Annales Henri Lebesgue
8 (2025) 769-810

ANNALES
HENRI LEBESGUE

PHILIPPE EYSSIDIEUX

L_2-CONSTRUCTIBLE COHOMOLOGY AND L_2-DE RHAM COHOMOLOGY FOR COHERENT $\mathcal{D}$-MODULES

COHOMOLOGIE CONSTRUCTIBLE L_2 ET COHOMOLOGIE DE DE RHAM L_2 POUR LES $\mathcal{D}$-MODULES COHÉRENTS

Abstract. — This article constructs Von Neumann invariants for constructible complexes and coherent $\mathcal{D}$-modules on compact complex manifolds, generalizing the work of the author on coherent L_2-cohomology. We formulate a conjectural generalization of Dingoyan's L_2-*Mixed Hodge structures* in terms of Saito's *Mixed Hodge Modules* and give partial results in this direction.

Résumé. — Cet article construit des invariants de Von Neumann pour les complexes constructibles et les $\mathcal{D}$-modules cohérents sur les variétés complexes compactes, généralisant le travail de l'auteur sur la cohomologie L_2 cohérente. On formule une généralisation conjecturale des structures de Hodge mixtes L_2 de Dingoyan pour les modules de Hodge mixtes de Saito et on donne des résultats partiels dans cette direction.

Keywords: complex manifolds, $\mathcal{D}$-modules, constructible sheaves, Hodge modules, mixed Hodge theory, Atiyah's L_2-index theorem, group Von Neumann algebras, L_2 Betti numbers.
2020 *Mathematics Subject Classification:* 32J27, 32C38, 32Q30, 14C30, 46L10, 58J22.
DOI: `https://doi.org/10.5802/ahl.248`
(*) This research was partially supported by the ANR projects Hodgefun ANR-16-CE40-0011-01 and GAG ANR-24-CE40-3526-01.

Introduction

This article is an extension of Dingoyan's L_2-*Mixed Hodge theory* [Din13] which was inspired by Gromov's influential article on Kähler-hyperbolic manifolds [Gro91].

Gromov found a way to use the L_2-De Rham theory of an infinite Galois covering space of a compact Kähler manifold X and obtain algebro-geometric restrictions if X is Kähler-hyperbolic, for instance a compact complex submanifold of a neat quotient of a bounded symmetric domain. This inspired the influential works of Campana [Cam94, Cam95] and Kollár [Kol93, Kol95] masterfully exploiting in Kähler geometry the striking ideas of [Ati76] to study compact Kähler manifolds with infinite fundamental group. Gromov's ideas were also extended in [Eys97] to polarized *Variations of Hodge Structures* (actually to harmonic bundles) on a compact Kähler manifold X. They were also extended in [Cam01, Eys00] to a theory of coherent L_2-cohomology in *Complex Analytic Geometry*. Some applications were given, say in [BDET24, Bra21, Eys98, Eys99, Tak99]. The present article generalizes this theory to constructible and $\mathcal{D}$-module coefficients, hence to Mixed Hodge Modules [Sai90], and generalizes Dingoyan's L_2-*Mixed Hodge structures*.

Let us describe our main constructions.

Let X be a compact complex manifold. Let $\pi : \widetilde{X} \to X$ be an infinite Galois covering space with $\mathrm{Deck}(\widetilde{X}/X) = \Gamma$.

If $F^\bullet$ is a bounded complex of $\mathbb{C}$-vector spaces with constructible cohomology on X, we construct, using a classical observation of Kashiwara, cohomology groups $\mathbb{H}^\bullet_{(2)}(\widetilde{X}, F^\bullet)$ that coincide in the case $F^\bullet = \mathbb{C}_X$ with the L_2-cohomology of $\widetilde{X}$, see [Lüc02]. They obey Atiyah's L_2 index theorem, Poincaré–Verdier duality and are compatible with proper morphisms of complex analytic spaces.

If $\mathcal{M}$ is a coherent $\mathcal{D}$-module on X, we define, using the construction of [Eys00], cohomology groups $\mathbb{H}^\bullet_{\mathrm{DR},(2)}(\widetilde{X}, \mathcal{M})$. They coincide in the case $\mathcal{M} = \mathcal{O}_X$ with the L_2-De Rham cohomology of $\widetilde{X}$ with respect to a Riemannian metric pulled back from X. If $\mathcal{M} = \mathcal{D} \otimes_{\mathcal{O}_X} \mathcal{F}$, $\mathcal{F}$ being a coherent analytic sheaf, they coincide with the L_2 cohomology groups $H^\bullet_2(\widetilde{X}, \mathcal{F} \otimes \omega^n_X)$ constructed in [Eys00]. They obey Atiyah's L_2-index theorem. We did not check except in the simplest cases whether they are compatible with proper holomorphic mappings and did not study duality questions.

When an isomorphism in the derived category of sheaves $\mathrm{rh} : F^\bullet \cong \mathrm{DR}(\mathcal{M})$ is given, $\mathcal{M}$ being holonomic, we construct a natural isomorphism $\mathrm{rh}_{(2)} : \mathbb{H}^\bullet_{(2)}(\widetilde{X}, F^\bullet) \cong \mathbb{H}^\bullet_{\mathrm{DR},(2)}(\widetilde{X}, \mathcal{M})$.

These cohomology groups are typically infinite dimensional quotients of Hilbertian Γ-modules by non necessarily closed submodules. They are also modules over $\mathcal{N}(\Gamma)$ the Von Neumann algebra of Γ. But one can be much more precise.

Given Γ a discrete countable group, the exact category of finite type projective Hilbert Γ-modules naturally embeds in a rather simple abelian category $E_f(\Gamma)$ due to Farber [Far96] and Lück [Lüc02] endowed with a faithful functor to $\mathrm{Mod}(\mathcal{N}(\Gamma))$. This abelian category has projective dimension one, its projective objects being finite type projective Hilbert Γ-modules. The preceding L_2-cohomology groups are in the essential image of the forgetful functor and the isomorphism $\mathrm{rh}_{(2)}$ lifts too.

We summarize these constructions in the following:

Theorem 0.1. — *Let X be a compact complex manifold and $\widetilde{X} \to X$ be a Galois covering with Galois group Γ. Let $MD(X)$ be the abelian category whose objects are triples*

$$\mathbb{M} = \left(\mathcal{M} = \mathbb{M}^{\mathrm{DR}}, P = \mathbb{M}^{\mathrm{Betti}}, \alpha\right)$$

where $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module admitting a good filtration, P is a perverse sheaf of $\mathbb{R}$-vector spaces and $\alpha : P \otimes_{\mathbb{R}} \mathbb{C} \to \mathrm{DR}(\mathcal{M})$ is an isomorphism in the derived category of sheaves and whose morphisms are the obvious ones.

There is a ∂-functor which, on the Betti side, is compatible with proper direct images, satisfies Atiyah's L_2 index theorem and Poincaré–Verdier duality:

$$L_2dR : D^b MD(X) \longrightarrow D^b E_f(\Gamma)$$

and for each $\mathbb{M} \in MD(X)$ and $q \in \mathbb{Z}$ functorial isomorphisms in $E_f(\Gamma)$

$$H^q(L_2dR(\mathbb{M})) \cong \mathbb{H}^q_{(2)}\left(\widetilde{X}, \mathbb{M}^{\mathrm{Betti}}\right) \cong \mathbb{H}^q_{\mathrm{DR},(2)}\left(\widetilde{X}, \mathbb{M}^{\mathrm{DR}}\right).$$

If X is a projective algebraic manifold every coherent $\mathcal{D}_X$-module admits a global good filtration (a fact the author has learned from talks given by B. Malgrange). The author does not believe admitting a good filtration is an essential restriction here.

For applications, it seems to be useful to consider the case X is only a compact complex-analytic space such that one can embed X in a complex manifold Z'. In that situation, one can construct, taking a regular neighborhood Z of X, an infinite Galois covering space $\pi : \widetilde{Z} \to Z$ with $\mathrm{Deck}(\widetilde{Z}/Z) = \Gamma$ and a Γ-equivariant embedding $\widetilde{X} \to \widetilde{Z}$ covering the closed embedding $X \to Z$. Theorem 0.1 extends to this situation restricting one's attention to modules on Z whose support is contained in X.

Saito's category of Mixed Hodge Modules $\mathrm{MHM}(X)$ [Sai90] is an abelian subcategory of $MD(X)$.

Corollary 0.2. — *Let X be a compact Kähler manifold and $\widetilde{X} \to X$ be a Galois covering with Galois group Γ.*

There is a ∂-functor which, on the Betti side, is compatible with proper direct images, satisfies Atiyah's L_2 index theorem and Poincaré–Verdier duality:

$$L_2dR : D^b \,\mathrm{MHM}(X) \longrightarrow D^b E_f(\Gamma)$$

and for each $\mathbb{M} \in \mathrm{MHM}(X)$ and $q \in \mathbb{Z}$ functorial isomorphisms in $E_f(\Gamma)$

$$H^q(L_2dR(\mathbb{M})) \cong \mathbb{H}^q_{(2)}\left(\widetilde{X}, \mathbb{M}^{\mathrm{Betti}}\right) \cong \mathbb{H}^q_{\mathrm{DR},(2)}\left(\widetilde{X}, \mathbb{M}^{\mathrm{DR}}\right).$$

These cohomology groups are endowed with a real structure, a real filtration W coming from the weight filtration on $\mathbb{M}^{\mathrm{Betti}}$ and a complex filtration F coming from Saito's Hodge filtration on $\mathbb{M}^{\mathrm{DR}}$. These filtrations and real structures are compatible with morphisms of Mixed Hodge Modules.

It is not clear whether these filtrations define a Mixed Hodge Structure. It seems difficult not to pass to reduced L^2-cohomology. Using an idea of Dingoyan [Din13], we conjecture:

Conjecture 0.3. — *Let $\mathcal{N}(\Gamma) \subset \mathcal{U}(\Gamma)$ be the algebra of affiliated operators. Let $\mathbb{M}$ be a Mixed Hodge module (resp. a pure Hodge Module).*

$$\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} H^q\left(\widetilde{X}, L_2 dR(\mathbb{M})\right)$$

carries a Mixed (resp. a pure) Hodge structure in the abelian category of real $\mathcal{U}(\Gamma)$-modules with finite Γ-dimension. It is given by the functorial filtrations in Corollary 0.2. The restrictions on the Hodge numbers are as in the compact case.

We will use the notation $H^q(L_2dR(\mathbb{M})) \overset{\text{not.}}{=} H^q(\widetilde{X}, L_2dR(\mathbb{M}))$ whenever it is necessary to emphasize that $\mathbb{M}$ lives on X and that we are considering the covering space $\widetilde{X} \to X$.

We do not understand Saito's theory well enough to dare conjecture a similar statement for the derived category of $\mathrm{MHM}(X)$.

Theorem 0.4. — *Conjecture 0.3 is true in the following cases:*

- *There is a closed complex submanifold $i : Z \hookrightarrow X$ and a smooth polarized $\mathbb{Q}$-VSH $(Z, \mathbb{V}, F, S)$ on Z such that $\mathbb{M} = \mathbb{M}_i(\mathbb{V})$ is the corresponding Hodge Module on X.*
- *There is an open embedding $j : U \hookrightarrow X$ such that $X \setminus U$ is a divisor with simple normal crossings and a smooth $\mathbb{Q}$-VSH $(X, \mathbb{V}, F, S)$ on X such that $\mathbb{M} = Rj_* j^{-1}\mathbb{M}_X(\mathbb{V})$.*
- *There is an open embedding $j : U \hookrightarrow X$ such that $X \setminus U$ is a divisor with simple normal crossings and a smooth $\mathbb{Q}$-VSH $(X, \mathbb{V}, F, S)$ on X such that $\mathbb{M} = Rj_! j^{-1}\mathbb{M}_X(\mathbb{V})$.*

The first case follows easily from [Eys97]. The second item in case $\mathbb{V} = \mathbb{Q}_X$ is in fact a reformulation of [Din13]. We nevertheless felt it was helpful to recast Dingoyan's results in our language. The third case does not follow from [Din13]. A more general result holds, it is enough that the Gr_W of the Mixed Hodge module is a direct sum of modules of the form $\mathbb{M}_i(\mathbb{V}^\alpha)$.

The methods we develop, see Section 6, reduce the problem to the case of pure polarizable Hodge modules. The case $\dim(X) = 1$ has been settled by B. Jean [Jea22], as a part of his PHD thesis under the direction of the author. The higher dimensional case is an open problem the author would like to attack using Jean's technical innovations.

The article is organized as follows. The Section 1 constructs L_p-constructible cohomology. The Section 2 constructs L_p-De Rham cohomology for coherent $\mathcal{D}$-modules on complex manifolds. The Section 3 reviews some facts on the homological algebra for $\mathcal{N}(\Gamma)$-modules and about $\mathcal{U}(\Gamma)$. The Section 4 lifts the L_2-cohomology theory to $E_f(\Gamma)$ and finishes the proof of Theorem 0.1. It gives a statement of a refined form of Conjecture 0.3 in terms of the reduced L^2 cohomology of Mixed Hodge Modules and a brief treatment of the singular case. The Section 5 studies analytic L^2 Hodge decomposition in the Kähler case. The Section 6 gives a proof of Theorem 0.4. The Appendix A gives more details on some technical facts the author preferred not to include in the text.

The article is a write-up of a project that was started 20 years ago with the definition of constructible L_2-cohomology in $E_f(\Gamma)$. The initial motivation was to extend [Gro91] to singular Kähler varieties, the role of the L_2-cohomology being played by the L_2-intersection cohomology defined here.

After [Din13] appeared, the scope of the project was extended to include *Mixed Hodge Modules*. The author has given a handful of seminar and conference talks on this project during these years and wishes to apologize for not having made a text available. At some point, it was a work in collaboration with P. Dingoyan, who withdrew from the projet. The author would like to address special thanks to him for many enlightening discussions.

In a forthcoming work in preparation, we will complete the task of extending [Gro91] to the singular case modulo analytic realizations of *Hodge Modules* in the spirit of [SZ21].

The author also thanks P. Bressler, S. Diverio, S. Guillermou, F. Ivorra, B. Jean, W. Lück, J. Tapia and C. Sabbah for valuable discussions, some of them 25 years ago, on topics related to this article. Many thanks to the referee for the many improvements he/she suggested.

1. Constructible L_p-cohomology

In the following $1 \leqslant p < +\infty$ will be a real number. No applicable results will be lost if one restricts oneself to the case $p = 2$. We also let $\mathbb{Q} \subset K \subset \mathbb{C}$ be a subfield of the complex numbers.

1.1. Equivariant constructible sheaves on Γ-simplicial complexes

In this section, we will recall basic well-known definitions, cf [KS90, Chapter VIII]. Our presentation relies on the observation that their constructions are so natural that they are equivariant under proper actions of discrete groups.

Let Γ be a discrete countable group. Let T be a paracompact topological space endowed with an action of Γ (by homeomorphisms). We denote by $\mathbf{Mod}_\Gamma(K_T)$, the category of Γ-equivariant sheaves of K-vector spaces[(1)]. Let A be an abelian category, we also call $D^b(A)$ its bounded derived category[(2)]. We use the shorter notation $D^b_{K,\Gamma}(T) := D^b\,\mathbf{Mod}_\Gamma(K_T)$. We shall drop dependance on K when $K = \mathbb{C}$.

A Γ-simplicial complex $\mathbb{S}$ is a locally finite simplicial complex endowed with a proper left action of Γ, i.e. $\mathbb{S} = (S, \Delta, i)$ where S is a non-empty set endowed with an action of Γ $i : \Gamma \to \mathfrak{S}(S)$ and Δ is a set of non-empty finite subsets of S, the *simplices* of $\mathbb{S}$, such that:

- For every element s of S, the singleton $\{s\}$ belongs to Δ.
- For every element σ of Δ, any non-empty subset τ of σ belongs to Δ.
- For every element s of S, the subset of Δ consisting of the simplices containing s is finite.

(1) A compatible action of Γ on a sheaf $\mathcal{S}$ is a continuous action on $Et(\mathcal{S})$ the étalé space of $\mathcal{S}$ such that the canonical local homeomorphism $Et(\mathcal{S}) \to T$ is Γ-equivariant.

(2) There is no need to restrict to the bounded derived category until Section 3.2, but we will not pursue more generality.

- Γ preserves Δ.
- Γ acts on S with finite stabilizers.

Obviously, Γ acts on Δ with finite stabilizers and Γ acts properly on the topological realization $|\mathbb{S}|$ of $\mathbb{S}$. $|\mathbb{S}|$ is a closed subspace of $\mathbb{R}^S$ (endowed with the product topology) decomposed as $|\mathbb{S}| = \bigcup_{\sigma\in\Delta} |\sigma|$ where

$$|\sigma| = \left\{ x \in \mathbb{R}^S \;\middle|\; x(p) = 0 \quad \text{if} \quad p \notin \sigma,\ x(p) > 0 \quad \text{if} \quad p \in \sigma, \quad \sum_{p\in\sigma} x(p) = 1 \right\}.$$

Say $\mathbb{S}$ is finite dimensional if $\sup_{\sigma\in\Delta} \mathrm{Card}(\sigma) < \infty$. Say $\mathbb{S}$ is cocompact if it is finite dimensional and $\Gamma\backslash S$ is finite. In this case, $\Gamma\backslash\Delta$ is finite and $\Gamma\backslash|\mathbb{S}|$ is compact.

A Γ-equivariant sheaf of K-vector spaces F on $|\mathbb{S}|$ is *weakly* $\mathbb{S}$-*constructible*, resp. $\mathbb{S}$-*constructible*, if for every simplex σ, $i_{|\sigma|}^{-1}F$ is a constant sheaf[(3)], resp. and for every x in $|\mathbb{S}|$, F_x is of finite dimension. The abelian category of $\mathbb{S}$-constructible (resp. weakly $\mathbb{S}$-constructible) equivariant sheaves will be denoted by $\mathbf{Cons}_{K,\Gamma}(\mathbb{S})$ (resp. $w\,\mathbf{Cons}_{K,\Gamma}(\mathbb{S})$). A complex of Γ-equivariant sheaves $F^\bullet$ with bounded cohomology (i.e.: an object of $D^b_\Gamma(|\mathbb{S}|)$) is called $\mathbb{S}$-constructible (resp. weakly $\mathbb{S}$-constructible) if its cohomology sheaves $\mathcal{H}^j(F^\bullet)$ are $\mathbb{S}$-constructible (resp. weakly $\mathbb{S}$-constructible). $\mathbb{S}$-constructible complexes (resp. weakly $\mathbb{S}$-constructible complexes) are the objects of a full thick triangulated subcategory $D^b_{\mathbb{S}-c,K,\Gamma}(|\mathbb{S}|)$ (resp. $D^b_{w-\mathbb{S}-c,K,\Gamma}(|\mathbb{S}|)$) of $D^b_{K,\Gamma}(|\mathbb{S}|)$.

Proposition 1.1. — *Let $\mathbb{S}$ be a finite dimensional Γ-simplicial complex. Then the natural functors*

$$D^b(w\,\mathbf{Cons}_{K,\Gamma}(\mathbb{S})) \longrightarrow D^b_{w-\mathbb{S}-c,K,\Gamma}(|\mathbb{S}|),\ D^b(\mathbf{Cons}_{K,\Gamma}(\mathbb{S})) \longrightarrow D^b_{\mathbb{S}-c,K,\Gamma}(|\mathbb{S}|)$$

are equivalences of triangulated categories.

Proof. — The proof of [KS90, Theorems 8.1.10 and 8.1.11 p. 326] (which is the special case where Γ is the trivial group) applies here mutatis mutandis. Actually, if the action of Γ is free, we can use the natural equivalence of categories between the various categories of Γ-equivariant sheaves on $|\mathbb{S}|$ and of sheaves on $\Gamma\backslash|\mathbb{S}|$ to formally reduce the statement to [KS90, Chapter VIII]. In the general case, observe that the functor β constructed [KS90, p. 324], which is right adjoint to the inclusion functor $w\,\mathbf{Cons}(\mathbb{S}) \to \mathbf{Mod}(|\mathbb{S}|)$, lifts to the category of Γ-equivariant sheaves. Then one can apply β at the level of complexes to construct an inverse equivalence of categories to the first functor. This equivalence maps $D^b_{\mathbb{S}-c,K,\Gamma}(|\mathbb{S}|)$ to $D^b_{\mathbf{Cons}_{K,\Gamma}(\mathbb{S})}(w\,\mathbf{Cons}_{K,\Gamma}(\mathbb{S}))$ thanks to the finite dimensionality hypothesis. Hence, the second equivalence follows the fact that $D^b(\mathbf{Cons}_{K,\Gamma}(\mathbb{S})) \to D^b_{\mathbf{Cons}_{K,\Gamma}(\mathbb{S})}(w\,\mathbf{Cons}_{K,\Gamma}(\mathbb{S}))$ is an equivalence, the main reason being that $\mathbf{Cons}_{K,\Gamma}(\mathbb{S})$ is stable by extensions in $w\,\mathbf{Cons}_{K,\Gamma}(\mathbb{S})$. □

1.2. L_p-cohomology for equivariant constructible sheaves. The case of simplicial complexes.

Let $\mathbb{S}$ be a finite dimensional Γ-simplicial complex. Consider the natural quotient map $\pi : |\mathbb{S}| \to \Gamma\backslash|\mathbb{S}|$. It is easy to see that $\pi_!$, the direct image with proper support,

(3) Whenever Z is a locally closed subset of X, we denote by $i_Z : Z \to X$ the resulting embedding.

is exact on $\mathbb{S}$-constructible sheaves[4]. On the category of equivariant $\mathbb{S}$-constructible sheaves, $\pi_!$ factorizes through the category of sheaves of left $K\Gamma$-modules on $\Gamma\backslash|\mathbb{S}|$.

