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ASYMPTOTIC STABILITY OF A WIDE CLASS OF STATIONARY SOLUTIONS FOR THE HARTREE AND SCHRÖDINGER EQUATIONS FOR INFINITELY MANY PARTICLES

STABILITÉ ASYMPTOTIQUE D'UNE LARGE CLASSE DE SOLUTIONS STATIONNAIRES POUR LES ÉQUATIONS DE HARTREE ET DE SCHRÖDINGER POUR UN NOMBRE INFINI DE PARTICULES

Abstract. — We consider the Hartree and Schrödinger equations describing the time evolution of wave functions of infinitely many interacting fermions in three-dimensional space. These equations can be formulated using density operators, and they have infinitely many

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stationary solutions. In this paper, we prove the asymptotic stability of a wide class of stationary solutions. We emphasize that our result includes Fermi gas at zero temperature. This is one of the most important steady states from the physics point of view; however, its asymptotic stability has been left open after the seminal work of Lewin and Sabin [LS14], which first formulated this stability problem and gave significant results.

Résumé. — Nous considérons les équations de Hartree et de Schrödinger décrivant l'évolution temporelle des fonctions d'onde d'un nombre infini de fermions en interaction dans un espace tridimensionnel. Ces équations peuvent être formulées à l'aide d'opérateurs de densité, et elles ont une infinité de solutions stationnaires. Dans cet article, nous prouvons la stabilité asymptotique d'une large classe de solutions stationnaires. Nous insistons sur le fait que notre résultat inclut le gaz de Fermi à température nulle. Il s'agit de l'un des états stationnaires les plus importants du point de vue de la physique ; cependant, sa stabilité asymptotique est restée ouverte après les travaux séminaux de Lewin et Sabin [LS14], qui ont formulé pour la première fois ce problème de stabilité et ont donné des résultats significatifs.

1. Introduction

In this paper, we study the following Hartree equation in three-dimensional space:

$$i\partial_t\gamma = [-\Delta + w * \rho_\gamma, \gamma], \quad \gamma : \mathbb{R} \to \mathcal{B}(L^2_x). \tag{NLH}$$

This is a nonlinear evolution equation for operator-valued functions. We denote the set of all bounded linear operators on $L^2_x := L^2(\mathbb{R}^3)$ by $\mathcal{B}(L^2_x)$, convolution in space by $*$ and commutator by $[\cdot,\cdot]$. We assume that w is a given finite signed Borel measure on $\mathbb{R}^3$. For $A \in \mathcal{B}(L^2_x)$, $\rho_A(x) : \mathbb{R}^3_x \to \mathbb{C}$ is the density function of A, that is, $\rho_A(x) := k(x,x)$, where $k(x,y)$ is the integral kernel of A. When $w = \pm\delta$, we should call (NLH) the cubic nonlinear Schrödinger equation; however, we consistently call (NLH) the Hartree equation in the rest of this paper.

We briefly explain the background and fundamental properties of (NLH) for convenience according to [LS15]. See [LS15, Introduction] for more details.

Wave functions of N fermions in three-dimensional space evolve according to a linear Schrödinger equation on $\mathbb{R}^{3N}$. The Hartree equation is one of the approximations of this equation:

$$\begin{aligned} &i\partial_t u_n(t,x) \\ &\quad = \left(-\Delta_x + w * \left(\sum_{j=1}^N |u_j(t,x)|^2\right)\right) u_n(t,x), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^3, \quad (n = 1,\dots N). \end{aligned} \tag{1.1}$$

If we try to deal with the case $N = \infty$, namely, the Hartree equation for infinitely many particles, it seems somewhat difficult to do our analysis in this form. Thus, it is a good idea to rewrite the original equation (1.1). For a solution to (1.1), $(u_n(t))_{n=1}^N \subset C(\mathbb{R}, L^2_x)$, define the operator-valued function $\gamma : \mathbb{R} \to \mathcal{B}(L^2_x)$ by

$$\gamma(t) = \sum_{n=1}^N |u_n(t)\rangle\langle u_n(t)|,$$

where $|f\rangle\langle g| : \phi \mapsto \langle g|\phi\rangle_{L^2_x} f$. It is quite easy to see that $\gamma(t)$ solves (NLH). (NLH) is more suitable for our analysis because it is not dependent on the number of particles N.

The following proposition is easy to prove, but it is one of the most essential properties of (NLH).

Proposition 1.1. — *Let w be a finite Borel measure on $\mathbb{R}^3$. Let $f \in L^1_\xi \cap L^\infty_\xi$. Let $\gamma_f := f(-i\nabla) = \mathcal{F}^{-1} f \mathcal{F}$. Then γ_f is a stationary solution to* (NLH).

By Proposition 1.1, we have infinitely many stationary solutions to (NLH). Furthermore, it is known that some f are important from the physics perspective:

$$f(\xi) = \chi_{\{|\xi|^2 \leqslant \mu\}}(\xi), \quad \text{(Fermi gas at zero temperature and } \mu > 0), \tag{1.2}$$

$$f(\xi) = \frac{1}{e^{(|\xi|^2-\mu)/T} + 1}, \quad \text{(Fermi gas at positive temperature } T \text{ and } \mu \in \mathbb{R}), \tag{1.3}$$

$$f(\xi) = \frac{1}{e^{(|\xi|^2-\mu)/T} - 1}, \quad \text{(Bose gas at positive temperature } T \text{ and } \mu < 0), \tag{1.4}$$

$$f(\xi) = e^{-(|\xi|^2-\mu)/T}, \quad \text{(Boltzmann gas at positive temperature } T \text{ and } \mu \in \mathbb{R}), \tag{1.5}$$

where χ_A is the indicator function of A and each μ is chemical potential. Proposition 1.1 tells us that (NLH) with non-trace class initial values is important since γ_f is not compact unless $f = 0$.

In this paper, we study the asymptotic stability of γ_f. Let $Q(t)$ be a perturbation from γ_f, that is, $Q(t) = \gamma(t) - \gamma_f$. Then we have

$$i\partial_t Q = [-\Delta + w * \rho_Q, \gamma_f + Q]. \tag{f-NLH}$$

We denote the Schatten α-class on a Hilbert space H by $\mathfrak{S}^\alpha(H)$, that is,

$$\|A\|_{\mathfrak{S}^\alpha} := \left(\mathrm{Tr}(|A|^\alpha)\right)^{\frac{1}{\alpha}}, \tag{1.6}$$

where Tr is the trace in H. We write $\mathfrak{S}^\alpha := \mathfrak{S}^\alpha(L^2_x)$. We refer to [Sim05] for more details on the Schatten classes. Note that $\mathrm{Tr}\, A$ is the total number of particles contained in the system for any density operator A, and $\mathrm{Tr}\, A = \infty$ at least formally if $A \in \mathfrak{S}^\alpha \setminus \mathfrak{S}^1$ for some $\alpha > 1$.

The well-posedness of the Cauchy problem of (NLH) with $\gamma(0) = \gamma_0$ has been studied by [BPF74, BPF76, Cha76, Zag92], and recently, the small data scattering is shown in [PS21]. More precisely, the above results dealt with more general nonlinear terms. However, they considered only trace class operators; their arguments are in the framework of finite particle systems. We need to emphasize that Lewin and Sabin *first* formulated (f-NLH) and gave significant results in [LS14, LS15]. In [LS15], they proved the local and global well-posedness of (f-NLH) with non-trace class initial data when the interaction potential w is in $L^1_x \cap L^\infty_x$. In [LS14], they showed asymptotic stability of γ_f for small initial perturbations Q_0 in two-dimensional space. They assumed that interaction potential w is in the Sobolev space $W^{1,1}(\mathbb{R}^2)$ and f is smooth. Chen, Hong and Pavlović extended the above results. In [CHP17], they proved the global well-posedness when the potential w is the delta measure and f is Fermi gas at zero temperature. In [CHP18], they extended the result in [LS14] and proved scattering when $d \geqslant 3$. However, they also assumed that f is smooth and w satisfies somewhat complicated conditions. The Hartree equation with a constant magnetic field also has infinitely many stationary solutions. In [Don21], Dong gave a well-posedness result of the Cauchy problem that initial data are given around these stationary solutions.

Strichartz estimates for orthonormal functions are one of the most essential tools for their analysis. This is first proven in [FLLS14], and developed in [BHL+19, FS17].

In relation to [LS14, LS15], Lewin and Sabin gave rigorous proof that the semi-classical limit of the Hartree equation for infinitely many particles is the Vlasov equation in [LS20]. And they get the global well-posed result for the Vlasov equation as a by-product.

We rewrote (1.1) in terms of density operators on L^2_x, and all of the known results mentioned above are based on this formulation; however, de Suzzoni gave an alternative formulation by random fields in [Suz15]. She clarified the correspondence with the formulation using density operators in detail. For example, we have infinitely many steady states corresponding to γ_f in random fields formulation. In [CdS20], Collot and de Suzzoni proved their asymptotic stability when $d \geqslant 4$, and they extended their result to $d = 2, 3$ in [CdS22]. Remarkably, the general finite Borel measure on $\mathbb{R}^3$ is allowed in [CdS22].

In the previous works, the *asymptotic* stability of γ_f given by (1.3), (1.4) and (1.5), which are physically important examples at positive temperature, was proved. However, the asymptotic stability of Fermi gas at zero temperature (1.2) has been left open since it was mentioned as an open question in the seminal work [LS14]. Moreover, no scattering result allows singular interaction potentials at the density operator level. In this paper, we prove the asymptotic stability of γ_f when f is in a wide class including $\chi_{\{|\xi|^2 \leqslant 1\}}$, allowing general finite Borel measure on $\mathbb{R}^3$.

1.1. Main result

We write $U(t) := e^{it\Delta}$. Define $A_\star B := ABA^*$ for two operators A and B. For $s \geqslant 0$ and $\alpha \in [1, \infty]$, we define the Schatten–Sobolev space $\mathcal{H}^{s,\alpha}$ by

$$\|A\|_{\mathcal{H}^{s,\alpha}} := \|\langle\nabla\rangle^s A \langle\nabla\rangle^s\|_{\mathfrak{S}^\alpha}. \tag{1.7}$$

We write $\mathcal{H}^s := \mathcal{H}^{s,2}$. For w and f, we define a linear operator $\mathcal{L}_1 = \mathcal{L}_1(f, w)$ by

$$\mathcal{L}_1[g](t) := \rho\left(i\int_0^t U(t-\tau)_\star[w * g(\tau), \gamma_f]d\tau\right). \tag{1.8}$$

Our main result is the following:

Theorem 1.2. — *Let w be a given finite signed Borel measure on $\mathbb{R}^3$. Let $f \in L^1_\xi \cap L^\infty_\xi$ satisfy $\|\langle\xi\rangle^2 f\|_{L^1_\xi \cap L^\infty_\xi} < \infty$. Assume that the operator $\mathcal{L}_1 = \mathcal{L}_1(f, w)$ satisfies*

$$\left\|(1 + \mathcal{L}_1)^{-1}\right\|_{\mathcal{B}(L^2_{t,x})} < \infty. \tag{1.9}$$

Then there exists $\varepsilon_0 > 0$ such that the following holds. If $\|Q_0\|_{\mathcal{H}^{1/2,3/2}} \leqslant \varepsilon_0$, then there exists a unique global solution $Q(t) \in C(\mathbb{R}, \mathcal{H}^{\frac{1}{2}})$ to (f-NLH) with $Q(0) = Q_0$ such that $\rho_Q \in L^2_t(\mathbb{R}, H^{\frac{1}{2}}_x)$. Furthermore, $Q(t)$ scatters; that is, there exist $Q_\pm \in \mathfrak{S}^3$ such that

$$U(-t)Q(t)U(t) \to Q_\pm \text{ in } \mathfrak{S}^3 \text{ as } t \to \pm\infty. \tag{1.10}$$

First, we discuss the invertibility of $1+\mathcal{L}_1$. Some sufficient conditions for $(1+\mathcal{L}_1)^{-1}\in\mathcal{B}(L^2_{t,x})$ are known. For example, see [LS14, Corollary 1]. Although the author could not find any clear statement, [LS14, Proposition 1] immediately implies the following:

Proposition 1.3. — *Let w be a finite signed Borel measure on $\mathbb{R}^3$. Let $f\in L^1_\xi\cap L^\infty_\xi$ be real-valued and radial. If*

$$\frac{\|\widehat{w}\|_{L^\infty_\xi}}{2\,|\mathbb{S}^{d-1}|}\int_{\mathbb{R}^d}\frac{\left|\check{f}(x)\right|}{|x|^{d-2}}dx<1\quad\text{or}\quad\left\|\frac{\widehat{w}(\xi)}{|\xi|}\right\|_{L^\infty_\xi}\int_0^\infty\left|\check{f}(r)\right|dr<1 \tag{1.11}$$

holds, then $(1+\mathcal{L}_1(f,w))^{-1}\in\mathcal{B}(L^2_{t,x})$.

Remark 1.4. — We identify a radial function $H:\mathbb{R}^d\to\mathbb{R}$ and $h:[0,\infty)\to\mathbb{R}$ such that $H(x)=h(|x|)$.

Our result includes the Fermi gas at zero temperature because we have the following criteria:

Proposition 1.5 ([Had23, Proposition 1.6]). — *Let w be a finite signed Borel measure on $\mathbb{R}^3$. Let $f(\xi)=\chi_{\{|\xi|^2\leqslant 1\}}$. If $\widehat{w}$ is real-valued and*

$$-\delta_0\leqslant\widehat{w}(\xi)\leqslant\frac{\delta_1}{\langle\log|\xi|\rangle} \tag{1.12}$$

for (small) absolute constants $\delta_0,\delta_1>0$, then $(1+\mathcal{L}_1(f,w))^{-1}\in\mathcal{B}(L^2_tL^2_x)$.

Now we give some remarks about our main results.

Remark 1.6. — When $f(\xi)=\chi_{\{|\xi|^2\leqslant 1\}}$, the scattering holds only in the focusing case by (1.12).

Remark 1.7. — If we include the delta measure as an interaction potential of (NLH), Theorem 1.2 is optimal, meaning cubic NLS in 3D is critical in $H^{\frac{1}{2}}_x$. However, it may be an interesting problem to improve the scattering norm or the function space that the solution $Q(t)$ belongs to at each time. In particular, the author suspects that it may be possible to replace $\mathfrak{S}^3$ in Theorem 1.2 by $\mathcal{H}^{\frac{1}{2},3}$.

1.2. Summary of ideas of the proof of the main result

The rough story of the proof is the same as that of [CHP18, LS14]. First, we reduce the Cauchy problem (f-NLH) with $Q(0)=Q_0$ to the nonlinear equation of the density function ρ_Q (see (IVP*)). After finding a unique global solution ρ_Q, we restore the scattering solution $Q(t)\in C(\mathbb{R},\mathcal{H}^{\frac{1}{2}})$. See Section 2 for more details. Most of our efforts are devoted to finding a unique global density function ρ_Q in $L^2_t(\mathbb{R},H^{\frac{1}{2}}_x)$.

One of the most essential tools in this paper is the Strichartz estimates for orthonormal functions; in particular, we essentially use the estimates proved by Bez, Hong, Lee, Nakamura and Sawano in [BHL+19]. This is the estimate for the free propagator $e^{it\Delta}$, but we need to extend their result to the general propagator $U_V(t)$ (see (2.1) and (IVP*)). To do so, according to [CHP18, LS14], we use the wave operator decomposition of $U_V(t)$ (see (4.30)). We need to emphasize that [BHL$^+$19] appeared after the important previous works [CHP18, LS14] were written, and the estimates established in [BHL$^+$19] enabled us to get the result in this paper.

Another important ingredient in this paper is Christ–Kiselev type lemma in Schatten classes (see Lemma 3.1). It is a direct corollary of the result in [GK70], and it was obtained much earlier than Christ–Kiselev's result [CK01]; therefore, we should call it Gohberg–Kreĭn theorem. Gohberg–Kreĭn theorem and the duality principle, first introduced in [FS17] to prove the orthonormal Strichartz estimates, is also very useful. The author would like to emphasize that both the duality principle and Gohberg–Kreĭn theorem are known results, but it was not noticed that combining them is useful as far as the author knows.

It is natural to find a density function ρ_Q such that $w * \rho_Q \in L^2_t(\mathbb{R}, H^{\frac{1}{2}}_x)$. However, we would like to deal with general finite signed Borel measures on $\mathbb{R}^3$; hence, we need to find ρ_Q in $L^2_t(\mathbb{R}, H^{\frac{1}{2}}_x)$. It means that we need to estimate $\|\rho(U_V(t)_\star Q_0)\|_{L^2_t H^{1/2}_x}$ (see (IVP*)). One of the most difficult parts of this paper is to bound this term, and the difficulty comes from $H^{\frac{1}{2}}_x$. If we replace $H^{\frac{1}{2}}_x$ by H^n_x with integer n, then we can bound it much easier because we have the formula $\nabla\rho_Q = \rho([\nabla, Q])$, and we can easily calculate $[\nabla, U_V(t)]$ explicitly; however, for the fractional derivatives, there does not exist this type of calculation as far as the author knows. Hence, we "interpolate" the bounds of $\|\rho(U_V(t)_\star Q_0)\|_{L^2_t L^2_x}$ and $\|\rho(U_V(t)_\star Q_0)\|_{L^2_t H^1_x}$, but things do not go so straightforwardly because the bounds of these terms are nonlinear with respect to V. To overcome this problem, we decompose $U_V(t)$ into wave operators and then multilinearize them. In this argument, Gohberg–Kreĭn theorem (and duality argument) plays an essential role.

Finally, we will estimate the terms including γ_f. In the first version of this paper, the author said "we can bound them by simple duality arguments". However, it is not true. We will explain this difficulty here. We often need to get estimate in the following form

$$\left\|\int_0^\infty dt U(t)^* \left(\langle\nabla\rangle^{\frac{1}{2}} V\right)(t) U(t) \int_0^t ds U(s)^* W(s) U(s)\right\|_{\mathfrak{S}^\alpha} \lesssim \|V\|_{L^2_t L^p_x} \left\|\langle\nabla\rangle^{\frac{1}{2}} W\right\|_{L^2_t L^q_x}. \tag{1.13}$$

In the most cases, it is pretty easy to get

$$\left\|\int_0^\infty dt U(t)^* \left(\langle\nabla\rangle^{\frac{1}{2}} V\right)(t) U(t) \int_0^\infty ds U(s)^* W(s) U(s)\right\|_{\mathfrak{S}^\alpha} \lesssim \|V\|_{L^2_t L^p_x} \left\|\langle\nabla\rangle^{\frac{1}{2}} W\right\|_{L^2_t L^q_x}. \tag{1.14}$$

However, the author do not know any way to conclude (1.13) from (1.14) directly. To overcome this problem, we first replace $\langle\nabla\rangle^{\frac{1}{2}}$ to $\langle\nabla\rangle^j$ with $j = 0, 1$, and then get similar estimates. Lastly, we interpolate them. See Section 5 for detailed arguments.

