta\left(\tau^{\frac{1}{3}}(x-x_0)\right) f(x)\right|^p dx\,dx_0,\end{aligned}$$

i.e. the L^p norm $\|f\|_{L^p(\Omega)}$ is equivalent to the norm

$$\tau^{\frac{d}{3p}} \left\| \left\| \eta\left(\tau^{\frac{1}{3}}(x-x_0)\right) f(x) \right\|_{L^p_x(\Omega)} \right\|_{L^p_{x_0}(\Omega)}.$$

This implies in particular that, for $p>2$,

$$\begin{aligned}(1.9)\quad \|f\|_{L^p_{\eta,\tau}(\Omega)} &= \tau^{\frac{d}{6}} \left\| \left\| \eta\left(\tau^{\frac{1}{3}}(x-x_0)\right) f(x) \right\|_{L^p_x(\Omega)} \right\|_{L^2_{x_0}(\Omega)} \\ &\leqslant C\tau^{\frac{d}{6}} \left\| \left\| \eta\left(\tau^{\frac{1}{3}}(x-x_0)\right) f(x) \right\|_{L^p_x(\Omega)} \right\|_{L^p_{x_0}(\Omega)} \leqslant C\tau^{\left(\frac{1}{2}-\frac{1}{p}\right)\frac{d}{3}} \|f\|_{L^p(\Omega)}.\end{aligned}$$

On the other hand, for $p>2$, by Minkowski's integral inequality ([Ste70], page 271), we have, for C independent of $\tau \geqslant \tau_0$,

$$\begin{aligned}(1.10)\quad \|f\|_{L^p(\Omega)} &= \left(\int_{x\in\Omega} |f(x)|^p\,dx\right)^{\frac{1}{p}} \\ &\leqslant C\left(\int_{x\in\Omega}\left(\tau^{\frac{d}{3}}\int_{x_0\in\Omega}\left|\eta\left(\tau^{\frac{1}{3}}(x-x_0)\right) f(x)\right|^2 dx_0\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}} \\ &\leqslant C\tau^{\frac{d}{6}}\left(\int_{x_0\in\Omega}\left(\int_{x\in\Omega}\left|\eta\left(\tau^{\frac{1}{3}}(x-x_0)\right) f(x)\right|^p dx\right)^{\frac{2}{p}} dx_0\right)^{\frac{1}{2}} = C\|f\|_{L^p_{\eta,\tau}(\Omega)}.\end{aligned}$$

Similarly, for $p<2$, we get, using Minkowski's integral inequality and the norm equivalence (1.8), that there exists a constant C independent of $\tau \geqslant \tau_0$, such that

$$(1.11)\qquad \frac{1}{C}\tau^{\left(\frac{1}{2}-\frac{1}{p}\right)\frac{d}{3}} \|f\|_{L^p(\Omega)} \leqslant \|f\|_{L^p_{\eta,\tau}(\Omega)} \leqslant C\|f\|_{L^p(\Omega)}.$$

Note that, of course, the estimates (1.9), (1.10), and (1.11) can be used to simplify the norms $L^p_{\eta,\tau}$ in the Carleman estimates (1.5) and (1.6) and replace them by the classical L^p norms.

Finally, let us point out that, for p, q, r in $[1,\infty]$ such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$, for $V \in L^q(\Omega)$ and $u \in L^p(\Omega)$, we have the following Hölder type estimate

$$(1.12)\quad \|Vu\|_{L^r_{\eta,\tau}(\Omega)} \leqslant \|u\|_{L^p_{\eta,\tau}(\Omega)} \sup_{x_0\in\Omega}\left\{\|V\|_{L^q\left(B\left(x_0,\tau^{-\frac{1}{3}}\right)\right)}\right\} \leqslant \|u\|_{L^p_{\eta,\tau}(\Omega)}\|V\|_{L^q(\Omega)}.$$

Application to the quantification of unique continuation with respect to lower order terms

Next, as a consequence of the Carleman estimates in Theorem 1.1, we will prove (in Section 8) the following result:

THEOREM 1.3. — *Let $d \geqslant 3$, $\Omega \subset \mathbb{R}^d$ be a bounded domain of class C^3, and ω be a non-empty open subset of Ω with $\overline{\omega} \subset \Omega$. Then there exists a constant $C = C(\Omega, \omega) > 0$ depending only on Ω and ω such that for any solution $u \in H_0^1(\Omega)$ of*

$$\Delta u = Vu + W_1 \cdot \nabla u + \operatorname{div}(W_2 u) \quad \text{in } \Omega,$$

with

$$V \in L^{q_0}(\Omega), \quad W_1 \in L^{q_1}\left(\Omega; \mathbb{C}^d\right), \quad W_2 \in L^{q_2}\left(\Omega; \mathbb{C}^d\right),$$

we have:

(1) *If $q_0 \in (\frac{d}{2}, \infty]$, $q_1 \in (\frac{3d-2}{2}, \infty]$ and $q_2 \in (\frac{3d-2}{2}, \infty]$ and*

$$\frac{1}{q_1} + \frac{1}{q_2} < 4\left(\frac{1 - \frac{1}{d}}{3d-2}\right),$$

the function u satisfies

$$\|u\|_{L^2(\Omega)} \leqslant C e^{C\left(\|V\|_{L^{q_0}(\Omega)}^{\gamma(q_0)} + \|W_1\|_{L^{q_1}(\Omega)}^{\delta(q_1)} + \|W_2\|_{L^{q_2}(\Omega)}^{\delta(q_2)} + \left(\|W_1\|_{L^{q_1}(\Omega)}\|W_2\|_{L^{q_2}(\Omega)}\right)^{\rho(q_1,q_2)}\right)} \|u\|_{L^{\frac{2d}{d-2}}(\omega)}, \tag{1.13}$$

with

$$\gamma(q) = \begin{cases} \dfrac{1}{\dfrac{3}{2}\left(1 - \dfrac{d}{2q}\right) + \dfrac{1}{2q}} & \text{if } q \geqslant d, \\[2ex] \dfrac{1}{\left(\dfrac{3}{4} + \dfrac{1}{2d}\right)\left(2 - \dfrac{d}{q}\right)} & \text{if } q \in \left(\dfrac{d}{2}, d\right], \end{cases} \qquad \delta(q) = \frac{2}{1 - \dfrac{3d-2}{2q}}, \qquad \rho(q_1, q_2) = \frac{1}{1 - \dfrac{1}{d} - \left(\dfrac{3}{4} - \dfrac{1}{2d}\right)\left(\dfrac{d}{q_1} + \dfrac{d}{q_2}\right)}.$$

(2) *If $q_0 \in (\frac{d}{2}, \infty]$, $q_1 \in (\frac{3d}{2}, \infty]$ and $q_2 \in (\frac{3d}{2}, \infty]$, the function u satisfies*

$$\|u\|_{L^2(\Omega)} \leqslant C e^{C\left(\|V\|_{L^{q_0}(\Omega)}^{\gamma(q_0)} + \|W_1\|_{L^{q_1}(\Omega)}^{\widetilde{\delta}(q_1)} + \|W_1\|_{L^{q_1}(\Omega)}^{\widetilde{\delta}(q_2)}\right)} \|u\|_{L^{\frac{2d}{d-2}}(\omega)}, \tag{1.14}$$

with

$$\widetilde{\delta}(q) = \frac{2}{1 - \dfrac{3d}{2q}}.$$

Remark 1.4. — Note that the conditions in item 1 and in item 2 do not overlap, in the sense that there are cases in which the conditions in item 2 are satisfied while conditions in item 1 are not (for instance $q_1 = q_2 = \frac{3d}{2} + \epsilon$ with $\epsilon > 0$ small), and reciprocally (for instance $q_1 = \frac{3d-2}{2} + \epsilon$ with $\epsilon > 0$ small and $q_2 = \infty$).

Several remarks are in order.

First, unique continuation is known to hold for general $V \in L^{q_0}(\Omega)$, $W_1 \in L^{q_1}(\Omega;\mathbb{C}^d)$, and $W_2 \in L^{q_2}(\Omega;\mathbb{C}^d)$ for $q_0 \geqslant \frac{d}{2}$, $q_1 \geqslant d$ and $q_2 \geqslant d$, see [Wol93], and [KT01] where even strong unique continuation is proved in that case when $q_0 > \frac{d}{2}$, $q_1 > d$ and $q_2 > d$. (These classes of integrability for the potentials are sharp, see [KT02].)

These unique continuation results require the use of a Carleman estimate and a delicate argument from harmonic analysis inspired by [Wol92], see also [KT01]. In this argument, the weight function in the Carleman estimate depends on the solution, making the quantification of unique continuation with respect to the norms of the potentials difficult to track. Another related result is the article [MV12], which quantifies unique continuation properties for the Laplacian operator with lower order terms in the sharp integrability class, but not with respect to the norms of the potentials. In fact, since this work is based on [KT01], as said above, it is not clear how the proof in [MV12] can be made quantitative with respect to the norms of the potentials.

Therefore, when trying to quantify the unique continuation property with respect to the norms of the lower order terms, the known results rely only on the use of a Carleman estimate, which, as pointed out in [BKRS88], does not allow to go beyond $W_1 \in L^{\frac{3d-2}{2}}(\Omega)$. This corresponds to what is done in [DZ19, Dav20] using L^p Carleman estimate. But the results in [DZ19] describing the maximal order of vanishing of solutions of elliptic equations require V and W_1 respectively in $L^{q_0}(\Omega)$ with $q_0 > d(3d-2)/(5d-2)$ and in $L^{q_1}(\Omega)$ with $q_1 > \frac{3d-2}{2}$, and $W_2 = 0$. Also note that [Dav20, Theorem 1], which applies when $W_1 = W_2 = 0$, exhibits the same dependence in the $L^{q_0}(\Omega)$ norm of V as in Theorem 1.3.

Let us also mention that taking L^2 Carleman estimates, one cannot reach the same integrability class as in our case, see for instance [DZZ08].

Finally, note that using a quantitative Caccioppoli inequality with singular lower order terms, see for instance [DZ19, Lemma 5], and Sobolev embedding, one can show that the inequalities (1.13) and (1.14) remain true by replacing $\|u\|_{L^{\frac{2d}{d-2}}(\omega)}$ by $\|u\|_{L^2(\omega_1)}$ for ω_1 an open subset satisfying $\overline{\omega} \subset \omega_1$ (Since ω is any arbitrary non-empty open set in Theorem 1.3, this is of course a harmless condition).

Let us also note that one can be slightly more precise in Theorem 1.3, in (1.13) and (1.14), by using the intermediate bound in (1.12) instead of the extremal one in the proof of Theorem 1.3.

Strategy of the proof of Theorem 1.1

In order to prove Theorem 1.1, we start with the easy geometric case of a vertical strip, with a linear weight function $x \mapsto x_1$, and a second order operator of the form $\Delta - x_1 \sum_{j=2}^d \lambda_j \partial_j^2$, see Section 2 for the statements.

Although this might seem at first to be a very specific case, we will check in Section 7 that this is not the case, due to the two following facts. First, if we localize the functions in a ball of radius sufficiently small, one can do a change of variables (in the spirit of the normal geodesic coordinates), such that the conjugated operator $e^{\tau\varphi}\Delta\,(e^{-\tau\varphi}\cdot)$ can be recast into the problem in the strip with an operator of the form $\Delta - x_1\sum_{j=2}^d \lambda_j\partial_j^2$ and the linear weight function $x \mapsto x_1$. Second, one can glue the local and boundary Carleman estimates obtained that way, and the localization terms introduced by the cut-off can be absorbed through that process if the localization is not too strong. Therefore, we have to balance the two processes, and to choose the localization rate appropriately. It turns out that a localization in balls of size $\tau^{-\frac{1}{3}}$ works.

Accordingly, most of the article in fact focuses on the proof of a Carleman estimate in the strip for an operator of the form $\Delta - x_1\sum_{j=2}^d \lambda_j\partial_j^2$ with linear weight $x \mapsto x_1$. We do that in several steps.

First, due to the specific geometric setting, one can perform a Fourier transform in the tangential variables (which are transverse to the gradient of the weight function, i.e. to the direction e_1), and construct explicitly a parametrix, see Section 3. In fact, this approach is inspired by [KT01, Sog89] and by recent works on Carleman estimates for Laplace operator with discontinuous conductivities, for instance [LRL13].

Once this is done, it is clear that we will have to get estimates on the operators appearing in the parametrix. Dealing with the Hilbertian norms can be done using classical multiplier type arguments and Parseval's identity, see Section 4.

It thus remains to understand how to get estimates on the operators appearing in the parametrix in $\mathscr{L}(L^{\frac{2d}{d+2}}(\Omega), L^{\frac{2d}{d-2}}(\Omega))$ for instance, and other operator norms involving non-Hilbertian spaces. In order to do this, we will rely on the Fourier restriction Stein–Tomas theorem, recalled in Theorem 5.1, see e.g. [Tom75], [Ste93, Theorem 2, p. 352], or [Sog17, Corollary 2.2.2]. Similarly, as in [BKRS88, KRS87], this approach will allow us to give an efficient manner to estimate the norm in $\mathscr{L}(L^{\frac{2d}{d+2}}(\mathbb{R}^{d-1}), L^{\frac{2d}{d-2}}(\mathbb{R}^{d-1}))$ (among others) of operators given in Fourier, see Section 5.2.

Using these results, and the explicit formula obtained for the parametrix, we manage to get L^p Carleman estimates in the strip for an operator of the form $\Delta - x_1\sum_{j=2}^d \lambda_j\partial_j^2$ with linear weight $x \mapsto x_1$.

Let us finally emphasize that we made the choice of presenting the proofs in a (hopefully) pedagogical manner, and thus of giving all the technical details required to get through the whole proofs. Therefore, some parts, for instance regarding the Hilbertian estimates or the Fourier restriction theorems, might seem merely classical, but we made the choice to present them nevertheless since we did not find them in the literature in the precise version we needed.

Outline

The rest of the paper is as follows. Section 2 is devoted to state Carleman estimates (namely Theorems 2.1 and 2.4) in the specific case of a vertical strip, with a linear weight function $x \mapsto x_1$, and for an operator of the form $\Delta - x_1\sum_{j=2}^d \lambda_j\partial_j^2$. Section 3 gives a parametrix of the conjugated operator $e^{\tau x_1}(\Delta - x_1\sum_{j=2}^d \lambda_j\partial_j^2)(e^{-\tau x_1}\cdot)$. Section 4 explains how to get Hilbertian estimates on the parametrix. Section 5 then recalls Fourier restriction theorems and explains how they can be used in our context to estimate $\mathscr{L}(L^{\frac{2d}{d+2}}(\mathbb{R}^{d-1}), L^{\frac{2d}{d-2}}(\mathbb{R}^{d-1}))$ norms (among others) of operators given in Fourier. We then derive all the estimates needed on the operators appearing in the parametrix in Section 6 and conclude the proof of Theorems 2.1 and 2.4. In Section 7, we explain how to derive the proof of Theorem 1.1 from Theorem 2.4. We then provide in Section 8 the proof of Theorem 1.3. Finally, in the Appendix, we provide some reminders of classical results, namely the Hardy–Littlewood–Sobolev theorem and the stationary phase lemma (the refined version in [ABZ17]). We also give the proof of a technical result of interpolation used in Section 7.

Notations

Here is a set of notations we will use throughout the article.

For every $x \in \mathbb{R}^d$, $x = (x_1, .., x_d)$, we set $x = (x_1, x')$, where $x' = (x_2, .., x_d) \in \mathbb{R}^{d-1}$.

The notations ∇ and Δ respectively stand for the gradient and the Laplacian with respect to $x = (x_1, .., x_d)$, and $\nabla' = (\partial_2, .., \partial_d)$ and $\Delta' = \sum_{j=2}^d \partial_j^2$ are, respectively, the tangential gradient and Laplacian operators.

In all the document except in Section 5, the Fourier transform is always taken to be the Fourier transform with respect to $x' = (x_2, .., x_d)$, and its dual variable $\xi' \in \mathbb{R}^{d-1}$ is then indexed by $\xi' = (\xi_2, .., \xi_d)$. The Fourier transform of a function $f \in \mathscr{S}(\mathbb{R}^{d-1})$ will be denoted by $\widehat{f}$:

$$\widehat{f}(\xi') = \frac{1}{(2\pi)^{\frac{d-1}{2}}} \int_{\mathbb{R}^{d-1}} e^{-\mathbf{i}x'\cdot\xi'} f(x')\, dx', \qquad \xi' \in \mathbb{R}^{d-1},$$

and is extended by duality as usual to any $f \in \mathscr{S}'(\mathbb{R}^{d-1})$.

Note that for a function f defined on $\mathbb{R}^d$ such that $f(x_1, \cdot) \in \mathscr{S}(\mathbb{R}^d)$, $\widehat{f}(x_1, \cdot)$ denotes the partial Fourier transform with respect to x', that is:

$$\widehat{f}(x_1, \xi') = \frac{1}{(2\pi)^{\frac{d-1}{2}}} \int_{\mathbb{R}^{d-1}} e^{-\mathbf{i}x'\cdot\xi'} f(x_1, x')\, dx', \qquad \xi' \in \mathbb{R}^{d-1}. \tag{1.15}$$

2. A Carleman estimate in a strip

In this section, we focus on the case of a strip

$$\Omega = (X_0, X_1) \times \mathbb{R}^{d-1}, \tag{2.1}$$

and on the following elliptic problem

$$\begin{cases} \Delta v - x_1 \sum_{j=2}^d \lambda_j \partial_j^2 v = f_2 + f_{2'_*} + \operatorname{div} F & \text{in } \Omega, \\ v(X_0, x') = g(x'), & \text{for } x' \in \mathbb{R}^{d-1}, \\ v(X_1, x') = 0, & \text{for } x' \in \mathbb{R}^{d-1}, \end{cases} \tag{2.2}$$

where

$$f_2 \in L^2(\Omega), \quad f_{2'_*} \in L^{\frac{2d}{d+2}}(\Omega), \quad F \in L^2\left(\Omega; \mathbb{C}^d\right), \quad \text{and} \quad g \in H^{\frac{1}{2}}\left(\mathbb{R}^{d-1}\right). \tag{2.3}$$

To be able to solve the elliptic problem (2.2), we assume the coercivity of the operator $-\Delta + x_1 \sum_{j=2}^d \lambda_j \partial_j^2$ in Ω, that is

$$\text{(2.4)} \quad \exists\, c_0 > 0, \quad \forall\, x_1 \in [X_0, X_1],\, \forall\, \xi \in \mathbb{R}^d, \quad \frac{1}{c_0^2}|\xi|^2 \leqslant \sum_{j=1}^d (1 - x_1\lambda_j)\,|\xi_j|^2 \leqslant c_0^2|\xi|^2,$$

where we have set $\lambda_1 = 0$ for convenience. Under condition (2.4) and the integrability and regularity assumptions (2.3), the problem (2.2) has a unique solution $v \in H^1(\Omega)$.

Our goal is to prove the following Carleman estimate:

Theorem 2.1. — *Let Ω be as in (2.1) with $X_0 < 0 < X_1$ and $\max\{|X_0|, |X_1|\} \leqslant 1$, and assume that the coefficients $(\lambda_j)_{j\in\{1,\cdots,d\}} \in \mathbb{R}^d$ satisfy $\lambda_1 = 0$, (2.4) and that there exist positive constants m_* and M_* such that*

$$\text{(2.5)} \qquad 0 < m_* \leqslant \min_{j\in\{2,\cdots,d\}} \lambda_j \leqslant \max_{j\in\{2,\cdots,d\}} \lambda_j \leqslant M_*.$$

Then there exists a constant $C > 0$ depending on c_0, m_* and M_* (independent of X_0, X_1) such that for all $(f_2, f_{2'_*}, F, g)$ as in (2.3), if the solution v of (2.2) satisfies $(\partial_1 v - F_1)(X_1, x') = 0$ for $x' \in \mathbb{R}^{d-1}$, then we have, for all $\tau \geqslant 1$,

$$\text{(2.6)} \quad \tau^{\frac{3}{2}} \left\| v e^{\tau x_1} \right\|_{L^2(\Omega)} + \tau^{\frac{1}{2}} \left\| v e^{\tau x_1} \right\|_{H^1(\Omega)} \leqslant C \Bigg(\left\| f_2 e^{\tau x_1} \right\|_{L^2(\Omega)} + \tau^{\frac{3}{4} - \frac{1}{2d}} \left\| f_{2'_*} e^{\tau x_1} \right\|_{L^{\frac{2d}{d+2}}(\Omega)} + \tau \left\| F e^{\tau x_1} \right\|_{L^2(\Omega)} + \tau^{\frac{3}{4}} \left\| g e^{\tau X_0} \right\|_{H^{\frac{1}{2}}(\{X_0\}\times\mathbb{R}^{d-1})} \Bigg),$$

and

$$\text{(2.7)} \quad \tau^{\frac{3}{4} + \frac{1}{2d}} \left\| v e^{\tau x_1} \right\|_{L^{\frac{2d}{d-2}}(\Omega)} \leqslant C \Bigg(\left\| f_2 e^{\tau x_1} \right\|_{L^2(\Omega)} + \tau^{\frac{3}{4} + \frac{1}{2d}} \left\| f_{2'_*} e^{\tau x_1} \right\|_{L^{\frac{2d}{d+2}}(\Omega)} + \tau \left\| F e^{\tau x_1} \right\|_{L^2(\Omega)} + \tau^{\frac{3}{4} + \frac{1}{2d}} \left\| g e^{\tau X_0} \right\|_{H^{\frac{1}{2}}(\{X_0\}\times\mathbb{R}^{d-1})} \Bigg).$$

Remark 2.2. — A solution v of (2.2) with $(f_2, f_{2'_*}, F, g)$ as in (2.3) only belongs a priori to $H^1(\Omega)$. Therefore, trace theorems do not allow defining directly its normal trace. However, $\nabla v - F$ satisfies $\nabla v - F \in L^2(\Omega; \mathbb{C}^d)$ and $\operatorname{div}(\nabla v - F) \in L^2(\Omega) + L^{\frac{2d}{d+2}}(\Omega)$ and it is easy to check that if $R \in L^2(\Omega; \mathbb{C}^d)$ and $\operatorname{div} R \in L^2(\Omega) + L^{\frac{2d}{d+2}}(\Omega)$, then $R \cdot n$ is well-defined as an element of $H^{-\frac{1}{2}}(\partial\Omega)$, see [BF12, Theorem III.2.43]. Therefore, the trace $(\nabla v - F) \cdot n$ is well-defined as an element of $H^{-\frac{1}{2}}(\partial\Omega)$.

Remark 2.3. — The strict positivity of the coefficients $(\lambda_j)_{j\in\{2,\cdots,d\}}$ guaranteed by condition (2.5) is the sub-ellipticity condition for the operator $-\Delta + x_1 \sum_{j=2}^d \lambda_j \partial_j^2$ with respect to the weight function $x \mapsto x_1$, see for instance [LRLR22, Part 1, Definition 3.30].

As one easily checks by working on w defined by $w(x) = e^{\tau x_1} v(x)$ in Ω, Theorem 2.1 is implied by the following result, whose proof is developed from Section 3 to Section 6:

Theorem 2.4. — *Let Ω be as in* (2.1) *with $X_0<0<X_1$ and* $\max\{|X_0|,|X_1|\}\leqslant 1$, *and assume that the coefficients $(\lambda_j)_{j\in\{1,\cdots,d\}}\in\mathbb{R}^d$ satisfy $\lambda_1=0$,* (2.4), *and* (2.5). *Then there exist constants $C>0$ and $\tau_0\geqslant 1$ depending on c_0, m_* and M_* (independent of X_0,X_1), such that for all $(f_2,f_{2'_*},F,g)$ as in* (2.3), *if the solution w of*

$$\begin{cases}\Delta w - x_1\sum_{j=2}^{d}\lambda_j\partial_j^2 w - 2\tau\partial_1 w+\tau^2 w = f_2+f_{2'_*}+\operatorname{div}F & \text{in } \Omega,\\ w(X_0,x')=g(x'), & \text{for } x'\in\mathbb{R}^{d-1},\\ w(X_1,x')=0, & \text{for } x'\in\mathbb{R}^{d-1},\end{cases}\tag{2.8}$$

satisfies $(\partial_1 w-F_1)(X_1,x')=0$ for $x'\in\mathbb{R}^{d-1}$, then for all $\tau\geqslant\tau_0$,

$$\begin{aligned}&\tau^{\frac32}\|w\|_{L^2(\Omega)}+\tau^{\frac12}\|\nabla w\|_{L^2(\Omega)}\\ &\quad\leqslant C\left(\|f_2\|_{L^2(\Omega)}+\tau^{\frac34-\frac{1}{2d}}\|f_{2'_*}\|_{L^{\frac{2d}{d+2}}(\Omega)}+\tau\|F\|_{L^2(\Omega)}+\tau^{\frac34}\|g\|_{H^{\frac12}(\{X_0\}\times\mathbb{R}^{d-1})}\right),\end{aligned}\tag{2.9}$$

and

$$\begin{aligned}&\tau^{\frac34+\frac{1}{2d}}\|w\|_{L^{\frac{2d}{d-2}}(\Omega)}\\ &\quad\leqslant C\left(\|f_2\|_{L^2(\Omega)}+\tau^{\frac34+\frac{1}{2d}}\|f_{2'_*}\|_{L^{\frac{2d}{d+2}}(\Omega)}+\tau\|F\|_{L^2(\Omega)}+\tau^{\frac34+\frac{1}{2d}}\|g\|_{H^{\frac12}(\{X_0\}\times\mathbb{R}^{d-1})}\right).\end{aligned}\tag{2.10}$$

In fact, the correspondence between Theorem 2.1 and 2.4 is given by

$$(f_2,f_{2'_*},F,g)\to\left((f_2-\tau F_1)e^{\tau x_1},f_{2'_*}e^{\tau x_1},Fe^{\tau x_1},ge^{\tau X_0}\right).$$

Theorem 2.1 and Theorem 2.4 are then completely equivalent, and we thus focus only on the latter.

3. Construction of the parametrix in the case of a strip

The goal of this section is to explicitly construct the solution w of (2.8) for $\tau\geqslant 1$, $(f_2,f_{2'_*},F,g)$ as in (2.3), under the assumptions that the domain Ω is a strip as in (2.1), and the coefficients $(\lambda_j)_{j\in\{1,\cdots,d\}}\in\mathbb{R}^d$ satisfy $\lambda_1=0$, (2.4) and (2.5).

In order to do that, we take the partial Fourier transform in the variable $x'\in\mathbb{R}^{d-1}$ of (2.8) with dual variable $\xi'\in\mathbb{R}^{d-1}$:

$$\begin{cases}(\partial_1-\tau)^2\widehat{w}-\sum_{j=2}^{d}(1-x_1\lambda_j)\xi_j^2\widehat{w}\\ \qquad\qquad =\widehat{f}_2+\widehat{f}_{2'_*}+\partial_1\widehat{F}_1+\mathbf{i}\sum_{j=2}^{d}\xi_j\widehat{F}_j & \text{for } (x_1,\xi')\in\Omega,\\ \widehat{w}(X_0,\xi')=\widehat{g}(\xi'), & \text{for } \xi'\in\mathbb{R}^{d-1},\\ \widehat{w}(X_1,\xi')=0, & \text{for } \xi'\in\mathbb{R}^{d-1}.\end{cases}\tag{3.1}$$

We then show the following:

Proposition 3.1. — *Let Ω be as in (2.1), and assume that the coefficients $(\lambda_j)_{j \in \{1, \cdots, d\}} \in \mathbb{R}^d$ satisfy $\lambda_1 = 0$, (2.4) and (2.5). We introduce the function $\psi : \overline{\Omega} \to \mathbb{R}$ defined by*

$$\psi(x_1, \xi') = \sqrt{\sum_{j=2}^{d} (1 - x_1 \lambda_j) \, \xi_j^2}, \qquad x_1 \in [X_0, X_1], \ \xi' \in \mathbb{R}^{d-1}. \tag{3.2}$$

For all $\tau \geqslant 1$, for all $(f_2, f_{2'_*}, F, g)$ as in (2.3), the solution w of (2.8) formally satisfies

$$w = K_{\tau,0} \, (f_2 + f_{2'_*}) + K_{\tau,1}(F_1) + \sum_{j=2}^{d} K_{\tau,j}(F_j) + R_\tau(w) + G_\tau(g) + H_\tau\Big((\partial_1 w - F_1)(X_1, \cdot)\Big), \tag{3.3}$$

where, using the partial Fourier transform (1.15), the operators $K_{\tau,j}$, for $j \in \{0, \cdots, d\}$, and R_τ are formally defined for f depending on $(x_1, x') \in \Omega$, by

$$\widehat{K_{\tau,j} f}(x_1, \xi') = \int_{y_1 \in (X_0, X_1)} k_{\tau,j}(x_1, y_1, \xi') \widehat{f}(y_1, \xi') \, dy_1, \qquad (x_1, \xi') \in \Omega, \tag{3.4}$$

$$\widehat{R_\tau f}(x_1, \xi') = \int_{y_1 \in (X_0, X_1)} r_\tau(x_1, y_1, \xi') \widehat{f}(y_1, \xi') \, dy_1, \qquad (x_1, \xi') \in \Omega, \tag{3.5}$$

with kernels given, for $(x_1, y_1, \xi') \in [X_0, X_1]^2 \times \mathbb{R}^{d-1}$, by

$$\begin{aligned} k_{\tau,0}(x_1, y_1, \xi') &= -1_{\psi(x_1,\xi') > \tau} \int_{X_0}^{\min\{x_1, y_1\}} e^{-\tau(y_1 - x_1) - \int_{\widetilde{x}_1}^{x_1} \psi(\widetilde{y}_1, \xi') \, d\widetilde{y}_1 - \int_{\widetilde{x}_1}^{y_1} \psi(\widetilde{y}_1, \xi') \, d\widetilde{y}_1} \, d\widetilde{x}_1 \\ &\quad + 1_{\psi(x_1,\xi') \leqslant \tau} 1_{y_1 > x_1} \int_{x_1}^{y_1} e^{-\tau(y_1 - x_1) + \int_{x_1}^{\widetilde{x}_1} \psi(\widetilde{y}_1, \xi') \, d\widetilde{y}_1 - \int_{\widetilde{x}_1}^{y_1} \psi(\widetilde{y}_1, \xi') \, dy_1} \, d\widetilde{x}_1, \end{aligned} \tag{3.6}$$

$$\begin{aligned} k_{\tau,1}(x_1, y_1, \xi') &= -1_{\psi(x_1,\xi') \leqslant \tau} \, 1_{y_1 > x_1} e^{-\tau(y_1 - x_1) + \int_{x_1}^{y_1} \psi(\widetilde{y}_1, \xi') \, d\widetilde{y}_1} \\ &\quad + 1_{\psi(x_1,\xi') > \tau} \, 1_{y_1 < x_1} e^{\tau(x_1 - y_1) - \int_{y_1}^{x_1} \psi(\widetilde{y}_1, \xi') \, d\widetilde{y}_1} + k_{\tau,0}(x_1, y_1, \xi') \Big(\tau + \psi(y_1, \xi')\Big), \end{aligned} \tag{3.7}$$

$$\begin{aligned} k_{\tau,j}(x_1, y_1, \xi') &= \mathbf{i} \xi_j k_{\tau,0}(x_1, y_1, \xi'), \qquad j \in \{2, \cdots, d\}, \\ r_\tau(x_1, y_1, \xi) &= k_{\tau,0}(x_1, y_1, \xi') \partial_1 \psi(y_1, \xi'). \end{aligned} \tag{3.8}$$

The operators G_τ and H_τ are formally given in Fourier for $g_0 \in \mathscr{S}(\mathbb{R}^{d-1})$ by

$$\begin{aligned} \widehat{G_\tau g_0}(x_1, \xi') &= g_\tau(x_1, \xi') \widehat{g}_0(\xi'), \qquad (x_1, \xi') \in \Omega, \\ \widehat{H_\tau g_0}(x_1, \xi') &= h_\tau(x_1, \xi') \widehat{g}_0(\xi'), \qquad (x_1, \xi') \in \Omega, \end{aligned} \tag{3.9}$$

where g_τ and r_τ are given, for $(x_1,\xi') \in [X_0,X_1]\times\mathbb{R}^{d-1}$, by

$$
\begin{aligned}
&g_\tau(x_1,\xi') = 1_{\psi(x_1,\xi')\geqslant\tau} e^{\tau(x_1-X_0)-\int_{X_0}^{x_1}\psi(\widetilde{y}_1,\xi')\,d\widetilde{y}_1},\\
(3.10)\quad &h_\tau(x_1,\xi') = 1_{\psi(x_1,\xi')>\tau}\int_{X_0}^{x_1} e^{-\tau(X_1-x_1)-\int_{\widetilde{x}_1}^{x_1}\psi(\widetilde{y}_1,\xi')\,d\widetilde{y}_1-\int_{\widetilde{x}_1}^{X_1}\psi(\widetilde{y}_1,\xi')\,d\widetilde{y}_1}d\widetilde{x}_1\\
&\qquad - 1_{\psi(x_1,\xi')\leqslant\tau}\int_{x_1}^{X_1} e^{-\tau(X_1-x_1)+\int_{x_1}^{\widetilde{x}_1}\psi(\widetilde{y}_1,\xi')\,d\widetilde{y}_1-\int_{\widetilde{x}_1}^{X_1}\psi(\widetilde{y}_1,\xi')\,d\widetilde{y}_1}d\widetilde{x}_1.
\end{aligned}
$$

Remark 3.2. — We emphasize that Proposition 3.1 is formal. We will prove later, in Theorem 4.1, Proposition 4.3 and in Proposition 6.2, that the operators $K_{\tau,0}$, $(K_{\tau,j})_{j\in\{1,\cdots,d\}}$, R_τ, G_τ and H_τ respectively belong to $\mathscr{L}(L^2(\Omega)+L^{\frac{2d}{d+2}}(\Omega);L^2(\Omega))$, $(\mathscr{L}(L^2(\Omega);L^2(\Omega)))^d$, $\mathscr{L}(L^2(\Omega);L^2(\Omega))$, $\mathscr{L}(H^1(\Omega);L^2(\Omega))$, $\mathscr{L}(H^{\frac12}(\partial\Omega);L^2(\Omega))$, and $\mathscr{L}(H^{-\frac12}(\Omega);L^2(\Omega))$. A simple density argument would then allow to justify rigorously formula (3.3).

Proof. — The basic strategy of proof of Proposition 3.1 consists in the factorization of the operator $(\partial_1-\tau)^2-\sum_{j=2}^d(1-x_1\lambda_j)\xi_j^2$:

$$(\partial_1-\tau)^2-\sum_{j=2}^d(1-x_1\lambda_j)\xi_j^2 = (\partial_1-\tau-\psi(x_1,\xi'))\,(\partial_1-\tau+\psi(x_1,\xi'))-\partial_1\psi(x_1,\xi'),$$

where ψ is the function introduced in (3.2), and the last term should be seen as a correction term.

