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QUASI-ISOMETRIC FREE GROUP REPRESENTATIONS INTO $\mathrm{SL}_3(\mathbb{R})$

REPRÉSENTATIONS QUASI-ISOMÉTRIQUES DE GROUPES LIBRES DANS $\mathrm{SL}_3(\mathbb{R})$

Abstract. — We study quasi-isometric representations of finitely generated non-abelian free groups into some higher rank semi-simple Lie groups which are not Anosov, nor approximated by Anosov. We show in some cases that these can be perturbed to be non-quasi-isometric, or to have some instability properties with respect to their action on the flag space.

Résumé. — Nous étudions des représentations quasi-isométriques de groupes libres non abéliens de type fini dans certains groupes de Lie semi-simples de rang supérieur, qui ne sont ni Anosov ni limites de représentations Anosov. Dans certains cas, nous montrons que ces représentations peuvent être perturbées de manière à perdre la propriété quasi-isométrique, ou à présenter des instabilités dans leur action sur la variété des drapeaux.

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1. Introduction

1.1. Main problem

This article attempts to make a contribution towards understanding the following:

Problem 1.1. — *Let $\rho : \Gamma \to \mathsf{G}$ be a representation from a finitely generated non-abelian free group Γ to a connected semi-simple Lie group G. If ρ is robustly quasi-isometric, does it follow that ρ is Anosov?*

We will now recall the necessary definitions. We say a property of a representation $\rho : \Gamma \to \mathsf{G}$ is *robust* if it holds in an open neighborhood of ρ, in the topology of pointwise convergence. This topology is equivalent to the one induced by evaluating a representation on a given finite generating set.

Quasi-isometric representations are defined as follows. Fixing $F \subset \Gamma$ a finite symmetric generating set of Γ[(1)], we recall that the *word length* of an element $\gamma \in \Gamma$ is defined by

$$|\gamma| := \min\{n \geqslant 1 : \gamma \in F^n\},$$

with the convention that $|1|$ (the word length of the neutral element) is 0, and $F^n := \{f_1 \cdots f_n : f_1, \ldots, f_n \in F\}$. Fixing any left-invariant Riemannian metric on G and letting dist denote the associated distance, a representation $\rho : \Gamma \to \mathsf{G}$ is said to be *quasi-isometric* if it is a quasi-isometric embedding of Γ, that is, there exist constants $a \geqslant 1$ and $b \geqslant 0$ such that

$$a^{-1}\left|\gamma^{-1}\eta\right| - b \leqslant \operatorname{dist}(\rho(\gamma), \rho(\eta)) \leqslant a\left|\gamma^{-1}\eta\right| + b, \tag{1.1}$$

for all $\gamma, \eta \in \Gamma$.

Finally, we recall the definition of Anosov representations. We fix a Cartan subspace $\mathfrak{a}$ of G, a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$, and a corresponding Cartan projection $a : \mathsf{G} \to \mathfrak{a}^+$. A representation is said to be *Anosov* if there exist constants $c, d > 0$ and a simple root α such that

$$c|\gamma| - d \leqslant \alpha(a(\rho(\gamma))), \tag{1.2}$$

for all $\gamma \in \Gamma$. In that case we say that ρ is $\{\alpha\}$-*Anosov*.

Anosov representations were introduced by Labourie [Lab06] in his study of the Hitchin component [Hit92]. They were further generalized to arbitrary word hyperbolic groups by Guichard–Wienhard [GW12]. If G is rank one, they coincide with convex co-compact representations. They are nowadays understood as the correct generalization of this notion to higher rank. See the surveys of Kassel [Kas18] and Wienhard [Wie18] for further information and motivations. In particular, being Anosov is a robust property [GW12, Lab06] (see also [BPS19, Corollary 5.10]). We point out that the original definition of Anosov representations is not the one given in Equation (1.2), but this is an equivalent one [BPS19, GGKW17, KLP17].

From Equation (1.2) one readily sees that Anosov representations are quasi-isometric. While in rank one these two notions coincide, in higher rank being quasi-isometric is not a robust property (see Subsection 1.3.1 below).

[(1)] Throughout, we assume that the rank of Γ as a free group is $\geqslant 2$.

Problem 1.1 was suggested in Bochi–Potrie–Sambarino [BPS19, Section 4.4] and then formalized in Potrie [Pot18, Section 4.3]. The question also appears in [Kas18, Section 8]. While the question makes sense for groups more general than free groups (and was posed in that way in the mentioned references), one needs to be careful about which groups can admit a positive answer. Indeed, recently Tsouvalas [Tso24] presented examples showing that the answer is negative for general hyperbolic groups, see Section 1.3.2 for further details.

1.2. Rank 1 groups and products

For G of rank 1, Problem 1.1 is trivial because quasi-isometric representations and Anosov representations coincide, see [GW12, Theorem 5.15]. However, the following question is still interesting:

Problem 1.2. — *Suppose that G is a rank 1 connected semi-simple Lie group and ρ is a robustly faithful representation of a non-abelian free group into G. Is ρ necessarily Anosov?*

A celebrated theorem by Sullivan gives a positive answer in a special case[(2)]:

Theorem 1.3 (Sullivan [Sul85, Theorem A]). — *Suppose that Γ is a finitely generated non-abelian free group and let $\rho : \Gamma \to \mathsf{SL}_2(\mathbb{C})$ be a robustly faithful representation. Then ρ is Anosov.*

In the case $\mathsf{G} = \mathsf{SL}_2(\mathbb{R})$ the answer to Problem 1.2 is still affirmative and is an elementary fact, see Section 2.1 for a proof[(3)]. For other rank 1 groups, such as $\mathsf{SO}_0(1, d)$ with $d > 3$, the problem remains open as far as the authors are aware (c.f. Kapovich [Kap08, Question 11.14]). We also mention here that for G of arbitrary rank, a robustly faithful representation of a non abelian free group into G is necessarily (robustly) discrete, see Glutsyuk [Glu11, Theorem 1.2].

In Section 2.2 we prove the following, which in the special case of certain products of rank one Lie groups could be considered as a generalization of both Problems 1.1 and 1.2:

Theorem 1.4. — *Suppose that Γ is a finitely generated non-abelian free group and $\mathsf{G} = \mathsf{SL}_2(\mathbb{R})^r \times \mathsf{SL}_2(\mathbb{C})^s$ for some non-negative integers r and s. Then $\rho : \Gamma \to \mathsf{G}$ is robustly faithful if and only if it is Anosov.*

We remark that in the context of Theorem 1.4 the fact that ρ is Anosov is equivalent to the existence of a projection onto some factor which is quasi-isometric. We also observe that Theorem 1.4 remains valid when replacing Γ by a surface group, see Remark 2.4.

(2) Note that Sullivan's Theorem holds not only for free groups, but is also true in a much more general setting.

(3) However for semi-groups the problem is more subtle and studied carefully by Avila–Bochi–Yoccoz [ABY10].

1.3. Free groups in $\mathsf{SL}_3(\mathbb{R})$

Let $s_1(g) \geqslant \cdots \geqslant s_d(g)$ denote the singular values of an element $g \in \mathsf{SL}_d(\mathbb{R})$ with respect to the standard Euclidean inner product(4). In particular, $s_1(g)$ is the standard operator norm of g.

For $\mathsf{G} = \mathsf{SL}_d(\mathbb{R})$, the left-invariant Riemannian distance $\operatorname{dist}(1, g)$ is comparable to $\log s_1(g)$, see e.g. [BPS19, Section 7.7]. In particular, recalling (1.1) we conclude that a representation $\rho : \Gamma \to \mathsf{SL}_d(\mathbb{R})$ is quasi-isometric if and only if

$$\log s_1(\rho(\gamma)) \geqslant a^{-1}|\gamma| - b \tag{1.3}$$

for all $\gamma \in \Gamma$ and for some $a \geqslant 1$ and $b \geqslant 0$.

On the other hand, for $\mathsf{G} = \mathsf{SL}_d(\mathbb{R})$ one may take the simple roots $\{\alpha_i\}_{i=1}^{d-1}$ so that they satisfy

$$\alpha_i(a(g)) = \log s_i(g) - \log s_{i+1}(g).$$

Hence, if the Anosov condition (1.2) is satisfied for some α_i, it is also satisfied by α_{d+1-i}. In particular, Anosov representations into $\mathsf{SL}_3(\mathbb{R})$ always satisfy the condition (1.2) for $\alpha = \alpha_1$.

1.3.1. Non-Anosov Examples

There exist quasi-isometric representations of the free group into $\mathsf{SL}_3(\mathbb{R})$ which are not in the closure of the set of Anosov representations. There are two types of examples. The first class was studied by Lahn [Lah23] and will be reviewed in Section 4.1. These representations are always reducible. In Section 4.4 we prove the following:

Theorem 1.5. — *Let $k > 2$ be an integer and Γ_k be a non abelian free group of rank k. There exists a quasi-isometric representation $\rho_k : \Gamma_k \to \mathsf{SL}_3(\mathbb{R})$ whose image is Zariski dense, and such that ρ_k is not accumulated by Anosov representations. Furthermore, the representations ρ_k are not robustly quasi-isometric.*

The examples in Theorem 1.5 are all constructed by restricting a particular quasi-isometric representation $\rho_2 : \Gamma_2 \to \mathsf{SL}_3(\mathbb{R})$ of a non abelian free group in two generators to an appropriate finite index subgroup. The representation ρ_2 is also not a limit of Anosov representations, but we don't know whether it is robustly quasi-isometric or not. Nevertheless, by applying a recent result by Dey–Hurtado [DH24] we can show that in any neighborhood of ρ_2 there exists a representation $\rho_2' : \Gamma_2 \to \mathsf{SL}_3(\mathbb{R})$ whose action on the space of full flags $\mathcal{F}$ of $\mathbb{R}^3$ is minimal(5). This is in contrast with what happens with the starting representation ρ_2, which in fact has a proper *limit set* in $\mathcal{F}$.

(4) Recall that the *singular values* of g are the square roots of the eigenvalues of gg^t.

(5) Recall that the space of full flags is the space of pairs (L, P) where L is a one dimensional subspace of $\mathbb{R}^3$, P is a two dimensional subspace, and $L \subset P$. The space of flags is included in, and inherits its topology from, the product $\mathbb{RP}^2 \times \mathcal{G}_2(\mathbb{R}^3)$, where $\mathbb{RP}^2$ is the real projective plane, and $\mathcal{G}_2(\mathbb{R}^3)$ is the Grassmannian manifold of two dimensional subspaces.

Problem 1.1 can be compared with the problem of understanding structural stability or robust transitivity of diffeomorphisms (see [BPS19, Section 4.4]). This is somewhat the viewpoint of Sullivan [Sul85]. The basic example ρ_2 alluded above, while we are not able to show that it is not robustly quasi-isometric, we can show that the action on the flag space is not structurally stable (in fact, the topology of the limit set changes by perturbation). This is also related to Dey–Hurtado [DH24, Question 1.1] which asks whether a discrete subgroup of $\mathsf{SL}_3(\mathbb{R})$ with full limit is necessarily a lattice. If that question admits a positive answer, it would imply that ρ_2 is not robustly faithful and discrete.

Before moving on to the statement of our main result we mention here that Guichard first constructed examples of quasi-isometric representations of a non-abelian free group into $\mathsf{SL}_2(\mathbb{R}) \times \mathsf{SL}_2(\mathbb{R})$ which are not stable under small deformations. In fact, they are accumulated by dense representations (see Guéritaud–Guichard–Kassel–Wienhard [GGKW17, Appendix A]). After that, Tsouvalas [Tso23] constructed examples of quasi-isometric representations into $\mathsf{SL}_d(\mathbb{R})$ which are not limits of Anosov representations, for some specific values of d and some specific word hyperbolic groups. In particular, [Tso23, Proposition 4.2] constructs examples of quasi-isometric representations of free groups into $\mathsf{SL}_6(\mathbb{R})$ which are not accumulated by Anosov representations. These examples are strongly irreducible. The basic example ρ_2 improves this result by finding Zariski dense examples in lower dimensions.

