Annales Henri Lebesgue
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ANNALES
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SCATTERING FOR DEFOCUSING CUBIC NLS UNDER LOCALLY DAMPED STRONG TRAPPING

SCATTERING POUR NLS CUBIQUE DÉFOCALISANTE AVEC DES TRAJECTOIRES CAPTÉES FORTES LOCALEMENT AMORTIES

ABSTRACT. — We are interested in the scattering problem for the cubic 3D nonlinear defocusing Schrödinger equation with variable coefficients. Previous scattering results for such problems address only the cases with constant coefficients or assume strong variants of the non-trapping condition, stating that all the trajectories of the Hamiltonian flow associated with the operator are escaping to infinity. In contrast, we consider the most general setting, where strong trapping, such as stable closed geodesics, may occur, but we introduce a compactly supported damping term localized in the trapping region, to explore how damping can mitigate the effects of trapping.

In addition to the challenges posed by the trapped trajectories, notably the loss of smoothing and of scale-invariant Strichartz estimates, difficulties arise from the damping itself, particularly since the energy is not, a priori, bounded. For $H^{1+\epsilon}$ initial data (chosen because the local-in-time theory is a priori no better than for 3D unbounded manifolds, where local well-posedness

Keywords: nonlinear Schrödinger equation, scattering, damping, trapped trajectories.
2020 *Mathematics Subject Classification:* 35Q55, 35B40.
DOI unassigned yet
(*) The work of B. S. is partially supported by ANR-11-LABX-0040-CIMI and the ANR project NQG ANR-23-CE40-0005.

of strong H^1 solutions is unavailable) we establish global existence and scattering in H^s for any $0 \leqslant s < 1$ in positive times, the inability to reach H^1 being related to the loss of smoothing due to trapping.

Résumé. — Nous nous intéressons au problème du scattering pour l'équation de Schrödinger non linéaire cubique défocalisante en dimension trois, avec coefficients variables. Les résultats de scattering disponibles dans la littérature ne concernent que le cas à coefficients constants, ou bien reposent sur des variantes fortes de la condition de non-piégeage, imposant que toutes les trajectoires du flot hamiltonien associé à l'opérateur s'échappent à l'infini. À l'inverse, nous considérons ici le cadre le plus général, où des trajectoires captées fortes peuvent à priori exister, et nous introduisons un terme d'amortissement à support compact, localisé dans la région piégeante, afin d'étudier dans quelle mesure cet amortissement peut compenser les effets des trajectoires captées.

Outre les difficultés engendrées par ces trajectoires (notamment la perte d'effet régularisant et l'absence d'estimations de Strichartz invariantes par changement d'échelle) l'amortissement lui-même entraîne des complications, en particulier du fait que l'énergie n'est pas a priori bornée. Pour des données initiales dans $H^{1+\varepsilon}$ (espace choisi car la théorie locale en temps n'est pas meilleure que dans le cas des variétés non bornées de dimension 3, où le caractère bien-posé local en temps de solutions fortes n'est pas disponible dans H^1) nous établissons l'existence globale et le scattering dans H^s pour tout $0 \leqslant s < 1$ en temps positif, l'impossibilité d'atteindre H^1 étant liée à la perte d'effet régularisant due aux trajectoires captées.

1. Introduction

We are interested in the following cubic defocusing nonlinear Schrödinger equation with a variable-coefficients Laplacian in divergence form and a compactly supported damping by a potential

$$\begin{cases} i\partial_t u + \Delta_G u + iau = |u|^2 u, \\ u(0) = u_0 \in H^s(\mathbb{R}^3). \end{cases} \tag{1.1}$$

Here Δ_G denotes

$$\Delta_G u := \operatorname{div}(G\nabla u),$$

with $G \in C^\infty(\mathbb{R}^{3\times 3}, \mathbb{R})$ symmetric and uniformly positive-definite, in the sense that

$$\exists\, c > 0, \quad \forall\, x \in \mathbb{R}^3, \quad \forall\, \xi \in \mathbb{R}^3, \quad G(x)\xi \cdot \xi \geqslant c|\xi|^2,$$

and $G = \mathrm{I}$ outside a compact set

$$G - \mathrm{I} \in C_c^\infty(\mathbb{R}^{3\times 3}, \mathbb{R}).$$

In addition, the damping potential a is non-negative and compactly supported

$$a \in C_c^\infty(\mathbb{R}^3, \mathbb{R}_+),$$

and in what follows, it will typically be active where $G - \mathrm{I} \neq 0$. The following quantities, respectively the total mass and energy, are conserved in the case $a = 0$ and therefore will play a crucial role:

$$\begin{aligned} M[u(t)] &:= \int |u(t,x)|^2 dx, \\ E[u(t)] &:= \frac{1}{2}\int G(x)\nabla u(t,x) \cdot \nabla \bar{u}(t,x)\, dx + \frac{1}{4}\int |u(t,x)|^4\, dx. \end{aligned}$$

We are interested in the forward global-in-time behavior of solutions, more specifically, in showing scattering to linear solutions.