We thus set, for all $x_1\in[X_0,X_1]$, $\xi'\in\mathbb{R}^{d-1}$,

$$\widehat{H}(x_1,\xi') = \widehat{f}_2(x_1,\xi')+\widehat{f}_{2'_*}(x_1,\xi')+\mathbf{i}\sum_{j=2}^d\xi_j\widehat{F}_j(x_1,\xi')+\partial_1\psi(x_1,\xi')\widehat{w}(x_1,\xi'),$$

so that equation $(3.1)_{(1)}$ can be rewritten as

$$(\partial_1-\tau-\psi(x_1,\xi'))\,(\partial_1-\tau+\psi(x_1,\xi'))\,\widehat{w} = \widehat{H}+\partial_1\widehat{F}_1, \quad \text{in } \Omega.$$

Accordingly, introducing the additional unknown $\widehat{z}(x_1,\xi')=(\partial_1-\tau+\psi(x_1,\xi'))\widehat{w}(x_1,\xi')$, equation (3.1) can be rewritten as a system of two first order ODE indexed by $\xi'\in\mathbb{R}^{d-1}$:

$$
(3.11)\qquad \begin{cases}
(\partial_1-\tau+\psi(x_1,\xi'))\,\widehat{w}(x_1,\xi') = \widehat{z}(x_1,\xi') & \text{in } \Omega,\\
(\partial_1-\tau-\psi(x_1,\xi'))\,\widehat{z}(x_1,\xi') = \widehat{H}(x_1,\xi')+\partial_1\widehat{F}_1(x_1,\xi') & \text{in } \Omega,\\
\widehat{w}(X_0,\xi') = \widehat{g}(\xi'), & \text{on } \mathbb{R}^{d-1},\\
\widehat{w}(X_1,\xi') = 0 & \text{on } \mathbb{R}^{d-1}.
\end{cases}
$$

Let $\xi'\in\mathbb{R}^{d-1}$. Solving $(3.11)_2$ from the right, which can be done easily by working on $(\widehat{z}-\widehat{F}_1)(\cdot,\xi')$, by Duhamel's formula we get, for $x_1\in(X_0,X_1)$,

$$
\begin{aligned}
(3.12)\quad \widehat{z}(x_1,\xi') &= e^{-\tau(X_1-x_1)-\int_{x_1}^{X_1}\psi(\widetilde{y}_1,\xi')\,d\widetilde{y}_1}\left(\partial_1\widehat{w}-\widehat{F}_1\right)(X_1,\xi')+\widehat{F}_1(x_1,\xi')\\
&\quad -\int_{x_1}^{X_1} e^{-\tau(y_1-x_1)-\int_{x_1}^{y_1}\psi(\widetilde{y}_1,\xi')\,d\widetilde{y}_1}\left(\widehat{H}(y_1,\xi')+(\tau+\psi(y_1,\xi'))\widehat{F}_1(y_1,\xi')\right)dy_1,
\end{aligned}
$$

where we did the additional remark that, using $(3.11)_{(1,4)}$, $\widehat{z}(X_1,\xi') = \partial_1\widehat{w}(X_1,\xi')$.

We then focus on the equations $(3.11)_{(1,3,4)}$ giving $\widehat{w}(\cdot,\xi')$ in terms of $\widehat{z}(\cdot,\xi')$. One should notice here that $(3.11)_{(3,4)}$ give two boundary conditions for a first order equation. Therefore, we do a choice when solving $(3.11)_{(1)}$ based on the fact that we want formulae involving only exponentials of nonpositive numbers. In order to do such a choice, we analyse the sign of the function $x_1 \mapsto -\tau + \psi(x_1,\xi')$. Due to conditions (2.4) and (2.5), the function $x_1 \mapsto \psi(x_1,\xi')$ is strictly decreasing on $[X_0,X_1]$. Therefore, the function $x_1 \mapsto -\tau + \psi(x_1,\xi')$ can vanish only once on $[X_0,X_1]$, and if it vanishes at some point $x_{\tau,\xi'} \in [X_0,X_1]$, it is positive in $[X_0,x_{\tau,\xi'})$ and negative for $x_1 \in (x_{\tau,\xi'},X_1]$.

Accordingly, for $x_1 \in [X_0,X_1]$ such that $\psi(x_1,\xi') \leqslant \tau$, we use the formula

$$\widehat{w}(x_1,\xi') = -\int_{x_1}^{X_1} e^{-\tau\left(\widetilde{x}_1 - x_1\right) + \int_{x_1}^{\widetilde{x}_1} \psi\left(\widetilde{y}_1,\xi'\right)\, d\widetilde{y}_1} \widehat{z}\left(\widetilde{x}_1,\xi'\right)\, d\widetilde{x}_1,$$

while for $x_1 \in [X_0,X_1]$ such that $\psi(x_1,\xi') > \tau$, we use the formula

$$\widehat{w}(x_1,\xi') = e^{\tau(x_1-X_0) - \int_{X_0}^{x_1} \psi\left(\widetilde{y}_1,\xi'\right)\, d\widetilde{y}_1} \widehat{g}(\xi') + \int_{X_0}^{x_1} e^{\tau(x_1-\widetilde{x}_1) - \int_{\widetilde{x}_1}^{x_1} \psi\left(\widetilde{y}_1,\xi'\right)\, d\widetilde{y}_1} \widehat{z}(\widetilde{x}_1,\xi')\, d\widetilde{x}_1.$$

These two formulae can be written in one under the form

$$\begin{aligned}(3.13)\quad \widehat{w}(x_1,\xi') = -1_{\psi(x_1,\xi')\leqslant\tau} \int_{x_1}^{X_1} e^{-\tau\left(\widetilde{x}_1 - x_1\right) + \int_{x_1}^{\widetilde{x}_1} \psi\left(\widetilde{y}_1,\xi'\right)\, d\widetilde{y}_1} \widehat{z}(\widetilde{x}_1,\xi')\, d\widetilde{x}_1 + 1_{\psi(x_1,\xi')>\tau} \\ \left(e^{\tau(x_1-X_0) - \int_{X_0}^{x_1} \psi\left(\widetilde{y}_1,\xi'\right)\, d\widetilde{y}_1} \widehat{g}(\xi') + \int_{X_0}^{x_1} e^{\tau(x_1-\widetilde{x}_1) - \int_{\widetilde{x}_1}^{x_1} \psi\left(\widetilde{y}_1,\xi'\right)\, d\widetilde{y}_1} \widehat{z}(\widetilde{x}_1,\xi')\, d\widetilde{x}_1 \right).\end{aligned}$$

The formulae given by Proposition 3.1 are then deduced by putting together formulae (3.12) and (3.13). Details are left to the reader. □

4. Hilbertian estimates

The goal of this section is to prove the following result:

Theorem 4.1. — *Let Ω be as in* (2.1) *with $X_0 < 0 < X_1$ and $\max\{|X_0|,|X_1|\} \leqslant 1$, and assume that the coefficients $(\lambda_j)_{j\in\{1,\cdots,d\}} \in \mathbb{R}^d$ satisfy $\lambda_1 = 0$,* (2.4) *and* (2.5).

Then there exist constants $C > 0$ and $\tau_0 \geqslant 1$ depending on c_0, m_* and M_* (independent of X_0, X_1) such that for all $\tau \geqslant \tau_0$, for all $(f_2,F,g_0,g_1) \in L^2(\Omega) \times L^2(\Omega;\mathbb{C}^d) \times H^{\frac12}(\mathbb{R}^{d-1}) \times H^{-\frac12}(\mathbb{R}^{d-1})$, the function w given by

$$(4.1)\qquad w = K_{\tau,0}(f_2) + K_{\tau,1}(F_1) + \sum_{j=2}^{d} K_{\tau,j}(F_j) + G_\tau(g_0) + H_\tau(g_1),$$

where $(K_{\tau,j})_{j\in\{0,\cdots,d\}}$, G_τ and H_τ are given by Proposition 3.1, satisfies:

$$\begin{aligned}(4.2)\quad \tau^{\frac32}\|w\|_{L^2(\Omega)} + \tau^{\frac12}\|\nabla' w\|_{L^2(\Omega)} \leqslant C\|f_2\|_{L^2(\Omega)} \\ + C\tau\|F\|_{L^2(\Omega)} + C\tau\|g_0\|_{H^{\frac12}\left(\mathbb{R}^{d-1}\right)} + C\tau\|g_1\|_{H^{-\frac12}\left(\mathbb{R}^{d-1}\right)}.\end{aligned}$$

Besides, for $w \in L^2(\Omega)$ satisfying $\nabla' w \in L^2(\Omega)$, for all $\tau \geqslant \tau_0$, $R_\tau(w)$ introduced in Proposition 3.1, satisfies

$$(4.3)\qquad \tau^{\frac32}\|R_\tau(w)\|_{L^2(\Omega)} + \tau^{\frac12}\|\nabla' R_\tau(w)\|_{L^2(\Omega)} \leqslant C\|\nabla' w\|_{L^2(\Omega)}.$$

Remark 4.2. — The estimate (4.2) above gives estimates on the norm of the operators $K_{\tau,j}$ for $j \in \{0, \cdots, d\}$ as operators in $\mathscr{L}(L^2(\Omega), L^2(\Omega))$, and on the norm of the operators G_τ and H_τ in, respectively $\mathscr{L}(H^{\frac{1}{2}}(\mathbb{R}^{d-1}), L^2(\Omega))$ and $\mathscr{L}(H^{-\frac{1}{2}}(\mathbb{R}^{d-1}), L^2(\Omega))$, but with no claim of optimality. In fact, as we will see later in Theorem 4.4, the estimate on the operator norm of G_τ is not sharp. On the contrary, the ones on $K_{\tau,j}$ for $j \in \{0, \cdots, d\}$ are sharp, see for instance [LRLR22, Part 1, Theorems 4.4, 4.5 and Theorem 4.10] regarding $K_{\tau,0}$.

In this section and in Section 6, all the constants C depend only on c_0 in (2.4) and m_*, M_* in (2.5), and this fact will not be mentioned in the sequel.

Proof. — We first remark that w as in (4.1) satisfies by construction the following version of (3.11):

$$(4.4)\quad \begin{cases} (\partial_1 - \tau + \psi(x_1,\xi'))\,\widehat{w}(x_1,\xi') = \widehat{z}(x_1,\xi') & \text{for } (x_1,\xi') \in \Omega_{1,\tau}, \\ (\partial_1 - \tau - \psi(x_1,\xi'))\,(\widehat{z} - \widehat{F}_1)(x_1,\xi') = \widehat{f}_2(x_1,\xi') & \\ \qquad\qquad + \mathbf{i}\sum_{j=2}^{d} \xi_j \widehat{F}_j(x_1,\xi') + (\tau + \psi(x_1,\xi'))\widehat{F}_1(x_1,\xi') & \text{for } (x_1,\xi') \in \Omega, \\ \widehat{w}(X_0,\xi') = \widehat{g_0}(\xi'), & \text{if } \psi(X_0,\xi') > \tau, \\ \widehat{w}(X_1,\xi') = 0 & \text{if } \psi(X_1,\xi') \leqslant \tau, \\ (\widehat{z} - \widehat{F}_1)(X_1,\xi') = \widehat{g}_1(\xi'), & \text{for } \xi' \in \mathbb{R}^{d-1}, \end{cases}$$

where $\Omega_{1,\tau} = (\{\psi(x_1,\xi') > \tau\} \cap \Omega) \cup (\{\psi(x_1,\xi') < \tau\} \cap \Omega)$. Note that, given $\xi' \in \mathbb{R}^{d-1}$, due to the conditions (2.4) and (2.5), there exists at most one element $x_1^*(\xi') \in [X_0, X_1]$ such that $\psi(x_1^*(\xi'),\xi') = 0$. Consequently, given $\xi' \in \mathbb{R}^{d-1}$, $\{x_1 \in (X_0,X_1),\, (x_1,\xi') \in \Omega_{1,\tau}\}$ is either the whole interval (X_0,X_1) or the union of two disjoint intervals $(X_0, x_1^*(\xi')) \cup (x_1^*(\xi'), X_1)$.

Since this system is now a family of ODE indexed by the tangential Fourier parameter $\xi' \in \mathbb{R}^{d-1}$, from now on, we see $\xi' \in \mathbb{R}^{d-1}$ as a free parameter.

We then perform estimates on $\widehat{z}(\cdot,\xi')$ using $(4.4)_{(2,5)}$ by setting $\widetilde{z}(\cdot,\xi') = \widehat{z}(\cdot,\xi') - \widehat{F}_1(\cdot,\xi')$, which satisfies:

$$\begin{cases} (\partial_1 - \tau - \psi(x_1,\xi'))\,\widetilde{z}(x_1,\xi') = \widehat{f}_2(x_1,\xi') & \\ \qquad\qquad + \mathbf{i}\sum_{j=2}^{d} \xi_j \widehat{F}_j(x_1,\xi') + (\tau + \psi(x_1,\xi'))\widehat{F}_1(x_1,\xi') & \text{in } \Omega, \\ \widetilde{z}(X_1,\xi') = \widehat{g}_1(\xi'), & \text{on } \mathbb{R}^{d-1}. \end{cases}$$

We then use a multiplier approach, taking the square of both sides and integrating in x_1:

$$\begin{aligned} &\int_{X_0}^{X_1} \left(|\partial_1 \widetilde{z}(x_1,\xi')|^2 + \Big((\tau + \psi(x_1,\xi'))^2 + \partial_1 \psi(x_1,\xi') \Big) |\widetilde{z}(x_1,\xi')|^2 \right) dx_1 \\ &\qquad + (\tau + \psi(X_0,\xi'))|\widetilde{z}(X_0,\xi')|^2 = (\tau + \psi(X_1,\xi'))|\widetilde{z}(X_1,\xi')|^2 \\ &\qquad + \int_{X_0}^{X_1} \left| \widehat{f}_2(x_1,\xi') + \mathbf{i}\sum_{j=2}^{d} \xi_j \widehat{F}_j(x_1,\xi') + (\tau + \psi(x_1,\xi'))\widehat{F}_1(x_1,\xi') \right|^2 dx_1 \\ &\leqslant (\tau + \psi(X_1,\xi'))|\widehat{g}_1(\xi')|^2 + (d+1) \int_{X_0}^{X_1} \left| \widehat{f}_2(x_1,\xi') \right|^2 dx_1 \end{aligned}$$

$$+ (d+1)\sum_{j=2}^{d}\int_{X_0}^{X_1} \left|\xi_j \widehat{F}_j(x_1,\xi')\right|^2 dx_1$$
$$+ (d+1)\int_{X_0}^{X_1} (\tau+\psi(x_1,\xi'))^2 \left|\widehat{F}_1(x_1,\xi')\right|^2 dx_1.$$

Note that, within the setting of Theorem 4.1, the function ψ defined as in (3.2) is such that there exists $C_1 > 0$ for which

$$\forall\, x_1 \in [X_0,X_1],\ \forall\, \xi' \in \mathbb{R}^{d-1},\ \frac{|\xi'|}{C_1} \leqslant \psi(x_1,\xi') \leqslant C_1|\xi'|, \qquad -C_1|\xi'| \leqslant \partial_1\psi(x_1,\xi') \leqslant -\frac{|\xi'|}{C_1}. \tag{4.5}$$

Accordingly, for $\tau \geqslant \tau_0$ large enough, there exists $C > 0$ such that for all $\xi' \in \mathbb{R}^{d-1}$ and $x_1 \in [X_0,X_1]$,

$$\frac{1}{C}(\tau+|\xi'|)^2 \leqslant (\tau+\psi(x_1,\xi'))^2 + \partial_1\psi(x_1,\xi') \leqslant C(\tau+|\xi'|)^2,$$
$$\frac{1}{C}(\tau+|\xi'|) \leqslant \tau+\psi(x_1,\xi') \leqslant C(\tau+|\xi'|).$$

Therefore, the above estimate yields:

$$\int_{X_0}^{X_1}(\tau+|\xi'|)^2|\widetilde{z}(x_1,\xi')|^2\,dx_1 \leqslant C(\tau+|\xi'|)|\widehat{g}_1(\xi')|^2 + C\int_{X_0}^{X_1}|\widehat{f}_2(x_1,\xi')|^2\,dx_1$$
$$+ C\sum_{j=2}^{d}\int_{X_0}^{X_1}\left|\xi_j\widehat{F}_j(x_1,\xi')\right|^2 dx_1 + C\int_{X_0}^{X_1}(\tau+|\xi'|)^2\left|\widehat{F}_1(x_1,\xi')\right|^2 dx_1.$$

Recalling that $\widetilde{z}(\cdot,\xi') = \widehat{z}(\cdot,\xi') - \widehat{F}_1(\cdot,\xi')$, we obtain

$$\int_{X_0}^{X_1}(\tau+|\xi'|)^2|\widehat{z}(x_1,\xi')|^2\,dx_1 \leqslant C(\tau+|\xi'|)|\widehat{g}_1(\xi')|^2 + C\int_{X_0}^{X_1}\left|\widehat{f}_2(x_1,\xi')\right|^2 dx_1$$
$$+ C\sum_{j=2}^{d}\int_{X_0}^{X_1}\left|\xi_j\widehat{F}_j(x_1,\xi')\right|^2 dx_1 + C\int_{X_0}^{X_1}(\tau+|\xi'|)^2\left|\widehat{F}_1(x_1,\xi')\right|^2 dx_1.$$

We then derive estimates on $\widehat{w}(\cdot,\xi')$ from the equation $(4.4)_{(1,3,4)}$, again by taking the square of both sides of $(4.4)_{(1)}$ and doing integration by parts. If for all $x_1 \in [X_0,X_0]$, $\psi(x_1,\xi') \neq \tau$, we do the computation at once by doing the integration by parts on (X_0,X_1), and if there exists $x_1^*(\xi') \in [X_0,X_1]$ such that $\psi(x_1^*(\xi'),\xi') = \tau$ (recall that such an $x_1^*(\xi')$ is necessarily unique), we do the computations on $[X_0,x_1^*(\xi'))$ and on $(x_1^*(\xi'),X_1]$, and we sum the estimates. In this latter case, $\partial_1\widehat{w}$ should be interpreted as $1_{x_1<x_1^*(\xi')}\partial_1\widehat{w}(x_1,\xi') + 1_{x_1>x_1^*(\xi')}\partial_1\widehat{w}(x_1,\xi')$. There is a priori no reason that this coincides with the derivative of $\widehat{w}(\cdot,\xi')$ in the sense of $\mathscr{D}'(X_0,X_1)$, which would require some continuity conditions on $\widehat{w}(x_1^*(\xi')^{\pm},\xi')$. We get:

$$\int_{X_0}^{X_1}\left(|\partial_1\widehat{w}(x_1,\xi')|^2 + \left((\tau-\psi(x_1,\xi'))^2 - \partial_1\psi(x_1,\xi')\right)|\widehat{w}(x_1,\xi')|^2\right)dx_1$$
$$+ (\psi(X_1,\xi')-\tau)\,|\widehat{w}(X_1,\xi')|^2 - (\psi(X_0,\xi')-\tau)\,|\widehat{w}(X_0,\xi')|^2 = \int_{X_0}^{X_1}|\widehat{z}(x_1,\xi')|^2\,dx_1.$$

Accordingly, using the boundary conditions $(4.4)_{(3,4)}$ and (4.5),

$$\int_{X_0}^{X_1} \left(|\partial_1 \widehat{w}(x_1,\xi')|^2 + \left((\tau - \psi(x_1,\xi'))^2 + |\xi'| \right) |\widehat{w}(x_1,\xi')|^2 \right) dx_1 \\ \leqslant C(\psi(X_0,\xi') - \tau) 1_{\psi(X_0,\xi') > \tau} |\widehat{g}_0(\xi')|^2 + C \int_{X_0}^{X_1} |\widehat{z}(x_1,\xi')|^2 \, dx_1.$$

Therefore, there exists $C > 0$ such that for all $\tau \geqslant \tau_0$ and $\xi' \in \mathbb{R}^{d-1}$,

$$\int_{X_0}^{X_1} \left(|\partial_1 \widehat{w}(x_1,\xi')|^2 + \left((\tau - \psi(x_1,\xi'))^2 + |\xi'| \right) |\widehat{w}(x_1,\xi')|^2 \right) dx_1 \\ \leqslant C(\psi(X_0,\xi') - \tau) 1_{\psi(X_0,\xi') > \tau} |\widehat{g}_0(\xi')|^2 + C \frac{1}{\tau + |\xi'|} |\widehat{g}_1(X_1,\xi')|^2 \\ + \frac{C}{(\tau + |\xi'|)^2} \left(\int_{X_0}^{X_1} \left| \widehat{f}_2(x_1,\xi') \right|^2 dx_1 + |\xi'|^2 \sum_{j=2}^{d} \int_{X_0}^{X_1} \left| \widehat{F}_j(x_1,\xi') \right|^2 dx_1 \right) \\ + C \int_{X_0}^{X_1} \left| \widehat{F}_1(x_1,\xi') \right|^2 dx_1.$$

We finally use that there exists a constant such that for all $\xi' \in \mathbb{R}^{d-1}$, $\tau \geqslant \tau_0$, and $x_1 \in [X_0, X_1]$,

$$\left((\tau - \psi(x_1,\xi'))^2 + |\xi'| \right) \geqslant \frac{1}{C} \left(\tau + \frac{|\xi'|^2}{\tau} \right), \quad (\psi(X_0,\xi') - \tau) 1_{\psi(X_0,\xi') > \tau} \leqslant C|\xi'|, \\ \frac{1}{\tau + |\xi'|} \leqslant \frac{1}{1 + |\xi'|}, \quad \frac{1}{(\tau + |\xi'|)^2} \leqslant \frac{1}{\tau^2}, \quad \frac{|\xi'|^2}{(\tau + |\xi'|)^2} \leqslant 1,$$

so that

$$\int_{X_0}^{X_1} \left(\tau \, |\widehat{w}(x_1,\xi')|^2 + |\partial_1 \widehat{w}(x_1,\xi')|^2 + \frac{1}{\tau} |\xi'|^2 \, |\widehat{w}(x_1,\xi')|^2 \right) dx_1 \\ \leqslant C|\xi'| \, |\widehat{g}_0(\xi')|^2 + \frac{C}{1 + |\xi'|} |\widehat{g}_1(\xi')|^2 + \frac{C}{\tau^2} \int_{X_0}^{X_1} \left| \widehat{f}_2(x_1,\xi') \right|^2 dx_1 \\ + C \sum_{j=1}^{d} \int_{X_0}^{X_1} \left| \widehat{F}_j(x_1,\xi') \right|^2 dx_1.$$

Integrating in $\xi' \in \mathbb{R}^{d-1}$ and using Parseval's identity, we derive (4.2).

To prove (4.3), we simply remark that $\widehat{R_\tau w} = \widehat{K_{\tau,0} f_2}$ with $\widehat{f}_2(x_1,\xi') = \partial_1 \psi(x_1,\xi') \, \widehat{w}(x_1,\xi')$, which clearly satisfies $\|f_2\|_{L^2(\Omega)} \leqslant C \|\nabla' w\|_{L^2(\Omega)}$. Accordingly the estimates on $K_{\tau,0}$ in (4.2) immediately provide (4.3). □

In view of the above computations, for w as in (4.1), we have good estimates on $\partial_1 \widehat{w}$ in $L^2(\Omega_{1,\tau})$, where

$$\text{(4.6)} \quad \Omega_{1,\tau} = \{(x_1,\xi') \in \Omega, \text{ with } \psi(x_1,\xi') > \tau\} \cup \{(x_1,\xi') \in \Omega, \text{ with } \psi(x_1,\xi') < \tau\}.$$

Indeed, from the above computations, for w as in (4.1), we have

$$\tau\|\partial_1\widehat{w}\|_{L^2(\Omega_{1,\tau})} \leqslant C\|f_2\|_{L^2(\Omega)} + C\tau\sum_{j=1}^{d}\|F_j\|_{L^2(\Omega)} + C\tau\|g_0\|_{H^{\frac{1}{2}}(\mathbb{R}^{d-1})} + C\tau\|g_1\|_{H^{-\frac{1}{2}}(\mathbb{R}^{d-1})}. \tag{4.7}$$

In particular, if one knows that $w\in H^1(\Omega)$, we get an estimate on $\partial_1 w$ in $L^2(\Omega)$.

Note that the above proof and the previous remark immediately give the following result, whose proof is left to the reader, since all solutions w of (2.8) with source terms $(f_2, f_{2'_*}, F, g)$ as in (2.3) belong to $H^1(\Omega)$, and the terms $\|\nabla' w\|^2_{L^2(\Omega)}$ coming from $R_\tau(w)$ and (4.3) can be easily absorbed by taking τ large enough:

Proposition 4.3. — *Let Ω be as in* (2.1) *with $X_0<0<X_1$ and* $\max\{|X_0|,|X_1|\}\leqslant 1$, *and assume that the coefficients $(\lambda_j)_{j\in\{1,\cdots,d\}}\in\mathbb{R}^d$ satisfy $\lambda_1=0$,* (2.4) *and* (2.5).

Then there exist constants $C>0$ and $\tau_0\geqslant 1$ depending on c_0, m_* and M_* (independent of X_0, X_1) such that for all $(f_2, f_{2'_*}, F, g)$ as in (2.3) *with $f_{2'_*}=0$, if the solution w of* (2.8) *satisfies $(\partial_1 w - F_1)(X_1,x')=0$ in $\mathbb{R}^{d-1}$, then for all $\tau\geqslant\tau_0$,*

$$w = K_{\tau,0}(f_2) + K_{\tau,1}(F_1) + \sum_{j=2}^{d} K_{\tau,j}(F_j) + R_\tau(w) + G_\tau(g),$$

where the operators $(K_{\tau,i})_{i\in\{0,\cdots,d\}}$, R_τ and G_τ are defined in Proposition 3.1, and

$$\tau^{\frac{3}{2}}\|w\|_{L^2(\Omega)} + \tau\|\partial_1 w\|_{L^2(\Omega)} + \tau^{\frac{1}{2}}\|\nabla' w\|_{L^2(\Omega)} \leqslant C\|f_2\|_{L^2(\Omega)} + C\tau\|F\|_{L^2(\Omega)} + C\tau\|g\|_{H^{\frac{1}{2}}(\mathbb{R}^{d-1})}.$$

We now check that the estimate on G_τ can indeed be improved:

Theorem 4.4. — *Let Ω be as in* (2.1) *with $X_0<0<X_1$ and* $\max\{|X_0|,|X_1|\}\leqslant 1$, *and assume that the coefficients $(\lambda_j)_{j\in\{1,\cdots,d\}}\in\mathbb{R}^d$ satisfy $\lambda_1=0$,* (2.4) *and* (2.5), *and let G_τ be the operator in* (3.9) *and* (3.10).

Then there exists a constant $C>0$ depending on c_0, m_* and M_* (independent of X_0, X_1) such that for all $\tau\geqslant 1$, for all $g\in H^{\frac{1}{2}}(\mathbb{R}^{d-1})$,

$$\tau^{\frac{3}{4}}\|G_\tau(g)\|_{L^2(\Omega)} + \left\|\widehat{\partial_1 G_\tau}(g)\right\|_{L^2(\Omega_{1,\tau})} + \tau^{-\frac{1}{4}}\|\nabla' G_\tau(g)\|_{L^2(\Omega)} \leqslant C\|g\|_{H^{\frac{1}{2}}(\mathbb{R}^{d-1})},$$

where $\Omega_{1,\tau}$ is defined by (4.6).

Remark 4.5. — Note that the estimates in Theorem 4.4 yield better estimates than the ones of Theorem 4.1 on the $\mathscr{L}(H^{\frac{1}{2}}(\mathbb{R}^{d-1}), L^2(\mathbb{R}^d))$ norms of the operators G_τ and $\nabla' G_\tau$, and are in agreement with the ones obtained in [IP03].

Proof. — For $\tau\geqslant 1$ and $\xi'\in\mathbb{R}^{d-1}$ such that $\psi(X_0,\xi')>\tau$, we introduce $x_1^*(\xi')\in(X_0,X_1]$ as the unique solution of $\psi(x_1^*(\xi'),\xi')=\tau$ if it exists, or $x_1^*(\xi')=X_1$ otherwise, and we compute

$$\int_{X_0}^{x_1^*(\xi')} |g_\tau(x_1,\xi')|^2\,dx_1, \text{ and } \int_{X_0}^{x_1^*(\xi')} |\xi'|^2\,|g_\tau(x_1,\xi')|^2\,dx_1, \text{ and } \int_{X_0}^{x_1^*(\xi')} |\partial_1 g_\tau(x_1,\xi')|^2\,dx_1.$$

In order to do that, we recall that, within the setting of Theorem 4.4, we have

$$\begin{aligned}
&\exists\, C>0, \forall\, \xi'\in\mathbb{R}^{d-1}, \forall\,(x_1,y_1)\in[X_0,X_1]^2 \text{ with } y_1<x_1,\\
&\qquad\qquad \psi(y_1,\xi') \geqslant \psi(x_1,\xi') + \frac{1}{C}|\xi'||x_1-y_1|,\\
&\exists\, C>0,\ \forall\,\xi'\in\mathbb{R}^{d-1},\ \forall\,\tau\geqslant 1,\ \forall\, x_1\in[X_0,x_1^*(\xi')],\\
&\qquad\qquad \frac{1}{C}|\xi'||x_1^*(\xi')-x_1| \leqslant \tau-\psi(x_1,\xi'),
\end{aligned}$$

so that

$$\begin{aligned}
&\int_{X_0}^{x_1^*(\xi')} |g_\tau(x_1,\xi')|^2\ dx_1\\
&\quad\leqslant \int_{X_0}^{x_1^*(\xi')} e^{2\tau(x_1-X_0)-2\int_{X_0}^{x_1}\psi(y_1,\xi')\,dy_1}\,dx_1\\
&\quad\leqslant \int_{X_0}^{x_1^*(\xi')} e^{2(\tau-\psi(x_1,\xi'))(x_1-X_0)-|\xi'|(x_1-X_0)^2/C}\,dx_1\\
&\quad\leqslant \int_{X_0}^{x_1^*(\xi')} e^{-2|\xi'|\left(x_1^*(\xi')-x_1\right)(x_1-X_0)/C-|\xi'|(x_1-X_0)^2/C}\,dx_1\\
&\quad\leqslant \int_{X_0}^{\left(X_0+x_1^*(\xi')\right)/2} e^{-2|\xi'|\left(x_1^*(\xi')-x_1\right)(x_1-X_0)/C}\,dx_1
+\int_{\left(X_0+x_1^*(\xi')\right)/2}^{x_1^*(\xi')} e^{-|\xi'|(x_1-X_0)^2/C}\,dx_1\\
&\quad\leqslant \int_{X_0}^{\left(X_0+x_1^*(\xi')\right)/2} e^{-|\xi'|\left(x_1^*(\xi')-X_0\right)(x_1-X_0)/(2C)}dx_1
+\int_{\left(X_0+x_1^*(\xi')\right)/2}^{x_1^*(\xi')} e^{-|\xi'|\left(x_1^*(\xi')-X_0\right)(x_1-X_0)/C}dx_1\\
&\quad\leqslant 2\int_{X_0}^{\left(X_0+x_1^*(\xi')\right)/2} e^{-|\xi'|\left(x_1^*(\xi')-X_0\right)(x_1-X_0)/(2C)}\,dx_1\\
&\quad\leqslant C\min\left\{\frac{1}{|\xi'||x_1^*(\xi')-X_0|}, |x_1^*(\xi')-X_0|\right\}.
\end{aligned}$$

It is then easy to check that, for $\tau \geqslant 1$ and $\xi' \in \mathbb{R}^{d-1}$ such that $\psi(X_0,\xi') > \tau$,

$$\begin{aligned}\int_{X_0}^{x_1^*(\xi')} &\left|g_\tau(x_1,\xi')\right|^2\,dx_1 |g(\xi')|^2\\ &\leqslant C\left(\sup_{\psi(X_0,\xi')>\tau}\left\{\min\left\{\frac{1}{|\xi'|^2|x_1^*(\xi')-X_0|},\frac{|x_1^*(\xi')-X_0|}{|\xi'|}\right\}\right\}\right)|\xi'||g(\xi')|^2\\ &\leqslant \frac{C}{\tau^{\frac{3}{2}}}\,|\xi'|\,\,|g(\xi')|^2.\end{aligned}$$

Integrating in $\xi' \in \mathbb{R}^{d-1}$ and using Parseval's identity, we obtain that

$$\tau^{\frac{3}{4}}\|G_\tau(g)\|_{L^2(\Omega)} \leqslant C\|g\|_{H^{\frac{1}{2}}\left(\mathbb{R}^{d-1}\right)}.$$

Similarly, we prove that

$$\begin{aligned}\int_{X_0}^{x_1^*(\xi')} &|\xi'|^2\left|g_\tau(x_1,\xi')\right|^2\,dx_1\,|g(\xi')|^2\\ &\leqslant C\left(\sup_{\psi(X_0,\xi')>\tau}\left\{\min\left\{\frac{1}{|x_1^*(\xi')-X_0|},|\xi'|\,|x_1^*(\xi')-X_0|\right\}\right\}\right)|\xi'|\,\,|g(\xi')|^2\\ &\leqslant C\tau^{\frac{1}{2}}\,|\xi'|\,\,|g(\xi')|^2\,,\end{aligned}$$

so that there exists a constant $C>0$ such that for all $g \in H^{\frac{1}{2}}(\mathbb{R}^{d-1})$,

$$\tau^{-\frac{1}{4}}\left\|\nabla' G_\tau(g)\right\|_{L^2(\Omega)} \leqslant C\|g\|_{H^{\frac{1}{2}}\left(\mathbb{R}^{d-1}\right)}.$$

We then check that

$$\partial_1 g_\tau(x_1,\xi') = (\tau - \psi(x_1,\xi'))\,g_\tau\left(x_1,\xi'\right).$$

Then, if $\xi' \in \mathbb{R}^{d-1}$ is such that $\psi(X_0,\xi') > \tau$ and $\psi(X_1,\xi') < \tau$, using that for all $x_1 \in [X_0, x_1^*(\xi'))$, $\tau - \psi(x_1,\xi') \leqslant C|\xi'||x_1^*(\xi') - x_1|$, we get

$$\begin{aligned}\int_{X_0}^{x_1^*(\xi')}\left|\partial_1 g_\tau(x_1,\xi')\right|^2\,dx_1 &\leqslant \int_{X_0}^{x_1^*(\xi')}\left(\tau-\psi(x_1,\xi')\right)^2 e^{2\tau(x_1-X_0)-2\int_{X_0}^{x_1}\psi(y_1,\xi')\,dy_1}\,dx_1\\ \leqslant C\int_{X_0}^{x_1^*(\xi')} &|\xi'|^2\left(x_1^*(\xi')-x_1\right)^2 e^{-2|\xi'|\left(x_1^*(\xi')-x_1\right)(x_1-X_0)/C-|\xi'|(x_1-X_0)^2/C}\,dx_1\\ \leqslant C\int_{X_0}^{\left(X_0+x_1^*(\xi')\right)/2} &|\xi'|^2\left(x_1^*(\xi')-X_0\right)^2 e^{-2|\xi'|\left(x_1^*(\xi')-X_0\right)(x_1-X_0)/C}\,dx_1\\ +C\int_{\left(X_0+x_1^*(\xi')\right)/2}^{x_1^*(\xi')} &|\xi'|^2\left(x_1^*(\xi')-x_1\right)^2 e^{-|\xi'|\left(x_1^*(\xi')-X_0\right)(x_1-X_0)/(2C)}\,dx_1\\ \leqslant C\int_{X_0}^{\left(X_0+x_1^*(\xi')\right)/2} &|\xi'|^2\left(x_1^*(\xi')-X_0\right)^2 e^{-|\xi'|\left(x_1^*(\xi')-X_0\right)(x_1-X_0)/(2C)}\,dx_1\\ \leqslant C\min&\left\{|\xi'|(x_1^*(\xi')-X_0),|\xi'|^2\left(x_1^*(\xi')-X_0\right)^3\right\}\\ \leqslant C\min&\left\{\psi(X_0,\xi')-\tau,\frac{(\psi(X_0,\xi')-\tau)^3}{|\xi'|}\right\}.\end{aligned}$$

In particular,

$$\begin{aligned}\int_{X_0}^{x_1^*(\xi')} \left|\partial_1 g_\tau(x_1,\xi')\right|^2\, dx_1 |g(\xi')|^2 \\ \leqslant C \left(\sup_{\psi(X_0,\xi')>\tau} \left\{ \min\left\{ \frac{\psi(X_0,\xi')-\tau}{|\xi'|}, \frac{(\psi(X_0,\xi')-\tau)^3}{|\xi'|^2} \right\}\right\}\right) |\xi'|\ \left|g(\xi')\right|^2 \\ \leqslant C\left|\xi'\right|\ \left|g(\xi')\right|^2.\end{aligned}$$

Similar estimates can be achieved for $\xi' \in \mathbb{R}^{d-1}$ such that $\psi(X_1,\xi') \geqslant \tau$, and details are left to the reader.