1.3.2. Derived from Barbot representations

Our main result is to provide a positive answer to Problem 1.1 for reducible representations into $\mathsf{SL}_3(\mathbb{R})$. As being Anosov and quasi-isometric are properties which are preserved under taking duals, we will always assume that our reducible representations preserve a hyperplane $P \subset \mathbb{R}^3$. Following Lahn [Lah23] we will call such a representation a *reducible suspension.*

For every reducible suspension $\rho : \Gamma \to \mathsf{SL}_3(\mathbb{R})$ we may define

$$\rho_P : \Gamma \longrightarrow \mathsf{SL}^{\pm}(P) : \ \rho_P(\gamma) := \frac{1}{|\det_P(\rho(\gamma))|^{\frac{1}{2}}} \cdot \rho(\gamma)|_P.$$

In the above formula, P is given the volume form inherited from the standard volume form in $\mathbb{R}^3$, $|\det_P(g)|$ denotes the Jacobian of a linear map g restricted to P, and $\mathsf{SL}^+(P) = \mathsf{SL}(P)$ (resp. $\mathsf{SL}^-(P)$) denotes the set of linear maps of P whose determinant is equal to 1 (resp. -1). A reducible suspension will be called *derived from Barbot* if ρ_P is quasi-isometric. The terminology is motivated by a well-known construction in hyperbolic dynamics. Indeed, Barbot [Bar10] first studied reducible suspensions which are moreover Anosov, and small enough deformations of them. However, the class of derived from Barbot representations is strictly more general: these are reducible suspensions which are not necessarily Anosov, but obtained from an Anosov one by a very large deformation. This is reminiscent to the construction of *derived from Anosov* diffeomorphisms [Mañ78].

In Section 5.1 we prove the following proposition.

Proposition 1.6. — *Let Γ be a finitely generated non-abelian free group and $\rho : \Gamma \to \mathsf{SL}_3(\mathbb{R})$ be a reducible suspension preserving some hyperplane P. Then the following are equivalent:*

(1) *the representation ρ is quasi-isometric,*
(2) *the representation ρ is derived from Barbot,*
(3) *the representation ρ is robustly quasi-isometric among representations preserving P.*

The above result is partially proved in [Lah23] in a more general setting. In particular, Lahn proves that a reducible suspension of a quasi-isometric representation is always quasi-isometric [Lah23, Proposition 3.4]. Moreover, if ρ is faithful and discrete then ρ_P is also faithful and discrete by [Lah23, Proposition 3.1]. Hence, the only novelty in Proposition 1.6 is the fact that ρ being quasi-isometric implies that ρ_P is also quasi-isometric (see Corollary 5.3), compare with [Lah23, Proposition 4.2].

The main result of this note is to provide an affirmative answer to Problem 1.1 for reducible suspensions. By Proposition 1.6, we may restrict our attention to derived from Barbot representations. Lahn [Lah23, Theorem 2] gives a description of which derived from Barbot representations are Anosov. Starting from this result we prove the following.

Theorem 1.7. — *Let Γ be a finitely generated non-abelian free group and $\rho : \Gamma \to \mathsf{SL}_3(\mathbb{R})$ be a derived from Barbot representation. Assume moreover that ρ_P preserves the orientation of P. Then ρ is robustly quasi-isometric if and only if it is Anosov.*

We point out here that examples of robustly quasi-isometric representations which fail to be Anosov were found recently by Tsouvalas [Tso24]. These are representations of free products of uniform lattices in $\mathsf{Sp}(n,1)$ into $\mathsf{SL}_d(\mathbb{R})$ for large enough d. Even though these representations admit non trivial deformations, these deformations are quite special thanks to Corlette's Archimedean Superrigidity [Cor92]. This is the main reason why we have decided to focus on free groups in Problem 1.1, in order to have enough flexibility to work with. We also believe that Problem 1.1 is interesting when replacing Γ by a (closed) surface group, as these are also quite flexible (c.f. Remark 2.4). In this direction, we mention that similar questions were studied by Danciger–Guéritaud–Kassel–Lee–Marquis [DGK+23] for some Coxeter groups embedded into some $\mathsf{SL}_d(\mathbb{R})$. In particular, in these cases the interior of the set of representations which arise as a reflection group is shown to coincide with some space of Anosov representations (see [DGK+23, Corollary 1.18]). We point out also that the authors found an example of a robustly faithful and discrete representation of a word hyperbolic (Coxeter) group which is not Anosov, see [DGK+23, Remark 1.19].

1.4. Organization of the paper

In Section 2 we prove Theorem 1.4, for which we first address Problem 1.2 for $\mathsf{G} = \mathsf{SL}_2(\mathbb{R})$ (Subsection 2.1). In Section 3 we recall generalities about quasi-isometric representations into $\mathsf{SL}_3(\mathbb{R})$, and prove Proposition 3.4. This proposition is one of the two main ingredients in the proof of Theorem 1.7, as it provides a sufficient condition for a representation not to be robustly quasi-isometric. Theorem 1.5 is proven in Section 4, more concretely in Section 4.4, after constructing the basic example ρ_2 in Subsections 4.2 and 4.3. Previous to that, in Section 4.1 we recall recent work by Lahn [Lah23] discussing reducible examples of quasi-isometric representations which are not limits of Anosov representations. In Section 5 we prove Theorem 1.7. We first prove Proposition 1.6 in Subsection 5.1. In Subsection 5.2 we prove Proposition 5.5, which is the second main ingredient in the proof of Theorem 1.7, as it shows that a non-Anosov reducible quasi-isometric representation can be perturbed so that a specific element has a repeated eigenvalue. We then observe in Subsection 5.3 that this proposition together with Proposition 3.4 readily imply Theorem 1.7.

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2. Products of SL_2's

2.1. Robust faithfulness in $\mathsf{SL}_2(\mathbb{R})$

Here we show that robustly faithful representations of a free group into $\mathsf{SL}_2(\mathbb{R})$ are Anosov (note that this does not follow from Sullivan's Theorem 1.3). This will be used in the proof of Theorem 1.4.

We begin with a preparatory lemma.

Lemma 2.1. — *Let g_0 and h_0 be non commuting hyperbolic elements in $\mathsf{SL}_2(\mathbb{R})$. Then the commutator*

$$[\cdot,\cdot] : \mathsf{SL}_2(\mathbb{R}) \times \mathsf{SL}_2(\mathbb{R}) \longrightarrow \mathsf{SL}_2(\mathbb{R}); [g,h] := ghg^{-1}h^{-1}$$

is a submersion at (g_0, h_0).

Proof. — This is a direct computation, we include a proof for completeness.

Let $\mathfrak{sl}_2(\mathbb{R})$ be the Lie algebra of $\mathsf{SL}_2(\mathbb{R})$, that is, the vector space of traceless 2×2 real matrices. For $g \in \mathsf{SL}_2(\mathbb{R})$, the tangent space $T_g\,\mathsf{SL}_2(\mathbb{R})$ is $g \cdot \mathfrak{sl}_2(\mathbb{R})$.

Let $X, Y \in \mathfrak{sl}_2(\mathbb{R})$. The derivative of $[\cdot,\cdot]$ at (g_0, h_0) evaluated on the pair $(g_0 \cdot X, h_0 \cdot Y)$ is given by

$$[g_0,h_0] \cdot \Big(\mathrm{Ad}_{h_0 g_0 h_0^{-1}}(X) + \mathrm{Ad}_{h_0 g_0}(Y) - \mathrm{Ad}_{h_0 g_0}(X) - \mathrm{Ad}_{h_0}(Y)\Big),$$

where $\mathrm{Ad} : \mathsf{SL}_2(\mathbb{R}) \to \mathrm{Aut}(\mathfrak{sl}_2(\mathbb{R}))$ is the adjoint representation. In particular, it suffices to show that the linear map

$$(X,Y) \longmapsto \mathrm{Ad}_{g_0}\Big(\mathrm{Ad}_{h_0^{-1}}(X) - X\Big) + (\mathrm{Ad}_{g_0}(Y) - Y)$$

is surjective.

Without loss of generality we may suppose that g_0 is diagonal, with eigenvalues $\mu \neq \pm 1$ and μ^{-1}. Then Ad_{g_0} is diagonalizable on the basis $\{\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right), \left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right), \left(\begin{smallmatrix}0 & 0\\ 1 & 0\end{smallmatrix}\right)\}$, with respective eigenvalues 1, μ^2 and μ^{-2}. The image of $Y \mapsto \mathrm{Ad}_{g_0}(Y) - Y$ is therefore

$$W_{g_0} := \mathrm{span}\left\{\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right\}.$$

Moreover, as h_0 is hyperbolic but not diagonal (because it does not commute with g_0), it follows easily that W_{g_0} is different from the image of

$$X \longmapsto \mathrm{Ad}_{g_0}\Big(\mathrm{Ad}_{h_0^{-1}}(X) - X\Big),$$

which is again two-dimensional. This completes the proof of Lemma 2.1. □

Proposition 2.2. — *Let Γ be a finitely generated non-abelian free group and $\rho : \Gamma \to \mathsf{SL}_2(\mathbb{R})$ a robustly faithful representation. Then ρ is Anosov (equivalently, quasi-isometric).*

Proof. — We first observe that ρ must be discrete. Otherwise, the Euclidean closure $\overline{\rho(\Gamma)}$ of $\rho(\Gamma)$ would be either $\mathsf{SL}_2(\mathbb{R})$ or contained in a (virtually) solvable subgroup of $\mathsf{SL}_2(\mathbb{R})$. The latter case is ruled out, as it would imply that ρ has a non trivial kernel. Hence $\overline{\rho(\Gamma)} = \mathsf{SL}_2(\mathbb{R})$ and there is an element $\gamma \in \Gamma \setminus \{1\}$ so that the trace $\mathrm{tr}(\rho(\gamma))$ belongs to the interval $(-2, 2)$. Since the trace function $\rho' \mapsto \mathrm{tr}(\rho'(\gamma))$ is a non-constant polynomial in the entries of the image of a free generating set of Γ, it must take a value of the form $2\cos(\frac{k\pi}{n})$ for some integers k and n and arbitrarily small perturbations ρ' of ρ. Thus we would get a small perturbation of ρ which is not faithful contradicting our assumption.

Now that we know that $\rho(\Gamma)$ is discrete, observe that the induced representation $\Gamma \to \mathsf{PSL}_2(\mathbb{R})$ is also discrete. It is moreover robustly faithful and, to finish the proof, it suffices to show that this induced representation is quasi-isometric. By abuse of notations, we still denote this induced representation by ρ.

Consider the hyperbolic surface $\mathbb{H}^2/\rho(\Gamma)$, which is geometrically finite as Γ is finitely generated (see e.g. [Kat92, Theorem 4.6.1]). We will show that every element in $\rho(\Gamma)$ is hyperbolic, and this will finish the proof (see e.g. [Dal11, Theorems 4.8 & 4.13 and Corollary 4.17]).

Suppose by contradiction that there is some parabolic element corresponding to some cusp in $\mathbb{H}^2/\rho(\Gamma)$. There are two cases, namely, either $\mathbb{H}^2/\rho(\Gamma)$ has genus $g = 0$, or $g > 0$.