Let $\infty > p \geqslant 1$ be a real number. The left and right regular representations, denoted by λ and ρ, on the vector space $l_p\Gamma$ of complex valued functions $(a_\gamma)_{\gamma\in\Gamma}$ defined on Γ such that $\sum_{\gamma\in\Gamma}|a_\gamma|^p < \infty$ define a bimodule over $\mathbb{C}\Gamma$. We call $Rl_p\Gamma$ the right Γ-module attached to ρ. In particular, given a sheaf F of left $K\Gamma$-modules on a topological space T, the tensor product $Rl_p\Gamma \otimes_{K\Gamma} F$ is a sheaf of left $\mathbb{C}\Gamma$-modules. Denote by $W_{l,p}(\Gamma)$ the bicommutant of $\lambda(\mathbb{Z}\Gamma)$ in the algebra of continuous linear endomorphisms of $l_p\Gamma$. Certainly $\mathbb{C}\Gamma \subset W_{l,p}(\Gamma)$ and the left action of $\mathbb{C}\Gamma$ extends by construction to a left action of $W_{l,p}(\Gamma)$ on $Rl_p\Gamma$. Hence $Rl_p\Gamma \otimes_{K\Gamma} F$ is actually a sheaf of $W_{l,p}(\Gamma)$-modules since $Rl_p\Gamma$ is a $(W_{l,p}(\Gamma), K\Gamma)$-bimodule (the first acts on the left, the second on the right).

Lemma 1.2. — *The functor $F \mapsto Rl_p\Gamma \otimes_{K\Gamma} \pi_! F$ is exact on $\mathbf{Cons}_{K,\Gamma}(\mathbb{S})$.*

Proof. — Since $\mathbb{C}$ is a K-vector space it is flat over K and $\pi_!$ commutes with $\otimes_K\mathbb{C}$. Hence, it is enough to prove exactness of $F_\mathbb{C} \mapsto Rl_p\Gamma \otimes_{\mathbb{C}\Gamma} \pi_! F_\mathbb{C}$ on $\mathbb{S}$-constructible equivariant sheaves of $\mathbb{C}$-vector spaces. Since the stalk at $p \in \Gamma\backslash|\mathbb{S}|$ of $\pi_!\mathcal{F}_\mathbb{C}$ is isomorphic to $\mathbb{C}[\Gamma/H_{\tilde p}]^{\oplus n}$ where n is a nonnegative integer and $H_{\tilde p}$ is the stabilizer of some lift $\tilde p \in |\mathbb{S}|$ of p, it follows that it is a projective module over $\mathbb{C}\Gamma$. Indeed, whenever H is a finite subgroup of Γ, the pull-back injection $i : \mathbb{C}[\Gamma/H] \to \mathbb{C}\Gamma$ has a right inverse $\pi((a_\gamma)_{\gamma\in\Gamma})_g = \frac{1}{(H)}\sum_{h\in H} a_{gh}$ which is equivariant for the left action of Γ. Exactness follows from the facts that stalks of tensor products are computed stalkwise, that $\pi_!$ is exact and that a short exact sequence of projective modules splits. ☐

Lemma 1.3. — *The functor $F \mapsto Rl_p\Gamma \otimes_{K\Gamma} \pi_! F$ is exact on $\mathbf{Mod}_\Gamma(K_{|\mathbb{S}|})$.*

Proof. — The above proof also works with a minor modification for all sheaves of K-vector spaces. ☐

Definition 1.4. — *Let $F^\bullet$ be an object of $D^b(\mathbf{Cons}_{K,\Gamma}(\mathbb{S}))$. Its k^{th} L_p- hypercohomology group is the $W_{l,p}(\Gamma)$-module*

$$\mathbb{H}^k_{(p)}(|\mathbb{S}|, F^\bullet) := \mathbb{H}^k(\Gamma\backslash|\mathbb{S}|, Rl_p\Gamma \otimes_{K\Gamma} \pi_! F^\bullet).$$

Lemma 1.5. — *The composition of the derived functor of $H^0(\Gamma\backslash|\mathbb{S}|, -)$ and of $Rl_p\Gamma \otimes_{K\Gamma} \pi_! -$ gives a ∂-functor[5] of triangulated categories*

$$\mathbb{H}^\bullet_{(p)}(|\mathbb{S}|, -) : D^b(\mathbf{Cons}_{K,\Gamma}(\mathbb{S})) \longrightarrow D^b\Big(\mathrm{Mod}_{W_{l,p}(\Gamma)}\Big),$$

where $\mathrm{Mod}_{W_{l,p}(\Gamma)}$ stands for the category of left $W_{l,p}(\Gamma)$-modules such that the L_p-hypercohomology groups are its cohomology objects.

---

(4) It is a direct consequence of [KS90, Proposition 8.1.4 p. 323] in case the action is free. The general case is easily taken care of by a barycentric subdivision argument.

(5) As in [Har66, p. 22], a ∂-functor of triangulated categories is an additive functor which commutes with the translation functor and respects distinguished triangles.

Proof. — The fact that it is a ∂-functor follows from the exactness of the functor $Rl_p\Gamma \otimes_{K\Gamma} \pi_! -$ and the property that sheaf cohomology on $\Gamma\backslash|\mathbb{S}|$ can be computed using the derived functor of $H^0(\Gamma\backslash|\mathbb{S}|, -)$, in particular for sheaves of $W_{l,p}(\Gamma)$-modules. □

This definition gives rise to the long exact sequence attached to a short exact sequence and to various spectral sequences generalizing it.

Thanks to Lemma 1.3, we also get with the same proof as Lemma 1.5:

Lemma 1.6. — *The same formula as in Definition 1.4 defines an extension of* $\mathbb{H}^*_{(p)}(|\mathbb{S}|, -)$ *to a* ∂*-functor*

$$\mathbb{H}^\bullet_{(p)}(|\mathbb{S}|, -) : D^b_{K,\Gamma}(|\mathbb{S}|) \longrightarrow D^b\Big(\mathrm{Mod}_{W_{l,p}(\Gamma)}\Big).$$

We don't use a different notation hoping this will not cause any confusion.

1.3. The subanalytic case

1.3.1. Subanalytic stratifications and constructible sheaves

A subanalytic[(6)] Γ-space is a Γ-space that can be realized as a locally closed Γ-invariant subanalytic subset of a real analytic manifold endowed with a proper real analytic action of Γ. A stratified subanalytic space $\mathbb{X}$ is a subanalytic space $\widetilde{X} := |\mathbb{X}|$ endowed with a locally finite partition $\widetilde{X} = \bigcup_i X_i$ in disjoint subanalytic submanifolds satisfying $X_i \cap \overline{X}_j \neq \emptyset \Rightarrow X_i \subset \overline{X}_j$. A stratified subanalytic Γ-space $\mathbb{X}$ is a proper analytic action of Γ on $\widetilde{X} = |\mathbb{X}|$ such that for every $g \in \Gamma$ and every point $x \in |\mathbb{X}|$ the germ at x of the stratification is carried by g to the germ at gx of the stratification. When the stratification comes from a Γ-simplicial complex, one calls it a triangulation. [KS90, Proposition 8.2.5] implies that, in the cocompact case, any Γ-stratification may be refined to a Γ-triangulation and that the Γ-triangulations form a cofinal system with respect to refinement.

The obvious extension of the definitions and notations of Section 1.1 will be left to the reader, the only change being that $\mathbb{X}$-constructible sheaves on $|\mathbb{X}|$ are now assumed to be locally constant along the strata of $\mathbb{X}$.

Let $\widetilde{X}$ be a subanalytic Γ-space. A sheaf of K-vector spaces F on $\widetilde{X}$ is called constructible if it is constructible with respect to some subanalytic stratification of $\widetilde{X}$. We denote by $\mathbb{R}\,\mathbf{Cons}_{K,\Gamma}(\widetilde{X})$ the category of equivariant constructible sheaves on $\widetilde{X}$ and by $D^b_{K,\mathbb{R}c,\Gamma}(\widetilde{X})$ the thick full subcategory of $D^b_{K,\Gamma}(\widetilde{X})$ consisting of complexes with bounded constructible cohomology. The equivariant version of [KS90, Theorem 8.4.5 p. 339] is easily checked to hold and can be stated as the following:

Proposition 1.7. — *The natural functor* $D^b(\mathbb{R}\,\mathbf{Cons}_{K,\Gamma}(\widetilde{X})) \to D^b_{\mathbb{R}c,K,\Gamma}(\widetilde{X})$ *is an equivalence of triangulated categories if* $\widetilde{X}$ *is cocompact.*

Proof. — I leave it to the reader to look through the arguments in [KS90] and observe they can be made equivariant. □

(6) Actually, "definable in a o-minimal structure" is the natural hypothesis.

1.3.2. Constructible L_p-cohomology

Proposition 1.8. — *Let $\widetilde{X}$ be a cocompact subanalytic Γ-space. Let $\mathcal{F}^\bullet$ be an object of $D^b_{\mathbb{R}-c,K,\Gamma}(\widetilde{X})$. Its kth L_p-hypercohomology group is the $W_{l,p(\Gamma)}$-module*

$$\mathbb{H}^k_{(p)}(\widetilde{X},\mathcal{F}^\bullet) := \mathbb{H}^k\Big(\Gamma\backslash\widetilde{X}, Rl_p\Gamma\otimes_{K\Gamma}\pi_!\mathcal{F}^\bullet\Big).$$

There is a ∂-functor of triangulated categories

$$\mathbb{H}^\bullet_{(p)}(\widetilde{X},-) : D^b_{\mathbb{R}-c,K,\Gamma}(\widetilde{X}) \longrightarrow D^b\Big(\mathrm{Mod}_{W_{l,p}(\Gamma)}\Big),$$

where $\mathrm{Mod}_{W_{l,p}(\Gamma)}$ *stands for the category of left $W_{l,p}(\Gamma)$-modules such that*

$$\mathbb{H}^k_{(p)}(\widetilde{X},\mathcal{F}^\bullet) = H^k\Big(\mathbb{H}^\bullet_{(p)}(\widetilde{X},\mathcal{F}^\bullet)\Big).$$

Proof. — We can replace $D^b_{\mathbb{R}-c,K,\Gamma}(\widetilde{X})$ by $D^b\mathbb{R}\,\mathbf{Cons}_{K,\Gamma}(\widetilde{X})$ since the natural functor is an equivalence by Proposition 1.1 and $Rl_p\Gamma\otimes_{K\Gamma}\pi_!\mathbb{C}\otimes-$ is exact by Lemma 1.6.
Since $D^b\mathbb{R}\,\mathbf{Cons}_{K,\Gamma}(\widetilde{X})$ is the limit of its full subcategories $D^b\,\mathbf{Cons}_{K,\Gamma}(\mathbb{X})$, $\mathbb{X}$ running through all subanalytic Γ-triangulations, this follows from Definition 1.4. ☐

1.4. Complex Analytic case

We assume here $K=\mathbb{C}$ and drop K from the notation.

Assume from now on that $\widetilde{X}$ is a cocompact complex Γ-space. The relevant stratifications are complex analytic stratification (by definition, a subanalytic stratification is complex analytic if so are the closures of the strata) and we say that an equivariant sheaf is constructible if it is constructible with respect to some complex analytic stratification and that a complex of equivariant sheaves is constructible if so are its cohomology sheaves. Then $\mathbf{Cons}_\Gamma(\widetilde{X})$ is a full abelian subcategory of $\mathbb{R}\,\mathbf{Cons}_\Gamma(\widetilde{X})$ stable by extensions, $D^b_{c,\Gamma}(\widetilde{X})$, the full subcategory of $D^b_{\mathbb{R}c,\Gamma}(\widetilde{X})$ whose cohomology objects are in $\mathbf{Cons}_\Gamma(\widetilde{X})$, is a thick triangulated subcategory and we have a natural ∂-functor

$$D^b\,\mathbf{Cons}_\Gamma(\widetilde{X}) \longrightarrow D^b_{c,\Gamma}(\widetilde{X}).$$

Remark 1.9. — This functor is an equivalence of categories if $\widetilde{X}$ is a Galois topological covering space of the analytization of a complex projective variety using GAGA and [Nor02].

Definition 1.10. — *We can restrict $\mathbb{H}^\bullet_{(p)}(\widetilde{X},-)$ to $D^b_{c,\Gamma}(\widetilde{X})$ to get the* constructible L_p-cohomology functor*:*

$$\mathbb{H}^\bullet_{(p)}(\widetilde{X},-) : D^b_{c,\Gamma}(\widetilde{X}) \longrightarrow D^b\Big(\mathrm{Mod}_{W_{l,p}(\Gamma)}\Big).$$

An important special case is L_p-intersection cohomology.

Definition 1.11. — *Let Z be a singular compact complex space and $\pi:\widetilde{Z}\to Z$ its universal covering space. Its k^{th} intersection L_p cohomology is the $W_{l,p}(\Gamma)$-module $\mathbb{H}^k_{(p)}(\widetilde{Z},\pi^{-1}\mathcal{IC}^\bullet_Z)$ where $\mathcal{IC}^\bullet_Z$ is the intersection cohomology sheaf of Z* [BBD82].

The initial impetus for this work was to formulate the following:

Conjecture 1.12. — *If $p=2$, and Z is a closed analytic subset of a compact Kähler hyperbolic manifold then $\mathbb{H}^k_{(2)}(\tilde{Z},\pi^{-1}\mathcal{IC}^\bullet_Z)=0$ for $k\neq \dim(Z)$.*

1.5. Real structures

The algebra $W_{l,p}(\Gamma)$ carries a real structure, namely a conjugate linear algebra involutive automorphism we shall denote by †. It comes from the real structure on $l^p\Gamma$ represented by the complex conjugation c which maps $f:\Gamma\to\mathbb{C}$ to its complex conjugate function. This complex conjugation lifts to a real structure on the bounded operators of $l^p\Gamma$ defined by $\Phi^\dagger=c\circ\Phi\circ c^{-1}$. Since the subalgebra $\mathbb{Z}\Gamma$ is invariant by †, so is its bicommutant and $W_{l,p}(\Gamma)$ is †-invariant. A real structure on a module over $W_{l,p}(\Gamma)$ is just a conjugate linear automorphism on the underlying $\mathbb{C}$-vector space compatible with †. For instance $l_p\Gamma$ has a real structure.

Modules with real structures form a $\mathbb{R}$-linear abelian category $\mathrm{Mod}_{W_{l,p}(\Gamma),\mathbb{R}}$ which has an exact faithful forgetful functor to $\mathrm{Mod}_{W_{l,p}(\Gamma),\mathbb{R}}$ and if $K\subset\mathbb{R}$ the functors of Proposition 1.8 and Definition 1.10 lift to $D^b(\mathrm{Mod}_{W_{l,p}(\Gamma),\mathbb{R}})$.

2. L^p-cohomology and differential operators for coherent sheaves

2.1. Coherent L^p-cohomology

Let X be a complex manifold and let $\mathcal{O}_X$ (resp. $\mathcal{D}_X$) denote its structure sheaf (resp. the sheaf of holomorphic differential operators). Let $\pi:\widetilde{X}\to X$ be a Galois topological covering space and $\Gamma=\mathrm{Gal}(\widetilde{X}/X)$ be its Galois group, acting on $\widetilde{X}$ on the left.

If $\mathcal{R}$ is a sheaf of rings on X, denote by $\mathbf{Mod}(\mathcal{R})$ the abelian category of sheaves of left $\mathcal{R}$-modules by $\mathrm{Hom}_{\mathcal{R}}$ its group of morphisms and by $\underline{\mathrm{Hom}}_{\mathcal{R}}$ the internal Hom bifunctor on $\mathbf{Mod}(\mathcal{R})$. When considering right $\mathcal{R}$-modules, we use the notations $\mathbf{Mod}(\mathcal{R}^o)$, $\mathrm{Hom}_{\mathcal{R}^o}$, $\underline{\mathrm{Hom}}_{\mathcal{R}^o}$. If R is a ring then we denote by R_X the sheaf of rings of locally constant functions with values in R.

Denote by $\mathbf{Coh}(\mathcal{O}_X)$ the full abelian subcategory of $\mathbf{Mod}(\mathcal{O}_X)$ whose objects are the coherent analytic sheaves of X. Denote by $\mathbf{Coh}(\mathcal{D}_X)$ (resp. $\mathbf{Hol}(\mathcal{D}_X)$) the full abelian subcategory of $\mathbf{Mod}(\mathcal{D}_X)$ whose objects are the coherent (resp. holonomic) $\mathcal{D}_X$-modules. A $\mathcal{O}_X$-module is quasi coherent if it is locally the limit of its coherent submodules.

For every $p\in[1,+\infty[$ and $\mathcal{F}$ a coherent analytic sheaf on X, [Eys00] (see also [Cam01]) constructs a subsheaf $l^p\pi_*\mathcal{F}\subset\pi_*\pi^{-1}\mathcal{F}$ which can be described locally as follows. Choose $\varphi:\mathcal{O}_X^{\oplus N}\to\mathcal{F}$ a presentation of $\mathcal{F}$ on a Stein open subset U such that $\pi^{-1}(U)=\Gamma\times U$ then:

$$l^p\pi_*\mathcal{F}(U)=\begin{Bmatrix}(s_\gamma)_{\gamma\in\Gamma}\in\mathcal{F}(U)^\Gamma, & \exists\, s'_\gamma\in\mathcal{O}_X^{\oplus N}(U)^\Gamma, & \varphi(s'_\gamma)=s_\gamma\\ \text{and}\quad \forall\, K\Subset U & \sum_{\gamma\in\Gamma}\int_K|s_\gamma|^p<+\infty & \end{Bmatrix}.$$

The independence on φ is checked in [Eys00]. Given $\phi : \mathcal{F} \to \mathcal{F}'$ a $\mathcal{O}_X$-linear morphism of coherent sheaves

$$\pi_*\pi^{-1}\phi : \pi_*\pi^{-1}\mathcal{F} \longrightarrow \pi_*\pi^{-1}\mathcal{F}'$$

maps $l^p\pi_*\mathcal{F}$ into $l^p\pi_*\mathcal{F}'$. Denote by $l^p\pi_*\phi : l^p\pi_*\mathcal{F} \to l^p\pi_*\mathcal{F}'$ the restriction of $\pi_*\pi^{-1}\phi$. The resulting functor $l^p\pi_* : \mathbf{Coh}(\mathcal{O}_X) \to \mathbf{Mod}(W_{l,p}(\Gamma) \otimes_{\mathbb{C}} \mathcal{O}_X)$ is exact [Eys00] and one can define

$$H^\bullet_{L^p}(\widetilde{X}, \mathcal{F}) := H^\bullet(X, l^p\pi_*\mathcal{F}).$$

Since $H^q_{L^p}(\widetilde{X}, \mathcal{F}) = 0$ for $q > \dim_{\mathbb{C}}(X)$ (at least when X is compact) this yields a good cohomology theory on $\mathbf{Coh}(\mathcal{O}_X)$, indeed a ∂-functor

$$D^b\,\mathbf{Coh}(\mathcal{O}_X) \longrightarrow D^b\,\mathrm{Mod}_{W_{l,p}(\Gamma)}\,.$$

Observe that if $\mathcal{F}$ is coherent $l^p\pi_*\mathcal{F} = l^p\pi_*\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{F}$. Hence, the functor $l^p\pi_*$ extends to $D^b(\mathbf{Mod}(\mathcal{O}_X))$ setting $l^p\pi_*\mathcal{L} := l^p\pi_*\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{L}$ thanks to:

Lemma 2.1. — *The functor $l^p\pi_* = l^p\pi_*\mathcal{O}_X \otimes_{\mathcal{O}_X}$ is exact on $\mathbf{Mod}(\mathcal{O}_X)$.*

Proof. — The problem is local. Since this functor is exact on $\mathbf{Coh}(\mathcal{O}_X)$, it follows from the fact that tensor products of sheaves commute with taking the stalks [God73, p. 137] that $Tor_1^{\mathcal{O}_{X,x}}((l^p\pi_*\mathcal{O}_X)_x, \mathcal{O}_{X,x}/I_x) = 0$ for every (finitely generated) ideal of $\mathcal{O}_{X,x}$. Hence $(l^p\pi_*\mathcal{O}_X)_x$ is a flat $\mathcal{O}_{X,x}$-module and exactness follows applying [God73, p. 137] once more. □

It should however be noted that if a sheaf $\mathcal{F}$ has two different $\mathcal{O}_X$-module structures, say $\mathcal{F}_1$ and $\mathcal{F}_2$, it may be the case that $l^p\pi_*\mathcal{F}_1 \neq l^p\pi_*\mathcal{F}_2$ as subsheaves of $\pi_*\pi^{-1}\mathcal{F}_1 = \pi_*\pi^{-1}\mathcal{F}_2 = \pi_*\pi^{-1}\mathcal{F}$.

Remark 2.2. — There is a natural structure of left $\mathcal{D}_X$-module on $l^p\pi_*\mathcal{O}_X$ hence $l^p\pi_*$ gives rise to an exact endofunctor of $\mathbf{Mod}(\mathcal{D}_X)$. When Γ is infinite, it does not preserve the full subcategory of coherent or quasicoherent modules.

2.2. Differential operators

We need to check that differential operators between quasicoherent analytic sheaves preserve $l^p\pi_*$.

