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HENRI LEBESGUE

SEBASTIAN BAADER

PETER FELLER

ASYMPTOTICS OF THE SMOOTH A_n-REALIZATION PROBLEM

ASYMPTOTIQUES DU PROBLÈME DE RÉALISATION LISSE DE A_n

ABSTRACT. — We solve an asymptotic variant of a smooth version of the A_n-realization problem for plane curves. As an application, we determine the cobordism distance between torus links of type $T(d,d)$ and $T(2,N)$ up to an error of at most $3d$. We also discuss the limits of knot theoretic approaches aimed at solving the A_n-realization problem.

RÉSUMÉ. — Nous résolvons une variante asymptotique d'une version lisse du problème de réalisation de A_n pour les courbes planes. Comme application, nous déterminons la distance de cobordisme entre les entrelacs toriques de type $T(d,d)$ et $T(2,N)$ avec une erreur d'au plus $3d$. Nous discutons également les limites des approches basées sur la théorie des noeuds visant à résoudre le problème de la réalisation de A_n.

1. Introduction

The algebraic A_n-realization problem asks for the minimal degree $d(n)$ of a polynomial $f(x,y) \in \mathbb{C}[x,y]$ that has an isolated singularity of type A_n at the origin [GLS98, GS20]. The minimal degree $d(n)$ is known to satisfy

$$\frac{7}{12} \leqslant \liminf_{n\to\infty} \frac{n}{d(n)^2} \leqslant \limsup_{n\to\infty} \frac{n}{d(n)^2} \leqslant \frac{3}{4}. \tag{1.1}$$

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The lower bound is due to Orevkov by a concrete construction, while the upper bound results from an analysis of the signature spectrum [Ore12].

Let $f(x,y) \in \mathbb{C}[x,y]$ be a square-free polynomial of degree d that has an isolated singularity of type A_n at the origin. In knot theoretic terms, as e.g. described in [Wal04, Chapter 5], this means that, for sufficiently small $\varepsilon > 0$, the intersection of the algebraic curve $f^{-1}(0) \subset \mathbb{C}^2$ with the sphere $S^3_\varepsilon \subset \mathbb{C}^2$ of radius ε around the origin is a torus link of type $T(2, n+1)$. Fixing a sufficiently small $\varepsilon > 0$ and adding a generic polynomial of degree d with small coefficients to $f(x,y)$, we obtain a polynomial $\widetilde{f}(x,y)$ of degree d such that $\widetilde{f}^{-1}(0) \subset \mathbb{C}^2$ is smooth, $\widetilde{f}^{-1}(0) \cap S^3_\varepsilon$ is still a torus link of type $T(2, n+1)$, and in addition the link at infinity ($\widetilde{f}^{-1}(0) \cap \partial S^3_R$ for large $R > 0$) is a torus link of type $T(d,d)$; see Remark 2.2. Taking into account the classic genus-degree formula for smooth algebraic curves, one finds that the curve $\widetilde{f}^{-1}(0)$ provides a smooth connected cobordism of Euler characteristic around $n - d^2$ between the two links $T(d,d)$ and $T(2, n+1)$; compare with Appendix A. In particular, if the upper bound of $\frac{3}{4}$ for the ratio $\frac{n}{d^2}$ were achieved, we would obtain a smooth cobordism of Euler characteristic around $n - d^2 = -\frac{1}{4}d^2$ between the links $T(d,d)$ and $T(2,n)$. The purpose of this note is to prove that asymptotically such a smooth cobordism actually exists.

Theorem 1.1. — *The maximal Euler characteristic $\chi(d)$ among all smooth connected cobordisms between the links $T(d,d)$ and $T(2, \lfloor \frac{3}{4}d^2 \rfloor)$ satisfies*

$$\lim_{d\to\infty} \frac{\chi(d)}{d^2} = -\frac{1}{4}.$$

Unlike the derivation of previous results on the cobordism distance between torus links [Baa12, BCG17, BFLZ19, Fel16, FP21], the proof of Theorem 1.1 is not based on combinatorial braid group considerations. Instead, a key input are examples of algebraic curves with a large number of A_m-singularities described by Hirano [Hir92]; see Section 2 for the an explicit family of these curves and how they are used to establish Theorem 1.1. It is surprising to the authors that algebraic curves considered over 30 years ago allow to build cobordisms that make the observations in this note possible.

In Section 3, as an application of Theorem 1.1 and its proof, we find that the link $T(2, \lfloor \frac{3}{4}d^2 \rfloor)$ is essentially the nearest link to $T(d,d)$ among all torus links of type $T(2,N)$, in terms of the smooth cobordism distance; compare Section 3. Here is the precise result.

Theorem 1.2. — *For all non-zero integers d and all integers N, the maximal Euler characteristic $\chi(d,N)$ among all smooth connected cobordisms between the links $T(d,d)$ and $T(2,N)$ has value around $-\frac{1}{4}d^2 - |\mathrm{N} - \mathrm{sign}(d)\frac{3}{4}d^2|$. More precisely,*

$$-\frac{d^2}{4} - \left|\mathrm{sign}(d)\frac{3}{4}d^2 - N\right| - 4|d| \leqslant \chi(d,N) \leqslant -\frac{d^2}{4} - \left|\mathrm{sign}(d)\frac{3}{4}d^2 - N\right| + 2|d|.$$

For context, we note that, of course, many values of $\chi(d,N)$ are known exactly. For example, in case $d > 0 > N$, the upper bound is an equality by the resolution of the local Thom conjecture [KM93]; see the last paragraph of the proof of Theorem 1.2. The point is that for many choices of d (in particular for $d \geqslant 10$ and $N \geqslant \frac{3d^2}{4}$), the exact value of $\chi(d,N)$ remains unknown and Theorem 1.2 constitutes the first time $\chi(d,N)$ is determined up to an error that is linear in $|d|$. We also note that Theorem 1.2 not only recovers Theorem 1.1 (by setting $N = \lfloor \frac{3d^2}{4} \rfloor$), but makes precise the rate of convergence in Theorem 1.1; see also (2.2).