We thus obtained that there exists a constant $C>0$ such that for all $g \in H^{\frac12}(\mathbb{R}^{d-1})$ and $\tau \geqslant 1$,

$$\left\| \partial_1 \widehat{G_\tau(g)} \right\|_{L^2(\Omega_{1,\tau})} \leqslant C \|g\|_{H^{\frac12}\left(\mathbb{R}^{d-1}\right)}.$$

This concludes the proof of Theorem 4.4. □

5. Fourier Restriction theorems and applications

In this section, we first recall the classical Fourier restriction theorem, and present a version adapted to our case. We will then explain how it can be applied to estimate the norms of operators of some specific forms, which will encompass the ones appearing in the parametrix provided in Proposition 3.1.

In this section, $n \geqslant 2$ and, for a function $f \in \mathscr{S}(\mathbb{R}^n)$, the Fourier transform $\widehat{f} \in \mathscr{S}(\mathbb{R}^n)$ is given by

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{x\in\mathbb{R}^n} e^{-\mathbf{i}x\cdot\xi} f(x)\,dx, \qquad \xi \in \mathbb{R}^n,$$

and is extended by duality to functions in $\mathscr{S}'(\mathbb{R}^n)$ as usual. We will see later that n in fact corresponds to $d-1$ in the applications we have in mind.

5.1. Fourier restriction theorems

We start by recalling the classical Stein–Tomas Fourier restriction theorem:

Theorem 5.1 ([Tom75], see also [Ste93, Theorem 2, p. 352]). — *Let $n \geqslant 2$, and $\mathbb{S}^{n-1}$ denote the unit sphere of $\mathbb{R}^n$.*

Then the map

$$\begin{cases} L^1(\mathbb{R}^n) \to L^2\left(\mathbb{S}^{n-1}\right) \\ f \mapsto \widehat{f}\,\Big|_{\mathbb{S}^{n-1}} \end{cases}$$

can be extended by continuity on $L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n)$, and there exists a constant $C>0$ such that for all $f \in L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n)$,

$$\left\|\widehat{f}\right\|_{L^2(\mathbb{S}^{n-1})} \leqslant C\|f\|_{L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n)}.$$

It is well-known that this restriction theorem can be extended to any hypersurface with non-vanishing Gaussian curvature, (see, for example, [Sog17, Corollary 2.2.2]).

In view of the formulae in Proposition 3.1, it is interesting for us to analyze Fourier restriction theorems on the family of surfaces

$$\Sigma_a = \{\xi \in \mathbb{R}^n,\, \psi(a,\xi) = 1\}, \qquad a \in [X_0, X_1], \tag{5.1}$$

where by analogy with the function ψ in (3.2) and the conditions (2.4), we have set

$$\psi(a,\xi) = \sqrt{\sum_{j=1}^n (1-a\lambda_j)\xi_j^2}, \qquad a \in [X_0, X_1],\ \xi \in \mathbb{R}^n. \tag{5.2}$$

where the family of coefficients $(\lambda_j)_{j\in\{1,\cdots,n\}}$ satisfies

$$\exists\, c_0 > 0, \quad \forall\, a \in [X_0, X_1], \forall\, \xi \in \mathbb{R}^n, \quad \frac{1}{c_0}|\xi|^2 \leqslant \sum_{j=1}^n (1-a\lambda_j)\,|\xi_j|^2 \leqslant c_0|\xi|^2. \tag{5.3}$$

Note that due to condition (5.3), for all $a \in [X_0, X_1]$, the surface Σ_a is an ellipsoid and thus [Sog17, Corollary 2.2.2] applies and yields that for all $a \in [X_0, X_1]$, the map $f \mapsto \widehat{f}\,|_{\Sigma_a}$ maps $L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n)$ to $L^2(\Sigma_a)$.

For our purpose, we need a slightly more refined version of this result, guaranteeing that the norm of this map is independent of $a \in [X_0, X_1]$.

Theorem 5.2. — *Let $n \geqslant 2$. Assume that the family of coefficients $(\lambda_j)_{j\in\{1,\cdots,n\}}$ satisfies (5.3) for some $c_0 > 0$. Then there exists a constant $C > 0$ depending only on c_0 (and n) such that for all $a \in [X_0, X_1]$, for all $f \in L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n)$,*

$$\left\|\widehat{f}\right\|_{L^2(\Sigma_a)} \leqslant C\|f\|_{L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n)}. \tag{5.4}$$

Note that the proof below follows the classical one of Theorem 5.1 and is mainly based on the stationary phase lemma.

Proof. — For $a \in [X_0, X_1]$, we denote by T_a the map $T_a : f \in L^1(\mathbb{R}^n) \mapsto \widehat{f}\,|_{\Sigma_a} \in L^2(\Sigma_a)$. We then consider its adjoint operator $T_a^\star : L^2(\Sigma_a) \to L^\infty(\mathbb{R}^n)$: for $g \in L^2(\Sigma_a)$,

$$T_a^\star g(x) = \int_{\omega\in\Sigma_a} e^{\mathbf{i}x\cdot\omega} g(\omega)\, d\Sigma_a(\omega), \qquad x \in \mathbb{R}^n.$$

The operator $T_a^\star T_a$ then maps $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ and, for $f \in L^1(\mathbb{R}^n)$,

$$T_a^\star T_a f(x) = \int_{\mathbb{R}^n}\int_{\omega\in\Sigma_a} e^{\mathbf{i}(x-\widetilde{x})\cdot\omega}\, d\Sigma_a(\omega) f(\widetilde{x})\, d\widetilde{x}, \qquad x \in \mathbb{R}^n. \tag{5.5}$$

Next, we will prove that the operator $T_a^\star T_a$ can in fact be extended as an operator from $L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n)$ to $L^{\frac{2(n+1)}{(n-1)}}(\mathbb{R}^n)$, uniformly with respect to $a \in [X_0, X_1]$. This will prove (5.4) since

$$\|T_a^\star T_a\|_{\mathscr{L}\left(L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n),\, L^{\frac{2(n+1)}{(n-1)}}(\mathbb{R}^n)\right)} = \|T_a\|^2_{\mathscr{L}\left(L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n),\, L^2(\Sigma_a)\right)}.$$

To start with, we parametrize the hypersurface Σ_a through several patches. We first remark that Σ_a can be mapped into the sphere $\mathbb{S}^{n-1}$ as follows. For $\omega \in \Sigma_a$, we define $\xi = G_a(\omega)$ by

$$\forall\, j \in \{1, \cdots, n\}, \quad \xi_j = \omega_j \sqrt{1 - a\lambda_j}.$$

We then choose a spherical cap

$$\mathscr{C}_n = \left\{ \xi \in \mathbb{S}^{n-1}; \quad \xi_n \geqslant \frac{1}{\sqrt{2n}} \right\}.$$

It is easy to check that, if, for $\epsilon \in \{-1, 1\}$ and $j \in \{1, \cdots, n\}$, $R_{\epsilon,j}$ denotes the rotation that maps the basis vector e_n to ϵe_j, and leaves all the vectors e_k for $k \neq j, n$ invariant, then the family of $R_{\epsilon,j}\mathscr{C}_n$ for $\epsilon \in \{-1, 1\}$ and $j \in \{1, \cdots, n\}$ covers the whole sphere. Therefore, there exists a partition of unity $(\chi_{\epsilon,j})_{\epsilon \in \{-1,1\},\, j \in \{1, \cdots, n\}}$ of the sphere $\mathbb{S}^{n-1}$ such that for each $\epsilon \in \{-1, 1\}$ and $j \in \{1, \cdots, n\}$, the function $\chi_{\epsilon,j}$ is smooth and compactly supported in $R_{\epsilon,j}\mathscr{C}_n$. Since by construction, $\sum_{\epsilon,j} \chi_{\epsilon,j}(\xi) = 1$ for all $\xi \in \mathbb{S}^{n-1}$, we have

$$\forall\, \omega \in \Sigma_a, \qquad \sum_{\epsilon \in \{-1,1\},\, j \in \{1, \cdots, n\}} \chi_{\epsilon,j}(G_a(\omega)) = 1.$$

Therefore,

$$T_a^\star T_a f(x) = \sum_{\epsilon \in \{-1,1\},\ j \in \{1, \cdots, n\}} \int_{\mathbb{R}^n} \int_{\omega \in \Sigma_a} \chi_{\epsilon,j}(G_a(\omega)) e^{\mathbf{i}\left(x - \widetilde{x}\right)\cdot\omega}\, d\Sigma_a(\omega) f(\widetilde{x})\, d\widetilde{x}, \quad x \in \mathbb{R}^n.$$

Besides, for all $\epsilon \in \{-1, 1\}$ and $j \in \{1, \cdots, n\}$, $\chi_{\epsilon,j} \circ G_a$ is supported in the set of all $\omega \in \Sigma_a$ such that $\epsilon\omega_j\sqrt{1 - a\lambda_j} \geqslant 1/\sqrt{2n}$, i.e. the pre-image of the cap $R_{\epsilon,j}(\mathscr{C}_n)$ by G_a, that we denote by $\mathscr{C}_{\epsilon,j}$. It is clear that this set can be parametrized by $\mathbb{R}^{n-1}$ as follows. Denoting $\check{\xi}_j = (\xi_1, \cdots, \xi_{j-1}, \xi_{j+1}, \cdots, \xi_n)$, we easily obtain

$$\mathscr{C}_{\epsilon,j} = \left\{ \left(\xi_1, .., \xi_{j-1}, \epsilon h_j\left(a, \check{\xi}_j\right), \xi_{j+1}, .., \xi_n\right); \quad \sum_{\substack{k=1\\k\neq j}}^{n} (1 - a\lambda_k)\xi_k^2 \leqslant 1 - \frac{1}{2n} \right\},$$

where the function h_j is defined by the formula

$$h_j\left(a, \check{\xi}_j\right) = \frac{1}{\sqrt{1 - a\lambda_j}} \sqrt{1 - \sum_{\substack{k=1\\k\neq j}}^{n} (1 - a\lambda_k)\xi_k^2}, \qquad a \in [X_0, X_1],\ \check{\xi}_j \in V_{a,j},$$

with $V_{a,j}$ given by

$$V_{a,j} = \left\{ \check{\xi}_j \in \mathbb{R}^{n-1}; \sum_{\substack{k=1\\k\neq j}}^{n} (1 - a\lambda_k)\xi_k^2 \leqslant 1 - \frac{1}{2n} \right\}.$$

Therefore, the study of $T_a^\star T_a$ is reduced to the study of the family of operators

$$f \mapsto \Bigg(x \mapsto \int_{\mathbb{R}^n} \int_{\check{\xi}_j \in V_{a,j}} \chi_{\epsilon,j}\left(G_a\left(H_{\epsilon,j}\left(a, \check{\xi}_j\right)\right)\right) e^{\mathbf{i}(\check{x}_j - \check{y}_j)\cdot\check{\xi}_j + \epsilon\mathbf{i}\left(x_j - \widetilde{x}_j\right) h\left(a, \check{\xi}_j\right)} \times \sqrt{1 + \left|\nabla h_j\left(a, \check{\xi}_j\right)\right|^2} d\check{\xi}_j f(y)\, dy \Bigg),$$

where $H_{\epsilon,j}(a,\check{\xi}_j) = \Big(\xi_1, .., \xi_{j-1}, \epsilon h_j(a,\check{\xi}_j), \xi_{j+1}, .., \xi_n\Big)$, for all $j \in \{1, \cdots, n\}$ and $\epsilon \in \{-1, 1\}$.

Thus, up to a renumbering of the coefficients, we can focus without loss of generality on the operator corresponding to $\epsilon = 1$ and $j = n$. Accordingly, we introduce the notation $x' = (x_1, \cdots, x_{n-1})$ and $\xi' = (\xi_1, \cdots, \xi_{n-1})$, and we consider the operator, defined for $f \in L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ by

$$\mathcal{T}_a f(x) = \int_{\mathbb{R}^n} \int_{\xi' \in \mathbb{R}^{n-1}} \chi\Big(G_a(\xi', h_n(a,\xi'))\Big) e^{\mathbf{i}\left(x'-\widetilde{x}'\right)\cdot\xi' + \mathbf{i}\left(x_n - \widetilde{x}_n\right) h_n(a,\xi')} \times \sqrt{1 + |\nabla h_n(a,\xi')|^2}\, d\xi' f(\widetilde{x})\, d\widetilde{x}, \tag{5.6}$$

where χ is a smooth function on the sphere $\mathbb{S}^{n-1}$ compactly supported in the spherical cap $\mathscr{C}_n$ and $\chi(G_a(\cdot, h_n(a,\cdot))$ is extended by 0 for $\xi' \notin V_{a,n}$.

We have reduced the proof of Theorem 5.2 to the proof of the fact that the maps $\mathcal{T}_a$ defined in (5.6) belong to the space $\mathscr{L}(L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n), L^{\frac{2(n+1)}{(n-1)}}(\mathbb{R}^n))$ uniformly with respect to $a \in [X_0, X_1]$.

In order to show this property, for $a \in [X_0, X_1]$ and $\delta \in \mathbb{R}$, we introduce the family of operators, defined from $L^1(\mathbb{R}^{n-1})$ to $L^\infty(\mathbb{R}^{n-1})$ by

$$\mathscr{T}_{a,\delta} f(x') = \int_{\mathbb{R}^{n-1}} \int_{\xi' \in \mathbb{R}^{n-1}} \chi\Big(G_a(\xi', h_n(a,\xi'))\Big) e^{\mathbf{i}\left(x'-\widetilde{x}'\right)\cdot\xi' + \mathbf{i}\delta h_n(a,\xi')} \times \sqrt{1 + |\nabla h_n(a,\xi')|^2}\, d\xi' f(\widetilde{x}')\, d\widetilde{x}',$$

for which we will show that there exists a constant $C > 0$ such that for all $a \in [X_0, X_1]$ and $\delta \in \mathbb{R}$,

$$\|\mathscr{T}_{a,\delta}\|_{\mathscr{L}\left(L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^{n-1}), L^{\frac{2(n+1)}{(n-1)}}(\mathbb{R}^{n-1})\right)} \leqslant C|\delta|^{-\frac{n-1}{n+1}}. \tag{5.7}$$

Indeed, if the estimate (5.7) holds, then Hardy–Littlewood–Sobolev theorem (recalled in Appendix in Theorem A.1) implies that, for $f \in L^1(\mathbb{R}^n) \cap L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n)$,

$$\begin{aligned}
\|\mathcal{T}_a f\|_{L^{\frac{2(n+1)}{(n-1)}}(\mathbb{R}^n)} &\leqslant \left\| \left\| \int_{\widetilde{x}_n \in \mathbb{R}} \mathscr{T}_{a, x_n - \widetilde{x}_n} f(\cdot, \widetilde{x}_n)\, d\widetilde{x}_n \right\|_{L_{x'}^{\frac{2(n+1)}{(n-1)}}(\mathbb{R}^{n-1})} \right\|_{L_{x_n}^{\frac{2(n+1)}{(n-1)}}(\mathbb{R})} \\
&\leqslant \left\| \int_{\widetilde{x}_n \in \mathbb{R}} \left\| \mathscr{T}_{a, x_n - \widetilde{x}_n} f(\cdot, \widetilde{x}_n) \right\|_{L_{x'}^{\frac{2(n+1)}{(n-1)}}(\mathbb{R}^{n-1})} d\widetilde{x}_n \right\|_{L_{x_n}^{\frac{2(n+1)}{(n-1)}}(\mathbb{R})} \\
&\leqslant C \left\| \int_{\widetilde{x}_n \in \mathbb{R}} |\widetilde{x}_n - x_n|^{-\frac{n-1}{n+1}} \|f(\cdot, \widetilde{x}_n)\|_{L_{\widetilde{x}'}^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^{n-1})} d\widetilde{x}_n \right\|_{L_{x_n}^{\frac{2(n+1)}{(n-1)}}(\mathbb{R})} \\
&\leqslant C \left\| \|f(\cdot, \widetilde{x}_n)\|_{L_{\widetilde{x}'}^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^{n-1})} \right\|_{L_{\widetilde{x}_n}^{\frac{2(n+1)}{(n+3)}}(\mathbb{R})} = C \|f\|_{L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n)},
\end{aligned}$$

since

$$\frac{n-1}{n+1} = 1 - \left(\frac{n+3}{2(n+1)} - \frac{n-1}{2(n+1)}\right).$$

We thus focus on the proof of estimate (5.7), which, as explained above, would conclude the proof of Theorem 5.2. This is done in three steps.

In the first step, we check that $\mathscr{T}_{a,\delta}$ maps $L^2(\mathbb{R}^{n-1})$ into itself with uniform bounds. Indeed, taking the Fourier transform $x' \to \xi'$ of $\mathbb{R}^{n-1}$, we easily get:

$$\widehat{\mathscr{T}_{a,\delta} f}(\xi') = \chi\Big(G_a(\xi', h_n(a,\xi'))\Big) e^{\mathbf{i}\delta h_n(a,\xi')} \sqrt{1 + |\nabla h_n(a,\xi')|^2}\, \widehat{f}(\xi'),$$

so that by Parseval's identity,

$$\|\mathscr{T}_{a,\delta}\|_{\mathscr{L}(L^2(\mathbb{R}^{n-1}))} \leqslant \left\| \chi\Big(G_a(\xi', h_n(a,\xi'))\Big) e^{\mathbf{i}\delta h_n(a,\xi')} \sqrt{1 + |\nabla h_n(a,\xi')|^2} \right\|_{L^\infty_{\xi'}(\mathbb{R}^{n-1})}.$$

We then immediately get that there exists a constant $C > 0$ depending only on c_0 in (5.3) such that for all $a \in [X_0, X_1]$ and $\delta \in \mathbb{R}$,

$$\|\mathscr{T}_{a,\delta}\|_{\mathscr{L}(L^2(\mathbb{R}^{n-1}))} \leqslant C. \tag{5.8}$$

In a second step, we check that $\mathscr{T}_{a,\delta}$ maps continuously $L^1(\mathbb{R}^{n-1})$ to $L^\infty(\mathbb{R}^{n-1})$ and get an estimate on its norm. In fact, we clearly have that

$$\begin{aligned} &\|\mathscr{T}_{a,\delta}\|_{\mathscr{L}(L^1(\mathbb{R}^{n-1}), L^\infty(\mathbb{R}^{n-1}))} \\ &\qquad \leqslant \sup_{\delta' \in \mathbb{R}^{n-1}} \left| \int_{\mathbb{R}^{n-1}} e^{\mathbf{i}\delta'\cdot\xi' + \mathbf{i}\delta h_n(a,\xi')} \chi\Big(G_a(\xi', h_n(a,\xi'))\Big) \sqrt{1 + |\nabla h_n(a,\xi')|^2}\, d\xi' \right|. \end{aligned} \tag{5.9}$$

Our goal is then to prove that there exists a constant $C > 0$ such that for all $a \in [X_0, X_1]$ and $\delta \in \mathbb{R}$,

$$\begin{aligned} \sup_{\delta' \in \mathbb{R}^{n-1}} \left| \int_{\mathbb{R}^{n-1}} e^{\mathbf{i}\delta'\cdot\xi' + \mathbf{i}\delta h_n(a,\xi')} \chi\Big(G_a(\xi', h_n(a,\xi'))\Big) \sqrt{1 + |\nabla h_n(a,\xi')|^2}\, d\xi' \right| \\ \leqslant \frac{C}{|\delta|^{\frac{n-1}{2}}}. \end{aligned} \tag{5.10}$$

For $\delta' \in \mathbb{R}^{n-1}$ and $\delta \in \mathbb{R}$, we define $\lambda > 0$ and $\omega \in \mathbb{S}^{n-1}$ by

$$\lambda = \sqrt{|\delta'|^2 + \delta^2} \quad \text{ and } \omega = \frac{1}{\lambda}(\delta', \delta).$$

Accordingly,

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} e^{\mathbf{i}\delta'\cdot\xi' + \mathbf{i}\delta h_n(a,\xi')} \chi\Big(G_a(\xi', h_n(a,\xi'))\Big) \sqrt{1 + |\nabla h_n(a,\xi')|^2} d\xi' \\ = \int_{\mathbb{R}^{n-1}} e^{\mathbf{i}\lambda\Phi(\omega,a,\xi')} \widetilde{\chi}(a,\xi') d\xi', \end{aligned}$$

where

$$\Phi(\omega,a,\xi') = \omega'\cdot\xi' + \omega_n h_n(a,\xi'), \text{ and } \quad \widetilde{\chi}(a,\xi') = \chi\Big(G_a(\xi', h_n(a,\xi'))\Big) \sqrt{1 + |\nabla h_n(a,\xi')|^2}.$$

Note that we immediately have that

$$\left| \int_{\mathbb{R}^{n-1}} e^{\mathbf{i}\lambda\Phi(\omega,a,\xi')} \widetilde{\chi}(a,\xi')\, d\xi' \right| \leqslant \sup_{a \in [X_0,X_1]} \|\widetilde{\chi}(a,\cdot)\|_{L^1(\mathbb{R}^{n-1})}, \tag{5.11}$$

so that we are only interested in large values of δ. It is then clear that we have to use the stationary phase lemma to get a suitable estimate on that quantity. Since we need to quantify properly in terms of the parameters $a \in [X_0, X_1]$ and $\delta \in \mathbb{R}$, we will use the refined version of [ABZ17, Theorem 1], recalled in the appendix in Theorem A.2.

Let $\Omega_n \in (0, \frac{1}{2})$ be such that

$$\Omega_n \sup_{a \in [X_0, X_1]} \|\nabla_{\xi'} h_n(a, \xi')\|_{L^\infty_{\xi'}(V_{a,n})} \leqslant \frac{1}{2}\sqrt{1 - \Omega_n^2}.$$

Then

$$\forall\, \omega \in \mathbb{S}^{n-1} \text{ with } |\omega_n| \leqslant \Omega_n, \qquad \inf_{a \in [X_0, X_1],\, \xi' \in V_{a,n}} |\nabla_{\xi'} \Phi(\omega, a, \xi')| \geqslant \frac{|\omega'|}{2}.$$

Therefore, if $\omega \in \mathbb{S}^{n-1}$ with $|\omega_n| \leqslant \Omega_n$, applying integration by parts based on the formula

$$e^{\mathbf{i}\lambda\Phi(\omega,a,\xi')} = \frac{1}{\mathbf{i}\lambda |\nabla_{\xi'} \Phi(\omega, a, \xi')|^2} \nabla_{\xi'} \Phi\left(\omega, a, \xi'\right) \cdot \nabla_{\xi'} e^{\mathbf{i}\lambda\Phi(\omega,a,\xi')},$$

we get that, for all $k \in \mathbb{N}$, there exists $C_k > 0$ and a decreasing function $\mathcal{F}_k : \mathbb{R} \to \mathbb{R}$ such that

$$\left| \int_{\mathbb{R}^{n-1}} e^{\mathbf{i}\lambda\Phi(\omega,a,\xi')} \widetilde{\chi}(a, \xi')\, d\xi' \right| \leqslant \frac{C_k}{|\lambda|^k} \mathcal{F}_k\left(\|\Phi(\omega, \cdot, \cdot)\|_{W^{k+1,\infty}([X_0,X_1] \times V_{a,n})} \right) \|\widetilde{\chi}\|_{W^{k+1,\infty}([X_0,X_1] \times V_{a,n})}.$$

Therefore, for all $k \in \mathbb{N}$, there exists a constant $C_k > 0$ such that for $\omega \in \mathbb{S}^{n-1}$ with $|\omega_n| \leqslant \Omega_n$ and $a \in [X_0, X_1]$,

$$\left| \int_{\mathbb{R}^{n-1}} e^{\mathbf{i}\lambda\Phi(\omega,a,\xi')} \widetilde{\chi}(a, \xi')\, d\xi' \right| \leqslant \frac{C_k}{|\delta|^k}. \tag{5.12}$$

It remains to analyze what happens when $\omega \in \mathbb{S}^{n-1}$ satisfies $|\omega_n| \geqslant \Omega_n$. There, we use that

$$\left(\operatorname{Hess}_{\xi'} \Phi\right)(\omega, a, \xi') = \omega_n \left(\operatorname{Hess}_{\xi'} h_n\right)(a, \xi').$$

Since there exists $a_0 > 0$ such that

$$\inf_{a \in [X_0, X_1]} \inf_{\xi' \in V_{a,n}} \left|\det\left(\operatorname{Hess}_{\xi'} h_n\right)(a, \xi')\right| \geqslant a_0.$$

a direct application of Theorem A.2 yields the existence of a constant $C > 0$ such that for all $a \in [X_0, X_1]$, for all $\omega \in \mathbb{S}^{n-1}$ with $|\omega_n| \geqslant \Omega_n$,

$$\left| \int_{\mathbb{R}^{n-1}} e^{\mathbf{i}\lambda\Phi(\omega,a,\xi')} \widetilde{\chi}(a, \xi')\, d\xi' \right| \leqslant \frac{C}{|\lambda|^{\frac{n-1}{2}}} \leqslant \frac{C}{|\delta|^{\frac{n-1}{2}}}. \tag{5.13}$$

Combining (5.11)–(5.12)–(5.13), we get (5.10), and thus from (5.9), the existence of a constant C such that for all $a \in [X_0, X_1]$ and $\delta \in \mathbb{R}$,

$$\|\mathscr{T}_{a,\delta}\|_{\mathscr{L}(L^1(\mathbb{R}^{n-1}), L^\infty(\mathbb{R}^{n-1}))} \leqslant \frac{C}{|\delta|^{\frac{n-1}{2}}}. \tag{5.14}$$

In a third and last step, we conclude the estimate (5.7) by M. Riesz interpolation theorem ([Sog17, Theorem 0.1.13]) combining (5.8) and (5.14). This concludes the proof of Theorem 5.2. □

5.2. Fourier multiplier operators

The goal of this section is to show how Theorem 5.2 can be applied to get estimates on some families of Fourier multipliers operators.

To be more precise, for $X_0 < X_1$ and coefficients $(\lambda_j)_{j\in\{1,\cdots,n\}}$ satisfying (5.3), we define ψ as in (5.2) and Σ_a the ellipsoid defined for $a \in [X_0, X_1]$ by (5.1).

For $a \in [X_0, X_1]$ and $k \in L^\infty(\mathbb{R}_+, L^\infty(\Sigma_a))$, we consider operators given as follows:

$$K_{a,k} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \qquad \text{given by } \widehat{K_{a,k}(f)}(\xi) = k\left(\psi(a,\xi), \frac{\xi}{\psi(a,\xi)}\right)\widehat{f}(\xi), \quad \xi \in \mathbb{R}^n. \tag{5.15}$$

We prove the following result:

Proposition 5.3. — *Let $n \in \mathbb{N}$, $n \geqslant 2$. Let $X_0 < X_1$, and the coefficients $(\lambda_j)_{j\in\{1,\cdots,n\}}$ satisfy (5.3). For $a \in [X_0, X_1]$, let ψ and Σ_a be as in (5.1)–(5.2). Then there exists a constant $C > 0$ such that, for all $a \in [X_0, X_1]$, for all $k \in L^\infty(\mathbb{R}_+, L^\infty(\Sigma_a))$,*

- *the Fourier multiplier operator $K_{a,k}$ in (5.15) maps $L^2(\mathbb{R}^n)$ to itself and*

$$\|K_{a,k}\|_{\mathscr{L}(L^2(\mathbb{R}^n))} \leqslant \|k\|_{L^\infty(\mathbb{R}_+, L^\infty(\Sigma_a))}. \tag{5.16}$$

- *if moreover, k satisfies*

$$\int_0^\infty \|k(\lambda,\cdot)\|_{L^\infty(\Sigma_a)} \lambda^{\frac{n-1}{n+1}}\, d\lambda < \infty,$$

the operator $K_{a,k}$ in (5.15) belongs to $\mathscr{L}(L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n), L^{\frac{2(n+1)}{(n-1)}}(\mathbb{R}^n))$ and

$$\|K_{a,k}\|_{\mathscr{L}\left(L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n), L^{\frac{2(n+1)}{(n-1)}}(\mathbb{R}^n)\right)} \leqslant C \int_0^\infty \|k(\lambda,\cdot)\|_{L^\infty(\Sigma_a)} \lambda^{\frac{n-1}{n+1}}\, d\lambda. \tag{5.17}$$

- *if moreover, k satisfies*

$$\int_0^\infty \|k(\lambda,\cdot)\|^2_{L^\infty(\Sigma_a)} \lambda^{\frac{n-1}{n+1}}\, d\lambda < \infty,$$

the operator $K_{a,k}$ in (5.15) belongs to

$$\mathscr{L}\left(L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n), L^2(\mathbb{R}^n)\right) \cap \mathscr{L}\left(L^2(\mathbb{R}^n), L^{\frac{2(n+1)}{(n-1)}}(\mathbb{R}^n)\right),$$

and

$$\|K_{a,k}\|_{\mathscr{L}\left(L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n), L^2(\mathbb{R}^n)\right)} \leqslant C\sqrt{\int_0^\infty \|k(\lambda,\cdot)\|^2_{L^\infty(\Sigma_a)} \lambda^{\frac{n-1}{n+1}}\, d\lambda}, \tag{5.18}$$

$$\|K_{a,k}\|_{\mathscr{L}\left(L^2(\mathbb{R}^n), L^{\frac{2(n+1)}{(n-1)}}(\mathbb{R}^n)\right)} \leqslant C\sqrt{\int_0^\infty \|k(\lambda,\cdot)\|^2_{L^\infty(\Sigma_a)} \lambda^{\frac{n-1}{n+1}}\, d\lambda}. \tag{5.19}$$

Remark 5.4. — The estimates of Proposition 5.3 will play a similar role as Sogge's spectral projection bounds [Sog88] for the spherical Laplacian to prove L^p Carleman estimates in the elliptic case as in [Jer86, Sog90, KT01].

Proof. — The proof of (5.16) follows immediately from Parseval's identity.

The proof of (5.17) is more subtle and is done in several steps. First, based on Theorem 5.2, we analyze, for $\lambda > 0$, the map $T_{a,\lambda} : f \in L^1(\mathbb{R}^n) \mapsto \widehat{f}|_{\Sigma_{a,\lambda}}$, where $\Sigma_{a,\lambda} = \{\xi \in \mathbb{R}^n,\ \psi(a,\xi) = \lambda\}$. We then explain how this yields estimate (5.17).

The first step is based on the fact that for $a \in [X_0, X_1]$ and $\lambda > 0$, for $f \in L^1(\mathbb{R}^n)$,

$$T^\star_{a,\lambda} T_{a,\lambda} f(x) = \int_{\widetilde{x} \in \mathbb{R}^n} \int_{\xi \in \Sigma_{a,\lambda}} e^{\mathbf{i}(x-\widetilde{x}).\xi}\, d\Sigma_{a,\lambda}(\xi) f(\widetilde{x})\, d\widetilde{x}, \qquad x \in \mathbb{R}^n.$$

Since the function $\xi \mapsto \psi(a,\xi)$ is homogeneous of degree 1, by using a scaling argument, we get

$$\begin{aligned} T^\star_{a,\lambda} T_{a,\lambda} f(x) &= \lambda^{n-1} \int_{\widetilde{x} \in \mathbb{R}^n} \int_{\omega \in \Sigma_{a,1}} e^{\mathbf{i}\lambda\left(x-\widetilde{x}\right)\cdot\omega}\, d\Sigma_{a,1}(\omega) f(\widetilde{x})\, d\widetilde{x} \\ &= \lambda^{-1} \int_{\widetilde{x} \in \mathbb{R}^n} \int_{\omega \in \Sigma_{a,1}} e^{\mathbf{i}\left(\lambda x-\widetilde{x}\right)\cdot\omega}\, d\Sigma_{a,1}(\omega) f\left(\frac{\widetilde{x}}{\lambda}\right) d\widetilde{x}, \qquad x \in \mathbb{R}^n. \end{aligned}$$

We thus obtain

$$\left(T^\star_{a,\lambda} T_{a,\lambda} f\right)(x) = \lambda^{-1} \left(T^\star_{a,1} T_{a,1} \left(f\left(\frac{\cdot}{\lambda}\right)\right)\right)(\lambda x), \qquad x \in \mathbb{R}^n.$$

From this identity, a simple scaling argument shows that for all $a \in [X_0, X_1]$ and all $\lambda > 0$,

$$\left\| T^\star_{a,\lambda} T_{a,\lambda} \right\|_{\mathscr{L}\left(L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n), L^{\frac{2(n+1)}{(n-1)}}(\mathbb{R}^n)\right)} \leqslant \lambda^{\frac{n-1}{n+1}} \left\| T^\star_{a,1} T_{a,1} \right\|_{\mathscr{L}\left(L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n), L^{\frac{2(n+1)}{(n-1)}}(\mathbb{R}^n)\right)}.$$

Since $T^\star_{a,1} T_{a,1} = T^\star_a T_a$ is the operator defined in (5.5), and since T_a belongs to $\mathscr{L}(L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n), L^2(\mathbb{R}^n))$ from Theorem 5.2, we deduce that there exists $C > 0$ such that for all $a \in [X_0, X_1]$, for all $\lambda > 0$,

$$\begin{aligned} &\left\| T^\star_{a,\lambda} T_{a,\lambda} \right\|_{\mathscr{L}\left(L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n), L^{\frac{2(n+1)}{(n-1)}}(\mathbb{R}^n)\right)} \leqslant C \lambda^{\frac{n-1}{n+1}}, \\ &\left\| T_{a,\lambda} \right\|_{\mathscr{L}\left(L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n), L^2(\Sigma_{a,\lambda})\right)} = \left\| T^\star_{a,\lambda} \right\|_{\mathscr{L}\left(L^2(\Sigma_{a,\lambda}), L^{\frac{2(n+1)}{(n-1)}}(\mathbb{R}^n)\right)} \leqslant C \lambda^{\frac{(n-1)}{2(n+1)}}. \end{aligned}$$

The second step then consists on rewriting the operator $K_{a,k}$ as follows:

$$K_{a,k} = \int_{\lambda > 0} T^\star_{a,\lambda} M_{a,k(\lambda,\cdot)} T_{a,\lambda}\, d\lambda,$$

where $M_{a,k(\lambda,\cdot)}$ is the operator defined from $L^2(\Sigma_{a,\lambda})$ to itself as follows: for $g \in L^2(\Sigma_{a,\lambda})$,

$$M_{a,k(\lambda,\cdot)} g(\xi) = k\left(\lambda, \frac{\xi}{\psi(a,\xi)}\right) g(\xi), \qquad \xi \in \Sigma_{a,\lambda}.$$

Accordingly, we have

$$\begin{aligned}
&\|K_{a,k}\|_{\mathscr{L}\left(L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n),L^{\frac{2(n+1)}{(n-1)}}(\mathbb{R}^n)\right)} \\
&\leqslant \int_{\lambda>0} \left\|T^{\star}_{a,\lambda} M_{a,k(\lambda,\cdot)} T_{a,\lambda}\right\|_{\mathscr{L}\left(L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n),L^{\frac{2(n+1)}{(n-1)}}(\mathbb{R}^n)\right)} d\lambda \\
&\leqslant \int_{\lambda>0} \left\|T^{\star}_{a,\lambda}\right\|_{\mathscr{L}\left(L^2(\Sigma_{a,\lambda}),L^{\frac{2(n+1)}{(n-1)}}(\mathbb{R}^n)\right)} \left\|M_{a,k(\lambda,\cdot)}\right\|_{\mathscr{L}\left(L^2(\Sigma_{a,\lambda})\right)} \\
&\qquad \|T_{a,\lambda}\|_{\mathscr{L}\left(L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n),L^2(\Sigma_{a,\lambda})\right)} d\lambda \\
&\leqslant \int_{\lambda>0} \|k(\lambda,\cdot)\|_{L^\infty(\Sigma_a)} \lambda^{\frac{n-1}{n+1}}\, d\lambda,
\end{aligned}$$

where we used the straightforward estimates:

$$\left\|M_{a,k(\lambda,\cdot)}\right\|_{\mathscr{L}\left(L^2(\Sigma_{a,\lambda})\right)} \leqslant \|k(\lambda,\cdot)\|_{L^\infty(\Sigma_a)}.$$

This concludes the proof of the estimate (5.17).