Recall from [Sai89] that for $\mathcal{L}, \mathcal{L}'$ two $\mathcal{O}_X$-modules $\mathrm{Diff_X}(\mathcal{L}, \mathcal{L}')$ is the image of the natural injective morphism:

$$\mathrm{Hom}_{\mathcal{D}_X^\circ}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X) \longrightarrow \mathrm{Hom}_{\mathbb{C}_X}(\mathcal{L}, \mathcal{L}')$$

given by the composition of the natural adjunction

$$\mathrm{Hom}_{\mathcal{D}_X^\circ}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X),$$

$\mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X$ being endowed with the *right* $\mathcal{O}_X$-module structure, with left composition by the natural $\mathbb{C}$-linear (actually left $\mathcal{O}_X$-linear) morphism

$$\nu_{\mathcal{L}'} : \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X \longrightarrow \mathcal{L}'$$

which maps $\ell \otimes P$ to $P(1)\ell$. One has $\nu_{\mathcal{L}'} = \mathcal{L}' \otimes_{\mathcal{O}_X} \nu_{\mathcal{O}_X}$ where $\nu_{\mathcal{O}_X} : \mathcal{D}_X \to \mathcal{O}_X$ is the naturel *left* $\mathcal{D}_X$-linear (hence left $\mathcal{O}_X$-linear) morphism mapping $P \in \mathcal{D}_X$ to $P(1) \in \mathcal{O}_X$.

Lemma 2.3. — *Assume $\mathcal{L}'$ is quasicoherent. Let $(\mathcal{L}'\otimes_{\mathcal{O}_X}\mathcal{D}_X)_l$ resp. $(\mathcal{L}'\otimes_{\mathcal{O}_X}\mathcal{D}_X)_r$ the left resp. the right $\mathcal{O}_X$-modules structures of $\mathcal{L}'\otimes_{\mathcal{O}_X}\mathcal{D}_X$. Then:*

$$l^p\pi_*(\mathcal{L}'\otimes_{\mathcal{O}_X}\mathcal{D}_X)_l = l^p\pi_*(\mathcal{L}'\otimes_{\mathcal{O}_X}\mathcal{D}_X)_r \subset \pi_*\pi^{-1}\mathcal{L}'\otimes\mathcal{D}_X.$$

Proof. — Let us begin by treating the case where $\mathcal{L}'=\mathcal{O}_X$. Then both $l^p(\pi_*(\mathcal{D}_X))_\sharp$, $\sharp=l,r$ are the increasing union of the subsheaves $l^p\pi_*(F_k\mathcal{D}_X)_\sharp$ where $F_k\mathcal{D}_X$ is the sub-$\mathcal{O}_X$-bimodule consisting of the holomorphic differential operators of degree $\leqslant k$. Hence it is enough to show that $l^p\pi_*(F_k\mathcal{D}_X)_l = l^p\pi_*(F_k\mathcal{D}_X)_r$. The problem being local assume we have a coordinate system on an open set U such that $\pi^{-1}(U)\simeq\Gamma\times U$. A section of $\pi_*\pi^{-1}F_k\mathcal{D}_X$ of the form $(\sum_{|\alpha|\leqslant k} f_{\alpha,\gamma}\partial^\alpha)$ is in $l^p(\pi_*(\mathcal{D}_X))_l$ iff, for all $K\Subset U$, $\sum_\gamma\int_K\sum_\alpha|f_{\alpha,\gamma}|^p<+\infty$ whereas a section $\pi_*\pi^{-1}F_k\mathcal{D}_X$ of the form $(\sum_{|\alpha|\leqslant k}\partial^\alpha g_{\alpha,\gamma})$ is in $l^p(\pi_*(\mathcal{D}_X))_r$ iff, for all $K\Subset U$,

$$\sum_\gamma\int_K\sum_\alpha|g_{\alpha,\gamma}|^p<+\infty.$$

Since $\partial^\alpha g = g\partial^\alpha+\sum_{\beta<\alpha}P_{\beta,\alpha}(g)\partial^\beta$ where $P_{\beta,\alpha}$ is a universal differential operator, the Cauchy inequality gives

$$\int_K\sum_\alpha|f_{\alpha,\gamma}|^p\leqslant C_{K,K'}\int_{K'}\sum_\alpha|g_{\alpha,\gamma}|^p$$

if $K'\Subset U$ is a compact neighborhood of K. Whence the inclusion $l^p\pi_*(F_k\mathcal{D}_X)_l\subset l^p\pi_*(F_k\mathcal{D}_X)_r$. The reverse inclusion follows by the same token.

This implies the lemma for $\mathcal{L}'$ a free $\mathcal{O}_X$-module of possibly infinite rank.

Now, for the general case. The statement being local, we may choose $\varphi:\mathcal{O}_X^N\to\mathcal{L}'$ a presentation, N being some cardinal. The definition implies that $l^p\pi_*(\mathcal{L}'\otimes_{\mathcal{O}_X}\mathcal{D}_X)_\sharp\in\pi_*\pi^{-1}\mathcal{L}'\otimes_{\mathcal{O}_X}\mathcal{D}_X$ is the image by $\pi_*\pi^{-1}\varphi$ of $l^p\pi_*(\mathcal{O}_X^N\otimes_{\mathcal{O}_X}\mathcal{D}_X)_\sharp$ in $\pi_*\pi^{-1}\mathcal{L}'\otimes_{\mathcal{O}_X}\mathcal{D}_X$. The lemma follows. ☐

Lemma 2.4. — *Under the hypothesis of Lemma 2.3, let $P\in\mathrm{Diff_X}(\mathcal{L},\mathcal{L}')$ and let $p\in\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L},\mathcal{L}'\otimes_{\mathcal{O}_X}\mathcal{D}_X)$ be the unique right $\mathcal{O}_X$-linear morphism such that $P=\nu_{\mathcal{L}'}\circ p$. Then*

$$l^p\pi_*P := \nu_{l^p\pi_*\mathcal{L}'}\circ l^p\pi_*p : l^p\pi_*\mathcal{L}\longrightarrow l^p\pi_*\mathcal{L}'$$

is the restriction of $\pi_*\pi^{-1}P$ and defines a $W_{l,p}(\Gamma)\otimes_{\mathbb{C}}\mathcal{O}_X$-linear morphism of sheaves.

The assignment $P\mapsto l^p\pi_*P$ defines an additive functor

$$l^p\pi_*:\mathbf{Qcoh}(\mathcal{O}_X,\mathrm{Diff_X})\longrightarrow\mathbf{Mod}\Big(\underline{W_{l,p}(\Gamma)}_X\Big)$$

where $\underline{W_{l,p}(\Gamma)}_X$ is the constant sheaf with constant value $W_{l,p}(\Gamma)$ and $\mathbf{Qcoh}(\mathcal{O}_X,\mathrm{Diff_X})$ is the additive category whose objects are quasi coherent $\mathcal{O}_X$-modules and whose morphisms are differential operators.

Proof. — The definition makes sense thanks to lemma 2.3. The statement is thus an easy consequence of the definition and of the properties of $l^p\pi_*$ described above. ☐

2.3. L_p De Rham cohomology

Let $\mathcal{M}$ be a (quasi) coherent $\mathcal{D}_X$-module viewed as a $\mathcal{O}_X$-module endowed with a flat connection $\nabla : \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^1_X$. The De Rham complex of $\mathcal{M}$ defined as:

$$\mathrm{DR}(\mathcal{M}) = \Big(\mathcal{M} \xrightarrow{\nabla} \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^1_X \xrightarrow{\nabla} \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^2_X \longrightarrow \dots\Big)[\dim X]$$

is a complex in $\mathbf{Qcoh}(\mathcal{O}_X, \mathrm{Diff_X})$. Applying the functor $l^p\pi_*$ we define the L^p De Rham complex $l^p\pi_* \mathrm{DR}(\mathcal{M})$ and the L^p De Rham cohomology:

$$H^\bullet_{\mathrm{DR},L^p}(\widetilde{X}, \mathcal{M}) := H^\bullet(X, l^p\pi_* \mathrm{DR}(\mathcal{M})).$$

We will not try to put more structure than the natural $W_{l,p}(\Gamma)$-module structure on these general L^p cohomology groups.

The L^p De Rham constructible cohomology groups come from a ∂-functor

$$H^\bullet_{\mathrm{DR},L^p,} : D^b(\mathbf{Coh}(\mathcal{D}_X)) \longrightarrow D^b\Big(\mathrm{Mod}_{W_{l,p}(\Gamma)}\Big).$$

Example 2.5. — If $\mathcal{F}$ is a quasi coherent $\mathcal{O}_X$-module,

$$H^\bullet_{\mathrm{DR},L^p}\Big(\widetilde{X}, \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}\Big) = H^\bullet_{L^p}\Big(\widetilde{X}, \mathcal{F} \otimes \omega_X\Big).$$

Proof. — The natural augmentation $\epsilon : \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \omega_X \to \mathcal{F} \otimes_{\mathcal{O}_X} \omega_X$ gives rise to a quasi-isomorphism $\mathrm{DR}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}) \xrightarrow{\epsilon} \mathcal{F} \otimes_{\mathcal{O}_X} \omega_X$. Locally it is a Koszul complex for the regular sequence $(\partial_{x_1}, \dots, \partial_{x_n})$. The same is actually true for its $l^p\pi_*$ and we get a quasi-isomorphism $l^p\pi_* \mathrm{DR}(\mathcal{F}) \xrightarrow{l^p\pi_*\epsilon} l^p\pi_*\mathcal{F} \otimes_{\mathcal{O}_X} \omega_X$. □

Example 2.6. — Denote by $\ell^p\pi_*\mathbb{C}_{\tilde{X}} \subset \pi_*\mathbb{C}_{\tilde{X}}$ the locally constant sheaf of $W_{l,p}(\Gamma)$-modules attached to the right regular representation of Γ in $L^p\Gamma$. Let $\mathbb{V}$ be a finite rank complex local system on X and $\mathcal{V}$ be the $\mathcal{D}_X$-module whose underlying finite rank locally free $\mathcal{O}_X$-module is $\mathbb{V} \otimes_{C_X} \mathcal{O}_X$ and holomorphic connection ∇ so that the natural morphism $\sigma : \mathbb{V} \to \mathcal{V}$ represents $\ker(\nabla)$.

Then $H^\bullet_{\mathrm{DR},L^p}(\widetilde{X}, \mathcal{V}) = H^{\bullet+\dim(X)}(X, l^p\pi_*\mathbb{C}_{\tilde{X}} \otimes_{\mathbb{C}_X} \mathbb{V}) = \mathbb{H}^{\bullet+\dim(X)}_{(p)}(\widetilde{X}, \mathbb{V})$.

Proof. — Left to the reader. □

Remark 2.7. — With the notation of Remark 2.2, $l^p\pi_* \mathrm{DR}(\mathcal{M}) = \mathrm{DR}(l^p\pi_*\mathcal{M})$.

2.4. Compatibility to the Riemann–Hilbert correspondence

Remark 2.8. — The natural sheaf monomorphism $i_\pi : \ell^p\pi_*\mathbb{C}_{\tilde{X}} \otimes_{\mathbb{C}_X} \mathcal{O}_X \to l^p\pi_*\mathcal{O}_X$ is not an epimorphism. One would need a completed tensor product of sheaves in locally convex topological vector spaces we will not try and discuss.

Proposition 2.9. — *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then the natural map $Rl^p\Gamma \otimes_{\mathbb{C}\Gamma} \pi_!\pi^{-1} \mathrm{DR}(\mathcal{M}) \to l^p\pi_* \mathrm{DR}(\mathcal{M})$ is a quasi-isomorphism.*

Proof. — The problem is local. Thus, we can assume $\mathcal{M}$ has a good filtration $F_\bullet$. Then $F_\bullet \mathrm{DR}(\mathcal{M})$ is a filtration of $\mathrm{DR}(\mathcal{M})$ by differential complexes of coherent sheaves. For $q \gg 0$, the natural morphism $F_q \mathrm{DR}(M) \to \mathrm{DR}(\mathcal{M})$ is a quasi-isomorphism [Sai89, Lemma 1.14] (see also [Bjö93, Lemma 1.5.6 p. 31]). Hence it is enough to prove that $Rl^p\Gamma \otimes_{\mathbb{C}\Gamma} \pi_!\pi^{-1}F_qDR(\mathcal{M}) \to l^p\pi_*F_qDR(\mathcal{M})$ is a quasi-isomorphism for such a $q \gg 0$.

Thanks to the Kashiwara constructibility theorem[(7)] [Kas80, MNM93], the cohomology of $F_qDR(\mathcal{M})$ is constructible. Choose U appropriately such that it is Stein and

$$H^i\Big(U, \mathcal{H}^j(\mathrm{DR}(\mathcal{M}))\Big) = 0$$

for $i > 0$ and all j. We also have $H^i(U, \mathcal{H}^j(F_qDR(\mathcal{M}))) = 0$ for $q \gg 0$, $i > 0$ and all j.

Note that the kernel and images of the differentials in $F_q\,\mathrm{DR}(\mathcal{M})$ have also vanishing cohomology in positive degree on U.

Under this hypothesis, we also have:

$$\mathcal{H}^{k-n}(\mathrm{DR}(\mathcal{M}))(U) = H^k\begin{pmatrix} \dots \longrightarrow F_{q+k-1}\mathcal{M}\otimes\Omega^{k-1}(U) \longrightarrow F_{q+k}\mathcal{M}\otimes\Omega^k(U) \\ \longrightarrow F_{q+k+1}\mathcal{M}\otimes\Omega^{k+1}(U) \longrightarrow \dots \end{pmatrix}.$$

Kashiwara's constructibility theorem also implies that $\dim \mathcal{H}^{k-n}(\mathrm{DR}(\mathcal{M}))(U)$ is finite. The U satisfying these properties form a basis of the topology of X.

We have to show that every element z of $\mathrm{Ker}(d) : F_{q+k}\mathcal{M}\otimes\Omega^k(U) \to F_{q+k+1}\mathcal{M}\otimes\Omega^{k+1}(U)$ can be decomposed as a sum $z = dt + g(h(z))$ where

$$g : \mathcal{H}^{k-n}(F_q\,\mathrm{DR}(\mathcal{M})) \longrightarrow \mathrm{Ker}(d) \subset F_{q+k}\mathcal{M}$$

is a section over U of the morphism of sheaves

$$h : \mathrm{Ker}\Big(d : F_{q+k}\mathcal{M}\otimes\Omega^k \longrightarrow F_{q+k+1}\mathcal{M}\otimes\Omega^{k+1}\Big) \longrightarrow \mathcal{H}^{k-n}(\mathrm{DR}(\mathcal{M}))$$

and t a section over U of $F_{q+k-1}\mathcal{M}\otimes\Omega^{k-1}$ with local L^p-estimates.

This means the following. The Fréchet structure of $\mathcal{G}(U)$ where $\mathcal{G}$ is coherent is given by an inverse limit of a countable family of L^p norms $(\|-\|_n)_{n\in\mathbb{N}}$ defined by integration on an exhaustive family of compact subsets of U if the sheaf is locally free, of quotient norms of such L_p norms in a locally presentation of the sheaf in general [Eys00]. A local L^p estimate is then, for all $n \in \mathbb{N}$, a series of estimates of the form:

$$\|t\|_n \leqslant C_n.\|z\|_{n'}$$

for some $n' \in \mathbb{N}$.

This follows from the continuity of d for this Fréchet structure, the fact that a continuous operator of Fréchet spaces has closed range if it has a finite dimensional kernel and the open mapping theorem for Fréchet spaces using a standard argument (cf. e.g. [Eys00, pp. 534-535]). □

Corollary 2.10. — *The natural map i_π induces a natural invertible transformation of functors on $\mathbf{Hol}(\mathcal{D}_X)$:*

$$\mathrm{rh}_{(p)} : H^\bullet_{\mathrm{DR},L^p}(\widetilde{X}, \_) \xleftarrow{\simeq} \mathbb{H}^{\bullet+\dim_{\mathbb{C}}(\tilde{X})}_{(p)}\Big(\widetilde{X}, \mathrm{DR}(\_)\Big).$$

Remark 2.11. — As in Section 1, we may work in the more general set-up of a proper action of Γ on a complex manifold $\widetilde{X}$ with cocompact quotient or even restrict our attention to cocompactly supported equivariant coherent $\mathcal{D}_X$ or $\mathcal{O}_X$-modules.

(7) Which is used implicitly in the statement of the proposition.

3. Farber's abelian category and its localisation

Up to this point, we were working with L^p-cohomology. Now, it is time to admit that unless $p = 2$ the objects we constructed are out of control.

We will change our notations and define $\mathcal{N}(\Gamma) = W_{l,2}(\Gamma)$ and survey some relevant homological algebraic aspects of modules over this operator algebra, which is known as the Von Neumann algebra of the group Γ.

3.1. Hilbert Γ-modules

Let us first briefly review a very nice construction due to Farber and Lück [Far96]. For a longer review adapted to our purposes, see [Eys00, pp. 539-544]. For a complete review, including applications in topology and algebra, see the bible of the subject [Lüc02].

Definition 3.1. — *A Hilbert Γ-module (resp. of finite type, resp. separable) is a topological $\mathbb{C}$-vector space with a continuous Γ-action which can be realized as a closed Γ-invariant subspace of $l_2\Gamma\widehat{\otimes}H$ where H is a Hilbert space (resp. of finite dimension, resp. separable).*

Lemma 3.2. — *The action of $\mathbb{C}\Gamma$ on a Hilbert Γ-module E extends uniquely to an action of the C^*-algebra $\mathcal{N}(\Gamma)$ in such a way that the image of $\mathcal{N}(\Gamma)$ is strongly closed in $\mathfrak{B}(E)$.*

Proposition 3.3. — *The following categories $E_f(\Gamma) \subset E(\Gamma)$:*

- *Objects of $E(\Gamma)$ are triples (E_1, E_2, e) where E_1 et E_2 are Hilbert Γ-module and e continuous Γ-equivariant linear map.*
- $\mathrm{Hom}_{E(\Gamma)}((E_1, F_2, e), (F_1, F_2, f))$ *is the set of pairs $(\phi_1 : E_1 \to F_1, \phi_2 : E_2 \to F_2)$ of continuous Γ-equivariant linear maps such that $\phi_2 e = f\phi_1$ under the equivalence relation $(\phi_1, \phi_2) \sim (\phi_1', \phi_2') \iff \exists\, T \in L_\Gamma(E_2, F_1),\ \phi_2' - \phi_2 = fT$.*
- *$E_f(\Gamma)$ is the full subcategory of $E(\Gamma)$ whose objects (E_1, E_2, e) have the property that E_2 is of finite type (E_1 is then also of finite type).*

are abelian categories of projective dimension one. The forgetful functor Φ from $E(\Gamma)$ to the category of $\mathcal{N}(\Gamma)$-modules defined by $\Phi((E_1, E_2, e)) := E_2/e(E_1)$ is faithful, respects direct sums, kernels and cokernels and is conservative.

Proof. — See [Eys00]. The main point is that the proof in [Far96] does not require finite type. □

Remark 3.4. — It is not clear to the author whether the forgetful functor Φ is fully faithful on $E_f(\Gamma)$. It is fully faithful on the full subcategory of projective modules thanks to [Gri66]. Fully faithfulness would follow if $\Phi(E)$ was a projective $\mathcal{N}(\Gamma)$-module whenever E is a finite type Hilbert Γ-module but it doesn't seem to be true.

The following corollary greatly simplifies our treatment:

Corollary 3.5. — *If $f^\bullet : K^\bullet \to L^\bullet$ is a continuous morphism of complexes of Hilbert Γ-modules whose terms are in $E(\Gamma)$, hence a morphism of complexes in $E(\Gamma)$, $f^\bullet$ induces a isomorphism in $D(E(\Gamma))$ if and only if $\Phi(f^\bullet)$ induces an algebraic isomorphism in cohomology.*

An object $X = (E_1, E_2, e)$ of $E_f(\Gamma)$ has two basic invariants. Its Von Neumann dimension $\dim_\Gamma X \in \mathbb{R}$ depends only on $P(X) = E_2/\overline{eE_1}$ and has properties similar to the dimension function of ordinary linear algebra and its Novikov–Shubin invariant $\mathrm{NoSh}(X) = (E_1, \overline{eE_1}, e)$.

Remark 3.6 (Tapia). — This construction of an abelian category is a special case of [BBD82, pp. 20, 40-41]. Actually Hilbert Γ-modules form an exact category, even a quasi abelian one as follows from [Sch99, section 3.2], which satisfies the conditions in [BBD82]. The same holds with $\mathcal{N}(\Gamma)$-Fréchet modules.

3.2. Γ-Fredholm Complexes

The main nice property complexes of Hilbert Γ-modules can have in general is being Γ-Fredholm.

Definition 3.7. — *A bounded complex of Hilbert Γ-modules (with a positive inner product) (C^k, d_k) is Γ-Fredholm if and only if the spectral family $E_\lambda^{dd^*+d^*d}$ satisfies $\exists\, \lambda > 0$ such that the image of E_λ has finite Γ-dimension.*

This notion depends on the notion of a Fredholm operator given in [Lüc02, Definition 1.20, p. 26]. It is invariant by quasi-isomorphisms in $E(\Gamma)$ thanks to [Lüc02, Theorem 2.19 p. 83]. There is a stronger notion.

Definition 3.8. — *A bounded complex of Hilbert Γ-modules (with a positive inner product) $(\overline{C}^k, \bar{d}_k)$ is strongly Γ-Fredholm if and only if it is quasi-isomorphic as a complex in $E(\Gamma)$ to another complex (C^k, d_k) whose spectral family $E_\lambda^{dd^*+d^*d}$ satisfies $\exists\, \lambda > 0$ such that the image of E_λ is a finitely generated Hilbert Γ-module.*

This is a stronger notion since a finite Γ-dimensional Hilbert module need not be finitely generated (e.g. for $\Gamma = \mathbb{Z}$).

