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CONSTRUCTION OF ISOZAKI–KITADA MODIFIERS FOR DISCRETE SCHRÖDINGER OPERATORS ON GENERAL LATTICES

CONSTRUCTION DE MODIFICATEURS D'ISOZAKI–KITADA POUR LES OPÉRATEURS DE SCHRÖDINGER DISCRETS SUR DES RÉSEAUX GÉNÉRAUX

Abstract. — We consider a scattering theory for difference operators on $\mathcal{H} = \ell^2(\mathbb{Z}^d; \mathbb{C}^n)$ perturbed with a long-range potential $V : \mathbb{Z}^d \to \mathbb{R}^n$. One of the motivating examples is discrete Schrödinger operators on $\mathbb{Z}^d$-periodic graphs. We construct time-independent modifiers, so-called Isozaki–Kitada modifiers, and we prove that the modified wave operators with the above-mentioned Isozaki–Kitada modifiers exist and that they are complete.

Keywords: long-range scattering theory, discrete Schrödinger operators, modified wave operators, time-independent modifiers.
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Résumé. — Nous considérons une théorie du scattering pour les opérateurs aux différences sur $\mathcal{H} = \ell^2(\mathbb{Z}^d;\mathbb{C}^n)$ perturbés par un potentiel à longue portée $V : \mathbb{Z}^d \to \mathbb{R}^n$. Un des exemples motivants est celui des opérateurs de Schrödinger discrets sur des graphes $\mathbb{Z}^d$-périodiques. Nous construisons des modificateurs indépendants du temps, appelés modificateurs d'Isozaki–Kitada, et nous prouvons que les opérateurs d'onde modifiés avec les modificateurs d'Isozaki–Kitada mentionnés ci-dessus existent et qu'ils sont complets.

1. Introduction

The aim of the present article is to construct a long-range scattering theory for difference operators on the space of vector-valued functions on $\mathbb{Z}^d$. This problem is motivated by discrete Schrödinger operators on an arbitrary non-primitive lattice, e.g., hexagonal lattice, diamond lattice, Kagome lattice and graphite (see [AIM16] for more examples). Note that the cases of primitive lattices and the hexagonal lattice are considered in [Tad19a, Tad19b], respectively.

Let $\mathcal{H} = \ell^2(\mathbb{Z}^d;\mathbb{C}^n)$, where d and n are positive integers. For $u \in \mathcal{H}$, we use the notation

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \quad u_j \in \ell^2\left(\mathbb{Z}^d\right) = \ell^2\left(\mathbb{Z}^d;\mathbb{C}\right).$$

We consider a generalized form of discrete Schrödinger operators on $\mathcal{H}$:

$$H = H_0 + V.$$

The unperturbed operator H_0 is defined as a convolution operator by $(f_{jk})_{1 \leqslant j,k \leqslant n}$, that is,

$$H_0 u = \begin{pmatrix} H_{0,11} & H_{0,12} & \cdots & H_{0,1n} \\ H_{0,21} & H_{0,22} & \cdots & H_{0,2n} \\ \vdots & \vdots & \ddots & \vdots \\ H_{0,n1} & H_{0,n2} & \cdots & H_{0,nn} \end{pmatrix} u, \quad u \in \mathcal{H},$$

$$H_{0,jk} u_k(x) = \sum_{y \in \mathbb{Z}^d} f_{jk}(x-y) u_k(y), \quad u_k \in \ell^2\left(\mathbb{Z}^d\right).$$

Here each $f_{jk} : \mathbb{Z}^d \to \mathbb{C}$ is a rapidly decreasing function, i.e.,

$$\sup_{x \in \mathbb{Z}^d} \langle x \rangle^m \left| f_{jk}(x) \right| < \infty$$

for any $m \in \mathbb{N}$, where $\langle x \rangle = (1+|x|^2)^{\frac{1}{2}}$. The perturbation V is a multiplication operator by $V = {}^t(V_1, \cdots, V_n) : \mathbb{Z}^d \to \mathbb{R}^n$,

$$V u(x) = \begin{pmatrix} V_1(x) u_1(x) \\ V_2(x) u_2(x) \\ \vdots \\ V_n(x) u_n(x) \end{pmatrix}, \quad u \in \mathcal{H}.$$

We denote the discrete Fourier transform by $\mathcal{F}$;

$$\mathcal{F}u(\xi) = \begin{pmatrix} Fu_1(\xi) \\ Fu_2(\xi) \\ \vdots \\ Fu_n(\xi) \end{pmatrix}, \quad \xi \in \mathbb{T}^d := [-\pi, \pi)^d,$$

$$Fu_j(\xi) = (2\pi)^{-\frac{d}{2}} \sum_{x \in \mathbb{Z}^d} e^{-ix\cdot\xi} u_j(x),$$

for $u \in \ell^1(\mathbb{Z}^d; \mathbb{C}^n)$. Then $\mathcal{F}$ is extended to a unitary operator from $\mathcal{H}$ onto $\widehat{\mathcal{H}} = L^2(\mathbb{T}^d; \mathbb{C}^n)$, and we denote its extension by the same symbol $\mathcal{F}$. We easily see that $\mathcal{F} \circ H_0 \circ \mathcal{F}^*$ is the multiplication operator on $\mathbb{T}^d$ by the matrix-valued function

$$H_0(\xi) = \begin{pmatrix} h_{11}(\xi) & h_{12}(\xi) & \cdots & h_{1n}(\xi) \\ h_{21}(\xi) & h_{22}(\xi) & \cdots & h_{2n}(\xi) \\ \vdots & \vdots & \ddots & \vdots \\ h_{n1}(\xi) & h_{n2}(\xi) & \cdots & h_{nn}(\xi) \end{pmatrix},$$

where

$$h_{jk}(\xi) := \sum_{x \in \mathbb{Z}^d} e^{-ix\cdot\xi} f_{jk}(x).$$

Since f_{jk}'s are assumed to be rapidly decreasing, h_{jk}'s are smooth functions on $\mathbb{T}^d$. Note that $\sigma(H_0) = \{\lambda \mid \det(H_0(\xi) - \lambda) = 0 \text{ for some } \xi \in \mathbb{T}^d\}$ and H_0 is a self-adjoint operator if and only if $H_0(\xi)$ is a symmetric matrix for any $\xi \in \mathbb{T}^d$, i.e., by the definition of $H_0(\xi)$,

$$\overline{f_{jk}(-x)} = f_{kj}(x), \quad x \in \mathbb{Z}^d, \ 1 \leqslant j, k \leqslant n. \tag{1.1}$$

In this paper, we assume the following assumption concerning the self-adjointness of H_0 and a long-range condition of V.

Assumption 1.1. —

(1) *f_{jk}'s are rapidly decreasing functions satisfying* (1.1)*.*
(2) *$V = {}^t(V_1, \cdots, V_n)$ has the following representation*

$$V = V_L + V_S,$$

where each entry of V_L is the same, i.e., $V_L = {}^t(V_\ell, \cdots, V_\ell)$ with some $V_\ell : \mathbb{Z}^d \to \mathbb{R}$. Furthermore, there exist $\rho > 0$ and $C, C_\alpha > 0$ such that

$$\left|\tilde{\partial}_x^\alpha V_\ell(x)\right| \leqslant C_\alpha \langle x \rangle^{-\rho-|\alpha|}, \tag{1.2}$$

$$|V_S(x)| \leqslant C\langle x \rangle^{-1-\rho} \tag{1.3}$$

for any $x \in \mathbb{Z}^d$ and $\alpha \in \mathbb{Z}_+^d$. Here $\tilde{\partial}_x^\alpha = \tilde{\partial}_{x_1}^{\alpha_1} \cdots \tilde{\partial}_{x_d}^{\alpha_d}$, $\tilde{\partial}_{x_j} V(x) = V(x) - V(x - e_j)$ is the difference operator with respect to the j^{th} variable.

Assumption 1.1 implies that V is a compact operator on $\mathcal{H}$ and hence

$$\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0),$$

where $\sigma_{\text{ess}}(H)$ (resp. $\sigma_{\text{ess}}(H_0)$) denotes the essential spectrum of H (resp. H_0).

We denote the union of Fermi surfaces corresponding to the energies in $\Gamma \subset \mathbb{R}$ by

$$\begin{aligned}\mathrm{Ferm}(\Gamma) &:= \left\{p = (\xi, \lambda) \in \mathbb{T}^d \times \Gamma \,\middle|\, \lambda \text{ is an eigenvalue of } H_0(\xi)\right\} \\ &= \left\{p = (\xi, \lambda) \in \mathbb{T}^d \times \Gamma \,\middle|\, \det\left(H_0(\xi) - \lambda\right) = 0\right\}.\end{aligned}$$

Before describing the main theorem, we prepare the notation of non-threshold energies.

Definition 1.2. — *$\lambda_0 \in \sigma(H_0)$ is said to be* a non-threshold energy of H_0 *if the following properties* (1) *and* (2) *hold:*

(1) *For any $\xi_0 \in \mathbb{T}^d$ such that* $\det(H_0(\xi_0) - \lambda_0) = 0$*, there exists an open neighborhood $G \subset \mathbb{T}^d \times \mathbb{R}$ of $p = (\xi_0, \lambda_0)$ such that* $\mathrm{Ferm}(\mathbb{R}) \cap G$ *has a graph representation, i.e.*

$$\mathrm{Ferm}(\mathbb{R}) \cap G = \{(\xi, \lambda(\xi)) \mid \xi \in U\} \tag{1.4}$$

with some $U \ni \xi_0$ and $\lambda \in C^\infty(U)$.