Next, we discuss the impact of Theorem 1.1 on knot theoretic strategies to approach the A_n-realization problem.

In light of the upper bound in (1.1), an interesting next step in the algebraic A_n-realization problem would be to find a constant $c < 3/4$ such that

$$\liminf_{n\to\infty} \frac{n}{d(n)^2} \leqslant c. \tag{1.2}$$

Theorem 1.1 has a consequence, which is arguably somewhat disappointing: an approach from knot concordance theory towards finding such a c using a certain type of concordance invariants is doomed to fail. This follows from the following corollary of Theorem 1.1 as we explain in detail in Appendix B.

Corollary 1.3. — *Every real-valued 1-Lipschitz concordance invariant I with $\lim_{m\to\infty} \frac{I(T(2,2m+1))}{g_4(T(2,2m+1))} = 1$ satisfies*

$$\liminf_{d\to\infty} \frac{I(T(d,d+1))}{g_4(T(d,d+1))} \geqslant \frac{1}{2}.$$

Examples of such invariants I include many classical and recent knot invariants (when appropriately normalized), for example Trotter's signature σ, Rasmussen's s, Ozsváth and Szabó's τ, Ozsváth, Stipsicz, and Szabó's $\Upsilon(t)$, and Hom and Wu's ν^+. We discuss such invariants I and a proof of Corollary 1.3 in Section 4.

However, Theorem 1.1 does not destroy all hope of using smooth concordance as an approach towards making progress on the A_n-realization problem. For context, we explain this in Appendix A, where we also make explicit a smooth analogue of the A_n-realization problem.

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2. Hirano's curves and the proof of Theorem 1.1

We consider links $L \subset S^3$ smooth non-empty 1-submanifolds and cobordisms between them. For us, a *cobordism* between two links $L_0, L_1 \subset S^3$ is a smooth oriented neatly embedded surface $F \subset S^3 \times [0,1]$ such that $\partial F = L_0 \times \{0\} \cup \overline{L_1} \times \{0\}$, where $\overline{L_1}$ denotes L_1 with reversed orientation. While all our cobordisms are smooth, we will add the adjective "smooth" in our statements, to avoid any confusion with the topological category that behaves fundamentally differently.

We derive Theorem 1.1 (and Theorem 1.2) from the following family of examples.

Lemma 2.1. — *For all integers $m \geqslant 1$, there exists a connected smooth cobordism of Euler characteristic $-m^2 - 2m$ between the links $T(2m,2m)$ and $T(2,3m^2)$.*

Lemma 2.1 follows from the existence of a particular family of projective algebraic curves: for every integer $m \geqslant 2$, there exists an irreducible projective algebraic curve in $\mathbb{CP}^2$ of degree $2m$ with exactly $N := 3m$ singularities, all of which are of type A_{m-1} [Hir92, Theorem 2]; see also [GS20, Theorem 3.2].

Proof of Lemma 2.1. — For $m = 1$, there even exists a smooth connected cobordism from $T(2,2)$ to $T(2,3)$ with Euler characteristic -1. Hence we consider the case of an integer $m \geqslant 2$. We consider the projective algebraic curve $C \subset \mathbb{CP}^2$ of degree $d \coloneqq 2m$ given by the irreducible homogeneous polynomial

$$F = (x^m + y^m + z^m)^2 - 4(x^m y^m + y^m z^m + z^m x^m) \in \mathbb{C}[x, y, z],$$

which was used by Hirano to prove [Hir92, Theorem 2]. A calculation reveals that C has $N := 3m$ singularities, all of which are of type A_{m-1}.

We obtain an affine algebraic curve in $\mathbb{C}^2$ with N-many A_{m-1} singularities by removing a generic projective line from $\mathbb{CP}^2$. We explain this in more detail in the rest of this paragraph. Pick a projective line L (a subvariety $L \subset \mathbb{CP}$ defined by $ax + by + cz = 0$ for some $[a : b : c] \in \mathbb{CP}^2$) that intersects C transversally (i.e. if $p \in C \cap L$, then p is a non-singular point of C and the tangent spaces of C and L at p span the entire tangent space of $\mathbb{CP}^2$ at p). Note that by Bézout's theorem $C \cap L$ consists of d points. We pick a linear transformation $A \in \mathrm{Gl}(3, \mathbb{C})$ that maps L to the line at infinity, which is defined by $z = 0$ and denoted by $\mathbb{CP}^1$. Now consider $\widetilde{F} = F \circ A^{-1}$, which is a homogeneous polynomial defining the curve $\widetilde{C}$ obtained from applying A to C, define $\widetilde{f} \coloneqq \widetilde{F}(x, y, 1)$, and take the desired affine algebraic curve in $\mathbb{C}^2 = \mathbb{CP}^2 \setminus \mathbb{CP}^1$ to be $\widetilde{f}^{-1}(0)$.