1.3. Organization of this paper

This paper is organized as follows. In Section 2, we explain the outline of the proof of the main result. We reduce the Cauchy problem (NLH) with $Q(0) = Q_0$ to the equation of density function ρ_Q. This is the density functional method. In Section 3, we prepare a lot of basic tools. It includes Christ–Kiselev type lemma, resolvent expansion of the propagator U_V, and so on. In Section 4, we prove the key Strichartz estimates in this paper. In Section 5, we show the global and unique existence of the density function ρ_Q. Finally, in Section 6, we restore the solution $Q(t)$ to the original Cauchy problem from its density function ρ_Q.

2. Outline of the proof of the main result

2.1. Reduction to the equation for density functions

Let $U_V(t,s)$ be the propagator for the following linear Schrödinger equation with a time-dependent potential:

$$(i\partial_t + \Delta - V(t,x))u = 0, \quad u(t,x) : \mathbb{R}^{1+3} \to \mathbb{C}, \quad V(t,x) : \mathbb{R}^{1+3} \to \mathbb{R}, \tag{2.1}$$

namely, for any $s \in \mathbb{R}$, $u(t) := U_V(t,s)u(s)$ is a solution to (2.1).

Our strategy is the same as that of [CHP18, LS14]. First, we find a solution to the nonlinear equation of density function ρ_Q:

$$\rho_Q = \rho(U_V(t)_\star Q_0) - \rho\left[i\int_0^t U_V(t,\tau)_\star[V(\tau),\gamma_f]d\tau\right], \tag{IVP}$$

$$V = w * \rho_Q. \tag{2.2}$$

If we find a solution ρ_Q to (IVP), then we can restore the solution to (NLH) with $Q(0) = Q_0$ by

$$Q(t) := U_V(t)_\star Q_0 - i\int_0^t U_V(t,\tau)_\star[V(\tau),\gamma_f]d\tau, \tag{2.3}$$

$$V := w * \rho_Q. \tag{2.4}$$

We can reduce the proof of our main result to finding the solution ρ_Q to (IVP) because we have the following lemma:

Lemma 2.1. — *Let $d = 3$. Let $\langle\xi\rangle^2 f(\xi) \in L^1_\xi \cap L^\infty_\xi$, $Q_0 \in \mathcal{H}^{\frac{1}{2}}$ and $V \in L^2_t(\mathbb{R}, H^{\frac{1}{2}}_x)$. Then $Q(t)$ defined by (2.3) is in $C(\mathbb{R}, \mathcal{H}^{\frac{1}{2}})$. Furthermore, $Q(t)$ scatters; that is, there exist $Q_\pm \in \mathfrak{S}^3$ such that*

$$U(-t)Q(t)U(t) \to Q_\pm \text{ in } \mathfrak{S}^3 \text{ as } t \to \pm\infty. \tag{2.5}$$

2.2. Set-up for the contraction mapping argument

We transform (IVP) to a more suitable form to estimate. Note that $U_V(t,s)$ satisfies

$$\begin{aligned} U_V(t,s) &= U(t-s) - i\int_s^t U(t-\tau)V(\tau)U_V(\tau,s)d\tau \\ &=: U(t-s) + D_V(t,s). \end{aligned} \tag{2.6}$$

Hence, we have

$$\begin{aligned} &-i\int_0^t U_V(t,\tau)_\star[V(\tau),\gamma_f]d\tau \\ &= -i\int_0^t \Big(U(t-\tau)+D_V(t,\tau)\Big)[V(\tau),\gamma_f]\Big(U(\tau-t)+D_V(\tau,t)\Big)d\tau \\ &= -i\int_0^t U(t-\tau)[V(\tau),\gamma_f]U(\tau-t)d\tau - i\int_0^t U(t-\tau)[V(\tau),\gamma_f]D_V(\tau,t)d\tau \\ &\quad - i\int_0^t D_V(t,\tau)[V(\tau),\gamma_f]U(\tau-t)d\tau - i\int_0^t D_V(t,\tau)[V(\tau),\gamma_f]D_V(\tau,t)d\tau \\ &=: L_1[V](t) + N_1[V](t) + N_2[V](t) + N_3[V](t). \end{aligned} \tag{2.7}$$

We rewrite (IVP) and obtain

$$\begin{aligned} \rho_Q &= \rho(U_V(t)_\star Q_0) + \rho(L_1[V]) + \rho(N_1[V]) + \rho(N_2[V]) + \rho(N_3[V]) \\ &= \rho(U_V(t)_\star Q_0) - \mathcal{L}_1[\rho_Q] + \rho(N_1[V]) + \rho(N_2[V]) + \rho(N_3[V]). \end{aligned} \tag{2.8}$$

Since we assumed $(1+\mathcal{L}_1)^{-1} \in \mathcal{B}(L^2_{t,x})$, we get

$$\rho_Q = (1+\mathcal{L}_1)^{-1}(\rho(U_V(t)_\star Q_0) + \rho(N_1[V]) + \rho(N_2[V]) + \rho(N_3[V])). \tag{IVP*}$$

Combining Theorem 2.1, we can reduce the proof of the main result to the following theorem:

Theorem 2.2. — *Under the same assumptions as in Theorem 1.2, there exists a unique global solution $\rho_Q \in L^2_t(\mathbb{R}, H^{\frac{1}{2}}_x)$ to* (IVP*).

3. Preliminaries

In this section, we consider general d-dimensional spaces because there is no benefit to limiting our argument to three-dimensional space.

3.1. Christ–Kiselev type lemma

First, we give a Christ–Kiselev type lemma in Schatten classes; however, the following result is already obtained in [GK70] before [CK01]. Therefore, we should call it Gohberg–Kreĭn theorem. We obtain the following result by applying [BS03, Theorem 7.2] (but originally by [GK70, Theorem III.6.2]), setting $E_t = F_t = P_{L^2((a,t);L^2_x)}$ in their notations, where P_A is a projection to A. Note that the integral kernel of $\mathbb{D} = \mathcal{J}_\theta^{E,F}\mathbb{T}$ is $\theta(t-\tau)K(t,\tau)$, where $\theta(x) = \chi_{\{x\geqslant 0\}}$.

Theorem 3.1 (Gohberg–Kreĭn). — *Let $d \geqslant 1$. Let $-\infty \leqslant a < b \leqslant \infty$, $1 < p < \infty$ and $\alpha \in (1,\infty)$. Let $(a,b)^2 \ni (t,\tau) \mapsto K(t,\tau) \in \mathcal{B}(L^2_x)$ be strongly continuous. Assume that we can write $\mathbb{T} \in \mathcal{B}(L^2_{t,x})$ by*

$$(\mathbb{T}g)(t,x) = \int_a^b K(t,\tau)g(\tau)d\tau. \tag{3.1}$$

If $\mathbb{T} \in \mathfrak{S}^\alpha(L^2_{t,x})$, then $\mathbb{D}$ defined by

$$(\mathbb{D}g)(t,x) = \int_a^t K(t,\tau)g(\tau)d\tau \tag{3.2}$$

is also in $\mathfrak{S}^\alpha(L^2_{t,x})$, and

$$\|\mathbb{D}\|_{\mathfrak{S}^\alpha\left(L^2_{t,x}\right)} \leqslant C_p\|\mathbb{T}\|_{\mathfrak{S}^\alpha\left(L^2_{t,x}\right)} \tag{3.3}$$

holds.

3.2. Duality principle and its application

In this subsection, we assume all Hilbert spaces are separable and infinite-dimensional. Let H, K be complex Hilbert spaces. Let $\alpha \in [1,\infty]$ and $A : H \to K$ be compact. We define Schatten α-norm by

$$\|A\|_{\mathfrak{S}^\alpha(H\to K)} := \begin{cases} \left(\mathrm{Tr}_H\left(|A^*A|^{\frac{\alpha}{2}}\right)\right)^{\frac{1}{\alpha}} & \text{if } 1 \leqslant \alpha < \infty, \\ \|A\|_{\mathcal{B}(H\to K)} & \text{if } \alpha = \infty. \end{cases} \tag{3.4}$$

Note that $\mathfrak{S}^\alpha(H \to H) = \mathfrak{S}^\alpha(H)$. Hölder's inequality for Schatten norm is well-known. If $\alpha, \alpha_0, \alpha_1 \in [0,\infty]$ satisfy $\frac{1}{\alpha} = \frac{1}{\alpha_0} + \frac{1}{\alpha_1}$, then it holds that

$$\|A_0A_1\|_{\mathfrak{S}^\alpha(H)} \leqslant \|A_0\|_{\mathfrak{S}^{\alpha_0}(H)}\|A_1\|_{\mathfrak{S}^{\alpha_1}(H)}. \tag{3.5}$$

Note that (3.5) implies the following. If $\alpha, \alpha_0, \alpha_1 \in [0,\infty]$ satisfy $\frac{1}{\alpha} = \frac{1}{\alpha_0} + \frac{1}{\alpha_1}$, then it follows that

$$\|A_0A_1\|_{\mathfrak{S}^\alpha(H_0\to H_2)} \leqslant \|A_0\|_{\mathfrak{S}^{\alpha_0}(H_1\to H_2)}\|A_1\|_{\mathfrak{S}^{\alpha_1}(H_0\to H_1)}. \tag{3.6}$$

In fact, by the singular value decomposition, we can write $A_0 = U\widetilde{A_0}$, where $\|\widetilde{A_0}\|_{\mathfrak{S}^{\alpha_0}(H_1)} = \|A_0\|_{\mathfrak{S}^{\alpha_0}(H_1\to H_2)}$ and $U : H_1 \to H_2$ is unitary. In the same way, $A_1 = \widetilde{A_1}V$, where $\|\widetilde{A_1}\|_{\mathfrak{S}^{\alpha_1}(H_1)} = \|A_1\|_{\mathfrak{S}^{\alpha_1}(H_0\to H_1)}$ and $V : H_0 \to H_1$ is unitary. Therefore, we obtain

$$\begin{aligned} \|A_0A_1\|_{\mathfrak{S}^\alpha(H_0\to H_2)} &= \|U\widetilde{A_0}\widetilde{A_1}V\|_{\mathfrak{S}^\alpha(H_0\to H_2)} = \|\widetilde{A_0}\widetilde{A_1}\|_{\mathfrak{S}^\alpha(H_1)} \\ &\leqslant \|\widetilde{A_0}\|_{\mathfrak{S}^{\alpha_0}(H_1)}\|\widetilde{A_1}\|_{\mathfrak{S}^{\alpha_1}(H_1)} = \|A\|_{\mathfrak{S}^{\alpha_0}(H_1\to H_2)}\|A_1\|_{\mathfrak{S}^{\alpha_1}(H_0\to H_1)}. \end{aligned} \tag{3.7}$$

For any $\alpha \in [1,\infty]$, define

$$\mathfrak{S}^\alpha := \mathfrak{S}^\alpha\left(L^2_x \to L^2_x\right), \quad \mathfrak{S}^\alpha_{t,x} := \mathfrak{S}^\alpha\left(L^2_{t,x} \to L^2_{t,x}\right), \tag{3.8}$$

$$\mathfrak{S}^\alpha_{x\to(t,x)} := \mathfrak{S}^\alpha\left(L^2_x \to L^2_{t,x}\right), \quad \mathfrak{S}^\alpha_{(t,x)\to x} := \mathfrak{S}^\alpha\left(L^2_{t,x} \to L^2_x\right). \tag{3.9}$$

Let $I \subset \mathbb{R}$ be an interval. Let $A(t) \in \mathcal{B}(L^2_x)$ for all $t \in I$ and $\sup_{t\in I} \|A(t)\|_{\mathcal{B}} < \infty$. Assume that $I \ni t \mapsto A(t) \in \mathcal{B}(L^2_x)$ be strongly continuous. We define $A^{\bigcirc} : L^2_x \to C_t(I, L^2_x)$ and $A^{\circledast} : L^1_t(I, L^2_x) \to L^2_x$ by

$$\left(A^{\bigcirc} u_0\right)(t,x) := (A(t)u_0)(x), \tag{3.10}$$

$$\left(A^{\circledast} f\right)(x) := \int_I A(\tau)^* f(\tau, x) d\tau. \tag{3.11}$$

Note that $A^{\bigcirc}$ and $A^{\circledast}$ are formally adjoint to each other. The following lemma is very important for our analysis, but it is essentially the same thing as [FS17, Lemma 3].

Lemma 3.2 (Duality principle). — *Let $p, q, \alpha \in [1,\infty]$. The following are equivalent:*

(1) *For any $\gamma \in \mathfrak{S}^\alpha$,*

$$\|\rho(A(t)_\star \gamma)\|_{L^p_t(I, L^q_x)} \leqslant C\|\gamma\|_{\mathfrak{S}^\alpha}. \tag{3.12}$$

(2) *For any $f \in L^{2p'}_t(I, L^{2q'}_x)$,*

$$\left\|f A^{\bigcirc}\right\|_{\mathfrak{S}^{2\alpha'}_{x\to(t,x)}} \leqslant C'\|f\|_{L^{2p'}_t\left(I, L^{2q'}_x\right)}. \tag{3.13}$$

(3) *For any $f \in L^{2p'}_t(I, L^{2q'}_x)$,*

$$\|A^{\circledast} f\|_{\mathfrak{S}^{2\alpha'}_{(t,x)\to x}} \leqslant C''\|f\|_{L^{2p'}_t\left(I, L^{2q'}_x\right)}. \tag{3.14}$$

Moreover, $\sqrt{C}$, C' and C'' coincide.

Proof. — Assume (1). Then we have

$$\begin{aligned} \left\|f A^{\bigcirc}\right\|^2_{\mathfrak{S}^{2\alpha'}_{x\to(t,x)}} &= \left\|A^{\circledast}|f|^2 A^{\bigcirc}\right\|_{\mathfrak{S}^{\alpha'}} \\ &= \sup\left\{\left|\mathrm{Tr}\left[\gamma A^{\circledast}|f|^2 A^{\bigcirc}\right]\right| : \|\gamma\|_{\mathfrak{S}^\alpha} \leqslant 1\right\} \\ &= \sup\left\{\left|\int_I \mathrm{Tr}\left[A(t)\gamma A(t)^*|f(t)|^2\right] dt\right| : \|\gamma\|_{\mathfrak{S}^\alpha} \leqslant 1\right\} \\ &\leqslant \sup\left\{\|\rho(A(t)_\star\gamma)\|_{L^p_t(I, L^q_x)}\|f\|^2_{L^{2p'}_t\left(I, L^{2q'}_x\right)} : \|\gamma\|_{\mathfrak{S}^\alpha} \leqslant 1\right\} \\ &\leqslant C\|f\|^2_{L^{2p'}_t\left(I, L^{2q'}_x\right)}. \end{aligned} \tag{3.15}$$

Therefore, we obtain

$$\left\|f A^{\bigcirc}\right\|_{\mathfrak{S}^{2\alpha'}_{x\to(t,x)}} \leqslant \sqrt{C}\|f\|_{L^{2p'}_t\left(I, L^{2q'}_x\right)}. \tag{3.16}$$

We can prove that (2) yields $C \leqslant (C')^2$ similarly. Since

$$\left\|f A^{\bigcirc}\right\|_{\mathfrak{S}^{2\alpha'}_{x\to(t,x)}} = \|A^{\circledast} f\|_{\mathfrak{S}^{2\alpha'}_{(t,x)\to x}}, \tag{3.17}$$

we completed the proof. □

The following lemma is sometimes useful:

LEMMA 3.3. — *Let $I \subset \mathbb{R}$ be an interval. Assume that*

$$\|\rho(A_n(t)_\star\gamma)\|_{L_t^{p_n}\left(I,L_x^{q_n}\right)} \leqslant C_n\|\gamma\|_{\mathfrak{S}^{\alpha_n}} \tag{3.18}$$

for $n = 0, 1$. Then we have

$$\left\|\rho(A_0(t)\gamma A_1(t)^*)\right\|_{L_t^p\left(I,L_x^q\right)} \leqslant \sqrt{C_0C_1}\|\gamma\|_{\mathfrak{S}^\alpha}, \tag{3.19}$$

where

$$\frac{1}{X} = \frac{1}{2}\left(\frac{1}{X_0} + \frac{1}{X_1}\right), \quad X = p, q, \alpha. \tag{3.20}$$

Proof. — Let $f \in L_t^{p'}(I, L_x^{q'})$. f can be decomposed as $f = f_0f_1$, where

$$\|f\|_{L_t^{p'}\left(I,L_x^{q'}\right)} = \|f_0\|_{L_t^{2p'_0}\left(I,L_x^{2q'_0}\right)}\|f_1\|_{L_t^{2p'_1}\left(I,L_x^{2q'_1}\right)}. \tag{3.21}$$

Then we have

$$\begin{aligned}\left|\int_I\int_{\mathbb{R}^d}\rho(A_0(t)\gamma A_1(t)^*)(x)f(t,x)dxdt\right| \qquad\qquad\qquad\qquad\\ \leqslant \left|\int_I \mathrm{Tr}(A_0(t)\gamma A_1(t)^*f(t,x))dt\right|\\ \leqslant \left|\int_I \mathrm{Tr}(A_1(t)^*f(t,x)A_0(t)\gamma)dt\right|\\ \leqslant \left\|A_1^{\circledast}fA_0^{\bigcirc}\right\|_{\mathfrak{S}^{\alpha'}}\|\gamma\|_{\mathfrak{S}^\alpha}\\ \leqslant \|\gamma\|_{\mathfrak{S}^\alpha}\|A_1^{\circledast}f_1\|_{\mathfrak{S}^{2\alpha'_1}_{(t,x)\to x}}\left\|f_0A_0^{\bigcirc}\right\|_{\mathfrak{S}^{2\alpha'_0}_{x\to(t,x)}}\\ \leqslant \sqrt{C_0C_1}\|\gamma\|_{\mathfrak{S}^\alpha}\|f\|_{L_t^{p'}L_x^{q'}}. \quad \square\end{aligned} \tag{3.22}$$

3.3. Resolvent expansion of the propagators

If $\sigma(\mu,\nu) := \frac{2}{\mu} + \frac{d}{\nu} = 2$, $\mu < \infty$ and $V \in L_t^\mu(\mathbb{R}, L_x^\nu)$, there exists a unitary propagator $U_V(t)$. Namely, U_V satisfies

$$U_V(t) = U(t) - i\int_0^t U(t-\tau)V(\tau)U_V(\tau)d\tau. \tag{3.23}$$

Let $V_0 := |V|^{1/2}$ and $V = V_0V_1$. Multiplication with V_0 yields

$$V_0(t)U_V(t) = V_0(t)U(t) - i\int_0^t V_0(t)U(t-\tau)V_1(\tau)V_0(\tau)U_V(\tau)d\tau. \tag{3.24}$$

Define $\mathcal{D}(V_0, V_1)$ by

$$(\mathcal{D}(V_0,V_1)f)(t,x) := -i\int_0^t V_0(t)U(t-\tau)(V_1(\tau)f(\tau))d\tau. \tag{3.25}$$

Then we can write (3.23) and (3.24) as

$$\begin{aligned}U_V^{\bigcirc} &= U^{\bigcirc} + \mathcal{D}(1,V_1)V_0U_V^{\bigcirc},\\ V_0U_V^{\bigcirc} &= V_0U^{\bigcirc} + \mathcal{D}(V_0,V_1)V_0U_V^{\bigcirc}.\end{aligned} \tag{3.26}$$

Hence, we have

$$(1-\mathcal{D}(V_0,V_1))V_0U_V^{\bigcirc}=V_0U^{\bigcirc}. \tag{3.27}$$

If $\mathcal{R}(V_0,V_1):=(1-\mathcal{D}(V_0,V_1))^{-1}\in B(L^2_{t,x})$, we have

$$V_0U_V^{\bigcirc}=\mathcal{R}(V_0,V_1)V_0U^{\bigcirc}. \tag{3.28}$$

Therefore, (3.26) and (3.28) imply

$$U_V^{\bigcirc}=U^{\bigcirc}+\mathcal{D}(1,V_1)\mathcal{R}(V_0,V_1)V_0U^{\bigcirc}. \tag{3.29}$$

The following lemma justifies (3.29).