(2) *Let ξ_0 be arbitrarily fixed so that* $\det(H_0(\xi_0) - \lambda_0) = 0$ *holds, and let $\lambda(\xi)$ be as in* (1.4)*. Then $\nabla_\xi \lambda(\xi_0) \neq 0$ holds.*

Remark 1.3. — There is a sufficient condition of non-threshold energies:

$$\nabla_\xi \det(H_0(\xi) - \lambda_0) \neq 0 \text{ for any } \xi \in \mathbb{T}^d \text{ such that } \det(H_0(\xi) - \lambda_0) = 0,$$

see Condition (A-3) in [AIM16, Sections 6 and 7]. The principal difference is that Definition 1.2 covers the case where $H_0(\xi)$ has degenerate eigenvalues but no branching occurs.

On the other hand, the set of non-threshold energies in the present paper can be smaller than that of [PR18, Definition 5.5]. Definition 1.2 is assumed in order for each eigenvalue and its associated eigenprojection of the Fourier symbol $H_0(\xi)$ of H_0 to be smooth with respect to ξ on the non-threshold energy level, which is needed to construct modifiers and to show some lemmas by the pseudodifferential calculus.

Let $\Gamma(H_0)$ be the set of non-threshold energies of H_0. Then $\Gamma(H_0)$ is an open set of $\mathbb{R}$ and $\Gamma(H_0) \subset \sigma(H_0)$. Note that H_0 has purely absolutely continuous spectrum on $\Gamma(H_0)$, i.e., $\sigma_{pp}(H_0) \cap \Gamma(H_0) = \sigma_{sc}(H_0) \cap \Gamma(H_0) = \phi$ (see Remark 3.2).

The main theorem of this paper is the following.

Theorem 1.4. — *Suppose Assumption 1.1 and $\Gamma \Subset \Gamma(H_0)$. Then there are bounded operators $J_\pm = J_{\pm,\Gamma}$ on $\mathcal{H}$, called Isozaki–Kitada modifiers, such that the modified wave operators exist:*

$$W_{\mathrm{IK}}^{\pm}(\Gamma) = \operatorname*{s-lim}_{t \to \pm\infty} e^{itH} J_\pm e^{-itH_0} E_{H_0}(\Gamma), \tag{1.5}$$

where E_{H_0} denotes the spectral measure of H_0, and that the following properties hold:

(i) *Intertwining property: $HW_{\mathrm{IK}}^{\pm}(\Gamma) = W_{\mathrm{IK}}^{\pm}(\Gamma)H_0$.*

(ii) *Partial isometries: $\|W_{\mathrm{IK}}^{\pm}(\Gamma)u\| = \|E_{H_0}(\Gamma)u\|$.*

(iii) *Completeness:* $\operatorname{Ran} W_{\mathrm{IK}}^{\pm}(\Gamma) = E_H(\Gamma)\mathcal{H}_{\mathrm{ac}}(H)$.

Here $\mathcal{H}_{\mathrm{ac}}(H)$ denotes the absolutely continuous subspace of H.

Various examples of unperturbed operators H_0 are given by Ando, Isozaki and Morioka [AIM16, Section 3]. Note that, if the perturbation V is short-range, i.e., $V_L = 0$, we can set $J_\pm = \mathrm{Id}_{\mathcal{H}}$, thus there exist the wave operators in this case. See [PR18] for short-range scattering theory for discrete Schrödinger operators on various lattices. We also note that a long-range scattering theory in the case of $n = 1$, e.g., discrete Schrödinger operators on square and triangular lattices, is considered by Nakamura [Nak14] and the author [Tad19a]. Moreover, Theorem 1.4 covers an arbitrary periodic lattice $\mathcal{L}$ with each primitive unit cell $\mathcal{L}/\Gamma$ containing finite elements, where $\Gamma \cong \mathbb{Z}^d$ denotes the transformation group associated to $\mathcal{L}$. In particular, it includes the result by the author [Tad19b], where a long-range scattering theory for discrete Schrödinger operators on the hexagonal lattice is studied. See also [DG97, RS79, Yaf10] and references therein for scattering theory of Schrödinger operators on $\mathbb{R}^d$.

The key idea of the present paper is to locally observe the eigenvalues $\lambda_k(\xi)$ of $H_0(\xi)$. This enables us to construct modifiers via the Hamilton flow on $T^*\mathbb{T}^d = \mathbb{R}^d \times \mathbb{T}^d$ associated with $\lambda_k(\xi) + \tilde{V}_\ell(x)$, where $\tilde{V}_\ell$ is a smooth extension of V_ℓ onto $\mathbb{R}^d$ satisfying $|\partial_x^\alpha \tilde{V}_\ell(x)| \leqslant C'_\alpha \langle x\rangle^{-\rho-|\alpha|}$, as well as to have the limiting absorption principle and the radiation estimate. For local observation of eigenvalues, we need to use the eigenprojections of $H_0(\xi)$ depending smoothly on ξ. We note that the concrete diagonalization of $H_0(\xi)$ is employed in [Tad19b], where the representation of diagonalization is smoothly defined globally away from the Dirac points.

The organization of this paper is as follows. We first prepare notations and properties of pseudodifference operators in Section 2. In Section 3, the limiting absorption principle and the propagation estimate for H are studied. We use the Mourre theory and a standard argument of the propagation of wave packets as in Yafaev [Yaf10, Chapter 10]. The construction of conjugate operators is essentially due to Parra and Richard [PR18]. Section 4 is devoted to constructing phase functions which are given as local solutions to eikonal equations corresponding to each fiber of eigenvalues of $H_0(\xi)$. The construction of phase functions is due to [Nak22]. In Section 5, using the phase functions in the previous section, we construct Isozaki–Kitada modifiers. Finally in Section 6, we use lemmas in the previous section to prove Theorem 1.4. The proof is based on Kato's smooth perturbation theory, and is an analogue of that in long-range scattering theory for Schrödinger operators on $\mathbb{R}^d$ (see [Yaf10]).

2. Preliminaries

2.1. Representations of fibers

Let Γ be as in Theorem 1.4, and let $I \Subset \Gamma(H_0)$ be fixed so that $\Gamma \Subset I \Subset \Gamma(H_0)$.

For each $p = (\xi_0, \lambda_0) \in \mathrm{Ferm}(\Gamma(H_0))$, let $G = G_p$ be as in Definition 1.2. Then $\{G_p\}_{p\in \mathrm{Ferm}(\Gamma(H_0))}$ is an open covering of $\mathrm{Ferm}(\Gamma(H_0))$. Since $\mathrm{Ferm}(\overline{I})$ is compact, we can take a finite family $\{G_j\}_{j=1}^J = \{G_{p_j}\}_{j=1}^J$ of open sets which covers $\mathrm{Ferm}(\overline{I})$.

Note that $\{G_j \cap \mathrm{Ferm}(\mathbb{R})\}_{j=1}^J$ is also a covering family of $\mathrm{Ferm}(\overline{I})$. Let G'_k, $k = 1, \dots, K$, be the connected components of $\cup_{j=1}^J G_j \cap \mathrm{Ferm}(\mathbb{R})$. We see that each G'_k

remains to have a graph representation

$$G'_k = \{(\xi, \lambda_k(\xi)) \,|\, \xi \in \mathcal{U}_k\} \tag{2.1}$$

with some open set $\mathcal{U}_k \subset \mathbb{T}^d$ and $\lambda_k \in C^\infty(\mathcal{U}_k)$. We denote by $P_k(\xi)$ the projection matrix onto $\mathrm{Ker}(H_0(\xi) - \lambda_k(\xi))$ for $\xi \in \mathcal{U}_k$. Then we have for $\psi \in C_c^\infty(I)$

$$\psi(H_0(\xi)) = \sum_{k=1}^{K} \psi(\lambda_k(\xi)) P_k(\xi) \chi_{\mathcal{U}_k}(\xi). \tag{2.2}$$

2.2. Pseudodifference calculus

For $a : \mathbb{Z}^d \times \mathbb{T}^d \to M_n(\mathbb{C}) \cong \mathbb{C}^{n\times n}$,

$$a(x, D_x)u(x) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{T}^d} e^{ix\cdot\xi} a(x,\xi) \mathcal{F}u(\xi) d\xi, \quad u \in \mathcal{H},$$

denotes the pseudodifference operator on $\mathbb{Z}^d$ with symbol $a(x,\xi)$. If a depends only on ξ, we denote by $a(D_x) = \mathcal{F}^* \circ a(\cdot) \circ \mathcal{F}$ the Fourier multiplier associated with $a(\xi)$ in short.

We cite a lemma concerning the pseudodifference calculus on $\mathcal{H}$ (see [RT09, Theorem 4.2.10] and the proof of [Tad19a, Lemma 2.2]).

Lemma 2.1. — *Let $a : \mathbb{Z}^d \times \mathbb{Z}^d \times \mathbb{T}^d \to M_n(\mathbb{C})$ be a smooth function with respect to $\mathbb{T}^d$, and let*

$$Au(x) = (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{y \in \mathbb{Z}^d} e^{i(x-y)\cdot\xi} a(x,y,\xi) u(y) d\xi.$$

Suppose that for any $\alpha \in \mathbb{Z}_+^d$

$$\sup_{(x,y,\xi) \in \mathbb{Z}^d \times \mathbb{Z}^d \times \mathbb{T}^d} \left| \partial_\xi^\alpha a(x,y,\xi) \right| < \infty. \tag{2.3}$$

Then A is a bounded operator on $\ell^2(\mathcal{H})$.

Let S^m be the symbol class of order $m \in \mathbb{R}$, i.e.,

$$S^m = \left\{ a : \mathbb{Z}^d \times \mathbb{T}^d \to M_n(\mathbb{C}) \;\middle|\; \begin{array}{l} a(x,\cdot) \in C^\infty\left(\mathbb{T}^d; M_n(\mathbb{C})\right),\ \forall\, x \in \mathbb{Z}^d, \\ \displaystyle\sup_{(x,\xi) \in \mathbb{Z}^d \times \mathbb{T}^d} \langle x \rangle^{-m+|\alpha|} \left| \tilde{\partial}_x^\alpha \partial_\xi^\beta a(x,\xi) \right| \\ < \infty,\ \forall\, \alpha, \beta \in \mathbb{Z}_+^d \end{array} \right\},$$

where $\tilde{\partial}_x^\alpha$ denotes the difference operator as in (1.2).