If $g = 0$, we may suppose that the cusp corresponds to a free generator of Γ. It can then be perturbed to a finite order element in $\mathsf{PSL}_2(\mathbb{R})$, contradicting the fact that ρ is robustly faithful. On the other hand, if $g > 0$ then the cusp corresponds to an element γ of Γ so that there is a free generating set

$$\{a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n\} \subset \Gamma$$

such that

$$\gamma = [a_1, b_1] \ldots [a_g, b_g] c_1 \ldots c_n.$$

We can assume that both $\rho(a_1)$ and $\rho(b_1)$ are hyperbolic, otherwise we conclude as in the previous case. Since ρ is faithful, by Lemma 2.1 we may perturb ρ only perturbing the image of a_1 and b_1, in such a way that $\rho(\gamma)$ becomes a finite order element of $\mathsf{PSL}_2(\mathbb{R})$. This finishes the proof of Proposition 2.2. □

2.2. Proof of Theorem 1.4

Theorem 1.4 follows from combining Proposition 2.2, Sullivan's Theorem 1.3 and the following general lemma.

Lemma 2.3. — *Suppose that Γ is a finitely generated non-abelian free group and $\mathsf{G} = \prod_{i=1}^{r} \mathsf{G}_i$ for some positive integer r and some Lie groups $\mathsf{G}_1, \dots, \mathsf{G}_r$. Then if $\rho = (\rho_1, \dots, \rho_r) : \Gamma \to \mathsf{G}$ is robustly faithful, there is some $i = 1, \dots, r$ such that ρ_i is robustly faithful.*

Proof. — By an inductive procedure, it suffices to show that either ρ_1 or $(\rho_2, \dots, \rho_r)$ is robustly faithful. Suppose by contradiction that both ρ_1 and $(\rho_2, \dots, \rho_r)$ are not. We may then find arbitrarily small perturbations ρ_1' and $(\rho_2', \dots, \rho_r')$ of ρ_1 and $(\rho_2, \dots, \rho_r)$ respectively, and elements $\gamma_1, \gamma_2 \neq 1$ in Γ so that

$$\rho_1'(\gamma_1) = 1 \quad \text{and} \quad \rho_i'(\gamma_2) = 1$$

for all $i = 2, \dots, r$.

The commutator $\gamma := [\gamma_1, \gamma_2]$ belongs to the kernel of $\rho' := (\rho_1', \dots, \rho_r')$, which is an arbitrarily small perturbation of ρ. Hence $\gamma = 1$ and there is some pair of integers such that $\gamma_1^n = \gamma_2^m$. It follows that γ_2^m belongs to the kernel of ρ', thus finding the desired contradiction. □

Remark 2.4. — We observe here that Theorem 1.4 remains valid when replacing Γ with the fundamental group of a closed orientable surface. Indeed, Proposition 2.2 remains true in that case thanks to Funar–Wolff [FW07, Theorem 1.1] (see also [DK06]). The corresponding statement for $\mathsf{SL}_2(\mathbb{C})$ is also true thanks to Sullivan [Sul85, Theorem A]. This shows the claim, as Lemma 2.3 is completely general.

3. Perturbed not quasi-isometric representations

In this section we prove a crucial result (Proposition 3.4) which gives a sufficient condition ensuring that a representation of a free group into $\mathsf{SL}_3(\mathbb{R})$ is not robustly quasi-isometric. The proof is given in Subsection 3.2. We begin by recalling general properties of quasi-isometric representations in Subsection 3.1.

3.1. General properties of quasi-isometric representations

Let ρ be a quasi-isometric representation of a finitely generated free group Γ into $\mathsf{GL}_d(\mathbb{R})$. It is not hard to see that every conjugate of ρ is also quasi-isometric. Moreover, ρ is necessarily discrete and faithful, as Γ is torsion free.

We record two other general properties quasi-isometric representations for future use.

Lemma 3.1. — *Let $\rho : \Gamma \to \mathsf{GL}_d(\mathbb{R})$ be a representation, where Γ is a finitely generated non abelian free group. Suppose that Γ_0 is a finite index subgroup of Γ. Then $\rho : \Gamma \to \mathsf{GL}_d(\mathbb{R})$ is quasi-isometric if and only if its restriction to Γ_0 is quasi-isometric.*

Proof. — Indeed, by the Švarc–Milnor lemma (see e.g. [BH99, Proposition 8.19]), Γ_0 is finitely generated and quasi-isometric to Γ. The lemma follows. ☐

The most important property that we will need is the following. Recall that an element $g \neq 1$ of $\mathsf{GL}_d(\mathbb{R})$ is said to be *unipotent* if all its eigenvalues are equal to 1.

Lemma 3.2. — *Let $\rho : \Gamma \to \mathsf{GL}_d(\mathbb{R})$ be a quasi-isometric representation, where Γ is a finitely generated non abelian free group. Then $\rho(\Gamma)$ does not contain unipotent elements.*

Proof. — Suppose by contradiction that $\rho(\gamma)$ is unipotent, for some $\gamma \in \Gamma$. It is conjugate to an upper triangular matrix. Moreover, for a given norm $\|\cdot\|$ on $\mathbb{R}^d$ the powers $\|\rho(\gamma^n)\|$ grow in a polynomial way as $n \to \infty$. This contradicts (1.3). ☐

3.2. A criterion for non quasi-isometry

For a matrix $g \in \mathsf{SL}_3(\mathbb{R})$ we let

$$\lambda_u(g) \geqslant \lambda_c(g) \geqslant \lambda_s(g)$$

be the moduli of the eigenvalues of g. Recall that g is said to be *loxodromic* if the inequalities above are all strict. In this case, the eigenvalues of g are real, and denoted respectively by $\mu_u(g), \mu_c(g)$ and $\mu_s(g)$. The corresponding eigenspaces are one dimensional and denoted by $E^u(g), E^c(g)$ and $E^s(g)$. We also let

$$E^{cu}(g) := E^u(g) \oplus E^c(g) \quad \text{and} \quad E^{cs}(g) := E^s(g) \oplus E^c(g),$$

which are respectively the attracting and repelling hyperplane of g acting on the Grassmannian $\mathcal{G}_2(\mathbb{R}^3)$.

We have the following general lemma.

Lemma 3.3. — *Let $g_0 \in \mathsf{SL}_3(\mathbb{R})$ be a loxodromic element and n be a non-zero integer. Then the map*

$$\mathsf{SL}_3(\mathbb{R}) \longrightarrow \mathsf{SL}_3(\mathbb{R}) : g \longmapsto g^n$$

sends every small enough neighborhood U of g_0 to a neighborhood $U^{(n)}$ of g_0^n.

Proof. — Indeed, every power of a loxodromic element is again loxodromic. Moreover, a neighborhood of a loxodromic element is parametrized by the choice of three lines E^u, E^c and E^s in general position, plus the choice of three non zero real numbers μ_u, μ_c and μ_s of different moduli, and whose product is equal to one. As the eigenlines of a power of g coincide with eigenlines of g, and the map

$$\mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R} \setminus \{0\} : \mu \longmapsto \mu^n$$

is open, the proof is complete. ☐

Proposition 3.4. — *Let Γ be a non-abelian free group and $\rho : \Gamma \to \mathsf{SL}_3(\mathbb{R})$ be a representation. Suppose that Γ admits a free generating set containing two elements a and b with the following properties:*

(1) *there exist non zero integers m and n so that the word $\omega := a^m b^n$ satisfies that for some non zero integer q and some $\mu \neq \pm 1$ the subspace $P_0 := \ker(\rho(\omega^q) - \mu)$ is two-dimensional. Moreover, the subspace $L_0 := \ker(\rho(\omega^q) - \mu^{-2})$ is one-dimensional.*

(2) *The matrices $\rho(a)$ and $\rho(b)$ are loxodromic and L_0 is contained either in the attracting plane $E^{cu}(\rho(a))$ of $\rho(a)$, or in the repelling one $E^{cs}(\rho(a))$.*

Then there exists a continuous path $\rho_t : \Gamma \to \mathsf{SL}_3(\mathbb{R})$ with $\rho_0 = \rho$ and a sequence $t_k \to 0$ such that for every k the element

$$\rho_{t_k}([\omega^q, a^{p_k} \omega^q a^{-p_k}])$$

is unipotent for some non zero integer p_k.

Proof. — Without loss of generality we assume that L_0 belongs to $E^{cu}(\rho(a))$.

Let us first observe that by taking an arbitrarily small perturbation of ρ if needed we may assume, in addition to the current hypotheses, that $E^c(\rho(a))$ is different from L_0 and $E^s(\rho(a))$ is not contained in P_0. Indeed, there exist arbitrarily small neighborhoods U of $\rho(b)$ so that $U^{(n)}$ is a neighborhood of $\rho(b^n)$ (Lemma 3.3). We may take an arbitrarily small perturbation $\rho'(a)$ of $\rho(a)$ so that $E^c(\rho'(a))$ is different from L_0, $E^s(\rho(a))$ does not belong to P_0, L_0 belongs to $E^{cu}(\rho'(a))$, and

$$\rho'(a)^{-m} \rho(a^m) \rho(b^n) \in U^{(n)}.$$

In particular, there is some $\rho'(b) \in U$ so that

$$\rho'(a)^{-m} \rho(a^m) \rho(b^n) = \rho'(b)^n.$$

By keeping all other generators of Γ fixed, this defines a perturbation ρ' of ρ for which $\rho'(\omega) = \rho(\omega)$, and ρ' is still in the hypotheses of the proposition, with the extra claimed genericity assumptions on $E^c(\rho'(a))$ and $E^s(\rho'(a))$. To lighten notations, we assume the representation ρ that we started with also satisfies these conditions.

Now fix any continuous path $t \mapsto \rho_t(a)$ passing through $\rho(a)$ at $t = 0$, which is constructed only deforming eigenline $E^u(\rho(a))$ in such a way that $t \mapsto E^u(\rho_t(a))$ intersects transversely $E^{cu}(\rho(a))$ precisely at $t = 0$. As above, we may find a continuous path $t \mapsto \rho_t(b)$ passing through $\rho(b)$ at $t = 0$ and such that $\rho_t(\omega) = \rho(\omega)$ for all t.

Fix a unit vector $v \in L_0$ and let $t \mapsto \theta_t \in (\mathbb{R}^3)^* \setminus \{0\}$ be a continuous path so that

$$\ker \theta_t = E^{cu}(\rho_t(a))$$

for all t. As $E^c(\rho(a)) = E^c(\rho_t(a)) \neq L_0$ and $L_0 \in E^{cu}(\rho(a))$, by our transversality assumption we may find arbitrarily small $t_- < 0 < t_+$ so that

$$\theta_{t_-}(v) \cdot \theta_{t_+}(v) < 0.$$

Without loss of generality we may assume

$$\theta_{t_-}(v) < 0 < \theta_{t_+}(v). \tag{3.1}$$

On the other hand, for every positive integer p we may take a continuous path $t \mapsto \theta_{t,p} \in (\mathbb{R}^3)^* \setminus \{0\}$ so that

$$\ker \theta_{t,p} = \rho_t(a^p) \cdot P_0$$

holds for all t. Note that, as $E^s(\rho(a)) = E^s(\rho_t(a)) \not\subset P_0$, for a given t we have

$$\rho_t(a^p) \cdot P_0 \longrightarrow E^{cu}(\rho_t(a)),$$

as $p \to \infty$. In particular, we may assume that $\theta_{t,p}$ is chosen in such a way that for every t one has

$$\theta_{t,p} \longrightarrow \theta_t$$

as $p \to \infty$. By Equation (3.1) we may find a large enough p such that

$$\theta_{t_-,p}(v) < 0 < \theta_{t_+,p}(v).$$

There exists then some $t_- < t_0 < t_+$ such that $\theta_{t_0,p}(v) = 0$ or, in other words,

$$L_0 \subset \rho_{t_0}(a^p) \cdot P_0.$$

In particular, $\rho_{t_0}(\omega^q) = \rho(\omega^q)$ preserves the hyperplane $\rho_{t_0}(a^p) \cdot P_0$, because it preserves L_0 and $P_0 \cap \rho_{t_0}(a^p) \cdot P_0$.