To motivate the problem, let us briefly review the related problems when $a = 0$. When $G = \mathrm{I}$ and $a = 0$, equation (1.1) has a rich history that we will not attempt to fully review here. In particular, the conservation laws together with the local theory give global-well-posedness when the data is in $H^1(\mathbb{R}^3)$, and it is known since the work of Ginibre and Velo [GV85] (see also [Nak99] for the 1d and 2d cases, and [Vis09] for extensions of these results) that solutions scatter in H^1 to linear solutions, in the sense that there exists $u_\pm \in H^1(\mathbb{R}^3)$ so that

$$\left\| u(t) - e^{it\Delta} u_\pm \right\|_{H^1(\mathbb{R}^3)} \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \pm\infty.$$

Whereas we are not aware of any scattering result in the case $G \neq \mathrm{I}$ and $a = 0$, scattering for $a = 0$ has been investigated in at least two related inhomogeneous situations: the case where $G = \mathrm{I}$, but with the equation posed outside an obstacle (i.e. in $\mathbb{R}^3 \backslash \Theta$, with Θ compact with smooth boundary) with Dirichlet boundary conditions [AS15, IP10, KVZ16a, KVZ16b, PV09, PV12, XZZ22], and the case where $G = \mathrm{I}$, but a perturbation Vu by a potential is added [BV16, FV18, Hon16, Laf16]. In both cases, scattering results rely crucially on strong *non-trapping* conditions. The most general version of this condition states that all the Hamiltonian trajectories (these are parametrized curves in $\mathbb{R}^3_x \times \mathbb{R}^3_\xi$) associated with the principal symbol of the operator having the role of Laplacian exit any compact of space in finite time. In the case of $G = \mathrm{I}$, these trajectories project in space to straight lines; outside an obstacle, reflections have to be taken into account, and these are the so-called generalized broken bi-characteristics. In the case of general $-\Delta_G$, the trajectory from (x_0, ξ_0) is the solutions of the Hamilton equation

$$\begin{aligned} \dot{x}(t) &= \nabla_\xi (G(x)\xi \cdot \xi) &&= 2G(x)\xi, \\ \dot{\xi}(t) &= -\nabla_x (G(x)\xi \cdot \xi) &&= -\sum_{1 \leqslant i,j \leqslant 3} \nabla G_{i,j}(x) \xi_i \xi_j, \end{aligned} \tag{1.2}$$

with $(x(0), \xi(0)) = (x_0, \xi_0)$. Such a non-trapping assumption and stronger repulsivity variants discussed momentarily play a crucial role in the two main ingredients used to show scattering: global Strichartz estimates and non-concentration estimates on the nonlinear solutions. Concerning the former, global-in-time Strichartz estimates typically hold in non-trapping situations. Indeed, at least in the case without boundaries, semiclassical Strichartz estimates (i.e., on time intervals of size $\sim h$ for data localized in frequencies $\sim h^{-1}$) hold without geometric assumption due to the semiclassical finite speed of propagation [BGT04b]. On the other hand, the non-trapping condition implies global-in-time smoothing (see e.g. [BGT04a, Section 2.3] for a proof from resolvent estimates, themselves tracing back to [Bur02, LP89, MS82, Vai89, VZ00]), which can in turn be used to deduce global-in-time Strichartz estimates from semiclassical ones [Iva10, ST02] (see also [BGH10, Bur03]). For the problem with boundaries, such global Strichartz estimates involve, in general loss of derivatives; however, this difficulty can be overcome using the global smoothing close to the (non-trapping) obstacle [PV09, PV12]. Non-concentration estimates, taking the form of Morawetz estimates such as originating from the work of Lin and Strauss [LS78], are the second crucial ingredient in a typical proof of scattering. These can for example be used in their modern interaction form [CKS+04, PV09], to show that the solution is in a global-in-time space-time Lebesgue space (which is then used, by interpolation with energy conservation arguments, to show that it is in a scale-invariant space-time Lebesgue space at the level we are interested in, which in turn implies scattering), or to rule-out compact flow solutions in a concentration-compactness/rigidity type argument by contradiction in the now classical Kenig–Merle scheme originating from [KM06]. Such Morawetz estimates rely on even stronger non-trapping conditions, namely *repulsivity conditions*, to ensure that the term arising from the perturbation in the computation has the right sign. For an obstacle, this is the assumption that it is star-shaped, i.e., $x \cdot n(x) \leqslant 0$ at the boundary; for a potential, it typically takes the form $x \cdot \nabla V \geqslant 0$ (or $(x \cdot \nabla V)_-$ small).