The following two assertions are analogous to the composition formula for pseudodifferential operators. See [RT09, Theorems 4.7.3 and 4.7.10] for the proofs.

Lemma 2.2. — *Let $a \in S^m$ and $b \in S^\ell$. Then $a(x,D_x)b(x,D_x) = c(x,D_x)$ with some $c \in S^{m+\ell}$ satisfying the asymptotic expansion*

$$c(x,\xi) - \sum_{|\alpha| \leqslant M} \partial_\xi^\alpha a(x,\xi) \tilde{\partial}_x^\alpha b(x,\xi) \in S^{m+\ell-M-1}$$

for any $M \in \mathbb{Z}_+$.

Lemma 2.3. — *Let $a \in S^m$. Then there exists $b \in S^m$ such that $a(x, D_x)^* = b(x, D_x)$ and $b(x,\xi) - a(x,\xi)^* \in S^{m-1}$.*

2.3. Kato's smooth perturbation theory

For a self-adjoint operator H and an H-bounded operator G, we say that G is H-smooth if

$$\frac{1}{2\pi} \sup_{\|u\|_{\mathcal{H}}=1, u \in D(H)} \int_{-\infty}^{\infty} \left\| G e^{-itH} u \right\|^2 dt < \infty. \tag{2.4}$$

For a Borel set $I \subset \mathbb{R}$, we say that G is H-smooth on I if $GE_H(I)$ is H-smooth, and we also say that G is locally H-smooth on I if G is H-smooth on I' for any $I' \Subset I$.

There are several conditions equivalent to (2.4) (see e.g. [Yaf92]), and the one we need in the following is:

$$\sup_{\lambda \in \mathbb{R}, \varepsilon > 0} \| G \delta_\varepsilon(\lambda, H) G^* \| < \infty, \tag{2.5}$$

where $\delta_\varepsilon(\lambda, H) = \frac{1}{2\pi i}\{(H - \lambda - i\varepsilon)^{-1} - (H - \lambda + i\varepsilon)^{-1}\}$.

3. Limiting absorption principle and radiation estimates

In this section, we consider the limiting absorption principle and radiation estimates for the proof of Theorem 1.4.

3.1. Limiting absorption principle

For a self-adjoint operator A and $m \in \mathbb{N}$, let

$$C^m(A) = \left\{ S \in \mathcal{B}(\mathcal{H}) \,\middle|\, \mathbb{R} \to \mathcal{B}(\mathcal{H}), t \mapsto e^{-itA} S e^{itA} \text{ is strongly of class } C^m \right\},$$

and $C^\infty(A) = \cap_{m \in \mathbb{N}} C^m(A)$. We denote by $\mathcal{C}^{1,1}(A)$ the set of the operators S satisfying

$$\int_0^1 \left\| e^{-itA} S e^{itA} + e^{itA} S e^{-itA} - 2S \right\| \frac{dt}{t^2} < \infty.$$

We set the Besov space

$$B := (\mathcal{D}(\langle x \rangle), \mathcal{H})_{\frac{1}{2},1},$$

where we have used the notation of real interpolation $(\cdot,\cdot)_{\theta,p}$ between Banach spaces (see [ABdMG96, Section 2.1]).

The following proposition is called the limiting absorption principle. The proof is given by the Mourre theory, and the construction of conjugate operators is essentially due to [PR18, Lemma 6.2].

Proposition 3.1. — *Suppose Assumption 1.1. Then:*

(1) *The set of eigenvalues of H is locally finite in $\Gamma(H_0)$ with counting multiplicities.*

(2) *For any $\lambda \in \Gamma(H_0)\backslash\sigma_{pp}(H)$, there exist the weak-* limits in $\mathcal{B}(B, B^*)$*

$$\mathrm{w}^*\text{-}\lim_{\varepsilon\to+0}(H-\lambda\mp i\varepsilon)^{-1}.$$

Moreover, each convergence is locally uniform in $\lambda \in \Gamma(H_0)\backslash\sigma_{pp}(H)$. In particular, for any $\Gamma \Subset \Gamma(H_0)\backslash\sigma_{pp}(H)$,

$$\sup_{\lambda\in\Gamma,\,\varepsilon>0}\left\|(H-\lambda\mp i\varepsilon)^{-1}\right\|_{\mathcal{B}(B,B^*)}<\infty. \tag{3.1}$$

Proof. — Let $\Gamma \Subset \Gamma(H_0)$ be arbitrarily fixed, and recall the representation (2.2). We set $\chi_k \in C_c^\infty(\mathcal{U}_k)$ so that $\chi_k = 1$ on $\lambda_k^{-1}(\Gamma)$. We also set the conjugate operator A by

$$\begin{aligned} A &= \sum_{k=1}^{K} P_k(D_x)\chi_k(D_x)i\left[\lambda_k(D_x), |x|^2\right]P_k(D_x)\chi_k(D_x) \\ &= \sum_{k=1}^{K} P_k(D_x)\chi_k(D_x)M_kP_k(D_x)\chi_k(D_x), \end{aligned}$$

where

$$M_k = x\cdot\nabla_\xi\lambda_k(D_x) + \nabla_\xi\lambda_k(D_x)\cdot x.$$

Now we employ the Mourre theory ([ABdMG96, Proposition 7.1.3, Corollary 7.2.11, Theorem 7.3.1], see also [Tad19b, Theorem A.1]). Then, since A is $\langle x\rangle$-bounded, it suffices to show that $H \in \mathcal{C}^{1,1}(A)$ and that, for any $\psi \in C_c^\infty(\Gamma)$, there exist $c > 0$ and a compact operator K such that the Mourre inequality holds:

$$\psi(H)i[H,A]\psi(H) \geqslant c\psi(H)^2 + K. \tag{3.2}$$

For the first assertion, we easily see $H_0 \in C^\infty(A)$, and $V \in \mathcal{C}^{1,1}(A)$ is proved by (1.2), (1.3) and Lemma 2.2 (see [PR18, Tad19b] for details of the proof).

For the proof of (3.2), we learn by Definition 1.2 (2) that

$$\begin{aligned} &\psi(H_0)i[H_0,A]\psi(H_0) \\ &\quad= 2\sum_{k=1}^{K} P_k(D_x)\psi\Big(\lambda_k(D_x)\Big)\chi_k(D_x)|\nabla_\xi\lambda_k(D_x)|^2P_k(D_x)\psi\Big(\lambda_k(D_x)\Big)\chi_k(D_x) \\ &\quad\geqslant c\sum_{k=1}^{K} P_k(D_x)\psi\Big(\lambda_k(D_x)\Big)^2\chi_k(D_x)^2 \geqslant c\psi(H_0)^2. \end{aligned} \tag{3.3}$$

It follows from (1.2) and (1.3) that $i[V, A]$ and $\psi(H)-\psi(H_0)$ are compact, and hence we have (3.2). ☐

Remark 3.2. — If we adopt the Mourre theory to $H = H_0$, (3.3) implies that H_0 has purely absolutely continuous spectrum on $\Gamma(H_0)$.

Since $B \supset \langle x\rangle^s\mathcal{H}$ and $B^* \subset \langle x\rangle^{-s}\mathcal{H}$ hold for any $s > \frac{1}{2}$ (see, e.g., [ABdMG96, Theorem 3.4.1]), (3.1) and the equivalence between (2.4) and (2.5) imply the following corollary.

Corollary 3.3. — *For any $s > \frac{1}{2}$, $\langle x\rangle^{-s}$ is locally H-smooth on $\Gamma(H_0)\backslash\sigma_{\mathrm{pp}}(H)$.*

3.2. Radiation estimates

In order to prove the existence and completeness of modified wave operators, we use, in addition to the limiting absorption principle, other propagation estimates called radiation estimates (see [Yaf10, Theorem 10.1.7]).

PROPOSITION 3.4. — *Let $\Gamma \Subset \Gamma(H_0)$ be fixed, and let $\lambda_k(\xi)$, $k = 1, \dots, K$, be as in* (2.1). *We set for $k = 1, \dots, K$ and $j = 1, \dots, d$,*

$$\nabla^{\perp}_{k,j} := \left\{ \left(\partial_{\xi_j}\lambda_k\right)(D_x) - \chi_{\{x\neq 0\}}|x|^{-2}x_j\langle x, (\nabla_\xi \lambda_k)(D_x)\rangle \right\} P_k(D_x)\chi_k(D_x),$$

where $\chi_k \in C_c^\infty(\mathcal{U}_k)$ is fixed arbitrarily so that $\chi_k = 1$ on $\lambda_k^{-1}(\Gamma)$. Then

$$\chi_{\{x\neq 0\}}|x|^{-\frac{1}{2}}\nabla^{\perp}_{k,j} \tag{3.4}$$

is locally H-smooth on $\Gamma(H_0)\backslash\sigma_{pp}(H)$.

Proof. — Fix $k = 1, \dots, K$. For simplicity of notation, we write λ, P, χ and $\nabla^\perp_j$ instead of λ_k, P_k, χ_k and $\nabla^\perp_{k,j}$, respectively.