Question 3.9. — *Can one drop the quasi-isomorphism? Perhaps the proof of* [Lüc02, Theorem 2.19 p. 83] *can be modified using the center-valued trace.*

Lemma 3.10. — *The homotopy category of bounded strongly Γ-Fredholm is equivalent to $D^b(E_f(\Gamma))$.*

Proof. — Since the full abelian subcategory $E_f(\Gamma) \subset E(\Gamma)$ has enough $E(\Gamma)$ projective and both have finite projective dimension $\psi : D^b(E_f(\Gamma)) \to D^b_{E_f(\Gamma)}(E(\Gamma))$ is an equivalence.

Certainly a strongly Fredholm complex has its cohomology in $E_f(\Gamma)$.

Quasi-isomorphisms in the homotopy category of complexes in $E(\Gamma)$ are exactly the homotopy classes of morphisms of complexes that are algebraic quasi isomorphisms thanks to the exactness of the faithful forgetful functor $E(\Gamma) \to \mathrm{Mod}_{\mathcal{N}(\Gamma)}$.

Since strongly Fredholm complexes are complex of projective objects in $E(\Gamma)$, the functor ψ' from the homotopy category of bounded above strongly Γ-Fredholm complexes to the derived category $D^bE(\Gamma)$ is fully faithful and takes its values in $D^b_{E_f(\Gamma)}(E(\Gamma))$. Since ψ is an equivalence, whose image is contained in the image of ψ', ψ' is essentially surjective. □

3.3. An equivalence of categories

There is a more algebraic approach to $E_f(\Gamma)$ [Lüc02, p. 288]. $\mathcal{N}(\Gamma)$ is a semihereditary [Lüc02, Theorem 6.7 p. 239] hence coherent ring. It turns out that $E_f(\Gamma)$ is equivalent to the abelian category of finitely presentable $\mathcal{N}(\Gamma)$-modules. But the equivalence in question, denote it by ν, is not given by Φ. Indeed it is constructed using the equivalence given by the functor on finite rank free $\mathcal{N}(\Gamma)$-modules defined by $M \mapsto l^2\Gamma \otimes_{\mathcal{N}(\Gamma)} M$. It is not obvious that it is an equivalence.

There is also a dimension theory for arbitrary $\mathcal{N}(\Gamma)$-modules which generalizes $\dim_\Gamma$ and more or less reduces the theory of L_2-Betti numbers and Novikov–Shubin invariants to algebra. However non zero $\mathcal{N}(\Gamma)$-modules of dimension 0 may exist in sharp contrast with projective Hilbert Γ-modules.

3.4. Affiliated Operators

The algebra of affiliated operators $\mathcal{U}(\Gamma)$ is a flat extension $\mathcal{N}(\Gamma) \subset \mathcal{U}(\Gamma)$ [Lüc02, Theorem 8.2.2] such that $\mathcal{U}(\Gamma) \otimes \mathrm{NoSh}(X) = 0$ whenever X is an object in $E_f(\Gamma)$. It is a coherent ring, even a Von Neumann regular one. So that finitely presented $\mathcal{U}(\Gamma)$-modules form an abelian category of projective dimension 0 (all objects are projective!). Furthermore $\dim_\Gamma$ extends to $\mathcal{U}(\Gamma)$-modules (no topological structure needed) in such a way that objects of the form $\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} E$ where E is a Hilbert Γ-module of finite type are finite dimensional, $\dim_\Gamma$ being preserved. In particular, for a complex $K^\bullet$ in $E_f(\Gamma)$,

$$H^q(\mathcal{U}(\Gamma) \otimes K^\bullet) = \mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} \overline{H}^q(K^\bullet) = \mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} P(H^q(K^\bullet)).$$

The author does not know how to use in *Complex Analytic Geometry* the torsion information lost in that process. So, forgetting about it seems to be appropriate from a pragmatic point of view.

The relevance of this localization process for Hodge theory is one of the main ideas of [Din13]. There, intermediate abelian categories fitting in a succession of exact functors of abelian categories

$$E_f(\Gamma) \xrightarrow{\text{faith.}} \mathrm{Mod}(\mathcal{N}(\Gamma)) \longrightarrow \mathrm{Mod}(\mathcal{N}(\Gamma))/\tau \longrightarrow \mathrm{Mod}(\mathcal{U}(\Gamma))$$

are introduced where τ is a torsion theory (or an appropriate Serre subcategory). Here, we will only consider the case $\tau = \tau_{\mathcal{U}(\Gamma)}$ in the notations of loc. cit.: when it is possible, we will work in $E_f(\Gamma)$ and when it becomes necessary, we will apply the functor $\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)}$.

However, the reason [Lüc02] introduces affiliated operators is his $[0,+\infty]$-valued dimension theory for $\mathcal{U}(\Gamma)$-modules which enables him to make the theory of L^2-Betti numbers more or less algebraic. This enables one to look at non-locally finite simplicial complexes like $K(\Gamma,1)$ simplifying Cheeger–Gromov's article [CG86]. One has however to do a minimal amount of functional analysis to prove the $\mathcal{U}(\Gamma)$-modules we encounter are finite dimensional or finitely generated projective. This is why we will not just apply the functor $\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}$ to the construction of the two preceding sections with $p=2$, although it is extremely tempting.

We conclude with the following lemmas:

Lemma 3.11. — *$\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}\nu$ is naturally equivalent to $\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}$ on $E_f(\Gamma)$.*

Proof. — This follows from the construction of ν and of the relation $\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)} l^2\Gamma=\mathcal{U}(\Gamma)$ which in turn follows from the realization of $\mathcal{U}(\Gamma)$ as an Ore localization of $\mathcal{N}(\Gamma)$ [Lüc02]. □

Lemma 3.12. — *Let E be an object of $E_f(\Gamma)$ endowed with 3 filtrations W,F,G. Then $\overline{F}$ and $\overline{G}$ are n-opposed on $\mathrm{Gr}^n_{\overline{W}}P(E)$ if and only if $\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}F$ and $\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)} G$ are n-opposed on $\mathrm{Gr}^n_{\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}W}\ \mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}E$.*

Proof. — Left to the reader. □

3.5. A natural question

The construction of a completely satisfying L_2 *Mixed Hodge Theory* might be eased by the use of further results from the theory of operators algebras. A salient feature of $\mathcal{U}(\Gamma)$ is that it is self injective which is exactly, according to a remark in [Lüc02], what is needed for neat duality statements. It would be helpful if the following question had a positive answer:

Question 3.13. — *Assume we have a complex of separable Hilbert Γ-modules (or Fréchet $\mathcal{N}(\Gamma)$-modules) whose cohomology is isomorphic as a $\mathcal{N}(\Gamma)$-module to the $\mathcal{N}(\Gamma)$-module underlying an object of $E_f(\Gamma)$. Is the complex strongly Γ-Fredholm?*

A slightly weaker statement is given in [Jea22] as a criterion for Γ-Fredholmness.

3.6. Real Structures

The $*$-algebras $\mathcal{N}(\Gamma)$ and $\mathcal{U}(\Gamma)$ carry a real structure, namely a conjugate linear algebra involutive automorphism † commuting with the conjugate linear algebra involutive anti-automorphism $*$, and a real structure on a module over these algebras is just a conjugate linear automorphism on the underlying vector space compatible with †. For instance $l_2\Gamma$ has a real structure. We will denote by $\mathbb{R}E_f(\Gamma)$ the category of formal quotients of real Hilbert Γ-modules.

4. Refined L^2-cohomology

4.1. Finiteness theorem for L^2 constructible cohomology

Proposition 4.1. — *Let $\mathbb{S}$ be a cocompact Γ-simplicial complex. There is a ∂-functor*

$$\mathbb{H}_2^*(|\mathbb{S}|,-) : D^b(\mathbf{Cons}_{K,\Gamma}(\mathbb{S})) \longrightarrow D^b(E_f(\Gamma))$$

such that, if the natural forgetful functor is denoted by

$$\Phi : D^b(E_f(\Gamma)) \longrightarrow D^b(\mathrm{Mod}(\mathcal{N}(\Gamma))),$$

we have with the notations of Definition 1.4:

$$\Phi \circ \mathbb{H}_2^*(|\mathbb{S}|,-) = \mathbb{H}_{(2)}^*(|\mathbb{S}|,-).$$

Proof. — First assume that $\mathbb{S}$ satisfies the following technical assumption: for every pair of distinct adjacent vertices $p, q \in S$, $\psi(p) \neq \psi(q)$ where $\psi : S \to \Gamma\backslash S$ is the quotient map. In particular, the set of vertices of a given simplex maps injectively to $\Gamma\backslash S$. Choose a well-ordering of $\Gamma\backslash S$. This provides each simplex σ with an order $<_\sigma$ on its vertices such that $<_{\gamma.\sigma}= \gamma(<_\sigma)$ and if $\tau \subset \sigma$, $<_\sigma |_\tau =<_\tau$. This defines a sign $\epsilon_{\tau,\sigma}$ for every pair of simplices $\tau \subset \sigma$ such that $\mathrm{Card}(\sigma) = \mathrm{Card}(\tau) + 1$, namely $\epsilon_{\tau,\sigma} = (-1)^\nu$ where $\phi : (\{0,\ldots,\mathrm{Card}(\sigma)-1\},<) \to (\sigma,<_\sigma)$ is the unique increasing bijection and $\sigma - \tau = \{\phi(\nu)\}$.

Let $\mathcal{F}$ be an object of $\mathbf{Cons}_\Gamma(\mathbb{S})$. For every simplex σ, set $U_\sigma := \bigcup_{\sigma\subset\tau} |\tau|$ and $F_\sigma := H^0(U_\sigma, \mathcal{F})$. For $\tau \subset \sigma$, the sheaf structure gives a map $\rho_{\tau,\sigma} : F_\tau \to F_\sigma$. Set $C_c^p(\mathbb{S},\mathcal{F}) := \bigoplus_{\mathrm{Card}(\sigma)=p+1} F_\sigma$ and for $f_\tau \in \mathcal{F}_\tau$,

$$df_\tau = \sum_{\tau\subset\sigma, \mathrm{Card}(\sigma)=\mathrm{Card}(\tau)-1} \epsilon_{\tau,\sigma}\rho_{\tau,\sigma}(f_\tau).$$

This defines a complex of Γ-modules $C_c^\bullet(\mathbb{S},\mathcal{F})$.

This complex is actually the Čech complex of $\pi_!\mathcal{F}$ in the covering $(U'_q)_{q\in\Gamma\backslash S}$ of $\Gamma\backslash|\mathbb{S}|$ where we define $U'_q = \pi(U(p))$ where $p \in S$ satisfies $\pi(p) = q$. Using [KS90, Proposition 8.1.4 p. 323], we see that $l_p\Gamma \otimes_{K\Gamma} C_c^\bullet(\mathbb{S},\mathcal{F})$ computes $H_{(p)}^\bullet(|\mathbb{S}|,\mathcal{F})$. In case $p = 2$; this complex is in fact a complex in $E_f(\Gamma)$.

This construction is obviously functorial, and taking the simple complex associated to a double complex one would construct the sought-for ∂-functor. The technical assumption on $\mathbb{S}$ is not always satisfied, but it holds for the barycentric subdivision $\beta\mathbb{S}$. We certainly have a fully faithful forgetful functor $\mathbf{Cons}_{K,\Gamma}(\mathbb{S}) \to \mathbf{Cons}_{K,\Gamma}(\beta\mathbb{S})$ and we define a functor between categories of complexes to be $s(l_2\Gamma \otimes_{K\Gamma} C^\bullet(\beta\mathbb{S},-))$. Passing to derived categories, it descends to $\mathbb{H}_2^*(|\mathbb{S}|,-)$. □

Proposition 4.2. — *Let $\widetilde{X}$ be a cocompact subanalytic Γ-space.*

There is a ∂-functor $\mathbb{H}_2^*(\widetilde{X},-) : D^b_{\mathbb{R}-c,\Gamma}(\widetilde{X})) \to D^b(E_f(\Gamma))$ such that one has $\Phi(\mathbb{H}_2^*(X,-)) = \mathbb{H}_{(2)}^*(X,-)$.

This functor enjoys the following properties:

- *(Leray spectral sequence) Given $\widetilde{Y}$ another cocompact subanalytic Γ-space, for every proper Γ-equivariant morphism $f : \widetilde{X} \to \widetilde{Y}$, $\mathbb{H}_2(\widetilde{X},-)$ and $\mathbb{H}_2(\widetilde{Y},-) \circ Rf_*$ are naturally isomorphic functors.*

- *(Atiyah's L_2 index theorem) If Γ is fixed point free on $\widetilde{X}$*

$$\sum_i (-1)^i \dim_\Gamma \mathbb{H}_2^i(\widetilde{X}, \mathcal{F}^\bullet) = \sum_i (-1)^i \dim_{\mathbb{C}} \mathbb{H}^i\Big(\Gamma\backslash\widetilde{X}, \Gamma\backslash\mathcal{F}^\bullet\Big).$$

Proof. — As in the proof of Proposition 1.8, the first part derives from Proposition 4.1. The proof of the additional statements is simpler than the proof of similar statements for coherent cohomology in [Eys00] and will not be given in detail. □

If $\widetilde{X}$ is complex analytic, we will restrict $\mathbb{H}_2^*(\widetilde{X}, -)$ to $D^b(\mathbf{Cons}_\Gamma(\widetilde{X}))$. We will also denote by $\overline{\mathbb{H}}_2^k(\widetilde{X}, -)$ the k^{th} reduced cohomology functor

$$\overline{\mathbb{H}}_2^k(\widetilde{X}, -) = P\Big(H^k(\mathbb{H}_2^*(\widetilde{X}, -)\Big).$$

It is a projective Hilbert Γ-module and one has

$$\dim_\Gamma \mathbb{H}_2^i(\widetilde{X}, \mathcal{F}^\bullet) = \dim_\Gamma \overline{\mathbb{H}}_2^i(\widetilde{X}, \mathcal{F}^\bullet).$$

4.2. Finiteness theorem for L^2 coherent De Rham cohomology

Assume in this subsection that X is a compact complex manifold. The L^2 coherent cohomology functor

$$H^\bullet_{L^2}(\widetilde{X}, -) : D^b\,\mathbf{Coh}(\mathcal{O}_X) \longrightarrow D^b(E_f(\Gamma))$$

defined in [Eys00, Théorème 5.3.8] comes from a $\mathbb{C}$-linear functor denoted by $\mathcal{C}$ from an ad hoc additive category A of coherent $\mathcal{O}_X$-modules endowed with inessential local data taking values in an ad hoc triangulated category with a localization which is naturally equivalent to $D^b(E_f(\Gamma))$. The resulting functor $C^b(A) \to D^b(E_f(\Gamma))$ factors through $D^b(\mathbf{Coh}(\mathcal{O}_X))$ and gives $H^\bullet_{L^2}(\widetilde{X}, -)$.

Up to some extra inessential technical auxiliary data, $\mathcal{C}$ can be identified with $\mathcal{C}'$ the Čech cohomology of $l^2\pi_*$ with respect to some Stein covering $\mathfrak{U}_0$ of X. More details are given in the Appendix for the reader's convenience.

LEMMA 4.3. — *The functor $\mathcal{C}'$ extends to $C^b\,\mathbf{Coh}(\mathcal{O}_X, \mathrm{Diff_X})$ the full subcategory of $C^b\mathbf{M}(\mathcal{O}_X, \mathrm{Diff_X})$ whose objects are differential complexes of coherent analytic sheaves as an additive functor. Moreover, for every object $\mathcal{F}^\bullet$ in $C^b\,\mathbf{Coh}(\mathcal{O}_X, \mathrm{Diff_X})$, the totalization of the bicomplex $\mathcal{C}'(\mathcal{F}^\bullet)$ is quasi-isomorphic to a bounded complex of finite type projective Hilbert Γ-modules.*

Proof. — Since differential operators between coherent analytic sheaves act continuously on the Fréchet space of their sections, the arguments of [Eys00] apply. □

Recall from [Sai89] that the correct notion of quasi isomorphism in the triangulated category $K^b\mathbf{M}(\mathcal{O}_X, \mathrm{Diff_X})$ of homotopy classes of bounded complexes in $\mathbf{M}(\mathcal{O}_X, \mathrm{Diff_X})$ are the differential quasi isomorphisms. We denote by dqi this localizing class which is a priori smaller than the class qi of sheaf-theoretic quasi-isomorphisms. The class dqi is needed to invert the De Rham functor and one has a ∂-functor

$$\nu : K^b\,\mathbf{Coh}(\mathcal{O}_X, \mathrm{Diff_X})_{\mathrm{dqi}} \longrightarrow K^b\,\mathbf{Coh}(\mathcal{O}_X, \mathrm{Diff_X})_{\mathrm{qi}}.$$

Here $K^b\,\mathbf{Coh}(\mathcal{O}_X, \mathrm{Diff_X})_{\mathrm{dqi}}$ stands for the essential image of $C^b\,\mathbf{Coh}(\mathcal{O}_X, \mathrm{Diff_X})$ (the essential image is a strictly full category).

Corollary 4.4. — *The functor $H^\bullet_{L^2}(\widetilde{X},-)$ extends to a ∂-functor*

$$H^\bullet_2(\widetilde{X},-) : K^b\,\mathbf{Coh}(\mathcal{O}_X,\mathrm{Diff}_\mathrm{X})_\mathrm{dqi} \longrightarrow D^b(E_f(\Gamma))$$

such that $\Phi\circ H^\bullet_{\mathrm{DR},2}(\widetilde{X},-)$ is naturally isomorphic to $H^\bullet(X,l^2\pi_*-)$.

Proof. — The functor $\mathcal{F}^\bullet\mapsto\mathcal{C}'(\mathcal{F}^\bullet)$ defined on $C^b\,\mathbf{Coh}(\mathcal{O}_X,\mathrm{Diff}_\mathrm{X})$ maps dqi isomorphisms, and actually qi isomorphisms, to quasi-isomorphisms and also sends a morphism homotopic to 0 to a morphism of complexes homotopic to 0. In particular it descends to a ∂-functor

$$K^b\Big(C^b\,\mathbf{Coh}(\mathcal{O}_X,\mathrm{Diff}_\mathrm{X})\Big)_\mathrm{dqi} \longrightarrow K^b(\mathrm{Mod}(\mathcal{N}(\Gamma)))_\mathrm{qi} = D^b(\mathrm{Mod}(\mathcal{N}(\Gamma)))$$

$\mathcal{C}(\mathcal{F}^\bullet)$ view a complex of $\mathcal{N}(\Gamma)$-modules computes $H^\bullet(X,l^2\pi_*\mathcal{F}^\bullet)$.

The quasi-isomorphism $C^b(E_f(\Gamma))\ni K\to\mathcal{C}'(\mathcal{F}^\bullet)$ constructed in Lemma 4.3 is not uniquely defined but the techniques in [Eys00], in particular Corollary 3.5, can be applied to check that

- K is essentially unique in $D^b(E_f(\Gamma))$,
- choosing arbitrarily one of them (with a quasiisomorphism to $\mathcal{C}'(\mathcal{F}^\bullet)$) and calling it $K(\mathcal{F}^\bullet)$, the assignment $\mathcal{F}^\bullet\to K(\mathcal{F}^\bullet)$ defines a functor

$$K^b\,\mathbf{Coh}(\mathcal{O}_X,\mathrm{Diff}_\mathrm{X})_\mathrm{dqi} \longrightarrow K^b(E_f(\Gamma))_\mathrm{qi} = D^b(E_f(\Gamma))$$

which is the required lift of $H^\bullet_{\mathrm{DR},2}(\widetilde{X},-)$ by the forgetful functor Φ. □

We will from now on make a technical assumption, namely that the coherent $\mathcal{D}_X$-modules we consider admit a global good filtration. A second possibility would be to work on the category $\mathrm{Filt}\,\mathbf{Mod}(\mathcal{D}_X)$.

Lemma 4.5. — *Let $\mathcal{K}^\bullet\in Ob(K^b\,\mathbf{Coh}(\mathcal{D}_X))$. Let $F_\bullet$ be a filtration of $K^\bullet$ inducing a good filtration on each term. Then $F_p\,\mathrm{DR}(\mathcal{K}^\bullet)\to\mathrm{DR}(\mathcal{K}^\bullet)$ is a quasi isomorphism for $p\gg 0$ and $F_p\,\mathrm{DR}(\mathcal{K}^\bullet)$ is independent of $p\gg 0$ up to a unique differential quasi-isomorphism hence defines unambiguously an object $\mathrm{DR}'(\mathcal{K}^\bullet)$ of $K^b\,\mathbf{Coh}(\mathcal{O}_X,\mathrm{Diff}_\mathrm{X})_\mathrm{dqi}$. This assignment is functorial.*

Proof. — See [Sai89], in particular for the functorial behaviour of this construction. □

It is tempting to believe that one could work with local good filtrations (actually with the local presentations inducing them) using simplicial gluing techniques as in [Eys00, Section 6] but we shall refrain from doing so.