On the other hand, the element $\rho_{t_0}(a^p \omega^q a^{-p})$ also preserves $\rho_{t_0}(a^p) \cdot P_0$ and furthermore acts as a scalar multiple of the identity on it. Hence, the commutator $\rho_{t_0}([\omega^q, a^p\omega^q a^{-p}])$ acts as the identity on it. As $\rho_{t_0}([\omega^q, a^p\omega^q a^{-p}])$ has determinant equal to 1, it is a unipotent element. ☐

4. Examples of quasi-isometric free groups in $\mathsf{SL}_3(\mathbb{R})$

We now discuss examples of quasi-isometric representations into $\mathsf{SL}_3(\mathbb{R})$. An important class is given by Anosov representations, for which we refer the reader to the surveys [Kas18, Wie18]. However, in this paper we are focusing on examples which are not Anosov, and not even limits of Anosov representations. We also focus on free groups, but many of the previously known results that we will quote here apply for more general groups.

4.1. Reducible examples

In this subsection we briefly review a special case of recent results by Lahn [Lah23]. Let $\rho : \Gamma \to \mathsf{SL}_3(\mathbb{R})$ be a reducible suspension. Recall that by definition this means that ρ preserves some hyperplane P. Up to conjugacy, we may write

$$\rho(\gamma) = \begin{pmatrix} e^{-\frac{1}{2}\varphi(\gamma)}\rho_P(\gamma) & \kappa(\gamma) \\ 0 & \pm e^{\varphi(\gamma)} \end{pmatrix}, \tag{4.1}$$

for some morphisms $\rho_P : \Gamma \to \mathsf{SL}_2^{\pm}(\mathbb{R})$ and $\varphi : \Gamma \to \mathbb{R}$, and some $\kappa : \Gamma \to \mathbb{R}^2$. We recall that ρ is said to be *derived from Barbot* if ρ_P is quasi-isometric.

Lahn [Lah23, Proposition 3.4] proves that if ρ is derived from Barbot, then it is necessarily quasi-isometric (see also Lemma 5.1 below). Moreover, Lahn characterizes which reducible suspensions are Anosov:

Theorem 4.1 (Lahn [Lah23, Theorem 2]). — *Let $\rho : \Gamma \to \mathsf{SL}_3(\mathbb{R})$ be a reducible suspension. Then ρ is Anosov if and only if*

$$\inf_{\gamma:\varphi(\gamma)\neq 0} \frac{\log(\lambda_u(\rho_P(\gamma)))}{|\varphi(\gamma)|} > \frac{3}{2},$$

where $\lambda_u(\rho_P(\gamma))$ denotes the largest modulus among the eigenvalues of $\rho_P(\gamma)$.

Building on Theorem 4.1 it is easy to construct examples of quasi-isometric reducible suspensions which are not accumulated by Anosov representations.

4.2. Constructing free groups

We now turn to non reducible examples, proving Theorem 1.5. The construction involves several intermediate elementary steps.

4.2.1. A modified ping-pong criterion

We start by giving a criterion which allows us to prove that a subgroup of $\mathsf{SL}_3(\mathbb{R})$ is freely generated and quasi-isometrically embedded. A similar variant of the ping-pong lemma may be found in Witte Morris [Mor15, Lemma 4.9.6].

We say a finite symmetric subset $F \subset \mathsf{SL}_3(\mathbb{R})$ satisfies condition $*$ if there exists $c > 1$, a one dimensional subspace L_0, and for each $g \in F$ a subset $C_g \subset \mathbb{RP}^2$ such that

(1) $C_g \cap C_h = \emptyset$ if $g \neq h$,
(2) $L_0 \notin \bigcup_{g\in F} C_g$,
(3) $g \cdot L_0 \in C_g$ for all $g \in F$,
(4) $g \cdot C_h \subset C_g$ whenever $g \neq h^{-1}$, and
(5) $\|g|_L\| \geqslant c$ for all $L \in \bigcup_{h\neq g^{-1}} C_h$.

Lemma 4.2 (Ping-pong variant). — *If $F \subset \mathsf{SL}_3(\mathbb{R})$ is finite, symmetric, and satisfies condition $*$ then F freely generates a free subgroup which is quasi-isometrically embedded in $\mathsf{SL}_3(\mathbb{R})$.*

Proof. — Let $n \geqslant 1$ and $g_1, \dots, g_n \in F$ be such that $g_k \neq g_{k+1}^{-1}$ for $k = 1, \dots, n-1$. We first prove that F freely generates a free group. To see this, observe that $g_n \cdot L_0 \in C_{g_n}$, and by induction it follows that $(g_1, \dots, g_n) \cdot L_0 \in C_{g_1}$. In particular this implies $(g_1, \dots, g_n) \cdot L_0 \neq L_0$ so that $g_1, \dots, g_n \neq 1$ as required.

We now show that the group generated by F is quasi-isometrically embedded. For this purpose we observe that for $k = 1, \dots, n-1$, since $(g_{k+1}, \dots, g_n) \cdot L_0 \in C_{g_{k+1}}$ and $g_k \neq g_{k+1}^{-1}$ we have $\|g_k|_{(g_{k+1},\dots,g_n)\cdot L_0}\| \geqslant c$. This implies

$$s_1(g_1, \dots, g_n) = \max_{L\in\mathbb{RP}^2} \|(g_1, \dots, g_n)|_L\| \geqslant \|(g_1, \dots, g_n)|_{L_0}\| \geqslant c^{n-1},$$

which concludes the proof since $c > 1$. ☐

4.2.2. Examples verifying condition $*$

We now produce a class of examples satisfying condition $*$. For this purpose we consider elements $g \in \mathsf{SL}_3(\mathbb{R})$ such that for some $\mu > 1$ and $m \in \mathbb{N}$ one has that $P_0 := \ker(g^m - \mu)$ is two-dimensional and $L_0 := \ker(g^m - \mu^{-2})$ is one-dimensional. For such an element and for a given finite g-invariant set $X \subset \mathbb{RP}^2$ whose elements are contained in P_0, we say that a set $A \subset \mathbb{RP}^2$ is a *prepared neighborhood of* X if X is contained in the interior of A, $g(A) = A$, and $A = \bigcup_{x \in A} x \oplus L_0$. One has the following.

Proposition 4.3. — *Let $g \in \mathsf{SL}_3(\mathbb{R})$ be as above and let $\mathcal{A} = \mathcal{A}_X$ be the set of prepared neighborhoods of a given finite subset $X \subset \mathbb{RP}^2$. Then $A_1 \cap \cdots \cap A_n \in \mathcal{A}$ for all $A_1, \ldots, A_n \in \mathcal{A}$ and $\bigcap_{A \in \mathcal{A}} A = \bigcup_{x \in X} x \oplus L_0$.*

big

Lemma 4.4. — *Let $f, g \in \mathsf{SL}_3(\mathbb{R})$ be such that:*
1. *$E^u, E^c, E^s \in \mathbb{RP}^2$ of f with respective eigenvalues of moduli $\lambda_u > \lambda_c > \lambda_s$.*
2. *There exist a positive integer m and $\mu > 1$ such that $P_0 := \ker(g^m - \mu)$ is two-dimensional and $L_0 := \ker(g^m - \mu^{-2})$ is one-dimensional.*
3. *The above subspaces satisfy $E^u, E^c, E^s \not\subset P_0$, $L_0 \not\subset E^{cu} := E^u \oplus E^c$, and $L_0 \not\subset E^{cs} := E^s \oplus E^c$.*
4. *For all $0 \leqslant k < m$ one has $g^k((E^u \oplus L_0) \cap P_0) \not\subset E^{cs} \cup E^{cu}$, and $g^k((E^s \oplus L_0) \cap P_0) \not\subset E^{cs} \cup E^{cu}$.*

Then there exists a positive integer n_0 such that $F_n := \{f^n, f^{-n}, g^n, g^{-n}\}$ satisfies condition $*$ for all $n > n_0$.

Proof. — Let $\angle(V, W)$ denote the minimal angle between two subspaces $V, W \subset \mathbb{R}^3$ when either V or W is one dimensional. For $r > 0$ and a subset $X \subset \mathbb{RP}^2$ we denote by $B(X, r)$ the set of $L \in \mathbb{RP}^2$ with $\angle(L', L) < r$ for some $L' \in X$.

Define

$$X := \bigcup_{k=0}^{m-1} g^k((E^u \oplus L_0) \cap P_0) \cup g^k((E^s \oplus L_0) \cap P_0) \subset \mathbb{RP}^2,$$

which by hypothesis is a finite set of lines contained in P_0 that is disjoint from $E^{cs} \cup E^{cu}$.

We consider the family $\mathcal{A} = \mathcal{A}_X$ of prepared neighborhoods of X (with respect to g). By Proposition 4.3 we may choose $A \in \mathcal{A}$ such that for some $\varepsilon_0 > 0$ the set $A \cap P_0$ is disjoint from $B(E^{cs} \cup E^{cu}, \varepsilon_0)$.

For $0 < \varepsilon < \varepsilon_0$ which will be specified later we define

$$C_g := A \cap B(P_0, \varepsilon), \qquad C_{g^{-1}} := A \cap B(L_0, \varepsilon)$$

and

$$C_f := B(E^u, \varepsilon), \qquad C_{f^{-1}} := B(E^s, \varepsilon).$$

By Hypothesis (3), if $\varepsilon > 0$ is small enough then the four sets above are pairwise disjoint. Moreover as A is g-invariant, by Hypothesis (2) we have

$$g^n \cdot (A \setminus C_{g^{-1}}) \subset C_g \quad \text{and} \quad g^{-n} \cdot (A \setminus C_g) \subset C_{g^{-1}}$$

for all n large enough.

Since A is a prepared neighborhood it contains a neighborhood of each point in $E^s \oplus L_0 \setminus \{L_0\}$, and similarly for $E^u \oplus L_0$. Therefore, by picking $\varepsilon > 0$ smaller if necessary, we may assume

$$C_f \cup C_{f^{-1}} \subset A.$$

Let $F := \{f, f^{-1}, g, g^{-1}\}$. By Hypothesis (2), we obtain $g^n \cdot (C_h) \subset C_g$ and

$$\inf_{L \in C_h} \|g^n|_L\| > 2$$

for all $h \neq g^{-1}$ and n large enough. Similarly, $g^{-n} \cdot (C_h) \subset C_{g^{-1}}$ and

$$\inf_{L \in C_h} \|g^n|_L\| > 2$$

for all $h \neq g$ and n large enough.

On the other hand, since $B(E^{cs}, \varepsilon_0) \cap (A \cap P_0) = \emptyset$, we may choose $\varepsilon > 0$ smaller if needed so that $C_g \cap B(E^{cs}, \varepsilon) = \emptyset$. By Hypothesis (3), we may also assume

$$C_f \cap B(E^{cs}, \varepsilon) = \emptyset \quad \text{and} \quad C_{g^{-1}} \cap B(E^{cs}, \varepsilon) = \emptyset.$$

Hence, for all n large enough and $h \in \{g, g^{-1}, f\}$ we have $f^n \cdot (C_h) \subset C_f$ and

$$\inf_{L \in C_h} \|f^n|_L\| > 2.$$

Similarly, one has $f^{-n} \cdot (C_h) \subset C_{f^{-1}}$ and

$$\inf_{L \in C_h} \left\| f^{-n} \right|_L \left\| > 2\right.$$

for $h \in \{f^{-1}, g, g^{-1}\}$ and all n large enough.