Let $a \in C^\infty(\mathbb{R}^d)$ be fixed so that $a(x) = |x|$ for $|x| \geqslant 1$, and let

$$a_j := \partial_{x_j} a, \quad v_j := \partial_{\xi_j}\lambda.$$

We set

$$\mathbb{A} := (P\chi)(D_x)\sum_{j=1}^d \left\{a_j(x)v_j(D_x) + v_j(D_x)a_j(x)\right\}(P\chi)(D_x).$$

Then $\mathbb{A}$ is a bounded symmetric operator. It follows from [Yaf10, Proposition 0.5.11] and Corollary 3.3 that we only have to show

$$(u, i[H, \mathbb{A}]u) \geqslant 2\sum_{j=1}^d \left\|\chi_{\{x\neq 0\}}|x|^{-1/2}\nabla^\perp_j u\right\|^2 - C\|\langle x\rangle^{-\mu}u\|^2 \tag{3.5}$$

with some $C > 0$, where $\mu = \min(\frac{\rho+1}{2}, 1) > \frac{1}{2}$.

We show (3.5) in the rest of the proof. The representation (2.2) implies

$$i[H_0, \mathbb{A}] = (P\chi)(D_x)\cdot M \cdot (P\chi)(D_x),$$

where

$$M = \sum_{j=1}^d \left\{ i\left[\lambda(D_x), a_j(x)\right]\cdot v_j(D_x) + v_j(D_x)\cdot i\left[\lambda(D_x), a_j(x)\right]\right\}.$$

It follows from Lemma 2.2 that, formally,

$$M = 2\sum_{j=1}^d\sum_{\ell=1}^d v_\ell(D_x)a_{j\ell}(x)v_j(D_x) + R_1,$$

where $a_{j\ell} := \partial_{x_\ell}\partial_{x_j}a$, and R_1 satisfies $\langle x\rangle^2(P\chi)(D_x)R_1(P\chi)(D_x) \in \mathcal{B}(\mathcal{H})$. Since for $|x| \geqslant 1$

$$a_{j\ell}(x) = \partial_{x_\ell}\partial_{x_j}\left(|x|\right) = -\frac{x_j x_\ell}{|x|^3} + \delta_{j\ell}|x|^{-1},$$

we learn

$$(3.6)\qquad \begin{aligned}(u, i[H_0, \mathbb{A}]u) = -2\sum_{j=1}^{d}\sum_{\ell=1}^{d}\left(u^\ell, \frac{x_j x_\ell}{|x|^3}\chi_{\{x\neq 0\}}u^j\right) \\ + 2\sum_{j=1}^{d}\left(u^j, |x|^{-1}\chi_{\{x\neq 0\}}u^j\right) + \Big((P\chi)(D_x)u, R_2(P\chi)(D_x)u\Big),\end{aligned}$$

where

$$u^j := (v_j P\chi)(D_x)u,$$

and

$$R_2 = R_1 + 2\sum_{j=1}^{d}\sum_{\ell=1}^{d} a_{j\ell}(0)v_\ell(D_x)\chi_{x=0}(x)v_j(D_x)$$

also satisfies $\langle x\rangle^2(P\chi)(D_x)R_2(P\chi)(D_x) \in \mathcal{B}(\mathcal{H})$.

On the other hand, a direct computation implies for $x \neq 0$

$$\begin{aligned}\left|\nabla_j^\perp u(x)\right|^2 = \left|u^j(x)\right|^2 - |x|^{-2}x_j\sum_{\ell=1}^{d} x_\ell\left(u^\ell(x)\overline{u^j(x)} + \overline{u^\ell(x)}u^j(x)\right) \\ + |x|^{-4}{x_j}^2\sum_{\ell=1}^{d}\sum_{m=1}^{d} x_\ell x_m \overline{u^\ell(x)}u^m(x).\end{aligned}$$

Summing up over $j = 1, \dots, d$, we learn

$$(3.7)\qquad \begin{aligned}\sum_{j=1}^{d}\left|\nabla_j^\perp u(x)\right|^2 &= \sum_{j=1}^{d}\left|u^j(x)\right|^2 - |x|^{-2}\sum_{j=1}^{d}\sum_{\ell=1}^{d} x_j x_\ell\left(u^\ell(x)\overline{u^j(x)} + \overline{u^\ell(x)}u^j(x)\right) \\ &\quad + |x|^{-2}\sum_{\ell=1}^{d}\sum_{m=1}^{d} x_\ell x_m\overline{u^\ell(x)}u^m(x) \\ &= \sum_{j=1}^{d}\left|u^j(x)\right|^2 - |x|^{-2}\sum_{\ell=1}^{d}\sum_{m=1}^{d} x_\ell x_m\overline{u^\ell(x)}u^m(x), \quad x \neq 0.\end{aligned}$$

Combining (3.7) with (3.6), we obtain

$$\begin{aligned}&(u, i[H, \mathbb{A}]u) \\ &\quad = 2\sum_{j=1}^{d}\left\|\chi_{\{x\neq 0\}}|x|^{-1/2}\nabla_j^\perp u\right\|^2 + \Big((P\chi)(D_x)u, R_2(P\chi)(D_x)u\Big) + (u, i[V, \mathbb{A}]u).\end{aligned}$$

We see that $\langle x\rangle^{1+\rho}[V, \mathbb{A}] \in \mathcal{B}(\mathcal{H})$ by (1.2), (1.3) and Lemma 2.2. Finally we obtain (3.5). □

4. Classical mechanics

In this section, we construct phase functions used for the definition of time-independent modifiers $J_\pm$ in (1.5). For the precise definition of $J_\pm$, see (6.1).

Let $\lambda_k(\xi) : \mathcal{U}_k \to \mathbb{R}$, $k = 1, \dots, K$, be the functions in (2.1). The next proposition concerns the classical scattering problem with respect to the Hamiltonian $\lambda_k(\xi)+\tilde{V}_\ell(x)$ on $T^*\mathcal{U}_k = \mathbb{R}^d_x \times \mathcal{U}_k$, where $\tilde{V}_\ell$ is a smooth extension of V_ℓ onto $\mathbb{R}^d$ such that $|\partial_x^\alpha \tilde{V}_\ell(x)| \leqslant C'_\alpha \langle x\rangle^{-\rho-|\alpha|}$ holds. See [Nak14, Lemma 2.1] for a concrete construction of $\tilde{V}_\ell$.

The proof of the following proposition is given by [Nak22, Section 2] (see also [IK85, Tad19a]).

Proposition 4.1. — *Let $\lambda_k(\xi) : \mathcal{U}_k \to \mathbb{R}$, $k = 1, \dots, K$, be fixed. Then for any open set $U \Subset \mathcal{U}_k$ and $\varepsilon \in (0, 2)$, there exist $R > 0$ and smooth functions $\varphi^k_\pm(x, \xi)$ defined on a neighborhood of*

$$D_{k,\pm} = \left\{(x, \xi) \in \mathbb{R}^d \times U \,\middle|\, |x| \geqslant R,\ \pm\cos(x, \nabla\lambda_k(\xi)) \geqslant -1 + \varepsilon\right\},$$

where

$$\cos(x, \nabla\lambda_k(\xi)) := \frac{x \cdot \nabla\lambda_k(\xi)}{|x||\nabla\lambda_k(\xi)|},$$

such that

$$\lambda_k\left(\nabla_x\varphi^k_\pm(x, \xi)\right) + \tilde{V}_\ell(x) = \lambda_k(\xi), \quad (x, \xi) \in D_{k,\pm}. \tag{4.1}$$

Furthermore, $\varphi^k_\pm$ satisfy for $(x, \xi) \in D_{k,\pm}$

$$\left|\partial_x^\alpha\partial_\xi^\beta \left[\varphi^k_\pm(x, \xi) - x\cdot\xi\right]\right| \leqslant C_{\alpha\beta}\langle x\rangle^{1-\rho-|\alpha|}, \tag{4.2}$$

$$\left|{}^t\nabla_x\nabla_\xi\varphi^k_\pm(x, \xi) - I\right| < \frac{1}{2}. \tag{4.3}$$

5. Construction of Isozaki–Kitada modifiers

Let $\Gamma \Subset \Gamma(H_0)$ be fixed. Let $\lambda_k \in C^\infty(\mathcal{U}_k)$, $k = 1, \dots, K$, be as in (2.1), and let $\varphi^k_\pm$ be the phase functions constructed in Proposition 4.1 with setting $\varepsilon = \frac{1}{4}$ and U so that $\lambda_k^{-1}(\Gamma) \Subset U \Subset \mathcal{U}_k$.

We take functions $\chi_k \in C_c^\infty(U; [0, 1])$, $\eta \in C^\infty(\mathbb{R}^d)$ and $\sigma_\pm \in C^\infty(\mathbb{R}; [0, 1])$ such that

$$\chi_k(\xi) = 1, \quad \xi \in \lambda_k^{-1}(\Gamma), \tag{5.1}$$

$$\eta(x) = \begin{cases} 1 & \text{if } |x| \geqslant 2R, \\ 0 & \text{if } |x| \leqslant R, \end{cases} \tag{5.2}$$

$$\sigma_\pm(\theta) = \begin{cases} 1 & \text{if } \pm\theta \geqslant \frac{1}{2}, \\ 0 & \text{if } \pm\theta \leqslant -\frac{1}{2}, \end{cases} \tag{5.3}$$

$$\sigma_+(\theta)^2 + \sigma_-(\theta)^2 = 1, \quad \theta \in \mathbb{R}, \tag{5.4}$$

where $R > 0$ is the constant in Proposition 4.1. Then we define the Isozaki–Kitada modifiers $J^k_\pm$ associated with the pair $(P_k, \lambda_k, \mathcal{U}_k)$ by

$$J^k_\pm u(x) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{T}^d} e^{i\varphi^k_\pm(x,\xi)} s^k_\pm(x, \xi)\mathcal{F}u(\xi)d\xi, \tag{5.5}$$

where

$$s_\pm^k(x,\xi) := \eta(x)\sigma_\pm\Big(\cos(x,\nabla\lambda_k(\xi))\Big)P_k(\xi)\chi_k(\xi).$$

We recall that $P_k(\xi)$ is the projection matrix onto $\mathrm{Ker}(H_0(\xi)-\lambda_k(\xi))$, and note that $\operatorname{supp} s_\pm^k \subset D_{k,\pm}$ holds. Their formal adjoints are given by

$$\left(J_\pm^k\right)^* u(x) = \mathcal{F}^*\left((2\pi)^{-\frac{d}{2}}\sum_{y\,\in\,\mathbb{Z}^d} e^{-i\varphi_\pm^k(y,\cdot)}s_\pm^k(y,\cdot)u(y)\right).$$

Direct computations imply

$$\sup_{(x,\xi)\,\in\,\mathbb{R}^d\times\mathbb{T}^d}\langle x\rangle^{|\alpha|}\left|\partial_x^\alpha\partial_\xi^\beta s_\pm^k(x,\xi)\right| < \infty, \tag{5.6}$$

in particular (2.3) holds.