We can thus define a ∂-functor

$$tDR : D^b\,\mathbf{Coh}(\mathcal{D}_X)_\text{good filt} \longrightarrow K^b\,\mathbf{Coh}(\mathcal{O}_X,\mathrm{Diff}_\mathrm{X})_\mathrm{dqi}$$

which is compatible to the restriction to $D^b\,\mathbf{Coh}(\mathcal{D}_X)$ of Saito's equivalence:

$$\mathrm{DR} : D^b\,\mathbf{Mod}(\mathcal{D}_X) \longrightarrow K^b\mathbf{M}(\mathcal{O}_X,\mathrm{Diff}_\mathrm{X})_\mathrm{dqi}$$

Proposition 4.6. — *The functor*

$$H^\bullet_{\mathrm{DR},2}(\widetilde{X},-) = H^\bullet_2(\widetilde{X},-)\circ tDR : D^b\,\mathbf{Coh}(\mathcal{D}_X)_\text{good filt} \longrightarrow D^b(E_f(\Gamma))$$

is a ∂-functor such that $\Phi\circ H^\bullet_{\mathrm{DR},2}(\widetilde{X},-)$ is naturally equivalent to the restriction of the functor $H^\bullet_{\mathrm{DR},L^2}(\widetilde{X},-)$.

Once again, one can define the reduced L_2 cohomology of $\mathcal{M}$ a holonomic $\mathcal{D}_X$-module admitting a good filtration (or a complex of such):

$$\overline{H}^k_{\mathrm{DR},2}(\widetilde{X},\mathcal{M}) = P\Big(H^k_{\mathrm{DR},2}(\widetilde{X},\mathcal{M})\Big).$$

Remark 4.7. — When $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, one can form the induced $\mathcal{D}_X$-module $\mathrm{ind}(\mathcal{F}) := \mathcal{D}_X \otimes \mathcal{F}$, a coherent $\mathcal{D}_X$-module with a global good filtration, and there is a morphism of complexes of sheaves $\mathrm{DR}(\mathrm{ind}(\mathcal{F})) \to \mathcal{F} \otimes \omega_X^n$ which is a quasi-isomorphism. We have a natural isomorphism in $D^b(E_f(\Gamma))$

$$H^\bullet_{\mathrm{DR},2}\Big(\widetilde{X},\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}\Big) \simeq H^q_2\Big(\widetilde{X},\mathcal{F}\otimes\omega_X^n\Big)$$

where we use the notation of [Eys00] for coherent L^2-cohomology.

4.3. L^2 Poincaré–Verdier Duality

Assume now that $\mathbb{S}$ is a cocompact (in particular finite dimensional) Γ-simplicial complex. The category $\mathbf{Cons}_{K,\Gamma}(\mathbb{S})$ can described combinatorially in terms of the poset $(\Sigma(\mathbb{S}),\leqslant)$ where $\Sigma(\mathbb{S})$ is the set of simplices and $\sigma \leqslant \tau$ if and only if σ is a face of $\overline{\tau}$ if and only if $\sigma \subset \tau$. This partial order is Γ-equivariant. Then $\mathbf{Cons}_{K,\Gamma}(\mathbb{S})$ identifies with the category of Γ-equivariant covariant functors

$$(\Sigma(\mathbb{S}),\leqslant) \longrightarrow \text{Finite-dimensional } K\text{-Vector Spaces}.$$

This is nothing but a reformulation of a part of the construction in the proof of Proposition 4.1. The data of $F \in \mathrm{Obb}(\mathbf{Cons}_{K,\Gamma}(\mathbb{S}))$ is thus equivalent to the data of maps $F_\tau \to F_\sigma$ when $\tau \leqslant \sigma$.

Poincaré–Verdier has an explicit combinatorial formulation [Cur18, Sch98, She85] which is presented very efficiently in the note [Cur14] and was apparently first observed by A. Shepard in his 1985 unpublished thesis under R. MacPherson's direction.

Before stating Shepard's result, we need to introduce some notation. Let $\sigma \in \Sigma(\mathbb{S})$ be a simplex. We denote by $\overline{\sigma}$ the closed subspace of $|\mathbb{S}|$ which is the image of the closed standard simplex corresponding to σ in $|\mathbb{S}|$. Denote by $\iota_{\overline{\sigma}} : \overline{\sigma} \to |\mathbb{S}|$ and by $\underline{K}_{\overline{\sigma}}$ the constant sheaf whose stalk at any point of $\overline{\sigma}$ is K. Then $\iota_{\sigma *}\underline{K}_{\overline{\sigma}}$ is a sheaf on $|\mathbb{S}|$ whose support is $\overline{\sigma}$.

Shepard's description of the Poincaré–Verdier duality can then be summarized as follows:

Proposition 4.8. — *Let F be a Γ-constructible sheaf. Then its Verdier dual is represented by a complex of injective Γ-constructible sheaves :*

$$\mathbb{D}(F) = \ldots \longrightarrow \bigoplus_{\dim(\sigma)=i} \iota_{\overline{\sigma}*}\underline{K}_{\overline{\sigma}} \otimes F_\sigma^\vee \xrightarrow{\partial} \bigoplus_{\dim(\tau)=i-1} \iota_{\overline{\tau}*}\underline{K}_{\overline{\tau}} \otimes F_\tau^\vee \longrightarrow \ldots$$

where

- *For every $i \in \mathbb{N}$, $\mathbb{D}^i(F) = \bigoplus \iota_{\overline{\sigma}*}\underline{K}_{\overline{\sigma}} \otimes F_\sigma^\vee$ is placed in degree $-i$.*
- *$V \mapsto V^\vee$ is the usual duality functor on finite-dimensional K-Vector Spaces.*

If $F^\bullet$ is a bounded complex of Γ-constructible sheaves $\mathbb{D}(F^\bullet)$ is represented by the totalisation of the double complex obtained by applying $\mathbb{D}$.

This proposition gives a lift of Verdier Duality to the category of complexes, before taking the derived category.

The differentials of $\mathbb{D}(F)$ are not described by the proposition. As a first approximation, we note that the global sections of $\mathbb{D}(F)$ give, up to the conversion from a cochain complex to a chain complex by changing the signs of the degrees of the chain groups, the homological chain complex of a certain cosheaf on the simplicial complex $|\mathbf{S}|$ which is dual to F. Indeed, if F is a Γ-constructible sheaf, we may construct $F^\vee$ by applying the usual duality functor to the covariant functor attached to F and get a Γ-constructible cosheaf, namely a *contravariant* functor

$$(\Sigma(\mathbb{S}), \leqslant) \longrightarrow \text{Finite-dimensionnal } K\text{-Vector Spaces}.$$

We may construct its homology chain complex $C_\bullet(\mathbb{S}, F^\vee)$, concentrated in positive degrees, and view it as a *cochain* complex $C_{-\bullet}(\mathbb{S}, F^\vee)$ concentrated in negative degrees. These are complexes of finitely generated projective $K\Gamma$-modules.

A more precise description will appear in the proof of the next lemma.

Lemma 4.9. — *There is a functorial monomorphism of complexes of $K\Gamma$-modules:*

$$C_{-\bullet}(\mathbb{S}, F^\vee) \longrightarrow C_c^\bullet(\mathbb{S}, \mathbb{D}(F))$$

such that the quotient complex is contractible.

Proof. — The complex $C_c^\bullet(\mathbb{S}, \mathbb{D}^i(F))$ is the same as

$$\bigoplus_{\dim(\sigma)=i} C^\bullet(\overline{\sigma}) \otimes F_\sigma^\vee$$

where $C^\bullet(\overline{\sigma})$ is the simplicial chain complex of the closed simplex $\overline{\sigma}$ converted to a cochain complex.

We have a natural cochain equivalence $K[0] \to C^\bullet(\overline{\sigma})$ sending 1 to the cochain which takes the value 1 on all vertices of $\overline{\sigma}$. Its cokernel is therefore contractible. This enables to construct a Γ-equivariant cochain equivalence:

$$\bigoplus_{\dim(\sigma)=i} F_\sigma^\vee[0] \longrightarrow \bigoplus_{\dim(\sigma)=i} C^\bullet(\overline{\sigma}) \otimes F_\sigma^\vee$$

which commutes with the natural boundaries. By construction, the cokernel is contractible as a complex of finite type projective $K\Gamma$-modules. This proves the lemma since $C_c^\bullet(\mathbb{S}, \mathbb{D}(F))$ is the simple complex attached to the double complex $C_c^\bullet(\mathbb{S}, \mathbb{D}^\bullet(F))$. □

Lemma 4.10. — *Let $F^\bullet$ be a bounded complex of Γ-constructible sheaves, then there is a functorial map of bounded complexes of projective Hilbert Γ-modules, inducing an isomorphism on cohomology, obtained by taking the simple complex attached to*

$$l^2\Gamma \otimes_{K\Gamma} C_{-\bullet}(\mathbb{S}, F^{\bullet\vee}) \longrightarrow l^2\Gamma \otimes_{K\Gamma} C_c^\bullet(\mathbb{S}, \mathbb{D}(F^\bullet)).$$

Proof. — Tensor the monomorphism given by lemma 4.9 by $l^2\Gamma$ to get an injective map of complexes such that the cokernel is contractible. Note that $l^2\Gamma\otimes_{K\Gamma}$ is an exact functor on projective $K\Gamma$-complexes. □

Lemma 4.11. — *Given $F^\bullet$ an object of $D^b(\mathbf{Cons}_{K,\Gamma}(\mathbb{S})$, there is a functorial perfect duality of projective Hilbert Γ-modules*

$$\overline{\mathbb{H}}_2^{\,i}(|\mathbb{S}|, F^\bullet \otimes_K \mathbb{C}) \otimes \overline{\mathbb{H}}_2^{\,-i}(|\mathbb{S}|, \mathbb{D}(F^\bullet) \otimes_K \mathbb{C}) \longrightarrow \mathbb{C}$$

which preserves the natural real structure if $K \subset \mathbb{R}$.

Proof. — This follows from the previous lemma and [Lüc02, Lemma 2.17 (2), p. 82]. □

Proposition 4.12. — *Notations of Proposition 4.2. There is a functorial perfect duality of projective Hilbert Γ-modules*

$$\overline{\mathbb{H}}_2^{\,i}\Big(\widetilde{X}, F^\bullet \otimes_K \mathbb{C}\Big) \otimes \overline{\mathbb{H}}_2^{\,-i}\Big(\widetilde{X}, R\underline{\mathrm{Hom}}^\bullet(F^\bullet, \omega^\bullet) \otimes_K \mathbb{C}\Big) \longrightarrow \mathbb{C}$$

which preserves the natural real structure if $K \subset \mathbb{R}$ where $\omega^\bullet = \mathbb{D}(\underline{K}_{\widetilde{X}})$ is the Verdier dualizing complex.

Proof. — This is a restatement of the previous lemma. Indeed, by construction, we have functorial isomorphisms of projective Γ-Hilbert modules:

$$\overline{\mathbb{H}}_2^{i}(|\mathbb{S}|, F^\bullet \otimes_K \mathbb{C}) \longrightarrow \overline{\mathbb{H}}_2^{i}\Big(\widetilde{X}, F^\bullet \otimes_K \mathbb{C}\Big),$$
$$\overline{\mathbb{H}}_2^{-i}(|\mathbb{S}|, \mathbb{D}(F^\bullet) \otimes_K \mathbb{C}) \longrightarrow \overline{\mathbb{H}}_2^{-i}\Big(\widetilde{X}, R\underline{\mathrm{Hom}}^\bullet(F^\bullet, \omega^\bullet) \otimes_K \mathbb{C}\Big).$$

□

Corollary 4.13. — $\dim_\Gamma \overline{\mathbb{H}}_2^{\,i}(\widetilde{X}, F^\bullet \otimes_K \mathbb{C}) = \dim_\Gamma \overline{\mathbb{H}}_2^{\,-i}(\widetilde{X}, R\underline{\mathrm{Hom}}^\bullet(F^\bullet, \omega^\bullet) \otimes_K \mathbb{C})$.

Remark 4.14. — It is extremely tempting to conjecture that, with the notations of [Eys00], we have a L_2-Serre Duality theorem for Coherent analytic sheaves on complex spaces stating that there is perfect duality:

$$\overline{H}_2^{q}(\widetilde{X}, \mathcal{F}) \otimes \overline{H}_2^{-q}\Big(\widetilde{X}, R\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_X}(\mathcal{F}, \Omega_X^\bullet)\Big) \longrightarrow \mathbb{C},$$

where $\Omega_X^\bullet$ is the dualizing complex and that this even holds for coherent $\mathcal{D}_X$-modules if X is smooth. Proving this conjecture is likely to be quite technical and does not seem promising for applications.

4.4. The comparison isomorphism

We need to show that the comparison isomorphism, a quasi-isomorphism of complexes of $\mathcal{N}(\Gamma)$-modules, lifts to a quasi isomorphism of bounded complexes in $E_f(\Gamma)$. This is not completely trivial. However, with the notations in Theorem 0.1:

Lemma 4.15. — *If Σ is a triangulation of X refining a stratification S of X and $\mathrm{DR}(\mathcal{M})$ has S-constructible cohomology, and if P is a bounded complex of Σ-constructible sheaves of $\mathbb{C}$-vector spaces then one can represent the quasi-isomorphism α by a morphism of complexes $\widetilde{\alpha} : P \to F_p\,\mathrm{DR}(\mathcal{M})$ composed with the natural quasi-isomorphism $F_p\,\mathrm{DR}(\mathcal{M}) \to \mathrm{DR}(\mathcal{M})$ for some $p \gg 1$.*

Proof. — This follows from [KS90, Proposition 8.1.9]. □

Corollary 4.16. — *If $\mathfrak{U}$ is a finite covering by Oka-Weil domains such that:*

- *it refines the covering $\mathfrak{V}$ of X by the stars of the vertices of Σ*
- *the non empty intersections are contractible,*

we have a quasi-isomorphism of $\mathcal{N}(\Gamma)$-Fréchet modules, the leftmost two being in $E_f(\Gamma)$:

$$\mathcal{C}^\bullet(\mathfrak{V}, P) \longleftarrow \mathcal{C}^\bullet(\mathfrak{U}, P) \xrightarrow{\tilde{\alpha}} \mathcal{C}^\bullet(\mathfrak{U}, F_p \operatorname{DR}(\mathcal{M})).$$

Now we have a model of $\mathbb{H}_{2,\mathrm{DR}}(\widetilde{X}, \mathcal{M}$ which is a bounded complex of finite type projective Hilbert Γ-modules with a quasi isomorphism:

$$L^\bullet \longrightarrow \mathcal{C}^\bullet(\mathfrak{U}, F_p \operatorname{DR}(\mathcal{M})).$$

Since the left hand side underlies a qhtf complex of Montelian modules, by [Eys00, Proposition 4.4.14], one constructs a morphism of complexes of projective Hilbert Γ-module which is a quasi isomorphism

$$\mathcal{C}^\bullet(\mathfrak{V}, P) \longrightarrow M^\bullet$$

and is the promised lift of rh_2 to an isomorphism in $D^b(E_f(\Gamma))$. The functoriality of the construction is left to the reader. This concludes the proof of Theorem 0.1.

4.5. L_2-cohomology of Mixed Hodge Modules

Now, X is a complex projective manifold.

Let $\mathbb{M}$ be a Mixed Hodge Module in the sense of [Sai90]. It is a triple

$$\Big((\mathcal{M}, F, W), \Big(\mathbb{M}^B, W^B\Big), \alpha\Big)$$

where:

- $(\mathcal{M}, F, W)$ is a bifiltered $\mathcal{D}_X$-module (which is regular holonomic),
- $(\mathcal{M}, F)$ is a good filtration,
- $\mathbb{M}^B$ is a perverse sheaf over $\mathbb{Q}$,
- W_B is a filtration of $\mathbb{M}^B$ in the abelian category of perverse sheaves,
- $\alpha : \operatorname{DR}(\mathcal{M}) \to \mathbb{M}^B \otimes_{\mathbb{Q}} \mathbb{C}$ an isomorphism in $D^b_c(X, \mathbb{C})$, actually a filtered quasi isomorphism if the weight filtrations are taken into account.

All these data satisfying some extremely non-trivial conditions which will not be needed here but are indispensable for proving the miraculous properties of Mixed Hodge Modules.

Definition 4.17. — *We can define in $E_f(\Gamma)$, a real structure and a real weight filtration W on the $\mathcal{N}(\Gamma)$-module $\mathbb{H}^k_2(\widetilde{X}, \pi^{-1}\mathbb{M}^B)$ by taking the image of the functorial morphism*

$$\mathbb{H}^k_2\Big(\widetilde{X}, \pi^{-1} W_\bullet \mathbb{M}^B\Big) \longrightarrow \mathbb{H}^k_2\Big(\widetilde{X}, \pi^{-1}\mathbb{M}^B\Big)$$

and a filtration $F_{\mathrm{DR},2}$ on $H^k_{\mathrm{DR},2}(\widetilde{X}, \mathcal{M})$ by taking the image of the natural map

$$H^k_2\Big(\widetilde{X}, F_\bullet \operatorname{DR}(\mathcal{M})\Big) \longrightarrow H^k_2\Big(\widetilde{X}, \operatorname{DR}(\mathcal{M})\Big).$$

Transporting the $F_{\mathrm{DR},2}$ filtration by the isomorphism rh_2 induced by α between these objects of $E_f\Gamma$, we get a real structure, a real W-filtration, the weight filtration, *and a complex filtration F, which we shall call the algebraically defined Hodge filtration, on* $H_2^k(\widetilde{X}, \mathbb{M}) := \mathbb{H}_2^k(\widetilde{X}, \pi^{-1}\mathbb{M}^B)$.

There is a perfect duality of the Hilbert Γ*-modules* $\overline{H}_2^q(\widetilde{X}, \mathbb{M})$ *and* $\overline{H}_2^{-q}(\widetilde{X}, \mathbb{D}(\mathbb{M}))$.

Proof. — This is a direct application of our construction. The last statement follows from Proposition 4.12. ☐

The following implies Conjecture 0.3 in the introduction.

Conjecture 4.18. — *After taking reduced cohomology and closure of* W, F*, the real structure, the weight filtration and the algebraically defined Hodge filtration on* $H_2^k(\widetilde{X}, \mathbb{M})$ *are the constituents of a functorial graded polarisable* Mixed Hodge structure *in the abelian category* $\mathbb{R}E_f(\Gamma)$.

The mixed Hodge numbers (namely the dimension of its $I^{p,q}$*) obey the same restrictions as in* [Del71, Del74, Sai90].

If $\mathbb{M}$ *is pure polarized,* L *is the cup product by a Hodge class, and* S *is a polarization defined by the combination of a Saito polarization and* L_2*-Poincaré Verdier duality,* $(\bigoplus_k \overline{H_2^k}(\widetilde{X}, \mathbb{M}), L, S)$ *is a polarized Hodge–Lefschetz in* $\mathbb{R}E_f(\Gamma)$ *in the sense of* [SS, Part 0, Chapter 3].

In the rest of the article we will see what can be done in that direction using only standard results. To establish the *Mixed Hodge structure*, it is enough to prove that, after tensoring with $\mathcal{U}(\Gamma)$, F and $F^\dagger$ become n-opposed in Gr_W^n, hence that the tensor product with $\mathcal{U}(\Gamma)$ is a $\mathcal{U}(\Gamma)$-*Mixed Hodge Structure* thanks to Lemma 3.12. The Hodge Lefschetz structure seems to require that the construction of the Hodge filtration is compatible with [CKS87, KK87]. The duality statement survives after tensoring with $\mathcal{U}(\Gamma)$ thanks to the duality anti-equivalence on finitely generated $\mathcal{U}(\Gamma)$-modules given by $M \mapsto M^\vee = \mathrm{Hom}_{\mathcal{U}(\Gamma)}(M, \mathcal{U}(\Gamma))$ (recall that $\mathcal{U}(\Gamma)$ is selfinjective and that all finitely generated $\mathcal{U}(\Gamma)$ modules are projective) the following form:

Lemma 4.19. — *There is natural isomorphism*

$$\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} \overline{H_2^k}(\widetilde{X}, \mathbb{M}) \longrightarrow (\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} \overline{H_2^{-k}}(\widetilde{X}, \mathbb{D}(\mathbb{M}))^\vee.$$

Proof. — This follows from Proposition 4.12. ☐

5. Analytical L_2 Hodge Structures

5.1. Complex polarized VHS on complete Kähler manifolds

Definition 5.1. — *Let* M *be a complex manifold. A quadruple* $(M, \mathbb{V}, F^\cdot, S)$ *is called a complex polarized variation of Hodge structure (a VHS) iff* $\mathbb{V}$ *is a flat bundle of finite dimensional complex vector spaces with flat connection* D*,* $F^\cdot$ *a decreasing filtration by holomorphic subbundles of* $\mathbb{V}$ *indexed by integers and* S *a flat non degenerate* $(-1)^w$*-hermitian pairing such that*

(1) *The* C^∞ *vector bundle* V *associated to* $\mathbb{V}$ *decomposes as a direct sum* $V = \bigoplus_{p+q=w} H^{p,q}$ *with* $F^P = \bigoplus_{p \geqslant P} H^{p,q}$.
(2) $p \neq r) \Rightarrow S(H^{p,q}, H^{r,s}) = 0$ *and* $(\sqrt{-1})^{p-q} S$ *is positive definite on* $H^{p,q}$.

(3) $D^{1,0}F^p \subset F^{p-1} \otimes \Omega^{1,0}_M$.

The subbundle $H^{p,q}$ can be given a holomorphic structure by the isomorphism $H^{p,q} \to F^p/F^{p+1}$. Denote by d''_p the corresponding Dolbeault operator and set $d'' = \bigoplus_p d''_p$. $D^{1,0}$ induces a C^∞-linear map $\nabla'_p : H^{p,q} \to H^{p-1,q+1} \otimes \Omega^1$ called the Gauss–Manin connection and set $\nabla' = \oplus_p \nabla'_p$. The Hermitian metric $H = \bigoplus_p (\sqrt{-1})^{p-q} S_{H^{p,q}}$ will be called the Hodge metric. The triple $(\mathbb{V}, d'', \nabla')$ is a Higgs bundle.