To conclude property $*$ is suffices to choose $L_0 \in A \cap \partial C_{g^{-1}}$ and set $c := 2$. $\square$

4.3. The basic example in two generators

From the general machinery of the previous section we can produce a basic example of a Zariski dense quasi-isometric representation of a free group in two generators into $\mathsf{SL}_3(\mathbb{R})$, which is not a limit of Anosov representations. We also show that arbitrary small perturbations of this example act minimally on the space of full flags $\mathcal{F}$ of $\mathbb{R}^3$.

Indeed, let

$$g = \begin{pmatrix} 2 & -2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & \frac{1}{8} \end{pmatrix}$$

and f be loxodromic chosen so that the hypotheses of Lemma 4.4 are satisfied. By Lemmas 4.2 and 4.4 we may take an odd positive integer n such that f^n, g^n freely generate a quasi-isometrically embedded free group Γ.

Let $\rho : \Gamma \to \mathsf{SL}_3(\mathbb{R})$ be the inclusion. We first claim that ρ is not accumulated by Anosov representations. To see this observe that, since n is odd, the only real eigenvalue of g^n is $\frac{1}{8^n}$. In particular if $\rho' : \Gamma \to \mathsf{SL}_3(\mathbb{R})$ is sufficiently close to the inclusion then $h = \rho'(g^n)$ also has a unique real eigenvalue and therefore h^k has two complex eigenvalues with equal modulus for all k. This implies that ρ' is not Anosov.

We will now prove that f can be chosen so that $\rho(\Gamma)$ is Zariski dense. For this, we apply a well-known criterion asserting that a subgroup of $\mathsf{SL}_d(\mathbb{R})$ is Zariski dense if and only if its adjoint action on the Lie algebra of $\mathsf{SL}_d(\mathbb{R})$ leaves no subspace invariant. This is quite direct in our case, as the adjoint action of a loxodromic element $f \in \mathsf{SL}_d(\mathbb{R})$ is diagonalizable thus, an invariant subspace of the adjoint action must be a direct sum of eigenspaces of the adjoint matrix Ad_f. Therefore, in our example we may consider f and g so that Ad_f and $\mathrm{Ad}_{gfg^{-1}}$ are diagonalizable in bases which are in general position, to conclude that the adjoint actions of f and gfg^{-1} do not share any invariant subspace.

This shows that the inclusion $\rho : \Gamma \to \mathsf{SL}_3(\mathbb{R})$ has Zariski dense image.

Finally, we show that ρ may be perturbed to act minimally on the space of full flags $\mathcal{F}$. Indeed, let $P_0 := \mathrm{span}\{e_1, e_2\}$ and $L_0 := \mathrm{span}\{e_3\}$, which are the invariant subspaces of g. Define the representation ρ' by letting $\rho'(f) := \rho(f)$ and $\rho'(g)$ preserve L_0 and P_0, but in such a way that $\rho'(g^n)|_{P_0}$ is not a scalar multiple of the identity for any integer $n \neq 0$. By Dey–Hurtado [DH24, Proposition 5.4], ρ' induces a minimal action on $\mathcal{F}$.

4.4. Proof of Theorem 1.5

We now build on the previous example to prove Theorem 1.5. We will need the following abstract lemma.

Lemma 4.5. — *Let Γ be the non-abelian free group in the generators a and b. Then for every $k > 2$ there exist a finite index subgroup $\Gamma_k \subset \Gamma$ and a free generating set $F_k = \{c_1, \dots, c_k\}$ of Γ_k such that*

$$c_1 = a,\ c_2 = b^2, \quad \textit{and} \quad c_3 = ba^pb^{-1},$$

for some positive integer p.

Proof. — We identify Γ with the fundamental group $\pi_1(X, x_0)$ of a wedge sum X of two oriented circles, glued about the base-point x_0, and labelled with letters a and b respectively. One may now construct a $(k-1)$-sheeted regular cover $\Pi : \widetilde{X} \to X$ together with a lifted base-point $\widetilde{x}_0 \in \Pi^{-1}(x_0)$ in such a way that $\pi_1(\widetilde{X}, \widetilde{x}_0)$ is freely generated by $c_1, \dots, c_k$ and for some $p \geqslant 1$ one has

$$\prod_*(c_1) = a,\ \prod_*(c_2) = b^2, \quad \text{and} \quad \prod_*(c_3) = ba^pb^{-1},$$

see Figure 4.1. This finishes the proof. □

We are now ready to construct our examples. Consider the free group $\langle f, g\rangle$ where f and g are as in Subsection 4.3, and let $\rho : \langle f, g\rangle \to \mathsf{SL}_3(\mathbb{R})$ be the inclusion representation. Fix $k > 2$ and Γ_k and F_k as in Lemma 4.5. The identification $\Gamma \cong \langle f, g\rangle$ given by

$$a \longmapsto g,\ b \longmapsto f$$

induces a quasi-isometric representation $\Gamma \to \mathsf{SL}_3(\mathbb{R})$ whose restriction to Γ_k we denote by ρ_k. By Lemma 3.1, ρ_k is quasi-isometric. Moreover, as $g \in \rho_k(\Gamma_k)$, the

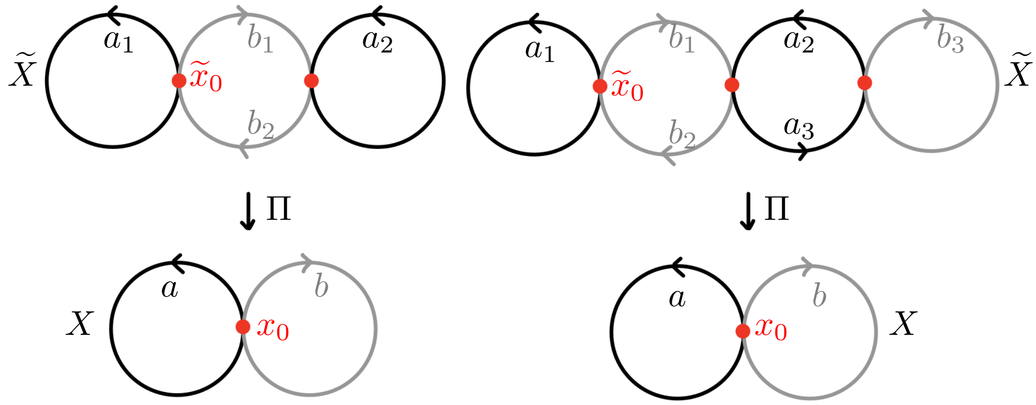

Figure 4.1. Proof of Lemma 4.5. On the left, the case $k = 3$. The loop a has two lifts a_1 and a_2 which are loops, while b lifts to two non-closed paths b_1 and b_2. The fundamental group $\pi_1(\widetilde{X}, \widetilde{x}_0)$ is freely generated by the homotopy classes of the concatenated paths a_1, $b_1 \star b_2$, and $b_1 \star a_2 \star \overline{b}_1$ (where $\overline{\beta}$ denotes the path β travelled with the opposite orientation). In this case, one may take $p = 1$. On the right, the case $k = 4$. Each generator in $\{a, b\}$ has two open lifts, and a closed one. A free generator of $\pi_1(\widetilde{X}, \widetilde{x}_0)$ is induced by a_1, $b_1 \star b_2$, $b_1 \star a_3 \star a_2 \star \overline{b}_1$, and $b_1 \star a_3 \star b_3 \star \overline{a}_3 \star \overline{b}_1$, so $p = 2$ in this case.

argument of Subsection 4.3 still applies to show that ρ_k is not a limit of Anosov representations. Observe moreover that ρ_k can be chosen to be Zariski dense.

To finish, we show that arbitrarily small perturbations of ρ_k contain unipotent elements, and therefore ρ_k is not robustly quasi-isometric (recall Lemma 3.2). These perturbations are constructed in two steps.

First, we consider an arbitrary small perturbation ρ'_k of ρ_k similar to what we did in Subsection 4.3. Namely, we let $\rho'_k(c_1)$ to act minimally on the projective line associated to P_0 and take $\rho'_k(c_i) := \rho_k(c_i)$ for all $i = 2, \ldots, k$. In particular, the induced ρ'_k-action of $\langle c_1, c_2\rangle \subset \Gamma_k$ on the Grassmannian $\mathcal{G}_2(\mathbb{R}^3)$ of two-dimensional subspaces of $\mathbb{R}^3$ is minimal.

On the other hand, observe that there is some positive integer q such that $\rho'_k(c_3^q)$ preserves a hyperplane $\widehat{P}$ on which it acts as a scalar multiple of the identity. We may consider a sequence $\{\gamma_n\} \subset \langle c_1, c_2\rangle$ such that

$$\rho'_k(\gamma_n) \cdot P_0 \longrightarrow \widehat{P}$$

as $n \to \infty$. In particular, we may find an arbitrary small perturbation ρ''_k of ρ'_k with $\rho''_k(c_i) = \rho'_k(c_i)$ for all $i \neq 3$ and such that, for an appropriate large enough n_0, the element $\rho''_k(c_3^q)$ preserves

$$\rho''_k(\gamma_{n_0}) \cdot P_0 = \rho'_k(\gamma_{n_0}) \cdot P_0$$

and acts as a scalar multiple of the identity on it. The commutator

$$\left[c_3^q, \gamma_{n_0} c_1 \gamma_{n_0}^{-1}\right]$$

has then unipotent image under ρ''_k.

5. Derived from Barbot representations

In this section we focus on reducible suspensions $\rho : \Gamma \to \mathsf{SL}_3(\mathbb{R})$, where Γ is a non abelian free group. We will use throughout the notations from Equation (4.1). Our main goal is to prove Theorem 1.7, for which we first need better understanding of which reducible suspensions are quasi-isometric (Proposition 1.6).

5.1. Proof of Proposition 1.6

The proof of Proposition 1.6 is split into Lemma 5.1 and Corollaries 5.3 and 5.4 below.

Lemma 5.1. — *Let Γ be a finitely generated non-abelian free group and $\rho : \Gamma \to \mathsf{SL}_3(\mathbb{R})$ be a reducible suspension. If ρ is derived from Barbot, then it is quasi-isometric.*

This is a special case of Lahn [Lah23, Proposition 3.4], we include a proof for completeness.

Proof of Lemma 5.1. — Let $\|\cdot\|$ denote the standard Euclidean norm on $\mathbb{R}^3$. By abuse of notations, we use the same symbol for the induced norm on P and for the corresponding operator norms. We need to show that $\|\rho(\gamma)\| \geqslant Be^{A|\gamma|}$ for all $\gamma \in \Gamma$ and some $A > 0$ and $B \leqslant 1$ (c.f. Equation (1.3)).

Now recalling Equation (4.1) and by definition of $\|\rho(\gamma)\|$ we have

$$\|\rho(\gamma)\| \geqslant \max\left\{e^{-\frac{1}{2}\varphi(\gamma)}\|\rho_P(\gamma)\|,\ e^{\varphi(\gamma)}\right\}$$

for all $\gamma \in \Gamma$. As ρ_P is quasi-isometric, Equation (1.3) applied to ρ_P gives

$$\|\rho(\gamma)\| \geqslant \max\left\{Be^{-\frac{1}{2}\varphi(\gamma)+A|\gamma|},\ e^{\varphi(\gamma)}\right\}$$

for all $\gamma \in \Gamma$ and some $A > 0$ and $B \leqslant 1$. Up to changing A by a smaller constant if necessary, we have $\|\rho(\gamma)\| \geqslant Be^{A|\gamma|}$ for all $\gamma \in \Gamma$ as desired. ☐

We now turn to the converse statement (Corollary 5.3 below).