The fact that $\widetilde{C}$ intersects $\mathbb{CP}^1$ transversally, implies that the link at infinity of $\widetilde{f}^{-1}(0)$ is the torus link $T(d, d)$, i.e. $S^3_R \cap \widetilde{f}^{-1}(0)$ has link type $T(d, d)$ for $R > 0$ large enough. Here $S^3_R \subset \mathbb{C}^2$ denotes the 3-sphere of radius R, i.e. the boundary of the 4-ball B^4_R of radius R with center the origin. In order to see this, consider a closed regular neighborhood $\nu(\mathbb{CP}^1)$ of $\mathbb{CP}^1 \subset \mathbb{CP}^2$. To be concrete, take $\nu(\mathbb{CP}^1)$ to be the complement of $\mathrm{int}(B^4_R) \subset \mathbb{C}^2 \subset \mathbb{CP}^2$ for some large $R > 0$. Such a neighborhood is diffeomorphic (via some ϕ) to the total space E of the once-twisted D^2-bundle over $S^2 \cong \mathbb{CP}^1$ (the D^2-bundle $\pi\colon E \to S^2$ with $\partial E = S^3$). Choosing $\nu(\mathbb{CP}^1)$ smaller if needed (that is increasing R), by transversality we can arrange for the diffeomorphism $\phi\colon \nu(\mathbb{CP}^1) \to E$ to map $\nu(\mathbb{CP}^1) \cap \widetilde{C}$ to d fibers $\pi^{-1}(p_1), \pi^{-1}(p_2), \ldots, \pi^{-1}(p_d)$ for points $p_1, \ldots, p_d \in S^2$. Since $\pi|_{\partial E}\colon \partial E \to S^2$ is the Hopf fibration (since, up to isomorphisms of bundles, there is only one S^1-bundle over S^2 with total space S^3), $T \coloneqq \pi|^{-1}_{\partial E}\{p_1, \ldots, p_d\} \subset \partial E = S^3$ is a $T(d, d)$ torus link. Recalling $\partial(\nu(\mathbb{CP}^1)) = S^3_R$, we see that ϕ yields a diffeomorphism of pairs between $(S^3_R, S^3_R \cap \widetilde{f}^{-1}(0)) = (S^3_R, \phi^{-1}(T))$ and (S^3, T). Hence, as desired, the link at infinity is the torus link $T(d, d)$.

We now use $\widetilde{f}^{-1}(0)$ to find the desired cobordism. For this let $s_1, \ldots, s_N$ denote the singular points of $\widetilde{f}^{-1}(0)$ and choose $\varepsilon > 0$ small enough that S^3_{ε, s_k} (the sphere of radius ε around s_k) intersects $\widetilde{f}^{-1}(0)$ transversally in $T(2, m)$. Let $W := B^4_R \setminus (\bigcup_{1 \leqslant k \leqslant N} \mathrm{int}(B^4_{\varepsilon, s_k}))$, where B^4_R denotes the close ball with boundary S^3_R and $\mathrm{int}(B^4_{\varepsilon, s_k})$ denotes the open ball with boundary S^3_{ε, s_k}. Towards finding our cobordism (a surface

F in $S^3 \times [0,1]$), we modify W to be diffeomorphic to $S^3 \times [0,1]$ by tubing the small boundary spheres together and track what happens to $\widetilde{f}^{-1}(0)$. For this, we pick $N-1$ pairwise disjoint embedded closed arcs $a_k \subset W \cap \widetilde{f}^{-1}(0)$ that start and end on $\partial W \setminus S^3_R = \bigcup_{1 \leqslant k \leqslant N} S^3_{\varepsilon, s_k}$ such that $(\partial W \setminus S^3_R) \cup \bigcup_{1 \leqslant k \leqslant N-1} a_k$ is connected. We take $\nu(a_k)$ to be a small closed tubular neighborhood of a_k such that its boundary intersects $\widetilde{f}^{-1}(0)$ transversally and the pair $(\nu(a_k), \widetilde{f}^{-1}(0) \cap \nu(a_k))$ is diffeomorphic to $(B^3 \times [0,1], ([-1,1] \times \{0\} \times \{0\}) \times [0,1])$, where B^3 denotes the closed unit ball in $\mathbb{R}^3$. We set X to be the closure in $\mathbb{C}^2$ of $W \setminus \bigcup_{1 \leqslant k \leqslant N} \nu(a_k)$. After smoothing corners, we pick a diffeomorphism from X to $S^3 \times [0,1]$ and let $F \subset S^3 \times [0,1]$ denote the image of $X \cap \widetilde{f}^{-1}(0)$ under said diffeomorphism.

Next, we determine the Euler characteristic of F and along the way observe that it is connected. We discuss the case that m is odd. A similar calculation works when m is even and yields the same result. We first note that C (as a topological surface) is connected, as is the case for all irreducible projective curves in $\mathbb{CP}^2$, and has genus $(d-1)(d-2)/2 - N\frac{m-1}{2}$. The latter can for example be seen by noting that a small generic deformation of C is a smooth algebraic curve of degree d, which has genus $(d-1)(d-2)/2$, where each of the N-many A_{m-1}-singularities contributes $\frac{m-1}{2}$ to the genus. $\widetilde{f}^{-1}(0)$ and $\widetilde{f}^{-1}(0) \cap W$ have the same genus as C (since they are obtained by removing discs), and $\widetilde{f}^{-1}(0) \cap X \cong F$ also has the same genus as C and is also connected since it is obtained from $\widetilde{f}^{-1}(0) \cap W$ by removing neighborhoods of embedded arcs that connect different boundary components. Thus, we find

$$\chi(F) = 2 - 2\frac{(d-1)(d-2) - N(m-1)}{2} - (d+1) = -(d-1)(d-1) + N(m-1),$$

because F is connected and has $d+1$ boundary components.

By construction, F is a connected smooth cobordism between $T(d,d)$ and a knot K given as the connected sum of N-many $T(2,m)$ torus links. Take F' to be a cobordism given by $N-1$ one-handles between K and $T(2,Nm)$; in particular, F' is connected and $\chi(F') = -N+1$.

Composing the two cobordisms F and F', we find a connected smooth cobordism from $T(d,d)$ to $T(2,Nm)$ with Euler characteristic

$$-(d-1)(d-1) + N(m-1) - N + 1 = -(d-1)(d-1) + (m-2)N + 1 = -m^2 - 2m. \quad \square$$

Proof of Theorem 1.1. — We fix an integer $d \geqslant 2$ and consider the largest integer m such that $d \geqslant D := 2m$. In other words, $D = d$ if d is even and $D = d-1$ if d is odd. We construct the desired cobordism as the composition of three cobordisms, F, G, and F'. The key contribution is H, which is given by Lemma 2.1, while F and F' are of a more cosmetic nature that help adapt the interesting cobordism H (H stands for Hirano) to start and end at the right links. We describe all three in detail.