Following Deligne, we define $E^{P,Q}(\mathbb{V}) = \bigoplus_{p+r=P, s+q=Q} H^{p,q} \otimes E^{r,s}$ and $D'' = d'' + \nabla'$. One also defines $E^k(\mathbb{V}) = \bigoplus_{r+s=k} \bigoplus_{p,q} H^{p,q} \otimes E^{r,s}$. It follows that $D'' E^{P,Q}(\mathbb{V}) \subset E^{P,Q+1}(\mathbb{V})$. Then, see [Zuc79], given any Kähler metric $\omega_{\tilde{X}}$ on $M = \widetilde{X}$, taking formal adjoints of differential operators with respect to this Kähler metric and the Hodge metric on $\mathbb{V}$, the usual Kähler identities hold.

If furthermore the metric $\omega_{\tilde{X}}$ is complete then the Dirac operators $D'' + \mathfrak{d}''$, $D + \mathfrak{d}, \dots$ and the Laplace operator $\Delta_D = 2\Delta_{D''} = 2\Delta_{D'}$ are formally self-adjoint unbounded operators on the Hilbert space of L_2 forms with values in $\mathbb{V}$, see e.g.: [Eys97, Section 5.1] in this case or [BL92] and the references therein for the general theory. Thanks to [Dem, Chapter VIII, Theorem 3.2], it also follows that the closure of D, D'', D' is given by the naive ansatz (namely the domain of D is the space of globally L_2 forms ϕ such that $D\phi$ taken in the sense of distributions is globally L_2), the Hilbert space adjoints of D, D'', D' are given by the naive adjoints (namely the domain of $\mathfrak{d}$ is the space of globally L_2 forms ϕ such that $\mathfrak{d}\phi$ taken in the sense of distributions is globally L_2) and that the L_2 decomposition theorem holds replacing images of D, D', D'' and their adjoints by their closure, namely we have an orthogonal decomposition:

$$L^2\left(\widetilde{X}, E^k(\mathbb{V})\right) = \mathcal{H}^k(\widetilde{X}, \mathbb{V}) \oplus \overline{\operatorname{Im}(D)} \oplus \overline{\operatorname{Im}(\mathfrak{d})},$$

where $\mathcal{H} := \ker(\Delta_D)$ is the space of L_2 harmonic forms and similarly for D''.

The L_2 De Rham complex $L^2\,\mathrm{DR}^\bullet(\widetilde{X}, \mathbb{V})$ (resp. its Dolbeault counterpart) is the complex of bounded linear operators obtained by restricting D (resp. D'') to its domain. The L_2 De Rham cohomology groups $\ker(D)/D\,\mathrm{Dom}(D)$ (resp. their L_2-Dolbeault counterparts) are not represented by harmonic forms but the reduced cohomology groups $\ker(D)/\overline{D\,\mathrm{Dom}(D)}$ (resp.) are.

Lemma 5.2. — *The k^{th} reduced L^2 cohomology of the complete Kähler manifold $\widetilde{X}$ with coefficients in the VHS $\mathbb{V}$ has a Hodge structure of weight $w + k$.*

Proof. — It follows from the fact that $\Delta_D = 2\Delta_{D''}$ commutes with the decomposition in (P, Q) type. □

Lemma 5.3. — *The Hodge–Lefschetz package holds for the reduced L^2 cohomology of the complete Kähler manifold $\widetilde{X}$ with coefficients in the VHS $\mathbb{V}$. More precisely $(\overline{H}^*_2(\widetilde{X}, \mathbb{V}), L)$ is a Hodge–Lefschetz structure polarized by $\int_{\widetilde{X}} S(- \wedge -)$ in the sense of* [SS, Part 0, Chapter 3].

Proof. — Since the reduced L_2 cohomology is represented by harmonic forms and the Kähler identities hold, the usual proof in the compact case [Zuc79], which precisely relies on that property and not on the finite dimensionality of the cohomology groups, applies. □

Since Δ_D is essentially self-adjoint there exists a spectral decomposition
$$\Delta_D = \int_0^\infty \lambda dE_\lambda$$
where $(E_\lambda)_{\lambda>0}$ is the spectral family of Δ_D, an increasing orthonormal projector-valued function on $[0,+\infty[$ converging strongly to Id. The support of this spectral projector valued measure dE_λ is the spectrum of Δ_D. E_0 is the Hilbert space projector on the closed subspace $\mathcal{H} := \ker(\Delta_D)$ and E_λ is the projector on the space of L_2 forms ϕ such that:
$$\forall\, n \in \mathbb{N} \quad \langle \Delta_D^n \phi, \phi\rangle \leqslant \lambda^n \langle \phi, \phi\rangle.$$
The E_λ commute with decomposition in (P,Q)-type and actually with all the operators $D, D', D'', \mathfrak{d}, \ldots, L, \Lambda$. The statement that E_λ commutes with a differential operator means in particular that it preserves its domain.

For future use, we record the following more precise notation, for every $\lambda > 0$:
$$E_\lambda^k(\widetilde{X}, \mathbb{V}) = \operatorname{Im}(E_\lambda) \cap L^2\operatorname{DR}^k(\widetilde{X}, \mathbb{V}).$$
This gives a subcomplex of the L^2 De Rham complex:
$$E_\lambda^\bullet(\widetilde{X}, \mathbb{V}) = \Big(\ldots E_\lambda^k(\widetilde{X}, \mathbb{V}) \xrightarrow{D} E_\lambda^{k+1}(\widetilde{X}, \mathbb{V}) \longrightarrow \ldots\Big).$$
This first order differential operators have closed range if and only if $E_\epsilon = E_0$ for some $\epsilon > 0$ if and only 0 is isolated in the spectrum of the Laplace operator. This fails for instance if $\widetilde{X}$ is the complex line and $\mathbb{V}$ is a the constant sheaf $\mathbb{C}_X$ on a compact Riemann surface X of genus 1.

The natural analog of the space of smooth forms in the compact case is the following subcomplex of $L^2\operatorname{DR}^\bullet(\widetilde{X}, \mathbb{V})$:
$$L^2\operatorname{DR}^\bullet_\infty(\widetilde{X}, \mathbb{V}) = \left(\bigoplus_k \bigcap_{n>0} \operatorname{Dom}\left(\Delta_D^n\Big|_{L^2(\tilde{X}, E^k)}\right), D\right).$$
It is a complex of $\mathcal{N}(\Gamma)$-Fréchet spaces and we have $L^2\operatorname{DR}^\bullet_\infty(\widetilde{X}, \mathbb{V}) \subset C^{\infty,\bullet}(\widetilde{X}, \mathbb{V})$ by standard elliptic estimates. The same construction works also for the Dolbeault complex. See [BL92] for a wider perspective.

Lemma 5.4. — *Assume $\lambda' > \lambda > 0$. Then the following inclusions of complexes:*
$$E_\lambda^\bullet(\widetilde{X}, \mathbb{V}) \subset E_{\lambda'}^\bullet(\widetilde{X}, \mathbb{V}) \subset L^2\operatorname{DR}^\bullet_\infty(\widetilde{X}, \mathbb{V}) \subset L^2\operatorname{DR}^\bullet(\widetilde{X}, \mathbb{V})$$
are quasi-isomorphisms. In fact $E_\lambda^\bullet(\widetilde{X}, \mathbb{V})$ is a homotopy retract of the three other complexes.

The same holds for the L^2-Dolbeault complex of a Γ-equivariant holomorphic hermitian vector bundle.

Proof. — Define $g = \int_\lambda^\infty \mu^{-1} dE_\mu$. Then g, a continuous linear operator, preserves all the 4 complexes above and so does $h = \mathfrak{d}g$. Now, one has $[D, h] = Id - E_\lambda$. The proof works for the Dolbeault complex too, using the Dolbeault Laplacian and $\mathfrak{d}''$. □

Hence $E_\lambda^\bullet(\widetilde{X}, \mathbb{V}) \to L^2\operatorname{DR}^\bullet(\widetilde{X}, \mathbb{V})$ is an isomorphism in the derived category of the abelian category of formal quotients of Hilbert spaces (aka separable Hilbert $\{1\}$-modules). And $E_\lambda^\bullet(\widetilde{X}, \mathbb{V}) \to L^2\operatorname{DR}^\bullet_\infty(\widetilde{X}, \mathbb{V})$ is an isomorphism in the derived category constructed in [Sch99].

We endow the 4 complexes in Lemma 5.4 with filtration induced by the Hodge filtration $F^p = \bigoplus_{P\geqslant p} E^{P,Q}(\mathbb{V})$ on $L^2(\widetilde{X}, E^k(\mathbb{V}))$. This filtration is in each degree a closed subspace which is furthermore a summand. Actually the first three complexes are bigraded in the usual fashion.

Lemma 5.5. — *The first two inclusions of Lemma 5.4 are filtered quasi-isomorphisms.*

Proof. — We have to prove that the maps between the F-exact sequences are isomorphic at the E_1 page. The usual proof does work perfectly well for the first three complexes. Indeed $\operatorname{Gr}_F E^\bullet_\lambda(\widetilde{X}, \mathbb{V}) = (E_{2\lambda}(\Delta_{D''}), D'')$ and $\operatorname{Gr}_F L^2 \operatorname{DR}^\bullet_\infty(\widetilde{X}, \mathbb{V}) = L^2 \operatorname{Dolb}^\bullet_\infty(\widetilde{X}, \mathbb{V})$ whose cohomology are isomorphic by the Dolbeault version of Lemma 5.4. □

Remark 5.6. — For the third one, it seems to be more delicate. One has:

$$\operatorname{Gr}_F L^2 \operatorname{DR}^\bullet_\infty(\widetilde{X}, \mathbb{V}) \subsetneq \operatorname{Gr}_F L^2 \operatorname{DR}^\bullet(\widetilde{X}, \mathbb{V}) \subsetneq L^2 \operatorname{Dolb}^\bullet(\widetilde{X}, \mathbb{V}).$$

Using $\mathfrak{d}''g$ as above, we obtain a quasi-isomorphism of the two extreme complexes with $\operatorname{Gr}_F E_\lambda(\widetilde{X}, \mathbb{V})$, hence the natural inclusion is a quasi isomorphism between them. The problem is that $\mathfrak{d}''g$ does not seem to preserve $\operatorname{Gr}^P_F L^2 \operatorname{DR}^\bullet(\widetilde{X}, \mathbb{V})$. This group contains $\operatorname{Dom}(D') \cap \operatorname{Dom}(D'')$ which is preserved but the inclusion may be strict.

However, the classical case applies without any modification under a strong hypothesis that fails in the simplest case of the universal covering space of a genus one curve:

Lemma 5.7. — *Zero is isolated in the spectrum of Δ_D if and only if $(E_0, 0) \subset (E_\lambda, D)$ is a quasi-isomorphism.*

Proof. — If 0 is isolated in the spectrum and $\epsilon > 0$ is the infimum of the spectrum we can improve of the proof of Lemma 5.4 and construct a bounded Green operator g on E_λ by the formula $g = \int_0^\infty \mu^{-1} dE_\mu$. Its operator norm is indeed $\leqslant \epsilon^{-1}$. It satisfies $\Delta_\bullet g = g\Delta = \operatorname{id}_{E_\lambda} - \prod_{\ker(\Delta)}$ where $\prod_{\ker(\Delta)}$ is the orthogonal projector to the space E_0 of L^2 harmonic forms. □

Lemma 5.8. — *If zero is isolated in the spectrum of Δ_D then:*

(1) *The decomposition theorem is valid without taking the closure of* $\operatorname{Im}(D)$, $\operatorname{Im}(\mathfrak{d})$. *Namely,* $\operatorname{Im}(D)$ *and* $\operatorname{Im}(\mathfrak{d})$ *are L^2-closed and:*

$$L^2\Big(\widetilde{X}, E^k(\mathbb{V})\Big) = \mathcal{H}^k(\widetilde{X}, \mathbb{V}) \oplus \operatorname{Im}(D) \oplus \operatorname{Im}(\mathfrak{d}),$$

and also we have an equivariant decomposition as a direct sum of closed Fréchet subspaces:

$$L^2 \operatorname{DR}^k_\infty(\widetilde{X}, \mathbb{V}) = \mathcal{H}^k(\widetilde{X}, \mathbb{V}) \oplus D\Big(L^2 \operatorname{DR}^{k-1}_\infty(\widetilde{X}, \mathbb{V})\Big) \oplus \mathfrak{d}\Big(L^2 \operatorname{DR}^{k+1}_\infty(\widetilde{X}, \mathbb{V})\Big).$$

(2) *The decomposition theorem for the L^2 Dolbeault complex is valid without taking the closure of* $\operatorname{Im}(D'')$, $\operatorname{Im}(\mathfrak{d}'')$.

(3) *The decomposition theorem for the L^2 D' complex is valid without taking the closure of* $\operatorname{Im}(D')$, $\operatorname{Im}(\mathfrak{d}')$.

(4) *The $D'D''$ lemma holds. Namely,*

$$\phi \in L^2\operatorname{DR}^k_\infty(\widetilde{X},\mathbb{V}) \cap \operatorname{Im}(D') \cap \operatorname{Im}(D'') \implies \exists\, \psi \in L^2\operatorname{DR}^{k-2}_\infty(\widetilde{X},\mathbb{V})\ \phi = D'D''\psi.$$

(5) *The Hodge to De Rham spectral sequence of $L^2\operatorname{DR}^k_\infty(\widetilde{X},\mathbb{V})$ degenerates at E_1 and D is F-strict.*

(6) *The Hodge to De Rham spectral sequence of $E^\bullet_\lambda(\widetilde{X},\mathbb{V})$ degenerates at E_1 and D is F-strict.*

Proof. — If 0 is isolated in the spectrum, we can construct a bounded Green operator G on L^2 k-forms by the formula $G = \int_\epsilon^\infty \mu^{-1} dE_\mu$. It satisfies $\Delta.G = G\Delta = \operatorname{Id} - \prod_{\ker(\Delta)}$ where $\prod_{\ker(\Delta)}$ is the orthogonal projector to the space of L^2 harmonic forms.

Granted this, the textbook proof of these statements in the compact case applies without any modification. □

5.2. Polarized VHS on Galois covering spaces of compact Kähler manifolds

Let X be a compact Kähler manifold and $(X,\mathbb{V},F^\cdot,S)$ be a polarized complex Variation of Hodge Structure of weight w. Assume $\widetilde{X}$ is a Galois covering space of X so that its Galois group Γ acts properly discontinuously by automorphisms on $(\widetilde{X},\pi^{-1}\omega_X,\pi^{-1}\mathbb{V},\pi^{-1}F^\cdot,\pi^{-1}S)$. Then it is easy to see that all the Hilbert spaces considered in the previous section are separable projective Γ-modules and as such are endowed with a $\mathcal{N}(\Gamma)$-module structure. Furthermore if the VHS is real the E_λ and the De Rham L_2 cohomology groups carry a natural real structure. Basic elliptic theory gives:

Proposition 5.9. — *The L^2-De Rham complex $L^2\operatorname{DR}^\bullet(\widetilde{X},\pi^{-1}\mathbb{V})$ is strongly Γ-Fredholm.*

Proof. — This is essentially in [Ati76]. One can construct a Γ equivariant parametrix namely a L^2 bounded Γ-equivariant operator

$$L^2\operatorname{DR}^\bullet\left(\widetilde{X},\pi^{-1}\mathbb{V}\right) \longrightarrow L^2\operatorname{DR}^\bullet\left(\widetilde{X},\pi^{-1}\mathbb{V}\right)[-1]$$

such that $[D,P] = I - S$ where S is a smoothing operator. This also follows from [Shu95] which applies to any elliptic complex (including the case of operators!). For the reader's convenience, we will however give an easy argument.

Let $(\phi_a)_{a\in A}$ be a finite family of smooth real functions such that

$$\sum_{a\in A}\phi_a^2 = 1$$

and $\operatorname{Supp}(\phi_a) \subset U_a$ where U_a is an open subset of X small enough so that $\pi^{-1}(U_a) \cong \Gamma \times U_a$.

If $\psi \in L^2\operatorname{DR}^k(\widetilde{X},\pi^{-1}\mathbb{V})$, $\phi_a\psi$ identifies with an element with compact support in $L^2\Gamma\widehat{\otimes}L^2\operatorname{DR}^k(U_a,\mathbb{V})$, where by $\widehat{\otimes}$ we denote completed Hilbert space tensor product, which we may extend by 0 to an element $\overline{\phi_a\psi}$ of $L^2\Gamma\widehat{\otimes}L^2\,\mathrm{DR}^k(X,\mathbb{V})$. By construction the map Φ:

$$L^2\,\mathrm{DR}^k\Big(\widetilde{X},\pi^{-1}\mathbb{V}\Big) \longrightarrow (L^2\Gamma)^A\widehat{\otimes}L^2\,\mathrm{DR}^k(X,\mathbb{V}) \quad \psi\longmapsto\Phi(\psi)=(\overline{\phi_a\psi})_{a\in A}$$

is a Γ-equivariant Hilbert space isometric (hence closed) embedding.

Let σ_{D+D^*} be the symbol of the operator $D+D^*$. For every ψ in the domain of $D+D^*$ on $L^2\,\mathrm{DR}^k(\widetilde{X},\pi^{-1}\mathbb{V})$ we have:

$$(D+D^*)\phi_a\psi=\phi_a(D+D^*)\psi+\pi^{-1}\sigma(d\phi_a)\psi.$$

Summing up, we obtain

$$\|(D+D^*)\psi\|+K\|\psi\|\geqslant\Big\|\mathrm{Id}_{L^2\Gamma^A}\otimes(D+D^*)(\overline{\phi_a\psi})_{a\in A}\Big\|\geqslant\|(D+D^*)\psi\|-K\|\psi\|$$

where $K=\mathrm{Card}(A)\max_{x\in X}\|\sigma_x\|$.

Assume now $\psi\in E^k_{\lambda^2}(\widetilde{X},\pi^{-1}\mathbb{V})$. Then $\|(D+D^*)\psi\|\leqslant\lambda\|\psi\|$. Hence

$$\Big\|\mathrm{Id}_{pr^k_\mu\circ\Phi}\otimes(D+D^*)(\overline{\phi_a\psi})_{a\in A}\Big\|\leqslant(\lambda+K)\|\psi\|..$$

Introduce the tensor product by $\mathrm{Id}_{(L^2\Gamma)^A}$ of the spectral projector E_μ of $(v,\mathbb{V}$:

$$pr^k_\mu:(L^2\Gamma)^A\widehat{\otimes}L^2\,\mathrm{DR}^k(X,\mathbb{V})\longrightarrow(L^2\Gamma)^A\widehat{\otimes}E^k_\mu(X,\mathbb{V}).$$

Then if $\sqrt{\mu}>\lambda+K$ we have $\|pr^k_\mu\circ\Phi(\psi)\|\geqslant\epsilon\|\psi\|$. for $\epsilon=\sqrt{\mu-\lambda-K}>0$.

It follows that we have a closed embedding of Hilbert Γ-modules:

$$pr^k_\mu\circ\Phi:E^k_\lambda\Big(\widetilde{X},\pi^{-1}\mathbb{V}\Big)\longrightarrow(L^2\Gamma)^A\widehat{\otimes}E^k_\mu(X,\mathbb{V}).$$

Since $E^k_\mu(X,\mathbb{V})$ is a finite dimensional vector space by standard elliptic theory it follows that $E^k_\lambda(\widetilde{X},\pi^{-1}\mathbb{V})$ is a finitely generated Hilbert Γ-module for every $\lambda\geqslant 0$. We conclude using:

Lemma 5.10. — *The L_2 De Rham and Dolbeault complexes are Γ-Fredholm (resp. strongly) if and only if there exists $\epsilon>0$ such that E_ϵ is a finite Γ-dimensional (resp. finite type) projective module.*

Proof. — This follows directly from Definitions 3.7 and 3.8. □

□

In the most general relevant case, Γ need not act in a cocompact fashion, hence we have to add the Γ-Fredholm hypothesis to state the following:

Lemma 5.11. — *Assume $L^2\,\mathrm{DR}^\bullet(\widetilde{X},\pi^{-1}\mathbb{V})$ is Γ-Fredholm. The following inclusions of complexes:*

$$E^\bullet_\lambda\Big(\widetilde{X},\pi^{-1}\mathbb{V}\Big)\subset E^\bullet_{\lambda'}\Big(\widetilde{X},\pi^{-1}\mathbb{V}\Big)\subset L^2\,\mathrm{DR}^\bullet_\infty\Big(\widetilde{X},\pi^{-1}\mathbb{V}\Big)\subset L^2\,\mathrm{DR}^\bullet\Big(\widetilde{X},\pi^{-1}\mathbb{V}\Big)$$

are quasi-isomorphisms of complexes of Hilbert (resp. Fréchet for the third one) Γ-modules and define the same element of $D^b(E_{\mathrm{sep}}(\Gamma))$ (resp. of the derived category of the exact category of $\mathcal{N}(\Gamma)$-Fréchet modules) all of whose cohomology groups have finite $\mathcal{N}(\Gamma)$-dimension.