Lemma 5.2. — *Let Γ be a finitely generated non-abelian free group and $\rho : \Gamma \to \mathsf{SL}_3(\mathbb{R})$ be a reducible suspension preserving some hyperplane P. If ρ is quasi-isometric, then for every $1 \neq \gamma \in \Gamma$ the element $\rho_P(\gamma)$ is hyperbolic.*

Proof. — We keep the notations from Equation (4.1) and suppose by contradiction that $-2 \leqslant \operatorname{tr}(\rho_P(\gamma)) \leqslant 2$ for some $\gamma \neq 1$. We fix any element $\eta \in \Gamma$ not belonging to the largest cyclic subgroup of Γ containing γ. We arrive to the desired contradiction by a case-by-case analysis.

On the one hand, we first note that $\rho_P(\gamma) \neq \pm\,\mathrm{id}$, otherwise the matrix $\rho([\gamma, \eta])$ would be unipotent contradicting Lemma 3.2. Hence, it only remains to rule out the possibility of $\rho_P(\gamma)$ being either parabolic or elliptic. We only treat the first case, the second one can be handled analogously.

Suppose then that $\mathrm{tr}(\rho_P(\gamma)) = \pm 2$ and $\rho_P(\gamma) \neq \pm\,\mathrm{id}$. Up to replacing γ by γ^2 if needed, we may assume $\mathrm{tr}(\rho_P(\gamma)) = 2$. Hence, we may replace ρ by a conjugate representation in order to have

$$\rho(\gamma) = \begin{pmatrix} e^{-\frac{1}{2}\varphi(\gamma)}\left(\begin{smallmatrix} 1 & \pm 1 \\ 0 & 1 \end{smallmatrix}\right) & \kappa(\gamma) \\ 0 & e^{\varphi(\gamma)} \end{pmatrix}.$$

Note then that $\varphi(\gamma)$ must be different from 0, otherwise we get a contradiction with Lemma 3.2. We write

$$\rho(\gamma) = \begin{pmatrix} \tau \cdot g & 0 \\ 0 & \tau^{-2} \end{pmatrix}$$

for simplicity, where $\tau := e^{-\frac{1}{2}\varphi(\gamma)}$ and $g := \left(\begin{smallmatrix} 1 & \pm 1 \\ 0 & 1 \end{smallmatrix}\right)$. Up to replacing γ by γ^{-1}, we may assume $\tau < 1$.

We now claim that the norm $\|\rho(\gamma^n \eta \gamma^{-n})\|$ does not grow exponentially as $n \to \infty$, which will give the desired contradiction. Indeed, we have

$$\rho(\eta) = \begin{pmatrix} \delta \cdot h & \kappa \\ 0 & \pm\delta^{-2} \end{pmatrix},$$

for some $h \in \mathsf{SL}_2^{\pm}(\mathbb{R})$, a real number $\delta > 0$, and a vector $\kappa \in \mathbb{R}^2$. Hence,

$$\rho\Big(\gamma^n \eta \gamma^{-n}\Big) = \begin{pmatrix} \delta \cdot g^n h g^{-n} & \tau^{3n} g^n \cdot \kappa \\ 0 & \pm\delta^{-2} \end{pmatrix}.$$

As the norms $\|g^n h g^{-n}\|$ and $\|g^n\|$ grow at most polynomially as $n \to \infty$ and $\tau < 1$, the proof is complete. ☐

Corollary 5.3. — *Let Γ be a finitely generated non-abelian free group and $\rho : \Gamma \to \mathsf{SL}_3(\mathbb{R})$ be a reducible suspension. If ρ is quasi-isometric, then it is derived from Barbot.*

Proof. — We need to show that ρ_P is quasi-isometric, for which we first show that it is faithful and discrete (compare with Lahn [Lah23, Proposition 3.1]). Indeed, Lemma 5.2 directly implies that ρ_P is faithful. With this at hand we can show that it is discrete. Otherwise, $\rho_P(\Gamma)$ would contain an elliptic element (c.f. the proof of Proposition 2.2), and this would contradict again Lemma 5.2.

Finally, by Lemma 3.1 in order to show that ρ_P is quasi-isometric it suffices to show that its restriction to

$$\Gamma_0 := \{\gamma \in \Gamma : \rho_P(\gamma) \in \mathsf{SL}_2(\mathbb{R})\}$$

is quasi-isometric. But note that the representation $\rho_P|_{\Gamma_0} : \Gamma_0 \to \mathsf{SL}_2(\mathbb{R})$ is faithful and discrete, and by Lemma 5.2 does not contain parabolic elements. As in the proof of Proposition 2.2, this implies that $\rho_P|_{\Gamma_0}$ is convex co-compact. ☐

To finish the proof of Proposition 1.6 we show the following.

Corollary 5.4. — *Let Γ be a finitely generated non-abelian free group and $\rho : \Gamma \to \mathsf{SL}_3(\mathbb{R})$ be a reducible suspension preserving some hyperplane P. Then ρ is derived from Barbot if and only if ρ is robustly quasi-isometric among representations preserving P.*

Proof. — This follows from Lemma 5.1 and Corollary 5.3, and the fact that quasi-isometric representations into $\mathsf{SL}^{\pm}(P)$ form a stable class. Indeed, every small perturbation of ρ preserving P induces a small perturbation of ρ_P and conversely. $\square$

5.2. Finding eigenvalues with repeated moduli

The goal of this subsection is to prove Proposition 5.5 below, which together with Proposition 3.4 is the main step in the proof of Theorem 1.7.

For $g \in \mathsf{SL}_3(\mathbb{R})$ preserving the hyperplane P we let

$$\lambda_1(g) \geqslant \lambda_2(g)$$

be the moduli of the eigenvalues of g restricted to P. We also let

$$\lambda_\perp(g) := \frac{1}{\lambda_1(g) \cdot \lambda_2(g)}$$

be the modulus of the complementary eigenvalue of g. Observe that $\lambda_\perp$ is a group morphism: if $h \in \mathsf{SL}_3(\mathbb{R})$ also preserves P, then

$$\lambda_\perp(gh) = \lambda_\perp(g) \cdot \lambda_\perp(h).$$

Note also that if $\lambda_1(g) > \lambda_2(g)$ then all the eigenvalues of g are real and the corresponding Jordan blocks have size at most two.

Proposition 5.5. — *Suppose that Γ is a non-abelian free group of rank $k \geqslant 2$, and let $\rho : \Gamma \to \mathsf{SL}_3(\mathbb{R})$ be a derived from Barbot representation preserving the hyperplane P. Assume moreover that ρ_P preserves the orientation of P. Suppose that ρ is not Anosov and fix some neighborhood $\mathscr{U}$ of ρ. Then there exists $\rho' \in \mathscr{U}$ preserving P, and free generators $a = c_1, b = c_2, c_3, \ldots, c_k$ of Γ such that the following holds:*

(1) *there exist non-zero integers n and m so that $\rho'(a^m b^n)$ satisfies*

$$\lambda_1(\rho'(a^m b^n)) = \lambda_\perp(\rho'(a^m b^n)) \neq 1.$$

(2) *The matrices $\rho'(a)$ and $\rho'(b)$ are loxodromic, and the eigenline L_0 corresponding to $\lambda_2(\rho'(a^m b^n))$ is contained in the repelling hyperplane $E^{cs}(\rho'(a))$.*

The proof of Proposition 5.5 involves several intermediate results. Roughly, the outline is the following. We first prove in Proposition 5.9 that Lahn's Theorem 4.1 implies that, after possibly perturbing ρ slightly inside the set of representations preserving P, there is a free generating set of Γ containing an element a so that

$$\lambda_2(\rho(a)) < \lambda_1(\rho(a)) < \lambda_\perp(\rho(a)).$$

We then show in Proposition 5.10 that we may also assume that a complementary generating element b satisfies

$$\lambda_2(\rho(b)) < \lambda_\perp(\rho(b)) < \lambda_1(\rho(b)).$$

The fact that ρ is derived from Barbot together with a continuity argument will then prove the statement.

We assume throughout that Γ is a non abelian free group of rank $k \geqslant 2$, and fix a free generating ordered set $F_0 \subset \Gamma$, which will serve as a reference to construct the one in Proposition 5.5. This choice induces a morphism

$$\psi_0 : \Gamma \longrightarrow \mathbb{Z}^k,$$

by identifying the abelianization $H^1(\Gamma) := \Gamma/[\Gamma,\Gamma]$ of Γ with $\mathbb{Z}^k$. We will say that two elements in Γ are *homologous* if they have the same image under ψ_0.

We have the following properties of derived from Barbot representations that we will use all the time (recall the notations from Equation (4.1)).

Remark 5.6. — Suppose that $\rho : \Gamma \to \mathsf{SL}_3(\mathbb{R})$ is a derived from Barbot representation preserving P. Then:

(1) for all $\gamma \in \Gamma$ we have

$$\lambda_1(\rho(\gamma)) > \lambda_2(\rho(\gamma)).$$

(2) The largest modulus among the eigenvalues of $\rho_P(\gamma)$ is $\lambda_\perp^{\frac{1}{2}}(\rho(\gamma)) \cdot \lambda_1(\rho(\gamma))$. In particular, as ρ_P is quasi-isometric we have

$$\inf_{\gamma\in\Gamma} \lambda_\perp^{\frac{1}{2}}(\rho(\gamma)) \cdot \lambda_1(\rho(\gamma)) > 1.$$

(3) Let $\gamma_1, \dots, \gamma_s \in \Gamma$ be elements so that the eigenlines (in P) of the $\rho_P(\gamma_i)$ are all different. Then

$$\frac{\lambda_1(\rho(\gamma_1^{m_1}, \dots, \gamma_s^{m_s}))}{\lambda_1(\rho(\gamma_1))^{m_1}, \dots, \lambda_1(\rho(\gamma_s))^{m_s}}$$

converges to some positive constant only depending on the eigenlines of the $\rho_P(\gamma_i)$, as $m_1, \dots, m_s \to +\infty$ (see e.g. [BQ16, Lemma 7.5]).

(4) Fix some ε and choose a generator $c_0 \in F_0$. We may consider another derived from Barbot representation, which is obtained by *scaling the c_0-action on P by the factor ε*. More precisely, let $\rho_\varepsilon : \Gamma \to \mathsf{SL}_3(\mathbb{R})$ be the representation given by

$$\rho_\varepsilon(c_0) := \begin{pmatrix} e^{-\frac{1}{2}(\varphi(c_0)+\varepsilon)}\rho_P(c_0) & \kappa(c_0) \\ 0 & \pm e^{\varphi(c_0)+\varepsilon} \end{pmatrix},$$

and $\rho_\varepsilon(c) := \rho(c)$ for all $c \in F_0 \setminus \{c_0\}$.

For $\gamma \in \Gamma$ there is an integer $p = p(c_0, \gamma)$ which is the coordinate of $\psi_0(\gamma)$ associated to the generator c_0. One has

$$\lambda_\perp(\rho_\varepsilon(\gamma)) = e^{\varepsilon p} \cdot \lambda_\perp(\rho(\gamma)) \quad \text{and} \quad \lambda_1(\rho_\varepsilon(\gamma)) = e^{-\frac{1}{2}\varepsilon p} \cdot \lambda_1(\rho(\gamma)).$$

In particular,

$$\frac{\lambda_\perp(\rho_\varepsilon(\gamma))}{\lambda_1(\rho_\varepsilon(\gamma))} = e^{\frac{3}{2}\varepsilon p} \cdot \frac{\lambda_\perp(\rho(\gamma))}{\lambda_1(\rho(\gamma))}.$$

Here is a consequence of Lahn's Theorem 4.1 that we will need in the future.