Let F be a smooth cobordism from $T(d,d)$ to $T(D,D)$ with

$$\chi(F) = (D-1)^2 - (d-1)^2 = \begin{cases} 0 & \text{if } d \text{ is even} \\ -4m+1 & \text{if } d \text{ is odd} \end{cases}$$

as follows: F is the identity cobordism if d is even, i.e. $F = T(d,d) \times [0,1]$, while, for d odd, F is any connected cobordism. We expand on the existence of such an ´F for d odd in the rest of this paragraph. For example, F can be obtained by a complex algebraic curve $C \in \mathbb{C}^2$ intersected with $B^4 \setminus \mathrm{int}(B_{\frac{1}{2}})$, where C is chosen to intersect S^3 and $S^3_{\frac{1}{2}}$ transversally in $T(d,d)$ and $T(D,D)$, respectively. (See for example Remark 2.2 for how such a curve C can be produced.) The local Thom conjecture implies that the Euler characteristic is as claimed. Here is a different, explicit construction that builds a handle decomposition of a cobordism as desired. Consider the following sequence of $4m = 2D = 2(d-1)$ positive d-braids:

$$\begin{aligned}\sigma_1\sigma_2,\ldots,\sigma_{d-2}\sigma_{d-1}(\sigma_1\sigma_2,\ldots,\sigma_{d-2})^{d-2},(\sigma_1\sigma_2,\ldots,\sigma_{d-2}\sigma_{d-1})^2(\sigma_1\sigma_2,\ldots,\sigma_{d-2})^{d-3},\\ (\sigma_1\sigma_2,\ldots,\sigma_{d-2}\sigma_{d-1})^3(\sigma_1\sigma_2,\ldots,\sigma_{d-2})^{d-4},\ldots,(\sigma_1\sigma_2,\ldots,\sigma_{d-2}\sigma_{d-1})^{d-1},\\ (\sigma_1\sigma_2,\ldots,\sigma_{d-2}\sigma_{d-1})^{d-1}\sigma_1,(\sigma_1\sigma_2,\ldots,\sigma_{d-2}\sigma_{d-1})^{d-1}\sigma_1\sigma_2,\ldots,\\ (\sigma_1\sigma_2,\ldots,\sigma_{d-2}\sigma_{d-1})^{d-1}\sigma_1\sigma_2,\ldots,\sigma_{d-2},\\ (\sigma_1\sigma_2,\ldots,\sigma_{d-2}\sigma_{d-1})^{d-1}\sigma_1\sigma_2,\ldots,\sigma_{d-1}\\ =(\sigma_1\sigma_2,\ldots,\sigma_{d-2}\sigma_{d-1})^{d}.\end{aligned}$$

The first braid has as its braid closure $T(D,D)$ and the last braid has $T(d,d)$ as its braid closure. Each differs from its predecessor by one more generator σ_i, hence their closures are related by a cobordism consisting of one 1-handle (in particular a cobordism of Euler characteristic -1). Composing these $4m-1$ cobordism yields the desired cobordism F. To see that F is connected, note that the $(d-1)^{\text{th}}$ braid has closure $T(d,d-1)$ (hence is connected) and by construction (only 1-handles) every point in F has at most as many connected component as this level set.

Let F' be a smooth cobordism from $T(2,3m^2)$ to $T(2,\lfloor\frac{3}{4}d^2\rfloor)$ with

$$\chi(F') = -\left\lfloor 3d^2/4\right\rfloor + 3m^2 = \begin{cases}0 & \text{if } d \text{ is even}\\ -3m & \text{if } d \text{ is odd}\end{cases}$$

as follows: if d is even, F' is the identity cobordism, while if d is odd, F' is any connected cobordism. Similar to F, if d is odd, such an F' can for example be given by an algebraic curve or an explicit construction as follows. Consider the following sequence of $3m+1 = \lfloor 3d^2/4\rfloor - 3m^2$ positive 2-braids:

$$\sigma_1^{3m^2},\sigma_1^{3m^2+1},\ldots,\sigma_1^{\left\lfloor 3d^2/4\right\rfloor},$$

for which the closure of the first and last braid are $T(2,3m^2)$ and $T(2,\lfloor 3d^2/4\rfloor)$, respectively. The closure of any two consecutive braids are related by a cobordism given by a one handle that is connected. Since one of the closures has two components while the other has one, all these $3m$ cobordisms are diffeomorphic to a 3-holed sphere; in particular they are connected and have Euler characteristic -1. Hence, their composition yields a connected cobordism F' of Euler characteristic $-3m$ as desired.

Composing the following three cobordisms

- F from $T(d,d)$ to $T(D,D)$,
- a cobordism H from $T(D,D)$ to $T(2,3m^2)$ as guaranteed to exist by Lemma 2.1,

- and F' from $T(2,3m^2)$ to $T(2,\lfloor\frac{3}{4}d^2\rfloor)$,

we find a connected cobordism G between $T(d,d)$ and $T(2,\lfloor\frac{3}{4}d^2\rfloor)$ with

$$\begin{aligned}\chi(G) &= \chi(F)+\chi(H)+\chi(F')\\ &= \begin{cases} 0-m^2-2m+0=-m^2-2m & \text{if } d \text{ is even}\\ -4m+1-m^2-2m-3m=-m^2-9m+1 & \text{if } d \text{ is odd}\end{cases}\end{aligned}$$

Therefore, we have

$$\chi(d)\geqslant \begin{cases} -m^2-2m=-\frac{d^2}{4}-d & \text{if } d \text{ is even}\\ -m^2-9m+1=-\frac{d^2}{4}-4d+5+\frac{1}{4} & \text{if } d \text{ is odd}\end{cases} \geqslant -\frac{d^2}{4}-4d.$$