In contrast, we are interested in the most general situation without any non-trapping assumption, hence with the Hamilton flow (1.2) associated with $-\Delta_G$ having possibly the strongest (i.e., the most stable) possible trapped trajectories, but we introduce a damping term iau, active where the trapping takes place, intending to understand how such a damping can mitigate the effects of trapping. Hence, the damping a will verify the following control condition:

$$\operatorname{supp}(G - \mathrm{I}) \subset \{a > 0\}. \tag{1.3}$$

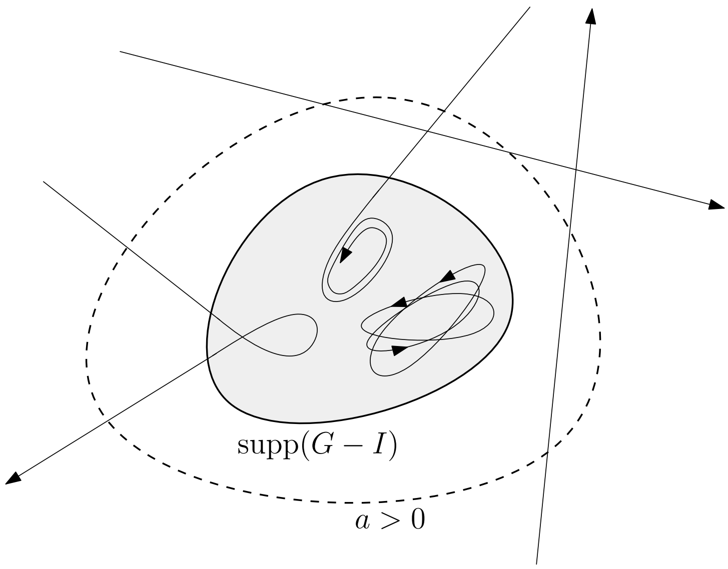

Figure 1.1. Schematic representation of the control condition (1.3) *and of typical Hamilton trajectories.*

The overall idea is that, at least at the L^2-level, the damping should allow recovering *some* global-in-time integrability in the region of trapping, where Morawetz-type arguments break down. On the other hand, numerous challenges are introduced *both* by the trapping and the damping, namely:

(1) *Loss of smoothing:* one of the most spectacular effects of the existence of stable trapped trajectories is that no smoothing effect, even with a loss of

derivative, can be true for the associated linear Schrödinger operator (see e.g. [Bur04, Remark 4.2]). Adding a damping term can permit to recover some global-in-time integrability at the L^2-level for linear solutions u_L, in the form that $\|u_L\|_{L^2(\mathbb{R}_+,B(0,R))} \lesssim_R \|u_0\|_{L^2(\mathbb{R}^3)}$ under the assumption that any trapped trajectory intersects $\{a > 0\}$ [AK07]; however damping by a potential doesn't permit to recover any smoothing (see also [AKR17, KR17]).

(2) *Loss in local-in-time Strichartz estimates:* Consequently, smoothing cannot be used to obtain even local-in-time scale-invariant Strichartz estimates from semiclassical estimates. Therefore, we have a priori no better than the same local-in-time Strichartz estimates as in a general manifold, which involves a loss of $\frac{1}{p}$ derivatives [BGT04b]. As a consequence, the local-well-posedness theory is a priori no better than in a general 3D manifold, for which the usual contraction principle gives well-posedness of strong $C([0,T],H^s)$ solutions only for $s > 1$. Recall, however, that [BGT04b] managed to show global existence and uniqueness of weak H^1 solutions in a bounded manifold, but we lack the appropriate compactness tools to replicate their argument in our unbounded setting.

(3) *No a priori uniform-in-time bound on the energy:* whereas the mass is non-increasing thanks to the damping, the derivative of the energy is a priori not signed, hence no a priori uniform-in-time bound on the energy is at hand.

(4) *Negative times:* Finally, let us mention that the linear operator is not bounded uniformly in negative times in $L^2(\mathbb{R}^3)$, hence it makes the use of the Duhamel formula involving the retarded evolution to show scattering from space-time Lebesgue bounds challenging.

We are now ready to introduce our main result. It shows that (1.1) is locally well-posed in $H^{1+\epsilon}$, and under the control condition (1.3), solutions are uniformly bounded forward in time in the energy space, and scatter in H^{1-}. This constitutes the first result of scattering *both* under strong geometrical trapping, and for non-constant damping, for any nonlinear dispersive equation.

Theorem 1.1. — *For any $\epsilon > 0$, equation (1.1) is locally well-posed in $H^{1+\epsilon}(\mathbb{R}^3)$. In addition, under the control condition (1.3), for any $u_0 \in H^{1+\epsilon}(\mathbb{R}^3)$, the associated forward maximal solution is global in positive times, verifies $\|u\|_{L^\infty(\mathbb{R}_+,H^1(\mathbb{R}^3))} \leqslant C\|u_0\|_{H^1(\mathbb{R}^3)}$, and scatters in $H^{1-}(\mathbb{R}^3)$ to a free linear wave, in the sense that there exists $u_+ \in H^1(\mathbb{R}^3)$ so that*

$$\forall\, s \in [0,1), \quad \left\|u(t) - e^{it\Delta}u_+\right\|_{H^s(\mathbb{R}^3)} \longrightarrow 0 \quad as \quad t \longrightarrow +\infty.$$

The fact that the well-posedness holds in $H^{1+\epsilon}$ is related to (2) above. On the other hand, we see the inability to reach scattering in H^1 as related to the loss of smoothing in the linear flow (1) due to trapping – in particular, if G induces a non-trapping Hamiltonian flow, both well-posedness and scattering hold in H^1, as discussed at the end of this introduction and shown in Appendix B.