Estimates (5.18)–(5.19) are based on the estimate (5.17) and the facts that the map $K_{a,k}$ satisfies $K^{\star}_{a,k} = K_{a,\overline{k}}$ when computing the adjoint with respect to the $L^2(\mathbb{R}^n)$ scalar product, and $K^{\star}_{a,k} K_{a,\overline{k}} = K_{a,|k|^2}$. Therefore,

$$\begin{aligned}
&\|K_{a,k}\|^2_{\mathscr{L}\left(L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n),L^2(\mathbb{R}^n)\right)} \\
&\qquad = \left\|K_{a,\overline{k}}\right\|^2_{\mathscr{L}\left(L^2(\mathbb{R}^n),L^{\frac{2(n+1)}{(n-1)}}(\mathbb{R}^n)\right)} \\
&\qquad = \left\|K_{a,|k|^2}\right\|_{\mathscr{L}\left(L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n),L^{\frac{2(n+1)}{(n-1)}}(\mathbb{R}^n)\right)} = \int_0^\infty \|k(\lambda,\cdot)\|^2_{L^\infty(\Sigma_a)}\, \lambda^{\frac{n-1}{n+1}}\, d\lambda,
\end{aligned}$$

which concludes the proof of Proposition 5.3, up to exchanging k and $\overline{k}$ in the above formulae. □

6. L^p-Estimates on the parametrix, and proofs of Theorem 2.1 and Theorem 2.4

This section is devoted to give estimates on the norms of the various operators appearing in Proposition 3.1, especially in the spaces $\mathscr{L}(L^p(\Omega), L^q(\Omega))$ for suitable values of p and q. This will be done in particular by using the results in Proposition 5.3 with $n = d - 1$ and the Hardy–Littlewood–Sobolev theorem (Theorem A.1).

We will also repeatedly use the straightforward lemma below, whose proof is left to the reader.

Lemma 6.1. —

(1) *For all* $\alpha \in \mathbb{R}$, *and* $a \in [\frac{1}{2}, 2]$, *there exists* $C > 0$ *such that for all* $\mu > 1$, $\int_1^\mu e^{a\lambda} \lambda^\alpha \, d\lambda \leqslant C e^{a\mu} (1+\mu)^\alpha$.
(2) *For all* $\alpha > -1$, *there exists* $C > 0$ *such that for all* $\mu > 0$, $\int_0^\mu e^{\lambda} \lambda^\alpha d\lambda \leqslant C e^\mu \frac{\mu^{\alpha+1}}{1+\mu}$.
(3) *For all* $\alpha > -1$, *there exists* $C > 0$ *such that for all* $\gamma > 0$, $\int_{\lambda > \gamma} e^{-\lambda} \lambda^\alpha d\lambda \leqslant C e^{-\gamma} (1+\gamma)^\alpha$.
(4) *For all* $\alpha < -1$, *there exists* $C > 0$ *such that for all* $\gamma > 0$, $\int_{\lambda > \gamma} e^{-\lambda} \lambda^\alpha d\lambda \leqslant C e^{-\gamma} \frac{\gamma^{\alpha+1}}{1+\gamma}$.
(5) *For all* $\alpha \in (-3, 0)$ *and* $a \in [\frac{1}{2}, 2]$, *there exists* $C > 0$ *such that for all* $\gamma > 0$, $\int_{\lambda > \gamma} (\lambda - \gamma)^2 e^{-a\lambda} \lambda^\alpha \, d\lambda \leqslant C \frac{e^{-a\gamma}}{1+\gamma^{-\alpha}}$.

In the whole section, we assume the setting of Theorem 2.1. Within this setting, with ψ defined as in (3.2), there exists $c_1 > 0$ depending only on c_0 in (2.4) such that

$$\forall\, (x_1, y_1) \in [X_0, X_1]^2, \; \forall\, \xi' \in \mathbb{R}^{d-1}, \quad \frac{1}{c_1} \psi(y_1, \xi') \leqslant \psi(x_1, \xi') \leqslant c_1 \psi(y_1, \xi'). \tag{6.1}$$

We also recall that in this section, all the constants C depend only on c_0 in (2.4) and m_*, M_* in (2.5).

6.1. Estimates on the operator $K_{\tau,0}$ in (3.4)–(3.6)

The goal of this section is to estimate the norm of the operator $K_{\tau,0}$ in (3.4)–(3.6), more precisely:

Proposition 6.2. — *Let* Ω *be as in* (2.1) *with* $X_0 < 0 < X_1$ *and* $\max\{|X_0|, |X_1|\} \leqslant 1$, *and assume that the coefficients* $(\lambda_j)_{j \in \{1, \cdots, d\}} \in \mathbb{R}^d$ *satisfy* $\lambda_1 = 0$, (2.4) *and* (2.5). *Then there exist* $C > 0$ *and* $\tau_0 \geqslant 1$ *independent of* X_0, X_1 *(and depending only on* c_0, m_* *and* M_* *in* (2.4) *and* (2.5)*), such that for all* $\tau \geqslant \tau_0$, *for all* $f \in L^{\frac{2d}{d+2}}(\Omega)$,

$$\|K_{\tau,0} f\|_{L^{\frac{2d}{d-2}}(\Omega)} + \tau^{\frac{3}{4} + \frac{1}{2d}} \|K_{\tau,0} f\|_{L^2(\Omega)} + \left\| \partial_1 \widehat{K_{\tau,0} f} \right\|_{L^2(\Omega_{1,\tau})} + \tau^{-\frac{1}{4} + \frac{1}{2d}} \|\nabla' K_{\tau,0} f\|_{L^2(\Omega)} \leqslant C \|f\|_{L^{\frac{2d}{d+2}}(\Omega)}, \tag{6.2}$$

and, for all $f \in L^2(\Omega)$,

$$\tau^{\frac{3}{4} + \frac{1}{2d}} \|K_{\tau,0} f\|_{L^{\frac{2d}{d-2}}(\Omega)} + \tau^{\frac{3}{2}} \|K_{\tau,0} f\|_{L^2(\Omega)} + \tau \left\| \partial_1 \widehat{K_{\tau,0} f} \right\|_{L^2(\Omega_{1,\tau})} + \tau^{\frac{1}{2}} \|\nabla' K_{\tau,0} f\|_{L^2(\Omega)} \leqslant C \|f\|_{L^2(\Omega)}, \tag{6.3}$$

with $\Omega_{1,\tau}$ *as in* (4.6).

Remark 6.3. — In the above estimates, we point out that the $\mathscr{L}(L^2(\Omega), L^{\frac{2d}{d-2}}(\Omega))$ and $\mathscr{L}(L^{\frac{2d}{d+2}}(\Omega), L^2(\Omega))$ bounds of the operator $K_{\tau,0}$ are estimated by a power of the Carleman parameter that depends on d. This fact, which does not occur for the Hilbertian estimates, has been already observed in several cases, and we refer for instance to [BKRS88, KT01, KT05, Sog89].

Proof. — In view of the results in Proposition 5.3, we first estimate weighted norms of $k_{\tau,0}(x_1, y_1, \cdot)$ for x_1 and y_1 in $[X_0, X_1]$ (recall the definition of $k_{\tau,0}$ in (3.6)). We also identify $\xi' \in \mathbb{R}^{d-1}$ with pairs $(\lambda, \omega') \in \mathbb{R}_+ \times \Sigma_{x_1}$, where $\Sigma_{x_1} = \{\omega' \in \mathbb{R}^{d-1}, \psi(x_1, \omega') = 1\}$, through the formula $\xi' = \lambda\omega'$, or equivalently $\lambda = \psi(x_1, \xi')$ and $\omega' = \xi'/\psi(x_1, \xi')$. With a slight abuse of notations, we denote $k_{\tau,0}$ similarly whether it is written in terms of $\xi' \in \mathbb{R}^{d-1}$ or in terms of $(\lambda, \omega') \in \mathbb{R}_+ \times \Sigma_{x_1}$, that is

$$k_{\tau,0}(x_1, y_1, \lambda, \omega') = k_{\tau,0}(x_1, y_1, \lambda\omega').$$

We begin with the following lemma:

Lemma 6.4. — *There exist constants $C > 0$ and $C_1 > 0$ independent of X_0, X_1 (and depending only on c_0, m_* and M_* in* (2.4) *and* (2.5)*), such that for all x_1 and y_1 in $[X_0, X_1]$, for all $\tau \geqslant 1$, and $\lambda > 0$,*

- *If $\lambda \leqslant \tau$, then the kernel $k_{\tau,0}$ defined in* (3.6) *satisfies*

$$\|k_{\tau,0}(x_1, y_1, \lambda, \cdot)\|_{L^\infty(\Sigma_{x_1})} \leqslant \begin{cases} C|y_1 - x_1|e^{-\tau|y_1-x_1|}, & \text{if } \lambda|y_1 - x_1| \leqslant 1, \\ \dfrac{C}{\lambda}e^{-(\tau-\lambda)|y_1-x_1|-\lambda(y_1-x_1)^2/C_1}, & \text{if } \lambda|y_1 - x_1| \geqslant 1. \end{cases} \tag{6.4}$$

- *If $\lambda \geqslant \tau$, then $k_{\tau,0}$ satisfies*

$$\|k_{\tau,0}(x_1, y_1, \lambda, \cdot)\|_{L^\infty(\Sigma_{x_1})} \leqslant \begin{cases} \dfrac{C}{\lambda}e^{-(\lambda-\tau)|x_1-y_1|-\lambda(x_1-y_1)^2/C_1}, & \text{if } y_1 < x_1, \\ \dfrac{C}{\lambda}e^{-(\lambda/C+\tau)|y_1-x_1|}, & \text{if } y_1 > x_1. \end{cases} \tag{6.5}$$

Setting, for x_1 and y_1 in $[X_0, X_1]$, $\xi' \in \mathbb{R}^{d-1}$,

$$\begin{aligned} k_{\tau,0,\partial_1}(x_1, y_1, \xi') &= -1_{x_1<y_1} e^{-\tau(y_1-x_1)-\int_{x_1}^{y_1}\psi(\widetilde{y}_1,\xi')\,d\widetilde{y}_1} \\ &- 1_{\psi(x_1,\xi')>\tau}(\tau - \psi(x_1, \xi'))s\int_{X_0}^{\min\{x_1,y_1\}} e^{-\tau(y_1-x_1)-\int_{\widetilde{x}_1}^{x_1}\psi(\widetilde{y}_1,\xi')\,d\widetilde{y}_1-\int_{\widetilde{x}_1}^{y_1}\psi(\widetilde{y}_1,\xi')\,d\widetilde{y}_1} d\widetilde{x}_1 \\ &+ 1_{\psi(x_1,\xi')\leqslant\tau}\, 1_{x_1<y_1}\,(\tau - \psi(x_1, \xi'))\int_{x_1}^{y_1} e^{-\tau(y_1-x_1)+\int_{x_1}^{\widetilde{x}_1}\psi(\widetilde{y}_1,\xi')\,d\widetilde{y}_1-\int_{\widetilde{x}_1}^{y_1}\psi(\widetilde{y}_1,\xi')\,d\widetilde{y}_1} d\widetilde{x}_1, \end{aligned} \tag{6.6}$$

$k_{\tau,0,\partial_1}$ satisfies the following bounds:

- *If $\lambda \leqslant \tau$,*

$$\|k_{\tau,0,\partial_1}(x_1, y_1, \lambda, \cdot)\|_{L^\infty(\Sigma_{x_1})} \leqslant Ce^{-(\lambda/C+\tau)|y_1-x_1|} + (\tau - \lambda)\|k_{\tau,0}(x_1, y_1, \lambda, \cdot)\|_{L^\infty(\Sigma_{x_1})}. \tag{6.7}$$

- *If $\lambda > \tau$,*

$$\|k_{\tau,0,\partial_1}(x_1, y_1, \lambda, \cdot)\|_{L^\infty(\Sigma_{x_1})} \leqslant \begin{cases} \dfrac{C}{\lambda}(\lambda - \tau)e^{-(\lambda-\tau)|x_1-y_1|-\lambda(x_1-y_1)^2/C_1}, & \text{if } y_1 < x_1, \\ Ce^{-(\lambda/C+\tau)(y_1-x_1)}, & \text{if } y_1 > x_1. \end{cases} \tag{6.8}$$

Remark 6.5. — The kernel $k_{\tau,0,\partial_1}$ corresponds to the kernel of $\partial_1 K_{\tau,0}$ in the following sense: for all $f \in L^2((X_0,X_1);L^2(\mathbb{R}^{d-1}))$, and all $(x_1,\xi') \in \Omega_{1,\tau}$,

$$\partial_1 \widehat{K_{\tau,0} f}(x_1,\xi') = \int_{y_1 \in (X_0,X_1)} k_{\tau,0,\partial_1}(x_1,y_1,\xi') \widehat{f}(y_1,\xi')\, dy_1. \tag{6.9}$$

Proof of Lemma 6.4. — Let us first prove (6.4) corresponding to $\lambda \leqslant \tau$. Let $\xi' \in \mathbb{R}^{d-1}$ and $x_1 \in [X_0,X_1]$ be such that $\lambda = \psi(x_1,\xi') \leqslant \tau$. We then have to estimate, for $y_1 > x_1$,

$$\int_{x_1}^{y_1} e^{-\tau(y_1-x_1)+\int_{x_1}^{\widetilde{x}_1} \psi(\widetilde{y}_1,\xi')\,d\widetilde{y}_1 - \int_{\widetilde{x}_1}^{y_1} \psi(\widetilde{y}_1,\xi')\,d\widetilde{y}_1}\, d\widetilde{x}_1.$$

For $y_1 \in [x_1,X_1]$, we introduce the map

$$\rho(x_1,\widetilde{x}_1,y_1,\xi') = \int_{x_1}^{\widetilde{x}_1} \psi(\widetilde{y}_1,\xi')\,d\widetilde{y}_1 - \int_{\widetilde{x}_1}^{y_1} \psi(\widetilde{y}_1,\xi')\,d\widetilde{y}_1,$$

which clearly satisfies $\rho(x_1,y_1,y_1,\xi') = -\rho(x_1,x_1,y_1,\xi') \geqslant |y_1-x_1|\lambda/c_1$, and $\partial_{\widetilde{x}_1}\rho(x_1,\widetilde{x}_1,y_1,\xi') = 2\psi(\widetilde{x}_1,\xi')$ and $\partial^2_{\widetilde{x}_1}\rho(x_1,\widetilde{x}_1,y_1,\xi') = 2\partial_1\psi(\widetilde{x}_1,\xi') < 0$ by (2.5). Therefore, by concavity in $\widetilde{x}_1$, for $x_1 < y_1$ and $\widetilde{x}_1 \in [x_1,y_1]$,

$$\begin{aligned}\rho(x_1,\widetilde{x}_1,y_1,\xi') &\leqslant \rho(x_1,y_1,y_1,\xi') - 2\psi(y_1,\xi')|y_1-\widetilde{x}_1| \\ &\leqslant \int_{x_1}^{y_1} \psi(\widetilde{y}_1,\xi')\,d\widetilde{y}_1 - \frac{2}{c_1}\lambda|y_1-\widetilde{x}_1|,\end{aligned}$$

where the last estimate follows from (6.1). Hence, we obtain

$$\begin{aligned}&\left|\int_{x_1}^{y_1} e^{-\tau(y_1-x_1)+\int_{x_1}^{\widetilde{x}_1} \psi(\widetilde{y}_1,\xi')\,d\widetilde{y}_1 - \int_{\widetilde{x}_1}^{y_1} \psi(\widetilde{y}_1,\xi')\,d\widetilde{y}_1}\, d\widetilde{x}_1\right| \\ &\qquad\leqslant e^{-\tau(y_1-x_1)} \int_{x_1}^{y_1} e^{\int_{x_1}^{y_1}\psi(\widetilde{y}_1,\xi')\,d\widetilde{y}_1 - 2\lambda|y_1-\widetilde{x}_1|/c_1}\, d\widetilde{x}_1 \\ &\qquad\leqslant e^{-\tau(y_1-x_1)+\int_{x_1}^{y_1}\psi(\widetilde{y}_1,\xi')\,d\widetilde{y}_1} \min\left\{|y_1-x_1|, \frac{c_1}{2\lambda}\right\}.\end{aligned}$$

We then use that the function $\widetilde{y}_1 \mapsto \psi(\widetilde{y}_1,\xi')$ is concave, so that for $x_1 < \widetilde{y}_1$, we have $\psi(\widetilde{y}_1,\xi') \leqslant \psi(x_1,\xi') + \partial_1\psi(x_1,\xi')(\widetilde{y}_1 - x_1)$. From (5.3) and (2.5), there exists a constant $c_2 > 0$ depending only on c_0, m_* and M_* such that $\partial_1\psi(x_1,\xi') \leqslant -c_2\psi(x_1,\xi') = -c_2\lambda$. Therefore, for $y_1 > x_1$, $\int_{x_1}^{y_1}\psi(\widetilde{y}_1,\xi')\,d\widetilde{y}_1 \leqslant (y_1-x_1)\lambda - c_2\lambda(y_1-x_1)^2$, and (6.4) follows immediately.

We then prove (6.5) corresponding to $\lambda > \tau$. Let $\xi' \in \mathbb{R}^{d-1}$ and $x_1 \in [X_0,X_1]$ be such that $\lambda = \psi(x_1,\xi') > \tau$. We then have to estimate, for $y_1 \in [X_0,X_1]$, the quantity

$$\int_{X_0}^{\min\{x_1,y_1\}} e^{-\tau(y_1-x_1)-\int_{\widetilde{x}_1}^{x_1} \psi(\widetilde{y}_1,\xi')\,d\widetilde{y}_1 - \int_{\widetilde{x}_1}^{y_1} \psi(\widetilde{y}_1,\xi')\,d\widetilde{y}_1}\, d\widetilde{x}_1.$$

As before, one easily checks that the map $\widetilde{x}_1 \mapsto -\int_{\widetilde{x}_1}^{x_1}\psi(\widetilde{y}_1,\xi')\,d\widetilde{y}_1 - \int_{\widetilde{x}_1}^{y_1}\psi(\widetilde{y}_1,\xi')\,d\widetilde{y}_1$ has derivative $2\psi(\widetilde{x}_1,\xi')$ and is thus strictly increasing and concave. Therefore, for all $\widetilde{x}_1 \in [X_0, \min\{x_1, y_1\}]$,

$$\begin{aligned} -\int_{\widetilde{x}_1}^{x_1} \psi(\widetilde{y}_1, \xi')\, d\widetilde{y}_1 - \int_{\widetilde{x}_1}^{y_1} \psi(\widetilde{y}_1, \xi')\, d\widetilde{y}_1 \\ \leqslant -\int_{\min\{x_1,y_1\}}^{\max\{x_1,y_1\}} \psi(\widetilde{y}_1, \xi')\, d\widetilde{y}_1 + 2\psi(\min\{x_1, y_1\}, \xi')(\widetilde{x}_1 - \min\{x_1, y_1\}) \\ \leqslant -\int_{\min\{x_1,y_1\}}^{\max\{x_1,y_1\}} \psi(\widetilde{y}_1, \xi')\, d\widetilde{y}_1 - \frac{2}{c_1}\lambda(\min\{x_1, y_1\} - \widetilde{x}_1). \end{aligned}$$

Accordingly,

$$\begin{aligned} \int_{X_0}^{\min\{x_1,y_1\}} e^{-\tau(y_1-x_1) - \int_{\widetilde{x}_1}^{x_1} \psi(\widetilde{y}_1,\xi')\, d\widetilde{y}_1 - \int_{\widetilde{x}_1}^{y_1} \psi(\widetilde{y}_1,\xi')\, d\widetilde{y}_1}\, d\widetilde{x}_1 \\ \leqslant e^{-\tau(y_1-x_1) - \int_{\min\{x_1,y_1\}}^{\max\{x_1,y_1\}} \psi(\widetilde{y}_1,\xi')\, d\widetilde{y}_1} \min\left\{ \frac{c_1}{2\lambda}, |\min\{x_1, y_1\} - X_0| \right\}. \end{aligned}$$

If $y_1 > x_1$, we simply use

$$-\int_{\min\{x_1,y_1\}}^{\max\{x_1,y_1\}} \psi\left(\widetilde{y}_1, \xi'\right)\, d\widetilde{y}_1 \leqslant -\frac{1}{c_1}(y_1 - x_1)\lambda.$$

If $y_1 < x_1$, we use that ψ is concave decreasing and thus for all $\widetilde{y}_1 \in [y_1, x_1]$, $\psi(\widetilde{y}_1, \xi') \geqslant \psi(x_1, \xi') + (\widetilde{y}_1 - x_1)\partial_1\psi(y_1, \xi')$. But there exists a constant $c_3 > 0$ depending on c_0, m_* and M_* such that $\partial_1\psi(y_1, \xi') \leqslant -\psi(x_1, \xi')/c_3$, so that we easily get

$$-\int_{\min\{x_1,y_1\}}^{\max\{x_1,y_1\}} \psi(\widetilde{y}_1, \xi')\, d\widetilde{y}_1 \leqslant -(y_1 - x_1)\lambda - \frac{1}{c_3}(y_1 - x_1)^2\lambda.$$

in this case. Combining the last three estimates immediately yields (6.5).

The proof of estimates (6.7)–(6.8) follows from the fact that, for $y_1 > x_1$,

$$\left\| e^{-\tau(y_1-x_1) - \int_{x_1}^{y_1} \psi(\widetilde{y}_1,\xi')\, d\widetilde{y}_1} \right\|_{L^\infty_{\omega'}(\Sigma_{x_1})} \leqslant e^{-(\tau+\lambda/c_1)(y_1-x_1)},$$

and from the estimates already proved above. Details are left to the reader. □

Using the bounds in Lemma 6.4, we prove the following lemma:

Lemma 6.6. — *There exists a constant $C > 0$ independent of X_0, X_1 (and depending only on c_0, m_* and M_* in* (2.4) *and* (2.5)*), such that for all x_1 and y_1 in*

$[X_0, X_1]$, *for all* $\tau \geqslant 1$,

$$\int_{\lambda>0} \|k_{\tau,0}(x_1,y_1,\lambda,\cdot)\|_{L^\infty(\Sigma_{x_1})}\ \lambda^{1-\frac{2}{d}}\,d\lambda \leqslant \frac{C}{|x_1-y_1|^{1-\frac{2}{d}}}, \tag{6.10}$$

$$\left(\int_{\lambda>0} \|k_{\tau,0}(x_1,y_1,\lambda,\cdot)\|^2_{L^\infty(\Sigma_{x_1})}\ \lambda^{1-\frac{2}{d}}\,d\lambda\right)^{\frac{1}{2}} \leqslant C\widetilde{k}_{\tau,0}(x_1-y_1), \tag{6.11}$$
$$\text{with } \widetilde{k}_{\tau,0} \in L^{\frac{d}{d-1}}(\mathbb{R}) \text{ and } \left\|\widetilde{k}_{\tau,0}\right\|_{L^{\frac{d}{d-1}}(\mathbb{R})} \leqslant C\tau^{-\frac{3}{4}-\frac{1}{2d}},$$

$$\left(\int_{\lambda>0} \|k_{\tau,0,\partial_1}(x_1,y_1,\lambda,\cdot)\|^2_{L^\infty(\Sigma_{x_1})}\ \lambda^{1-\frac{2}{d}}\,d\lambda\right)^{\frac{1}{2}} \leqslant \frac{C}{|x_1-y_1|^{1-\frac{1}{d}}}, \tag{6.12}$$

$$\left(\int_{\lambda>0} \|\lambda\omega' k_{\tau,0}(x_1,y_1,\lambda,\omega')\|^2_{L^\infty_{\omega'}(\Sigma_{x_1})}\ \lambda^{1-\frac{2}{d}}\,d\lambda\right)^{\frac{1}{2}} \leqslant \frac{C}{|x_1-y_1|^{1-\frac{1}{d}}} + \check{k}_{\tau,0}(x_1-y_1), \tag{6.13}$$
$$\text{with } \check{k}_{\tau,0} \in L^{\frac{d}{d-1}}(\mathbb{R}) \text{ and } \left\|\check{k}_{\tau,0}\right\|_{L^{\frac{d}{d-1}}(\mathbb{R})} \leqslant C\tau^{\frac{1}{4}-\frac{1}{2d}}.$$

Proof of Lemma 6.6. — We start by simply noticing that we can always impose that C_1 in Lemma 6.4 is large enough to get for all $(x_1, y_1) \in [X_0, X_1]^2$, that $|x_1 - y_1|^2/C_1 \leqslant |x_1 - y_1|/4$. This can be done by assuming for instance $C_1 \geqslant 8$ since $|X_0|, |X_1| \leqslant 1$. This will make some of the estimates below easier to prove properly. For convenience, this constant C_1 will next be denoted by C, similarly as generic constants which depend only on the dimension and the parameters c_0, m_* and M_* in (2.4), and (2.5).

Proof of (6.10). We decompose the integral in the left-hand side of (6.10) in three terms more suitable to use the results in Lemma 6.4. The first is easily estimated as follows:

$$\begin{aligned}
&\int_0^{\min\{\tau,1/|y_1-x_1|\}} \|k_{\tau,0}(x_1,y_1,\lambda,\cdot)\|_{L^\infty(\Sigma_{x_1})}\ \lambda^{1-\frac{2}{d}}\,d\lambda \\
&\qquad \leqslant C\int_0^{\min\{\tau,1/|y_1-x_1|\}} |y_1-x_1|e^{-\tau|y_1-x_1|}\,\lambda^{1-\frac{2}{d}}\,d\lambda \\
&\qquad \leqslant C|y_1-x_1|e^{-\tau|y_1-x_1|}\min\left\{\tau,\frac{1}{|y_1-x_1|}\right\}^{2-\frac{2}{d}} \\
&\qquad \leqslant C|y_1-x_1|^{-1+\frac{2}{d}}e^{-\tau|y_1-x_1|}\min\{\tau|y_1-x_1|,1\}^{2-\frac{2}{d}} \leqslant C|y_1-x_1|^{-1+\frac{2}{d}}.
\end{aligned}$$

Next if $1/|y_1 - x_1| < \tau$, that is $\tau|y_1 - x_1| > 1$, we get

$$\begin{aligned}\int_{1/|y_1-x_1|}^{\tau} \|k_{\tau,0}(x_1, y_1, \lambda, \cdot)\|_{L^\infty\left(\Sigma_{x_1}\right)} \lambda^{1-\frac{2}{d}}\, d\lambda \\ \leqslant C \int_{1/|y_1-x_1|}^{\tau} e^{-(\tau-\lambda)|y_1-x_1|-\lambda|y_1-x_1|^2/C_1} \lambda^{-\frac{2}{d}}\, d\lambda \\ \leqslant C e^{-\tau|y_1-x_1|} \int_{1/|y_1-x_1|}^{\tau} e^{\lambda|y_1-x_1|} \lambda^{-\frac{2}{d}}\, d\lambda \\ \leqslant C|y_1 - x_1|^{-1+\frac{2}{d}} e^{-\tau|y_1-x_1|} \int_1^{\tau|y_1-x_1|} e^{\lambda} \lambda^{-\frac{2}{d}}\, d\lambda \\ \leqslant C|y_1 - x_1|^{-1+\frac{2}{d}} \frac{1}{(1+\tau|y_1-x_1|)^{\frac{2}{d}}} \leqslant C|y_1 - x_1|^{-1+\frac{2}{d}},\end{aligned}$$

where, from the fourth to the fifth lines, we have used Lemma 6.1 (1). Finally, we also have, for $y_1 > x_1$,

$$\begin{aligned}\int_{\lambda>\tau} \|k_{\tau,0}(x_1, y_1, \lambda, \cdot)\|_{L^\infty\left(\Sigma_{x_1}\right)} \lambda^{1-\frac{2}{d}}\, d\lambda \\ \leqslant C \int_{\lambda>\tau} e^{-(\lambda/C+\tau)(y_1-x_1)} \lambda^{-\frac{2}{d}}\, d\lambda \\ \leqslant C e^{-\tau(y_1-x_1)} \int_{\lambda>0} e^{-\lambda(y_1-x_1)/C} \lambda^{-\frac{2}{d}}\, d\lambda \\ \leqslant C e^{-\tau(y_1-x_1)} |y_1 - x_1|^{-1+\frac{2}{d}} \leqslant C|y_1 - x_1|^{-1+\frac{2}{d}}.\end{aligned}$$

Accordingly, estimate (6.10) holds for $y_1 > x_1$. Then for $y_1 < x_1$, it only remains to prove the following estimate, in which we use Lemma 6.1 (3):

$$\begin{aligned}\int_{\lambda>\tau} \|k_{\tau,0}(x_1, y_1, \lambda, \cdot)\|_{L^\infty\left(\Sigma_{x_1}\right)} \lambda^{1-\frac{2}{d}}\, d\lambda \\ \leqslant C \int_{\lambda>\tau} e^{-(\lambda-\tau)(x_1-y_1)-\lambda(x_1-y_1)^2/C_1} \lambda^{-\frac{2}{d}}\, d\lambda \\ \leqslant C e^{\tau(x_1-y_1)} \int_{\lambda>\tau} e^{-\lambda(x_1-y_1)} \lambda^{-\frac{2}{d}}\, d\lambda \\ \leqslant C\, |x_1 - y_1|^{-1+\frac{2}{d}}\, e^{\tau(x_1-y_1)} \int_{\lambda>\tau(x_1-y_1)} e^{-\lambda} \lambda^{-\frac{2}{d}}\, d\lambda \\ \leqslant C\, |x_1 - y_1|^{-1+\frac{2}{d}}.\end{aligned}$$

This concludes the proof of (6.10) for $y_1 < x_1$ as well. □

Proof of (6.11). Of course, the proof of (6.11) is very similar to the one of (6.10). We first have

$$\begin{aligned}\int_0^{\min\{\tau,1/|y_1-x_1|\}} &\|k_{\tau,0}(x_1,y_1,\lambda,\cdot)\|^2_{L^\infty(\Sigma_{x_1})}\,\lambda^{1-\frac{2}{d}}\,d\lambda\\ &\leqslant C\int_0^{\min\{\tau,1/|y_1-x_1|\}} |y_1-x_1|^2 e^{-2\tau|y_1-x_1|}\,\lambda^{1-\frac{2}{d}}\,d\lambda\\ &\leqslant C|y_1-x_1|^2 e^{-2\tau|y_1-x_1|}\min\left\{\tau,\frac{1}{|y_1-x_1|}\right\}^{2-\frac{2}{d}}\\ &\leqslant C\tau^{-\frac{2}{d}}\left((\tau|y_1-x_1|)^{\frac{2}{d}}e^{-2\tau|y_1-x_1|}\min\left\{\tau|y_1-x_1|,1\right\}^{2-\frac{2}{d}}\right).\end{aligned}$$

Next if $1/|y_1-x_1|<\tau$, that is $\tau|y_1-x_1|>1$, using Lemma 6.1 (1), we have

$$\begin{aligned}\int_{1/|y_1-x_1|}^{\tau} &\|k_{\tau,0}(x_1,y_1,\lambda,\cdot)\|^2_{L^\infty(\Sigma_{x_1})}\,\lambda^{1-\frac{2}{d}}\,d\lambda\\ &\leqslant C\int_{1/|y_1-x_1|}^{\tau} e^{-2(\tau-\lambda)|y_1-x_1|-2\lambda(y_1-x_1)^2/C_1}\lambda^{-1-\frac{2}{d}}\,d\lambda\\ &\leqslant Ce^{-2\tau|y_1-x_1|}\int_{1/|y_1-x_1|}^{\tau} e^{2\lambda\left(|y_1-x_1|-(y_1-x_1)^2/C_1\right)}\lambda^{-1-\frac{2}{d}}\,d\lambda\\ &\leqslant C|y_1-x_1|^{\frac{2}{d}}e^{-2\tau|y_1-x_1|}\int_1^{\tau|y_1-x_1|} e^{2\lambda(1-|y_1-x_1|/C_1)}\lambda^{-1-\frac{2}{d}}\,d\lambda\\ &\leqslant C|y_1-x_1|^{\frac{2}{d}}\frac{e^{-\tau(y_1-x_1)^2/C}}{(1+\tau|y_1-x_1|)^{1+\frac{2}{d}}}\\ &\leqslant C\tau^{-\frac{2}{d}}\frac{e^{-\tau(y_1-x_1)^2/C}}{(1+\tau|y_1-x_1|)}.\end{aligned}$$