To find an upper bound on $\chi(d)$, we employ Murasugi's signature obstruction on the Euler characteristic of cobordism between links [Mur65]. Using

$$\sigma(T(d,d))=-\left\lfloor\frac{d^2-1}{2}\right\rfloor \text{ and } \sigma(T(2,k))=-k+1 \quad \text{[GLM81, Theorem 5.2]}, \tag{2.1}$$

for all positive integers d and k, we find

$$\chi(d)\overset{\text{[Mur65]}}{\leqslant}\sigma\left(T\left(2,\left\lfloor\frac{3}{4}d^2\right\rfloor\right)\right)-\sigma(T(d,d))\overset{(2.1)}{=}\left\lfloor\frac{d^2-1}{2}\right\rfloor-\left\lfloor\frac{3}{4}d^2\right\rfloor+1\leqslant-\left\lfloor\frac{d^2}{4}\right\rfloor+1.$$

In conclusion, we have shown

$$-\frac{d^2}{4}-4d\leqslant\chi(d)\leqslant-\left\lfloor\frac{d^2}{4}\right\rfloor+1; \tag{2.2}$$

in particular, this establishes $\lim_{d\to\infty}\frac{\chi(d)}{d^2}=-\frac{1}{4}$. □

We end this section with a remark about rearranging curves in $\mathbb{C}^2$ such that their link at infinity is $T(d,d)$ without changing the singularity at the origin, which we have used in the first paragraph of the introduction. The argument is very similar to the one from the second paragraph of the proof of Lemma 2.1.

Remark 2.2. — Let $C\subset\mathbb{C}^2$ be a reduced[(1)] algebraic curve of degree d and fix some $p\in C$. We claim by adding a small degree d polynomial to a defining square-free polynomial f of C, we can change C to a reduced algebraic curve $\widetilde{C}$ with the same singularity at p such that the link at infinity of $\widetilde{C}$ is $T(d,d)$. This can be done similarly as argued in the second paragraph of the proof of Lemma 2.1. Compare also with [Fel16, Section 2].

Considering the closure in $\mathbb{CP}^2$ (by homogenizing to a 3-variable polynomial F of degree d with $F(x,y,1)=f(x,y)$), choosing a generic projective line, and then composing the defining equation F with a linear transformation of $\mathbb{CP}^2$ that maps this line to the line at infinity $\{[x:y:z]\mid z=1\}$ and fixes p, we find a new defining equation $\widetilde{F}$ such that setting $\widetilde{f}(x,y)=\widetilde{F}(x,y,1)$ yields the defining equation of an algebraic curve $\widetilde{C}$ as desired. To guarantee that the coefficients between f and $\widetilde{f}$ vary little, choose the generic projective line to be given by an equation $ax+by+cz=0$ with a and b close to 0 and c close to 1 and choose the linear transformation close

(1) Reduced simply amounts to the defining polynomial f being square-free.

to the identity (say as an element in $\mathrm{Gl}_3(\mathbb{C})$). Finally, we note that a further small change of the coefficients of $\widetilde{f}$ (respectively $\widetilde{F}$) does not change the fact that the link at infinity is $T(d,d)$ since the corresponding projective curve transversally intersects the line at infinity.

In the second paragraph of the introduction, we further wanted a smooth algebraic curve. For this we note that adding a generic constant arranges that $\widetilde{f}^{-1}(0)$ is smooth. Choosing the constant small (compared to an ε with $S^3_{\varepsilon'} \pitchfork \widetilde{f}^{-1}(0)$ being a torus link of type $T(2,n+1)$ for all $0<\varepsilon'\leqslant\varepsilon$) assures that $\widetilde{f}^{-1}(0)\cap S^3_\varepsilon$ remains a torus link of type $T(2,n+1)$. Also, as argued at the end of the last paragraph, the link at infinity remains $T(d,d)$ if the constant is chosen sufficiently small.

3. Cobordism distance between torus links of type $T(d,d)$ and $T(2,N)$

Theorem 1.1 and its proof combined with the link signature bound on the Euler characteristic of cobordisms and the resolution of the local Thom conjecture allows to determine the smooth cobordism distance between the link $T(d,d)$ and all torus links $T(2,N)$ up to an error of at most $3d$. This is the content of Theorem 1.2, which we prove below, after a comment on the cobordism distance.

One may define the cobordism distance between two links L_0 and L_1 as minus the maximal Euler characteristic of all smooth cobordisms F between L_0 and L_1 for which each component of F has boundary components in both $S^3\times\{0\}$ and $S^3\times\{1\}$. With this definition the cobordism distance is an integer valued metric on the set of smooth concordance classes of links, justifying the name “distance”. Alternatively, the first author considered a variation when investigating cobordism distance between positive torus links using scissor equivalences; see [Baa12]. In case of two non-isotopic non-trivial positive torus links, whatever variation of the definition is chosen, the cobordism distance is realized by a connected smooth cobordism. For clarity of presentation, we abstain from using the term cobordism distance in our statements and instead consider connected smooth cobordisms, as done in the statement of Theorem’s 1.1 and 1.2.

Proof of Theorem 1.2. — Without loss of generality, take d to be positive. In fact, we consider the case when $d\geqslant 2$, since the case $d=1$ is immediate from the local Thom conjecture; compare (3.1) below.

We first discuss the case $N\geqslant 0$. Using a cobordism between $T(d,d)$ and $T(2,\lfloor\frac{3}{4}d^2\rfloor)$ that realizes $\chi(d)$ and composing it with a connected cobordism between $T(2,\lfloor\frac{3}{4}d^2\rfloor)$ to $T(2,N)$ of Euler characteristic $-|\lfloor\frac{3}{4}d^2\rfloor-N|$, yields $-|\lfloor\frac{3}{4}d^2\rfloor-N|+\chi(d)\leqslant\chi(d,N)$. Combined with

$$-\left|\left\lfloor\frac{3}{4}d^2\right\rfloor-N\right|-\frac{d^2}{4}-4d\overset{(2.2)}{\leqslant}-\left|\left\lfloor\frac{3}{4}d^2\right\rfloor-N\right|+\chi(d),$$