We now give some ideas of the proof. The first preliminary step, carried out in Section 2, is to show local well-posedness together with a conditional global-well-posedness result. Strichartz estimates for the undamped linear flow from [BGT04b] give local well-posedness in $H^{1+\epsilon}$. We will show in a later key step that solutions are uniformly bounded forward in time in the energy space, but for $H^{1+\epsilon}$ well-posed solutions, this is a priori not enough to obtain global existence. However, from an adaptation of the argument used in [BGT04b] to show the existence of a unique global weak H^1 solution, we can show that a uniform bound on the energy implies global forward existence of $H^{1+\epsilon}$ well-posed solutions.

Scattering in H^{1-} will be obtained by interpolation from scattering in $H^{\frac{1}{2}}$. Scattering in $H^{\frac{1}{2}}$ in turn will be a consequence of a local energy decay estimate (without gain of regularity) for solutions of (1.1) once forward global well-posedness is established, namely, we will show that:

$$\|u\|_{L^2(\mathbb{R}_+,H^1(B(0,R)))} \lesssim_R \|u_0\|_{H^1(\mathbb{R}^3)}. \tag{1.4}$$

We show both the above and a uniform bound on the energy, hence global well-posedness, in one go. This will be the main result of Section 3. The idea is the following. We first remark that, from the energy law,

$$E[u(t)] \lesssim E[u(0)] + \|u\|_{L^2([0,t],L^2(B(0,R)))}, \tag{1.5}$$

for $R > 0$ large enough, hence it suffices to show decay of the local mass,

$$\|u\|_{L^2(\mathbb{R}_+,L^2(B(0,R)))} < \infty,$$

to obtain a uniform energy bound. In order to do so, we use a Morawetz-type estimate, which makes precisely appear the localized mass as the term coming from the linear part of the equation. More precisely, we perform a Morawetz computation (on the maximal time of existence) with linear weight $\langle x\rangle$, perturbatively with respect to $G = \mathrm{I}$, hence with an unsigned error term localized on $\operatorname{supp}(G - \mathrm{I})$. The localized mass term arising in the computation is then

$$\lambda(t) := \int_0^t \int \langle x\rangle^{-7}|u|^2 dxdt.$$

We control the energy by λ, and, thanks to the energy law and the control condition, we are able to control the error terms *up to lower order terms in* λ. We show in such a way that λ verifies an inequality of the form

$$\lambda(t)^2 - b\lambda(t) - c \leqslant 0,$$

and is therefore bounded. The uniform energy bound and global existence follow, and coming back to the Morawetz computation and plugging the information that $\sup_t \lambda(t) < \infty$, we obtain local energy decay (1.4).

Once (1.5) is established, the second ingredient is a bilinear (or interaction) Morawetz estimate in the spirit of [CKS$^+$04, PV09], showing that solutions to (1.1) are in $L^4(\mathbb{R}_+, L^4(\mathbb{R}^3))$, obtained in Section 4. The idea is to perform the computation perturbatively with respect to $G = \mathrm{I}$, hence with error terms again localized in $\operatorname{supp}(G - I)$. But as $\operatorname{supp}(G - I)$ is compact, we can control these terms thanks to the local energy decay (1.5), and the global space-time bound follows.

To conclude from a global space-time bound, one typically takes advantage of the Duhamel formula and global Strichartz estimates to construct u_+. However, in our setting, this direct approach is hopeless, as it would require global estimates backward in time for the linear group. One cannot consider the damping iau as a perturbation and put it in the source term either, as the linear group would now be $e^{it\Delta_G}$, for which no global estimate is at hand due to trapping. To overcome these difficulties, we cut the solution in two parts: a part χu localized close to $\operatorname{supp} a$, and a part $(1-\chi)u$ away from it. From local energy decay, one can show that $\chi u \to 0$ in any $H^s(\mathbb{R}^3)$, $s<1$. On the other hand, $(1-\chi)u$ verifies a nonlinear Schrödinger equation with the free Laplacian $-\Delta$, and one can show scattering in $H^{\frac{1}{2}}$ of this part by expressing it thanks to Duhamel with the free linear group $e^{it\Delta}$. This involves control of the nonlinearity $u|u|^2$, which comes from the L^4L^4-bound, and control of the commutant $[\Delta,\chi]u$ arising in the equation, which follows thanks to local energy decay. Once scattering in $H^{\frac{1}{2}}$ is established, it can be improved to H^{1-} by interpolation with the uniform bound on energy, and the proof is completed. This final step is carried out in Section 5.

Remark that scattering of the part $(1-\chi)u$ cannot be improved, before interpolation, beyond $H^{\frac{1}{2}}(\mathbb{R}^3)$ with our method, as it would require a bound on $[\Delta,\chi]u$ at a higher level of regularity, which is not at hand. One cannot, either, improve the result a posteriori by cutting again the solution in the spirit of [PV09], as this approach requires local smoothing for the linear group, which does not hold in our case due to trapping. This is the reason why we relate the inability to reach H^1 in our scattering result to the loss of smoothing.