Proof. — That the inclusions are quasi-isomorphisms of complexes has been proved in the preceding few paragraphs. Except for the third one they are complexes of Hilbert Γ-modules. The first two are complexes of projective Hilbert Γ-modules. The Fredholm condition means that the first one is a complex of finite type projective Hilbert Γ-modules for $\lambda > 0$ small enough. In particular its cohomology groups are in $E_f(\Gamma)$ and have finite $\mathcal{N}(\Gamma)$-dimension. However having finite $\mathcal{N}(\Gamma)$-dimension is an algebraic property of $\mathcal{N}(\Gamma)$-modules [Lüc02, Definition 6.6 p. 239]. □

Lemma 5.12. — *Assume $L^2\,\mathrm{DR}^\bullet(\widetilde{X},\pi^{-1}\mathbb{V})$ is Γ-Fredholm.*

(1) *The decomposition theorem is valid. Namely, we have a direct sum decomposition of $\mathcal{U}(\Gamma)$ modules*

$$F\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)} L^2\Big(\widetilde{X},E^k(\pi^{-1}\mathbb{V})\Big)=\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}\mathcal{H}^k\Big(\widetilde{X},\pi^{-1}\mathbb{V}\Big)\oplus\mathrm{Im}(D)\oplus\mathrm{Im}(\mathfrak{d}).$$

(2) *The decomposition theorem for the corresponding $\mathcal{U}(\Gamma)$-Dolbeault complexes is valid.*
(3) *The decomposition theorem for the $\mathcal{U}(\Gamma)$-D' complexes is valid.*
(4) $\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}E_0^\bullet(\widetilde{X},\pi^{-1}\mathbb{V})\subset\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}E_\lambda^\bullet(\widetilde{X},\pi^{-1}\mathbb{V})$ *is a filtered quasi-isomorphism where $\lambda > 0$.*
(5) *The Hodge to De Rham spectral sequence of $\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}E_\lambda^\bullet(\widetilde{X},\pi^{-1}\mathbb{V})$ degenerates at E_1 and D is F-strict.*
(6) *The Hodge to De Rham spectral sequence of $\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}L^2\,\mathrm{DR}_\infty^\bullet(\widetilde{X},\pi^{-1}\mathbb{V})$ degenerates at E_1 and D is F-strict.*
(7) *Dingoyan's $D'D''$ lemma holds. Namely,*

$$\phi\in L^2\,\mathrm{DR}^k(\widetilde{X},\mathbb{V})\cap\mathrm{Im}(D')\cap\mathrm{Im}(D'')\Longrightarrow\exists\,\psi\;\exists\,u\in\mathcal{U}(\Gamma)^\times\;u\phi=D'D''\psi.$$

Proof. — As in [Din13], (1), (2), (3) follow from [Din13, Lemme 2.15] and the fact these complexes are Γ-Fredholm. Observing that the formation of the cohomology of a complex commutes with $\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}$, since $\mathcal{N}(\Gamma)\to\mathcal{U}(\Gamma)$ is flat, (4) follows from the fact that we have an isomorphism on cohomology after tensoring with $\mathcal{U}(\Gamma)$, and we also have an isomorphism on cohomology after passing to Gr_F since the L^2-Dolbeault complex is Γ-Fredholm too (Γ-Fredholmness means that E_λ has finite Γ-dimension for $\lambda > 0$ small enough). (5) follows from (4) and the fact that the statement is trivially true for $\lambda = 0$ and invariant by filtered quasi-isomorphism. (6) follows in the same way from Lemma 5.5 and (5). (7) follows by an easy adaptation of the argument of [Din13, Lemma 3.13]. □

Theorem 5.13. — *Under the same Γ-Fredholm hypothesis, the k^{th} cohomology group of $\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}L^2\,\mathrm{DR}_\infty^\bullet(\widetilde{X},\pi^{-1}\mathbb{V})$ carries a $\mathcal{U}(\Gamma)$-Hodge structure of weight $k+w$ which we call the analytic Hodge filtration.*
It gives rise to a weight $w+k$ real Hodge structure on $\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}H_2^k(\widetilde{X},\mathbb{M})$.
Every Kähler class on X induces a Hodge–Lefschetz isomorphism

$$L^k:\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}H_2^{\dim(X)-k}(\widetilde{X},\mathbb{V})\longrightarrow\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}H_2^{\dim(X)+k}(\widetilde{X},\mathbb{V}).$$

Proof. — The Γ-Fredholm condition is the unique new ingredient relative to the case where Γ is finite and X is compact that is used in [Din13] to construct the

$\mathcal{U}(\Gamma)$-Hodge structure in case $\mathbb{V} = \mathbb{C}$. He also gives a proof, based on the same Γ-Fredholm condition, of the fact that the natural morphism

$$\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} H_2^{\dim(X)-k}(\widetilde{X}, \mathbb{V}) \longrightarrow \mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} \overline{H_2^{\dim(X)-k}}(\widetilde{X}, \mathbb{V})$$

is an isomorphism in the case $\mathbb{V} = \mathbb{C}$. So this Hodge–Lefschetz isomorphism is a consequence of the Hodge–Lefschetz isomorphism for the reduced cohomology, which follows from the fact it is represented by harmonic forms. □

5.3. $\mathcal{U}(\Gamma)$-Hodge Complex

In this paragraph, we compare these analytic L_2-cohomology groups with the previous $\mathcal{D}_X$-module theoretic and combinatorial constructions.

We can construct on X a resolution of $l^2\pi_*\pi^{-1}\mathbb{V}$ by the sheafified L^2 De Rham complex. This is a complex of sheaves $l^2\,\mathrm{DR}^\bullet(\mathbb{V})$ whose value over $U \subset X$ is given in degree k by

$$l^2\,\mathrm{DR}^\bullet(\mathbb{V})(U) = \left\{\omega \in L^2_{\mathrm{loc}}\left(\pi^{-1}(U), E^k(\pi^{-1}\mathbb{V})\right) \ \ \forall\, K \Subset U \ \int_K \|\omega\|^2 + \|D\omega\|^2 < +\infty\right\}.$$

One can construct a polarizable Hodge module $\mathbb{M} = \mathbb{M}_X(\mathbb{V})$ such that $\mathbb{M}^B = \mathbb{V}[\dim(X)]$, with a trivial W-filtration, the underlying filtered $\mathcal{D}_X$-module $\mathcal{V} = \mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_X$ endowed with the filtration $F^\cdot$ made increasing, the underlying perverse sheaf is $\mathbb{V}[\dim(X)]$ and the comparison morphism α is the usual resolution $\mathbb{V}[\dim(X)] \to \mathrm{DR}(\mathcal{V})$.

Proposition 5.14. — *There is a natural filtered quasi isomorphism of complexes in $E(\Gamma)$*

$$CD_2 : (\mathcal{C}(\mathrm{DR}(\mathcal{V}), F_\bullet) \longrightarrow \left(L^2\,\mathrm{DR}^\bullet_\infty\left(\widetilde{X}, \pi^{-1}\mathbb{V}\right), F_\bullet\right)$$

such that the composition with the comparison morphism induced by α:

$$\mathrm{rh}_2 : \mathbb{H}_2^\bullet\left(\widetilde{X}, \pi^{-1}\mathbb{V}\right) \longrightarrow \mathcal{C}(\mathrm{DR}(\mathcal{V})) \simeq H^\bullet_{\mathrm{DR},2}(\widetilde{X}, \mathcal{V})$$

is the Čech–De Rham comparison isomorphism.

Proof. — First observe that CD_2 indeed maps into $L^2\,\mathrm{DR}^\bullet_\infty$ and that there is indeed a morphism of complexes. Then, it is a routine task to check that these maps are quasi-isomorphisms and that they have the stated compatibility. It is a filtered quasi-isomorphism thanks to the Čech–Dolbeault isomorphism for $\mathrm{Gr}^F\,\mathcal{V}$ described in the appendix. □

Proposition 5.15. — *Consider a locally finite covering $\mathfrak{U}$ of X by small enough Oka–Weil domains, we can define a morphism of filtered $\mathcal{N}(\Gamma)$-Fréchet complexes*

$$\left(\mathcal{C}^\bullet\left(\mathfrak{U}, l^2\pi_*\,\mathrm{DR}\left(\mathbb{M}_X^{\mathrm{DR}}(\mathbb{V})\right)\right), F\right) \longrightarrow \left(L^2\,\mathrm{DR}^\bullet_\infty\left(\widetilde{X}, \pi^{-1}\mathbb{V}\right), F_\bullet\right)$$

which is a filtered quasi isomorphism of complexes of $\mathcal{N}(\Gamma)$-modules.

Proof. — The preceding argument gives also this. □

Using the flatness of $\mathcal{N}(\Gamma) \subset \mathcal{U}(\Gamma)$, we deduce:

Corollary 5.16. — *Tensoring by $\mathcal{U}(\Gamma)$, we obtain a filtered morphism of complexes of $\mathcal{U}(\Gamma)$-modules*

$$\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} \left(\mathcal{C}^\bullet\left(\mathfrak{U}, l^2\pi_* \operatorname{DR}\left(\mathbb{M}_X^{\mathrm{DR}}(\mathbb{V})\right)\right), F\right) \longrightarrow \mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} \left(L^2 \operatorname{DR}^\bullet_\infty\left(\widetilde{X}, \pi^{-1}\mathbb{V}\right), F_\bullet\right)$$

which is a filtered quasi isomorphism.

Before summarizing the outcome of the discussion, we need the following definition [Del74]:

Definition 5.17. — *A Hodge complex of $\mathcal{U}(\Gamma)$-modules with real structure and weight w is a triple: $(A^\bullet, (B^\bullet, F), \gamma)$ where $A^\bullet$ is a complex of $\mathcal{U}(\Gamma)$-modules with real structures, $\gamma : A^\bullet \to B^\bullet$ an isomorphism in the derived category of $\mathcal{U}(\Gamma)$-modules such that the F-spectral sequence degenerates at E_1 and the pair of filtrations $(F, \overline{F})$ on $H^k(B^\bullet)$ is a Hodge structure of weight $w + k$ in the category of $\mathcal{U}(\Gamma)$-modules.*

We say $(B^\bullet, F)$ underlies a Hodge complex of $\mathcal{U}(\Gamma)$-modules with real structure and weight w if it can be completed to such a triple.

Theorem 5.18. — *Let X be a compact Kähler manifold and $(X, \mathbb{V}, F^\cdot, S)$ be a polarized complex Variation of Hodge Structure of weight w.*

1. *The F-spectral sequence of $\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} (\mathcal{C}^\bullet(\mathfrak{U}, l^2\pi_* \operatorname{DR}(\mathbb{M}_X^{\mathrm{DR}}(\mathbb{V}))), F)$ degenerates at E_1.*
2. *The algebraically defined Hodge filtration gives rise to a weight $w + k$ real Hodge structure on $\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} H^k_2(\widetilde{X}, \mathbb{V})$ which coincides with the analytically defined one.*
3. *$\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} (\mathcal{C}^\bullet(\mathfrak{U}, l^2\pi_* \operatorname{DR}(\mathbb{M}_X^{\mathrm{DR}}(\mathbb{V}))), F)$ underlies a weight w Hodge complex of $\mathcal{U}(\Gamma)$-modules with real structure and finite $\mathcal{U}(\Gamma)$-dimensional cohomology objects.*
4. *Every Kähler class on X induces a Hodge–Lefschetz isomorphism*

$$L^k : \mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} H_2^{\dim(X)-k}(\widetilde{X}, \mathbb{V}) \longrightarrow \mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} H_2^{\dim(X)+k}(\widetilde{X}, \mathbb{V}).$$

Proof. — Immediate. We use the natural quasi-isomorphism $\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} \mathrm{rh}_2$ to construct the Hodge complex in the third statement. Actually, the cohomology objects are in the abelian category of finitely presented $\mathcal{U}(\Gamma)$-modules. □

5.4. Using analytic realizations of Hodge Modules

Assume now that $U \subset X$ is a Kähler snc compactification and let $\widetilde{X} \to X$ be a Galois covering space. Endow U with a Poincaré Kähler metric ω_U [CKS87, KK87, Zuc79]. The lift $\omega_{\widetilde{U}}$ of ω_U to $\widetilde{U} := U \times_X \widetilde{X}$ is a complete Kähler metric which is Poincaré with respect to the partial Kähler snc compactification $\widetilde{U} \subset \widetilde{X}$. Consider $(U, \mathbb{V}, F^\cdot, S)$ a (say real) polarized VHS on U with quasi unipotent monodromy. Recall the fundamental result of [CKS87, KK87] that the sheafified L^2 De Rham complex $\mathcal{DR}^\bullet_2(U, \mathbb{V})$ with respect to the Poincaré metric on U is a fine model of the perverse sheaf $\mathcal{IC}_X(\mathbb{V})$.

A trivial modification of the definition in [KK87] replacing $U \to X$ with $\widetilde{U} \to X$ yields sheaves on X we shall denote by $l^2\pi_*\mathcal{DR}_2^k(\widetilde{U}, \pi^{-1}\mathbb{V})$ and the operators D', D'', D between these sheaves on X and we have:

Proposition 5.19. — *$l^2\pi_*\mathcal{DR}_2^\bullet(\widetilde{U}, \pi^{-1}\mathbb{V})$ is a fine model of $l^2\pi_*\pi^{-1}\mathcal{IC}_X(\mathbb{V})$.*

Proof. — Same method as in the proof of Proposition 2.9. Left to the reader. □

Corollary 5.20. — *Under the current assumptions, the k^{th} cohomology of the L_2 De Rham complex of $\widetilde{U}$ with values in $\pi^{-1}\mathbb{V}$ in the Poincaré metric $\omega_{\widetilde{U}}$ is isomorphic as a $\mathcal{N}(\Gamma)$-module to the $\mathcal{N}(\Gamma)$-module underlying $\mathbb{H}_2^k(\widetilde{X}, \pi^{-1}\mathcal{IC}_X(\mathbb{V}))$.*

Corollary 5.21. — *If the L^2 De Rham complex is Γ-Fredholm, the reduced k^{th} cohomology group twisted by $\mathcal{U}(\Gamma)$,*

$$\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} \mathbb{H}_2^k\Big(\widetilde{X}, \pi^{-1}\mathcal{IC}_X(\mathbb{V})\Big) = H^k\Big(X, \mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} l^2\pi_*\pi^{-1}\mathcal{IC}_X(\mathbb{V})\Big)$$

carries a natural real $\mathcal{U}(\Gamma)$-Hodge structure of weight $k+w$.

Proof. — Immediate. □

We do not have a proof of the Γ-Fredholmness in this case. It is not clear whether the filtrations of this analytically defined Hodge Structure coincide with the algebraic ones we have constructed in this article.

6. Proof of Theorem 0.4

Let X be a compact Kähler manifold and $\mathfrak{U}$ be a finite covering of X by sufficiently small Oka–Weil domains.

6.1. Direct image by a closed immersion

Let $i : Z \to X$ be a closed immersion of a smooth compact complex manifold and $(Z, \mathbb{V}, F^\cdot, S)$ be a polarized complex Variation of Hodge Structure of weight w. The case when $Z = X$ follows from Theorem 5.18.

Then $i_*^{\text{MHM}}\mathbb{M}_Z(\mathbb{V}) = \mathbb{M}_i(\mathbb{V})$. The filtered $\mathcal{D}_X$-module $(\mathbb{M}_i^{\text{DR}}(\mathbb{V}), F)$ can be computed as $i_+(\mathbb{M}_Z^{\text{DR}}(\mathbb{V}), F)$. $\text{DR}(\mathbb{M}_i^{\text{DR}}(\mathbb{V}))$ is not equal to $i_*\,\text{DR}(\mathbb{M}_Z^{\text{DR}}(\mathbb{V}))$ except if $Z = X$ where $i_* = Ri_*$ is the ordinary sheaf theoretic direct image. Nevertheless, we can prove:

Lemma 6.1. — *$\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} (R\Gamma(X, l^2\pi_*\mathbb{M}^B), (\mathcal{C}^\bullet(\mathfrak{U}, l^2\pi_*\mathbb{M}_i^{\text{DR}}(\mathbb{V}), F), \text{rh}_2)$ is a $\mathcal{U}(\Gamma)$ Hodge Complex.*

Proof. — For $q \gg 1$, there is a filtered quasi isomorphism

$$F_q\Big(\text{DR}\Big(\mathbb{M}_i^{\text{DR}}(\mathbb{V})\Big), F\Big) \longrightarrow \Big(\text{DR}\Big(\mathbb{M}_i^{\text{DR}}(\mathbb{V})\Big), F\Big)$$

and there is a (differential) filtered quasi-isomorphism of bounded differential complexes coherent sheaves of

$$i_*\,\text{DR}\Big(\mathbb{M}_Z^{\text{DR}}(\mathbb{V})\Big) \longrightarrow F_q\Big(\text{DR}\Big(\mathbb{M}_i^{\text{DR}}(\mathbb{V})\Big), F\Big).$$

On the other hand, one does have $i_*\mathbb{V}[\dim(Z)] = \mathbb{M}_i^B(\mathbb{V})$ and the comparison isomorphism satisfies $\alpha_{\mathbb{M}_i(\mathbb{V})} = i_*\alpha \circ \eta$.

Consider a locally finite covering $\mathfrak{U}$ of X by small enough Oka–Weil domains. We have a canonical identification of filtered $\mathcal{N}(\Gamma)$-Fréchet complexes

$$\Big(\mathcal{C}^\bullet\Big(\mathfrak{U}, l^2\pi_* i_* \operatorname{DR}\Big(\mathbb{M}_Z^{\mathrm{DR}}(\mathbb{V})\Big)\Big), F\Big) = \Big(\mathcal{C}^\bullet\Big(i^{-1}\mathfrak{U}, l^2\pi_* \operatorname{DR}\Big(\mathbb{M}_Z^{\mathrm{DR}}(\mathbb{V})\Big)\Big), F\Big)$$

in particular it is a filtered quasi isomorphism.

It follows that there is a filtered quasi isomorphism

$$\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} \Big(\mathcal{C}^\bullet\Big(i^{-1}\mathfrak{U}, l^2\pi_* \operatorname{DR}\Big(\mathbb{M}_Z^{\mathrm{DR}}(\mathbb{V})\Big)\Big), F\Big) \longrightarrow \mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} \Big(\mathcal{C}^\bullet\Big(\mathfrak{U}, l^2\pi_*\mathbb{M}_i^{\mathrm{DR}}(\mathbb{V})\Big), F\Big).$$

which is compatible with the comparison isomorphism. Hence, the Lemma 6.1 is a consequence of Theorem 5.18. □

It follows from the construction that the real Hodge structure on $\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} H^q(\widetilde{X}, L^2dR(\mathbb{M}_i^{\mathrm{DR}}(\mathbb{V}))$ is the same as the one on $\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} H^q(\widetilde{Z}, L^2dR(\mathbb{M}_Z^{\mathrm{DR}}(\mathbb{V}))$. The first part of Theorem 0.4 is proved.

6.2. Comparison with Dingoyan's work

In this subsection, we finish the proof of the second and third cases of Theorem 0.4.

Definition 6.2 ([Del74]). — *A Mixed Hodge complex of $\mathcal{U}(\Gamma)$-modules with real structure is a triple $((A^\bullet, W), (B^\bullet, W, F), \gamma)$ where $A^\bullet$ is a biregular increasingly filtered complex of $\mathcal{U}(\Gamma)$-modules with real structure, $\gamma : (A^\bullet, W) \to (B^\bullet, W)$ an isomorphism in the filtered derived category of $\mathcal{U}(\Gamma)$-modules such that, for all $k \in \mathbb{Z}$,*

$$\Big(\operatorname{Gr}_W^k A^\bullet, \Big(\operatorname{Gr}_W^k B^\bullet, \operatorname{Gr}_W^k F\Big), \operatorname{Gr}_W^k \gamma\Big)$$

is Hodge complexes of weight k with real structure.

Lemma 6.3 ([Del74]). — *If $((A, W), (B, W, F), \gamma)$ is a* Mixed Hodge Complex *of $\mathcal{U}(\Gamma)$-modules with real structure, for all $n \in \mathbb{Z}$*

$$\Big(H^n(A), \operatorname{Im}\Big(H^n(W) \longrightarrow H^n(A)\Big), H^n(\gamma)^{-1}\Big(F_{H^n(B)}\Big), \Big(H^n(\gamma)^{-1}\Big(F_{H^n(B)}\Big)\Big)^\dagger\Big)$$

where $F_{H^n(B)} = \operatorname{Im}(H^n(F) \to H^n(B))$ is a real $\mathcal{U}(\Gamma)$ mixed Hodge structure.

Lemma 6.4. — *Let X be a compact Kähler manifold and $\mathbb{M}$ such that the Gr_W^k satisfy Conjecture 0.3. Then $\mathbb{M}$ satisfies Conjecture 0.3.*

Proof. — Under these hypotheses, we see immediately that:

$$\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} \Big(R\Gamma\Big(X, l^2\pi_*\mathbb{M}^B, W\Big), \quad \Big(\mathcal{C}^\bullet\Big(\mathfrak{U}, l^2\pi_*\mathbb{M}_i^{\mathrm{DR}}(\mathbb{V}), W, F\Big), \mathrm{rh}_2\Big)\Big)$$

is a $\mathcal{U}(\Gamma)$-*Mixed Hodge Complex.* □

Thanks to [Sai90] (see [Sch99, Example 5.4] for one smooth divisor) the second case of Theorem 0.4 follows from the first case and Lemma 6.4. Indeed the weight graded pieces of $Rj_*j^{-1}M_X(\mathbb{V})$ are sums of Hodge modules of the form $\mathbb{M}_i(\mathbb{V})$ with i an immersion of a closed smooth submanifold.

The third case of Theorem 0.4 also follows using some of the properties of Verdier duality on *Mixed Hodge Modules*, see [Sai90]. Indeed

$$Rj_!j^{-1}\mathbb{M}_X(\mathbb{V}) = \mathbb{D}Rj_*j^{-1}M_X(\mathbb{V}^\vee)$$

hence $Rj_!j^{-1}\mathbb{M}_X(\mathbb{V})$ is a *Mixed Hodge Module*. Furthermore, one has

$$\mathbb{D}\Big(\operatorname{Gr}^k_W \mathbb{M}\Big) = \operatorname{Gr}^{-k}_W(\mathbb{D}(\mathbb{M}))$$

and $\mathbb{D}(\mathbb{M}_i(\mathbb{V})) = \mathbb{M}_i(\mathbb{V}^\vee)$.