we find the desired lower bound

$$-\frac{d^2}{4}-\left|\frac{3}{4}d^2-N\right|-4d\leqslant\chi(d,N).$$

For the upper bound, in case $N \geqslant \frac{3}{4}d^2$, we recall the signature bound

$$\chi(d,N) \overset{\text{[Mur65]}}{\leqslant} -\sigma(T(d,d)) + \sigma(T(2,N)) \overset{(2.1)}{=} \left\lfloor \frac{d^2-1}{2} \right\rfloor - N + 1 \leqslant -\left|-\frac{d^2}{2} + N\right| + \frac{1}{2}$$

and apply $|-\frac{d^2}{2}+N| = |\frac{d^2}{4} + (-\frac{3}{4}d^2+N)| = \frac{d^2}{4} + |-\frac{3}{4}d^2+N|$ to find

$$\chi(d,N) \leqslant -\frac{d^2}{4} - \left|-\frac{3}{4}d^2 + N\right| + \frac{1}{2},$$

as desired. If instead, $0 \leqslant N < \frac{3}{4}d^2$, we use the following triangle inequality for the cobordism distance $\chi(d,N) \leqslant \chi(d,1) - \chi(1,N)$ in combination with the following consequence of the local Thom conjecture [KM93, Corollary 1.3]:

$$\chi(d,1) = -(|d|-1)^2 \text{ and } \chi(1,N) = -||N|-1|, \tag{3.1}$$

for all integers N and non-zero integers d. We find

$$\begin{aligned}\chi(d,N) &\leqslant -(d-1)^2 + (N-1)\\ &= -d^2 + N + 2d - 2\\ &= -\frac{d^2}{4} - \left|\frac{3}{4}d^2 - N\right| + 2d - 2,\end{aligned}$$

where we combined the triangle inequality and (3.1) to see the inequality.

Finally, if $N \leqslant -1$, then

$$\chi(d,N) = \chi(d,1) + \chi(1,N) \overset{(3.1)}{=} -(d-1)^2 + N + 1 = -\frac{d^2}{4} - \left|\frac{3}{4}d^2 - N\right| + 2d,$$

where the first equality is a consequence of the local Thom conjecture [KM93, Corollary 1.3]. □

4. 1-Lipschitz concordance invariants

We call a real-valued knot invariant $I\colon \mathfrak{K}nots \to \mathbb{R}$ a *1-Lipschitz concordance invariant* if $|I(K) - I(J)| \leqslant g_4(J\# -K)$ for all $K, J \in \mathfrak{K}nots$, where $\mathfrak{K}nots$ denotes the set of isotopy classes of knots and $-K$ denotes the reverse of the mirror of K.

Most classically, Trotter's signature $-\sigma/2$ is an example [Mur65, Tro62], but also Ozsváth and Szabó's τ [OS03] and Rasmussen's $-s/2$ [Ras10] (and more generally all slice-torus invariants as studied in [Lew14, Liv04]), and Ozsváth, Stipsicz, and Szabó's $-\Upsilon(t)/t$ [OSS17]. All of these are also additive under connected sum. A none-additive example is Hom and Wu's ν^+ [HW16]. These examples of 1-Lipschitz concordances invariants satisfy $|I(T_{2,2m+1})| = m$ for $m \in \mathbb{N}$. For such I, as a consequence of Theorem 1.1, we find Corollary 1.3, which can be paraphrased to say that $|I(T(d,d+1))|$ is at least half of the genus of $T(d,d+1)$ asymptotically for large d. This might be of independent interest, but for us this is actually a negative result since it shows that a certain approach towards making progress on the A_n-realization problem can not work; see Appendix B, where we make this statement precise.

Proof of Corollary 1.3. — Fix a positive integer d and write $N := \lfloor \frac{3}{4}d^2 \rfloor$. By composing a connected cobordism of Euler characteristic $\chi(d)$ between $T(d,d)$ and $T(2,N)$ with a connected cobordism of Euler characteristic $1-d$ between $T(d,d+1)$ and $T(d,d)$, we find a connected cobordism of Euler characteristic $\chi(d)-(d-1)$ between $T(d,d+1)$ and $T(2,N)$. If N is odd, we take F to be this cobordism and write $K = T(2,N)$, if not we take F to be a connected cobordism between $T(d,d+1)$ and $T(2,N+1)$ of Euler characteristic $\chi(d)-d$ and write $K = T(2,N+1)$. In both cases, F has genus $\lceil \frac{-\chi(d)+(d-1)}{2} \rceil$; hence, $g_4(T(d,d+1)\# -K \leqslant \lceil \frac{-\chi(d)+(d-1)}{2} \rceil$. We complete the proof by the following calculation, which uses Theorem 1.1 for the first equality and the assumption $\lim_{m\to\infty} \frac{I(T(2,2m+1))}{g_4(T(2,2m+1))} = 1$ for the last equality:

$$\begin{aligned}
\frac{1}{4} &= \lim_{d\to\infty} \frac{-\chi(d)}{d^2} = \lim_{d\to\infty} \frac{\left\lceil \frac{-\chi(d)+(d-1)}{2} \right\rceil}{d^2/2} = \lim_{d\to\infty} \frac{\left\lceil \frac{-\chi(d)+(d-1)}{2} \right\rceil}{g_4(T(d,d+1))} \\
&= \lim_{d\to\infty} \frac{\left\lceil \frac{-\chi(d)+(d-1)}{2} \right\rceil}{g_4(T(d,d+1))} = \liminf_{d\to\infty} \frac{\left\lceil \frac{-\chi(d)+(d-1)}{2} \right\rceil}{g_4(T(d,d+1))} \\
&\geqslant \liminf_{d\to\infty} \frac{I(K) - I(T(d,d+1))}{g_4(T(d,d+1))} \\
&\geqslant \liminf_{d\to\infty} \frac{I(K)}{g_4(T(d,d+1))} - \liminf_{d\to\infty} \frac{I(T(d,d+1))}{g_4(T(d,d+1))} \\
&= \liminf_{d\to\infty} \frac{I(K)}{d^2/2} - \liminf_{d\to\infty} \frac{I(T(d,d+1))}{g_4(T(d,d+1))} \\
&= \frac{3}{4} - \liminf_{d\to\infty} \frac{I(T(d,d+1))}{g_4(T(d,d+1))}.
\end{aligned}$$