LEMMA 3.4. — *Let $d\geqslant 1$. Let $\mu\in[1,\infty)$ and $\nu\in[1,\infty]$ satisfy $\sigma(\mu,\nu)=2$. Then there exists a monotone increasing function $\varphi:[0,\infty)\to[0,\infty)$ such that for any $V\in L^\mu_t(\mathbb{R},L^\nu_x)$*

$$\|\mathcal{R}(V_0,V_1)\|_{\mathcal{B}\left(L^2_{t,x}\right)}\leqslant\varphi\left(\|V\|_{L^\mu_tL^\nu_x}\right) \tag{3.30}$$

holds, where $V_0:=|V|^{1/2}$ and $V=V_0V_1$.

Remark 3.5. — In this paper, we denote monotone increasing functions by φ. We use the same symbol φ for different monotone increasing functions. This is like the constant C. We do not distinguish a monotone increasing function from another one.

Proof. — It follows from the standard Strichartz estimates that

$$\|\mathcal{D}(V_0,V_1)\|_{\mathcal{B}\left(L^2_{t,x}\right)}\leqslant C_0\|V\|_{L^\mu_tL^\nu_x}. \tag{3.31}$$

Hence, if $\|V\|_{L^p_tL^q_x}\leqslant 1/(2C_0)=:\delta$, then we have

$$\|\mathcal{D}(V_0,V_1)\|_{\mathcal{B}\left(L^2_{t,x}\right)}\leqslant\frac{1}{2}, \tag{3.32}$$

$$\|\mathcal{R}(V_0,V_1)\|_{\mathcal{B}\left(L^2_{t,x}\right)}=\left\|(1-\mathcal{D}(V_0,V_1))^{-1}\right\|_{\mathcal{B}\left(L^2_{t,x}\right)}\leqslant 2. \tag{3.33}$$

Thus, we can assume that $\|V\|_{L^\mu_tL^\nu_x}\geqslant\delta$. There exists $0=T_0<T_1<\cdots<T_N<T_{N+1}=\infty$ such that

$$\frac{\delta}{2}<\|V\|_{L^\mu_t(I_k,L^\nu_x)}<\delta \text{ for } k=0,\ldots,N-1, \tag{3.34}$$

$$\|V\|_{L^\mu_t(I_N,L^\nu_x)}<\delta, \tag{3.35}$$

where $I_k=(T_k,T_{k+1})$. Therefore, we can solve the equation

$$(1-\mathcal{D}(V_0,V_1))u=f \tag{3.36}$$

uniquely on I_0. We denote this solution by $u_0\in L^2_t(I_0,L^2_x)$. Assume that there exists a unique solution on $[0,T_k]=I_0\cup\cdots\cup I_{k-1}$, denoted by $u_{k-1}\in L^2_t([0,T_k],L^2_x)$. We consider the equation of u on $I_k=[T_k,T_{k+1}]$:

$$\begin{aligned} u(t)+i\int_{T_k}^t V_0(t)U(t-\tau)V_1(\tau)u(\tau)d\tau \\ =f(t)-i\int_0^{T_k}V_0(t)U(t-\tau)V_1(\tau)u_{k-1}(\tau)d\tau. \end{aligned} \tag{3.37}$$

This is solvable; that is, a unique solution exists $u_k \in L^2_t(I_k, L^2_x)$ to (3.37). Let $u_k(t) = u_{k-1}(t)$ for $t \in [0, T_k]$. Then $u_k \in L^2_t([0, T_{k+1}], L^2_x)$ is the solution to (3.36). We have a unique solution to (3.36) on $u \in L^2_t([0, \infty), L^2_x)$ inductively. We can similarly extend this solution to $u \in L^2_t(\mathbb{R}, L^2_x)$.

Now we estimate the operator norm of $\mathcal{R}(V_0, V_1)$. We have

$$\|\mathcal{R}(V_0, V_1)f\|_{L^2_{t,x}} = \|u\|_{L^2_t(\mathbb{R}, L^2_x)} \tag{3.38}$$

$$\leqslant \|u\|_{L^2_t([0,\infty), L^2_x)} + \|u\|_{L^2_t((-\infty,0], L^2_x)}. \tag{3.39}$$

By (3.37), we have

$$\|u_k\|_{L^2_t(I_k, L^2_x)} \lesssim \|f\|_{L^2_{t,x}} + C_0\|V\|_{L^\mu_t L^\nu_x}\|u_{k-1}\|_{L^2_t([0,T_k], L^2_x)}; \tag{3.40}$$

hence we obtain

$$\|u_k\|_{L^2_t([0,T_{k+1}], L^2_x)} \lesssim \|f\|_{L^2_{t,x}} + \langle C_0\|V\|_{L^\mu_t L^\nu_x}\rangle\|u_{k-1}\|_{L^2_t([0,T_k], L^2_x)}. \tag{3.41}$$

Hence it follows from (3.40) that

$$\|u\|_{L^2_t([0,\infty), L^2_x)} \lesssim \langle C_0\|V\|_{L^\mu_t L^\nu_x}\rangle^N \|f\|_{L^2_{t,x}}. \tag{3.42}$$

Since

$$N\left(\frac{\delta}{2}\right)^\mu \leqslant \|V\|^\mu_{L^\mu_t(\mathbb{R}, L^\nu_x)}, \tag{3.43}$$

we have

$$\|u\|_{L^2_t([0,\infty), L^2_x)} \lesssim \langle C_0\|V\|_{L^\mu_t L^\nu_x}\rangle^{(2/\delta)^\mu \|V\|^\mu_{L^\mu_t L^\nu_x}} \|f\|_{L^2_{t,x}}. \tag{3.44}$$

We can bound $\|u\|_{L^2((-\infty,0], L^2_x)}$ similarly. From the above, we have

$$\|\mathcal{R}(V_0, V_1)\|_{\mathcal{B}\left(L^2_{t,x}\right)} \lesssim \langle C_0\|V\|_{L^\mu_t L^\nu_x}\rangle^{C\|V\|^\mu_{L^\mu_t L^\nu_x}}. \tag{3.45}$$

□

3.4. Propagators with time-dependent potentials

Let $p, q \in (1, \infty)$ and $\alpha \in [1, \infty]$. We call (p, q, α) admissible when $\frac{1}{\alpha} \geqslant \frac{1}{dp} + \frac{1}{q}$ and $\alpha < p$ hold. The following result is the most essential tool in this paper:

Theorem 3.6 ([FLLS14, Theorem 1]; [FS17, Theorem 8]; [BHL+19, Theorem 1.5]). *Let $d \geqslant 1$. Let (p, q, α) be admissible and $\sigma(p, q) = \frac{2}{p} + \frac{d}{q} = d - 2s$ for $s \in (0, d/2)$. Then it holds that*

$$\|\rho(U(t)_\star Q_0)\|_{L^p_t L^q_x} \lesssim \|\langle\nabla\rangle^s_\star Q_0\|_{\mathfrak{S}^\alpha}. \tag{3.46}$$

By Theorems 3.1, 3.6 and Lemma 3.2, we obtain the following corollary.

Corollary 3.7. — *Let $d \geqslant 1$. Let (p_n, q_n, α_n) be admissible and $\sigma(p_n, q_n) = d$ for $n = 0, 1$. Define*

$$\mathcal{D}(g_0, g_1) = \int_0^t g_0(t)U(t-\tau)g_1(\tau)d\tau. \tag{3.47}$$

Then we have

$$\|\mathcal{D}(g_0,g_1)\|_{\mathfrak{S}^{\alpha'}_{t,x}} \lesssim \prod_{n=0,1} \|g_n\|_{L_t^{2p'_n} L_x^{2q'_n}}, \tag{3.48}$$

where $\frac{1}{\alpha'} = \frac{1}{2\alpha'_0} + \frac{1}{2\alpha'_1}$.

Proof. — First, we consider the operator

$$\mathbb{T}(g_0,g_1) = \int_0^\infty g_0(t)U(t-\tau)g_1(\tau)d\tau = g_0 U^{\bigcirc} U^{\circledast} g_1. \tag{3.49}$$

By Lemma 3.2 and Theorem 3.6, we obtain

$$\|\mathbb{T}(g_0,g_1)\|_{\mathfrak{S}^{\alpha'}_{t,x}} \leqslant \left\|g_0 U^{\bigcirc}\right\|_{\mathfrak{S}^{2\alpha'_0}_{x\to(t,x)}} \|U^{\circledast} g_1\|_{\mathfrak{S}^{2\alpha'_1}_{(t,x)\to x}} \tag{3.50}$$

$$\lesssim \prod_{n=0,1} \|g_n\|_{L_t^{2p'_n} L_x^{2q'_n}}. \tag{3.51}$$

Therefore, we get the desired estimate by Theorem 3.1. ☐

We extend Theorem 3.6 with $s=0$ as follows.

Lemma 3.8. — *Let $d \geqslant 1$ and $0 \in I \subset \mathbb{R}$ be an interval. Let (p,q,α) be admissible and $\sigma(p,q)=d$. Let $\mu \in [1,\infty)$ and $\nu \in [1,\infty]$ satisfy $\sigma(\mu,\nu)=2$. Then there exists a monotone increasing function $\varphi : [0,\infty) \to [0,\infty)$ such that*

$$\|\rho(U_V(t)_\star \gamma)\|_{L_t^p(I,L_x^q)} \leqslant \varphi\left(\|V\|_{L_t^\mu(I,L_x^\nu)}\right) \|\gamma\|_{\mathfrak{S}^\alpha}. \tag{3.52}$$

Proof. — We use Lemma 3.2. Let $f \in L_t^{2p'}(I, L_x^{2q'})$. By (3.29), we have

$$\left\|fU_V^{\bigcirc}\right\|_{\mathfrak{S}^{2\alpha'}_{x\to(t,x)}} \leqslant \left\|fU^{\bigcirc}\right\|_{\mathfrak{S}^{2\alpha'}_{x\to(t,x)}} + \left\|\mathcal{D}(f,V_1)\mathcal{R}(V_0,V_1)V_0U^{\bigcirc}\right\|_{\mathfrak{S}^{2\alpha'}_{x\to(t,x)}}. \tag{3.53}$$

On the one hand, Lemma 3.2 and Theorem 3.6 yield

$$\left\|fU^{\bigcirc}\right\|_{\mathfrak{S}^{2\alpha'}_{x\to(t,x)}} \lesssim \|f\|_{L_t^{2p'}\left(I,L_x^{2q'}\right)}. \tag{3.54}$$

On the other hand, we have by Lemma 3.4

$$\begin{aligned} \left\|\mathcal{D}(f,V_1)\mathcal{R}(V_0,V_1)V_0U^{\bigcirc}\right\|_{\mathfrak{S}^{2\alpha'}_{x\to(t,x)}} &\leqslant \|\mathcal{D}(f,V_1)\|_{\mathfrak{S}^{2\alpha'}_{t,x}} \|\mathcal{R}(V_0,V_1)\|_{\mathcal{B}(L^2_x)} \left\|V_0U^{\bigcirc}\right\|_{\mathcal{B}\left(L^2_x\to L^2_{t,x}\right)} \\ &\leqslant \|f\|_{L_t^{2p'}\left(I,L_x^{2q'}\right)} \|V\|_{L_t^\mu L_x^\nu} \varphi\left(\|V\|_{L_t^\mu L_x^\nu}\right), \end{aligned} \tag{3.55}$$

where we estimated $\|\mathcal{D}(f,V_1)\|_{\mathfrak{S}^{2\alpha'}_{t,x}}$ in the same way as the proof of Corollary 3.7. ☐

Note that the "usual" Strichartz estimates also hold:

Lemma 3.9. — *Let $d \geqslant 3$. Let $p,q,\widetilde{p},\widetilde{q} \in [2,\infty]$ satisfy $\sigma(p,q)=\sigma(\widetilde{p},\widetilde{q})=d/2$. Let $\mu \in [1,\infty)$ and $\nu \in [1,\infty]$ satisfy $\sigma(\mu,\nu)=2$. Then there exists a monotone* *increasing function* $\varphi : [0, \infty) \to [0, \infty)$ *such that*

$$\|U_V(t)u_0\|_{L_t^p L_x^q} \leqslant \varphi\left(\|V\|_{L_t^\mu L_x^\nu}\right) \|u_0\|_{L_x^2}, \tag{3.56}$$

$$\left\|\int_{\mathbb{R}} U_V(t,\tau) f(\tau) d\tau\right\|_{L_t^p L_x^q} \leqslant \varphi\left(\|V\|_{L_t^\mu L_x^\nu}\right) \|f\|_{L_t^{\widetilde{p}'} L_x^{\widetilde{q}'}}, \tag{3.57}$$

$$\left\|\int_0^t U_V(t,\tau) f(\tau) d\tau\right\|_{L_t^p L_x^q} \leqslant \varphi\left(\|V\|_{L_t^\mu L_x^\nu}\right) \|f\|_{L_t^{\widetilde{p}'} L_x^{\widetilde{q}'}}. \tag{3.58}$$

Proof. — The usual Strichartz estimates and (3.29) immediately imply (3.56). (3.56) and usual duality argument imply (3.57). By the same argument that led to (3.29), we have

$$U_V(t,\tau)f = U(t-\tau)f - i\int_\tau^t U(t-\tau_1)V_1(\tau_1)(Tf)(\tau_1,\tau)d\tau_1, \tag{3.59}$$

where $(Tf)(\tau_1,\tau) = \mathcal{R}(V_0,V_1)(V_0U(\cdot-\tau)f)(t)$. The standard Strichartz estimates and (3.59) imply (3.58). □

The following lemma is also useful:

Lemma 3.10. — *Let* $d \geqslant 3$. *Let* $s \in [0,\infty)$. *Let* $\mu, \nu, \widetilde{\nu} \in (1,\infty)$ *satisfy* $\frac{1}{\nu} = \frac{1}{\widetilde{\nu}} + \frac{s}{d}$ *and* $\sigma(\mu,\widetilde{\nu}) = 2$. *Let* (p,q) *be an admissible pair of the standard Strichartz estimate; that is,* $p, q \in [2,\infty]$ *and* $\sigma(p,q) = d/2$. *Moreover, we assume that* $\frac{1}{\nu} < \frac{1}{2} + \frac{1}{d}$ *when* $\frac{1}{\mu} + \frac{1}{p} < \frac{1}{2}$. *Then there exists a monotone increasing function* $\varphi : [0,\infty) \to [0,\infty)$ *such that*

$$\left\|\langle\nabla\rangle^s U_V(t)\langle\nabla\rangle^{-s}u_0\right\|_{L_t^p L_x^q} \leqslant \varphi\left(\|V\|_{L_t^\mu H_x^{s,\nu}}\right) \|u_0\|_{L_x^2}. \tag{3.60}$$

Proof. — First, we assume

$$\frac{1}{2} \leqslant \frac{1}{p} + \frac{1}{\mu} \leqslant 1. \tag{3.61}$$

Since $V \in L_t^\mu H_x^{s,\nu} \hookrightarrow L_t^\mu L_x^{\widetilde{\nu}}$ and $\sigma(\mu,\widetilde{\nu}) = 2$, the propagator $U_V(t)$ is well-defined. We have

$$\begin{aligned}
&\left\|\langle\nabla\rangle^s U_V(t)\langle\nabla\rangle^{-s}u_0\right\|_{L_t^p L_x^q} \\
&\quad\leqslant \|U(t)u_0\|_{L_t^p L_x^q} + \left\|\int_0^t U(t-\tau)\langle\nabla\rangle^s\left(V(\tau)U_V(\tau)\langle\nabla\rangle^{-s}u_0\right)d\tau\right\|_{L_t^p L_x^q} \\
&\quad\leqslant C_0\|u_0\|_{L_x^2} + C_0\left\|\langle\nabla\rangle^s\left(V(t)U_V(t)\langle\nabla\rangle^{-s}u_0\right)\right\|_{L_t^{r'} L_x^{c'}} \\
&\quad\leqslant C_0\|u_0\|_{L_x^2} + CC_0\|V\|_{L_t^\mu H_x^{s,\nu}}\left\|\langle\nabla\rangle^s U_V(t)\langle\nabla\rangle^{-s}u_0\right\|_{L_t^p L_x^q},
\end{aligned} \tag{3.62}$$

where

$$\frac{1}{r} := 1 - \frac{1}{\mu} - \frac{1}{p}, \quad \frac{1}{c} := 1 - \frac{1}{\widetilde{\nu}} - \frac{1}{q}. \tag{3.63}$$

(r,c) is an admissible pair of the standard Strichartz estimates. Therefore, we get

$$\left\|\langle\nabla\rangle^s U_V(t)\langle\nabla\rangle^{-s}u_0\right\|_{L_t^p L_x^q} \leqslant 2C_0\|u_0\|_{L_x^2} \tag{3.64}$$

when $\|V\|_{L_t^\mu H_x^{s,\nu}} \leqslant \delta$ with sufficiently small $\delta > 0$. Let $\|V\|_{L_t^\mu([0,\infty),H_x^{s,\nu})} > \delta$. Then there exist $0 = T_0 < T_1 < \cdots < T_N < T_{N+1} = \infty$ such that

$$\frac{\delta}{2} \leqslant \|V\|_{L_t^\mu\left(I_n,H_x^{s,\nu}\right)} \leqslant \delta, \text{ for } 0 \leqslant n \leqslant N; \quad \|V\|_{L_t^\mu\left(I_{N+1},H_x^{s,\nu}\right)} \leqslant \delta, \tag{3.65}$$

where $I_n := [T_n, T_{n+1}]$. First, we have

$$\left\|\langle\nabla\rangle^s U_V(t)\langle\nabla\rangle^{-s}u_0\right\|_{L_t^p\left(I_0,L_x^q\right)} \leqslant 2C_0\|u_0\|_{L_x^2}. \tag{3.66}$$

Let $V_1(t) := V(t)\chi_{I_1}(t)$. Since $U_V(t,T_1) = U_{V_1}(t,T_1)$ for $t \in I_1$ and $U_{V_1}(t) = U(t)$ for $t \in I_0$, we have

$$U_V(t) = U_{V_1}(t,T_1)U_V(T_1) = U_{V_1}(t)U(-T_1)U_V(T_1). \tag{3.67}$$

Hence, we have

$$\begin{aligned}\left\|\langle\nabla\rangle^s U_V(t)\langle\nabla\rangle^{-s}u_0\right\|_{L_t^p\left(I_1,L_x^q\right)} &\leqslant 2C_0\left\|\langle\nabla\rangle^s U_V(T_1)\langle\nabla\rangle^{-s}u_0\right\|_{L_x^2}\\ &\leqslant (2C_0)^2\|u_0\|_{L_x^2}.\end{aligned} \tag{3.68}$$

In the same way, we obtain

$$\left\|\langle\nabla\rangle^s U_V(t)\langle\nabla\rangle^{-s}u_0\right\|_{L_t^p\left(I_n,L_x^q\right)} \leqslant (2C_0)^{n+1}\|u_0\|_{L_x^2}, \tag{3.69}$$

which implies

$$\begin{aligned}\left\|\langle\nabla\rangle^s U_V(t)\langle\nabla\rangle^{-s}u_0\right\|_{L_t^p\left([0,\infty),L_x^q\right)} &\leqslant \sum_{n=0}^{N}(2C_0)^{n+1}\|u_0\|_{L_x^2}\\ &\leqslant N(2C_0)^{N+1}\|u_0\|_{L_x^2}.\end{aligned} \tag{3.70}$$

$N \lesssim \|V\|_{L_t^\mu H_x^{s,\nu}}^\mu$ yields the desired estimates.