Finally, we also have, for $y_1>x_1$,

$$\begin{aligned}\int_{\lambda>\tau} &\|k_{\tau,0}(x_1,y_1,\lambda,\cdot)\|^2_{L^\infty(\Sigma_{x_1})}\,\lambda^{1-\frac{2}{d}}\,d\lambda\\ &\leqslant C\int_{\lambda>\tau} e^{-2(\lambda/C+\tau)(y_1-x_1)}\lambda^{-1-\frac{2}{d}}\,d\lambda\\ &\leqslant Ce^{-2\tau(y_1-x_1)}\int_{\lambda>\tau} e^{-2\lambda(y_1-x_1)/C}\lambda^{-1-\frac{2}{d}}\,d\lambda\\ &\leqslant Ce^{-2\tau(y_1-x_1)}\int_{\lambda>\tau}\lambda^{-1-\frac{2}{d}}\,d\lambda\\ &\leqslant Ce^{-2\tau(y_1-x_1)}\tau^{-\frac{2}{d}}.\end{aligned}$$

On the other hand, for $x_1 > y_1$, we have that

$$\begin{aligned}\int_{\lambda>\tau} \|k_{\tau,0}(x_1,y_1,\lambda,\cdot)\|^2_{L^\infty(\Sigma_{x_1})} \lambda^{1-\frac{2}{d}}\,d\lambda \\ &\leqslant C\int_{\lambda>\tau} e^{-2(\lambda-\tau)(x_1-y_1)-2\lambda(x_1-y_1)^2/C_1}\lambda^{-1-\frac{2}{d}}\,d\lambda \\ &\leqslant Ce^{2\tau(x_1-y_1)}\int_{\lambda>\tau} e^{-2\lambda\left((x_1-y_1)+(x_1-y_1)^2/C_1\right)}\lambda^{-1-\frac{2}{d}}\,d\lambda \\ &\leqslant Ce^{2\tau(x_1-y_1)}|x_1-y_1|^{\frac{2}{d}}\int_{\lambda>\tau|x_1-y_1|} e^{-2\lambda\left(1+(x_1-y_1)^2/C_1\right)}\lambda^{-1-\frac{2}{d}}\,d\lambda,\end{aligned}$$

so that using Lemma 6.1 (4), for $x_1 > y_1$, we obtain

$$\int_{\lambda>\tau} \|k_{\tau,0}(x_1,y_1,\lambda,\cdot)\|^2_{L^\infty(\Sigma_{x_1})} \lambda^{1-\frac{2}{d}}\,d\lambda \leqslant C\tau^{-\frac{2}{d}}\frac{e^{-2\tau(x_1-y_1)^2/C_1}}{1+\tau(x_1-y_1)}.$$

Therefore, combining the above estimates, we have

$$\begin{aligned}&\left(\int_{\lambda>0} \|k_{\tau,0}(x_1,y_1,\lambda,\cdot)\|^2_{L^\infty(\Sigma_{x_1})} \lambda^{1-\frac{2}{d}}\,d\lambda\right)^{\frac{1}{2}} \leqslant C\widetilde{k}_{\tau,0}(x_1-y_1), \\ &\text{where } \widetilde{k}_{\tau,0}(z_1) = 1_{|z_1|<1/\tau}\tau^{-\frac{1}{d}} + 1_{|z_1|>1/\tau}\tau^{-\frac{1}{d}}\left((\tau|z_1|)^{\frac{1}{d}}e^{-\tau|z_1|} + \frac{e^{-\tau|z_1|^2/C}}{(\tau|z_1|)^{\frac{1}{2}}}\right).\end{aligned} \tag{6.14}$$

Easy computations then yield $\|\widetilde{k}_{\tau,0}\|_{L^{\frac{d}{d-1}}(\mathbb{R})} \leqslant C\tau^{-\frac{3}{4}-\frac{1}{2d}}$ as announced. □

Proof of (6.12). We have

$$\int_{\lambda>0} e^{-2(\lambda/C+\tau)|y_1-x_1|}\lambda^{1-\frac{2}{d}}\,d\lambda \leqslant e^{-2\tau|y_1-x_1|}|y_1-x_1|^{-2+\frac{2}{d}} \leqslant |y_1-x_1|^{-2+\frac{2}{d}}.$$

We also have, as before,

$$\begin{aligned}\int_0^{\min\{\tau,1/|y_1-x_1|\}} (\tau-\lambda)^2|y_1-x_1|^2 e^{-2\tau|y_1-x_1|}\,\lambda^{1-\frac{2}{d}}\,d\lambda \\ &\leqslant \tau^2\int_0^{\min\{\tau,1/|y_1-x_1|\}} |y_1-x_1|^2 e^{-2\tau|y_1-x_1|}\,\lambda^{1-\frac{2}{d}}\,d\lambda \\ &\leqslant C\tau^2|y_1-x_1|^2 e^{-2\tau|y_1-x_1|}\min\left\{\tau,\frac{1}{|y_1-x_1|}\right\}^{2-\frac{2}{d}} \\ &\leqslant C|y_1-x_1|^{\frac{2}{d}}\tau^2 e^{-2\tau|y_1-x_1|}\min\{\tau|y_1-x_1|,1\}^{2-\frac{2}{d}} \\ &\leqslant C|y_1-x_1|^{-2+\frac{2}{d}}(\tau|y_1-x_1|)^2 e^{-2\tau|y_1-x_1|}\min\{\tau|y_1-x_1|,1\}^{2-\frac{2}{d}} \\ &\leqslant C|y_1-x_1|^{-2+\frac{2}{d}}.\end{aligned}$$

If $1/|y_1-x_1| < \tau$, that is $\tau|y_1-x_1| > 1$, we distinguish the cases $1/|y_1-x_1| \leqslant \tau/2$ and $1/|y_1-x_1| \geqslant \tau/2$.

If $1/|y_1 - x_1| \in [\tau/2, \tau]$, we have

$$\begin{aligned}
&\int_{1/|y_1-x_1|}^{\tau} (\tau-\lambda)^2 e^{-2(\tau-\lambda)|y_1-x_1|-2\lambda(y_1-x_1)^2/C_1} \lambda^{-1-\frac{2}{d}}\, d\lambda \\
&\qquad \leqslant C e^{-2\tau|y_1-x_1|} \int_{\tau/2}^{\tau} (\tau-\lambda)^2 e^{2\lambda\left(|y_1-x_1|-(y_1-x_1)^2/C_1\right)} \lambda^{-1-\frac{2}{d}}\, d\lambda \\
&\qquad \leqslant C e^{-2\tau|y_1-x_1|} \tau^{-1-\frac{2}{d}} \int_{\tau/2}^{\tau} (\tau-\lambda)^2 e^{2\lambda\left(|y_1-x_1|-(y_1-x_1)^2/C_1\right)}\, d\lambda \\
&\qquad \leqslant C e^{-2\tau|y_1-x_1|} \frac{\tau^{-1-\frac{2}{d}} e^{2\tau\left(|y_1-x_1|-(y_1-x_1)^2/C_1\right)}}{|y_1-x_1|^3} \\
&\qquad \leqslant C \frac{\tau^{-1-\frac{2}{d}} e^{-2\tau(y_1-x_1)^2/C_1}}{|y_1-x_1|^3} \\
&\qquad \leqslant C|y_1-x_1|^{-2+\frac{2}{d}} (\tau|y_1-x_1|)^{-1-\frac{2}{d}} e^{-2\tau(y_1-x_1)^2/C} \leqslant C|y_1-x_1|^{-2+\frac{2}{d}}.
\end{aligned}$$

If $1/|y_1 - x_1| \leqslant \tau/2$, we split the integral into two parts, $\int_{1/|y_1-x_1|}^{\tau/2}$ and $\int_{\tau/2}^{\tau}$. The second integral has been estimated above, and, using Lemma 6.1 (1), the first one is estimated as follows:

$$\begin{aligned}
&\int_{\lambda \in (1/|y_1-x_1|, \tau/2)} (\tau-\lambda)^2 e^{-2(\tau-\lambda)|y_1-x_1|-2\lambda(y_1-x_1)^2/C} \lambda^{-1-\frac{2}{d}}\, d\lambda \\
&\qquad \leqslant e^{-2\tau|y_1-x_1|} \tau^2 \int_{1/|y_1-x_1|}^{\tau/2} e^{2\lambda\left(|y_1-x_1|-(y_1-x_1)^2/C\right)} \lambda^{-1-2/d} d\lambda \\
&\qquad \leqslant C e^{-2\tau|y_1-x_1|} \tau^2 |y_1-x_1|^{\frac{2}{d}} \int_{1}^{\tau|y_1-x_1|/2} e^{2\lambda(1-(y_1-x_1)/C)} \lambda^{-1-\frac{2}{d}}\, d\lambda \\
&\qquad \leqslant C|y_1-x_1|^{\frac{2}{d}} \tau^2 e^{-\tau|y_1-x_1|} (\tau|y_1-x_1|)^{-1-\frac{2}{d}} \\
&\qquad \leqslant C|y_1-x_1|^{-2+\frac{2}{d}} (\tau|y_1-x_1|)^{1-\frac{2}{d}} e^{-\tau|y_1-x_1|} \leqslant C|y_1-x_1|^{-2+\frac{2}{d}}.
\end{aligned}$$

It remains to estimate, for $y_1 < x_1$,

$$\begin{aligned}
&\int_{\lambda > \tau} (\lambda-\tau)^2 e^{-2(\lambda-\tau)(x_1-y_1)-2\lambda(x_1-y_1)^2/C} \lambda^{-1-\frac{2}{d}}\, d\lambda \\
&\leqslant C e^{2\tau(x_1-y_1)} \int_{\lambda > \tau} (\lambda-\tau)^2 e^{-2\lambda((x_1-y_1)+(x_1-y_1)^2/C)} \lambda^{-1-\frac{2}{d}}\, d\lambda \\
&\leqslant C e^{2\tau(x_1-y_1)} |x_1-y_1|^{-2+\frac{2}{d}} \int_{\lambda > \tau|x_1-y_1|} (\lambda-\tau|x_1-y_1|)^2 e^{-2\lambda(1+(x_1-y_1)/C)} \lambda^{-1-\frac{2}{d}}\, d\lambda.
\end{aligned}$$

We now use Lemma 6.1 (5) and obtain for $x_1 > y_1$,

$$\begin{aligned}
&\int_{\lambda > \tau} (\lambda-\tau)^2 e^{-2(\lambda-\tau)(x_1-y_1)-2\lambda(x_1-y_1)^2/C} \lambda^{-1-\frac{2}{d}}\, d\lambda \\
&\qquad\qquad \leqslant \frac{C|x_1-y_1|^{-2+\frac{2}{d}} e^{-2\tau(x_1-y_1)^2/C}}{1+(\tau|x_1-y_1|)^{1+\frac{2}{d}}} \leqslant C|x_1-y_1|^{-2+\frac{2}{d}}.
\end{aligned}$$

□

Proof of (6.13). We have

$$\begin{aligned}
\int_0^{\min\{\tau,1/|y_1-x_1|\}} &\|\lambda\omega' k_{\tau,0}(x_1,y_1,\lambda,\omega')\|^2_{L^\infty_{\omega'}(\Sigma_{x_1})}\ \lambda^{1-\frac{2}{d}}\,d\lambda \\
&\leqslant C\int_0^{\min\{\tau,1/|y_1-x_1|\}} |y_1-x_1|^2 e^{-2\tau(y_1-x_1)}\,\lambda^{3-\frac{2}{d}}\,d\lambda \\
&\leqslant C|y_1-x_1|^2 e^{-2\tau(y_1-x_1)}\min\left\{\tau,\frac{1}{|y_1-x_1|}\right\}^{4-\frac{2}{d}} \\
&\leqslant C|y_1-x_1|^{-2+\frac{2}{d}}e^{-2\tau(y_1-x_1)}\min\{\tau|y_1-x_1|,1\}^{4-\frac{2}{d}} \leqslant C|y_1-x_1|^{-2+\frac{2}{d}}.
\end{aligned}$$

If $1/|y_1-x_1|<\tau$, that is $\tau|y_1-x_1|>1$, using Lemma 6.1 (1), we obtain

$$\begin{aligned}
\int_{1/|y_1-x_1|}^{\tau} &\|\lambda\omega' k_{\tau,0}(x_1,y_1,\lambda,\omega')\|^2_{L^\infty_{\omega'}(\Sigma_{x_1})}\ \lambda^{1-\frac{2}{d}}\,d\lambda \\
&\leqslant C\int_{1/|y_1-x_1|}^{\tau} e^{-2(\tau-\lambda)|y_1-x_1|-2\lambda(y_1-x_1)^2/C}\lambda^{1-\frac{2}{d}}\,d\lambda \\
&\leqslant Ce^{-2\tau|y_1-x_1|}\int_{1/|y_1-x_1|}^{\tau} e^{2\lambda\left(|y_1-x_1|-(y_1-x_1)^2/C\right)}\lambda^{1-\frac{2}{d}}\,d\lambda \\
&\leqslant C|y_1-x_1|^{-2+\frac{2}{d}}e^{-2\tau|y_1-x_1|}\int_1^{\tau|y_1-x_1|} e^{2\lambda(1-|y_1-x_1|/C)}\lambda^{1-\frac{2}{d}}\,d\lambda \\
&\leqslant C|y_1-x_1|^{-2+\frac{2}{d}}e^{-\tau(y_1-x_1)^2/C}(\tau|y_1-x_1|)^{1-\frac{2}{d}} \\
&\leqslant C\tau^{1-\frac{2}{d}}e^{-\tau(y_1-x_1)^2/C}|y_1-x_1|^{-1}.
\end{aligned}$$

Finally, using Lemma 6.1 (3), we also have, for $y_1>x_1$,

$$\begin{aligned}
\int_{\lambda>\tau} &\|\lambda\omega' k_{\tau,0}(x_1,y_1,\lambda,\omega')\|^2_{L^\infty_{\omega'}(\Sigma_{x_1})}\ \lambda^{1-\frac{2}{d}}\,d\lambda \\
&\leqslant C\int_{\lambda>\tau} e^{-2(\lambda/C+\tau)(y_1-x_1)}\lambda^{1-\frac{2}{d}}\,d\lambda \\
&\leqslant Ce^{-2\tau(y_1-x_1)}\int_{\lambda>\tau} e^{-2\lambda(y_1-x_1)/C}\lambda^{1-\frac{2}{d}}\,d\lambda \\
&\leqslant Ce^{-2\tau(y_1-x_1)}|y_1-x_1|^{-2+\frac{2}{d}}\int_{\lambda>\tau(y_1-x_1)} e^{-2\lambda/C}\lambda^{1-\frac{2}{d}}\,d\lambda \\
&\leqslant Ce^{-2\tau(y_1-x_1)}|y_1-x_1|^{-2+\frac{2}{d}}(1+(\tau|y_1-x_1|))^{1-\frac{2}{d}} \\
&\leqslant C|y_1-x_1|^{-2+\frac{2}{d}}.
\end{aligned}$$

Similarly, for $x_1>y_1$,

$$\begin{aligned}
\int_{\lambda>\tau} &\|\lambda\omega' k_{\tau,0}(x_1,y_1,\lambda,\omega')\|^2_{L^\infty_{\omega'}(\Sigma_{x_1})}\ \lambda^{1-\frac{2}{d}}\,d\lambda \\
&\leqslant C\int_{\lambda>\tau} e^{-2(\lambda-\tau)(x_1-y_1)-2\lambda(x_1-y_1)^2/C}\lambda^{1-\frac{2}{d}}\,d\lambda \\
&\leqslant Ce^{2\tau(x_1-y_1)}\int_{\lambda>\tau} e^{-2\lambda\left((x_1-y_1)+(x_1-y_1)^2/C\right)}\lambda^{1-\frac{2}{d}}\,d\lambda \\
&\leqslant Ce^{2\tau(x_1-y_1)}|x_1-y_1|^{-2+\frac{2}{d}}\int_{\lambda>\tau|x_1-y_1|} e^{-2\lambda(1+(x_1-y_1)/C)}\lambda^{1-\frac{2}{d}}\,d\lambda,
\end{aligned}$$

so that using Lemma 6.1 (3), we get, for $x_1 > y_1$,

$$\begin{aligned}\int_{\lambda > \tau} &\|\lambda\omega' k_{\tau,0}(x_1, y_1, \lambda, \cdot)\|^2_{L^\infty\left(\Sigma_{x_1}\right)} \lambda^{1-\frac{2}{d}}\, d\lambda \\ &\leqslant C|y_1 - x_1|^{-2+\frac{2}{d}} e^{-2\tau(y_1-x_1)^2/C}(1+\tau|y_1-x_1|)^{1-\frac{2}{d}} \\ &\leqslant C\left(1_{|y_1-x_1|\leqslant 1/\tau}|y_1-x_1|^{-2+\frac{2}{d}} + 1_{|y_1-x_1|\geqslant 1/\tau}\tau^{1-\frac{2}{d}}|y_1-x_1|^{-1}e^{-2\tau(y_1-x_1)^2/C}\right).\end{aligned}$$

Therefore, by combining the above estimates, we have

$$\begin{aligned}\left(\int_{\lambda > 0} \|\lambda\omega' k_{\tau,0}(x_1, y_1, \lambda, \cdot)\|^2_{L^\infty(\Sigma_{x_1})} \lambda^{1-\frac{2}{d}}\, d\lambda\right)^{\frac{1}{2}} \\ \leqslant C\left(\frac{1}{|x_1-y_1|^{1-\frac{1}{d}}} + \check{k}_{\tau,0}(x_1-y_1)\right), \\ \text{where } \check{k}_{\tau,0}(z_1) = 1_{|z_1|>1/\tau}\tau^{\frac{1}{2}-\frac{1}{d}}e^{-\tau z_1^2/C}|z_1|^{-\frac{1}{2}}.\end{aligned}$$

Easy computations then give $\|\check{k}_{\tau,0}(z_1)\|_{L^{\frac{d}{d-1}}(\mathbb{R})} \leqslant C\tau^{\frac{1}{4}-\frac{1}{2d}}$, as announced. This concludes the proof of Lemma 6.6. □

Now we are in position to conclude the proof of Proposition 6.2. First, from Proposition 5.3 with $n = d-1$, estimate (6.10) and the one-dimensional Hardy–Littlewood–Sobolev inequality (recall Theorem A.1), we have, for $f \in L^{\frac{2d}{d+2}}(\Omega)$,

$$\begin{aligned}&\|K_{\tau,0}f\|_{L^{\frac{2d}{d-2}}(\Omega)} \\ &\leqslant \left\| \|K_{\tau,0}f(x_1,\cdot)\|_{L^{\frac{2d}{d-2}}_{x'}(\mathbb{R}^{d-1})}\right\|_{L^{\frac{2d}{d-2}}_{x_1}(X_0,X_1)} \\ &\leqslant \left\|\int_{X_0}^{X_1}\left(\int_\lambda \|k_{\tau,0}(x_1,y_1,\lambda,\cdot)\|_{L^\infty(\Sigma_{x_1})}\lambda^{1-\frac{2}{d}}\, d\lambda\right)\|f(y_1,\cdot)\|_{L^{\frac{2d}{d+2}}_{y'}(\mathbb{R}^{d-1})}\, dy_1\right\|_{L^{\frac{2d}{d-2}}_{x_1}(X_0,X_1)} \\ &\leqslant C\left\|\int_{X_0}^{X_1}\frac{1}{|x_1-y_1|^{1-\frac{2}{d}}}\|f(y_1,\cdot)\|_{L^{\frac{2d}{d+2}}_{y'}(\mathbb{R}^{d-1})}\, dy_1\right\|_{L^{\frac{2d}{d-2}}_{x_1}(X_0,X_1)} \\ &\leqslant C\left\|\|f(y_1,\cdot)\|_{L^{\frac{2d}{d+2}}_{y'}(\mathbb{R}^{d-1})}\right\|_{L^{\frac{2d}{d+2}}_{y_1}(X_0,X_1)} = C\|f\|_{L^{\frac{2d}{d+2}}(\Omega)}.\end{aligned}$$

Using the estimate (6.11) in Proposition 5.3, and Young's inequality, we have, for $f \in L^{\frac{2d}{d+2}}(\Omega)$,

$$\begin{aligned}&\|K_{\tau,0}f\|_{L^2(\Omega)} \\ &\leqslant \left\|\|K_{\tau,0}f(x_1,\cdot)\|_{L^2_{x'}\left(\mathbb{R}^{d-1}\right)}\right\|_{L^2_{x_1}(X_0,X_1)} \\ &\leqslant \left\|\int_{X_0}^{X_1}\left(\int_\lambda \|k_{\tau,0}(x_1,y_1,\lambda,\cdot)\|^2_{L^\infty(\Sigma_{x_1})}\lambda^{1-\frac{2}{d}}\, d\lambda\right)^{\frac{1}{2}}\|f(y_1,\cdot)\|_{L^{\frac{2d}{d+2}}_{y'}(\mathbb{R}^{d-1})}\, dy_1\right\|_{L^2_{x_1}(X_0,X_1)}\end{aligned}$$

$$\leqslant C \left\| \int_{X_0}^{X_1} \widetilde{k}_{\tau,0}(x_1-y_1) \left\| f(y_1,\cdot) \right\|_{L^{\frac{2d}{d+2}}_{y'}\left(\mathbb{R}^{d-1}\right)} dy_1 \right\|_{L^2_{x_1}(X_0,X_1)}$$

$$\leqslant C \left\| \widetilde{k}_{\tau,0} \right\|_{L^{\frac{d}{d-1}}(\mathbb{R})} \left\| \left\| f(y_1,\cdot) \right\|_{L^{\frac{2d}{d+2}}_{y'}\left(\mathbb{R}^{d-1}\right)} \right\|_{L^{\frac{2d}{d+2}}_{y_1}(X_0,X_1)} \leqslant C\tau^{-\frac{3}{4}-\frac{1}{2d}} \left\| f \right\|_{L^{\frac{2d}{d+2}}(\Omega)} .$$

Similarly, for $f \in L^2(\Omega)$, we get

$$\left\| K_{\tau,0} f \right\|_{L^{\frac{2d}{d-2}}(\Omega)} \leqslant \tau^{-\frac{3}{4}-\frac{1}{2d}} \left\| f \right\|_{L^2(\Omega)} . \tag{6.15}$$

Using the relation (6.9), the estimate (6.12) in Proposition 5.3, and Hardy–Littlewood–Sobolev theorem (Theorem A.1), we get, for all $f \in L^{\frac{2d}{d+2}}(\Omega)$,

$$\left\| \partial_1 \widehat{K_{\tau,0} f} \right\|_{L^2(\Omega_{1,\tau})} \leqslant C \left\| f \right\|_{L^{\frac{2d}{d+2}}(\Omega)} .$$

Using the estimate (6.13) in Proposition 5.3, Hardy–Littlewood–Sobolev theorem (Theorem A.1) and Young's inequality, we get, for all $f \in L^{\frac{2d}{d+2}}(\Omega)$,

$$\left\| \nabla' K_{\tau,0} f \right\|_{L^2(\Omega)} \leqslant C\tau^{\frac{1}{4}-\frac{1}{2d}} \left\| f \right\|_{L^{\frac{2d}{d+2}}(\Omega)} .$$

The above estimates allow to conclude the estimate (6.2).

Estimate (6.3) simply consists in the combination of (6.15), Theorem 4.1, and estimate (4.7). ☐

6.2. Estimates on the operator $K_{\tau,1}$

The goal of this section is to estimate the norm of the operator $K_{\tau,1}$ in (3.4)–(3.7), more precisely:

Proposition 6.7. — *Let Ω be as in (2.1) with $X_0 < 0 < X_1$ and $\max\{|X_0|, |X_1|\} \leqslant 1$, and assume that the coefficients $(\lambda_j)_{j \in \{1,\cdots,d\}} \in \mathbb{R}^d$ satisfy $\lambda_1 = 0$, (2.4) and (2.5). Then there exist $C > 0$ and $\tau_0 \geqslant 1$ independent of X_0, X_1 (and depending only on c_0, m_* and M_* in (2.4) and (2.5)), such that for all $\tau \geqslant \tau_0$ and for all $f \in L^2(\Omega)$,*

$$\tau^{-\frac{1}{4}+\frac{1}{2d}} \left\| K_{\tau,1} f \right\|_{L^{\frac{2d}{d-2}}(\Omega)} + \tau^{\frac{1}{2}} \left\| K_{\tau,1} f \right\|_{L^2(\Omega)} + \left\| \partial_1 \widehat{K_{\tau,1} f} \right\|_{L^2(\Omega_{1,\tau})} + \tau^{-\frac{1}{2}} \left\| \nabla' K_{\tau,1} f \right\|_{L^2(\Omega)} \leqslant C \|f\|_{L^2(\Omega)}.$$

Proof. — We use the same notations as in the proof of the previous proposition. From the definition of $k_{\tau,1}$ in (3.7), adapting the proof of Lemma 6.6 to $k_{\tau,1}$, we can easily deduce the following result, whose proof is left to the reader:

Lemma 6.8. — *There exists a constant $C > 0$ independent of X_0, X_1 (and depending only on c_0, m_* and M_* in (2.4) and (2.5)), such that for all x_1 and y_1 in $[X_0, X_1]$, for all $\tau \geqslant 1$, and $\lambda > 0$,*

$$\begin{aligned} &\| k_{\tau,1}(x_1, y_1, \lambda, \cdot) \|_{L^\infty\left(\Sigma_{x_1}\right)} \\ &\qquad \leqslant C e^{-|\tau-\lambda||y_1-x_1| - \lambda(y_1-x_1)^2/C} + C(\tau+\lambda) \| k_{\tau,0}(x_1, y_1, \lambda, \cdot) \|_{L^\infty\left(\Sigma_{x_1}\right)}. \end{aligned} \tag{6.16}$$

We shall then prove the following lemma:

Lemma 6.9. — *There exists a constant $C > 0$ independent of X_0, X_1 (and depending only on c_0, m_* and M_* in (2.4) and (2.5)), such that for all x_1 and y_1 in $[X_0, X_1]$, for all $\tau \geqslant 1$,*

$$\left(\int_{\lambda>0} \|k_{\tau,1}(x_1,y_1,\lambda,\cdot)\|^2_{L^\infty(\Sigma_{x_1})} \lambda^{1-\frac{2}{d}}\,d\lambda\right)^{\frac12} \leqslant C\frac{1}{|x_1-y_1|^{1-\frac1d}} + \widetilde{k}_{\tau,1}(x_1-y_1), \tag{6.17}$$
$$\text{with } \widetilde{k}_{\tau,1} \in L^{\frac{d}{d-1}}(\mathbb{R}) \text{ and } \left\|\widetilde{k}_{\tau,1}\right\|_{L^{\frac{d}{d-1}}(\mathbb{R})} \leqslant C\tau^{\frac14-\frac{1}{2d}}.$$

Proof. — Using Lemma 6.1 (2), we get

$$\begin{aligned}
&\int_0^\tau e^{-2(\tau-\lambda)|y_1-x_1|-2\lambda(y_1-x_1)^2/C}\lambda^{1-\frac2d}\,d\lambda\\
&\qquad\leqslant Ce^{-2\tau|y_1-x_1|}\int_0^\tau e^{2\lambda\left(|y_1-x_1|-(y_1-x_1)^2/C\right)}\lambda^{1-\frac2d}\,d\lambda\\
&\qquad\leqslant C|y_1-x_1|^{-2+\frac2d}e^{-2\tau|y_1-x_1|}\int_0^{\tau|y_1-x_1|} e^{2\lambda(1-|y_1-x_1|/C)}\lambda^{1-\frac2d}\,d\lambda\\
&\qquad\leqslant C|y_1-x_1|^{-2+\frac2d}e^{-2\tau(y_1-x_1)^2/C}\frac{(\tau|y_1-x_1|)^{2-\frac2d}}{1+\tau|y_1-x_1|}\\
&\qquad\leqslant C\tau^{1-\frac2d}|y_1-x_1|^{-1}e^{-2\tau(y_1-x_1)^2/C}.
\end{aligned}$$

Similarly, using Lemma 6.1 (3), we get

$$\begin{aligned}
&\int_\tau^\infty e^{-2(\lambda-\tau)|y_1-x_1|-2\lambda(y_1-x_1)^2/C}\lambda^{1-\frac2d}\,d\lambda\\
&\quad\leqslant Ce^{2\tau|y_1-x_1|}\int_\tau^\infty e^{-2\lambda\left(|y_1-x_1|+(y_1-x_1)^2/C\right)}\lambda^{1-\frac2d}\,d\lambda\\
&\quad\leqslant C|y_1-x_1|^{-2+\frac2d}e^{-2\tau|y_1-x_1|}\int_{\tau|y_1-x_1|}^\infty e^{-2\lambda(1+|y_1-x_1|/C)}\lambda^{1-\frac2d}\,d\lambda\\
&\quad\leqslant C|y_1-x_1|^{-2+\frac2d}e^{-2\tau(y_1-x_1)^2/C}(1+\tau|y_1-x_1|)^{1-\frac2d}\\
&\quad\leqslant C\left(1_{|y_1-x_1|\leqslant1/\tau}|y_1-x_1|^{-2+\frac2d}+1_{|y_1-x_1|\geqslant1/\tau}\tau^{1-\frac2d}|y_1-x_1|^{-1}e^{-2\tau(y_1-x_1)^2/C}\right).
\end{aligned}$$

Now, from (6.11), (6.13), (6.16) and the explicit formula of $\widetilde{k}_{\tau,0}$ in (6.14), we get

$$\begin{aligned}
&\left(\int_{\lambda>0}(\tau+\lambda)^2\|k_{\tau,0}(x_1,y_1,\lambda)\|^2\ \lambda^{1-\frac2d}d\lambda\right)^{\frac12}\\
&\qquad\qquad\leqslant C\left(\frac{1}{|x_1-y_1|^{1-\frac1d}}+\tau\widetilde{k}_{\tau,0}(x_1-y_1)+\check{k}_{\tau,0}(x_1-y_1)\right).
\end{aligned}$$

We then easily obtain (6.17). □

We now conclude the proof of Proposition 6.7. As in the proof of Proposition 6.2, the estimate (6.17) easily implies that there exists $C > 0$ such that for all $\tau \geqslant 1$ and for all $f \in L^2(\Omega)$,

$$\tau^{-\frac14+\frac{1}{2d}}\|K_{\tau,1}f\|_{L^{\frac{2d}{d-2}}(\Omega)} \leqslant C\|f\|_{L^2(\Omega)}.$$

The estimates on $K_{\tau,1}$ and $\nabla' K_{\tau,1}$ as operators from $L^2(\Omega)$ to $L^2(\Omega)$, and on $\partial_1 \widehat{K_{\tau,1}}$ from $L^2(\Omega)$ to $L^2(\Omega_{1,\tau})$ can then be deduced immediately from Theorem 4.1 and (4.7). □

6.3. Estimates on the operator $K_{\tau,j}$ for $j \geqslant 2$

The goal of this section is to estimate the norm of the operator $K_{\tau,j}$ in (3.4)–(3.8) for $j \geqslant 2$, more precisely:

Proposition 6.10. — *Let Ω be as in* (2.1) *with $X_0 < 0 < X_1$ and $\max\{|X_0|, |X_1|\} \leqslant 1$, and assume that the coefficients $(\lambda_j)_{j \in \{1,\cdots,d\}} \in \mathbb{R}^d$ satisfy $\lambda_1 = 0$,* (2.4) *and* (2.5)*. Then there exist $C > 0$ and $\tau_0 \geqslant 1$ independent of X_0, X_1 (and depending only on c_0, m_* and M_* in* (2.4) *and* (2.5)*), such that for all $j \in \{2, \cdots, d\}$, for all $\tau \geqslant \tau_0$ and for all $f \in L^2(\Omega)$,*

$$\tau^{-\frac{1}{4}+\frac{1}{2d}} \|K_{\tau,j} f\|_{L^{\frac{2d}{d-2}}(\Omega)} + \tau^{\frac{1}{2}} \|K_{\tau,j} f\|_{L^2(\Omega)} + \left\|\partial_1 \widehat{K_{\tau,j} f}\right\|_{L^2(\Omega_{1,\tau})} + \tau^{-\frac{1}{2}} \|\nabla' K_{\tau,j} f\|_{L^2(\Omega)} \leqslant C\|f\|_{L^2(\Omega)}.$$

Proof. — We start by noticing that there exists a constant $C > 0$ such that for all $j \in \{2, \cdots, d\}$, for all x_1 and y_1 in $[X_0, X_1]$, for all $\tau \geqslant 1$, and $\lambda > 0$,

$$\|k_{\tau,j}(x_1, y_1, \lambda, \cdot)\|_{L^\infty\left(\Sigma_{x_1}\right)} \leqslant C\lambda \|k_{\tau,0}(x_1, y_1, \lambda, \cdot)\|_{L^\infty\left(\Sigma_{x_1}\right)}.$$

From (6.13), there exists a constant $C > 0$ such that for all x_1 and y_1 in $[X_0, X_1]$, for all $\tau \geqslant 1$, and for all $j \in \{2, \cdots, d\}$,

$$\left(\int_{\lambda > 0} \|k_{\tau,j}(x_1, y_1, \lambda, \cdot)\|^2_{L^\infty(\Sigma_{x_1})} \lambda^{1-\frac{2}{d}}\, d\lambda\right)^{\frac{1}{2}} \leqslant C \frac{1}{|x_1 - y_1|^{1-\frac{1}{d}}} + \check{k}_{\tau,0}(x_1 - y_1), \tag{6.18}$$

$$\text{with } \check{k}_{\tau,0} \in L^{\frac{d}{d-1}}(\mathbb{R}) \text{ and } \left\|\check{k}_{\tau,1}\right\|_{L^{\frac{d}{d-1}}(\mathbb{R})} \leqslant C\tau^{\frac{1}{4}-\frac{1}{2d}}.$$

Accordingly, there exists $C > 0$, such that for all $j \in \{2, \cdots, d\}$, for all $\tau \geqslant 1$ and for all $f \in L^2(\Omega)$,

$$\tau^{-\frac{1}{4}+\frac{1}{2d}} \|K_{\tau,j} f\|_{L^{\frac{2d}{d-2}}(\Omega)} \leqslant C\|f\|_{L^2(\Omega)}.$$

We then conclude Proposition 6.10 by combining this estimate with the ones in Theorem 4.1 and (4.7). □

6.4. Estimates on the operator G_τ in (3.9)–(3.10)

Proposition 6.11. — *Let Ω be as in* (2.1) *with $X_0 < 0 < X_1$ and $\max\{|X_0|, |X_1|\} \leqslant 1$, and assume that the coefficients $(\lambda_j)_{j \in \{1,\cdots,d\}} \in \mathbb{R}^d$ satisfy $\lambda_1 = 0$,* (2.4) *and* (2.5)*. Then there exists $C > 0$ independent of X_0, X_1 (and depending only on c_0, m_* and M_* in* (2.4) *and* (2.5)*), such that for all $\tau \geqslant 1$, for all $g \in H^{\frac{1}{2}}(\mathbb{R}^{d-1})$,*

$$\tau^{\frac{3}{4}} \|G_\tau(g)\|_{L^2(\Omega)} + \left\|\widehat{\partial_1 G_\tau}(g)\right\|_{L^2(\Omega_{1,\tau})} + \tau^{-\frac{1}{4}} \|\nabla' G_\tau(g)\|_{L^2(\Omega)} + \|G_\tau g\|_{L^{\frac{2d}{d-2}}(\Omega)} \leqslant C\|g\|_{H^{\frac{1}{2}}\left(\mathbb{R}^{d-1}\right)}. \tag{6.19}$$

Proof. — Note that all the terms in (6.19) involving Hilbertian norms have already been estimated in Theorem 4.4, so only the estimate on G_τ in the $\mathscr{L}(H^{\frac{1}{2}}(\mathbb{R}^{d-1}), L^{\frac{2d}{d-2}}(\Omega))$-norm remains to prove.