□

Appendix A. Context: a smooth analogue of the A_n-realization problem and limitations of Theorem 1.1

Let us be exact in determining the Euler characteristic of the cobordism provided by $\widetilde{f}$ between $T(d,d)$ and $T(2,n+1)$ from the second paragraph of the introduction, which we earlier found to be around $n-d^2$. Its Euler characteristic is $n-(d-1)^2$, as we explain in the rest of this paragraph. Take $C \subset \mathbb{CP}^2$ to be the closure of $\widetilde{f}^{-1}(0)$, i.e. the projective algebraic curve given by the homogenization of $\widetilde{f}$. We note that C is a smooth curve (this follows, since all points of C in $\mathbb{C}^2 \subseteq \mathbb{CP}^2$ are non-singular by the choice of $\widetilde{f}$ and from the fact that the link at infinity is $T(d,d)$ we find that C has d non-singular points on $\mathbb{CP}^1 := \mathbb{CP}^2 \setminus \mathbb{C}^2$) of degree d; hence, it is a closed genus $(d-1)(d-2)/2$ surface and thus $\widetilde{f}^{-1}(0)$ is a d-times punctured genus $(d-1)(d-2)/2$ surface. The link $T(2,n+1)$ separates $\widetilde{f}^{-1}(0)$ into two pieces, one of which (the bounded one) is diffeomorphic to the Milnor fiber F of the A_n-singularity, i.e. a connected surface with first Betti number (aka its Milnor number) equal to n. Hence, the Euler characteristic of the cobordism is

$$\chi\left(\widetilde{f}^{-1}(0)\right) - \chi(F) = (-(d-1)(d-2) + 2 - d) - (-n+1) = n - (d-1)^2.$$

Motivated by the above calculation, we let $d_{\mathrm{sm}}(n)$ denote the smallest integer such that there exists a connected smooth cobordism of Euler characteristic $n - (d_{\mathrm{sm}}(n)-1)^2$ between $T(d_{\mathrm{sm}}(n), d_{\mathrm{sm}}(n))$ and $T(2,n+1)$. Equivalently, invoking the resolution of the local Thom conjecture, $d_{\mathrm{sm}}(n)$ is the smallest integer among the positive integers d such that there exists a χ-maximizing smooth connected cobordism $C \subset S^3 \times [-1,1]$ between $T(d,d)$ and the unknot U with $S^3 \times \{0\} \pitchfork C = T(2,n+1)$. In other words, $d_{\mathrm{sm}}(n)$ is the smallest integer among the d with

$$d_{\mathrm{cob}}(T(d,d),U) = d_{\mathrm{cob}}(T(d,d),T(2,n+1)) + d_{\mathrm{cob}}(T(2,n+1),U),$$

where d_{cob} denotes the cobordism distance between links; see also [Fel16, Observation 5].

The problem of determining $d_{\mathrm{sm}}(n)$ can be understood as a smooth analogue of the A_n-realization problem. Certainly, by the above calculation, one has $d_{\mathrm{sm}}(n) \leqslant d(n)$, but it is even conceivable that the following question has a positive answer: is $d_{\mathrm{sm}}(n) = d(n)$ for all $n \in \mathbb{N}$? This question appears to be folklore among some knot theorists, but no answer is in sight. In any case, since $d_{\mathrm{sm}}(n) \leqslant d(n)$, every constant c with

$$\liminf_{n\to\infty} \frac{n}{d_{\mathrm{sm}}(n)^2} \leqslant c, \tag{A.1}$$

also satisfies (1.2). Therefore, a positive answer to the following smooth concordance question would constitute progress on the algebraic A_n-realization problem.

Question A.1. — *Does there exists a constant $c < \frac{3}{4}$ that satisfies*

$$\liminf_{n\to\infty} \frac{n}{d_{\mathrm{sm}}(n)^2} \leqslant c?$$

While we suspect that the answer is no, in fact, we suspect $\limsup_{n\to\infty} \frac{n}{d_{\mathrm{sm}}(n)^2} = \frac{3}{4}$, we do not know. In particular, we note that Theorem 1.1 and its proof do *not* directly provide insight into Question A.1 since the cobordisms between $T(d,d)$ and $T(2, \lfloor \frac{3d^2}{4} \rfloor)$ we use have Euler characteristic strictly less than $\lfloor \frac{3d^2}{4} \rfloor - (d-1)^2$. However, Theorem 1.1 does show that certain asymptotic values of certain knot invariants cannot be used to answer Question A.1. We explain the latter below in Appendix B

For context, we also note that the for $d_{\mathrm{sm}}(n)$ in place of $d(n)$ the upper bounds from (1.1) also hold, while the lower bound is in fact better; see [Ore12, Theorem 3.13]:

$$\frac{2}{3} \leqslant \liminf_{n\to\infty} \frac{n}{d_{\mathrm{sm}}(n)^2} \leqslant \limsup_{n\to\infty} \frac{n}{d_{\mathrm{sm}}(n)^2} \leqslant \frac{3}{4}. \tag{A.2}$$

Appendix B. How not to resolve the A_n-realization problem

One may wonder what kind of invariants could help to answer Question A.1.

The following observation provides upper bounds on the asymptotic value of $\frac{n}{d_{\mathrm{sm}}(n)^2}$ (and hence $\frac{n}{d(n)^2}$).