Next, we consider the case $\frac{1}{\mu} + \frac{1}{p} \leqslant \frac{1}{2}$. Let $\frac{1}{r} = \frac{1}{2} - \frac{1}{d}$. Then the endpoint Strichartz estimates ([KT98]) imply

$$\begin{aligned}&\left\|\langle\nabla\rangle^s U_V(t)\langle\nabla\rangle^{-s}u_0\right\|_{L_t^pL_x^q}\\ &\quad\leqslant \|U(t)u_0\|_{L_t^pL_x^q} + \left\|\int_0^t U(t-\tau)\langle\nabla\rangle^s\left(V(\tau)U_V(\tau)\langle\nabla\rangle^{-s}u_0\right)d\tau\right\|_{L_t^pL_x^q}\\ &\quad\lesssim \|u_0\|_{L_x^2} + \left\|\langle\nabla\rangle^s\left(V(t)U_V(t)\langle\nabla\rangle^{-s}u_0\right)\right\|_{L_t^2L_x^{r'}}\\ &\quad\lesssim \|u_0\|_{L_x^2} + \|\langle\nabla\rangle^s V\|_{L_t^\mu L_x^\nu}\left\|\langle\nabla\rangle^s U_V(t)\langle\nabla\rangle^{-s}u_0\right\|_{L_t^{\widetilde{p}}L_x^{\widetilde{q}}},\end{aligned} \tag{3.71}$$

where $(\widetilde{p},\widetilde{q})$ is an admissible pair of the standard Strichartz estimates. Since $\frac{1}{\mu}+\frac{1}{\widetilde{p}} = \frac{1}{2}$, we conclude that

$$\left\|\langle\nabla\rangle^s U_V(t)\langle\nabla\rangle^{-s}u_0\right\|_{L_t^pL_x^q} \leqslant \varphi\left(\|V\|_{L_t^\mu L_x^\nu}\right)\|u_0\|_{L_x^2}. \tag{3.72}$$

Finally, we consider the case $\frac{1}{\mu}+\frac{1}{p}\geqslant 1$. We have

$$\begin{aligned}(3.73)\quad &\left\|\langle\nabla\rangle^s U_V(t)\langle\nabla\rangle^{-s}u_0\right\|_{L_t^pL_x^q}\\ &\leqslant \|U(t)u_0\|_{L_t^pL_x^q}+\left\|\int_0^t U(t-\tau)\langle\nabla\rangle^s(V(\tau)U_V(\tau)\langle\nabla\rangle^{-s}u_0)d\tau\right\|_{L_t^pL_x^q}\\ &\lesssim \|u_0\|_{L_x^2}+\left\|\langle\nabla\rangle^s(V(t)U_V(t)\langle\nabla\rangle^{-s}u_0)\right\|_{L_t^1L_x^2}\\ &\lesssim \|u_0\|_{L_x^2}+\|\langle\nabla\rangle^s V\|_{L_t^\mu L_x^\nu}\left\|\langle\nabla\rangle^s U_V(t)\langle\nabla\rangle^{-s}u_0\right\|_{L_t^{\widetilde{p}}L_x^{\widetilde{q}}},\end{aligned}$$

where $(\widetilde{p},\widetilde{q})$ is an admissible pair of the standard Strichartz estimates. Since $\frac{1}{\mu}+\frac{1}{\widetilde{p}}=1$, we conclude that

$$(3.74)\qquad \left\|\langle\nabla\rangle^s U_V(t)\langle\nabla\rangle^{-s}u_0\right\|_{L_t^pL_x^q}\leqslant\varphi\left(\|V\|_{L_t^\mu L_x^\nu}\right)\|u_0\|_{L_x^2}. \qquad \square$$

4. Strichartz estimates for density functions

In this section, we prove the key estimates in this paper.

Theorem 4.1. — *Let $d\geqslant 3$. Let $0\leqslant\widetilde{s}\leqslant s\leqslant 1$. Let $p,q,\mu,\nu,\alpha\in(1,\infty)$ satisfy $\sigma(p,q)=d-s$ and $\sigma(\mu,\nu)=2+\widetilde{s}$. Let (p,q,α) be admissible. Assume that there exist $p_j,q_j,\alpha_j\in(1,\infty)$ for $j=0,1,2$ and $\mu_j,\nu_j\in(1,\infty)$ for $j=0,1$ such that*

$$(4.1)\quad \sigma(p_j,q_j)=d-j,\quad (p_j,q_j,\alpha_j)\text{ are admissible for } j=0,1,2,$$

$$(4.2)\quad \sigma(\mu_0,\nu_0)=2,\quad \sigma(\mu_1,\nu_1)=2+s_1:=2+\frac{\widetilde{s}}{s},\quad 1+s_1<\frac{2}{\mu_1}+\frac{d}{2},$$

$$(4.3)\quad \frac{1}{X}=\frac{1-s}{X_0}+\frac{s}{X_1},\quad X=p,q,\mu,\nu,\alpha,\quad \frac{1}{Y_1}=\frac{1}{2}\left(\frac{1}{Y_0}+\frac{1}{Y_2}\right),\quad Y=p,q,\alpha.$$

Then there exists a monotone increasing function $\varphi:[0,\infty)\to[0,\infty)$ such that

$$(4.4)\qquad \|\rho(U_V(t)_\star Q_0)\|_{L_t^pH_x^{s,q}}\leqslant\varphi(\|V\|_{L_t^\mu H_x^{s,\nu}})\|Q_0\|_{\mathcal{H}^{s,\alpha}},$$

$$\begin{aligned}(4.5)\quad &\|\rho(U_V(t)_\star Q_0)-\rho(U_W(t)_\star Q_0)\|_{L_t^pH_x^{s,q}}\\ &\leqslant\varphi\left(\|V\|_{L_t^\mu H_x^{s,\nu}}+\|W\|_{L_t^\mu H_x^{s,\nu}}\right)\|V-W\|_{L_t^\mu H_x^{s,\nu}}\|Q_0\|_{\mathcal{H}^{s,\alpha}}.\end{aligned}$$

The following corollary is enough for our analysis:

Corollary 4.2. — *Let $d=3$. Then there exists a monotone increasing function $\varphi:[0,\infty)\to[0,\infty)$ such that*

$$(4.6)\qquad \|\rho(U_V(t)_\star Q_0)\|_{L_t^2H_x^{1/2}}\leqslant\varphi\left(\|V\|_{L_t^2H_x^{1/2}}\right)\|Q_0\|_{\mathcal{H}^{1/2,3/2}},$$

$$\begin{aligned}(4.7)\quad &\left\|\rho(U_V(t)_\star Q_0)-\rho(U_W(t)_\star Q_0)\right\|_{L_t^2H_x^{1/2}}\\ &\leqslant\varphi\left(\|V\|_{L_t^2H_x^{1/2}}+\|W\|_{L_t^2H_x^{1/2}}\right)\|V-W\|_{L_t^2H_x^{1/2}}\|Q_0\|_{\mathcal{H}^{1/2,3/2}}.\end{aligned}$$

Proof. — Define exponents as follows:

$$s = \widetilde{s} = \frac{1}{2}, \quad p = q = \mu = \nu = 2, \quad \alpha = \frac{3}{2}, \tag{4.8}$$

$$p_0 = q_0 = \frac{5}{3}, \quad \alpha_0 = \frac{5}{4}, \quad p_1 = q_1 = \frac{5}{2}, \quad \alpha_1 = \frac{15}{8}, \quad p_2 = q_2 = 5, \quad \alpha_2 = \frac{15}{4}, \tag{4.9}$$

$$\mu_0 = \mu_1 = 2, \quad \nu_0 = 3, \quad \nu_1 = \frac{3}{2}. \tag{4.10}$$

Then Theorem 4.1 implies the result. □

Before proving the above theorem, we define some notations. For $\mathbf{F} = (F_1, \dots, F_n) \in \mathcal{S}(\mathbb{R}^d)^n$, define

$$\mathcal{W}_n[\mathbf{F}](t) = (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \mathcal{U}[F_1](t_1) \cdots \mathcal{U}[F_n](t_n), \tag{4.11}$$

where $\mathcal{U}[V](t) := U(t)^* V(t) U(t)$. And we define multilinear operators by

$$T_{n,m}[\mathbf{F}, Q_0, \mathbf{G}](t) := \rho\Big(U(t)\mathcal{W}_n[F](t) Q_0 \mathcal{W}_m[G](t)^* U(t)^*\Big) \tag{4.12}$$

for $\mathbf{F} \in \mathcal{S}(\mathbb{R}^d)^n$ and $\mathbf{G} \in \mathcal{S}(\mathbb{R}^d)^m$. We have

Lemma 4.3. — *Under the same assumptions as in Theorem 4.1, The following multilinear estimates hold:*

$$\|T_{n,m}[\mathbf{F}, Q_0, \mathbf{G}]\|_{L_t^p H_x^{s,q}} \lesssim C_0^{n+m} \prod_{j=1}^n \|F_j\|_{L_t^\mu H_x^{s,\nu}} \|Q_0\|_{\mathcal{H}_x^{s,\alpha}} \prod_{k=1}^m \|G_k\|_{L_t^\mu H_x^{s,\nu}}. \tag{4.13}$$

Proof. — It suffices to prove the following two estimates:

$$\|T_{n,m}[\mathbf{F}, Q_0, \mathbf{G}]\|_{L_t^{p_0} L_x^{q_0}} \lesssim C_0^{n+m} \prod_{j=1}^n \|F_j\|_{L_t^{\mu_0} L_x^{\nu_0}} \|Q_0\|_{\mathfrak{S}_x^{\alpha_0}} \prod_{k=1}^m \|G_k\|_{L_t^{\mu_0} L_x^{\nu_0}}, \tag{4.14}$$

$$\|T_{n,m}[\mathbf{F}, Q_0, \mathbf{G}]\|_{L_t^{p_1} H_x^{1,q_1}} \lesssim C_0^{n+m} \prod_{j=1}^n \|F_j\|_{L_t^{\mu_1} H_x^{1,\nu_1}} \|Q_0\|_{\mathcal{H}_x^{1,\alpha_1}} \prod_{k=1}^m \|G_k\|_{L_t^{\mu_1} H_x^{1,\nu_1}}. \tag{4.15}$$

We only prove (4.15) because (4.14) can be shown similarly. We have

$$\begin{aligned} &\|T_{n,m}[\mathbf{F}, Q_0, \mathbf{G}]\|_{L_t^{p_1} H_x^{1,q_1}} \\ &\qquad \sim \|T_{n,m}[\mathbf{F}, Q_0, \mathbf{G}]\|_{L_t^{p_1} L_x^{q_1}} + \|\nabla T_{n,m}[\mathbf{F}, Q_0, \mathbf{G}]\|_{L_t^{p_1} L_x^{q_1}}. \end{aligned} \tag{4.16}$$

Let $f \in L_t^{2p_1'} L_x^{2q_1'}$. Note that $\langle\nabla\rangle = \langle\nabla\rangle^{-1} - L\nabla$ with $L := \nabla\langle\nabla\rangle^{-1}$. We have

$$\begin{aligned} f(t)U(t)\mathcal{W}_n[\mathbf{F}](t)\langle\nabla\rangle^{-1} &= f(t)U(t)\langle\nabla\rangle^{-1}\left(\langle\nabla\rangle^{-1} - L\nabla\right)\mathcal{W}_n[\mathbf{F}](t)\langle\nabla\rangle^{-1} \\ &=: W_1 - W_2. \end{aligned} \tag{4.17}$$

We can assume $n \geqslant 1$ because when $n = 0$, we obtain (4.25) by Theorem 3.6. W_1 is a good term, and the problem is W_2. We have

$$\begin{aligned} W_2 = f(t)U(t)\langle\nabla\rangle^{-1} L \sum_{k=1}^n \mathcal{W}_n[\mathbf{F}^k](t)\langle\nabla\rangle^{-1} \\ + f(t)U(t)\langle\nabla\rangle^{-1} L \mathcal{W}_n[\mathbf{F}](t)\nabla\langle\nabla\rangle^{-1}, \end{aligned} \tag{4.18}$$

where $\mathbf{F}^k = (F_1, \cdots, F_{k-1}, \nabla F_k, F_{k+1}, \cdots, F_n)$. The last term is harmless. Let $F_{1,0} := |F_1|^{1/2}$ and $F_1 = F_{1,0}F_{1,1}$. Note that $L_t^{\mu_1} H_x^{1,\nu_1} \hookrightarrow L_t^{\mu_1} L_x^{\widetilde{\nu_1}}$ with $\sigma(\mu_1, \widetilde{\nu_1}) = 2$. Then we have

$$\begin{aligned}(4.19)\quad &\Big\| f(t)U(t)\langle\nabla\rangle^{-1} L \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \mathcal{U}[F_1](t) \\ &\qquad\qquad \cdots \mathcal{U}[\nabla F_k](t_k) \cdots \mathcal{U}[F_n](t_n)\langle\nabla\rangle^{-1}\Big\|_{\mathfrak{S}^{2\alpha_1'}_{x\to(t,x)}} \\ &\leqslant \Big\| f(t)U(t)\langle\nabla\rangle^{-1} L \int_0^t dt_1 U(t_1) F_{1,0}(t_1)\Big\|_{\mathfrak{S}^{2\alpha_1'}_{t,x}} \\ &\quad \times \Big\| F_{1,1}(t_1)U(t_1) \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \mathcal{U}[F_1](t) \cdots \mathcal{U}[\nabla F_k](t_k) \\ &\qquad\qquad \cdots \mathcal{U}[F_n](t_n)\langle\nabla\rangle^{-1}\Big\|_{\mathcal{B}\left(L_x^2 \to L_{t,x}^2\right)} \\ &\lesssim C_0 \|f\|_{L_t^{2p_1'} L_x^{2q_1'}} \|F\|^{1/2}_{L_t^{\mu_1} L_x^{\widetilde{\nu_1}}} \Big\| F_{1,1}(t_1)U(t_1) \int_0^{t_1} dt_2 \mathcal{U}[F_2](t_1) \\ &\quad \times \cdots \int_0^{t_{k-1}} dt_k \mathcal{U}[\nabla F_k](t_k) \cdots \int_0^{t_{n-1}} dt_n \mathcal{U}[F_n](t_n)\langle\nabla\rangle^{-1}\Big\|_{\mathcal{B}\left(L_x^2 \to L_{t,x}^2\right)} =: I.\end{aligned}$$

To justify the last inequality, we used Theorem 3.1. Moreover, we have

$$\begin{aligned}(4.20)\quad & I \leqslant C_0^k \|f\|_{L_t^{2p_1'} L_x^{2q_1'}} \prod_{j=1}^{k-1} \|F_j\|_{L_t^{\mu_1} L_x^{\widetilde{\nu_1}}} \\ &\times \Big\| (\nabla F_k)(t_k) U(t_k) \int_0^{t_k} dt_{k+1} \mathcal{U}[F_{k+1}](t_{k+1}) \cdots \\ &\qquad\qquad \int_0^{t_{n-1}} dt_n \mathcal{U}[F_n](t_n)\langle\nabla\rangle^{-1}\Big\|_{\mathcal{B}\left(L_x^2 \to L_t^{\widetilde{r}'} L_x^{\widetilde{c}'}\right)} \\ &\leqslant C_0^k \|f\|_{L_t^{2p_1'} L_x^{2q_1'}} \prod_{j=1}^{k} \|F_j\|_{L_t^{\mu_1} H_x^{1,\nu_1}} \\ &\quad \times \Big\| U(t_k) \int_0^{t_k} dt_{k+1} \mathcal{U}[F_{k+1}](t_{k+1}) \cdots \int_0^{t_{n-1}} dt_n \mathcal{U}[F_n](t_n)\langle\nabla\rangle^{-1}\Big\|_{\mathcal{B}\left(L_x^2 \to L_t^r L_x^{c^*}\right)} \\ &\leqslant C_0^k \|f\|_{L_t^{2p_1'} L_x^{2q_1'}} \prod_{j=1}^{k} \|F_j\|_{L_t^{\mu_1} H_x^{1,\nu_1}} \\ &\quad \times \Big\| \langle\nabla\rangle U(t_k) \int_0^{t_k} dt_{k+1} \mathcal{U}[F_{k+1}](t_{k+1}) \cdots \int_0^{t_{n-1}} dt_n \mathcal{U}[F_n](t_n)\langle\nabla\rangle^{-1}\Big\|_{\mathcal{B}\left(L_x^2 \to L_t^r L_x^c\right)},\end{aligned}$$

where $\widetilde{r}, \widetilde{c}, r, c$ are defined as follows. Fix $r \geqslant 2$ such that $\frac{1}{2} - \frac{1}{\mu_1} < \frac{1}{r} < \min(\frac{d}{4} - \frac{s_1}{2}, 1 - \frac{1}{\mu_1})$. Define $\widetilde{r} \in [2, \infty]$ by $\frac{1}{\widetilde{r}'} = \frac{1}{\mu_1} + \frac{1}{r}$. Then there exists $c, \widetilde{c} \in [2, \infty]$ such that