Corollary 5.7. — *Let $\rho : \Gamma \to \mathsf{SL}_3(\mathbb{R})$ be a derived from Barbot representation preserving the hyperplane P. Suppose that ρ is not Anosov and fix some neighborhood $\mathscr{U}$ of ρ. Then there exists $\rho' \in \mathscr{U}$ preserving P and $\gamma \in \Gamma$ so that*

$$\lambda_1(\rho'(\gamma)) < \lambda_\perp(\rho'(\gamma)).$$

Proof. — By Theorem 4.1 we may find a sequence $\gamma_n \in \Gamma$ with $\varphi(\gamma_n) > 0$ and some $d \leqslant 1$ so that

$$\lim_{n\to+\infty} \frac{\log \lambda_1(\rho(\gamma_n))}{\log \lambda_\perp(\rho(\gamma_n))} = d.$$

If $d < 1$ there is nothing to prove, so we assume

$$\lim_{n\to+\infty} \frac{\log \lambda_1(\rho(\gamma_n))}{\log \lambda_\perp(\rho(\gamma_n))} = 1. \tag{5.1}$$

Observe that $\psi_0(\gamma_n) \neq 0$ for all n, as $\log \lambda_\perp(\rho(\gamma_n)) > 0$. The proof will be split into several cases.

First assume that the sequence $\{\log \lambda_\perp(\rho(\gamma_n))\}_n$ is bounded away from 0 and ∞. Up to replacing γ_n by appropriate conjugates if needed, this implies that the sequence $\{\gamma_n\}_n$ is bounded, and therefore it has some subsequence equal to some $\gamma \in \Gamma$. By Equation (5.1) we have then

$$\frac{\log \lambda_1(\rho(\gamma))}{\log \lambda_\perp(\rho(\gamma))} = 1.$$

As $\psi_0(\gamma) \neq 0$, we may scale the action of some element in F_0 to obtain the desired result (c.f. Remark 5.6).

Secondly, observe that by Equation (5.1) and Remark 5.6 no subsequence of $\{\log \lambda_\perp(\rho(\gamma_n))\}_n$ converges to 0. Hence, to finish the proof it remains to treat the case $\log \lambda_\perp(\rho(\gamma_n)) \to \infty$ as $n \to \infty$.

Write $\psi_0(\gamma_n) = (p_{1,n}, \dots, p_{k,n})$ for all n. Up to reordering the generator F_0 we may assume that $|p_{1,n}| \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\log \lambda_\perp(\rho(\gamma_n))}{p_{1,n}} = \alpha, \tag{5.2}$$

for some $\alpha \in \mathbb{R}$.

Up to taking a further subsequence if necessary, we may assume that there is some small ε so that $\varepsilon \cdot p_{1,n} > 0$ for all n. We now scale the action of the first generator of F_0 as in Remark 5.6. We obtain

$$\limsup_{n\to\infty} \frac{\log \lambda_1(\rho_\varepsilon(\gamma_n))}{\log \lambda_\perp(\rho_\varepsilon(\gamma_n))} \leqslant \limsup_{n\to\infty} \frac{\log \lambda_1(\rho(\gamma_n))}{\varepsilon \cdot p_{1,n} + \log \lambda_\perp(\rho(\gamma_n))},$$

which by Equations (5.1) and (5.2) is equal to

$$\frac{1}{\varepsilon/\alpha + 1} < 1.$$

This finishes the proof of Corollary 5.7. □

The result above suggests us to introduce the set

$$V_\rho := \{\gamma \in \Gamma : \lambda_1(\rho(\gamma)) < \lambda_\perp(\rho(\gamma))\},$$

where ρ is derived from Barbot. Note that

$$V_\rho^{-1} = \{\gamma \in \Gamma : \lambda_\perp(\rho(\gamma)) < \lambda_2(\rho(\gamma))\}.$$

Observe that V_ρ is invariant under conjugacy, and under taking positive powers. We also have the following, which says that decreasing the ρ_P-length of a curve keeping the homology class fixed keeps you inside V_ρ.

Lemma 5.8. — *Let $\rho : \Gamma \to \mathsf{SL}_3(\mathbb{R})$ be a derived from Barbot representation preserving the hyperplane P. Consider homologous elements $\gamma_0, \gamma_1 \in \Gamma$ so that $\gamma_0 \in V_\rho$ and*

$$\frac{\lambda_1(\rho(\gamma_1))}{\lambda_2(\rho(\gamma_1))} \leqslant \frac{\lambda_1(\rho(\gamma_0))}{\lambda_2(\rho(\gamma_0))}.$$

Then γ_1 belongs to V_ρ.

Proof. — Indeed, on the one hand as $\psi_0(\gamma_1) = \psi_0(\gamma_0)$ we have $\lambda_\perp(\rho(\gamma_1)) = \lambda_\perp(\rho(\gamma_0))$. Also, this fact together with the inequality in the statement implies

$$\lambda_2(\rho(\gamma_0)) \leqslant \lambda_2(\rho(\gamma_1)) < \lambda_1(\rho(\gamma_1)) \leqslant \lambda_1(\rho(\gamma_0)).$$

This proves the lemma. □

From the above lemma we manage to find generating elements in $V_{\rho'}$, for a small perturbation ρ' of ρ (this is the only place in our argument were we need to assume that the restriction ρ_P preserves the orientation of P).

Proposition 5.9. — *Let $\rho : \Gamma \to \mathsf{SL}_3(\mathbb{R})$ be a derived from Barbot representation preserving the hyperplane P. Assume moreover that ρ_P preserves the orientation of P. Suppose that ρ is not Anosov and fix some neighborhood $\mathscr{U}$ of ρ. Then there exist some $\rho' \in \mathscr{U}$ preserving P and a free generating set $F = \{a = c_1, c_2, \dots, c_k\} \subset \Gamma$ such that $a \in V_{\rho'}$.*

To prove Proposition 5.9 we look at the following useful object. Consider the convex co-compact (orientable) hyperbolic surface $S := \mathbb{H}^2/\rho_P(\Gamma)$. We let $C \subset S$ be the convex core, which is a compact hyperbolic surface with geodesic boundary containing all closed geodesics of S.

For a given non-zero $h \in H^1(\Gamma) = \Gamma/[\Gamma,\Gamma]$ we consider the infimum $\|h\|$ of lengths of multi-curves in C representing the homology class h. The Arzelá–Ascoli theorem implies that this is actually a minimum. Let m_h be a minimizing multi-curve. Its components are not all reduced to points, otherwise we would have $h = 0$. In particular, $\|h\| > 0$ and we may assume that all the components of m_h are (non constant) closed curves. Each of these curves are geodesics, as they minimize the length in its homotopy class. Further, these geodesics are simple (considered possibly with multiplicity) and their union is disjoint, c.f. McShane–Rivin [MR95, Theorem 4.1]. None of these simple geodesics is homologically trivial, otherwise it could be removed from m_h reducing the length.

Proof of Proposition 5.9. — By Corollary 5.7 we may assume that there is some $\gamma_0 \in V_\rho$. As a first step we prove that there is an element in V_ρ which induces a simple closed geodesic in C. Afterwards we show that this suffices to prove the statement.

Let us then prove the first claim, that is, we will find a simple geodesic in V_ρ. For this purpose, we consider the homology class $h := \psi_0(\gamma_0)$, which is non-zero as $\gamma_0 \in V_\rho$. In particular, if our element $\gamma_0 \in V_\rho$ is itself a minimizing multi-curve, then as it is a curve, it is simple and there is nothing to prove. As a consequence, to prove our first claim it remains to treat the case in which

$$\|\psi_0(\gamma_0)\| < \ell(\gamma_0),$$

where for $\gamma \in \Gamma$ we let

$$\ell(\gamma) := \log\left(\frac{\lambda_1(\rho(\gamma))}{\lambda_2(\rho(\gamma))}\right)$$

denote the length of the closed geodesic in C associated to $\gamma \in \Gamma$. We then have

$$\lim_{k\to+\infty} \ell\left(\gamma_0^k\right) - k \cdot \|\psi_0(\gamma_0)\| = \lim_{k\to+\infty} k \cdot \left(\ell(\gamma_0) - \|\psi_0(\gamma_0)\|\right) = +\infty.$$

On the other hand, let $\gamma_1, \ldots, \gamma_s$ be the (disjoint) simple closed geodesics supporting a length minimizing multi-curve in $\psi_0(\gamma_0)$ (considered possibly with multiplicity). For each positive integer k consider the curve $\widehat{\gamma}_k := \gamma_1^k, \ldots, \gamma_s^k$, which is homologous to γ_0^k. By Remark 5.6 we have

$$\limsup_{k\to+\infty}(\ell(\widehat{\gamma}_k) - k \cdot \|\psi_0(\gamma_0)\|) < +\infty.$$

By Lemma 5.8 we conclude that $\widehat{\gamma}_k \in V_\rho$ for every k large enough. That is,

$$1 > \frac{\lambda_1(\rho(\widehat{\gamma}_k))}{\lambda_\perp(\rho(\widehat{\gamma}_k))} = \frac{\lambda_1(\rho(\widehat{\gamma}_k))}{\lambda_1(\rho(\gamma_1))^k \ldots \lambda_1(\rho(\gamma_s))^k} \cdot \left(\frac{\lambda_1(\rho(\gamma_1))}{\lambda_\perp(\rho(\gamma_1))}\right)^k \cdots \left(\frac{\lambda_1(\rho(\gamma_s))}{\lambda_\perp(\rho(\gamma_s))}\right)^k$$

for all k large enough. Applying again Remark 5.6 we conclude that $\gamma_i \in V_\rho$ for some $i = 1, \ldots, s$, thus proving our first claim.

From the discussion above, we may assume that $\gamma_0 \in V_\rho$ induces in fact a simple (non homologically trivial) closed geodesic in $C \subset S$. As we now show this implies that γ_0 belongs to a free generating set of Γ, thus finishing the proof.

Indeed, this claim is a general fact about (orientable) topological surfaces of negative Euler characteristic. Let $g \geqslant 0$ be the genus of C and $n \geqslant 1$ be the number of boundary components. If γ_0 induces a boundary curve of C, then as it is non homologically trivial, we necessarily have $n \geqslant 2$. In this case γ_0 belongs to a free generating set of Γ and there is nothing to prove. More generally, if γ_0 is non separating, by classification of surfaces it also belongs to a free generating set. Hence we assume from now on that γ_0 separates C in two compact hyperbolic surfaces C_1 and C_2. In particular, Γ splits as the amalgamated product

$$\Gamma_1 \star_{\langle\gamma_0\rangle} \Gamma_2,$$

where Γ_i denotes the fundamental group of C_i.

We observe that both C_1 and C_2 have more than one boundary components, as γ_0 is not homologically trivial in C. Hence, for $i = 1, 2$ the group Γ_i admits a free generating set $F_i := \{a_1, b_1, \ldots, a_{g_i}, b_{g_i}, c_1 \ldots, c_{n_i}\}$ for some $g_i \geqslant 0$ and $n_i \geqslant 1$ so that

$$\gamma_0 = \left(\prod_{j=1}^{g_i}[a_j, b_j]\right) c_1, \ldots, c_{n_i}.$$

In particular,

$$F_i' := \{a_1, b_1, \ldots, a_{g_i}, b_{g_i}, c_1 \ldots, c_{n_i-1}, \gamma_0\}$$

is a free generating set of Γ_i. Hence, $F := F_1' \cup F_2'$ is a free generating set of Γ and contains $\gamma_0 \in V_\rho$. □

We now show that a complementary generating element can be assumed to be outside $V_{\rho'} \cup V_{\rho'}^{-1}$.