For $g \in H^{\frac{1}{2}}(\mathbb{R}^{d-1})$, we introduce $g_0 \in L^2(\mathbb{R}^{d-1})$ so that $\widehat{g}_0(\xi') = |\xi'|^{\frac{1}{2}}\widehat{g}(\xi')$, and we notice that $G_\tau(g) = G_{\tau,0}(g_0)$, where $G_{\tau,0}$ is given as follows:

$$\widehat{G_{\tau,0}g_0}(x_1,\xi') = g_{\tau,0}(x_1,\xi')\widehat{g}_0(\xi'), \qquad (x_1,\xi') \in [X_0,X_1]\times\mathbb{R}^{d-1},$$

with $g_{\tau,0}(x_1,\xi') = g_\tau(x_1,\xi')/|\xi'|^{\frac{1}{2}}$, and g_τ as in (3.10). It is then clear that the $\mathscr{L}(H^{\frac{1}{2}}(\mathbb{R}^{d-1}), L^{\frac{2d}{d-2}}(\Omega))$-norm of G_τ coincides with the $\mathscr{L}(L^2(\mathbb{R}^{d-1}), L^{\frac{2d}{d-2}}(\Omega))$-norm of $G_{\tau,0}$.

To estimate the $\mathscr{L}(L^2(\mathbb{R}^{d-1}), L^{\frac{2d}{d-2}}(\Omega))$-norm of $G_{\tau,0}$, we compute $G_{\tau,0}^\star$, where the adjoint is given with respect to the $L^2(\Omega)$ scalar product: for $f \in L^2(\Omega)$,

$$\widehat{G_{\tau,0}^\star f}(\xi') = \int_{x_1\in[X_0,X_1]} g_{\tau,0}(x_1,\xi')\widehat{f}(x_1,\xi')\,dx_1, \qquad \xi'\in\mathbb{R}^{d-1}.$$

Accordingly, for $f \in L^2(\Omega)$,

$$\widehat{G_{\tau,0}G_{\tau,0}^\star} f(x_1,\xi') = \int_{y_1\in[X_0,X_1]} g_{\tau,0}(x_1,\xi')\, g_{\tau,0}(y_1,\xi')\,\widehat{f}(y_1,\xi')\,dy_1, \qquad (x_1,\xi')\in\Omega.$$

Our next goal is to check that the operator $G_{\tau,0}G_{\tau,0}^\star$ belongs to $\mathscr{L}(L^{\frac{2d}{d+2}}(\Omega), L^{\frac{2d}{d-2}}(\Omega))$.

In order to do that, as previously, we first check that there exists a constant C such that for all $x_1\in[X_0,X_1]$, for all $\lambda \geqslant 0$,

$$\|g_{\tau,0}(x_1,\lambda,\cdot)\|_{L^\infty(\Sigma_{x_1})} \leqslant 1_{\lambda>\tau}\frac{C}{\lambda^{\frac{1}{2}}} e^{-(\lambda-\tau)(x_1-X_0)-\lambda(x_1-X_0)^2/C}. \tag{6.20}$$

Consequently, for $x_1\in[X_0,X_1]$ and $y_1\in[X_0,X_1]$,

$$\begin{aligned}
&\left\|G_{\tau,0}G_{\tau,0}^\star f(x_1,\cdot)\right\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^{d-1})}\\
&\quad\leqslant \int_{y_1\in[X_0,X_1]}\int_\tau^\infty \|g_{\tau,0}(x_1,\lambda,\cdot)\|_{L^\infty(\Sigma_{x_1})}\|g_{\tau,0}(y_1,\lambda,\cdot)\|_{L^\infty(\Sigma_{x_1})}\lambda^{1-\frac{2}{d}}d\lambda\\
&\qquad\qquad \|f(y_1,\cdot)\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^{d-1})}dy_1\\
&\quad\leqslant \int_{y_1\in[X_0,X_1]}\int_\tau^\infty e^{-(\lambda-\tau)((x_1-X_0)+(y_1-X_0))-\lambda\left((x_1-X_0)^2+(y_1-X_0)^2\right)/C}\lambda^{-\frac{2}{d}}d\lambda\\
&\qquad\qquad \|f(y_1,\cdot)\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^{d-1})}dy_1\\
&\quad\leqslant \int_{y_1\in[X_0,X_1]}((x_1-X_0)+(y_1-X_0))^{-1+\frac{2}{d}}\,\frac{e^{-\tau\left((x_1-X_0)^2+(y_1-X_0)^2\right)/C}}{1+(\tau(x_1-X_0+y_1-X_0))^{\frac{2}{d}}}\\
&\qquad\qquad \|f(y_1,\cdot)\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^{d-1})}dy_1,
\end{aligned}$$

where we have used Lemma 6.1 item (3). With straightforward bounds, we thus get

$$\left\|G_{\tau,0}G_{\tau,0}^\star f(x_1,\cdot)\right\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^{d-1})} \leqslant \int_{y_1\in[X_0,X_1]}(y_1-x_1)^{-1+\frac{2}{d}}\,\|f(y_1,\cdot)\|_{L^{2d/(d+2}(\mathbb{R}^{d-1})}\,dy_1,$$

and then Hardy–Littlewood–Sobolev theorem (Theorem A.1) implies that

$$\left\|G_{\tau,0}G^{\star}_{\tau,0}f\right\|_{L^{\frac{2d}{d-2}}(\Omega)} \leqslant C\|f\|_{L^{\frac{2d}{d+2}}(\Omega)}.$$

It follows that the $\mathscr{L}(L^2(\mathbb{R}^{d-1}), L^{\frac{2d}{d-2}}(\Omega))$-norm of $G_{\tau,0}$ is bounded by a constant independent of τ as announced, and thus this is also the case for the $\mathscr{L}(H^{\frac{1}{2}}(\mathbb{R}^{d-1}), L^{\frac{2d}{d-2}}(\Omega))$-norm of G_τ. □

Remark 6.12. — One may wonder why the above proof does not rely on the estimate (5.19) directly. This is due to the fact that it would correspond to a limit case. Indeed, from the estimate(6.20) and Proposition 5.3, for $x_1 \in [X_0, X_1]$,

$$\begin{aligned}
&\|G_{\tau,0}g_0(x_1,\cdot)\|_{L^{\frac{2d}{d-2}}\left(\mathbb{R}^{d-1}\right)} \\
&\qquad\leqslant C\|g_0\|_{L^2\left(\mathbb{R}^{d-1}\right)}\sqrt{\int_\tau^\infty \|g_{\tau,0}(x_1,\lambda,\cdot)\|^2_{L^\infty(\Sigma_{x_1})}\,\lambda^{1-\frac{2}{d}}\,d\lambda} \\
&\qquad\leqslant C\|g_0\|_{L^2\left(\mathbb{R}^{d-1}\right)}\sqrt{\int_\tau^\infty e^{-2(\lambda-\tau)(x_1-X_0)-2\lambda(x_1-X_0)^2/C}\lambda^{-\frac{2}{d}}\,d\lambda} \\
&\qquad\leqslant C\|g_0\|_{L^2\left(\mathbb{R}^{d-1}\right)}|x_1-X_0|^{-(d-2)/(2d)}\frac{e^{-\tau(x_1-X_0)^2/C}}{1+(\tau(x_1-X_0))^{\frac{1}{d}}},
\end{aligned}$$

where we used item (3) in Lemma 6.1. But the function

$$x_1 \mapsto |x_1-X_0|^{-(d-2)/(2d)}\frac{e^{-\tau(x_1-X_0)^2/C}}{1+(\tau(x_1-X_0))^{\frac{1}{d}}}$$

does not belong to $L^{\frac{2d}{d-2}}(X_0,X_1)$, and so we cannot conclude directly that $G_{\tau,0}g_0$ belongs to $L^{\frac{2d}{d-2}}(\Omega)$.

6.5. Estimates on the operator R_τ in (3.5)

For $f \in H^1(\Omega)$, using that $R_\tau(f) = K_{\tau,0}(g)$ with $\widehat{g}(x_1,\xi') = \partial_1\psi(x_1,\xi')\widehat{f}(x_1,\xi')$, and thus with $\|g\|_{L^2(\Omega)} \leqslant C\|\nabla' f\|_{L^2(\Omega)}$, we immediately deduce the following result from (6.3):

Proposition 6.13. — *Let Ω be as in (2.1) with $X_0<0<X_1$ and $\max\{|X_0|,|X_1|\} \leqslant 1$, and assume that the coefficients $(\lambda_j)_{j\in\{1,\cdots,d\}} \in \mathbb{R}^d$ satisfy $\lambda_1 = 0$, (2.4) and (2.5). Then there exist $C > 0$ and $\tau_0 \geqslant 1$ independent of X_0, X_1 (and depending only on c_0, m_* and M_* in (2.4) and (2.5)), such that for all $\tau \geqslant \tau_0$, for all $f \in H^1(\Omega)$,*

$$\tau^{\frac{3}{4}+\frac{1}{2d}}\|R_\tau f\|_{L^{\frac{2d}{d-2}}(\Omega)} + \tau^{\frac{3}{2}}\|R_\tau f\|_{L^2(\Omega)} + \tau\left\|\partial_1\widehat{R_\tau f}\right\|_{L^2(\Omega_{1,\tau})} + \tau^{\frac{1}{2}}\|\nabla' R_{\tau,0}f\|_{L^2(\Omega)} \leqslant C\|\nabla' f\|_{L^2(\Omega)}.$$

6.6. Proofs of Theorem 2.4 and Theorem 2.1

Within the setting of Theorem 2.4, we use Proposition 3.1 to write the solution w of (2.8) satisfying $(\partial_1 w - F_1)(X_1,\cdot) = 0$ in $\mathbb{R}^{d-1}$ under the form (3.3). Since $w \in H^1(\Omega)$, using the various estimates in Propositions 6.2, 6.7, 6.10, 6.11 and 6.13, we obtain, on one hand,

$$(6.21)\quad \tau^{\frac{3}{2}}\|w\|_{L^2(\Omega)} + \tau^{\frac{3}{4}}\|\partial_1 w\|_{L^2(\Omega)} + \tau^{\frac{1}{2}}\left\|\nabla' w\right\|_{L^2(\Omega)} \leqslant C\Big(\|f_2\|_{L^2(\Omega)} + \tau^{\frac{3}{4}-\frac{1}{2d}}\left\|f_{2'_*}\right\|_{L^{\frac{2d}{d+2}}(\Omega)} + \tau\|F\|_{L^2(\Omega)} + \tau^{\frac{3}{4}}\|g\|_{H^{\frac{1}{2}}\left(\{X_0\}\times\mathbb{R}^{d-1}\right)} + \|\nabla' w\|_{L^2(\Omega)}\Big),$$

and, on the other hand,

$$(6.22)\quad \tau^{\frac{3}{4}+\frac{1}{2d}}\|w\|_{L^{\frac{2d}{d-2}}(\Omega)} \leqslant C\Big(\|f_2\|_{L^2(\Omega)} + \tau^{\frac{3}{4}+\frac{1}{2d}}\left\|f_{2'_*}\right\|_{L^{\frac{2d}{d+2}}(\Omega)} + \tau\|F\|_{L^2(\Omega)} + \tau^{\frac{3}{4}+\frac{1}{2d}}\|g\|_{H^{\frac{1}{2}}\left(\{X_0\}\times\mathbb{R}^{d-1}\right)} + \|\nabla' w\|_{L^2(\Omega)}\Big).$$

We then simply take $\tau \geqslant \tau_0$ with $\tau_0 \geqslant 1$ large enough in order to absorb the last term in the right-hand side of (6.21) by $\tau^{\frac{1}{2}}\|\nabla' w\|_{L^2(\Omega)}$, and we get

$$(6.23)\quad \tau^{\frac{3}{2}}\|w\|_{L^2(\Omega)} + \tau^{\frac{3}{4}}\|\partial_1 w\|_{L^2(\Omega)} + \tau^{\frac{1}{2}}\left\|\nabla' w\right\|_{L^2(\Omega)} \leqslant C\Big(\|f_2\|_{L^2(\Omega)} + \tau^{\frac{3}{4}-\frac{1}{2d}}\left\|f_{2'_*}\right\|_{L^{\frac{2d}{d+2}}(\Omega)} + \tau\|F\|_{L^2(\Omega)} + \tau^{\frac{3}{4}}\|g\|_{H^{\frac{1}{2}}\left(\{X_0\}\times\mathbb{R}^{d-1}\right)}\Big),$$

that is estimate (2.9).

Therefore, using (6.23), the last term in (6.22) can be removed, thus yielding (2.10).

In order to prove Theorem 2.1, we simply use the correspondence $w(x) = e^{\tau x_1} v(x)$ for $x \in \Omega$. This proves (2.6) and (2.7) for $\tau \geqslant \tau_0$. We then deduce (2.6) and (2.7) for any $\tau \geqslant 1$ by changing the constant if necessary through straightforward bounds on $x_1 \mapsto \exp(\tau x_1)$ for $\tau \in [1,\tau_0]$.

7. General geometrical setting: proof of Theorem 1.1

Here, we provide a proof of Theorem 1.1 using Fourier techniques as we did earlier, following the approach developed in [DE23], and adapted to the case of source terms in $H^{-1}(\Omega)$ and in $L^{\frac{2d}{d+2}}(\Omega)$. This approach is based on a localization argument and a gluing argument, as it is usually done for Carleman estimates. The originality here is that we will localize the functions in balls of size $\tau^{-\frac{1}{3}}$, that is, depending on the Carleman parameter τ. Doing that choice allows to somehow approximate the weight function φ by its quadratic approximation, and to reduce the problem through a suitable change of variables to the case of a strip with linear coefficients as in Theorem 2.4 (see Lemma 7.1 and its proof).

For $\tau \geqslant 1$, we introduce

$$w = e^{\tau\varphi}u, \qquad \widetilde{f}_2 = e^{\tau\varphi}(f_2 - \tau\nabla\varphi\cdot F), \qquad \widetilde{f}_{2'_*} = e^{\tau\varphi} f_{2'_*}, \qquad \widetilde{F} = e^{\tau\varphi}F, \qquad \widetilde{g} = g,$$

so that the function u solves (1.4) if and only if w solves

$$\begin{cases} \Delta w - 2\tau\nabla\varphi\cdot\nabla w + \tau^2|\nabla\varphi|^2 w - \tau\Delta\varphi w = \widetilde{f}_2 + \widetilde{f}_{2'_*} + \operatorname{div}\left(\widetilde{F}\right), & \text{in } \Omega, \\ w = \widetilde{g} & \text{on } \partial\Omega. \end{cases} \tag{7.1}$$

7.1. Local estimates

Our first step is to introduce a local version of (7.1). Namely, for $x_0 \in \overline{\Omega}\setminus\omega$, we introduce $\eta_{x_0}(x)$ a cut-off function, which will be made more precise in (7.5), and set

$$w_{x_0}(x) = \eta_{x_0}(x) w(x), \qquad x \in \Omega,$$

which solves

$$\begin{cases} \Delta w_{x_0} - 2\tau\nabla\varphi\cdot\nabla w_{x_0} + \tau^2|\nabla\varphi|^2 w_{x_0} = f_{2,x_0} + f_{2'_*,x_0} + \operatorname{div}(F_{x_0}), & \text{in } \Omega, \\ w_{x_0} = \widetilde{g}_{x_0} & \text{on } \partial\Omega, \end{cases} \tag{7.2}$$

where

$$f_{2,x_0} = \eta_{x_0}\widetilde{f}_2 - \nabla\eta_{x_0}\cdot\widetilde{F} + \tau\Delta\varphi w_{x_0} + 2\nabla\eta_{x_0}\cdot\nabla w + \Delta\eta_{x_0} w - 2\tau\nabla\varphi\cdot\nabla\eta_{x_0} w,$$
$$f_{2'_*,x_0} = \eta_{x_0}\widetilde{f}_{2'_*}, \qquad F_{x_0} = \eta_{x_0}\widetilde{F}, \qquad g_{x_0} = \eta_{x_0}\widetilde{g}.$$

We claim that, provided the localization is strong enough, we can get a local Carleman estimate:

Lemma 7.1. — *There exist constants $C>0$ and $\tau_0 \geqslant 1$ (depending only on α, β, $\|\varphi\|_{C^3(\overline{\Omega})}$) such that for all $\tau \geqslant \tau_0$, for all $x_0 \in \overline{\Omega}\setminus\omega$, and for all*

$$f_{2,x_0} \in L^2(\Omega), \quad f_{2'_*,x_0} \in L^{\frac{2d}{d+2}}(\Omega), \quad F_{x_0} \in L^2(\Omega;\mathbb{C}^d), \quad \text{and} \quad g_{x_0} \in H^{\frac12}(\partial\Omega),$$

and w_{x_0} satisfying (7.2) and supported in $B(x_0,\tau^{-\frac13})\cap\overline{\Omega}$, we have

$$\tau^{\frac32}\|w_{x_0}\|_{L^2(\Omega)} + \tau^{\frac12}\|\nabla w_{x_0}\|_{L^2(\Omega)} + \tau^{\frac34-\frac{1}{2d}}\|w_{x_0}\|_{L^{\frac{2d}{d-2}}(\Omega)} \leqslant C\left(\|f_{2,x_0}\|_{L^2(\Omega)} + \tau^{\frac34-\frac{1}{2d}}\|f_{2'_*,x_0}\|_{L^{\frac{2d}{d+2}}(\Omega)} + \tau\|F_{x_0}\|_{L^2(\Omega)} + \tau^{\frac34}\|g_{x_0}\|_{H^{\frac12}(\partial\Omega)}\right), \tag{7.3}$$

and

$$\tau^{\frac34+\frac{1}{2d}}\|w_{x_0}\|_{L^{\frac{2d}{d-2}}(\Omega)} + \tau^{\frac32}\|w_{x_0}\|_{L^2(\Omega)} + \tau^{\frac12}\|\nabla w_{x_0}\|_{L^2(\Omega)} \leqslant C\left(\|f_{2,x_0}|_{L^2(\Omega)} + \tau^{\frac34+\frac{1}{2d}}\|f_{2'_*,x_0}\|_{L^{\frac{2d}{d+2}}(\Omega)} + \tau\|F_{x_0}\|_{L^2(\Omega)} + \tau^{\frac34+\frac{1}{2d}}\|g_{x_0}\|_{H^{\frac12}(\partial\Omega)}\right). \tag{7.4}$$

The proof of Lemma 7.1 is postponed to the next section, and will be based on a suitable change of variables and the Carleman estimate in Theorem 2.4.

In the following, we will thus choose the localization as follows:

$$\eta_{x_0}(x) = \eta\left(\tau^{\frac13}(x-x_0)\right), \qquad x \in \mathbb{R}^d, \tag{7.5}$$

where η is a non-negative smooth radial function (in $\mathscr{C}_c^\infty(\mathbb{R}^d)$) such that $\eta(\rho)=1$ for $|\rho| \leqslant \frac12$ and vanishing outside the unit ball, so that Lemma 7.1 applies to w_{x_0}, and the estimates (7.3) and (7.4) hold uniformly with respect to $x_0 \in \overline{\Omega}\setminus\omega$.

It thus remains to prove Lemma 7.1, which is done in Section 7.2, and to show how to glue the estimates in Lemma 7.1 to conclude Theorem 1.1, which is explained in Section 7.3.

7.2. Proof of Lemma 7.1: A suitable change of coordinates

The proof of Lemma 7.1 mainly reduces to a suitable change of variables allowing to link the Carleman estimates (7.3) and (7.4) in small balls around x_0 with the Carleman estimates (2.9)-(2.10) in the strip proved in Theorem 2.4.

We let $x_0 \in \overline{\Omega} \setminus \omega$, and we introduce $L_1 \in \mathbb{R}^d$ and $A_1 \in \mathbb{R}^{d\times d}$ as follows:

$$L_1 = \nabla\varphi(x_0) \in \mathbb{R}^d, \quad A_1 = \operatorname{Hess}\varphi(x_0) \in \mathbb{R}^{d\times d}.$$

The bilinear form

$$\xi \in \mathbb{R}^d \mapsto (\operatorname{Hess}\varphi(x_0))\xi\cdot\xi$$

is symmetric on $\mathbb{R}^d$ and on $\operatorname{Span}\{L_1\}^\perp$. Accordingly, there exists a family of orthogonal vectors $(L_j)_{j\in\{2,\cdots,d\}}$ of $\operatorname{Span}\{L_1\}^\perp$ which diagonalizes this form, that we normalize so that for all $j \in \{2,\cdots,d\}$, $|L_j| = |L_1|$. Since the family $(L_j)_{j\in\{2,\cdots,d\}}$ of $\operatorname{Span}\{L_1\}^\perp$ diagonalizes the form $\xi \mapsto (\operatorname{Hess}\varphi(x_0))\xi\cdot\xi$ in $\operatorname{Span}\{L_1\}^\perp$, for all $j\in\{2,\cdots,d\}$, there exist α_j and μ_j in $\mathbb{R}$ such that

$$(\operatorname{Hess}\varphi(x_0))L_j = \mu_j L_j + \alpha_j L_1, \qquad j\in\{2,\cdots,d\}.$$

Note that by symmetry of $\operatorname{Hess}\varphi(x_0)$, we then necessarily have

$$(\operatorname{Hess}\varphi(x_0))L_1 = \mu_1 L_1 + \sum_{k\geqslant 2}\alpha_k L_k,$$

where

$$\mu_1 = \frac{1}{|L_1|^2}(\operatorname{Hess}\varphi(x_0))L_1\cdot L_1 = \frac{1}{|\nabla\varphi(x_0)|^2}(\operatorname{Hess}\varphi(x_0))\nabla\varphi(x_0)\cdot\nabla\varphi(x_0).$$

For $j\in\{2,\cdots,d\}$, we then introduce the self-adjoint matrix $A_j\in\mathbb{R}^{d\times d}$ defined by

$$\left\{\begin{array}{ll} A_jL_1 = -\alpha_jL_1 - \mu_jL_j, & \\ A_jL_k = \alpha_kL_j - \alpha_jL_k, & \text{if } k\in\{2,\cdots,d\}\setminus\{j\}, \\ A_jL_j = -\mu_jL_1 + \sum_{k\geqslant 2}\alpha_kL_k. & \end{array}\right. \tag{7.6}$$

(It is easy to check that the matrix A_j defined that way is indeed symmetric.)

We shall then introduce the following change of coordinates for x in a neighbourhood of x_0:

$$\begin{aligned} & y_1(x) = \varphi(x) - \varphi(x_0), \\ & \text{for } j\in\{2,\cdots,d\},\ y_j(x) = L_j\cdot(x-x_0) + \frac{1}{2}A_j(x-x_0)\cdot(x-x_0). \end{aligned}$$

By construction, there exists a neighbourhood, whose size depends on the C^2 norm of φ only, such that $x\mapsto y(x)$ is a local diffeomorphism between a neighbourhood $\mathcal{V}$ of x_0 in $\overline{\Omega}\setminus\omega$ and a neighbourhood of 0, that we call Ω_y, and which may thus contain the image of a part of $\partial\Omega$. If it exists, we will denote it by $\Gamma_y = y(\partial\Omega\cap\mathcal{V})$.

For τ large enough, we can ensure that the ball of center x_0 and radius $\tau^{-\frac{1}{3}}$, when intersected with $\overline{\Omega}$, is included in a set on which $x \mapsto y(x)$ is a diffeomorphism, and its image is included in a ball $B(0, C\tau^{-\frac{1}{3}})$.

Therefore, for w_{x_0} solving (7.2), we set

$$\check{w}(y) = w_{x_0}(x) \quad \text{for} \quad y = y(x),$$

Explicit computations then give that $\check{w}$ satisfies, for $x \in \Omega \cap B(x_0, \tau^{-\frac{1}{3}})$,

$$\sum_{j,k=1}^{d} b_{j,k}(x)\partial_{y_j}\partial_{y_k}\check{w}(y(x)) + \nabla_y \check{w}(y(x)) \cdot \Delta_x y(x) - 2\tau \sum_{j=1}^{d} c_j(x)\partial_{y_j}\check{w}(y(x)) + \tau^2|\nabla\varphi(x)|^2\check{w}(y(x)) = \left(\Delta w_{x_0} - 2\tau\nabla\varphi \cdot \nabla w_{x_0} + \tau^2|\nabla\varphi|^2 w_{x_0}\right)(x), \tag{7.7}$$

where

$$b_{j,k}(x) = \nabla_x y_j(x) \cdot \nabla_x y_k(x), \quad \text{and} \quad c_j(x) = \sum_{i=1}^{d} \partial_i\varphi(x)\partial_i y_j(x).$$

We then remark that $c_j(x) = b_{j,1}(x)$ and that $b_{j,k}(x) = b_{k,j}(x)$ for all x. We now briefly analyze the coefficients $b_{j,k}$. By construction of the coordinates $(y_j)_{j \in \{1, \cdots, d\}}$, we easily check that for $(j,k,\ell) \in \{1, \cdots, d\}^2$,

$$\begin{aligned} b_{j,k}(x_0) &= |L_1|^2\delta_{j,k}, \\ \partial_\ell b_{j,k}(x_0) &= \sum_{i=1}^{d}(\partial_\ell\partial_i y_j(x_0)\partial_i y_k(x_0) + \partial_i y_j(x_0)\partial_\ell\partial_i y_k(x_0)) \\ &= \sum_{i=1}^{d}(A_j e_i \cdot e_\ell)(L_k \cdot e_i) + (L_j \cdot e_i)(A_k e_i \cdot e_\ell) = (A_j L_k + A_k L_j) \cdot e_\ell, \end{aligned}$$

so that we have in particular that

$$L_\ell \cdot \nabla b_{j,k}(x_0) = (A_j L_k + A_k L_j) \cdot L_\ell,$$

For convenience, we also write separately

$$|\nabla\varphi(x_0)|^2 = |L_1|^2, \quad \text{and} \quad \partial_\ell\left(|\nabla\varphi|^2\right)(x_0) = 2\sum_i \partial_\ell\partial_i\varphi(x_0)\partial_i\varphi(x_0) = 2A_1L_1 \cdot e_\ell.$$

We can thus analyze $b_{j,k}/|\nabla\varphi|^2$ close to $x = x_0$:

$$\begin{aligned} \frac{b_{j,k}}{|\nabla\varphi|^2}(x_0) &= \delta_{j,k}, \\ L_\ell \cdot \nabla\left(\frac{b_{j,k}}{|\nabla\varphi|^2}\right)(x_0) &= \frac{1}{|L_1|^2}\left((A_jL_k + A_kL_j) \cdot L_\ell - 2\delta_{j,k}A_1L_1 \cdot L_\ell\right). \end{aligned}$$

In particular, since $A_jL_k + A_kL_j = 0$ for all $j,k \in \{1, \cdots, d\}$ with $j \neq k$,

$$\forall\, j,k \in \{1, \cdots, d\} \text{ with } j \neq k, \forall\, \ell \in \{1, \cdots, d\}, \quad L_\ell \cdot \nabla\left(\frac{b_{j,k}}{|\nabla\varphi|^2}\right)(x_0) = 0.$$

When $j = k = 1$, it is obvious that

$$\forall\, \ell \in \{1, \cdots, d\}, \quad L_\ell \cdot \nabla\left(\frac{b_{1,1}}{|\nabla\varphi|^2}\right)(x_0) = 0.$$

When $j = k \geqslant 2$, the choices (7.6) yield

$$L_\ell \cdot \nabla \left(\frac{b_{j,j}}{|\nabla \varphi|^2} \right)(x_0) = 0 \text{ when } \ell \in \{2, \cdots, d\},$$

and

$$L_1 \cdot \nabla \left(\frac{b_{j,j}}{|\nabla \varphi|^2} \right)(x_0) = -\frac{2}{|L_1|^2} \left(A_1 L_j \cdot L_j + A_1 L_1 \cdot L_1 \right).$$

Consequently, as a consequence of Taylor expansion of $b_{j,k}/|\nabla \varphi|^2$ close to $x = x_0$,

$$\frac{b_{j,k}(x)}{|\nabla \varphi(x)|^2} = \delta_{j,k} \left(1 - \lambda_j y_1(x)\right) + \mathcal{O}\left(|x - x_0|^2\right),$$

where

$$\lambda_1 = 0, \quad \lambda_j = \frac{2}{|L_1|^2}(A_1 L_j \cdot L_j + A_1 L_1 \cdot L_1) \quad \text{for } j \in \{2, \cdots, d\}. \tag{7.8}$$

Accordingly, setting

$$\check{b}_{j,k}(y) = \frac{b_{j,k}(x)}{|\nabla \varphi(x)|^2} \quad \text{for } y = y(x),$$

we have

$$\check{b}_{j,k}(y) = \delta_{j,k}(1 - \lambda_j y_1) + \mathcal{O}(|y|^2) \quad \text{and} \quad \nabla_y \left(\check{b}_{j,k}(y) - \delta_{j,k} \left(1 - \lambda_j y_1\right) \right) = \mathcal{O}(|y|).$$

Thus, using that $\check{w}$ is supported in $B(0, C\tau^{-\frac{1}{3}})$, writing

$$\begin{aligned} \sum_{j,k=1}^d \check{b}_{j,k}(y) \partial_{y_j} \partial_{y_k} \check{w}(y) - \sum_{j=1}^d (1 - \lambda_j y_1) \partial_{y_j}^2 \check{w} \\ = \sum_{j,k=1}^d \partial_{y_j} \left(\left(\check{b}_{j,k}(y) - \delta_{j,k}(1 - \lambda_j y_1) \right) \partial_{y_k} \check{w} \right) \\ - \sum_{j,k=1}^d \partial_{y_j} \left(\check{b}_{j,k}(y) - \delta_{j,k} \left(1 - \lambda_j y_1\right) \right) \partial_{y_k} \check{w}, \end{aligned}$$

we get that

$$\sum_{j,k=1}^d \check{b}_{j,k}(y) \partial_{y_j} \partial_{y_k} \check{w}(y) - \sum_{j=1}^d \left(1 - \lambda_j y_1\right) \partial_{y_j}^2 \check{w} = \check{f}_{2,a} + \operatorname{div}\left(\check{F}_a\right),$$

where

$$\begin{aligned} \check{f}_{2,a}(y) &= -\sum_{j,k=1}^d \partial_{y_j} \left(\check{b}_{j,k}(y) - \delta_{j,k} \left(1 - \lambda_j y_1\right) \right) \partial_{y_k} \check{w}, \\ \check{F}_{a,j}(y) &= \sum_{k=1}^d \left(\check{b}_{j,k}(y) - \delta_{j,k} \left(1 - \lambda_j y_1\right) \right) \partial_{y_k} \check{w} \end{aligned}$$

satisfy

$$\left\| \check{f}_{2,a} \right\|_{L^2(\Omega_y)} \leqslant C\tau^{-\frac{1}{3}} \left\| \nabla_y \check{w} \right\|_{L^2(\Omega_y)}, \quad \text{and} \quad \left\| \check{F}_a \right\|_{L^2(\Omega_y)} \leqslant C\tau^{-\frac{2}{3}} \left\| \nabla_y \check{w} \right\|_{L^2(\Omega_y)}.$$

Similarly,

$$-2\tau \sum_{j=1}^{d} \check{b}_{1,j}(y)\partial_{y_j}\check{w}(y) + 2\tau\partial_{y_1}\check{w}(y) = \check{f}_{2,b}(y),$$

with

$$\left\|\check{f}_{2,b}\right\|_{L^2(\Omega_y)} \leqslant C\tau^{\frac{1}{3}} \left\|\nabla_y \check{w}\right\|_{L^2(\Omega_y)}.$$

Finally,

$$\check{f}_{2,c}(y) = -\frac{1}{|\nabla\varphi(x(y))|^2}\nabla_y\check{w}(y)\cdot\Delta_x y(x(y)),$$

where $x(y)$ denotes the inverse of the change of variables $x \mapsto y(x)$, also satisfies

$$\left\|\check{f}_{2,c}\right\|_{L^2(\Omega_y)} \leqslant C \left\|\nabla_y \check{w}\right\|_{L^2(\Omega_y)}.$$

We then set

$$\rho_{j,k}(y) = \frac{\partial_{x_k} y_j(x(y))}{|\nabla\varphi(x(y))|^2},$$

and introduce

$$\check{f}_2(y) = \frac{1}{|\nabla\varphi(x(y))|^2} f_{2,x_0}(x(y)) - \sum_{j,k} \partial_{y_j}\rho_{k,j} F_{x_0,k}(x(y)) + \check{f}_{2,a}(y) + \check{f}_{2,b}(y) + \check{f}_{2,c}(y),$$

$$\check{f}_{2'_*}(y) = \frac{1}{|\nabla\varphi(x(y))|^2} f_{2'_*,x_0}(x(y)),$$

$$\check{F}_j(y) = \sum_{k=1}^{d} \rho_{k,j}(y) F_{x_0,k}(x(y)) + \check{F}_{j,a}(y), \qquad j \in \{1,\cdots,d\},$$

and we get that $\check{w}$ satisfies in Ω_y,

$$\sum_{j=1}^{d}(1-\lambda_j y_1)\partial_{y_j}^2\check{w} - 2\tau\partial_{y_1}\check{w} + \tau^2\check{w} = \check{f}_2 + \check{f}_{2'_*} + \operatorname{div}_y \check{F}.$$

If Γ_y is not empty, then we simply recall that the weight function φ has been chosen such that $\varphi = 0$ on the boundary of $\partial\Omega$. In particular, the set Γ_y is simply parametrized by $y_1(x) = Y_0$ for some $Y_0 = -\varphi(x_0) \leqslant 0$, and Ω can be locally defined by $y_1 > Y_0$. Thus, in this case, the equation of $\check{w}$ should be completed with

$$\check{w}(Y_0, y') = \check{g}(y'), \qquad \text{for } y' \in \mathbb{R}^{d-1},$$

where

$$\check{g}(y') = g_{x_0}(x(Y_0, y')), \qquad \text{for } y' \in \mathbb{R}^{d-1} \text{ such that } (Y_0, y') \in \Gamma_y.$$

Due to the form of Ω_y and the fact that we are considering functions which are supported in sets included in balls of the form $B(0, C\tau^{-\frac{1}{3}})$, we can then simply extend all the source terms in a strip of the form $[Y_0, Y_1] \times \mathbb{R}^{d-1}$, where the functions are extended by 0 outside Ω_y, and $\check{w}$ then satisfies

$$
(7.9)\quad \begin{cases} \Delta_y \check{w} - y_1 \sum_{j=2}^{d} \lambda_j \partial_{y_j}^2 \check{w} - 2\tau \partial_{y_1} \check{w} + \tau^2 \check{w} \\ \qquad\qquad = -\check{f}_2 + \check{f}_{2'_*} + \operatorname{div}_y \check{F} & \text{in } (Y_0, Y_1) \times \mathbb{R}^{d-1}, \\ \check{w}\,(Y_0, y') = \check{g}(y'), & \text{for } y' \in \mathbb{R}^{d-1}, \\ \check{w}\,(Y_1, y') = 0, & \text{for } y' \in \mathbb{R}^{d-1}, \\ \partial_{y_1} \check{w}\,(Y_1, y') = 0, & \text{for } y' \in \mathbb{R}^{d-1}. \end{cases}
$$