$\sigma(r,c)=\sigma(\widetilde{r},\widetilde{c})=\frac{d}{2}$. By direct calculation, we obtain

$$\frac{1}{c^*}:=\frac{1}{\widetilde{c}'}-\frac{1}{\nu_1}=\frac{1}{c}-\frac{s_1}{d}>0. \tag{4.21}$$

Since

$$\left\|\int_0^t U(t-\tau)(g(\tau)h(\tau))d\tau\right\|_{L_t^r H_x^{1,c}}\leqslant C_0\|g\|_{L_t^{\mu_1}H_x^{1,\nu_1}}\|h\|_{L_t^r H_x^{1,c}}, \tag{4.22}$$

we have

$$\begin{aligned}\left\|\langle\nabla\rangle U(t_k)\int_0^{t_k}dt_{k+1}\mathcal{U}[F_{k+1}](t_{k+1})\cdots\int_0^{t_{n-1}}dt_n\mathcal{U}[F_n](t_n)\langle\nabla\rangle^{-1}\right\|_{\mathcal{B}\left(L_x^2\to L_t^r L_x^c\right)}&\\ \leqslant C_0^{n-k}\prod_{j=k+1}^{n}\|F_j\|_{L_t^{\mu_1}H_x^{1,\nu_1}}&.\end{aligned} \tag{4.23}$$

From the above, we obtain

$$\left\|f(t)U(t)\langle\nabla\rangle^{-1}L\mathcal{W}_n[\mathbf{F}^k](t)\langle\nabla\rangle^{-1}\right\|_{\mathfrak{S}^{2\alpha_1'}_{x\to(t,x)}}\lesssim C_0^n\|f\|_{L_t^{2p_1'}L_x^{2q_1'}}\prod_{j=1}^{n}\|F_j\|_{L_t^{\mu_1}H_x^{1,\nu_1}}, \tag{4.24}$$

which yields

$$\left\|f(t)U(t)\mathcal{W}_n[\mathbf{F}](t)\langle\nabla\rangle^{-1}\right\|_{\mathfrak{S}^{2\alpha_1'}_{x\to(t,x)}}\lesssim C_0^n\|f\|_{L_t^{2p_1'}L_x^{2q_1'}}\prod_{j=1}^{n}\|F_j\|_{L_t^{\mu_1}H_x^{1,\nu_1}}. \tag{4.25}$$

Therefore, we conclude that

$$\begin{aligned}&\|T_{n,m}[\mathbf{F},Q_0,\mathbf{G}]\|_{L_t^{p_1}L_x^{q_1}}\\&\qquad\lesssim C_0^{n+m}\left(\prod_{j=1}^{n}\|F_j\|_{L_t^{\mu_1}H_x^{1,\nu_1}}\right)\|Q_0\|_{\mathcal{H}^{1,\alpha_1}}\left(\prod_{k=1}^{m}\|G_k\|_{L_t^{\mu_1}H_x^{1,\nu_1}}\right).\end{aligned} \tag{4.26}$$

Finally, we estimate $\|\nabla T_{n,m}[\mathbf{F},Q_0,\mathbf{G}]\|_{L_t^{p_1}L_x^{q_1}}$. In the completely same way as the above, we get

$$\begin{aligned}&\|T_{n,m}[\mathbf{F},Q_0,\mathbf{G}]\|_{L_t^{p_2}L_x^{q_2}}\\&\qquad\lesssim C_0^{n+m}\left(\prod_{j=1}^{n}\|F_j\|_{L_t^{\mu_1}H_x^{1,\nu_1}}\right)\|Q_0\|_{\mathcal{H}^{1,\alpha_2}}\left(\prod_{k=1}^{m}\|G_k\|_{L_t^{\mu_1}H_x^{1,\nu_1}}\right),\end{aligned} \tag{4.27}$$

$$\begin{aligned}&\left\|\rho\left[\nabla U(t)\mathcal{W}_n[\mathbf{F}](t)\langle\nabla\rangle^{-1}Q_0\langle\nabla\rangle^{-1}\mathcal{W}_n[\mathbf{G}](t)^*U(t)^*\nabla\right]\right\|_{L_t^{p_0}L_x^{q_0}}\\&\qquad\lesssim C_0^{n+m}\left(\prod_{j=1}^{n}\|F_j\|_{L_t^{\mu_1}H_x^{1,\nu_1}}\right)\|Q_0\|_{\mathfrak{S}^{\alpha_0}}\left(\prod_{k=1}^{m}\|G_k\|_{L_t^{\mu_1}H_x^{1,\nu_1}}\right).\end{aligned} \tag{4.28}$$

Therefore, we obtain by Lemma 3.3

$$(4.29)\quad \begin{aligned}&\|\nabla T_{n,m}[\mathbf{F},Q_0,\mathbf{G}]\|_{L_t^{p_1}L_x^{q_1}}\\ &\leqslant \|\rho\,[\nabla U(t)\mathcal{W}_n[\mathbf{F}](t)Q_0\mathcal{W}_m[\mathbf{G}](t)^*U(t)]\|_{L_t^{p_1}L_x^{q_1}}\\ &+\|\rho\,[U(t)\mathcal{W}_n[\mathbf{F}](t)Q_0\mathcal{W}_m[\mathbf{G}](t)^*U(t)\nabla]\|_{L_t^{p_1}L_x^{q_1}}\\ &\lesssim C_0^{n+m}\left(\prod_{j=1}^n\|F_j\|_{L_t^{\mu_1}H_x^{1,\nu_1}}\right)\|Q_0\|_{\mathcal{H}^{1,\alpha_1}}\left(\prod_{k=1}^m\|G_k\|_{L_t^{\mu_1}H_x^{1,\nu_1}}\right).\quad\square\end{aligned}$$

Proof of Theorem 4.1. — First, we prove (4.4). We have the series decomposition of $U_V(t)$:

$$(4.30)\qquad U_V(t)=\sum_{n=0}^\infty U(t)\mathcal{W}_V^{(n)}(t),$$

where $\mathcal{W}_V^{(n)}(t) := \mathcal{W}_n[\mathbf{V}](t)$ and $\mathbf{V} := (V,\ldots,V)$. If $\|V\|_{L_t^\mu H_x^{s,\nu}} < 1/(2C_0) =: \delta$, Lemma 4.3 implies

$$(4.31)\quad \begin{aligned}\|\langle\nabla\rangle^s\rho[U_V(t)_\star Q_0]\|_{L_t^pL_x^q} &\leqslant \sum_{n,m=1}^\infty\left\|\langle\nabla\rangle^s\rho\left[U(t)\mathcal{W}_V^{(n)}(t)Q_0\mathcal{W}_V^{(m)}(t)^*U(t)^*\right]\right\|_{L_t^pL_x^q}\\ &\lesssim \sum_{n,m=0}^\infty C_0^{n+m}\|V\|^{n+m}_{L_t^\mu H_x^{s,\nu}}\|Q_0\|_{\mathcal{H}_x^{s,\alpha}}\\ &\leqslant C\|Q_0\|_{\mathcal{H}_x^{s,\alpha}}.\end{aligned}$$

Let $\|V\|_{L_t^\mu([0,\infty),H_x^{s,\nu})} > \delta$. We can divide the interval $[0,\infty)$ such that

$$(4.32)\quad 0=T_0<T_1<\cdots<T_N<T_{N+1}=\infty,\quad I_k:=[T_k,T_{k+1}]\text{ for }k=0,\ldots N,$$

$$(4.33)\quad \frac{\delta}{2}\leqslant\|V\|_{L_t^\mu(I_k,H_x^{s,\nu})}\leqslant\delta\text{ for }k=0,\ldots,N-1,\quad \|V\|_{L_t^\mu(I_N,H_x^{s,\nu})}\leqslant\delta.$$

Define $V_k(t) := V(t)\chi_{I_k}(t)$. Then (3.67), Lemma 3.10 and $N\lesssim\|V\|^\mu_{L_t^\mu H_x^{s,\nu}}$ yield

$$(4.34)\quad \begin{aligned}&\|\langle\nabla\rangle^s\rho[U_V(t)_\star Q_0]\|_{L_t^p\left([0,\infty),L_x^q\right)}\\ &\leqslant\sum_{k=0}^N\left\|\langle\nabla\rangle^s\rho\left[U_{V_k}(t)_\star\left(U_{V_k}(T_k)^*U_V(T_k)Q_0U_V(T_k)^*U_{V_k}(T_k)\right)\right]\right\|_{L_t^p\left(I_k,L_x^q\right)}\\ &\leqslant C\sum_{k=0}^N\|U(T_k)^*U_V(T_k)Q_0U_V(T_k)^*U(T_k)\|_{\mathcal{H}^{s,\alpha}}\\ &\leqslant C\sum_{k=0}^N\left\|\langle\nabla\rangle^sU_V(T_k)\langle\nabla\rangle^{-s}\right\|^2_{\mathcal{B}}\|Q_0\|_{\mathcal{H}_x^{s,\alpha}}\\ &\leqslant\varphi\left(\|V\|_{L_t^\mu H_x^{s,\nu}}\right)(N+1)\|Q_0\|_{\mathcal{H}_x^{s,\alpha}}\\ &\leqslant\varphi\left(\|V\|_{L_t^\mu H_x^{s,\nu}}\right)\|Q_0\|_{\mathcal{H}_x^{s,\alpha}}.\end{aligned}$$

Finally, we prove (4.5). It suffices to show that

$$\|\langle\nabla\rangle^s\rho((U_V(t)-U_W(t))Q_0U_V(t))\|_{L_t^pL_x^q} \leqslant \varphi\left(\|V\|_{L_t^\mu H_x^{s,\nu}}+\|W\|_{L_t^\mu H_x^{s,\nu}}\right)\|V-W\|_{L_t^\mu H_x^{s,\nu}}\|Q_0\|_{\mathcal{H}^{s,\alpha}}. \tag{4.35}$$

We have

$$\begin{aligned}\|\langle\nabla\rangle^s\rho\left((U_V(t)-U_W(t))Q_0U_V(t)\right)\|_{L_t^pL_x^q} &\leqslant \sum_{n,m=0}^{\infty}\left\|\langle\nabla\rangle^s\rho\left(U(t)\left(\mathcal{W}_V^{(n)}(t)-\mathcal{W}_W^{(n)}(t)\right)Q_0\mathcal{W}_V^{(m)}(t)^*U(t)^*\right)\right\|_{L_t^pL_x^q}\\ &\leqslant \sum_{n,m=0}^{\infty}\sum_{k=1}^{n}\left\|\langle\nabla\rangle^s\rho\left(U(t)\mathcal{W}_n[\mathbf{X}_k](t)Q_0\mathcal{W}_m[\mathbf{V}](t)^*U(t)^*\right)\right\|_{L_t^pL_x^q},\end{aligned} \tag{4.36}$$

where $\mathbf{V}=(V,\dots,V)$, $\mathbf{X}_k=((\mathbf{X}_k)_1,\dots,(\mathbf{X}_k)_n)$ and

$$(\mathbf{X}_k)_j = V \text{ if } 1\leqslant j\leqslant k-1, \tag{4.37}$$

$$(\mathbf{X}_k)_j = V-W \text{ if } j=k, \tag{4.38}$$

$$(\mathbf{X}_k)_j = W \text{ if } k+1\leqslant j\leqslant n. \tag{4.39}$$

By Lemma 4.3, if $\|V\|_{L_t^\mu H_x^{s,\nu}}$ and $\|W\|_{L_t^\mu H_x^{s,\nu}}$ are sufficiently small, we obtain

$$\begin{aligned}\left\|\langle\nabla\rangle^s\rho\Big((U_V(t)-U_W(t))Q_0U_V(t)\Big)\right\|_{L_t^pL_x^q} &\lesssim \|V-W\|_{L_t^\mu H_x^{s,\nu}}\|Q_0\|_{\mathcal{H}^{s,\alpha}}\sum_{n,m=0}^{\infty}\sum_{k=1}^{n}C_0^{m+n}\|V\|_{L_t^\mu H_x^{s,\nu}}^{m+k-1}\|W\|_{L_t^\mu H_x^{s,\nu}}^{n-k}\\ &\lesssim \|V-W\|_{L_t^\mu H_x^{s,\nu}}\|Q_0\|_{\mathcal{H}^{s,\alpha}}.\end{aligned} \tag{4.40}$$

The same argument as we used to prove (4.4) yields

$$\left\|\langle\nabla\rangle^s\rho((U_V(t)-U_W(t))Q_0U_V(t))\right\|_{L_t^pL_x^q} \leqslant \varphi\left(\|V\|_{L_t^\mu H_x^{s,\nu}}\right)\|V-W\|_{L_t^\mu H_x^{s,\nu}}\|Q_0\|_{\mathfrak{S}^{s,\alpha}}. \quad\square \tag{4.41}$$

5. Proof of Theorem 2.2

In this section, we give a proof of Theorem 2.2 by giving a nonlinear estimates:

LEMMA 5.1. — *Let $d=3$. If $\|\langle\xi\rangle^2f(\xi)\|_{L_\xi^1\cap L_\xi^\infty}<\infty$, then there exists a monotone increasing function $\varphi:[0,\infty)\to[0,\infty)$ such that*

$$\|\rho(N_1[V])\|_{L_t^2H_x^{1/2}} \lesssim_f \varphi\left(\|V\|_{L_t^2H_x^{1/2}}\right)\|V\|_{L_t^2H_x^{1/2}}^2 \tag{5.1}$$

$$\|\rho(N_2[V])\|_{L_t^2H_x^{1/2}} \lesssim_f \varphi\left(\|V\|_{L_t^2H_x^{1/2}}\right)\|V\|_{L_t^2H_x^{1/2}}^2 \tag{5.2}$$

$$\|\rho(N_3[V])\|_{L_t^2H_x^{1/2}} \lesssim_f \varphi\left(\|V\|_{L_t^2H_x^{1/2}}\right)\|V\|_{L_t^2H_x^{1/2}}^3. \tag{5.3}$$

See (2.7) for the definition of N_1, N_2 and N_3.

Lemma 5.2. — *Let $d=3$. If $\|\langle\xi\rangle^2 f(\xi)\|_{L^1_\xi\cap L^\infty_\xi}<\infty$, then there exists a monotone increasing function $\varphi:[0,\infty)\to[0,\infty)$ such that*

$$\begin{aligned}(5.4)\quad &\Big\|\rho(N_1[V])-\rho(N_1[W])\Big\|_{L^2_tH^{1/2}_x}\\ &\lesssim_f \varphi\left(\|V\|_{L^2_tH^{1/2}_x}+\|W\|_{L^2_tH^{1/2}_x}\right)\left(\|V\|_{L^2_tH^{1/2}_x}+\|W\|_{L^2_tH^{1/2}_x}\right)\|V-W\|_{L^2_tH^{1/2}_x},\end{aligned}$$

$$\begin{aligned}(5.5)\quad &\Big\|\rho(N_2[V])-\rho(N_2[W])\Big\|_{L^2_tH^{1/2}_x}\\ &\lesssim_f \varphi\left(\|V\|_{L^2_tH^{1/2}_x}+\|W\|_{L^2_tH^{1/2}_x}\right)\left(\|V\|_{L^2_tH^{1/2}_x}+\|W\|_{L^2_tH^{1/2}_x}\right)\|V-W\|_{L^2_tH^{1/2}_x},\end{aligned}$$

$$\begin{aligned}(5.6)\quad &\Big\|\rho(N_3[V])-\rho(N_3[W])\Big\|_{L^2_tH^{1/2}_x}\\ &\lesssim_f \varphi\left(\|V\|_{L^2_tH^{1/2}_x}+\|W\|_{L^2_tH^{1/2}_x}\right)\left(\|V\|^2_{L^2_tH^{1/2}_x}+\|W\|^2_{L^2_tH^{1/2}_x}\right)\|V-W\|_{L^2_tH^{1/2}_x}.\end{aligned}$$

First, we prove that Lemmas 5.1 and 5.2 imply Theorem 2.2.