Observation B.1. — *Let $I\colon \mathfrak{K}nots \to \mathbb{R}$ be a 1-Lipschitz concordance invariant with* $\lim_{m\to\infty}\frac{I(T(2,2m+1))}{g_4(T(2,2m+1))}=1$. *Setting*

$$c' := \liminf_{d\to\infty}\frac{I(T(d,d+1))}{g_4(T(d,d+1))} \quad \textit{and} \quad c'' := \limsup_{d\to\infty}\frac{I(T(d,d+1))}{g_4(T(d,d+1))},$$

we find

$$\liminf_{n\to\infty}\frac{n}{d_{\mathrm{sm}}(n)^2}\leqslant\frac{1+c'}{2} \quad \textit{and} \quad \limsup_{n\to\infty}\frac{n}{d_{\mathrm{sm}}(n)^2}\leqslant\frac{1+c''}{2}.$$

We note that in the assumption of Observation B.1 and similarly in Corollary 1.3, the limit could be replaced with lim inf since $\frac{I(K)}{g_4(K)}\leqslant 1$.

At first sight Observation B.1 looks like a promising approach towards answering Question A.1. For example, the upper bound in (1.1) and (A.2) immediately follows using $I=-\sigma/2$ since $\lim_{d\to\infty}\frac{-\sigma(T(d,d+1))/2}{g_4(T(d,d+1))}=\frac{1}{2}$. In fact, this upper bound via the signature and Observation B.1 is essentially how the upper bound via the signature spectrum (mentioned in the first paragraph of the introduction) works.

The bad news is that, by Corollary 1.3, for every 1-Lipschitz concordance invariant $I\colon \mathfrak{K}nots\to\mathbb{R}$ with $\lim_{m\to\infty}\frac{I(T(2,2m+1))}{g_4(T(2,2m+1))}=1$, we have

$$\liminf_{d\to\infty} I(T_{d,d+1})/g_4(T(d,d+1))\geqslant\frac{1}{2}.$$

This means, there is no I that can be plugged into Observation B.1 to improve the upper bound of $\frac{3}{4}$ on any of the quantities

$$\liminf_{n\to\infty}\frac{n}{d(n)^2},\quad \liminf_{n\to\infty}\frac{n}{d_{\mathrm{sm}}(n)^2},\quad \limsup_{n\to\infty}\frac{n}{d(n)^2},\quad\text{and}\quad \limsup_{n\to\infty}\frac{n}{d_{\mathrm{sm}}(n)^2}.$$

It remains to prove Observation B.1.

Proof of Observation B.1. — We discuss only the inequality involving lim inf, the other follows by a similar argument. Fix integers $n,d>0$, where we take n to be even. Assume that there exists a connected cobordism of Euler characteristic $n-(d-1)^2$ between $T(d,d)$ and $T(2,n+1)$. Then there exists a connected cobordism of Euler characteristic $n-(d-1)d$ between $T(d,d+1)$ and $T(2,n+1)$. This cobordism has genus $\frac{(d-1)d-n}{2}$; hence, $\frac{(d-1)d}{2}-\frac{n}{2}\geqslant -I(T(d,d+1))+I(T(2,n+1))$, and we have

$$1+\frac{I(T(d,d+1))}{g_4(T(d,d+1))}\geqslant\frac{\frac{n}{2}+I(T(2,n+1))}{\frac{(d-1)d}{2}}=\frac{n+2I(T(2,n+1))}{(d-1)d}. \tag{B.1}$$

Taking lim inf, we find

$$\begin{aligned}1+c' &= 1+\liminf_{d\to\infty}\frac{I(T(d,d+1))}{g_4(T(d,d+1))} &&= 1+\liminf_{n\to\infty\,|\,n\text{odd}}\frac{I(T(d_{\mathrm{sm}}(n),d_{\mathrm{sm}}(n)+1))}{g_4(T(d_{\mathrm{sm}}(n),d_{\mathrm{sm}}(n)+1))}\\ &\overset{\text{(B.1)}}{\geqslant}\liminf_{n\to\infty\,|\,n\text{odd}}\frac{n+2I(T(2,n+1)}{(d_{\mathrm{sm}}(n)-1)d_{\mathrm{sm}}(n)} &&= \liminf_{n\to\infty}\frac{n+2I(T(2,n+1))}{(d_{\mathrm{sm}}(n)-1)d_{\mathrm{sm}}(n)}\\ &=\liminf_{n\to\infty}\frac{n+2I(T(2,n+1))}{d_{\mathrm{sm}}(n)^2} &&= \liminf_{n\to\infty}\frac{2n}{d_{\mathrm{sm}}(n)^2},\end{aligned}$$

which completes the proof. We comment on why the equalities hold. The first one is by definition of c'. For the second one, $\leqslant$ is clear, but not needed. We argue for $\geqslant$.

By (A.2) we know that for every large d there exists an even n with $|d_{\mathrm{sm}}(n)-d| \leqslant 2\sqrt{d}$. Picking $d' = d_{\mathrm{sm}}(n)$ for some such n, we have

$$\Big|I(T(d,d+1)) - I(T(d,d+1))\Big|, \quad \Big|g_4(T(d,d+1)) - g_4(T(d,d+1))\Big| \leqslant O\Big((d-d')^2\Big) \leqslant O(d),$$

and, since $g_4(T(d,d+1))$ grows quadratically in d, $\leqslant$ (in fact $=$) follows. The third to last equality follows by a similar argument using that every $d_{\mathrm{sm}}(n)$ for n odd is linearly (in $d_{\mathrm{sm}}(n)$) close to $d_{\mathrm{sm}}(n \pm 1)$. The second to last equality is clear since the two denominators are only $d_{\mathrm{sm}}(n)$ apart but both grow quadratically. Finally, the last equation follows from $\lim_{m\to\infty} \frac{I(T(2,2m+1))}{g_4(T(2,2m+1))} = 1$. □

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Sebastian BAADER
Universität Bern,
Sidlerstrasse 5,
3012 Bern (Switzerland)
sebastian.baader@unibe.ch

Peter FELLER
Université de Neuchâtel,
Rue Emile-Argand 11,
2000 Neuchâtel (Switzerland)
peter.feller@unine.ch