The next lemma follows from an analogue of the argument of calculus of Fourier integral operators (see [IK85, Tad19a]).

Lemma 5.1. — *Let $k=1,\dots,K$ be fixed, and let $\rho>0$ be the constant in Assumption 1.1(2). Then:*

(1) *$J_\pm^k$ are bounded operators on $\mathcal{H}$.*

(2) *The operators*

$$\langle x\rangle^\rho\left(J_\pm^k\left(J_\pm^k\right)^* - s_\pm^k(x,D_x)s_\pm^k(x,D_x)^*\right), \tag{5.7}$$

$$\langle x\rangle^\rho\left(\left(J_\pm^k\right)^* J_\pm^k - s_\pm^k(x,D_x)^*s_\pm^k(x,D_x)\right) \tag{5.8}$$

are bounded on $\mathcal{H}$.

(3) *For any $q\geqslant 0$,*

$$\langle x\rangle^{-q}J_\pm^k\langle x\rangle^q, \tag{5.9}$$

is bounded on $\mathcal{H}$.

(4) *Suppose that $\psi=\psi(\xi)\in C^\infty(\mathbb{T}^d;M_n(\mathbb{C}))$ commutes with $s_\pm^k(x,\xi)$ for any $(x,\xi)\in\mathbb{Z}^d\times\mathbb{T}^d$. Then*

$$\langle x\rangle^\rho\left[J_\pm^k,\psi(D_x)\right] \tag{5.10}$$

is bounded on $\mathcal{H}$. In particular, $[J_\pm^k,\psi(D_x)]$ are compact.

(5) *If $k\neq\ell$, then $J_\pm^k(J_\pm^\ell)^*=0$, and $(J_\pm^k)^*J_\pm^\ell$ are compact on $\mathcal{H}$.*

Proof. — (1) We compute

$$J_\pm^k\left(J_\pm^k\right)^* u(x) = (2\pi)^{-d}\int_{\mathbb{T}^d}\sum_{y\,\in\,\mathbb{Z}^d} e^{i\left(\varphi_\pm^k(x,\xi)-\varphi_\pm^k(y,\xi)\right)}s_\pm^k(x,\xi)s_\pm^k(y,\xi)u(y)d\xi.$$

We set $\varphi_\pm^k(x,\xi)-\varphi_\pm^k(y,\xi)=(x-y)\cdot\zeta(\xi;x,y)$, where

$$\zeta(\xi;x,y) := \int_0^1 \nabla_x\varphi_\pm^k(y+\theta(x-y),\xi)d\theta.$$

Then Proposition 4.1 implies that the mapping $\xi \mapsto \zeta(\xi; x, y)$ is a diffeomorphism from U into $\zeta(U)$ for any $x, y \in \mathbb{Z}^d$. Thus we have

$$J_\pm^k \left(J_\pm^k\right)^* u(x) = (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{y \in \mathbb{Z}^d} e^{i(x-y)\cdot\zeta} t_\pm^k(x, y, \zeta) u(y) d\zeta,$$

where

$$t_\pm^k(x, y, \zeta) := s_\pm^k(x, \xi(\zeta; x, y)) s_\pm^k(y, \xi(\zeta; x, y)) \left|\det\left(\frac{d\xi}{d\zeta}\right)\right|.$$

Since $|\frac{d\zeta}{d\xi}(\xi) - I| < \frac{1}{2}$ by Proposition 4.1, (5.6) implies $|\partial_\zeta^\alpha t_\pm^k(x, y, \zeta)| \leqslant C_\alpha$ for any α. Therefore $J_\pm^k$ are bounded by Lemma 2.1.

(2) The same argument as in (1) implies

$$\left(J_\pm^k \left(J_\pm^k\right)^* - s_\pm^k(x, D_x) s_\pm^k(x, D_x)^*\right) u(x) = (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{y \in \mathbb{Z}^d} e^{i(x-y)\cdot\zeta} r(x, y, \zeta) u(y) d\zeta,$$

where

$$r(x, y, \zeta) = t_\pm^k(x, y, \zeta) - s_\pm^k(x, \zeta) s_\pm^k(y, \zeta).$$

Since $|\partial_\xi^\alpha r(x, \zeta, y)| \leqslant C'_\alpha \langle x\rangle^{-\rho}$, Lemma 2.1 implies the boundedness of (5.7).

The other case (5.8) can be treated similarly if we consider the justification of PDO calculus; the argument using Poisson's summation formula as in [Nak22, Lemma 7.1] (see also [Tad19a, Lemma 2.3]) implies

$$\begin{aligned}
&\mathcal{F}\left(J_\pm^k\right)^* J_\pm^k \mathcal{F}^* f(\xi) \\
&\qquad = (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} e^{i\left(-\varphi_\pm^k(x,\xi) + \varphi_\pm^k(x,\eta)\right)} s_\pm^k(x, \xi) s_\pm^k(x, \eta) f(\eta) d\eta dx + K_1 f(\xi), \\
&\mathcal{F} s_\pm^k(x, D_x)^* s_\pm^k(x, D_x) \mathcal{F}^* f(\xi) \\
&\qquad = (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} e^{ix\cdot(-\xi+\eta)} s_\pm^k(x, \xi) s_\pm^k(x, \eta) f(\eta) d\eta dx + K_2 f(\xi),
\end{aligned}$$

where K_j, $j = 1, 2$, is a smoothing operator in the sense that $\langle D_x\rangle^N K_j \in \mathcal{B}(\mathcal{H})$ for any $N > 0$. Then by changing variables $x \mapsto \int_0^1 \nabla_\xi \varphi_\pm^k(x, \xi + \theta(\eta - \xi)) d\theta$, PDO calculus on $\mathbb{T}^d$ implies the boundedness of (5.8).

(3) By a complex interpolation argument, it suffices to show (5.9) for $q \in 2\mathbb{Z}_+$. Note that for $\alpha \in \mathbb{Z}_+^d$

$$\begin{aligned}
J_\pm^k x^\alpha u(x) &= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{T}^d} e^{i\varphi_\pm^k(x,\xi)} s_\pm^k(x, \xi) i^{|\alpha|} \partial_\xi^\alpha \mathcal{F} u(\xi) d\xi \\
&= (-i)^{|\alpha|} (2\pi)^{-\frac{d}{2}} \int_{\mathbb{T}^d} \partial_\xi^\alpha \left(e^{i\varphi_\pm^k(x,\xi)} s_\pm^k(x, \xi)\right) \mathcal{F} u(\xi) d\xi.
\end{aligned}$$

Then we learn for any $N \in \mathbb{Z}_+$,

$$J_\pm^k \langle x\rangle^{2N} u(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{T}^d} e^{i\varphi_\pm^k(x,\xi)} \left(L^N s_\pm^k\right)(x, \xi) \mathcal{F} u(\xi) d\xi,$$

where $L := \langle \nabla_\xi \varphi_\pm^k\rangle^2 - i\Delta_\xi \varphi_\pm^k - 2i\langle \nabla_\xi \varphi_\pm^k, \nabla_\xi\rangle - \Delta_\xi$. Since

$$\left|\partial_\xi^\beta \left(L^N s_\pm^k\right)(x, \xi)\right| \leqslant C_{p,\beta,N} \langle x\rangle^{2N}$$

for any $\beta \in \mathbb{Z}_+^d$, we have the boundedness of (5.9).

(4) It suffices to show the boundedness of $\langle D_\xi\rangle^\rho[\widehat{J}_\pm^k, \psi(\xi)]$ as an operator on $L^2(\mathbb{T}^d;\mathbb{C}^n)$, where $\widehat{J}_\pm^k := \mathcal{F} J_\pm^k \mathcal{F}^*$. A direct computation implies

$$\begin{aligned}\langle D_\xi\rangle^\rho &\left[\widehat{J}_\pm^k, \psi(\xi)\right] f(\xi)\\ &= (2\pi)^{-d} \sum_{x\in\mathbb{Z}^d} \int_{\mathbb{T}^d} e^{i\left(-x\cdot\xi+\varphi_\pm^k(x,\eta)\right)} \langle x\rangle^\rho \Big(\psi(\eta)-\psi(\xi)\Big) s_\pm^k(x,\eta) f(\eta) d\eta\\ &= (2\pi)^{-d} \sum_{x\in\mathbb{Z}^d} \int_{\mathbb{T}^d} e^{i\left(-x\cdot\xi+\varphi_\pm^k(x,\eta)\right)} \langle x\rangle^\rho \Psi_1(x,\eta) s_\pm^k(x,\eta) f(\eta) d\eta\\ &\quad + (2\pi)^{-d} \sum_{x\in\mathbb{Z}^d} \int_{\mathbb{T}^d} e^{i\left(-x\cdot\xi+\varphi_\pm^k(x,\eta)\right)} \langle x\rangle^\rho \Psi_2(x,\xi,\eta) s_\pm^k(x,\eta) f(\eta) d\eta,\end{aligned}$$

where

$$\begin{aligned}\Psi_1(x,\eta) &:= \psi(\eta) - \psi\left(\nabla_x\varphi_\pm^k(x,\eta)\right),\\ \Psi_2(x,\xi,\eta) &:= \psi\left(\nabla_x\varphi_\pm^k(x,\eta)\right) - \psi(\xi).\end{aligned}$$