Proof. — Let

$$(5.7)\quad \Phi[\rho_Q](t):=(1+\mathcal{L}_1)^{-1}\Big(\rho(U_V(t)_\star Q_0)+\rho(N_1[V])+\rho(N_2[V])+\rho(N_3[V])\Big),$$

$$(5.8)\quad E(R):=\left\{\rho_Q\in L^2_t\left(\mathbb{R},H^{\frac12}_x\right):\|\rho_Q\|_{L^2_tH^{1/2}_x}\leqslant R\right\},$$

where $V=w*\rho_Q$ and $R>0$ will be taken later. We prove $\Phi:E(R)\to E(R)$ is a contraction map. Corollary 4.2, Lemma 5.1 and Young's inequality imply

$$\begin{aligned}(5.9)\quad &\|\Phi[\rho_Q]\|_{L^2_tH^{1/2}_x}\\ &\leqslant \Big\|(1+\mathcal{L}_1)^{-1}\Big\|_{\mathcal{B}(L^2_{t,x})}\Big(\|\rho(U_V(t)_\star Q_0)\|_{L^2_tH^{1/2}_x}+\|\rho(N_1[V])\|_{L^2_tH^{1/2}_x}\\ &\qquad+\|\rho(N_2[V])\|_{L^2_tH^{1/2}_x}+\|\rho(N_3[V])\|_{L^2_tH^{1/2}_x}\Big)\\ &\leqslant C_f\varphi\left(\|V\|_{L^2_tH^{1/2}_x}\right)\left(\|V\|^2_{L^2_tH^{1/2}_x}+\|V\|^3_{L^2_tH^{1/2}_x}+\|Q_0\|_{\mathcal{H}^{1/2,3/2}}\right)\\ &\leqslant C_f\varphi(R)\left(R^2+R^3+\varepsilon_0\right).\end{aligned}$$

Hence, if $R>0$ and $\varepsilon_0>0$ are sufficiently small, then we obtain

$$(5.10)\quad \|\Phi[\rho_Q]\|_{L^2_{t,x}}\leqslant R.$$

Therefore, the map $\Phi:E(R)\to E(R)$ is well-defined. We can prove the contraction and the uniqueness of the solution by Corollary 4.2 and Lemma 5.2. ☐

In the rest of this section, we prove only (5.2) because we can prove the other estimates in the same way. Let

$$\begin{aligned}(5.11)\quad N_2[V](t)&=i\int_0^t D_V(t,\tau)\gamma_f V(\tau)U(\tau-t)d\tau-i\int_0^t D_V(t,\tau)V(\tau)\gamma_f U(\tau-t)d\tau\\ &=:N_2^1[V](t)-N_2^2[V](t).\end{aligned}$$

5.1. Estimate for N_2^1

First, note that we have the wave operator decomposition

$$U_V(t,\tau) = U(t)\sum_{n=0}^{\infty}\mathcal{W}_V^{(n)}(t,\tau)U(\tau)^*, \tag{5.12}$$

where

$$\mathcal{W}_V^{(0)}(t,\tau) = I, \tag{5.13}$$

$$\mathcal{W}_V^{(n)}(t,\tau) = (-i)^n\int_\tau^t dt_1\cdots\int_\tau^{t_{n-1}} dt_n U(t)\mathcal{U}[V](t_1)\cdots\mathcal{U}[V](t_n)U(\tau)^* \tag{5.14}$$

and $\mathcal{U}[F] := U(t)^*F(t)U(t)$. Hence, it follows that

$$\begin{aligned}
&\rho\left(N_2^1[V](t)\right)\\
&= \int_0^t\int_0^\tau \rho\Big[U_V(t,\tau)V(\tau)U(\tau-\tau_1)\gamma_f V(\tau_1)U(\tau_1-t)\Big]d\tau_1 d\tau\\
&= \sum_{n=0}^{\infty}\int_0^t\int_0^\tau \rho\Big[U(t)\mathcal{W}_V^{(n)}(t,\tau)U(\tau)^*V(\tau)U(\tau-\tau_1)\gamma_f V(\tau_1)U(\tau_1-t)\Big]d\tau_1 d\tau\\
&=: \sum_{n=0}^{\infty}\mathcal{N}_n(V).
\end{aligned} \tag{5.15}$$

Define

$$\begin{aligned}
&\mathcal{W}^{(n)}[g_1,\dots,g_n](t,\tau)\\
&\qquad := (-i)^n\int_\tau^t dt_1\int_\tau^{t_1} dt_2\cdots\int_\tau^{t_{n-1}} dt_n U(\tau)\mathcal{U}[g_1](t_1)\cdots\mathcal{U}[g_n](t_n)U(\tau)^*,
\end{aligned} \tag{5.16}$$

and

$$\begin{aligned}
&\mathcal{N}_n[g_1,\dots,g_{n+2}](t)\\
&:= \int_0^t d\tau\int_0^\tau d\tau_1\rho\Big[U(t)\mathcal{W}^{(n)}[g_1,\cdots,g_n](t,\tau)\\
&\qquad\qquad U(\tau)^*g_{n+1}(\tau)U(\tau-\tau_1)\gamma_f g_{n+2}(\tau_1)U(\tau_1-t)\Big]\\
&= (-i)^n\int_0^t dt_1\int_0^{t_1} dt_2\cdots\int_0^{t_{n-1}} dt_n\int_0^{t_n} d\tau\int_0^\tau d\tau_1\\
&\quad\times\rho\Big[U(t)\mathcal{U}[g_1](t_1)\cdots\mathcal{U}[g_n](t_n)U(\tau)^*g_{n+1}(\tau)U(\tau-\tau_1)\gamma_f g_{n+2}(\tau_1)U(\tau_1-t)\Big].
\end{aligned} \tag{5.17}$$

5.1.1. With no derivative I

Let $W \in C_c^\infty((0,\infty)\times\mathbb{R}^3)$. On the one hand, we have

$$\begin{aligned}(5.18)\quad &\left|\int_0^\infty\int_{\mathbb{R}^3} W(t,x)\mathcal{N}_n[g_1,\dots,g_{n+2}](t,x)dxdt\right| \\ &= \Bigg|\mathrm{Tr}\Bigg[\int_0^\infty U(t)^*W(t,x)U(t)dt \\ &\qquad \int_0^t U(t_1)^*g_1(t_1)U(t_1)dt_1\cdots\int_0^{t_{n-1}} U(t_n)^*g_n(t_n)U(t_n)dt_n \\ &\qquad \times\int_0^{t_n} U(\tau)^*g_{n+1}(\tau)U(\tau)d\tau\int_0^\tau U(\tau_1)^*\gamma_f g_{n+2}(\tau_1)U(\tau_1)d\tau_1\Bigg]\Bigg| \\ &= \Bigg|\mathrm{Tr}\Bigg[\int_0^\infty U(t)^*W(t,x)U(t)dt\langle\nabla\rangle^{-1}\langle\nabla\rangle\int_0^t U(t_1)^*g_1(t_1)U(t_1)dt_1\times\cdots \\ &\qquad \cdots\times\int_0^{t_n} U(\tau)^*g_{n+1}(\tau)U(\tau)d\tau\int_0^\tau U(\tau_1)^*\gamma_f g_{n+2}(\tau_1)U(\tau_1)d\tau_1\Bigg]\Bigg|.\end{aligned}$$

Since $\langle\nabla\rangle = \langle\nabla\rangle^{-1}+\nabla\cdot\nabla\langle\nabla\rangle^{-1}$, we get

$$\begin{aligned}(5.19)\quad &\left|\int_0^\infty\int_{\mathbb{R}^3} W(t,x)\mathcal{N}_n[g_1,\dots,g_{n+2}](t,x)dxdt\right| \\ &\qquad\qquad \leqslant C_0^{n+2}\|W\|_{L_t^2L_x^2}\prod_{j=1}^{n+1}\|g_j\|_{L_t^2H_x^1}\|g_{n+2}\|_{L_t^2L_x^2}.\end{aligned}$$

From the above, by the duality argument, we obtain

$$(5.20)\qquad \|\mathcal{N}_n[g_1,\dots,g_{n+2}]\|_{L_t^2L_x^2}\leqslant C_0^{n+2}\prod_{j=1}^{n+1}\|g_j\|_{L_t^2H_x^1}\|g_{n+2}\|_{L_t^2L_x^2}.$$

On the other hand, we obtain

$$\begin{aligned}(5.21)\quad &\left|\int_0^\infty\int_{\mathbb{R}^3} W(t,x)\mathcal{N}_n[g_1,\dots,g_{n+2}](t,x)dxdt\right| \\ &= \Bigg|\mathrm{Tr}\Bigg[\langle\nabla\rangle^{-1}\int_0^\infty U(t)^*W(t,x)U(t)dt\int_0^t U(t_1)^*g_1(t_1)U(t_1)dt_1\times\cdots \\ &\qquad \cdots\times\int_0^{t_n} U(\tau)^*g_{n+1}(\tau)U(\tau)d\tau\int_0^\tau U(\tau_1)^*\gamma_f g_{n+2}(\tau_1)U(\tau_1)d\tau_1\langle\nabla\rangle\Bigg]\Bigg|.\end{aligned}$$

Again, by the same argument as above and the duality argument, we obtain

$$(5.22)\qquad \|\mathcal{N}_n[g_1,\dots,g_{n+2}]\|_{L_t^2L_x^2}\leqslant C_0^{n+2}\prod_{j=1}^{n+1}\|g_j\|_{L_t^2L_x^2}\|g_{n+2}\|_{L_t^2H_x^1}.$$

Note that we used Gohberg–Kreĭn theorem several times to get (5.22). By interpolating (5.20) and (5.22), we obtain

$$(5.23)\qquad \|\mathcal{N}_n[g_1,\dots,g_{n+2}]\|_{L_t^2L_x^2}\leqslant C_0^{n+2}\prod_{j=1}^{n+2}\|g_j\|_{L_t^2H_x^{1/2}}.$$

5.1.2. With no derivative II

Next, we prove

$$\|\mathcal{N}_n[g_1,\dots,g_{n+2}]\|_{L^{5/3}_{t,x}} \leqslant C_0^{n+2}\prod_{j=1}^{n}\|g_j\|_{L^2_tL^3_x}\|g_{n+1}\|_{L^{20/11}_{t,x}}\|g_{n+2}\|_{L^{20/11}_{t,x}}. \tag{5.24}$$

Let $W\in C_c^\infty((0,\infty)\times\mathbb{R}^3)$. Let $W^0:=|W|^{\frac12}$ and $W=W^0W^1$. Define $g^0_{n+1}:=|g|^{\frac{4}{11}}$ and $g_1=g_1^0g_1^1$. Similarly, define $g^1_{n+2}:=|g|^{\frac{4}{11}}$ and $g_{n+2}=g^0_{n+2}g^1_{n+2}$. Also, define $g_j^0:=|g_j|^{\frac12}$ and $g_j=g_j^0g_j^1$ for $j=1,\dots,n$. Then, by Theorem 3.1, we have

$$\begin{aligned}
&\left|\int_0^\infty\int_{\mathbb{R}^3}W(t,x)\mathcal{N}_n[g_1,\dots,g_{n+2}](t,x)dxdt\right| \\
&= \left|\operatorname{Tr}\left[\int_0^\infty U(t)^*W(t,x)U(t)dt\int_0^t U(t_1)^*g_1(t_1)U(t_1)dt_1\right.\right.\\
&\qquad \cdots\int_0^{t_{n-1}}U(t_n)^*g_n(t_n)U(t_n)dt_n\\
&\qquad \left.\left.\times\int_0^{t_n}U(\tau)^*g_{n+1}(\tau)U(\tau)d\tau\int_0^\tau U(\tau_1)^*\gamma_f g_{n+2}(\tau_1)U(\tau_1)d\tau_1\right]\right|\\
&\leqslant \left\|U^{\circledast}W^0\right\|_{\mathfrak{S}^{10}_{t,x\to x}}\left\|W^1(t)U(t)\int_0^t U(\tau)^*g_1^0(\tau)d\tau\right\|_{\mathfrak{S}^{10}_{t,x}}\times\cdots\\
&\qquad\cdots\times\left\|g_n^1(t_n)U(t_n)\int_0^{t_n}U(\tau)^*g^0_{n+1}(\tau)d\tau\right\|_{\mathfrak{S}^{10}_{t,x}}\\
&\qquad\times\left\|g^1_{n+1}(\tau)U(\tau)\int_0^\tau d\tau_1\gamma_fU(\tau_1)g^0_{n+2}(\tau_1)\right\|_{\mathfrak{S}^{5/3}_{t,x}}\left\|g^1_{n+2}U^{\bigcirc}\right\|_{\mathfrak{S}^{10}_{x\to t,x}}\\
&\lesssim_f C_0^{n+2}\left\|W^0\right\|_{L^5_{t,x}}\left\|W^1\right\|_{L^5_{t,x}}\prod_{j=1}^n\|g_j\|_{L^2_tL^3_x}\left\|g^0_{n+1}\right\|_{L^5_{t,x}}\\
&\qquad \|g^1_{n+1}\|_{L^{20/7}_{t,x}}\left\|g^0_{n+2}\right\|_{L^{20/7}_{t,x}}\left\|g^1_{n+2}\right\|_{L^5_{t,x}}\\
&= C_0^{n+2}\|W\|_{L^{5/2}_{t,x}}\prod_{j=1}^n\|g_j\|_{L^2_tL^3_x}\|g_{n+1}\|_{L^{20/11}_{t,x}}\|g_{n+2}\|_{L^{20/11}_{t,x}}.
\end{aligned}\tag{5.25}$$

5.1.3. With derivative

Finally, we prove that

$$\begin{aligned}&\|\nabla\mathcal{N}_n[g_1,\dots,g_{n+2}]\|_{L^{5/2}_{t,x}}\\
&\qquad\leqslant C_0^{n+2}\prod_{j=1}^n\|\langle\nabla\rangle g_j\|_{L^2_tL^{3/2}_x}\|\langle\nabla\rangle g_{n+1}\|_{L^{20/9}_{t,x}}\|\langle\nabla\rangle g_{n+2}\|_{L^{20/9}_{t,x}}.\end{aligned}\tag{5.26}$$

Let $W \in C_c^\infty((0,\infty)\times\mathbb{R}^3)$. Let $f_0 := |f|^{\frac{1}{2}}$ and $f = f_0 f_1$. Then we have

$$\begin{aligned}(5.27)\quad &\left|\int_0^\infty \int_{\mathbb{R}^3} dt(\nabla W)(t,x)\mathcal{N}_1[g_1,\dots,g_{n+2}](t,x)dxdt\right| \\ &\leqslant \Big|\mathrm{Tr}\Big[\int_0^\infty dt U(t)^* W(t) U(t)\langle\nabla\rangle^{-1}\int_0^t dt_1\langle\nabla\rangle U(t_1)^* g_1(t_1)U(t_1) \\ &\qquad \cdots \int_0^{t_1} dt_n U(t_n)^* g_n(t_n) U(t_n) \\ &\quad \times \int_0^{t_n} d\tau U(\tau)^* g_{n+1}(\tau)U(\tau)\int_0^{t_{n+1}} d\tau_1 U(\tau_1)^*\gamma_f g_{n+2}(\tau_1)U(\tau_1)\nabla\Big]\Big| \\ &\quad + \Big|\mathrm{Tr}\Big[\int_0^\infty dt\langle\nabla\rangle^{-1}U(t)^* W(t)U(t)\nabla\int_0^t dt_1 U(t_1)^* g_1(t_1)U(t_1) \\ &\qquad \cdots \int_0^{t_1} dt_n U(t_n)^* g_n(t_n) U(t_n) \\ &\quad \times \int_0^{t_n} d\tau U(\tau)^* g_{n+1}(\tau)U(\tau)\int_0^{t_{n+1}} d\tau_1 U(\tau_1)^*\gamma_f g_{n+2}(\tau_1)U(\tau_1)\langle\nabla\rangle\Big]\Big| =: A + B.\end{aligned}$$

Since it follows from Theorem 3.6 and Lemma 3.3, that

$$\|\rho(U(t)_\star\gamma_0)\|_{L^{5/2}_{t,x}} \lesssim \|\langle\nabla\rangle\gamma_0\|_{\mathfrak{S}^{15/8}}, \tag{5.28}$$

$$\|\rho(U(t)_\star\gamma_0)\|_{L^{20/11}_{t,x}} \lesssim \left\|\langle\nabla\rangle^{\frac{1}{4}}\gamma_0\right\|_{\mathfrak{S}^{15/11}}, \tag{5.29}$$

$$\langle\nabla\rangle = \langle\nabla\rangle^{-1} - \nabla\cdot\nabla\langle\nabla\rangle^{-1}, \tag{5.30}$$

the duality argument and the usual Strichartz estimates implies

$$A \leqslant C_0^{n+2}\|W\|_{L^{5/3}_{t,x}}\prod_{j=1}^n \|\langle\nabla\rangle g_j\|_{L^2_t L^{3/2}_x}\|\langle\nabla\rangle g_{n+1}\|_{L^{20/9}_{t,x}}\|\langle\nabla\rangle g_{n+2}\|_{L^{20/9}_{t,x}}, \tag{5.31}$$

$$B \leqslant C_0^{n+2}\|W\|_{L^{5/3}_{t,x}}\prod_{j=1}^n \|\langle\nabla\rangle g_j\|_{L^2_t L^{3/2}_x}\|\langle\nabla\rangle g_{n+1}\|_{L^{20/9}_{t,x}}\|\langle\nabla\rangle g_{n+2}\|_{L^{20/9}_{t,x}}, \tag{5.32}$$

which yields the desired estimate.

5.1.4. Conclusion

By interpolating (5.24) and (5.26), we obtain

$$\left\||\nabla|^{\frac{1}{2}}\mathcal{N}_n[g_1,\dots,g_{n+2}]\right\|_{L^2_t L^2_x} \leqslant C_0^{n+2}\prod_{j=1}^{n+2}\|g_j\|_{L^2_t H^{1/2}_x}. \tag{5.33}$$

By (5.23) and (5.33), we get

$$\|\mathcal{N}_n[g_1,\dots,g_{n+2}]\|_{L^2_t H^{1/2}_x} \leqslant C_0^{n+2}\prod_{j=1}^{n+2}\|g_j\|_{L^2_t H^{1/2}_x}. \tag{5.34}$$

Therefore, we have

$$\begin{aligned}\left\|N_2^1(V)\right\|_{L_t^2 H_x^{1/2}} &\leqslant \sum_{n=0}^{\infty}\|\mathcal{N}_n(V)\|_{L_t^2 H_x^{1/2}} = \sum_{n=0}^{\infty}\|\mathcal{N}_n[V,\dots,V]\|_{L_t^2 H_x^{1/2}} \\ &\leqslant \sum_{n=0}^{\infty} C_0^{n+2}\|V\|_{L_t^2 H_x^{1/2}}^{n+2} \lesssim \|V\|_{L_t^2 H_x^{1/2}}^2\end{aligned} \tag{5.35}$$

for sufficiently small $\|V\|_{L_t^2 H_x^{1/2}}$. We can prove (5.2) in the same argument as in the proof of Lemmas 3.4, 3.10 and Theorem 4.1.

5.2. Estimate for N_2^2

By (5.12), we have

$$\begin{aligned}&N_2^2(V) \\ &= \sum_{n=0}^{\infty}\int_0^t\int_0^\tau \rho\Big[U(t)\mathcal{W}_V^{(n)}(t,\tau)U(\tau)^*V(\tau)U(\tau-\tau_1)V(\tau_1)\gamma_f U(\tau_1-t)\Big]d\tau_1 d\tau \\ &= \sum_{n=0}^{\infty}\mathcal{M}_n(V).\end{aligned} \tag{5.36}$$

Define $\mathcal{M}_n[g_1,\dots,g_{n+2}]$ in the same manner as $\mathcal{N}_n[g_1,\dots,g_{n+2}]$.