Appendix A. Fréchet Sheaves and the functor $\mathcal{C}$

A.1. Čech model

Given B a Banach algebra and X a second countable locally compact topological space, a B-Fréchet sheaf is just a sheaf taking its values in the category of Fréchet spaces with a continuous action of B. A coherent analytic sheaf on a complex analytic space is a $\mathbb{C}$-Fréchet sheaf [GR65, Chapter VIII], in fact a sheaf of Fréchet modules over the structure sheaf which is a $\mathbb{C}$-Fréchet sheaf of algebras, see [Hou73, Sch94].

Given $\mathcal{F}$ a Γ-equivariant coherent analytic sheaf on a proper Γ-complex manifold $\widetilde{X}$, the sheaf $l^2\pi_*\mathcal{F}$ is a Fréchet sheaf of $\mathcal{N}(\Gamma)$-modules as follows from the construction in [Cam01, Eys00] but it is not Montel in the sense of [GR65] when Γ is infinite. There must be a good concept of Γ-Montel sheaves (see [Eys00] for the corresponding notion of Γ-compactness and cp. [GR65, p. 235]) but we don't want to try and develop it. It will be enough for our present purposes to use the ad hoc theory given in [Eys00].

For a locally finite covering $\mathfrak{U}$ of $\Gamma\backslash\widetilde{X}$ by small enough Oka–Weil domains [GR65, p. 211] we can define $\mathcal{C}^\bullet(\mathfrak{U}, l^2\pi_*\mathcal{F})$ the Čech complex of $l^2\pi_*\mathcal{F}$. Here, an open subset Ω is *small enough* if and only if the preimage in $\widetilde{X}$ is a disjoint union of open subsets finite over Ω.

By a standard application of Leray's theorem, $\mathcal{C}^\bullet(\mathfrak{U}, l^2\pi_*\mathcal{F})$ computes the cohomology of $l^2\pi_*\mathcal{F}$.

The complex $\mathcal{C}^\bullet(\mathfrak{U}, l^2\pi_*\mathcal{F})$ does not depend on $\mathfrak{U}$ in the derived category D of the exact category of $\mathcal{N}(\Gamma)$-Fréchet modules (cf. Remark 3.6) hence in the derived category of the abelian category of $\mathcal{N}(\Gamma)$-modules.

In [Eys00] it is proved that the functor $D^b\,\mathbf{Coh}(\mathcal{O}_X) \to D^b\operatorname{Mod}(\mathcal{N}(\Gamma))$ defined by $\mathcal{F} \mapsto \mathcal{C}^\bullet(\mathfrak{U}, l^2\pi_*\mathcal{F})$ at the level of complexes lifts uniquely and functorially to $D^b(E_f(\Gamma))$ under the natural functor $D^b(E_f(\Gamma)) \to D^b\operatorname{Mod}(\mathcal{N}(\Gamma))$ if $\Gamma\backslash\widetilde{X}$ is compact.

The main ingredient is a construction of a quasi-isomorphism

$$K^\bullet \longrightarrow \mathcal{C}^\bullet\Big(\mathfrak{U}, l^2\pi_*\mathcal{F}\Big)$$

from a bounded complex of projective finite-type Hilbert Γ-modules $K^\bullet$.

To this end, one uses that the Γ-Fréchet space $l^2\pi_*\mathcal{F}(\Omega)$ of an open subset $\Omega \subset X$ is the inverse limit of a sequence of Hilbert Γ-modules with Γ-compact transition maps and the fact that Hilbert Γ-modules are projective in $E(\Gamma)$.

Actually one has to do something slightly more complicated, one has a germ at $t = 0$ of an increasing family of coverings $\mathfrak{U}_t$ defined for $t \geqslant 0$ such that $\mathfrak{U} = \mathfrak{U}_0$ and:

$$C_*(t) = \mathcal{C}^\bullet\Big(\mathfrak{U}_t, l^2\pi_*\mathcal{F}\Big) \longrightarrow \mathcal{C}^\bullet\Big(\mathfrak{U}_{t'}, l^2\pi_*\mathcal{F}\Big)$$

is a quasi-isomorphism for $0 \leqslant t' \leqslant t$ and there is a germ at $t = 0$ of a family of Hilbert Γ-modules $(C(t))_{t>0}$ endowed with morphisms $C(t) \to C(t')$ for $t \geqslant t'$ that are Γ-compact for $t > t'$ and $C_*(t) = \varprojlim_{t'<t} C(t')$ if $t > 0$. The family $(C(t))_{t>0}$ is an inessential auxiliary datum but it is instrumental to the construction of $K^\bullet$, its essential uniqueness in the derived category $D^b(E_f(\Gamma))$ and the functorial properties of the construction. At least when $\mathcal{F}$ is locally free, $C(t)$ is the subspace of L_2 Čech cochains in $C_*(t)$, L_2 being measured with respect to a smooth volume form.

The pair $((C_*(t))_{t\geqslant 0}, (C(t))_{t>0})$ is an object of an ad hoc additive category MontMod of so-called qhtf Montelian modules, introduced in [Eys00], which contain finite complexes of $E_f(\Gamma)$ as a full subcategory. The property of family $(C_*(t), C(t))_{t>0}$ that is called qhtf in loc. cit. is that the mappings $C_*(t) \to C_*(t')$ are quasi isomorphisms for all $t \geqslant t' \geqslant 0$. The homotopy category of qhtf Montelian modules localized with respect to a specific class of quasi-isomorphisms is naturally equivalent to $D^b(E_f(\Gamma))$. Unfortunately to construct a Montelian module structure one had to add yet another inessential set of auxiliary data to $\mathcal{F}$, namely a locally free resolution defined on each open subset of $\mathfrak{U}_\epsilon$ for some $\epsilon > 0$: these data are the objects of the category A alluded to in Section 4.2.

The motivation in [Eys00] was to do the above construction in the greater possible generality and define Novikov–Shubin invariants for coherent analytic sheaves(8). Indeed, there, X is a complex analytic space that needs not be smooth nor reduced. Once one knows that

$$\dim_{\mathcal{N}(\Gamma)} H^q_2\Big(\widetilde{X}, l^2\pi_*\mathcal{F}\Big) < +\infty,$$

e.g. that it lies in the essential image of $E_f(\Gamma)$, for all $\mathcal{F} \in \mathbf{Coh}(\mathcal{O}_X)$, a purely algebraic property, the special model of the Leray spectral sequence used in [Eys00, Section 6.1] whose main feature is that it comes from a $E_f(\Gamma)$-spectral sequence starting at the E_1-page, can be replaced with the usual Leray spectral sequence whose terms will have finite $\dim_{\mathcal{N}(\Gamma)}$-dimension as a consequence of the above finiteness statement. This is the main ingredient of the reduction to the case of a vector bundle over a smooth complex manifold of the coherent sheaf version of Atiyah's L_2-index theorem given in [Eys00, Theorem 6.2.1]. The case of a vector bundle on a smooth complex manifold is a special case of Atiyah's L_2-index theorem. Lück's dimension theory and the many remarkable facts collected in his systematic exposition [Lüc02] were not available when [Eys00] was written. This is unfortunate since it would have led to major simplifications.

(8) If one is ready to sacrifice some generality and assume X is a complex projective manifold a much simpler construction is given in [Eys99].

Most of the abstract nonsense in [Eys00] needed to establish the functoriality of the lift to $E_f(\Gamma)$ can also be eliminated tensoring everything with $\mathcal{U}(\Gamma)$ if one is ready to forget about Novikov–Shubin invariants.

A.2. Dolbeault model

In the case where $\mathcal{F}$ is locally free and $\widetilde{X}$ is smooth there is a much better way to proceed. Define

$$\mathrm{Dolb}_2^k(\widetilde{X},\mathcal{F}) = \left\{ s \in L^2_{\mathrm{loc}}\Big(\widetilde{X}, \mathcal{E}^{0,q}(\mathcal{F})\Big), \quad \int_{\widetilde{X}} \|s\|^2 + \|\bar{\partial} s\|^2 < +\infty \right\}$$

where the norms and volume form are computed with respect to a Γ-equivariant hermitian metric on $\mathcal{F}$ and a Γ-equivariant hermitian metric on $\widetilde{X}$

Indeed it is very natural to use the natural map of complexes of $\mathcal{N}(\Gamma)$-Fréchet modules which will be referred to as CD_2, the Čech–Dolbeault comparison map:

$$\begin{aligned} \mathcal{C}^\bullet\Big(\mathfrak{U}, l^2\pi_*\mathcal{F}\Big) &\longrightarrow \mathrm{Dolb}_2^\bullet(\widetilde{X},\mathcal{F})) \\ &= \Big(\ldots \longrightarrow \mathrm{Dolb}_2^k(\widetilde{X},\mathcal{F})) \xrightarrow{\bar{\partial}} \mathrm{Dolb}_2^{k+1}(\widetilde{X},\mathcal{F}) \longrightarrow \ldots\Big) \end{aligned}$$

attached to a smooth partition of unity (ϕ_α) subordinate to $\mathfrak{U}$ sending the q-cochain $(s_{\alpha_0\alpha(1)\ldots\alpha(q)})_{|\alpha|=q}$ to the twisted $(0,q$ form:

$$\sum_{|\alpha|=q} s_{\alpha_0\alpha(1)\ldots\alpha(q)} \bar{\partial}\phi_{\alpha(0)} \wedge \ldots \bar{\partial}\phi_{\alpha(q-1)}.\phi_{\alpha(q)}.$$

The complex $\mathrm{Dolb}_2^\bullet(\widetilde{X},\mathcal{F}))$ is complex of separable Γ-Hilbert modules and the resulting map $K^\bullet \to \mathrm{Dolb}_2^\bullet(\widetilde{X},\mathcal{F}))$ is an algebraic isomorphism induced by a continuous maps at level of the representatives hence $\mathrm{Dolb}_2^\bullet(\widetilde{X},\mathcal{F})) \in D^b_{E_f(\Gamma)}E_{\mathrm{sep}}(\Gamma)$ and is quasi-isomorphic to $\mathcal{C}(\mathcal{F}) \in D^b E_f(\Gamma)$.

A.3. Čech–De Rham comparison map

In a similar fashion, if $\mathbb{V}$ is a local system on X, and the intersections of elements of $\mathfrak{U}$ are contractible, we can construct, as in [Dod77], a quasi-isomorphism of separable projective Hilbert Γ-modules in the essential image of $D^b E_f(\Gamma)$

$$CDR_2 : \mathcal{C}^\bullet\Big(\mathfrak{U}, l^2\pi_*\mathbb{V}\Big) \longrightarrow L^2\,\mathrm{DR}^\bullet(\widetilde{X},\mathcal{V}).$$

Given a smooth partition of unity (ϕ_α) subordinate to $\mathfrak{U}$, it is defined by sending the Čech q-cochain $(s_{\alpha_0\alpha(1)\ldots\alpha(q)})_{|\alpha|=q}$ to the twisted q form:

$$\sum_{|\alpha|=q} s_{\alpha_0\alpha(1)\ldots\alpha(q)} d\phi_{\alpha(0)} \wedge \ldots d\phi_{\alpha(q-1)}.\phi_{\alpha(q)}.$$

BIBLIOGRAPHY

[Ati76] Michael F. Atiyah, *Elliptic operators, discrete groups and Von Neumann algebras*, Colloque “Analyse et Topologie” en l’honneur de Henri Cartan, Société Mathématique de France, 1976, pp. 43–72. ↑770, 798

[BBD82] Alexander A. Beĭlinson, Joseph Bernstein, and Pierre Deligne, *Faisceaux Pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Société Mathématique de France, 1982, pp. 3–171. ↑777, 784

[BDET24] Francesco Bei, Simone Diverio, Philippe Eyssidieux, and Stefano Trapani, *Weakly Kähler hyperbolic manifolds and the Green–Griffiths–Lang conjecture*, J. Reine Angew. Math. **807** (2024), 257–297. ↑770

[Bjö93] Jan-Erik Björk, *Analytic D-modules and applications*, Mathematics and its Applications (Dordrecht), vol. 247, Kluwer Academic Publishers, 1993. ↑781

[BL92] Jochen Brüning and Matthias Lesch, *Hilbert complexes*, J. Funct. Anal. **108** (1992), no. 1, 88–132. ↑795, 796

[Bra21] Lukas Braun, *The local fundamental group of a Kawamata log terminal singularity is finite*, Invent. Math. **226** (2021), no. 3, 845–896. ↑770

[Cam94] Fredéric Campana, *Remarks on the universal covering of compact Kähler manifolds*, Bull. Soc. Math. Fr. **122** (1994), no. 2, 255–284. ↑770

[Cam95] ———, *Fundamental group and positivity of cotangent bundles of compact Kähler manifolds*, J. Algebr. Geom. **4** (1995), no. 3, 487–502. ↑770

[Cam01] ———, *Cohomologie L^2 sur les revêtements d’une variété complexe compacte*, Ark. Mat. **39** (2001), no. 2, 263–282. ↑770, 778, 805

[CG86] Jeff Cheeger and Mikhael L. Gromov, *L_2-cohomology and group cohomology*, Topology **25** (1986), 189–215. ↑786

[CKS87] Eduardo Cattani, Aroldo Kaplan, and Wilfried Schmid, *L_2 and intersection cohomology for a polarizable Variation of Hodge Structure*, Invent. Math. **87** (1987), 217–252. ↑794, 802

[Cur14] Justin Curry, *Sheaves, Cosheaves and Verdier Duality*, `https://arxiv.org/abs/1303.3255v2`, 2014. ↑790

[Cur18] ———, *Dualities between cellular sheaves and cosheaves*, J. Pure Appl. Algebra **222** (2018), no. 4, 966–993. ↑790

[Del71] Pierre Deligne, *Théorie de Hodge II*, Publ. Math., Inst. Hautes Étud. Sci. **40** (1971), 5–57. ↑794

[Del74] ———, *Théorie de Hodge III*, Publ. Math., Inst. Hautes Étud. Sci. **44** (1974), 5–77. ↑794, 802, 804

[Dem] Jean-Pierre Demailly, *Complex Analytic and Differential Geometry*, `https://www-fourier.univ-grenoble-alpes.fr/~demailly/manuscripts/`. ↑795

[Din13] Pascal Dingoyan, *Some mixed Hodge structures on l^2-cohomology groups of coverings of Kähler manifolds*, Math. Ann. **357** (2013), no. 3, 1119–1174. ↑770, 771, 772, 773, 785, 800

[Dod77] Jozef Dodziuk, *De Rham–Hodge theory for L^2-cohomology of infinite coverings*, Topology **16** (1977), 157–165. ↑807

[Eys97] Philippe Eyssidieux, *La caractéristique d’Euler du complexe de Gauss–Manin*, J. Reine Angew. Math. **490** (1997), 155–212. ↑770, 772, 795

[Eys98] ———, *Un théorème de Nakai–Moishezon pour certaines classes de type* $(1,1)$, prépublication no. 133 du Laboratoire Emile Picard, `https://arxiv.org/abs/math/9811065`, 1998. ↑770

[Eys99] ———, *Systèmes linéaires adjoints L_2*, Ann. Inst. Fourier **49** (1999), no. 1, 141–176. ↑770, 806

[Eys00] ———, *Invariants de Von Neumann des faisceaux analytiques cohérents*, Math. Ann. **317** (2000), no. 3, 527–566. ↑770, 778, 779, 782, 783, 788, 789, 790, 792, 793, 805, 806, 807

[Far96] Michael S. Farber, *Homological algebra of Novikov–Shubin invariants and Morse inequalities*, Geom. Funct. Anal. **6** (1996), no. 4, 628–665. ↑770, 783

[God73] Roger Godement, *Topologie algébrique et théorie des faisceaux*, 3e ed., Publications de l'Institut de mathématique de l'Université de Strasbourg, vol. 13, Hermann, 1973. ↑779

[GR65] Robert C. Gunning and Hugo Rossi, *Analytic functions of several complex variables*, Prentice Hall Series in Modern Analysis, Prentice Hall, 1965. ↑805

[Gri66] Ernest L. jun. Griffin, *Everywhere defined Linear transformations affiliated to the rings of operators*, Pac. J. Math. **18** (1966), 489–493. ↑783

[Gro91] Mikhael L. Gromov, *Kähler hyperbolicity and L_2-Hodge theory*, J. Differ. Geom. **33** (1991), no. 1, 263–292. ↑770, 773

[Har66] Robin Hartshorne, *Residues and duality*, Lecture Notes in Mathematics, vol. 20, Springer, 1966. ↑775

[Hou73] Christian Houzel, *Espaces analytiques relatifs et théorèmes de finitude*, Math. Ann. **205** (1973), 13–54. ↑805

[Jea22] Bastien Jean, *L^2-cohomology of a variation of Hodge structure for an infinite covering of an open curve ramified at infinity*, `https://arxiv.org/abs/2211.11605`, 2022. ↑772, 786

[Kas80] Masaki Kashiwara, *Faisceaux constructibles et systèmes holonomes d'équations aux dérivées partielles linéaires à points singuliers réguliers*, Séminaire Goulaouic–Schwartz (1979-1980), Séminaire Équations aux dérivées partielles, École Polytechnique, Centre de Mathématiques, 1980, Exposé 19, pp. 1–6. ↑782

[KK87] Masaki Kashiwara and Takahiro Kawai, *The Poincaré lemma for variations of polarized Hodge structure*, Publ. Res. Inst. Math. Sci. **23** (1987), no. 2, 345–407. ↑794, 802, 803

[Kol93] János Kollár, *Shafarevich maps and plurigenera of algebraic varieties*, Invent. Math. **113** (1993), no. 1, 177–215. ↑770

[Kol95] ———, *Shafarevich maps and Automorphic Forms*, M. B. Porter Lectures, Princeton University Press, 1995. ↑770

[KS90] Masaki Kashiwara and Pierre Schapira, *Sheaves on Manifolds*, Grundlehren der Mathematischen Wissenschaften, vol. 292, Springer, 1990. ↑773, 774, 775, 776, 787, 792

[Lüc02] Wolfgang Lück, *L^2-invariants: Theory and Applications to Geometry and K-theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 44, Springer, 2002. ↑770, 783, 784, 785, 786, 792, 800, 806

[MNM93] Zoghman Mebkhout and Luis Narváez-Macarro, *The constructibility theorem of Kashiwara*, Images directes et constructibilité. Cours d'été du CIMPA "Éléments de la théorie des systèmes différentiels", août et septembre 1990, Nice, France, Hermann, 1993, pp. 47–98. ↑782

[Nor02] Madhav V. Nori, *Constructible Sheaves*, Proceedings of the international colloquium on algebra, arithmetic and geometry, Mumbai, India, January 4–12, 2000. Parts I and II, Narosa Publishing House; Tata Institute of Fundamental Research, 2002, pp. 471–491. ↑777

[Sai89] Morihiko Saito, *Induced $\mathcal{D}$-modules and differential complexes*, Bull. Soc. Math. Fr. **117** (1989), no. 3, 361–387. ↑779, 781, 788, 789

[Sai90] ———, *Mixed Hodge Modules*, Publ. Res. Inst. Math. Sci. **26** (1990), 221–333. ↑770, 771, 793, 794, 805

[Sch94] Jean-Pierre Schneiders, *A Coherence criterion for Fréchet Modules*, Index Theorem for elliptic pairs, Astérisque, vol. 224, Société Mathématique de France, 1994, pp. 99–113. ↑805

[Sch98] Peter Schneider, *Verdier Duality on the building*, J. Reine Angew. Math. **494** (1998), 205–218. ↑790

[Sch99] Jean-Pierre Schneiders, *Quasi abelian categories and sheaves*, Mémoires de la Société Mathématique de France. Nouvelle Série, vol. 76, Société Mathématique de France, 1999. ↑784, 796, 805

[She85] Allen D. Shepard, *A cellular description of the derived category of a stratified space*, Ph.D. thesis, Brown University, Providence, Rhode Island, USA, 1985. ↑790

[Shu95] Mikhail A. Shubin, L^2 *Riemann–Roch operators for elliptic operators*, Geom. Funct. Anal. **5** (1995), no. 2, 482–527. ↑798

[SS] Claude Sabbah and Christian Schnell, *The MHM project Version 2*, Notes available on C. Sabbah's homepage `https://www.cmls.polytechnique.fr/cmat/sabbah/programmes.html`, version of 2022/01/13. ↑794, 795

[SZ21] Junchao Shentu and Chen Zhao, L^2 *representations of Hodge Modules*, `https://arxiv.org/abs/2103.04030`, 2021. ↑773

[Tak99] Shigeharu Takayama, *Nonvanishing theorems on an algebraic variety with large fundamental group*, J. Algebr. Geom. **8** (1999), no. 1, 181–195. ↑770

[Zuc79] Steven Zucker, *Hodge theory with degenerating coefficients:* L_2 *cohomology in the Poincaré metric*, Ann. Math. (2) **109** (1979), 415–476. ↑795, 802

Manuscript received on 14th December 2022,
revised on 20th December 2024,
accepted on 13th March 2025.

Recommended by Editors C. Sorger and C. Xu.

eISSN: 2644-9463
This journal is a member of Centre Mersenne.

Philippe EYSSIDIEUX
Univ. Grenoble Alpes,
CNRS, Institut Fourier,
38000 Grenoble (France)
philippe.eyssidieux@univ-grenoble-alpes.fr