The first term is treated similarly to (2), since $|\partial_\eta^\alpha \Psi_1(x,\eta)| \leqslant C_\alpha \langle x\rangle^{-\rho}$ by (4.2). For the second term, we first employ the argument in the proof of boundedness of (5.8) to replace the summation over $\mathbb{Z}^d$ by the integral on $\mathbb{R}^d$ modulo smoothing operators. Then, since

$$\Psi_2(x,\xi,\eta) = \left(\nabla_x\varphi_\pm^k(x,\eta)-\xi\right)\cdot \int_0^1 \nabla_\xi\psi\left(\xi+\theta\left(\nabla_x\varphi_\pm^k(x,\eta)-\xi\right)\right) d\theta,$$

we have

$$\begin{aligned}(2\pi)^{-d} &\int_{\mathbb{R}^d}\int_{\mathbb{T}^d} e^{i\left(-x\cdot\xi+\varphi_\pm^k(x,\eta)\right)} \langle x\rangle^\rho \Psi_2(x,\xi,\eta) s_\pm^k(x,\eta) f(\eta) d\eta dx\\ &= i(2\pi)^{-d} \int_{\mathbb{R}^d}\int_{\mathbb{T}^d} e^{i\left(-x\cdot\xi+\varphi_\pm^k(x,\eta)\right)} a(\xi,\eta,x) f(\eta) d\eta dx,\end{aligned}$$

where

$$a(\xi,\eta,x) = \nabla_x \cdot \left(\langle x\rangle^\rho s_\pm^k(x,\eta) \int_0^1 \nabla_\xi\psi\left(\xi+\theta\left(\nabla_x\varphi_\pm^k(x,\eta)-\xi\right)\right) d\theta\right)$$

satisfies $|\partial_\xi^\alpha \partial_\eta^\beta \partial_x^\gamma a(\xi,\eta,x)| \leqslant C_{\alpha,\beta,\gamma}$. Finally we apply [AF78, Theorem 2.1] to obtain the boundedness of the second term.

(5) The first assertion follows from $s_k^\pm(x,\xi) s_\ell^\pm(y,\xi) = 0$ for any x, y and ξ. For the second assertion, we set $\psi_k \in C^\infty(\mathbb{T}^d; M_n(\mathbb{C}))$ so that $\psi_k(\xi) = P_k(\xi)$ on $\operatorname{supp}\chi_k$. Then we use the equality $J_\pm^k = J_\pm^k \psi_k(D_x)$ and compactness of $[J_\pm^k, \psi_k(D_x)]$, which follows from (4). □

Now we prove the existence of the following (inverse) local wave operators

$$W^\pm(\mathcal{J}) := \operatorname*{s-lim}_{t\to\pm\infty} e^{itH}\mathcal{J}e^{-itH_0}E_{H_0}(\Gamma), \tag{5.11}$$

$$I^\pm(\mathcal{J}) := \operatorname*{s-lim}_{t\to\pm\infty} e^{itH_0}\mathcal{J}^* e^{-itH}E_H^{\mathrm{ac}}(\Gamma), \tag{5.12}$$

for $\mathcal{J} = J_\#^k$ with $k = 1,\dots,K$ and $\# \in \{+,-\}$. Note that, if $\mathcal{J}$ is compact, then $W^\pm(\mathcal{J}) = I^\pm(\mathcal{J}) = 0$.

We set $\tilde{\chi}_k \in C_c^\infty(\mathcal{U}_k)$ so that $\tilde{\chi}_k = 1$ on $\operatorname{supp}\chi_k$. Since

$$(P_k\tilde{\chi}_k)(D_x)J_{\#}^k - J_{\#}^k = \left[(P_k\tilde{\chi}_k)(D_x), J_{\#}^k\right]$$

is compact by Lemma 5.1 (4), we have

$$W^{\pm}\left(J_{\#}^k\right) = W^{\pm}\Big((P_k\tilde{\chi}_k)(D_x)J_{\#}^k\Big),$$
$$I^{\pm}\left(J_{\#}^k\right) = I^{\pm}\Big((P_k\tilde{\chi}_k)(D_x)J_{\#}^k\Big),$$

and thus it suffices to show the existence of (5.11) and (5.12) for

$$\mathcal{J} = (P_k\tilde{\chi}_k)(D_x)J_{\#}^k.$$

Lemma 5.2. —

$$\Big(H(P_k\tilde{\chi}_k)(D_x)J_{\pm}^k - (P_k\tilde{\chi}_k)(D_x)J_{\pm}^k H_0\Big)u(x) = (2\pi)^{-d}\int_{\mathbb{T}^d}\sum_{y\in\mathbb{Z}^d} e^{i\left(\varphi_{\pm}^k(x,\xi)-y\cdot\xi\right)}a_{\pm}^k(x,\xi)u(y)d\xi,$$

where

$$a_{\pm}^k(x,\xi) = -i\eta(x)\sigma_{\pm}'\Big(\cos(x,\nabla_\xi\lambda_k(\xi))\Big)\frac{|\nabla_\xi\lambda_k(\xi)|^2 - |x|^{-2}(x\cdot\nabla_\xi\lambda_k(\xi))^2}{|x||\nabla_\xi\lambda_k(\xi)|}P_k(\xi)\chi_k(\xi) + r_{\pm}^k(x,\xi) \tag{5.13}$$

and $|\partial_\xi^\beta r_{\pm}^k(x,\xi)| \leqslant C_\beta\langle x\rangle^{-\min(1+\rho,2)}$.

Proof.

Step 1. — Let

$$g(x) := (2\pi)^{-d}\int_{\mathbb{T}^d} e^{ix\cdot\xi}H_0(\xi)P_k(\xi)\tilde{\chi}_k(\xi)d\xi$$
$$= (2\pi)^{-d}\int_{\mathbb{T}^d} e^{ix\cdot\xi}\lambda_k(\xi)P_k(\xi)\tilde{\chi}_k(\xi)d\xi.$$

Then we learn

$$H_0(P_k\tilde{\chi}_k)(D_x)J_{\pm}^k u(x) = (2\pi)^{-d}\int_{\mathbb{T}^d}\sum_{y\in\mathbb{Z}^d} e^{i\left(\varphi_{\pm}^k(x,\xi)-y\cdot\xi\right)}a_{\pm}^{k,1}(x,\xi)u(y)d\xi,$$

where

$$a_{\pm}^{k,1}(x,\xi) = \sum_{y\in\mathbb{Z}^d} g(y)e^{i\left(\varphi_{\pm}^k(x-y,\xi)-\varphi_{\pm}^k(x,\xi)\right)}s_{\pm}^k(x-y,\xi)$$
$$= \sum_{y\in\mathbb{Z}^d} g(y)e^{-iy\cdot\nabla_x\varphi_{\pm}^k(x,\xi)}(1+R(x,y,\xi))s_{\pm}^k(x-y,\xi),$$

and

$$R(x,y,\xi) := \exp\left[i\left(\varphi_{\pm}^k(x-y,\xi) - \varphi_{\pm}^k(x,\xi) + y\cdot\nabla_x\varphi_{\pm}^k(x,\xi)\right)\right] - 1.$$

Since

$$\begin{aligned}\Big|\partial_\xi^\beta\Big[\varphi_\pm^k(x-y,\xi)-\varphi_\pm^k(x,\xi)+y\cdot\nabla_x\varphi_\pm^k(x,\xi)\Big]\Big|\\ &=\left|y\cdot\int_0^1\partial_\xi^\beta\left(\nabla_x\varphi_\pm^k(x,\xi)-\nabla_x\varphi_\pm^k(x-\theta y,\xi)\right)d\theta\right|\\ &=\left|y\cdot\int_0^1\left(\int_0^1\partial_\xi^\beta\nabla_x^2\varphi_\pm^k(x-\phi\theta y,\xi)d\phi\right)\theta y d\theta\right|\\ &\leqslant C_\beta\langle x\rangle^{-1-\rho}\langle y\rangle^{3+\rho},\end{aligned}$$

we learn $|\partial_\xi^\beta R(x,y,\xi)|\leqslant C'_\beta\langle x\rangle^{-1-\rho}\langle y\rangle^{(3+\rho)\max\{1,|\beta|\}}$, and thus

$$\left|\partial_\xi^\beta\sum_{y\in\mathbb{Z}^d}g(y)e^{-iy\cdot\nabla_x\varphi_\pm^k(x,\xi)}R(x,y,\xi)s_\pm^k(x-y,\xi)\right|\leqslant C''_\beta\langle x\rangle^{-1-\rho}.$$

Furthermore, since (5.6) implies the similar inequality

$$\begin{aligned}\Big|\partial_\xi^\beta\Big[s_\pm^k(x-y,\xi)-s_\pm^k(x,\xi)+y\cdot\nabla_x s_\pm^k(x,\xi)\Big]\Big|\\ &=\left|y\cdot\int_0^1\left(\int_0^1\partial_\xi^\beta\nabla_x^2 s_\pm^k(x-\phi\theta y,\xi)d\phi\right)\theta y d\theta\right|\leqslant C_\beta\langle x\rangle^{-2}\langle y\rangle^4,\end{aligned}$$

we have

$$\begin{aligned}&\sum_{y\in\mathbb{Z}^d}g(y)e^{-iy\cdot\nabla_x\varphi_\pm^k(x,\xi)}s_\pm^k(x-y,\xi)\\ &\quad=\sum_{y\in\mathbb{Z}^d}g(y)e^{-iy\cdot\nabla_x\varphi_\pm^k(x,\xi)}\left(s_\pm^k(x,\xi)-y\cdot\nabla_x s_\pm^k(x,\xi)\right)+O\left(\langle x\rangle^{-2}\right)\\ &\quad=(\lambda_k P_k\tilde\chi_k)\left(\nabla_x\varphi_\pm^k(x,\xi)\right)s_\pm^k(x,\xi)-i\nabla_\xi(\lambda_k P_k\tilde\chi_k)\left(\nabla_x\varphi_\pm^k(x,\xi)\right)\cdot\nabla_x s_\pm^k(x,\xi)\\ &\qquad+O\left(\langle x\rangle^{-2}\right).\end{aligned}$$