To conclude this introduction, we now discuss other related problems, generalizations presented in Appendices A and B, and open questions. Observe that the problem we are interested in can be seen as related to the stabilization and controllability of the equation. Stabilization is typically expected under the geometric control condition, stating that every Hamiltonian trajectory intersects the control region $\{a>0\}$. In unbounded settings, stabilization results therefore typically assume that the control is active at infinity: $\{a>0\} \subset \mathbb{R}^d \backslash B(0,R)$, $R \gg 1$. We refer, for example, to [BCCF24, CCÖ+20, DGL06, Lau10, LBM25, RZ10, YNC21], and references therein, for the reader interested in the stabilization problem. In contrast, in our case (1.3), the control is active only locally near the perturbation: it constitutes the simplest example of an *exterior* control condition, stating that any trajectory either goes to infinity, or intersects the control region. The case of a constant damping $a \equiv 1$ (with constant coefficients $G = \mathrm{I}$) enjoys the best of both worlds, and, by conjugating with an exponential, one can show exponential scattering in H^1 [Inu19].

Remark that our result is new even in the case $G=\mathrm{I}$. In this case, and actually, under a general mild trapping assumption (i.e. for G such that $e^{it\Delta_G}$ verifies global, scale-invariant Strichartz estimates and a local-energy decay estimate, see Assumption B.1) which holds in particular in the non-trapping case, we can show scattering up to $H^1(\mathbb{R}^3)$ – still under the control condition (1.3). This is done in Appendix B.

Finally, observe that the nonlinear parts of our arguments work just as well in the case of the equation posed outside an obstacle. However, the best general local Strichartz estimates (i.e., without any geometric conditions) for the linear Schrödinger equation in a setting with boundaries known at the moment are not enough to obtain a well-posedness theory such as presented in Section 2, hence the result obtained is conditional. This is illustrated in Appendix A.

We finish this introduction by stating some open questions related to our problem. First, we don't know whether scattering in H^{1-} is sharp. Going to H^1, or giving a counter-example, would likely involve a better understanding of the related *linear* damped flow. In particular, could one show better small-time Strichartz estimates for the damped flow than for the undamped one, under the exterior control condition, or is there a counter-example? Related to this question, could one show that the loss of smoothing for the damped flow in the results of [AK07, AKR17, KR17] is sharp? This is expected, but no counterexample seems to be known. Coming back to the non-linear equation, it is natural to expect an analogous result, under a natural mass-energy threshold, for the focusing equation. Showing such a result would likely involve profile decompositions in the spirit of [KM06], which will be challenging to carry out in our setting, in particular due to the non-self-adjointness of the problem, and the fact that the linear operator is not bounded in negative times. Finally, observe that even in the non-trapping case, without damping, all the known scattering results involve a stronger repulsivity assumption mentioned earlier. In particular, it is natural to expect scattering for the undamped equation in any non-trapping geometry, but this is not known (the restriction to repulsive geometries is mostly due to the rigidity of the Morawetz computations in their various avatars used to show scattering). In the spirit of this paper, we expect an even stronger conjecture to hold: scattering (at least in H^{1-}) under the sole exterior control condition (any trajectory either meets the control condition or goes to infinity). This last question is, however, still far from reach.

Notations

We write $a \lesssim b$ to indicate that there exists a universal constant $C > 0$ such that $a \leqslant Cb$. We define $H^{1-}(\mathbb{R}^3)$ as $H^{1-}(\mathbb{R}^3) := \bigcap_{0\leqslant s<1} H^s(\mathbb{R}^3)$ with $H^0(\mathbb{R}^3) := L^2(\mathbb{R}^3)$.

2. Preliminaries

2.1. Strichartz Estimates for the undamped linear Schrödinger flow

We will say that the couple (p, q) is admissible whenever

$$2 \leqslant p, q \leqslant \infty, \quad \frac{2}{p} + \frac{3}{q} = \frac{3}{2}. \tag{2.1}$$

From [BGT04b], we have local-in-time Strichartz estimates for the *undamped* linear flow $e^{it\Delta_G}$ with a loss of $\frac{1}{p}$ derivatives:

Proposition 2.1. — *For any $T > 0$, any $s \geqslant 0$ and any admissible couple (p, q), there exists $C > 0$ such that, for any $u_0 \in H^{\frac{1}{p}+s}(\mathbb{R}^3)$ and any $f \in L^1((0,T), H^{\frac{1}{p}+s}(\mathbb{R}^3))$,*

$$\left\| e^{it\Delta_G} u_0 \right\|_{L^p((0,T),W^{s,q}(\mathbb{R}^3))} \leqslant C \|u_0\|_{H^{\frac{1}{p}+s}(\mathbb{R}^3)}, \tag{2.2}$$

and

$$\left\| \int_0^t e^{i(t-\tau)\Delta_G} f \right\|_{L^p((0,T),W^{s,q}(\mathbb{R}^3))} \leqslant C\|f\|_{L^1\left((0,T),H^{\frac{1}{p}+s}(\mathbb{R}^3)\right)}. \tag{2.3}$$

Proof. — These estimates are shown in [BGT04b, Theorem 1]. Albeit [BGT04b, Theorem 1] is stated for the Laplace–Beltrami operator $-\Delta_g$ in a compact manifold $(\mathcal{M}, g)$, the proof applies equally well in our setting – note in particular that a generalization to the Laplace operator associated to a Riemannian metric in $\mathbb{R}^d$ is given as [BGT04b, Appendix A.1], which is very close to our setting.