In the following, for convenience, we also write Ω_y for the strip $(Y_0, Y_1) \times \mathbb{R}^{d-1}$. Now, we come back to the definition of λ_j in (7.8) and remark that the condition (1.3), when taken at $x = x_0$, is equivalent to the condition (2.5). Accordingly, the Carleman estimates in Theorem 2.4 apply: for $\tau \geqslant \tau_0$, we have

$$
\begin{aligned}
&\tau^{\frac{3}{2}} \|\check{w}\|_{L^2(\Omega_y)} + \tau^{\frac{1}{2}} \|\nabla \check{w}\|_{L^2(\Omega_y)} + \tau^{\frac{3}{4}-\frac{1}{2d}} \|\check{w}\|_{L^{\frac{2d}{d-2}}(\Omega_y)} \\
&\quad \leqslant C \left(\left\|\check{f}_2\right\|_{L^2(\Omega_y)} + \tau^{\frac{3}{4}-\frac{1}{2d}} \left\|\check{f}_{2'_*}\right\|_{L^{\frac{2d}{d+2}}(\Omega_y)} + \tau \left\|\check{F}\right\|_{L^2(\Omega_y)} + \tau^{\frac{3}{4}} \|\check{g}\|_{H^{\frac{1}{2}}(\{Y_0\}\times\mathbb{R}^{d-1})} \right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\tau^{\frac{3}{4}+\frac{1}{2d}} \|\check{w}\|_{L^{\frac{2d}{d-2}}(\Omega_y)} + \tau^{\frac{3}{2}} \|\check{w}\|_{L^2(\Omega_y)} + \tau^{\frac{1}{2}} \|\nabla \check{w}\|_{L^2(\Omega_y)} \\
&\leqslant C \left(\left\|\check{f}_2\right\|_{L^2(\Omega_y)} + \tau^{\frac{3}{4}+\frac{1}{2d}} \left\|\check{f}_{2'_*}\right\|_{L^{\frac{2d}{d+2}}(\Omega_y)} + \tau \left\|\check{F}\right\|_{L^2(\Omega_y)} + \tau^{\frac{3}{4}+\frac{1}{2d}} \|\check{g}\|_{H^{\frac{1}{2}}(\{Y_0\}\times\mathbb{R}^{d-1})} \right).
\end{aligned}
$$

We then simply remark that, from the expression of $\check{f}_2, \check{f}_{2'_*}, \check{F}$ and $\check{g}$,

$$
\begin{aligned}
&\left\|\check{f}_2\right\|_{L^2(\Omega_y)} + \tau^{\frac{3}{4}-\frac{1}{2d}} \left\|\check{f}_{2'_*}\right\|_{L^{\frac{2d}{d+2}}(\Omega_y)} + \tau \left\|\check{F}\right\|_{L^2(\Omega_y)} + \tau^{\frac{3}{4}} \|\check{g}\|_{H^{\frac{1}{2}}(\{Y_0\}\times\mathbb{R}^{d-1})} \\
&\quad \leqslant C \Big(\|f_{2,x_0}\|_{L^2(\Omega)} + \tau^{\frac{3}{4}-\frac{1}{2d}} \|f_{2'_*,x_0}\|_{L^{\frac{2d}{d+2}}(\Omega)} + \tau \|F_{x_0}\|_{L^2(\Omega)} + \tau^{\frac{3}{4}} \|g_{x_0}\|_{H^{\frac{1}{2}}(\partial\Omega)} \\
&\qquad\qquad + \tau^{\frac{1}{3}} \|\nabla_y \widetilde{w}\|_{L^2(\Omega_y)} \Big),
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\|\check{f}_2\right\|_{L^2(\Omega_y)} + \tau^{\frac{3}{4}+\frac{1}{2d}} \left\|\check{f}_{2'_*}\right\|_{L^{\frac{2d}{d+2}}(\Omega_y)} + \tau \left\|\check{F}\right\|_{L^2(\Omega_y)} + \tau^{\frac{3}{4}+\frac{1}{2d}} \|\check{g}\|_{H^{\frac{1}{2}}(\{Y_0\}\times\mathbb{R}^{d-1})} \\
&\quad \leqslant C \Big(\|f_{2,x_0}\|_{L^2(\Omega)} + \tau^{\frac{3}{4}+\frac{1}{2d}} \|f_{2'_*,x_0}\|_{L^{\frac{2d}{d+2}}(\Omega)} + \tau \|F_{x_0}\|_{L^2(\Omega)} + \tau^{\frac{3}{4}+\frac{1}{2d}} \|g_{x_0}\|_{H^{\frac{1}{2}}(\partial\Omega)} \\
&\qquad\qquad + \tau^{\frac{1}{3}} \|\nabla_y \widetilde{w}\|_{L^2(\Omega_y)} \Big).
\end{aligned}
$$

Accordingly, taking $\tau_0 \geqslant 1$ larger if necessary, we get for all $\tau \geqslant \tau_0$,

$$
\begin{aligned}
&\tau^{\frac{3}{2}} \|\check{w}\|_{L^2(\Omega_y)} + \tau^{\frac{1}{2}} \|\nabla \check{w}\|_{L^2(\Omega_y)} + \tau^{\frac{3}{4}-\frac{1}{2d}} \|\check{w}\|_{L^{\frac{2d}{d-2}}(\Omega_y)} \\
&\quad \leqslant C \left(\left\|\check{f}_2\right\|_{L^2(\Omega_y)} + \tau^{\frac{3}{4}-\frac{1}{2d}} \left\|\check{f}_{2'_*}\right\|_{L^{\frac{2d}{d+2}}(\Omega_y)} + \tau \left\|\check{F}\right\|_{L^2(\Omega_y)} + \tau^{\frac{3}{4}} \|\check{g}\|_{H^{\frac{1}{2}}(\{Y_0\}\times\mathbb{R}^{d-1})} \right),
\end{aligned}
$$

and

$$\tau^{\frac{3}{4}+\frac{1}{2d}} \|\check{w}\|_{L^{\frac{2d}{d-2}}(\Omega_y)} + \tau^{\frac{3}{2}} \|\check{w}\|_{L^2(\Omega_y)} + \tau^{\frac{1}{2}} \|\nabla \check{w}\|_{L^2(\Omega_y)}$$
$$\leqslant C \left(\left\|\check{f}_2\right\|_{L^2(\Omega_y)} + \tau^{\frac{3}{4}+\frac{1}{2d}} \left\|\check{f}_{2'_*}\right\|_{L^{\frac{2d}{d+2}}(\Omega_y)} + \tau \left\|\check{F}\right\|_{L^2(\Omega_y)} + \tau^{\frac{3}{4}+\frac{1}{2d}} \|\check{g}\|_{H^{\frac{1}{2}}\left(\{Y_0\}\times\mathbb{R}^{d-1}\right)} \right).$$

Undoing the change of variables on the left-hand side, we easily deduce the estimates (7.3) and (7.4).

The fact that the constants above do not depend on $x_0 \in \overline{\Omega} \setminus \omega$ can be tracked in the above proof: it comes from uniformity properties of the diffeomorphism $x \mapsto y$, and relies heavily on the uniform bounds (1.2)–(1.3), on the fact that $\varphi \in C^3(\overline{\Omega})$, and that the constants in Theorem 2.4 depend only on c_0, m_* and M_* in (2.4), and (2.5) for $X_0 < 0 < X_1$ with $|X_0|, |X_1| \leqslant 1$. This ends the proof of Lemma 7.1.

7.3. A gluing argument: end of the proof of Theorem 1.1

We then perform a gluing argument, which essentially consists in integrating the local Carleman estimates (7.3) and (7.4), or rather the square of these estimates, with respect to $x_0 \in \overline{\Omega} \setminus \omega$, in order to deduce estimates (1.5) and (1.6), respectively. We will only explain how to deduce estimate (1.5) from the estimate (7.3), since the other argument is completely similar.

We thus start from (7.3): There exist constants $C > 0$ and $\tau_0 \geqslant 1$ such that for all $x_0 \in \overline{\Omega} \setminus \omega$ and $\tau \geqslant \tau_0$,

$$\tau^3 \|w_{x_0}\|^2_{L^2(\Omega)} + \tau^{\frac{3}{2}-\frac{1}{d}} \|w_{x_0}\|^2_{L^{\frac{2d}{d-2}}(\Omega)} + \tau \|\nabla w_{x_0}\|^2_{L^2(\Omega)}$$
$$\leqslant C \left(\|f_{2,x_0}\|^2_{L^2(\Omega)} + \tau^{\frac{3}{2}-\frac{1}{d}} \|f_{2'_*,x_0}\|^2_{L^{\frac{2d}{d+2}}(\Omega)} + \tau^2 \|F_{x_0}\|^2_{L^2(\Omega)} + \tau^{\frac{3}{2}} \|g_{x_0}\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \right).$$

Using the explicit expressions of the source terms, we obtain:

$$\tau^3 \|w_{x_0}\|^2_{L^2(\Omega)} + \tau^{\frac{3}{2}-\frac{1}{d}} \|w_{x_0}\|^2_{L^{\frac{2d}{d-2}}(\Omega)} + \tau \|\nabla w_{x_0}\|^2_{L^2(\Omega)}$$
$$\leqslant C \left(\left\|\eta_{x_0} \widetilde{f}_2\right\|^2_{L^2(\Omega)} + \tau^{\frac{3}{2}-\frac{1}{d}} \left\|\eta_{x_0} \widetilde{f}_{2'_*}\right\|^2_{L^{\frac{2d}{d+2}}(\Omega)} + \tau^2 \left\|\eta_{x_0} \widetilde{F}\right\|^2_{L^2(\Omega)} + \tau^{\frac{3}{2}} \|\eta_{x_0} \widetilde{g}\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \right)$$
$$+ C \left(\left\|\nabla \eta_{x_0} \cdot \widetilde{F}\right\|^2_{L^2(\Omega)} + \tau^2 \|w_{x_0}\|^2_{L^2(\Omega)} + \|\nabla \eta_{x_0} \cdot \nabla w\|^2_{L^2(\Omega)} + \|\Delta \eta_{x_0} w\|^2_{L^2(\Omega)} \right.$$
$$\left. + \tau^2 \||\nabla \eta_{x_0}| w\|^2_{L^2(\Omega)} \right).$$

By taking $\tau_0 \geqslant 1$ larger if necessary (which can be done uniformly in $x_0 \in \overline{\Omega} \setminus \omega$), we can absorb the term $\tau^2 \|w_{x_0}\|^2_{L^2(\Omega)}$, and we get for all $x_0 \in \overline{\Omega} \setminus \omega$, for all $\tau \geqslant \tau_0$,

$$\tau^3 \|w_{x_0}\|^2_{L^2(\Omega)} + \tau^{\frac{3}{2}-\frac{1}{d}} \|w_{x_0}\|^2_{L^{\frac{2d}{d-2}}(\Omega)} + \tau \|\nabla w_{x_0}\|^2_{L^2(\Omega)}$$
$$\leqslant C \left(\left\|\eta_{x_0} \widetilde{f}_2\right\|^2_{L^2(\Omega)} + \tau^{\frac{3}{2}-\frac{1}{d}} \left\|\eta_{x_0} \widetilde{f}_{2'_*}\right\|^2_{L^{\frac{2d}{d+2}}(\Omega)} + \tau^2 \left\|\eta_{x_0} \widetilde{F}\right\|^2_{L^2(\Omega)} + \tau^{\frac{3}{2}} \|\eta_{x_0} \widetilde{g}\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \right)$$
$$+ C \left(\left\|\nabla \eta_{x_0} \cdot \widetilde{F}\right\|^2_{L^2(\Omega)} + \|\nabla \eta_{x_0} \cdot \nabla w\|^2_{L^2(\Omega)} + \|\Delta \eta_{x_0} w\|^2_{L^2(\Omega)} + \tau^2 \||\nabla \eta_{x_0}| w\|^2_{L^2(\Omega)} \right).$$

Now, integrating in x_0 on $\Omega \setminus \omega$ and using Fubini's identity for the Hilbertian norms, we get

$$\begin{aligned}\tau^3 \int_\Omega \rho_0(x)|w(x)|^2\,dx + \tau \int_\Omega \rho_0(x)|\nabla w(x)|^2\,dx + \tau^{\frac{3}{2}-\frac{1}{d}} \int_{\Omega\setminus\omega} \|\eta_{x_0} w\|^2_{L^{\frac{2d}{d-2}}(\Omega)}\,dx_0 \\ \leqslant C\left(\int_\Omega \rho_0(x)\left|\widetilde{f}_2(x)\right|^2\,dx + \int_\Omega (\tau^2\rho_0(x)+\rho_{r,1}(x))\left|\widetilde{F}(x)\right|^2\,dx\right) \\ + C\left(\tau^{\frac{3}{2}-\frac{1}{d}}\int_{\Omega\setminus\omega}\left\|\eta_{x_0}\widetilde{f}_{2'_*}\right\|^2_{L^{\frac{2d}{d+2}}(\Omega)}\,dx_0 + \tau^{\frac{3}{2}}\int_{\Omega\setminus\omega}\|\eta_{x_0}\widetilde{g}\|^2_{H^{\frac{1}{2}}(\partial\Omega)}\,dx_0\right) \\ + C\left(\int_\Omega\left(\rho_{r,2}(x)+\tau^2\rho_{r,1}(x)\right)|w(x)|^2\,dx + \int_\Omega \rho_{r,1}(x)|\nabla w(x)|^2\,dx\right),\end{aligned}$$

where the weights ρ_0, $\rho_{r,i}$ are defined as follows:

$$\rho_0(x)=\int_{\Omega\setminus\omega}|\eta_{x_0}(x)|^2\,dx_0,\quad \rho_{r,1}(x)=\int_{\Omega\setminus\omega}|\nabla\eta_{x_0}(x)|^2\,dx_0,\quad \rho_{r,2}(x)=\int_{\Omega\setminus\omega}|\Delta\eta_{x_0}(x)|^2\,dx_0.$$

Taking an open subset ω_a such that $\overline{\omega} \subset \omega_a$ and $\overline{\omega_a} \subset \omega_1$, it is easy to check from the choice (7.5) that

$$\forall\, x \in \overline{\Omega}\setminus\omega_a,\quad |\rho_0(x)| \geqslant \frac{\tau^{-\frac{d}{3}}}{3}\,\|\eta\|^2_{L^2}, \tag{7.10}$$
$$\forall\, x\in\Omega,\quad |\rho_0(x)| \leqslant C\tau^{-\frac{d}{3}},$$

$$\begin{aligned}\forall\, x\in\Omega,\quad |\rho_{r,1}(x)| \leqslant C\tau^{\frac{2}{3}-\frac{d}{3}},\\ \forall\, x\in\Omega,\quad |\rho_{r,2}(x)| \leqslant C\tau^{\frac{4}{3}-\frac{d}{3}}.\end{aligned} \tag{7.11}$$

Thus, for τ large enough,

$$\begin{aligned}\tau^3\int_{\Omega\setminus\omega_a}|w(x)|^2\,dx + \tau\int_{\Omega\setminus\omega_a}|\nabla w(x)|^2\,dx + \tau^{\frac{3}{2}+\frac{d}{3}-\frac{1}{d}}\int_{\Omega\setminus\omega}\|\eta_{x_0}w\|^2_{L^{\frac{2d}{d-2}}(\Omega)}\,dx_0 \\ \leqslant C\int_\Omega\left|\widetilde{f}_2(x)\right|^2\,dx + C\left(\tau^2\int_\Omega\left|\widetilde{F}(x)\right|^2\,dx + \tau^{\frac{3}{2}+\frac{d}{3}-\frac{1}{d}}\int_{\Omega\setminus\omega}\left\|\eta_{x_0}\widetilde{f}_{2'_*}\right\|^2_{L^{\frac{2d}{d+2}}(\Omega)}\,dx_0\right) \\ + C\left(\tau^{\frac{3}{2}+\frac{d}{3}}\int_{\Omega\setminus\omega}\|\eta_{x_0}\widetilde{g}\|^2_{H^{\frac{1}{2}}(\partial\Omega)}\,dx_0 + \tau^{\frac{8}{3}}\int_\omega|w(x)|^2\,dx + \tau^{\frac{2}{3}}\int_\omega|\nabla w(x)|^2\,dx\right).\end{aligned}$$

We then add

$$\tau^3\int_{\omega_a}|w(x)|^2\,dx + \tau\int_{\omega_a}|\nabla w(x)|^2\,dx + \tau^{\frac{3}{2}+\frac{d}{3}-\frac{1}{d}}\int_{\omega_a}\|\eta_{x_0}w\|^2_{L^{\frac{2d}{d-2}}(\Omega)}\,dx_0$$

to both sides of the previous estimate and get

$$\begin{aligned}(7.12)\quad &\tau^3\int_\Omega |w(x)|^2\,dx+\tau\int_\Omega|\nabla w(x)|^2\,dx+\tau^{\frac32+\frac d3-\frac1d}\int_\Omega \|\eta_{x_0}w\|^2_{L^{\frac{2d}{d-2}}(\Omega)}\,dx_0\\ &\leqslant C\left(\int_\Omega\left|\widetilde f_2(x)\right|^2\,dx+\tau^2\int_\Omega\left|\widetilde F(x)\right|^2\,dx+\tau^{\frac32+\frac d3-\frac1d}\int_{\Omega\setminus\omega}\left\|\eta_{x_0}\widetilde f_{2'_*}\right\|^2_{L^{\frac{2d}{d+2}}(\Omega)}\,dx_0\right)\\ &\quad+C\left(\tau^{\frac32+\frac d3}\int_{\Omega\setminus\omega}\|\eta_{x_0}\widetilde g\|^2_{H^{\frac12}(\partial\Omega)}\,dx_0+\tau^3\int_{\omega_a}|w(x)|^2\,dx+\tau\int_{\omega_a}|\nabla w(x)|^2\,dx\right)\\ &\qquad+C\tau^{\frac32+\frac d3-\frac1d}\int_{\omega_a}\|\eta_{x_0}w\|^2_{L^{\frac{2d}{d-2}}(\Omega)}\,dx_0.\end{aligned}$$

We claim the following lemma, proven in Appendix B:

Lemma 7.2. — *With η_{x_0} as in (7.5), there exists a constant $C>0$ such that for all $\tau\geqslant\tau_0$ and $g\in H^{\frac12}(\partial\Omega)$,*

$$(7.13)\qquad \tau^{\frac32+\frac d3}\int_{\Omega\setminus\omega}\|\eta_{x_0}g\|^2_{H^{\frac12}(\partial\Omega)}\,dx_0\leqslant C\left(\tau^{\frac32}\|g\|^2_{H^{\frac12}(\partial\Omega)}+\tau^{11/6}\|g\|^2_{L^2(\partial\Omega)}\right).$$

We thus have the bound

$$\tau^{\frac32+\frac d3}\int_{\Omega\setminus\omega}\|\eta_{x_0}\widetilde g\|^2_{H^{\frac12}(\partial\Omega)}\,dx_0\leqslant C\left(\tau^{\frac32}\|\widetilde g\|^2_{H^{\frac12}(\partial\Omega)}+\tau^{11/6}\|\widetilde g\|^2_{L^2(\partial\Omega)}\right).$$

Now, $\widetilde g$ is the trace of the function $w\in H^1(\Omega)$. Taking $X\in W^{1,\infty}(\Omega;\mathbb R^d)$ such that $X\cdot n=1$ on $\partial\Omega$,

$$\begin{aligned}\|\widetilde g\|^2_{L^2(\partial\Omega)}=\int_\Omega \operatorname{div}\left(X|w|^2\right)\,dx&=\int_\Omega\operatorname{div}(X)|w|^2\,dx+2\int_\Omega X\cdot\nabla w w\,dx\\ &\leqslant C\|w\|^2_{L^2(\Omega)}+\|w\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)}.\end{aligned}$$

It follows that the term $\tau^{11/6}\|\widetilde g\|^2_{L^2(\partial\Omega)}$ can in fact be absorbed by the left-hand side of (7.12) by taking τ_0 larger if necessary, and we obtain:

$$\begin{aligned}&\tau^3\int_\Omega |w(x)|^2\,dx+\tau\int_\Omega|\nabla w(x)|^2\,dx+\tau^{\frac32+\frac d3-\frac1d}\int_\Omega \|\eta_{x_0}w\|^2_{L^{\frac{2d}{d-2}}(\Omega)}\,dx_0\\ &\leqslant C\Bigg(\int_\Omega\left|\widetilde f_2(x)\right|^2dx+\tau^2\int_\Omega\left|\widetilde F(x)\right|^2dx+\tau^{\frac32}\,\|\widetilde g\|^2_{H^{\frac12}(\partial\Omega)}\\ &\quad+\tau^{\frac32+\frac d3-\frac1d}\int_{\Omega\setminus\omega}\left\|\eta_{x_0}\widetilde f_{2'_*}\right\|^2_{L^{\frac{2d}{d+2}}(\Omega)}\,dx_0+\tau^3\int_{\omega_a}|w(x)|^2\,dx\\ &\quad+\tau\int_{\omega_a}|\nabla w(x)|^2\,dx+\tau^{\frac32+\frac d3-\frac1d}\int_{\omega_a}\|\eta_{x_0}w\|^2_{L^{\frac{2d}{d-2}}(\Omega)}\,dx_0\Bigg).\end{aligned}$$

We then go back to the variable u and get the following estimate:

$$\begin{aligned}(7.14)\quad & \tau^3 \left\|e^{\tau\varphi}u\right\|^2_{L^2(\Omega)} + \tau \left\|e^{\tau\varphi}\nabla u\right\|^2_{L^2(\Omega)} + \tau^{\frac{3}{2}+\frac{d}{3}-\frac{1}{d}} \int_\Omega \left\|\eta_{x_0} u e^{\tau\varphi}\right\|^2_{L^{\frac{2d}{d-2}}(\Omega)} dx_0 \\ & \leqslant C\Big(\left\|e^{\tau\varphi}f_2\right\|^2_{L^2(\Omega)} + \tau^{\frac{3}{2}+\frac{d}{3}-\frac{1}{d}} \int_\Omega \left\|\eta_{x_0} f_{2'_*} e^{\tau\varphi}\right\|^2_{L^{\frac{2d}{d+2}}(\Omega)} dx_0 + \tau^2 \left\|e^{\tau\varphi}F\right\|^2_{L^2(\Omega)} \\ & \quad + \tau^{\frac{3}{2}} \left\|e^{\tau\varphi}g\right\|^2_{H^{\frac{1}{2}}(\partial\Omega)} + \tau^3 \left\|e^{\tau\varphi}u\right\|^2_{L^2(\omega_a)} \\ & \quad + \tau \left\|e^{\tau\varphi}\nabla u\right\|^2_{L^2(\omega_a)} + \tau^{\frac{3}{2}+\frac{d}{3}-\frac{1}{d}} \int_{\omega_a} \left\|\eta_{x_0} u e^{\tau\varphi}\right\|^2_{L^{\frac{2d}{d-2}}(\Omega)} dx_0\Big).\end{aligned}$$

We finally explain how to remove the term $\tau\|e^{\tau\varphi}\nabla u\|^2_{L^2(\omega_a)}$ from the right-hand side of (7.14). In order to do that, we choose an open subset ω_b of Ω such that $\overline{\omega_a} \subset \omega_b$ and $\overline{\omega}_b \subset \omega_1$, and a smooth compactly supported function η_ω, taking value 1 in ω_a and vanishing in $\overline{\Omega} \setminus \omega_b$. We then multiply (1.4) by $\eta_\omega u e^{2\tau\varphi}$, which yields:

$$\begin{aligned}\int_\Omega \eta_\omega |\nabla u|^2 e^{2\tau\varphi}\, dx = \int_\Omega \eta_\omega f_2 u e^{2\tau\varphi}\, dx + \int_\Omega \eta_\omega f_{2'_*} u e^{2\tau\varphi}\, dx - \int_\Omega F \cdot \nabla\left(\eta_\omega e^{2\tau\varphi}\right) u\, dx \\ - \int_\Omega F \cdot \nabla u\, \eta_\omega e^{2\tau\varphi}\, dx + \frac{1}{2}\int_\Omega \Delta\left(\eta_\omega e^{2\tau\varphi}\right) |u|^2\, dx.\end{aligned}$$

Using the bound,

$$\left|\int_\Omega F \cdot \nabla u \eta_\omega e^{2\tau\varphi}\, dx\right| \leqslant \frac{1}{2}\|F e^{\tau\varphi}\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega \eta_\omega |\nabla u|^2 e^{2\tau\varphi}\, dx,$$

we easily get

$$\begin{aligned}\left\|e^{\tau\varphi}\nabla u\right\|^2_{L^2(\omega_a)} \leqslant C\left\|f_2 e^{\tau\varphi}\right\|_{L^2(\Omega)} \left\|u e^{\tau\varphi}\right\|_{L^2(\omega_b)} + C\left\|F e^{\tau\varphi}\right\|^2_{L^2(\Omega)} \\ + C\tau \left\|F e^{\tau\varphi}\right\|_{L^2(\Omega)} \left\|u e^{\tau\varphi}\right\|_{L^2(\omega_b)} + C\tau^2 \left\|u e^{\tau\varphi}\right\|^2_{L^2(\omega_b)} + C\left|\int_\Omega \eta_\omega f_{2'_*} u e^{2\tau\varphi}\, dx\right|.\end{aligned}$$

Only the last term is unusual. In order to estimate it, we remark that, for all $x \in \omega_b$ (recall that ω_1, hence ω_b, is at a positive distance from $\partial\Omega$), taking $\tau_0 \geqslant 1$ larger if necessary, for all $\tau \geqslant \tau_0$,

$$\int_{x_0 \in \Omega} \eta_{x_0}(x)^2\, dx_0 = \|\eta\|^2_{L^2\left(\mathbb{R}^d\right)} \tau^{-\frac{d}{3}}.$$

Accordingly,

$$\begin{aligned}\int_\Omega \eta_\omega f_{2'_*} u e^{2\tau\varphi}\, dx &= \frac{\tau^{\frac{d}{3}}}{\|\eta\|^2_{L^2(\mathbb{R}^d)}} \int_{x_0\, \mathrm{i}\, n\Omega} \int_{x \in \Omega} \eta_\omega f_{2'_*} u e^{2\tau\varphi} \eta_{x_0}(x)^2\, dx\, dx_0 \\ &\leqslant C\tau^{\frac{d}{3}} \int_{x_0 \in \Omega} \left\|\eta_{x_0} f_{2'_*} e^{\tau\varphi}\right\|_{L^{\frac{2d}{d+2}}(\Omega)} \left\|\eta_{x_0} u e^{\tau\varphi}\right\|_{L^{\frac{2d}{d-2}}(\omega_b)} dx_0 \\ &\leqslant C\tau^{\frac{d}{3}} \left\|\eta_{x_0} f_{2'_*} e^{\tau\varphi}\right\|_{L^2_{x_0}\left(\Omega; L_x^{\frac{2d}{d+2}}(\Omega)\right)} \left\|\eta_{x_0} u e^{\tau\varphi}\right\|_{L^2_{x_0}\left(\Omega; L_x^{\frac{2d}{d-2}}(\omega_b)\right)}.\end{aligned}$$

One then easily gets that

$$\begin{aligned}
&\tau \left\|e^{\tau\varphi}\nabla u\right\|_{L^2(\omega_a)}^2 \\
&\quad\leqslant C\Bigg(\left\|e^{\tau\varphi} f_2\right\|_{L^2(\Omega)}^2 + \tau^{\frac{3}{2}+\frac{d}{3}-\frac{1}{d}} \left\|\eta_{x_0} f_{2'_*} e^{\tau\varphi}\right\|_{L^2_{x_0}(\Omega; L_x^{\frac{2d}{d+2}}(\Omega))}^2 + \tau^2 \left\|e^{\tau\varphi} F\right\|_{L^2(\Omega)}^2 \\
&\qquad + \tau^{\frac{3}{2}} \left\|e^{\tau\varphi} g\right\|_{H^{\frac{1}{2}}(\partial\Omega)}^2 + \tau^3 \left\|e^{\tau\varphi} u\right\|_{L^2(\omega_b)}^2 + \tau^{\frac{3}{2}+\frac{d}{3}-\frac{1}{d}} \left\|\eta_{x_0} u e^{\tau\varphi}\right\|_{L^2_{x_0}\left(\Omega; L_x^{\frac{2d}{d-2}}(\omega_b)\right)}^2\Bigg),
\end{aligned}$$

which concludes the proof of (1.5) since we obviously have

$$\left\|\eta_{x_0} u e^{\tau\varphi}\right\|_{L^2_{x_0}\left(\Omega; L_x^{\frac{2d}{d-2}}(\omega_b)\right)}^2 \leqslant \left\|\eta_{x_0} u e^{\tau\varphi}\right\|_{L^2_{x_0}\left(\omega_1; L_x^{\frac{2d}{d-2}}(\omega_1)\right)}^2$$

for τ_0 sufficiently large so that $\tau_0^{\frac{1}{3}} \geqslant 1/d(\omega_b, \Omega \setminus \omega_1)$.

8. Proof of Theorem 1.3: Quantitative unique continuation

First, by restricting ω if necessary, we assume that ω is a non-empty open subset of Ω with $\overline{\omega} \subset \Omega$. Then, for ω_0 a non-empty open subset such that $\overline{\omega_0} \subset \omega$, there exists a function φ satisfying conditions (1.2)–(1.3) in ω_0 (see for instance [FI96, Lemma 1.1] or [LRLR22, Proposition 3.31]), so that the Carleman estimates (1.5)–(1.6) in Theorem 1.1 with $\omega_1 = \omega$ hold.

For $V \in L^{q_0}(\Omega)$, $W_1 \in L^{q_1}(\Omega; \mathbb{C}^d)$, and $W_2 \in L^{q_2}(\Omega; \mathbb{C}^d)$, we consider decompositions of the form

$$\begin{aligned}
V &= V_{\frac{d}{2}} + V_d + V_\infty, && \text{with } V_{\frac{d}{2}} \in L^{\frac{d}{2}}(\Omega),\ V_d \in L^d(\Omega),\ V_\infty \in L^\infty(\Omega), \\
W_1 &= W_{1,d} + W_{1,\infty}, && \text{with } W_{1,d} \in L^d\left(\Omega; \mathbb{C}^d\right),\ W_{1,\infty} \in L^\infty\left(\Omega; \mathbb{C}^d\right), \\
W_2 &= W_{2,d} + W_{2,\infty}, && \text{with } W_{2,d} \in L^d\left(\Omega; \mathbb{C}^d\right),\ W_{2,\infty} \in L^\infty\left(\Omega; \mathbb{C}^d\right),
\end{aligned}$$

which will be made precise later.