Thus we obtain

$$\begin{aligned}a_\pm^{k,1}(x,\xi)=(\lambda_k P_k\tilde\chi_k)\left(\nabla_x\varphi_\pm^k(x,\xi)\right)s_\pm^k(x,\xi)\\ -i\nabla_\xi(\lambda_k P_k\tilde\chi_k)\left(\nabla_x\varphi_\pm^k(x,\xi)\right)\cdot\nabla_x s_\pm^k(x,\xi)+O\left(\langle x\rangle^{-\min(1+\rho,2)}\right).\end{aligned}$$

Similar computations imply that

$$V(P_k\tilde\chi_k)(D_x)J_\pm^k u(x)=(2\pi)^{-d}\int_{\mathbb{T}^d}\sum_{y\in\mathbb{Z}^d}e^{i\left(\varphi_\pm^k(x,\xi)-y\cdot\xi\right)}a_\pm^{k,2}(x,\xi)u(y)d\xi,$$

$$(P_k\tilde\chi_k)(D_x)J_\pm^k H_0 u(x)=(2\pi)^{-d}\int_{\mathbb{T}^d}\sum_{y\in\mathbb{Z}^d}e^{i\left(\varphi_\pm^k(x,\xi)-y\cdot\xi\right)}a_\pm^{k,3}(x,\xi)u(y)d\xi,$$

where

$$
\begin{aligned}
&a_\pm^{k,2}(x,\xi)\\
&= V(x)\left((P_k\tilde\chi_k)\Big(\nabla_x\varphi_\pm^k(x,\xi)\Big)s_\pm^k(x,\xi) - i\nabla_\xi(P_k\tilde\chi_k)\Big(\nabla_x\varphi_\pm^k(x,\xi)\Big)\cdot\nabla_x s_\pm^k(x,\xi)\right)\\
&\qquad + O\left(\langle x\rangle^{-\rho-\min(1+\rho,2)}\right),\\
&a_\pm^{k,3}(x,\xi)\\
&= \lambda_k(\xi)(P_k\tilde\chi_k)\Big(\nabla_x\varphi_\pm^k(x,\xi)\Big)s_\pm^k(x,\xi) - i\lambda_k(\xi)\nabla_\xi(P_k\tilde\chi_k)\Big(\nabla_x\varphi_\pm^k(x,\xi)\Big)\cdot\nabla_x s_\pm^k(x,\xi)\\
&\qquad + O\left(\langle x\rangle^{-\min(1+\rho,2)}\right).
\end{aligned}
$$

Step 2. — Step 1 implies

$$
\begin{aligned}
&a_\pm^k(x,\xi)\\
&\quad= (P_k\tilde\chi_k)\Big(\nabla_x\varphi_\pm^k(x,\xi)\Big)s_\pm^k(x,\xi)\Big(\lambda_k\Big(\nabla_x\varphi_\pm^k(x,\xi)\Big) + V(x) - \lambda_k(\xi)\Big)\\
&\qquad - i\nabla_\xi(P_k\tilde\chi_k)\Big(\nabla_x\varphi_\pm^k(x,\xi)\Big)\cdot\nabla_x s_\pm^k(x,\xi)\Big(\lambda_k\Big(\nabla_x\varphi_\pm^k(x,\xi)\Big) + V(x) - \lambda_k(\xi)\Big)\\
&\qquad - i(P_k\tilde\chi_k)\Big(\nabla_x\varphi_\pm^k(x,\xi)\Big)\nabla_\xi\lambda_k\Big(\nabla_x\varphi_\pm^k(x,\xi)\Big)\cdot\nabla_x s_\pm^k(x,\xi)\\
&\qquad + O\left(\langle x\rangle^{-\min(1+\rho,2)}\right).
\end{aligned}
$$

The first and second terms are of order $\langle x\rangle^{-1-\rho}$ by (4.1) and (1.3). Moreover simple computations imply that, setting $v := \nabla_\xi\lambda_k(\xi)$,

$$
\nabla_x s_\pm^k(x,\xi) = \eta(x)\sigma_\pm'\Big(\cos(x,v)\Big)\left(\frac{1}{|x||v|}v - \frac{x\cdot v}{|x|^3|v|}x\right)P_k(\xi)\chi_k(\xi) + O\left(\langle x\rangle^{-\infty}\right),
$$

and therefore

$$
\begin{aligned}
a_\pm^k(x,\xi) &= -i(P_k\tilde\chi_k)(\xi)\nabla_\xi\lambda_k(\xi)\cdot\nabla_x s_\pm^k(x,\xi) + O\left(\langle x\rangle^{-\min(1+\rho,2)}\right)\\
&= -i\eta(x)\sigma_\pm'\Big(\cos(x,v)\Big)\left(\frac{|v|}{|x|} - \frac{(x\cdot v)^2}{|x|^3|v|}\right)P_k(\xi)\chi_k(\xi) + O\left(\langle x\rangle^{-\min(1+\rho,2)}\right).
\end{aligned}
$$

Here we have used (4.2) in the first equality to replace $\nabla_x\varphi_\pm^k(x,\xi)$ by ξ. □

Proposition 5.3. — *For any $k = 1,\dots,K$, there exist the limits* (5.11) *and* (5.12) *with $\mathcal{J} = J_\pm^k$.*

Proof. — We only prove the existence of (5.11), since the other is done in the same way.

We may assume $\rho < 1$ without loss of generality. The standard argument of existence of (modified) wave operators (see, e.g., [Yaf10, Lemmas 10.2.1 and 10.2.2, Theorem 0.5.4] and [RS78, Theorem XIII. 24]) implies that it suffices to prove that $H(P_k\tilde\chi_k)(D_x)J_\pm^k - (P_k\tilde\chi_k)(D_x)J_\pm^k H_0$ is a finite sum of the form $G_j^* B_j G_j'$ with G_j (resp. G_j') being H-(resp. H_0-) smooth in Γ and $B_j \in \mathcal{B}(\mathcal{H})$.

We set

$$a_j^k(x,\xi) = \eta(x)|x|^{-\frac{1}{2}} \left(\partial_{\xi_j}\lambda_k(\xi) - |x|^{-2}x_j\left(x\cdot\nabla_\xi\lambda_k(\xi)\right)\right) P_k(\xi)\tilde{\chi}_k(\xi),$$
$$b_\pm^k(x,\xi) = -i\eta(x)\sigma'_\pm\Big(\cos(x,\nabla_\xi\lambda_k(\xi))\Big)P_k(\xi)\chi_k(\xi).$$

Then we observe that

$$a_j^k(x,D_x) = \eta(x)|x|^{-\frac{1}{2}}\nabla_{k,j}^\perp,$$

where $\nabla_{k,j}^\perp$ is as in Proposition 3.4. Moreover we have by the definition (5.13) of $a_\pm^k(x,\xi)$

$$a_\pm^k(x,\xi) = b_\pm^k(x,\xi)\sum_{j=1}^d a_j^k(x,\xi)^2 + r_\pm^k(x,\xi),$$

where $\partial_\xi^\alpha r_\pm^k(x,\xi) = O(\langle x\rangle^{-2})$.

We take functions $\tilde{\tilde{\chi}}_k \in C_c^\infty(\mathcal{U}_k)$ and $\tilde{\sigma}_\pm(\theta)\in C^\infty(\mathbb{R})$ such that

$$\tilde{\sigma}_\pm(\theta) = \begin{cases} 1 & \text{if } \pm\theta \geqslant -\frac{1}{2}, \\ 0 & \text{if } \pm\theta \geqslant -\frac{3}{4}, \end{cases}$$
$$\tilde{\tilde{\chi}}_k(\xi) = 1, \quad \xi\in\operatorname{supp}\tilde{\chi}_k.$$

We set

$$\tilde{s}^k(x,\xi) = \eta(x)P_k(\xi)\tilde{\tilde{\chi}}_k(\xi),$$
$$\tilde{\varphi}_\pm^k(x,\xi) = \eta(x)\tilde{\sigma}_\pm\Big(\cos(x,\nabla\lambda_k(\xi))\Big)\varphi_\pm^k(x,\xi) + \Big(1-\eta(x)\tilde{\sigma}_\pm\Big(\cos(x,\nabla\lambda_k(\xi))\Big)\Big)x\cdot\xi,$$

and

$$\tilde{J}_\pm^k u(x) = (2\pi)^{-\frac{d}{2}}\int_{\mathbb{T}^d} e^{i\tilde{\varphi}_\pm^k(x,\xi)}\tilde{s}^k(x,\xi)\mathcal{F}u(\xi)d\xi,$$
$$A_{\pm,j}^k u(x) = (2\pi)^{-\frac{d}{2}}\int_{\mathbb{T}^d} e^{i\tilde{\varphi}_\pm^k(x,\xi)}a_j^k(x,\xi)\mathcal{F}u(\xi)d\xi,$$
$$C_{\pm,j}^k u(x) = (2\pi)^{-\frac{d}{2}}\int_{\mathbb{T}^d} e^{i\varphi_\pm^k(x,\xi)}b_\pm^k(x,\xi)a_j^k(x,\xi)^2\mathcal{F}u(\xi)d\xi.$$