We briefly sketch the proof of [BGT04b, Theorem 1] for the interested reader, then explain the (minimal) changes in our setting. The authors work for frequency-localized data $\psi(-h^2\Delta_g)u_0$, and show, thanks to a WKB approximation on the kernel of the associated operator, that the semiclassical Schrödinger flow verifies the dispersive estimate $|e^{it\Delta_g}\psi(-h^2\Delta_g)u_0(x)| \lesssim t^{-\frac{d}{2}}\|u_0\|_{L^1}$ in times $\lesssim h$ – this is the key [BGT04b, Lemma 2.5]. By the now classical [KT98], this implies Strichartz estimates for $e^{it\Delta_g}\psi(-h^2\Delta_g)$ (including endpoint) in time intervals of size $\sim h$, and hence for $e^{it\Delta_g}\psi(-h^2\Delta_g)$ in time intervals of size ~ 1 but with a loss $h^{-1/p}$, which corresponds to the derivative loss $H^{\frac{1}{p}}$ after re-summation of all frequencies.

The proof of [BGT04b, Theorem 1] therefore uses two main ingredients: a frequency localization $\psi(-h^2\Delta_g)$ and its properties when acting on spaces L^p, and a WKB construction used to show a dispersive estimate $L^1 \to L^\infty$ in times ~ 1 for the semiclassical Schrödinger flow $e^{ith\Delta_g}$.

In the case of the Laplace operator associated with a Riemannian metric in $\mathbb{R}^d$ presented in [BGT04b, Appendix A.1], the main difference is the different status of the L^p spaces in $\mathbb{R}^d$: for this reason, the authors show the dispersive estimate for a localization in frequencies $\Psi(hD)$, that is, $|e^{it\Delta_g}\Psi(hD)u_0(x)| \lesssim t^{-\frac{d}{2}}\|u_0\|_{L^1}$ in times $\lesssim h$ – this is [BGT04b, Lemma A.3], the analogue of the key [BGT04b, Lemma 2.5]. The localization $\Psi(hD)$ is a good approximation to $\psi(-h^2\Delta_g)$ by [BGT04b, Proposition A.1, Corollary A.2], and the Strichartz estimates follow as in the compact case thanks to this supplementary ingredient. The proof of the dispersive estimate [BGT04b, Lemma A.3] itself is almost verbatim the same as in the compact case, to the difference that the solutions to the eikonal and transport equations are now defined globally in space for small times.

In our case, the Strichartz estimates follow almost verbatim as in [BGT04b, Appendix A.1] recapped above. Indeed, $-\Delta_G$ satisfies the assumptions of [BGT04b, Proposition A.1], hence [BGT04b, Proposition A.1, Corollary A.2] apply. On the other hand, the same WKB method applies (without the need to work in coordinate patch) in the same way to show the analogue of [BGT04b, Lemma 2.5], by solving the eikonal and transport equations that now write

$$\begin{aligned} \partial_s\phi + G\nabla\phi\cdot\nabla\phi &= 0,\\ \partial_s a_0 + 2G\nabla\phi\cdot\nabla a_0 + \Delta_G(\phi)a_0 &= 0,\\ \partial_s a_j + 2G\nabla\phi\cdot\nabla a_j + \Delta_G(\phi)a_j &= -\Delta_G(a_{j-1})\ j \geqslant 1, \end{aligned}$$

which are the analogue to [BGT04b, (2.19)–(2.20)–(2.21)], and the end of the proof follows as at the end of [BGT04b, Appendix A.1]. □

Remark 2.2 (Global Strichartz estimates with dramatic loss for the damped flow). Recall that, in the non-trapping case, the frequency-localized Strichartz estimates of [BGT04b] can be combined with local smoothing to obtain global Strichartz estimates without loss of derivative [Iva10, ST02]. This strategy doesn't apply in our case as because of trapping, local smoothing is lost. For the damped linear flow however (which we will denote $S(t)$ for the sake of this remark), even if smoothing is not at hand, the damping term permits to recover some global-in-time integrability at the L^2-level: $\|S(t)u_0\|_{L^2(\mathbb{R}_+,B(0,R))} \lesssim_R \|u_0\|_{L^2(\mathbb{R}^3)}$ under the assumption that any trapped trajectory intersects $\{a > 0\}$ [AK07]. This can be combined with local Strichartz estimates for $S(t)$ (which hold from Proposition 2.1 by absorption by viewing iau as a source term) to obtain global Strichartz estimates for the damped linear flow $S(t)$ with a loss of $\frac{1}{p} + \frac{1}{2}$ derivatives:

$$\|S(t)u_0\|_{L^p(\mathbb{R}_+,L^q(\mathbb{R}^3))} \lesssim \|u_0\|_{H^{\frac{1}{p}+\frac{1}{2}}(\mathbb{R}^3)},$$

such a loss, however, seems too dramatic to be useful for our purposes, and we don't use such estimates in this paper.