5.2.1. With no derivative I

In the same way as we proved (5.23), we have

$$\|\mathcal{M}_n[g_1,\dots,g_{n+2}]\|_{L_t^2 L_x^2} \leqslant C_0^{n+2}\prod_{j=1}^{n+2}\|g_j\|_{L_t^2 H_x^{1/2}}. \tag{5.37}$$

5.2.2. With no derivative II

In the same way as we proved (5.24), we get

$$\|\mathcal{M}_n[g_1,\dots,g_{n+2}]\|_{L_{t,x}^{20/9}} \leqslant C_0^{n+2}\prod_{j=1}^{n}\|g_j\|_{L_x^{5/2}}\|g_{n+1}\|_{L_{t,x}^{5/2}}\|g_{n+2}\|_{L_{t,x}^{20/11}}. \tag{5.38}$$

5.2.3. With derivatives

We will prove

$$\begin{aligned}&\|\nabla\mathcal{M}_n[g_1,\dots,g_{n+2}]\|_{L_{t,x}^{20/11}} \\ &\qquad\leqslant C_0^{n+2}\prod_{j=1}^{n}\|\langle\nabla\rangle g_j\|_{L_{t,x}^{5/3}}\|\langle\nabla\rangle g_{n+1}\|_{L_{t,x}^{5/3}}\|\langle\nabla\rangle g_{n+2}\|_{L_{t,x}^{20/9}}.\end{aligned} \tag{5.39}$$

Lemma 5.3. — *Let $X_1,\dots,X_n$ and X be Banach spaces. Assume that bilinear maps $B, B_< : L_t^{p_1}(\mathbb{R}_+,X_1)\times\cdots\times L_t^{p_n}(\mathbb{R}_+,X_n)\to L_t^p(\mathbb{R}_+,X)$ are defined by*

$$B(g_1,\dots,g_n) := \int_0^\infty dt_1 K_1(t,t_1)g_1(t_1)\cdots\int_0^\infty dt_n K_n(t_{n-1},t_n)g_n(t_n), \tag{5.40}$$

$$B_<(g_1,\dots,g_n) := \int_0^t dt_1 K_1(t,t_1)g_1(t_1)\cdots\int_0^{t_{n-1}} dt_n K_n(t_{n-1},t_n)g_n(t_n). \tag{5.41}$$

If $p>p_1$ and $1/p_j+1/p_{j+1}>1$ for all $j=1,\cdots,n-1$, then we have

$$\begin{aligned}\|B(g_1,\dots,g_n)\|_{L_t^pX} &\lesssim \prod_{j=1}^n \|g_j\|_{L_t^{p_j}X_j}\\ \implies \|B_<(g_1,\dots,g_n)\|_{L_t^pX} &\lesssim \prod_{j=1}^n \|g_j\|_{L_t^{p_j}X_j}\end{aligned} \tag{5.42}$$

Idea of the proof. — We can assume that $n=2$. Define

$$\widetilde{B}_<(g_1,g_2) := \int_0^\infty dt_1 K_1(t,t_1)g_1(t_1)\int_0^{t_1} dt_2 K_2(t_1,t_2)g_2(t_2). \tag{5.43}$$

By the proof of the Christ–Kiselev lemma by the Whitney decomposition (see [Tao00]), we have

$$B \text{ is bounded } \implies \widetilde{B}_< \text{ is bounded }. \tag{5.44}$$

By the usual Christ–Kiselev lemma (see [CK01]), we get

$$\widetilde{B}_< \text{ is bounded } \implies B_< \text{ is bounded.} \tag{5.45}$$ □

Hence, it suffices to prove

$$\|\nabla\mathcal{M}_n^*[g_1,\dots,g_{n+2}]\|_{L_{t,x}^{20/11}} \lesssim \|\langle\nabla\rangle g_1\|_{L_{t,x}^{5/3}}\cdots\|\langle\nabla\rangle g_{n+1}\|_{L_{t,x}^{5/3}}\|\langle\nabla\rangle g_{n+2}\|_{L_{t,x}^{20/9}}, \tag{5.46}$$

where

$$\begin{aligned}\mathcal{M}_n^*[g_1,\dots,g_{n+2}](t) := (-i)^n\int_0^\infty dt_1\cdots\int_0^\infty dt_n\int_0^\infty d\tau\int_0^\infty d\tau_1\\ \times\rho\Big[U(t)\mathcal{U}[g_1](t_1)\cdots\mathcal{U}[g_n](t_n)U(\tau)^*g_{n+1}(\tau)U(\tau-\tau_1)g_{n+2}(\tau_1)\gamma_f U(\tau_1-t)\Big].\end{aligned} \tag{5.47}$$

For simplicity, we only consider $\mathcal{M}_0^*[g_1,g_2]$; we can deal with the general case in the completely same way. Let $W\in C_c^\infty((0,\infty)\times\mathbb{R}^3)$. Then we have

$$\begin{aligned}&\left|\int_0^\infty dt\int_{\mathbb{R}^3} dx W(t,x)\nabla\mathcal{M}_0^*[g_1,g_2](t,x)\right|\\ &\qquad\leqslant \left\|\langle\nabla\rangle^{-\frac34-\varepsilon}\int_0^\infty dt U(t)^*\nabla W(t)U(t)\langle\nabla\rangle^{-\frac34}\right\|_{\mathfrak{S}^{20/7}}\\ &\times\left\|\langle\nabla\rangle^{\frac34}\int_0^\infty dt_1 U(t_1)^*g_1(t_1)U(t_1)\int_0^\infty dt_2 U(t_2)^*g_2(t_2)\gamma_f U(t_2)^*\langle\nabla\rangle^{\frac34+\varepsilon}\right\|_{\mathfrak{S}^{20/13}} = AB.\end{aligned} \tag{5.48}$$

By [CHP18, Theorem 3.1], we have

$$\left\||\nabla|^{\frac32}\rho(U(t)_\star\gamma)\right\|_{L_{t,x}^2} \lesssim \left\|\langle\nabla\rangle^{1+\varepsilon}\gamma\langle\nabla\rangle\right\|_{\mathfrak{S}^2}. \tag{5.49}$$

By interpolating

$$\|\rho(U(t)_\star\gamma)\|_{L^{5/3}_{t,x}} \lesssim \|\gamma\|_{\mathfrak{S}^{5/4}}, \tag{5.50}$$

$$\|\nabla\rho(U(t)_\star\gamma)\|_{L^{5/3}_{t,x}} \lesssim \|\langle\nabla\rangle\gamma\langle\nabla\rangle\|_{\mathfrak{S}^{5/4}}, \tag{5.51}$$

we get

$$\left\||\nabla|^{\frac{1}{2}}\rho(U(t)_\star\gamma)\right\|_{L^{5/3}_{t,x}} \lesssim \left\|\langle\nabla\rangle^{\frac{1}{2}}\gamma\langle\nabla\rangle^{\frac{1}{2}}\right\|_{\mathfrak{S}^{5/4}}. \tag{5.52}$$

By interpolating (5.49) and (5.52), we obtain

$$\||\nabla|\rho(U(t)_\star\gamma)\|_{L^{20/11}_{t,x}} \lesssim \left\|\langle\nabla\rangle^{\frac{3}{4}+\varepsilon}\gamma\langle\nabla\rangle^{\frac{3}{4}}\right\|_{\mathfrak{S}^{20/13}}. \tag{5.53}$$

Note that the above γ does not need to be self-adjoint. Hence we have

$$A \lesssim \|W\|_{L^{20/9}_{t,x}}. \tag{5.54}$$

Next, we estimate B. On the one hand, we have

$$\begin{aligned} B_1 &:= \left\|\int_0^\infty dt_1 U(t_1)^* g_1(t_1)U(t_1)\int_0^\infty dt_2 U(t_2)^* g_2(t_2)\gamma_f U(t_2)^*\langle\nabla\rangle^{\frac{3}{4}+\varepsilon}\right\|_{\mathfrak{S}^{20/13}} \\ &\lesssim \left\|\int_0^\infty dt_1 U(t_1)^* g_1(t_1)U(t_1)\langle\nabla\rangle^{-1}\int_0^\infty dt_2 U(t_2)^*(\nabla g_2)(t_2)\gamma_f U(t_2)^*\langle\nabla\rangle^{\frac{3}{4}+\varepsilon}\right\|_{\mathfrak{S}^{20/13}} \\ &+ \left\|\int_0^\infty dt_1 U(t_1)^* g_1(t_1)U(t_1)\langle\nabla\rangle^{-1}\int_0^\infty dt_2 U(t_2)^* g_2(t_2)\nabla\gamma_f U(t_2)^*\langle\nabla\rangle^{\frac{3}{4}+\varepsilon}\right\|_{\mathfrak{S}^{20/13}}. \end{aligned} \tag{5.55}$$

Since Lemma 3.3 implies

$$\|\rho(U(t)_\star\gamma)\|_{L^{5/2}_t L^{60/29}_x} \lesssim \left\|\gamma\langle\nabla\rangle^{\frac{3}{4}}\right\|_{\mathfrak{S}^{60/37}}, \tag{5.56}$$

$$\|\rho(U(t)_\star\gamma)\|_{L^{20/11}_{t,x}} \lesssim \left\|\gamma\langle\nabla\rangle^{\frac{1}{4}}\right\|_{\mathfrak{S}^{15/11}}, \tag{5.57}$$

we have

$$\begin{aligned} &\left\|\int_0^\infty dt_1 U(t_1)^* g_1(t_1)U(t_1)\langle\nabla\rangle^{-1}\int_0^\infty dt_2 U(t_2)^* g_2(t_2)\nabla\gamma_f U(t_2)^*\langle\nabla\rangle^{\frac{3}{4}+\varepsilon}\right\|_{\mathfrak{S}^{20/13}} \\ &\quad\leqslant \left\|\int_0^\infty U(t_1)^* g_1(t_1)U(t_1)dt_1\langle\nabla\rangle^{-\frac{3}{4}}\right\|_{\mathfrak{S}^{60/23}} \\ &\qquad\qquad \left\|\langle\nabla\rangle^{-\frac{1}{4}}\int_0^\infty U(t_2)^* g_2(t_2)\gamma_f\nabla U(t_2)^*\langle\nabla\rangle^{\frac{3}{4}+\varepsilon}dt_2\right\|_{\mathfrak{S}^{15/4}} \\ &\quad\lesssim \|g_1\|_{L^{5/3}_t L^{60/31}_x}\|g_2\|_{L^{20/9}_{t,x}}\left\|\langle\xi\rangle^2 f(\xi)\right\|_{L^\infty_\xi} \\ &\quad\lesssim \|\langle\nabla\rangle g_1\|_{L^{5/3}_{t,x}}\|g_2\|_{L^{20/9}_{t,x}}\left\|\langle\xi\rangle^2 f(\xi)\right\|_{L^\infty_\xi}. \end{aligned} \tag{5.58}$$

By interpolating

$$\||\nabla|\rho(U(t)_\star\gamma)\|_{L^2_{t,x}} \lesssim \left\|\langle\nabla\rangle^{\frac{1}{2}}\gamma\langle\nabla\rangle^{1+\varepsilon}\right\|_{\mathfrak{S}^2} \tag{5.59}$$

$$\left\||\nabla|^{\frac{1}{2}}\rho(U(t)_\star\gamma)\right\|_{L^{5/3}_{t,x}} \lesssim \left\|\langle\nabla\rangle^{\frac{1}{2}}\gamma\langle\nabla\rangle^{\frac{1}{2}}\right\|_{\mathfrak{S}^{5/4}}, \tag{5.60}$$

we obtain

$$\left\||\nabla|^{\frac{3}{4}}\rho(U(t)_\star\gamma)\right\|_{L^{20/11}_{t,x}} \lesssim \left\|\langle\nabla\rangle^{\frac{1}{2}}\gamma\langle\nabla\rangle^{\frac{3}{4}+\varepsilon}\right\|_{\mathfrak{S}^{20/13}}. \tag{5.61}$$

Hence, we conclude that

$$\left\|\langle\nabla\rangle^{-\frac{1}{2}}\int_0^\infty U(t_2)^*(\nabla g_2)(t_2)\gamma_f U(t_2)^*\langle\nabla\rangle^{\frac{3}{4}+\varepsilon}dt_2\right\|_{\mathfrak{S}^{20/7}} \lesssim \left\|\langle\nabla\rangle^{\frac{1}{4}}g_2\right\|_{L^{20/9}_{t,x}}\left\|\langle\xi\rangle^2 f(\xi)\right\|_{L^\infty_\xi}. \tag{5.62}$$

Therefore, by using

$$\|\rho(U(t)_\star\gamma)\|_{L_t^{5/2}L_x^{30/17}} \lesssim \left\|\gamma\langle\nabla\rangle^{\frac{1}{2}}\right\|_{\mathfrak{S}^{10/7}}, \tag{5.63}$$

we have

$$\begin{aligned}
&\left\|\int_0^\infty dt_1 U(t_1)^* g_1(t_1)U(t_1)\langle\nabla\rangle^{-1}\right.\\
&\qquad\left.\int_0^\infty dt_2 U(t_2)^*(\nabla g_2)(t_2)\gamma_f U(t_2)^*\langle\nabla\rangle^{\frac{3}{4}+\varepsilon}\right\|_{\mathfrak{S}^{20/13}}\\
&\lesssim \left\|\int_0^\infty dt_1 U(t_1)^* g_1(t_1)U(t_1)\langle\nabla\rangle^{-\frac{1}{2}}\right\|_{\mathfrak{S}^{10/3}}\\
&\qquad\left\|\langle\nabla\rangle^{-\frac{1}{2}}\int_0^\infty dt_2 U(t_2)^*(\nabla g_2)(t_2)\gamma_f U(t_2)^*\langle\nabla\rangle^{\frac{3}{4}+\varepsilon}\right\|_{\mathfrak{S}^{20/7}}\\
&\lesssim \|g_1\|_{L_t^{5/3}L_x^{30/13}}\left\|\langle\nabla\rangle^{\frac{1}{4}}g_2\right\|_{L^{20/9}_{t,x}}\left\|\langle\xi\rangle^2 f(\xi)\right\|_{L^\infty_\xi}\\
&\lesssim \|\langle\nabla\rangle g_1\|_{L^{5/3}_{t,x}}\left\|\langle\nabla\rangle^{\frac{1}{4}}g_2\right\|_{L^{20/9}_{t,x}}\left\|\langle\xi\rangle^2 f(\xi)\right\|_{L^\infty_\xi}.
\end{aligned} \tag{5.64}$$

From the above, we get

$$B_1 \lesssim \|\langle\nabla\rangle g_1\|_{L^{5/3}_{t,x}}\left\|\langle\nabla\rangle^{\frac{1}{4}}g_2\right\|_{L^{20/9}_{t,x}}\left\|\langle\xi\rangle^2 f(\xi)\right\|_{L^\infty_\xi}. \tag{5.65}$$

On the other hand, we have

$$\begin{aligned}
&B_2\\
&:= \left\|\nabla\int_0^\infty dt_1 U(t_1)^* g_1(t_1)U(t_1)\int_0^\infty dt_2 U(t_2)^* g_2(t_2)\gamma_f U(t_2)^*\langle\nabla\rangle^{\frac{3}{4}+\varepsilon}\right\|_{\mathfrak{S}^{20/13}}\\
&\leqslant \left\|\int_0^\infty dt_1 U(t_1)^*(\nabla g_1)(t_1)U(t_1)\int_0^\infty dt_2 U(t_2)^* g_2(t_2)\gamma_f U(t_2)^*\langle\nabla\rangle^{\frac{3}{4}+\varepsilon}\right\|_{\mathfrak{S}^{20/13}}\\
&\quad+\left\|\int_0^\infty dt_1 U(t_1)^* g_1(t_1)U(t_1)\int_0^\infty dt_2 U(t_2)^*(\nabla g_2)(t_2)\gamma_f U(t_2)^*\langle\nabla\rangle^{\frac{3}{4}+\varepsilon}\right\|_{\mathfrak{S}^{20/13}}\\
&\quad+\left\|\int_0^\infty dt_1 U(t_1)^* g_1(t_1)U(t_1)\int_0^\infty dt_2 U(t_2)^* g_2(t_2)\nabla\gamma_f U(t_2)^*\langle\nabla\rangle^{\frac{3}{4}+\varepsilon}\right\|_{\mathfrak{S}^{20/13}}\\
&= C_1 + C_2 + C_3.
\end{aligned} \tag{5.66}$$

For C_1, we obtain

$$
\begin{aligned}
C_1 \lesssim & \left\| \int_0^\infty dt_1 U(t_1)^* (\nabla g_1)(t_1) U(t_1) \langle \nabla \rangle^{-1} \right. \\
& \qquad \left. \int_0^\infty dt_2 U(t_2)^* (\nabla g_2)(t_2) \gamma_f U(t_2)^* \langle \nabla \rangle^{\frac{3}{4}+\varepsilon} \right\|_{\mathfrak{S}^{20/13}} \\
& + \left\| \int_0^\infty dt_1 U(t_1)^* (\nabla g_1)(t_1) U(t_1) \langle \nabla \rangle^{-1} \right. \\
& \qquad \left. \int_0^\infty dt_2 U(t_2)^* g_2(t_2) \nabla \gamma_f U(t_2)^* \langle \nabla \rangle^{\frac{3}{4}+\varepsilon} \right\|_{\mathfrak{S}^{20/13}}
\end{aligned} \tag{5.67}
$$

Since it follows from Theorem 3.6 and Lemma 3.3 that

$$\|\rho(U(t)_\star \gamma)\|_{L_x^{5/2}} \lesssim \|\gamma \langle \nabla \rangle\|_{\mathfrak{S}^{15/8}}, \tag{5.68}$$

$$\|\rho(U(t)_\star \gamma)\|_{L_t^{20/11} L_x^{30/19}} \lesssim \|\gamma\|_{\mathfrak{S}^{60/49}}, \tag{5.69}$$

we conclude that

$$
\begin{aligned}
& \left\| \int_0^\infty dt_1 U(t_1)^* (\nabla g_1)(t_1) U(t_1) \langle \nabla \rangle^{-1} \right. \\
& \qquad \left. \int_0^\infty dt_2 U(t_2)^* g_2(t_2) \nabla \gamma_f U(t_2)^* \langle \nabla \rangle^{\frac{3}{4}+\varepsilon} \right\|_{\mathfrak{S}^{20/13}} \\
& \lesssim \left\| \int_0^\infty dt_1 U(t_1)^* (\nabla g_1)(t_1) U(t_1) \langle \nabla \rangle^{-1} \right\|_{\mathfrak{S}^{15/7}} \\
& \qquad \left\| \int_0^\infty dt_2 U(t_2)^* g_2(t_2) \nabla \gamma_f U(t_2)^* \langle \nabla \rangle^{\frac{3}{4}+\varepsilon} \right\|_{\mathfrak{S}^{60/11}} \\
& \lesssim \|\langle \nabla \rangle g_1\|_{L_{t,x}^{5/3}} \|g_2\|_{L_t^{20/9} L_x^{30/11}} \left\| \langle \xi \rangle^2 f(\xi) \right\|_{L_\xi^\infty} \\
& \lesssim \|\langle \nabla \rangle g_1\|_{L_{t,x}^{5/3}} \|\langle \nabla \rangle g_2\|_{L_{t,x}^{20/9}} \left\| \langle \xi \rangle^2 f(\xi) \right\|_{L_\xi^\infty}.
\end{aligned} \tag{5.70}
$$