Proposition 5.10. — *Let $\rho : \Gamma \to \mathsf{SL}_3(\mathbb{R})$ be a derived from Barbot representation preserving the hyperplane P. Assume moreover that ρ_P preserves the orientation of P. Suppose that ρ is not Anosov and fix some neighborhood $\mathscr{U}$ of ρ. Then there exist some $\rho' \in \mathscr{U}$ preserving P and a free generating set $\{a = c_1, b = c_2, c_3, \dots, c_k\} \subset \Gamma$ such that*

$$a \in V_{\rho'} \quad \text{and} \quad b \notin V_{\rho'} \cup V_{\rho'}^{-1}.$$

Proof. — By Proposition 5.9, up to replacing ρ by a small perturbation if needed we may find a free generating set $F = \{a = c_1, c_2, \dots, c_k\}$ of Γ so that $a \in V_\rho$. If for some $i = 2, \dots, k$ we have that c_i does not belong to $V_\rho \cup V_\rho^{-1}$ there is nothing to prove, so we assume that this is not the case. Let $b := c_2$. We will construct an automorphism σ of Γ such that the generating set $\sigma(F)$ satisfies

$$\sigma(a) \in V_\rho \quad \text{and} \quad \sigma(b) \notin V_\rho \cup V_\rho^{-1}.$$

The construction is inductive. For the first step, note that up to replacing b by b^{-1} we may assume $b \in V_\rho$. In particular, both $\lambda_\perp(\rho(a))$ and $\lambda_\perp(\rho(b))$ are strictly larger than 1. Up to composing with the automorphism of Γ that permutes a with b and fixes all the other generators if needed, we may assume that

$$\tau(a) \leqslant \tau(b) < 1,$$

where $\tau : \Gamma \to \mathbb{R}_{>0}$ is the morphism given by

$$\tau(\gamma) := \lambda_\perp(\rho(\gamma))^{-1}.$$

This gives us an automorphism σ_1 such that both $\sigma_1(a)$ and $\sigma_1(b)$ belong to V_ρ, and moreover

$$\tau(\sigma_1(a)) \leqslant \tau(\sigma_1(b)) < 1.$$

Let now n be an integer $\geqslant 1$. We assume by induction that we have constructed an automorphism σ_n of Γ such that both $\sigma_n(a)$ and $\sigma_n(b)$ belong to V_ρ, $\tau(\sigma_n(a)) \leqslant \tau(\sigma_n(b)) < 1$, and moreover

$$\sigma_n(a) = \sigma_{n-1}(b).$$

We find a positive integer p_n so that

$$\tau(\sigma_n(b)) < \tau\Big(\sigma_n(ab^{-p_n})\Big) \leqslant 1,$$

and we define the automorphism σ_{n+1} of Γ by letting

$$\sigma_{n+1}(a) := \sigma_n(b) \quad \text{and} \quad \sigma_{n+1}(b) := \sigma_n\Big(ab^{-p_n}\Big),$$

and keeping fixed all the other generators. Note then that $\sigma_{n+1}(a) \in V_\rho$. Hence, if $\sigma_{n+1}(b) \notin V_\rho \cup V_\rho^{-1}$ we are done. If this is not the case, we continue the induction. We only have to show that the process stops at some finite step.

Suppose by contradiction that this is not the case. We get an infinite sequence of automorphisms $\{\sigma_n\}_{n\geqslant 1}$ such that for all n both $\sigma_n(a)$ and $\sigma_n(b)$ belong to V_ρ, $\tau(\sigma_n(a)) \leqslant \tau(\sigma_n(b)) < 1$, and $\sigma_{n+1}(a) = \sigma_n(b)$. Furthermore, by construction the sequence $\{\tau(\sigma_n(b))\}$ is strictly increasing and therefore converges. We claim that in fact converges to 1. Indeed, note that $\{\tau(\sigma_n(a))\}_{n\geqslant 1}$ converges to the same limit, and

$$\frac{\tau(\sigma_n(a))}{\tau(\sigma_n(b))} = \tau\Big(\sigma_n\Big(ab^{-p_n}\Big)\Big) \cdot \tau\Big(\sigma_n\Big(b^{p_n-1}\Big)\Big) \leqslant \tau(\sigma_{n+1}(b)),$$

for all n. This shows the claim.

We then have

$$\lim_{n\to\infty} \lambda_\perp(\rho(\sigma_n(b))) = 1.$$

But by Remark 5.6 we have

$$\limsup_{n\to\infty} \frac{\lambda_1(\rho(\sigma_n(b)))}{\lambda_\perp(\rho(\sigma_n(b)))} = \limsup_{n\to\infty} \frac{\lambda_\perp^{\frac{1}{2}}(\rho(\sigma_n(b))) \cdot \lambda_1(\rho(\sigma_n(b)))}{\lambda_\perp^{\frac{3}{2}}(\rho(\sigma_n(b)))} > 1.$$

We may then find some integer $n > 1$ such that

$$\frac{\lambda_1(\rho(\sigma_n(b)))}{\lambda_\perp(\rho(\sigma_n(b)))} > 1.$$

By construction we also have $\lambda_\perp(\rho(\sigma_n(b))) > 1$, and this shows that $\sigma_n(b) \notin V_\rho \cup V_\rho^{-1}$, contradicting our assumptions. $\square$

We can finally prove the desired result of this subsection.

Proof of Proposition 5.5. — By Proposition 5.10 there is a free generating set F of Γ and $a, b \in F$ so that, after possibly perturbing ρ inside $\mathscr{U}$, we have

$$\lambda_2(\rho(a)) < \lambda_1(\rho(a)) < \lambda_\perp(\rho(a)),$$

and

$$\lambda_2(\rho(b)) \leqslant \lambda_\perp(\rho(b)) \leqslant \lambda_1(\rho(b)).$$

In particular, $P = E^{cs}(\rho(a))$. Moreover, as $\lambda_2(\rho(b)) < \lambda_1(\rho(b))$ we can perturb slightly $\rho(b)$ to assume that this matrix is also loxodromic, that is, the inequalities above are strict.

Now, by Remark 5.6 we have that

$$\frac{\lambda_1(\rho(a^m b^n))}{\lambda_1(\rho(a))^m \lambda_1(\rho(b))^n}$$

converges to some positive constant only depending on $\rho_P(a)$ and $\rho_P(b)$, as $m, n \to +\infty$. Hence, as

$$\frac{\lambda_1(\rho(a^m b^n))}{\lambda_\perp(\rho(a^m b^n))} = \frac{\lambda_1(\rho(a^m b^n))}{\lambda_1(\rho(a))^m \cdot \lambda_1(\rho(b))^n} \cdot \left(\frac{\lambda_1(\rho(a))}{\lambda_\perp(\rho(a))}\right)^m \cdot \left(\frac{\lambda_1(\rho(b))}{\lambda_\perp(\rho(b))}\right)^n$$

and as $\lambda_\perp(\rho(a)) > \lambda_1(\rho(a))$ and $\lambda_1(\rho(b)) > \lambda_\perp(\rho(b))$, we find a constant $C > 1$ so that there are sequences $m_k, n_k \to +\infty$ such that

$$\frac{\lambda_1(\rho(a^{m_k} b^{n_k}))}{\lambda_\perp(\rho(a^{m_k} b^{n_k}))} \in \left[C^{-1}, C\right]$$

for all k.

Fix a small $\varepsilon_0 > 0$ so that for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ the representation ρ_ε obtained by scaling the a-action on P belongs to $\mathscr{U}$ (recall Remark 5.6). We have

$$\frac{\lambda_1\Big(\rho_\varepsilon(a^{m_k}b^{n_k})\Big)}{\lambda_\perp\Big(\rho_\varepsilon(a^{m_k}b^{n_k})\Big)} = e^{-\frac{3}{2}m_k\varepsilon} \cdot \frac{\lambda_1\Big(\rho(a^{m_k}b^{n_k})\Big)}{\lambda_\perp\Big(\rho(a^{m_k}b^{n_k})\Big)} \in \Big[C^{-1}e^{-\frac{3}{2}m_k\varepsilon}, Ce^{-\frac{3}{2}m_k\varepsilon}\Big]$$

for all k. As m_k can be taken to be arbitrarily large, by continuity we find some ε such that for some $m = m_k$ and $n = n_k$ we have

$$\frac{\lambda_1(\rho_\varepsilon(a^m b^n))}{\lambda_\perp(\rho_\varepsilon(a^m b^n))} = 1.$$

Note moreover that $\lambda_1(\rho_\varepsilon(a^m b^n)) \neq 1$, because ρ_ε still preserves P and $\rho_P = (\rho_\varepsilon)_P$ is quasi-isometric. In particular, $L_0 \subset P = E^{cs}(\rho_\varepsilon(a))$. Moreover, as the deformation ρ_ε is arbitrarily small and $\rho(a)$ and $\rho(b)$ were loxodromic, so are $\rho_\varepsilon(a)$ and $\rho_\varepsilon(b)$. This finishes the proof. ☐

5.3. Proof of Theorem 1.7

We are now ready to prove Theorem 1.7, which will be a consequence of Propositions 3.4 and 5.5.

Hence, we fix a derived from Barbot representation ρ of a free group Γ, so that ρ_P preserves the orientation of P. We suppose moreover that ρ is not Anosov. We will find arbitrarily small perturbations of ρ which are not quasi-isometric.

Indeed, by Proposition 5.5, up to replacing ρ by an arbitrarily small deformation still preserving P, and letting $\omega := a^m b^n$ for appropriate integers m and n we have:

(1) the restriction of $\rho(\omega)$ to some hyperplane P_0 is either diagonalizable with eigenvalues of equal modulus $\lambda_1 = \lambda_\perp \neq 1$, or it is a Jordan block of eigenvalue μ, for some $\mu \neq \pm 1$.
(2) The matrices $\rho(a)$ and $\rho(b)$ are loxodromic.
(3) The complementary eigenline L_0 of $\rho(\omega)$ is contained in $E^{cs}(\rho(a))$.

In order to apply Proposition 3.4, we now show that we may find an arbitrarily small perturbation ρ' of ρ, not preserving P anymore, and for which condition (1) is replaced by

$$P_0 = \ker(\rho'(\omega^q) - \mu^q),$$

for some non-zero integer q and $\mu \neq \pm 1$, while still keeping conditions (2) and (3) unchanged. Indeed, if the restriction of $\rho'(\omega)$ to P_0 is diagonalizable then $\rho' = \rho$ and $q = 2$ do the job. In contrast, if it is a Jordan block instead, by Lemma 3.3 there exist arbitrarily small neighborhoods U of $\rho(b)$ so that $U^{(n)}$ is a neighborhood of $\rho(b^n)$. In particular, $\rho(a^m)U^{(n)}$ is a neighborhood of $\rho(\omega)$. Fix an element $h \in \rho(a^m)U^{(n)}$ preserving the splitting $L_0 \oplus P_0$ and for which

$$P_0 = \ker(h^q - \mu^q)$$

for some non-zero integer q. We may take $\rho'(b) \in U$ such that

$$\rho(a^m)\rho'(b^n) = h.$$

Hence, letting $\rho'(a) := \rho(a)$ proves the claim.

Finally, by Proposition 3.4 the representation ρ' is accumulated by representations containing unipotent elements. Lemma 3.2 yields the result.

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