2.2. Local, and conditionally global, well-posedness theory

We start by showing local well-posedness of solutions in $H^{1+\varepsilon}(\mathbb{R}^3)$ for any $\varepsilon > 0$ by the classical contraction principle.

Proposition 2.3. — *For any $\varepsilon > 0$, the followings holds. There exists $p > 2$ so that for any $u_0 \in H^{1+\varepsilon}(\mathbb{R}^3)$ there exists $T > 0$ and a unique solution $u \in C([0,T], H^{1+\varepsilon}(\mathbb{R}^3)) \cap L^p([0,T], L^\infty(\mathbb{R}^3))$ to (1.1). In addition:*

(1) *If $\|u_0\|_{H^{1+\varepsilon}(\mathbb{R}^3)}$ is bounded, then T is bounded below.*
(2) *The map $u_0 \in H^{1+\varepsilon}(\mathbb{R}^3) \mapsto u \in C([0,T], H^{1+\varepsilon}(\mathbb{R}^3))$ is continuous.*
(3) *If $u_0 \in H^m(\mathbb{R}^3)$ for $m > 1+\varepsilon$, then $u \in C([0,T], H^m(\mathbb{R}^3))$.*

In particular, from (1), if $T_{\max} > 0$ denotes the maximal forward time of existence, either $T_{\max} = +\infty$ or

$$\lim_{t \to T_{\max}^-} \|u(t)\|_{H^{1+\varepsilon}(\mathbb{R}^3)} = \infty. \tag{2.4}$$

Proof. — The proof is similar to [BGT04b, Proposition 3.1] dealing with iau as an additional source term. We fix $\varepsilon > 0$ and any (p,q) satisfying $p, q > 2$ and

$$s := 1 + \varepsilon > \frac{3}{2} - \frac{1}{p}, \quad \frac{2}{p} + \frac{3}{q} = \frac{3}{2}. \tag{2.5}$$

Observe that (2.5) implies that

$$\sigma := s - \frac{1}{p} > \frac{3}{q},$$

which yields the Sobolev embedding

$$W^{\sigma,q}(\mathbb{R}^3) \longrightarrow L^\infty(\mathbb{R}^3). \tag{2.6}$$

We define

$$X_T := L^\infty\left((0,T), H^{1+\varepsilon}(\mathbb{R}^3)\right) \cap L^p\left((0,T), W^{\sigma,q}(\mathbb{R}^3)\right),$$

equipped with

$$\|u\|_{X_T} := \|u\|_{L^\infty((0,T),H^{1+\varepsilon}(\mathbb{R}^3))} + \|u\|_{L^p((0,T),W^{s,q}(\mathbb{R}^3))},$$

where $T > 0$ is to be chosen later. For any $t \in (0,T)$ and any $u_0 \in H^{1+\varepsilon}(\mathbb{R}^3)$, we define the map

$$\phi(u_0,t)(u) = e^{it\Delta_G}u_0 - i\int_0^t e^{i(t-s)\Delta_G}\left(|u(s)|^2u(s) - iau(s)\right) ds.$$

We will show that for T small enough, the map ϕ is a contraction on a ball centered at the origin to itself. Indeed, by Strichartz estimates (2.2), (2.3), Hölder's inequality, Sobolev embedding (2.6), and the fractional Leibniz rule, we have

$$\begin{aligned}
&\|\phi(u_0,.)(u)\|_{X_T} \\
&\qquad \leqslant C\|u_0\|_{H^{1+\varepsilon}(\mathbb{R}^3)} + C\left\||u|^2u - iau\right\|_{L^1((0,T),H^{1+\varepsilon}(\mathbb{R}^3))} \\
&\qquad \leqslant C\|u_0\|_{H^{1+\varepsilon}(\mathbb{R}^3)} \\
&\qquad\quad + C\|u\|_{L^\infty((0,T),H^{1+\varepsilon}(\mathbb{R}^3))}\left(T\|a\|_{W^{1+\varepsilon,\infty}(\mathbb{R}^3)} + T^{\frac{p-2}{p}}\|u\|^2_{L^p((0,T),L^\infty(\mathbb{R}^3))}\right) \\
&\qquad \leqslant C\|u_0\|_{H^{1+\varepsilon}(\mathbb{R}^3)} + C_a\left(T + T^{\frac{p-2}{p}}\right)\left(\|u\|_{X_T} + \|u\|^3_{X_T}\right).
\end{aligned}$$