In particular, applying (1.5) for u solution of (1.4), with $f_{2'_*} = V_{\frac{d}{2}} u + V_d u + W_{1,d} \cdot \nabla u$, $f_2 = V_\infty u + W_{1,\infty} \cdot \nabla u$, and using (1.9) and (1.12), we get the existence of $C_1 > 0$ such that for $\tau \geqslant \tau_0$:

$$
\begin{aligned}
(8.1)\quad & \tau^{\frac{3}{2}}\|e^{\tau\varphi}u\|_{L^2(\Omega)}+\tau^{\frac{1}{2}}\|e^{\tau\varphi}\nabla u\|_{L^2(\Omega)}\\
&\leqslant C_1\left\|e^{\tau\varphi}(V_\infty u+W_{1,\infty}\cdot\nabla u)\right\|_{L^2(\Omega)}\\
&\quad + C_1\tau^{\frac{3}{4}-\frac{1}{2d}}\left\|e^{\tau\varphi}\left(V_{\frac{d}{2}}u+V_d u+W_{1,d}\cdot\nabla u\right)\right\|_{L_{\eta,\tau}^{\frac{2d}{d+2}}(\Omega)}\\
&\quad + C_1\tau\left\|e^{\tau\varphi}\left(W_{2,d}u+W_{2,\infty}u\right)\right\|_{L^2(\Omega)}\\
&\quad + C_1\left(\tau^{\frac{3}{2}}\left\|e^{\tau\varphi}u\right\|_{L^2(\omega)}+\tau^{\frac{3}{4}}\left\|e^{\tau\varphi}u\right\|_{L_{\eta,\tau}^{\frac{2d}{d-2}}(\omega)}\right)\\
&\leqslant C_1\left(\|V_\infty\|_{L^\infty(\Omega)}+\tau^{\frac{3}{4}-\frac{1}{2d}}\|V_d\|_{L^d(\Omega)}+\tau\|W_{2,\infty}\|_{L^\infty(\Omega)}\right)\|e^{\tau\varphi}u\|_{L^2(\Omega)}\\
&\quad + C_1\left(\|W_{1,\infty}\|_{L^\infty(\Omega)}+\tau^{\frac{3}{4}-\frac{1}{2d}}\|W_{1,d}\|_{L^d(\Omega)}\right)\|e^{\tau\varphi}\nabla u\|_{L^2(\Omega)}\\
&\quad + C_1\left(\tau^{\frac{3}{4}-\frac{1}{2d}}\left\|V_{\frac{d}{2}}\right\|_{L^{\frac{d}{2}}(\Omega)}+\tau\|W_{2,d}\|_{L^d(\Omega)}\right)\|e^{\tau\varphi}u\|_{L_{\eta,\tau}^{\frac{2d}{d-2}}(\Omega)}\\
&\quad + C_1\tau^{\frac{3}{2}}\|e^{\tau\varphi}u\|_{L^{\frac{2d}{d-2}}(\omega)}.
\end{aligned}
$$

Similarly, applying (1.6) with $f_{2'_*}=V_{\frac{d}{2}}u+W_{1,d}\cdot\nabla u$, $f_2=V_d u+V_\infty u+W_{1,\infty}\cdot\nabla u$, we obtain the existence of a constant $C_2>0$ such that for $\tau\geqslant\tau_0$,

$$
\begin{aligned}
(8.2)\quad & \tau^{\frac{3}{4}+\frac{1}{2d}}\|e^{\tau\varphi}u\|_{L_{\eta,\tau}^{\frac{2d}{d-2}}(\Omega)}\leqslant C_2\left(\|V_\infty\|_{L^\infty(\Omega)}+\tau\|W_{2,\infty}\|_{L^\infty(\Omega)}\right)\|e^{\tau\varphi}u\|_{L^2(\Omega)}\\
&\quad + C_2\left(\|W_{1,\infty}\|_{L^\infty(\Omega)}+\tau^{\frac{3}{4}+\frac{1}{2d}}\|W_{1,d}\|_{L^d(\Omega)}\right)\|e^{\tau\varphi}\nabla u\|_{L^2(\Omega)}\\
&\quad + C_2\left(\|V_d\|_{L^d(\Omega)}+\tau^{\frac{3}{4}+\frac{1}{2d}}\|V_{\frac{d}{2}}\|_{L^{\frac{d}{2}}(\Omega)}+\tau\|W_{2,d}\|_{L^d(\Omega)}\right)\|e^{\tau\varphi}u\|_{L_{\eta,\tau}^{\frac{2d}{d-2}}(\Omega)}\\
&\quad + C_2\tau^{\frac{3}{2}}\left\|e^{\tau\varphi}u\right\|_{L^{\frac{2d}{d-2}}(\omega)}.
\end{aligned}
$$

Thus, if we have

$$
2C_1\left(\|V_\infty\|_{L^\infty(\Omega)}+\tau^{\frac{3}{4}-\frac{1}{2d}}\|V_d\|_{L^d(\Omega)}+\tau\|W_{2,\infty}\|_{L^\infty(\Omega)}\right)\leqslant\tau^{\frac{3}{2}}, \tag{8.3}
$$

$$
2C_1\left(\|W_{1,\infty}\|_{L^\infty(\Omega)}+\tau^{\frac{3}{4}-\frac{1}{2d}}\|W_{1,d}\|_{L^d(\Omega)}\right)\leqslant\tau^{\frac{1}{2}}, \tag{8.4}
$$

$$
2C_2\left(\|V_d\|_{L^d(\Omega)}+\tau^{\frac{3}{4}+\frac{1}{2d}}\left\|V_{\frac{d}{2}}\right\|_{L^{\frac{d}{2}}(\Omega)}+\tau\|W_{2,d}\|_{L^d(\Omega)}\right)\leqslant\tau^{\frac{3}{4}+\frac{1}{2d}}, \tag{8.5}
$$

estimates (8.1)–(8.2) yield:

$$
\begin{aligned}
&\tau^{\frac{3}{2}}\left\|e^{\tau\varphi}u\right\|_{L^2(\Omega)}+\tau^{\frac{1}{2}}\left\|e^{\tau\varphi}\nabla u\right\|_{L^2(\Omega)}\\
&\quad\leqslant 2C_1\left(\tau^{\frac{3}{4}-\frac{1}{2d}}\left\|V_{\frac{d}{2}}\right\|_{L^{\frac{d}{2}}(\Omega)}+\tau\|W_{2,d}\|_{L^d(\Omega)}\right)\|e^{\tau\varphi}u\|_{L_{\eta,\tau}^{\frac{2d}{d-2}}(\Omega)}+2C_1\tau^{\frac{3}{2}}\left\|e^{\tau\varphi}u\right\|_{L^{\frac{2d}{d-2}}(\omega)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\tau^{\frac{3}{4}+\frac{1}{2d}}\left\|e^{\tau\varphi}u\right\|_{L_{\eta,\tau}^{\frac{2d}{d-2}}(\Omega)}\leqslant 2C_2\left(\|V_\infty\|_{L^\infty(\Omega)}+\tau\|W_{2,\infty}\|_{L^\infty(\Omega)}\right)\|e^{\tau\varphi}u\|_{L^2(\Omega)}\\
&+2C_2\left(\|W_{1,\infty}\|_{L^\infty(\Omega)}+\tau^{\frac{3}{4}+\frac{1}{2d}}\|W_{1,d}\|_{L^d(\Omega)}\right)\|e^{\tau\varphi}\nabla u\|_{L^2(\Omega)}+2C_2\tau^{\frac{3}{2}}\left\|e^{\tau\varphi}u\right\|_{L^{\frac{2d}{d-2}}(\omega)}.
\end{aligned}
$$

Following, under conditions (8.3)–(8.4)–(8.5), we also have

$$\begin{aligned}
&\tau^{\frac{3}{2}} \left\|e^{\tau\varphi}u\right\|_{L^2(\Omega)} + \tau^{\frac{1}{2}} \left\|e^{\tau\varphi}\nabla u\right\|_{L^2(\Omega)} \\
&\quad\leqslant 4C_1C_2\tau^{-\frac{3}{4}-\frac{1}{2d}} \left(\tau^{\frac{3}{4}-\frac{1}{2d}} \left\|V_{\frac{d}{2}}\right\|_{L^{\frac{d}{2}}(\Omega)} + \tau\|W_{2,d}\|_{L^d(\Omega)}\right) \\
&\qquad\qquad \left(\|V_\infty\|_{L^\infty(\Omega)} + \tau\|W_{2,\infty}\|_{L^\infty(\Omega)}\right) \|e^{\tau\varphi}u\|_{L^2(\Omega)} \\
&\quad + 4C_1C_2\tau^{-\frac{3}{4}-\frac{1}{2d}} \left(\tau^{\frac{3}{4}-\frac{1}{2d}}\|V_{\frac{d}{2}}\|_{L^{\frac{d}{2}}(\Omega)} + \tau\|W_{2,d}\|_{L^d(\Omega)}\right) \\
&\qquad\qquad \left(\|W_{1,\infty}\|_{L^\infty(\Omega)} + \tau^{\frac{3}{4}+\frac{1}{2d}}\|W_{1,d}\|_{L^d(\Omega)}\right) \times \|e^{\tau\varphi}\nabla u\|_{L^2(\Omega)} \\
&\quad +2C_1 \left(C_2 \left(\tau^{\frac{3}{4}-\frac{1}{2d}}\|V_{\frac{d}{2}}\|_{L^{\frac{d}{2}}(\Omega)} + \tau\|W_{2,d}\|_{L^d(\Omega)}\right) \tau^{\frac{3}{4}} + \tau^{\frac{3}{2}}\right) \|e^{\tau\varphi}u\|_{L^{\frac{2d}{d-2}}(\omega)}.
\end{aligned}$$

Note that this estimate yields an observation estimate if the additional following conditions are also satisfied:

$$\begin{aligned}
8C_1C_2\tau^{-\frac{3}{4}-\frac{1}{2d}} &\left(\tau^{\frac{3}{4}-\frac{1}{2d}} \left\|V_{\frac{d}{2}}\right\|_{L^{\frac{d}{2}}(\Omega)} + \tau\|W_{2,d}\|_{L^d(\Omega)}\right) \\
&\times \left(\|V_\infty\|_{L^\infty(\Omega)} + \tau\|W_{2,\infty}\|_{L^\infty(\Omega)}\right) \leqslant \tau^{\frac{3}{2}},
\end{aligned} \tag{8.6}$$

$$\begin{aligned}
8C_1C_2\tau^{-\frac{3}{4}-\frac{1}{2d}} &\left(\tau^{\frac{3}{4}-\frac{1}{2d}} \left\|V_{\frac{d}{2}}\right\|_{L^{\frac{d}{2}}(\Omega)} + \tau\|W_{2,d}\|_{L^d(\Omega)}\right) \\
&\times \left(\|W_{1,\infty}\|_{L^\infty(\Omega)} + \tau^{\frac{3}{4}+\frac{1}{2d}}\|W_{1,d}\|_{L^d(\Omega)}\right) \leqslant \tau^{\frac{1}{2}}.
\end{aligned} \tag{8.7}$$

Indeed, in this case, one would obtain

$$\begin{aligned}
&\tau^{\frac{3}{2}}\|e^{\tau\varphi}u\|_{L^2(\Omega)} + \tau^{\frac{1}{2}}\|e^{\tau\varphi}\nabla u\|_{L^2(\Omega)} \\
&\quad\leqslant 2C_1 \left(C_2 \left(\tau^{\frac{3}{4}-\frac{1}{2d}} \left\|V_{\frac{d}{2}}\right\|_{L^{\frac{d}{2}}(\Omega)} + \tau\|W_{2,d}\|_{L^\infty(\Omega)}\right) \tau^{\frac{3}{4}} + \tau^{\frac{3}{2}}\right) \|e^{\tau\varphi}u\|_{L^{\frac{2d}{d-2}}(\omega)}.
\end{aligned} \tag{8.8}$$

Therefore, our next step is to understand how, given $V \in L^{q_0}(\Omega)$, $W_1 \in L^{q_1}(\Omega;\mathbb{C}^d)$, and $W_2 \in L^{q_2}(\Omega;\mathbb{C}^d)$, one can minimize the value of τ for which we can find decompositions such that conditions (8.3)–(8.4)–(8.5)–(8.6)–(8.7) are satisfied.

Before going further, let us remark that (8.3)–(8.4)–(8.5)–(8.6)–(8.7) are satisfied provided, for $c_0 > 0$ small enough,

$$\begin{cases}
\|V_\infty\|_{L^\infty(\Omega)} \leqslant c_0\tau^{\frac{3}{2}}, \quad \|V_d\|_{L^d(\Omega)} \leqslant c_0\tau^{\frac{3}{4}+\frac{1}{2d}}, \quad \left\|V_{\frac{d}{2}}\right\|_{L^{\frac{d}{2}}(\Omega)} \leqslant c_0, \\
\|W_{1,\infty}\|_{L^\infty(\Omega)} + \|W_{2,\infty}\|_{L^\infty(\Omega)} \leqslant c_0\tau^{\frac{1}{2}}, \\
\tau^{\frac{1}{4}-\frac{1}{2d}} \left(\|W_{1,d}\|_{L^d(\Omega)} + \|W_{2,d}\|_{L^d(\Omega)}\right) \leqslant c_0, \\
\tau^{\frac{1}{2}}\|W_{1,d}\|_{L^d(\Omega)}\|W_{2,d}\|_{L^d(\Omega)} \leqslant c_0.
\end{cases} \tag{8.9}$$

The case $V \in L^{q_0}(\Omega)$ with $q_0 \in (\frac{d}{2}, d]$. For $V \in L^{q_0}(\Omega)$ with $q_0 \in (\frac{d}{2}, d]$ and $\lambda_0 > 0$ to be chosen later, we set $V_{\frac{d}{2}} = = V1_{|V|>\lambda_0}$, $V_d = V1_{|V|\leqslant\lambda_0}$, $V_\infty = 0$, for which we have the following estimates:

$$\lambda_0^{q_0-\frac{d}{2}} \left\|V_{\frac{d}{2}}\right\|^{\frac{d}{2}}_{L^{\frac{d}{2}}(\Omega)} \leqslant \|V\|^{q_0}_{L^{q_0}(\Omega)}, \qquad \|V_d\|^d_{L^d(\Omega)} \leqslant \lambda_0^{d-q_0}\|V\|^{q_0}_{L^{q_0}(\Omega)}.$$

Similarly, for $W_1 \in L^{q_1}(\Omega)$ and $W_2 \in L^{q_2}(\Omega)$ and λ_1, λ_2 positive numbers to be chosen later, we set $W_{1,d} = W_1 1_{|W_1| > \lambda_1}$, $W_{1,\infty} = W_1 1_{|W_1| \leqslant \lambda_1}$, and $W_{2,d} = W_2 1_{|W_2| > \lambda_2}$, $W_{2,\infty} = W_2 1_{|W_2| \leqslant \lambda_2}$, for which we have the estimates:

$$\left\{ \begin{array}{ll} \lambda_1^{q_1 - d} \|W_{1,d}\|_{L^d(\Omega)}^d \leqslant \|W_1\|_{L^{q_1}(\Omega)}^{q_1}, & \|W_{1,\infty}\|_{L^\infty(\Omega)} \leqslant \lambda_1, \\ \lambda_2^{q_2 - d} \|W_{2,d}\|_{L^d(\Omega)}^d \leqslant \|W_2\|_{L^{q_2}(\Omega)}^{q_2}, & \|W_{2,\infty}\|_{L^\infty(\Omega)} \leqslant \lambda_2. \end{array} \right. \tag{8.10}$$

Conditions (8.9) are thus satisfied provided

$$\left\{ \begin{array}{l} \lambda_0^{1-\frac{q_0}{d}} \|V\|_{L^{q_0}(\Omega)}^{\frac{q_0}{d}} \leqslant c_0 \tau^{\frac{3}{4}+\frac{1}{2d}}, \quad \lambda_0^{1-2\frac{q_0}{d}} \|V\|_{L^{q_0}(\Omega)}^{\frac{2q_0}{d}} \leqslant c_0, \\ \lambda_1 + \lambda_2 \leqslant c_0 \tau^{\frac{1}{2}}, \quad \tau^{\frac{1}{4}-\frac{1}{2d}} \left(\lambda_1^{1-\frac{q_1}{d}} \|W_1\|_{L^{q_1}(\Omega)}^{\frac{q_1}{d}} + \lambda_2^{1-\frac{q_2}{d}} \|W_2\|_{L^{q_2}(\Omega)}^{\frac{q_2}{d}} \right) \leqslant c_0, \\ \tau^{\frac{1}{2}} \lambda_1^{1-\frac{q_1}{d}} \|W_1\|_{L^{q_1}(\Omega)}^{\frac{q_1}{d}} \lambda_2^{1-\frac{q_2}{d}} \|W_2\|_{L^{q_2}(\Omega)}^{\frac{q_2}{d}} \leqslant c_0. \end{array} \right. \tag{8.11}$$

We then choose $\lambda_0 = \tau^{\alpha_0} \|V\|_{L^{q_0}(\Omega)}$, $\lambda_1 = \tau^{\alpha_1} \|W_1\|_{L^{q_1}(\Omega)}$ and $\lambda_2 = \tau^{\alpha_2} \|W_2\|_{L^{q_2}(\Omega)}$ for some real parameters α_0, α_1, α_2, so that conditions (8.11) yield:

$$\left\{ \begin{array}{l} \tau^{\alpha_0\left(1-\frac{q_0}{d}\right)-\frac{3}{4}-\frac{1}{2d}} \|V\|_{L^{q_0}(\Omega)} \leqslant c_0, \quad \tau^{\alpha_0\left(1-2\frac{q_0}{d}\right)} \|V\|_{L^{q_0}(\Omega)} \leqslant c_0, \\ \tau^{\alpha_1 - \frac{1}{2}} \|W_1\|_{L^{q_1}(\Omega)} + \tau^{\alpha_2 - \frac{1}{2}} \|W_2\|_{L^{q_2}(\Omega)} \leqslant c_0, \\ \tau^{\frac{1}{4}-\frac{1}{2d}+\alpha_1\left(1-\frac{q_1}{d}\right)} \|W_1\|_{L^{q_1}(\Omega)} + \tau^{\frac{1}{4}-\frac{1}{2d}+\alpha_2\left(1-\frac{q_2}{d}\right)} \|W_2\|_{L^{q_2}(\Omega)} \leqslant c_0, \\ \tau^{\frac{1}{2}+\alpha_1\left(1-\frac{q_1}{d}\right)+\alpha_2\left(1-\frac{q_2}{d}\right)} \|W_1\|_{L^{q_1}(\Omega)} \|W_2\|_{L^{q_2}(\Omega)} \leqslant c_0. \end{array} \right. \tag{8.12}$$

For $q_0 \in (\frac{d}{2}, d]$, $q_1 > \frac{3d-2}{2}$, $q_2 > \frac{3d-2}{2}$ with $\frac{1}{q_1} + \frac{1}{q_2} < 4(1 - \frac{1}{d})/(3d-2)$, we choose $\alpha_0 = (\frac{3}{4} + \frac{1}{2d})\frac{d}{q_0}$, $\alpha_1 = (\frac{3}{4} - \frac{1}{2d})\frac{d}{q_1}$, $\alpha_2 = (\frac{3}{4} - \frac{1}{2d})\frac{d}{q_2}$, so that system (8.12) is satisfied provided, for some C large enough,

$$\left\{ \begin{array}{l} \tau^{(2-\frac{d}{q_0})\left(\frac{3}{4}+\frac{1}{2d}\right)} \geqslant C \|V\|_{L^{q_0}(\Omega)}, \\ \tau^{\frac{1}{2}-\left(\frac{3}{4}-\frac{1}{2d}\right)\frac{d}{q_1}} \geqslant C \|W_1\|_{L^{q_1}(\Omega)}, \\ \tau^{\frac{1}{2}-\left(\frac{3}{4}-\frac{1}{2d}\right)\frac{d}{q_2}} \geqslant C \|W_2\|_{L^{q_2}(\Omega)}, \\ \tau^{1-\frac{1}{d}-\left(\frac{3}{4}-\frac{1}{2d}\right)\left(\frac{d}{q_1}+\frac{d}{q_2}\right)} \geqslant C \|W_1\|_{L^{q_1}(\Omega)} \|W_2\|_{L^{q_2}(\Omega)}, \end{array} \right. \tag{8.13}$$

that is, with the notations of Theorem 1.3,

$$\tau \geqslant C \left(\|V\|_{L^{q_0}(\Omega)}^{\gamma(q_0)} + \|W_1\|_{L^{q_1}(\Omega)}^{\delta(q_1)} + \|W_2\|_{L^{q_2}(\Omega)}^{\delta(q_2)} + \left(\|W_1\|_{L^{q_1}(\Omega)} \|W_2\|_{L^{q_2}(\Omega)} \right)^{\rho(q_1,q_2)} \right). \tag{8.14}$$

For $q_0 \in (\frac{d}{2}, d]$, $q_1 > \frac{3d}{2}$, and $q_2 > \frac{3d}{2}$, one can alternatively choose $\alpha_0 = (\frac{3}{4} + \frac{1}{2d})\frac{d}{q_0}$, $\alpha_1 = \frac{3d}{4q_1}$, $\alpha_2 = \frac{3d}{4q_2}$, so that system (8.12) is satisfied provided, for some C large enough,

$$\tau^{\left(2-\frac{d}{q_0}\right)\left(\frac{3}{4}+\frac{1}{2d}\right)} \geqslant C \|V\|_{L^{q_0}(\Omega)}, \quad \tau^{\frac{1}{2}-\frac{3d}{4q_1}} \geqslant C \|W_1\|_{L^{q_1}(\Omega)}, \quad \tau^{\frac{1}{2}-\frac{3d}{4q_2}} \geqslant C \|W_2\|_{L^{q_2}(\Omega)},$$

that is, with the notations of Theorem 1.3,

$$\tau \geqslant C \left(\|V\|_{L^{q_0}(\Omega)}^{\gamma(q_0)} + \|W_1\|_{L^{q_1}(\Omega)}^{\widetilde{\delta}(q_1)} + \|W_2\|_{L^{q_2}(\Omega)}^{\widetilde{\delta}(q_2)} \right). \tag{8.15}$$

Taking τ large enough that saturates condition (8.14), respectively condition (8.15), bounding the weight function $e^{\tau\varphi}$ from below and from above in (8.8), we easily deduce Theorem 1.3 item 1, respectively item 2, for $q_0 \in (\frac{d}{2}, d]$.

The case $V \in L^{q_0}(\Omega)$ with $q_0 \in [d,\infty]$. For $V \in L^{q_0}(\Omega)$ with $q_0 \in [d,\infty]$ and $\lambda_0 > 0$ to be chosen later, we set $V_{\frac{d}{2}} = 0$, $V_d = V 1_{|V|>\lambda_0}$, $V_\infty = V 1_{|V|\leqslant\lambda_0}$, for which we have the following estimates:

$$\lambda_0^{q_0-d}\|V_d\|^d_{L^d(\Omega)} \leqslant \|V\|^{q_0}_{L^{q_0}(\Omega)}, \qquad \|V_\infty\|_{L^\infty(\Omega)} \leqslant \lambda_0.$$

The potentials $W_1 \in L^{q_1}(\Omega)$ and $W_2 \in L^{q_2}(\Omega)$ are decomposed as before $W_1 = W_{1,d} + W_{1,\infty}$, $W_2 = W_{2,d} + W_{2,\infty}$ with the estimates (8.10) for positive parameters λ_1 and λ_2 to be chosen later. Similarly as before, conditions (8.9) are thus satisfied provided

$$\text{(8.16)}\quad \begin{cases} \lambda_0^{1-\frac{q_0}{d}}\|V\|^{\frac{q_0}{d}}_{L^{q_0}(\Omega)} \leqslant c_0\tau^{\frac{3}{4}+\frac{1}{2d}}, \quad \lambda_0 \leqslant c_0\tau^{\frac{3}{2}}, \\ \lambda_1+\lambda_2 \leqslant c_0\tau^{\frac12}, \quad \tau^{\frac14-\frac{1}{2d}}\left(\lambda_1^{1-\frac{q_1}{d}}\|W_1\|^{\frac{q_1}{d}}_{L^{q_1}(\Omega)} + \lambda_2^{1-\frac{q_2}{d}}\|W_2\|^{\frac{q_2}{d}}_{L^{q_2}(\Omega)}\right) \leqslant c_0, \\ \tau^{\frac12}\lambda_1^{1-\frac{q_1}{d}}\|W_1\|^{\frac{q_1}{d}}_{L^{q_1}(\Omega)}\lambda_2^{1-\frac{q_2}{d}}\|W_2\|^{\frac{q_2}{d}}_{L^{q_2}(\Omega)} \leqslant c_0. \end{cases}$$

Similarly as before, for $q_0 \in [d,\infty]$, $q_1 > \frac{3d-2}{2}$, $q_2 > \frac{3d-2}{2}$ with $\frac{1}{q_1}+\frac{1}{q_2} < 4(1-\frac{1}{d})/(3d-2)$, we choose $\alpha_0 = (\frac34-\frac{1}{2d})\frac{d}{q_0}$, $\alpha_1 = (\frac34-\frac{1}{2d})\frac{d}{q_1}$, $\alpha_2 = (\frac34-\frac{1}{2d})\frac{d}{q_2}$, and $\lambda_0 = \tau^{\alpha_0}\|V\|_{L^{q_0}(\Omega)}$, $\lambda_1 = \tau^{\alpha_1}\|W_1\|_{L^{q_1}(\Omega)}$ and $\lambda_2 = \tau^{\alpha_2}\|W_2\|_{L^{q_2}(\Omega)}$. We then deduce that system (8.16) is satisfied provided, for some C large enough,

$$\text{(8.17)}\quad \begin{cases} \tau^{\left(2-\frac{d}{q_0}\right)\left(\frac34-\frac{1}{2d}\right)} \geqslant C\|V\|_{L^{q_0}(\Omega)}, \\ \tau^{\frac12-\left(\frac34-\frac{1}{2d}\right)\frac{d}{q_1}} \geqslant C\|W_1\|_{L^{q_1}(\Omega)}, \\ \tau^{\frac12-\left(\frac34-\frac{1}{2d}\right)\frac{d}{q_2}} \geqslant C\|W_2\|_{L^{q_2}(\Omega)}, \\ \tau^{1-\frac1d-\left(\frac34-\frac{1}{2d}\right)\left(\frac{d}{q_1}+\frac{d}{q_2}\right)} \geqslant C\|W_1\|_{L^{q_1}(\Omega)}\|W_2\|_{L^{q_2}(\Omega)}, \end{cases}$$

that is, (8.14) with the notations of Theorem 1.3.

Here again, for $q_0 \in [d,\infty]$, $q_1 > \frac{3d}{2}$, and $q_2 > \frac{3d}{2}$, one can alternatively choose $\alpha_0 = (\frac34-\frac{1}{2d})\frac{d}{q_0}$, $\alpha_1 = \frac{3d}{4q_1}$, $\alpha_2 = \frac{3d}{4q_2}$, so that conditions (8.16) are implied by

$$\tau^{\left(\frac32-\frac{d}{q_0}\right)\left(\frac34+\frac{1}{2d}\right)} \geqslant C\|V\|_{L^{q_0}(\Omega)}, \quad \tau^{\frac12-\frac{3d}{4q_1}} \geqslant C\|W_1\|_{L^{q_1}(\Omega)}, \quad \tau^{\frac12-\frac{3d}{4q_2}} \geqslant C\|W_2\|_{L^{q_2}(\Omega)},$$

that is (8.15) with the notations of Theorem 1.3.

We then deduce Theorem 1.3 in the case $q \geqslant d$ immediately from (8.8) as in the case $q \leqslant d$.

Remark 8.1. — In fact, if we focus on the conditions

$$\text{(8.18)}\quad \begin{cases} \lambda_1+\lambda_2 \leqslant c_0\tau^{\frac12}, \quad \tau^{\frac14-\frac{1}{2d}}\left(\lambda_1^{1-\frac{q_1}{d}}\|W_1\|^{\frac{q_1}{d}}_{L^{q_1}(\Omega)} + \lambda_2^{1-\frac{q_2}{d}}\|W_2\|^{\frac{q_2}{d}}_{L^{q_2}(\Omega)}\right) \leqslant c_0, \\ \tau^{\frac12}\lambda_1^{1-\frac{q_1}{d}}\|W_1\|^{\frac{q_1}{d}}_{L^{q_1}(\Omega)}\lambda_2^{1-\frac{q_2}{d}}\|W_2\|^{\frac{q_2}{d}}_{L^{q_2}(\Omega)} \leqslant c_0, \end{cases}$$

which appear in the second and third lines of system (8.11) and (8.16), and choose $\lambda_1 = \tau^{\alpha_1}\|W_1\|_{L^{q_1}(\Omega)}$ and $\lambda_2 = \tau^{\alpha_2}\|W_2\|_{L^{q_2}(\Omega)}$, one can find τ large enough so that system (8.18) is satisfied provided

$$\tag{8.19} \begin{cases} \alpha_1 < \dfrac{1}{2}, \quad \alpha_2 < \dfrac{1}{2}, \\ \alpha_1\Big(\dfrac{q_1}{d} - 1\Big) > \dfrac{1}{4} - \dfrac{1}{2d}, \quad \alpha_2\Big(\dfrac{q_2}{d} - 1\Big) > \dfrac{1}{4} - \dfrac{1}{2d}, \\ \qquad\qquad \alpha_1\Big(\dfrac{q_1}{d} - 1\Big) + \alpha_2\Big(\dfrac{q_2}{d} - 1\Big) > \dfrac{1}{2}. \end{cases}$$

Indeed, in this case, it suffices to take, for a sufficiently large constant C,

$$\tag{8.20} \tau \geqslant C\left(\|W_1\|_{L^{q_1}(\Omega)}^{a_{q_1}(\alpha_1)} + \|W_2\|_{L^{q_2}(\Omega)}^{a_{q_2}(\alpha_2)} + \Big(\|W_1\|_{L^{q_1}(\Omega)}\|W_2\|_{L^{q_2}(\Omega)}\Big)^{b(\alpha_1,\alpha_2)}\right),$$

with

$$a_q(\alpha) = \max\left\{\frac{1}{\frac{1}{2} - \alpha}, \frac{1}{\alpha\left(\frac{q}{d} - 1\right) - \frac{1}{4} + \frac{1}{2d}}\right\},$$

$$b_{q_1,q_2}(\alpha_1,\alpha_2) = \frac{1}{\alpha_1\left(\frac{q_1}{d} - 1\right) + \alpha_2\left(\frac{q_2}{d} - 1\right) - \frac{1}{2}}.$$

Although it is rather easy to check that the system (8.19) admits solutions (α_1, α_2) if q_1 and q_2 satisfy $q_1 > \frac{3d}{2} - 1$, $q_2 > \frac{3d}{2} - 1$ and $q_1 + q_2 > 3d$, it is not clear how to choose α_1 and α_2 satisfying (8.19) to minimize τ in (8.20). We have thus decided in the above proof of Theorem 1.3 to restrict ourselves to the case in which both terms in $a_q(\alpha)$ are equal (this choice corresponds to item 1 in Theorem 1.3), or to consider, instead of (8.18), the sufficient conditions

$$\lambda_1 + \lambda_2 \leqslant c_0\tau^{\frac{1}{2}}, \quad \tau^{\frac{1}{4}}\left(\lambda_1^{1-\frac{q_1}{d}}\|W_1\|_{L^{q_1}(\Omega)}^{\frac{q_1}{d}} + \lambda_2^{1-\frac{q_2}{d}}\|W_2\|_{L^{q_2}(\Omega)}^{\frac{q_2}{d}}\right) \leqslant c_0,$$

this choice yielding to item (2) in Theorem 1.3.

Appendix A. Reminder of some classical results in harmonic analysis

We start by recalling the classical Hardy–Littlewood–Sobolev theorem.

Theorem A.1 ([Hör90, Theorem 4.5.3.] Hardy–Littlewood–Sobolev theorem). *Let $n \in \mathbb{N}$. For $(p,q,r) \in (1,\infty)^3$ such that*

$$\frac{1}{r} = 1 - \left(\frac{1}{p} - \frac{1}{q}\right),$$

there exists a constant $C_{p,q,n}$ such that for all $f \in L^p(\mathbb{R}^n)$,

$$\left\| x \mapsto \int_{\mathbb{R}^n} |x-y|^{-\frac{n}{r}} f(y)\,dy \right\|_{L^q(\mathbb{R}^n)} \leqslant C_{p,q,n}\|f\|_{L^p(\mathbb{R}^n)}.$$

In the article, we have also used the stationary phase lemma. Although it is a very classical lemma of harmonic analysis, we used the following version proved in [ABZ17], which presents the advantage of quantifying precisely the constants in the stationary phase lemma:

Theorem A.2 ([ABZ17, Theorem 1] Stationary phase lemma). — *Let* $\Phi \in C^\infty(\mathbb{R}^d;\mathbb{R})$, $b \in C_0^\infty(\mathbb{R}^d;\mathbb{R})$. *For* $\lambda \in \mathbb{R}$, *we introduce*

$$I_{\Phi,b}(\lambda) = \int_{\mathbf{R}^d} e^{i\lambda\Phi(\xi)} b(\xi) d\xi.$$

Set $K = \operatorname{supp} b$ *and let* V *be an open neighbourhood of* K, *and use the following notations:*

- $\mathcal{M}_{d+2} := \sum_{2 \leqslant |\alpha| \leqslant d+2} \sup_{\xi \in V} \left| D_\xi^\alpha \Phi(\xi) \right|$,
- $\mathcal{N}_{d+1} := \sum_{|\alpha| \leqslant d+1} \sup_{\xi \in K} \left| D_\xi^\alpha b(\xi) \right|$,

and assume that there exists $a_0 > 0$ *such that for all* $\xi \in V$, $|\det(\operatorname{Hess}\Phi(\xi))| \geqslant a_0$, *where* $\operatorname{Hess}\Phi(\xi)$ *denotes the Hessian matrix of* Φ *at* ξ.

There exists a constant C *independent of* (Φ, b) *satisfying the above assumptions, such that for all* $\lambda \geqslant 1$,

$$|I_{\Phi,b}(\lambda)| \leqslant \frac{C}{a_0^{1+d}} \left(1 + \mathcal{M}_{d+2}^{\frac{d}{2}+d^2} \right) \mathcal{N}_{d+1} \lambda^{-\frac{d}{2}}.$$

Appendix B. Proof of Lemma 7.2

Proof. — The proof of Lemma 7.2 relies on a suitable interpolation estimate.

First, for $\tau \geqslant \tau_0$, we define the operator $\Lambda_\tau : L^2(\partial\Omega) \to L^2(\Omega \setminus \omega; L^2(\partial\Omega))$ by

$$\Lambda_\tau g(x_0) = \eta_{x_0,\tau}(\cdot) g(\cdot), \qquad \text{for } x_0 \in \Omega \setminus \omega,$$

where we recall that $\eta_{x_0,\tau}$ is the function given by $\eta_{x_0,\tau}(x) = \eta(\tau^{\frac{1}{3}}(x - x_0))$ for $x \in \mathbb{R}^d$, for a smooth compactly supported function η.

Using (7.10)–(7.11), it is easy to check that there exists a constant $C > 0$ such that

$$\forall\, g \in L^2(\partial\Omega), \quad \|\Lambda_\tau g\|^2_{L^2(\Omega\setminus\omega; L^2(\partial\Omega))} \leqslant C\tau^{-\frac{d}{3}} \|g\|^2_{L^2(\partial\Omega)},$$
$$\forall\, g \in H^1(\partial\Omega), \quad \|\Lambda_\tau g\|^2_{L^2(\Omega\setminus\omega; H^1(\partial\Omega))} \leqslant C\tau^{-\frac{d}{3}} \left(\|g\|^2_{H^1(\partial\Omega)} + \tau^{\frac{2}{3}} \|g\|^2_{L^2(\partial\Omega)} \right).$$

We can then deduce easily the estimate (7.13). More precisely, by interpolation, Λ_τ maps $H^{\frac{1}{2}}(\partial\Omega)$ to $L^2(\Omega \setminus \omega; H^{\frac{1}{2}}(\partial\Omega))$. To estimate the operator in this norm with appropriate powers of τ, we proceed as follows. We let $(\Phi_j)_{j \in \mathbb{N}}$ be the basis of eigenfunctions of the Laplace Beltrami operator $-\Delta$ on $\partial\Omega$, with corresponding eigenvalues $(\lambda_j^2)_{j \in \mathbb{N}}$, which are non-negative and going to infinity. Accordingly, for $g = \sum_j a_j \Phi_j$, the $L^2(\partial\Omega)$, $H^1(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$ norms of g can be read as, respectively, $\|(a_j)\|_{\ell^2(\mathbb{N})}$, $\|(a_j(\lambda_j + 1))\|_{\ell^2(\mathbb{N})}$, and $\|(a_j(\lambda_j + 1)^{\frac{1}{2}})\|_{\ell^2(\mathbb{N})}$.

Writing $g \in H^{\frac{1}{2}}(\partial\Omega)$ under the form $g = \sum_{j \in \mathbb{N}} a_j \Phi_j$, we then introduce the function

$$f(z) = \sum_{j \in \mathbb{N}} a_j (\lambda_j + 1 + \tau^{\frac{1}{3}})^{\frac{1}{2} - z} \Lambda_\tau \Phi_j, \qquad z \in \mathbb{C} \text{ with } \Re(z) \in [0,1].$$

The function f is holomorphic in $\{z \in \mathbb{C} \text{ with } \Re(z) \in (0,1)\}$ with values in $L^2(\Omega \setminus \omega; L^2(\partial\Omega))$, $f(\frac{1}{2}) = \Lambda_\tau g$, and

$$\forall\, \beta \in \mathbb{R}, \quad \|f(i\beta)\|^2_{L^2(\Omega\setminus\omega;L^2(\partial\Omega))} \leqslant C\tau^{-\frac{d}{3}} \sum_j |a_j|^2 \left(\lambda_j + 1 + \tau^{\frac{1}{3}}\right),$$

$$\forall\, \beta \in \mathbb{R}, \quad \|f(1+i\beta)\|^2_{L^2(\Omega\setminus\omega;H^1(\partial\Omega))} \leqslant C\tau^{-\frac{d}{3}} \sum_j |a_j|^2 \left(\lambda_j + 1 + \tau^{\frac{1}{3}}\right).$$

Since $L^2(\Omega \setminus \omega; H^{\frac{1}{2}}(\partial\Omega)) = [L^2(\Omega \setminus \omega; L^2(\partial\Omega)), L^2(\Omega \setminus \omega; H^1(\partial\Omega))]_{\frac{1}{2}}$, we deduce from the above estimates that there exists $C > 0$ such that for all $g \in H^{\frac{1}{2}}(\partial\Omega)$ and $\tau \geqslant \tau_0$,

$$\|\Lambda_\tau g\|^2_{L^2\left(\Omega\setminus\omega;H^{\frac{1}{2}}(\partial\Omega)\right)} \leqslant C\tau^{-\frac{d}{3}} \left(\|g\|^2_{H^{\frac{1}{2}}(\partial\Omega)} + \tau^{\frac{1}{3}}\|g\|^2_{L^2(\partial\Omega)}\right).$$

This concludes the proof of Lemma 7.2. □

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