Then it follows from the same argument as Lemma 5.1 (2) that

$$\tilde{J}_\pm^k(\tilde{J}_\pm^k)^* = \tilde{s}^k(x,D_x)^2 + R_{\pm,j,1}^k,$$
$$(\tilde{J}_\pm^k)^*A_{\pm,j}^k = a_j^k(x,D_x) + R_{\pm,j,2}^k,$$
$$(\tilde{J}_\pm^k)^*C_{\pm,j}^k = a_j^k(x,D_x)b_\pm^k(x,D_x)a_j^k(x,D_x) + R_{\pm,j,3}^k,$$

where $\langle x\rangle^{\frac{1+\rho}{2}}R_{\pm,j,\ell}^k\langle x\rangle^{\frac{1+\rho}{2}} \in \mathcal{B}(\mathcal{H})$, $\ell = 1, 2, 3$. Moreover we learn by the argument in Lemma 5.1 (4) that

$$\tilde{s}^k(x,D_x)^2A_{\pm,j}^k = A_{\pm,j}^k + R_{\pm,j,4}^k,$$
$$\tilde{s}^k(x,D_x)^2C_{\pm,j}^k = C_{\pm,j}^k + R_{\pm,j,5}^k,$$

where $\langle x\rangle^{\frac{1+\rho}{2}} R^k_{\pm,j,\ell}\langle x\rangle^{\frac{1+\rho}{2}} \in \mathcal{B}(\mathcal{H})$, $\ell = 4, 5$. Thus we have, modulo operators of the form $\langle x\rangle^{-\frac{1+\rho}{2}} B\langle x\rangle^{-\frac{1+\rho}{2}}$ with $B \in \mathcal{B}(\mathcal{H})$,

$$\begin{aligned}
H(P_k\tilde{\chi}_k)(D_x)J^k_\pm - (P_k\tilde{\chi}_k)(D_x)J^k_\pm H_0 &\equiv \sum_{j=1}^d C^k_{\pm,j} \\
&\equiv \sum_{j=1}^d \tilde{s}^k(x,D_x)^2 C^k_{\pm,j} \\
&\equiv \sum_{j=1}^d \tilde{J}^k_\pm(\tilde{J}^k_\pm)^* C^k_{\pm,j} \\
&\equiv \sum_{j=1}^d \tilde{J}^k_\pm a^k_j(x,D_x) b^k_\pm(x,D_x) a^k_j(x,D_x) \\
&\equiv \sum_{j=1}^d \tilde{J}^k_\pm(\tilde{J}^k_\pm)^* A^k_{\pm,j} b^k_\pm(x,D_x) a^k_j(x,D_x) \\
&\equiv \sum_{j=1}^d \tilde{s}^k(x,D_x)^2 A^k_{\pm,j} b^k_\pm(x,D_x) a^k_j(x,D_x) \\
&\equiv \sum_{j=1}^d A^k_{\pm,j} b^k_\pm(x,D_x) a^k_j(x,D_x).
\end{aligned}$$

Since $b^k_\pm(x,D_x) \in \mathcal{B}(\mathcal{H})$ and Proposition 3.4 implies $a^k_j(x,D_x)$ is H_0-smooth on Γ, it remains to prove that $A^k_{\pm,j}$ is H-smooth on Γ. However, the proof is completed if we observe that $a^k_j(x,D_x)$ and $\langle x\rangle^{\frac{1+\rho}{2}}$ are H-smooth on Γ and that

$$\left(A^k_{\pm,j}\right)^* A^k_{\pm,j} = a^k_j(x,D_x)^* a^k_j(x,D_x) + R''_j,$$

where $\langle x\rangle^{\frac{1+\rho}{2}} R''_j \langle x\rangle^{\frac{1+\rho}{2}} \in \mathcal{B}(\mathcal{H})$. □

6. Proof of Theorem 1.4

We set

$$J_\pm := \sum_{k=1}^K J^k_\pm, \tag{6.1}$$

where $J^k_\pm$'s are given by (5.5). Then Proposition 5.3 implies the existence of the modified wave operators (1.5). The proof of the intertwining property is skipped since it is easily proved.

Proposition 6.1. — $W^\pm(J_\mp) = I^\pm(J_\mp) = 0$.

Proof. — For the first assertion, it suffices to prove $\lim_{t\to\pm\infty} J^k_\mp e^{-itH_0}u = 0$ for any u satisfying

$$(P_k\chi_k)(D_x)u = u.$$

We easily see that

$$J_{\mp}^{k} e^{-itH_0} u(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{T}^d} e^{i\left(\varphi_{\mp}^{k}(x,\xi) - t\lambda_k(\xi)\right)} \eta(x) \sigma_{\mp}\Big(\cos(x, \nabla\lambda_k(\xi))\Big) \mathcal{F}u(\xi) d\xi.$$

The estimate (4.2) and the conditions (5.2) and (5.3) imply there is a constant $c > 0$ such that on the support of the integrand

$$\begin{aligned}
\left|\nabla_\xi \varphi_{\mp}^{k}(x,\xi) - t\nabla\lambda_k(\xi)\right| &\geqslant |x - t\nabla\lambda_k(\xi)| - \left|x - \nabla_\xi \varphi_{\mp}^{k}(x,\xi)\right| \\
&\geqslant \sqrt{\frac{1 - \cos(x, \pm\nabla\lambda_k(\xi))}{2}} |x||t\nabla\lambda_k(\xi)| - C\langle x\rangle^{1-\rho} \\
&\geqslant c\Big(|x| + |t||\nabla\lambda_k(\xi)|\Big)
\end{aligned}$$

for sufficiently large $\pm t \geqslant 0$. The non-stationary phase method implies that

$$\left|J_{\mp}^{k} e^{-itH_0} u(x)\right| \leqslant C_N (1 + |x| + |t|)^{-N}, \quad x \in \mathbb{Z}^d, \ \pm t \geqslant 0,$$

for any $N \geqslant 1$. Thus we obtain $\|W^{\pm}(J_{\mp})u\| = 0$.

For the other assertion $I^{\pm}(J_{\mp}) = 0$, the intertwining property implies

$$I^{\pm}(\mathcal{J}) = I^{\pm}(\mathcal{J}) E_H(\Gamma) = E_{H_0}(\Gamma) I^{\pm}(\mathcal{J}).$$

Thus we learn that for any $v \in \mathcal{H}$

$$\begin{aligned}
\Big(I^{\pm}(J_{\mp})u, v\Big) &= \Big(E_{H_0}(\Gamma) I^{\pm}(J_{\mp})u, v\Big) \\
&= \lim_{t\to\pm\infty} \Big(e^{itH_0} J_{\mp}^{*} e^{-itH} E_H^{\mathrm{ac}}(\Gamma)u, E_{H_0}(\Gamma)v\Big) \\
&= \lim_{t\to\pm\infty} \Big(E_H^{\mathrm{ac}}(\Gamma)u, e^{itH} J_{\mp} e^{-itH_0} E_{H_0}(\Gamma)v\Big) \\
&= \Big(E_H^{\mathrm{ac}}(\Gamma)u, W^{\pm}(J_{\mp})v\Big) \\
&= 0
\end{aligned}$$

by the first assertion. □

Proposition 6.2. — *For any $u \in \mathcal{H}$,*

$$\left\|W^{\pm}(J_{\pm})u\right\| = \|E_{H_0}(\Gamma)u\|, \tag{6.2}$$

$$\left\|I^{\pm}(J_{\pm})u\right\| = \|E_H^{\mathrm{ac}}(\Gamma)u\|. \tag{6.3}$$

Proof. — We learn

$$\left\|W^{\pm}(\mathcal{J})u\right\|^2 = \lim_{t\to\pm\infty} \left\|\mathcal{J} e^{-itH_0} E_{H_0}(\Gamma)u\right\|^2 = \lim_{t\to\pm\infty} (u_t, \mathcal{J}^{*}\mathcal{J}u_t),$$

where $u_t := e^{-itH_0}E_{H_0}(\Gamma)u$. Thus Lemmas 2.3, 2.2, 5.1 (2), (5) and (2.2), (5.1), (5.4) imply

$$
\begin{aligned}
&\left\|W^{\pm}(J_+)u\right\|^2 + \left\|W^{\pm}(J_-)u\right\|^2 \\
&\qquad = \lim_{t\to\pm\infty}\left(u_t, \left(J_+^*J_+ + J_-^*J_-\right)u_t\right) \\
&\qquad = \lim_{t\to\pm\infty}\left(u_t, \left(\sum_{k=1}^{K}\left(J_+^k\right)^* J_+^k + \left(J_-^k\right)^* J_-^k\right)u_t\right) \\
&\qquad = \lim_{t\to\pm\infty}\left(u_t, \left(\sum_{k=1}^{K} s_+^k(x,D_x)s_+^k(x,D_x)^* + s_-^k(x,D_x)s_-^k(x,D_x)^*\right)u_t\right) \\
&\qquad = \lim_{t\to\pm\infty}\left(u_t, \eta(x)^2\sum_{k=1}^{K}\left(P_k\chi_k^2\right)(D_x)u_t\right) \\
&\qquad = \lim_{t\to\pm\infty}\left(u_t, \eta(x)^2 u_t\right) \\
&\qquad = \left\|E_{H_0}(\Gamma)u\right\|^2.
\end{aligned}
$$

Here we have used (2.2) and (5.2) to obtain $\sum_{k=1}^{K}(P_k\chi_k^2)(D_x)E_{H_0}(\Gamma) = E_{H_0}(\Gamma)$ and compactness of $1-\eta(x)^2$. Therefore we have the first equality (6.2) by Proposition 6.1.

The other equality (6.3) is obtained by the similar argument and the compactness of $\psi(H) - \psi(H_0)$ for $\psi \in C_c^\infty(\mathbb{R})$. ☐

It remains to prove the completeness of (1.5). However it is proved by the existence of $I^{\pm}(J_\pm)$ and (6.3).

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