Since it follows from Theorem 3.6 and Lemma 3.3 that

$$\|\rho(U(t)_\star \gamma)\|_{L_x^{5/2}} \lesssim \|\gamma \langle \nabla \rangle\|_{\mathfrak{S}^{15/8}}, \tag{5.71}$$

$$\|\rho(U(t)_\star \gamma)\|_{L_{t,x}^{20/11}} \lesssim \|\gamma \langle \nabla \rangle^{\frac{1}{4}}\|_{\mathfrak{S}^{15/11}}, \tag{5.72}$$

we get

$$
\begin{aligned}
& \left\| \int_0^\infty dt_1 U(t_1)^* (\nabla g_1)(t_1) U(t_1) \langle \nabla \rangle^{-1} \right. \\
& \qquad \left. \int_0^\infty dt_2 U(t_2)^* (\nabla g_2)(t_2) \gamma_f U(t_2)^* \langle \nabla \rangle^{\frac{3}{4}+\varepsilon} \right\|_{\mathfrak{S}^{20/13}} \\
& \qquad\qquad \lesssim \|\langle \nabla \rangle g_1\|_{L_{t,x}^{5/3}} \|\langle \nabla \rangle g_2\|_{L_{t,x}^{20/9}} \left\| \langle \xi \rangle^2 f(\xi) \right\|_{L_\xi^\infty}.
\end{aligned} \tag{5.73}
$$

From the above, we have

$$C_1 \lesssim \|\langle\nabla\rangle g_1\|_{L^{5/3}_{t,x}}\|\langle\nabla\rangle g_2\|_{L^{20/9}_{t,x}} \left\|\langle\xi\rangle^2 f(\xi)\right\|_{L^\infty_\xi}. \tag{5.74}$$

In the same way, we obtain

$$C_3 \leqslant \|\langle\nabla\rangle g_1\|_{L^{5/3}_{t,x}}\|\langle\nabla\rangle g_2\|_{L^{20/9}_{t,x}} \left\|\langle\xi\rangle^3 f(\xi)\right\|_{L^\infty_\xi} \tag{5.75}$$

For C_2, we have

$$\begin{aligned} C_2 \leqslant &\left\|\int_0^\infty dt_1 U(t_1)^* g_1(t_1) U(t_1)\langle\nabla\rangle^{-\frac{1}{2}}\right\|_{\mathfrak{S}^{10/3}} \\ &\times \left\|\int_0^\infty dt_2 \langle\nabla\rangle^{\frac{1}{2}} U(t_2)^*(\nabla g_2)(t_2)\gamma_f U(t_2)^*\langle\nabla\rangle^{\frac{3}{4}+\varepsilon}\right\|_{\mathfrak{S}^{20/7}}. \end{aligned} \tag{5.76}$$

Since

$$\|\rho(U(t)_\star\gamma)\|_{L^{5/2}_t L^{30/17}_x} \lesssim \left\|\gamma\langle\nabla\rangle^{\frac{1}{2}}\right\|_{\mathfrak{S}^{10/7}}, \tag{5.77}$$

we have

$$\left\|\int_0^\infty dt_1 U(t_1)^* g_1(t_1) U(t_1)\langle\nabla\rangle^{-\frac{1}{2}}\right\|_{\mathfrak{S}^{10/3}} \lesssim \|g_1\|_{L^{5/3}_t L^{30/13}_x} \lesssim \|\langle\nabla\rangle g_1\|_{L^{5/3}_{t,x}}. \tag{5.78}$$

Note that

$$\begin{aligned} &\left\|\langle\nabla\rangle^{\frac{1}{2}}\int_0^\infty dt_2 U(t_2)^*(\nabla g_2)(t_2)\gamma_f U(t_2)^*\langle\nabla\rangle^{\frac{3}{4}+\varepsilon}\right\|_{\mathfrak{S}^{20/7}} \\ &\quad\lesssim \left\|\langle\nabla\rangle^{-\frac{1}{2}}\int_0^\infty dt_2 U(t_2)^*(|\nabla|^2 g_2)(t_2)\gamma_f U(t_2)^*\langle\nabla\rangle^{\frac{3}{4}+\varepsilon}\right\|_{\mathfrak{S}^{20/7}} \\ &\qquad + \left\|\langle\nabla\rangle^{-\frac{1}{2}}\int_0^\infty dt_2 U(t_2)^*(\nabla g_2)(t_2)\nabla\gamma_f U(t_2)^*\langle\nabla\rangle^{\frac{3}{4}+\varepsilon}\right\|_{\mathfrak{S}^{20/7}}. \end{aligned} \tag{5.79}$$

By interpolating

$$\||\nabla|\rho(U(t)_\star\gamma)\|_{L^2_{t,x}} \lesssim \left\|\langle\nabla\rangle^{\frac{1}{2}}\gamma\langle\nabla\rangle^{1+\varepsilon}\right\|_{\mathfrak{S}^2} \tag{5.80}$$

$$\||\nabla|^{\frac{1}{2}}\rho(U(t)_\star\gamma)\|_{L^{5/3}_{t,x}} \lesssim \left\|\langle\nabla\rangle^{\frac{1}{2}}\gamma\langle\nabla\rangle^{\frac{1}{2}}\right\|_{\mathfrak{S}^{5/4}}, \tag{5.81}$$

we obtain

$$\left\||\nabla|^{\frac{3}{4}}\rho(U(t)_\star\gamma)\right\|_{L^{20/11}_{t,x}} \lesssim \left\|\langle\nabla\rangle^{\frac{1}{2}}\gamma\langle\nabla\rangle^{\frac{3}{4}+\varepsilon}\right\|_{\mathfrak{S}^{20/13}}. \tag{5.82}$$

Hence, we have

$$\begin{aligned} &\left\|\langle\nabla\rangle^{-\frac{1}{2}}\int_0^\infty dt_2 U(t_2)^*\left(|\nabla|^2 g_2\right)(t_2)\gamma_f U(t_2)^*\langle\nabla\rangle^{\frac{3}{4}+\varepsilon}\right\|_{\mathfrak{S}^{20/7}} \\ &\qquad\lesssim \left\||\nabla|^{\frac{5}{4}} g_2\right\|_{L^{20/9}_{t,x}} \left\|\langle\xi\rangle^2 f(\xi)\right\|_{L^\infty_\xi} \\ &\qquad\lesssim \left\|\langle\nabla\rangle^{\frac{5}{4}} g_2\right\|_{L^{20/9}_{t,x}} \left\|\langle\xi\rangle^2 f(\xi)\right\|_{L^\infty_\xi}. \end{aligned} \tag{5.83}$$

From the above, we conclude that

$$C_2 \lesssim \|\langle\nabla\rangle g_1\|_{L^{5/3}_{t,x}}\|\langle\nabla\rangle^{\frac{5}{4}} g_2\|_{L^{20/9}_{t,x}}\|\langle\xi\rangle^3 f(\xi)\|_{L^\infty_\xi}. \tag{5.84}$$

Finally, interpolating

$$
\begin{aligned}
(5.85)\quad B_1 &= \left\| \int_0^\infty dt_1 U(t_1)^* g_1(t_1) U(t_1) \int_0^\infty dt_2 U(t_2)^* g_2(t_2) \gamma_f U(t_2)^* \langle\nabla\rangle^{\frac{3}{4}+\varepsilon} \right\|_{\mathfrak{S}^{20/13}} \\
&\lesssim \|\langle\nabla\rangle g_1\|_{L^{5/3}_{t,x}} \left\|\langle\nabla\rangle^{\frac{1}{4}} g_2\right\|_{L^{20/9}_{t,x}} \left\|\langle\xi\rangle^2 f(\xi)\right\|_{L^\infty_\xi}, \\
(5.86)\quad B_2 &= \left\| \nabla \int_0^\infty dt_1 U(t_1)^* g_1(t_1) U(t_1) \int_0^\infty dt_2 U(t_2)^* g_2(t_2) \gamma_f U(t_2)^* \langle\nabla\rangle^{\frac{3}{4}+\varepsilon} \right\|_{\mathfrak{S}^{20/13}} \\
&\lesssim \|\langle\nabla\rangle g_1\|_{L^{5/3}_{t,x}} \left\|\langle\nabla\rangle^{\frac{5}{4}} g_2\right\|_{L^{20/9}_{t,x}} \left\|\langle\xi\rangle^3 f(\xi)\right\|_{L^\infty_\xi}.
\end{aligned}
$$

we obtain

$$
\begin{aligned}
(5.87)\quad B = \Bigg\| \langle\nabla\rangle^{\frac{3}{4}} \int_0^\infty dt_1 U(t_1)^* g_1(t_1) U(t_1) & \\
\int_0^\infty dt_2 U(t_2)^* g_2(t_2) \gamma_f U(t_2)^* \langle\nabla\rangle^{\frac{3}{4}+\varepsilon} \Bigg\|_{\mathfrak{S}^{20/13}} & \\
\lesssim \|\langle\nabla\rangle g_1\|_{L^{5/3}_{t,x}} \|\langle\nabla\rangle g_2\|_{L^{20/9}_{t,x}} \left\|\langle\xi\rangle^3 f(\xi)\right\|_{L^\infty_\xi}. &
\end{aligned}
$$

5.2.4. Conclusion

By interpolating (5.38) and (5.39), we obtain

$$
(5.88)\qquad \left\| |\nabla|^{\frac{1}{2}} \mathcal{M}_1[g_1, \dots, g_{n+2}] \right\|_{L^2_t L^2_x} \leqslant C_0^{n+2} \prod_{j=1}^{n+2} \|g_j\|_{L^2_t H^{1/2}_x}.
$$

By (5.37) and (5.88), we get

$$
(5.89)\qquad \|\mathcal{M}_1[g_1, \dots, g_{n+2}]\|_{L^2_t H^{1/2}_x} \leqslant C_0^{n+2} \prod_{j=1}^{n+2} \|g_j\|_{L^2_t H^{1/2}_x}.
$$

Therefore, we have

$$
\begin{aligned}
(5.90)\qquad \|N_2^2(V)\|_{L^2_t H^{1/2}_x} &\leqslant \sum_{n=0}^\infty \|\mathcal{M}_n(V)\|_{L^2_t H^{1/2}_x} = \sum_{n=0}^\infty \|\mathcal{M}_n[V, \dots, V]\|_{L^2_t H^{1/2}_x} \\
&\leqslant \sum_{n=0}^\infty C_0^{n+2} \|V\|^{n+2}_{L^2_t H^{1/2}_x} \lesssim \|V\|^2_{L^2_t H^{1/2}_x}
\end{aligned}
$$

for sufficiently small $\|V\|_{L^2_t H^{1/2}_x}$. We can prove (5.2) in the same argument as in the proof of Lemmas 3.4, 3.10 and Theorem 4.1.

6. Proof of Lemma 2.1

Proof. — Let

$$
\begin{aligned}
(6.1)\qquad Q(t) &= U_V(t)_\star Q_0 - i \int_0^t U_V(t,\tau)_\star [V(\tau), \gamma_f] d\tau \\
(6.2)\qquad &=: Q_1(t) + Q_2(t).
\end{aligned}
$$

First we consider $Q_1(t)$. We have

$$\langle\nabla\rangle^{\frac{1}{2}}Q_1(t)\langle\nabla\rangle^{\frac{1}{2}} = \left(\langle\nabla\rangle^{\frac{1}{2}}U_V(t)\langle\nabla\rangle^{-\frac{1}{2}}\right)_\star\left(\langle\nabla\rangle^{\frac{1}{2}}Q_0\langle\nabla\rangle^{\frac{1}{2}}\right). \tag{6.3}$$

It is easily proven by (3.23) that $\langle\nabla\rangle^{\frac{1}{2}}U_V(t)\langle\nabla\rangle^{-\frac{1}{2}}$ is strongly continuous. Therefore, $Q_1(t)\in C(\mathbb{R},\mathcal{H}^{\frac{1}{2}})$. Next we consider $Q_2(t)$. We obtain

$$\begin{aligned}Q_2(t+h)-Q_2(t) &= -i\int_t^{t+h}U_V(t+h,\tau)[V(\tau),\gamma_f]U_V(\tau,t+h)d\tau\\ &\quad -i\int_0^t\Big(U_V(t+h,\tau)-U_V(t,\tau)\Big)[V(\tau),\gamma_f]U_V(\tau,t+h)d\tau\\ &\quad -i\int_0^t U_V(t,\tau)[V(\tau),\gamma_f]\Big(U_V(\tau,t+h)-U_V(\tau,t)\Big)d\tau\\ &=: A+B+C.\end{aligned} \tag{6.4}$$

Lemma 3.10 and Kato–Seiler–Simon inequality ([SS75]; see also [Sim05, Theorem 4.1]) imply

$$\begin{aligned}&\left\|\langle\nabla\rangle^{\frac{1}{2}}A\langle\nabla\rangle^{\frac{1}{2}}\right\|_{\mathfrak{S}^2}\\ &\leqslant\int_t^{t+h}\left\|\langle\nabla\rangle^{\frac{1}{2}}U_V(t+h,\tau)\langle\nabla\rangle^{-\frac{1}{2}}\right\|_{\mathcal{B}(L^2_x)}\left\|\langle\nabla\rangle^{\frac{1}{2}}[V(\tau),\gamma_f]\langle\nabla\rangle^{\frac{1}{2}}\right\|_{\mathfrak{S}^2}\\ &\qquad\left\|\langle\nabla\rangle^{-\frac{1}{2}}U_V(\tau,t)\langle\nabla\rangle^{\frac{1}{2}}\right\|_{\mathcal{B}(L^2_x)}d\tau\\ &\leqslant\varphi\left(\|V\|_{L^2_tH^{1/2}_x}\right)\|\langle\xi\rangle f(\xi)\|_{L^2_\xi}\int_t^{t+h}\|V(\tau)\|_{H^{1/2}_x}d\tau\to 0\text{ as }h\to 0.\end{aligned} \tag{6.5}$$

By the similar argument and the fact that $\langle\nabla\rangle^{\frac{1}{2}}U_V(t)\langle\nabla\rangle^{-\frac{1}{2}}$ is strongly continuous, we have

$$\left\|\langle\nabla\rangle^{\frac{1}{2}}B\langle\nabla\rangle^{\frac{1}{2}}\right\|_{\mathfrak{S}^2}\to 0\text{ as }h\to 0,\quad \left\|\langle\nabla\rangle^{\frac{1}{2}}C\langle\nabla\rangle^{\frac{1}{2}}\right\|_{\mathfrak{S}^2}\to 0\text{ as }h\to 0, \tag{6.6}$$

which yields $Q_2(t)\in C(\mathbb{R},\mathcal{H}^{\frac{1}{2}})$. Therefore, we obtain $Q(t)\in C(\mathbb{R},\mathcal{H}^{\frac{1}{2}})$.

Next, we prove the scattering. First we consider $Q_1(t)\in C(\mathbb{R},\mathcal{H}^{\frac{1}{2}})$. We have

$$\begin{aligned}&\|U(-t)Q_1(t)U(t)-U(-s)Q_1(s)U(s)\|_{\mathfrak{S}^3}\\ &\leqslant\|U(-t)U_V(t)-U(-s)U_V(s)\|_{\mathcal{B}}\|Q_0\|_{\mathfrak{S}^3}\\ &\qquad+\|Q_0\|_{\mathfrak{S}^3}\left\|U_V(t)^*U(t)-U_V(s)^*U(s)\right\|_{\mathcal{B}}.\end{aligned} \tag{6.7}$$

Since

$$\begin{aligned}\|(U(-t)U_V(t)-U(-s)U_V(s))u_0\|_{L^2_x} &= \left\|\int_s^t U(\tau)^*V(\tau)U_V(\tau)u_0d\tau\right\|_{L^2_x}\\ &\lesssim\|V(\tau)U_V(\tau)u_0\|_{L^1_\tau([s,t],L^2_x)}\\ &\leqslant\|V\|_{L^2_\tau([s,t],L^3_x)}\|U_V(\tau)u_0\|_{L^2_\tau L^6_x}\\ &\leqslant\varphi\left(\|V\|_{L^2_\tau L^3_x}\right)\|V\|_{L^2_\tau([s,t],L^3_x)}\|u_0\|_{L^2_x},\end{aligned} \tag{6.8}$$

we obtain

$$\|U(-t)U_V(t)-U(-s)U_V(s)\|_{\mathcal{B}}\leqslant\varphi(\|V\|_{L^2_\tau L^3_x})\|V\|_{L^2_\tau([s,t],L^3_x)}\to 0\text{ as }t,s\to\infty. \tag{6.9}$$

Therefore, we get

$$\|U(-t)Q_1(t)U(t) - U(-s)Q_1(s)U(s)\|_{\mathfrak{S}^3} \to 0 \text{ as } t, s \to \infty. \tag{6.10}$$

Next we consider $Q_2(t) \in C(\mathbb{R}, \mathcal{H}^{\frac{1}{2}})$. We have

$$\begin{aligned}(6.11)\quad &\|U(-t)Q_2(t)U(t) - U(-s)Q_2(s)U(s)\|_{\mathfrak{S}^3} \\ &\leqslant \|U(-t)U_V(t) - U(-s)U_V(s)\|_{\mathcal{B}} \left\| \int_0^t U_V(\tau)^*[V(\tau), \gamma_f]U_V(\tau)d\tau \right\|_{\mathfrak{S}^3} \\ &\quad + \left\| \int_s^t U_V(\tau)^*[V(\tau), \gamma_f]U_V(\tau)d\tau \right\|_{\mathfrak{S}^3} \\ &\quad + \left\| \int_0^s U_V(\tau)^*[V(\tau), \gamma_f]U_V(\tau)d\tau \right\|_{\mathfrak{S}^3} \|U(-t)U_V(t) - U(-s)U_V(s)\|_{\mathcal{B}}.\end{aligned}$$

Let $V_0 := |V|^{\frac{2}{5}}$ and $V = V_0V_1$. Then we have

$$\begin{aligned}(6.12)\quad \left\| \int_I U_V(t)^*V(t)\gamma_f U_V(t)dt \right\|_{\mathfrak{S}^3} &\leqslant \left\| \int_I dt U_V(t)^*V_0(t) \right\|_{\mathfrak{S}^{10}_{(t,x)\to x}} \|V_1\gamma_f U_V(t)\|_{\mathfrak{S}^{30/7}_{x\to(t,x)}} \\ &\leqslant \varphi\left(\|V\|_{L^2_t L^3_x}\right) \left\| \langle \xi \rangle^{\frac{1}{2}} f(\xi) \right\|_{L^2_\xi} \|V\|_{L^2_t(I, L^2_x)}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}(6.13)\quad &\left\| \int_s^t U_V(\tau)^*[V(\tau), \gamma_f]U_V(\tau)d\tau \right\|_{\mathfrak{S}^3} \\ &\qquad \leqslant \varphi\left(\|V\|_{L^2_\tau L^3_x}\right) \left\| \langle \xi \rangle^{\frac{1}{2}} f(\xi) \right\|_{L^2_\xi} \|V\|_{L^2_\tau([s,t], L^2_x)} \to 0 \text{ as } t, s \to \infty.\end{aligned}$$

It follows from (6.9) and (6.13) that

$$\|U(-t)Q_2(t)U(t) - U(-s)Q_2(s)U(s)\|_{\mathfrak{S}^3} \to 0 \text{ as } t, s \to \infty. \tag{6.14}$$

□

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