Since $p - 2 > 0$, there exists $T > 0$ small enough so that ϕ send the ball of X_T, $B := B_{X_T}(0, \frac{C}{2}\|u_0\|_{H^{1+\epsilon}(\mathbb{R}^3)})$, to itself. In addition, we have similarly

$$\begin{aligned}
&\|\phi(u) - \phi(v)\|_{X_T} \\
&\qquad \lesssim \left(T + T^{\frac{p-2}{p}}\right)\left(1 + \|u\|^2_{L^\infty((0,T),H^{1+\varepsilon}(\mathbb{R}^3))} + \|v\|^2_{L^\infty((0,T),H^{1+\varepsilon}(\mathbb{R}^3))}\right)\|u - v\|_{X_T},
\end{aligned}$$

which implies that the map ϕ is a contraction on B for $T > 0$ small enough. Thus ϕ admits a fixed point, which is a solution to (1.1). Uniqueness and properties (1)–(2)–(3) are shown similarly. □

Remark 2.4. — When $G = \mathrm{I}$, the proof of the local well-posedness is a direct consequence of Kato's method, as in e.g. [Caz03, Section 4.4]. In particular, we can choose $\varepsilon = 0$.

In the sequel of the paper, we will show that solutions are uniformly bounded forward in time in the energy space. As we handle $H^{1+\epsilon}$ well-posed solutions given by Proposition 2.3, this is a priori not enough to obtain global existence from the local well-posedness theory alone. But we can show that such a bound on the energy indeed implies global existence:

Proposition 2.5. — *Assume that the following holds: there exists a continuous function $C : \mathbb{R}_+ \to \mathbb{R}_+^*$ so that, for any $\epsilon > 0$ and $u_0 \in H^{1+\epsilon}(\mathbb{R}^3)$, if u is the solution to* (1.1) *on* $[0,T]$ *with data u_0, then $\sup_{[0,T]}\|u\|_{H^1(\mathbb{R}^3)} \leqslant C(\|u_0\|_{H^1(\mathbb{R}^3)})$. Then, the maximal solutions to* (1.1) *are global forward in time.*

Proof. — Denote $s := 1 + \epsilon$. In view of the local well-posedness theory given by Proposition 2.3, it suffices to show that $\|u(t)\|_{H^s(\mathbb{R}^3)}$ is bounded in bounded (positive) times.

Step 1: Smooth solutions. — We begin by dealing with smooth solutions. The proof is contained in [BGT04b, Section 3.3], dealing with the damping iau as an additional "non-linearity". We sketch the argument. As already observed, [BGT04b] applies to our setting to show frequency localized Strichartz estimates for $e^{it\Delta_G}$. From these estimates, one can show the analogue of [BGT04b, Lemma 3.6] to our setting: namely, there exists $C > 0$ so that for any $a < b$ with $b - a < 1$ and $u \in C([a,b], H^1(\mathbb{R}^3))$ a solution to

$$i\partial_t u + \Delta_G u = -iau + |u|^2 u,$$

one has

$$\|u_h\|_{L^2([a,b],L^6(\mathbb{R}^3))} \leqslant Ch^{\frac{1}{2}}\|u_h\|_{L^2([a,b],H^1(\mathbb{R}^3))} + Ch^{\frac{1}{2}+\epsilon}\Lambda\left(\|u\|_{L^\infty([a,b],H^1(\mathbb{R}^3))}\right), \tag{2.7}$$

for some $\epsilon > 0$, where $\Lambda : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is non-decreasing, and u_h denotes a localization of u to frequencies $\sim h^{-1}$ $(0 < h < 1)$. Indeed, an inspection of the proof of [BGT04b, Lemma 3.6] reveals that in addition to the frequency localized Strichartz estimates, the only other ingredient is that the source term $f := -iau + |u|^2 u$ $(= G'(|u|^2)u$ in [BGT04b]) verifies

$$|\nabla f| \leqslant C\langle u\rangle^{2m-2}|\nabla u|$$

for some $m < \frac{5}{2}$, which is verified in our case with $m = 2$. Next, in the same way as [BGT04b, p. 589-590], by a well-chosen summation, it follows that any solution which is uniformly bounded in $H^1(\mathbb{R}^3)$ verifies

$$\|u\|_{L^2([0,T],L^\infty(\mathbb{R}^3))} \leqslant C\left(\left(T\log\left(2 + \|u\|_{L^2([0,T],H^s(\mathbb{R}^3))}\right)\right)^{\frac{1}{2}} + 1\right). \tag{2.8}$$

But by the Duhamel formula, together with standard nonlinear estimates

$$\|u\|_{L^\infty([0,T],H^s(\mathbb{R}^3))} \leqslant \|u_0\|_{H^s(\mathbb{R}^3)} + C\int_0^T \left(\|u(t)\|^2_{L^\infty(\mathbb{R}^3)} + 1\right)\|u(t)\|_{H^s(\mathbb{R}^3)}dt.$$

If u is a smooth solution, the above combined with (2.8) shows by a Gronwall argument that $\|u(t)\|_{H^s(\mathbb{R}^3)}$ is bounded in bounded times. It follows that any smooth, $H^s(\mathbb{R}^3)$ solution is global and in $L^\infty_{\text{loc}}H^s$. For future reference, note that we obtain a bound on $\|u\|_{L^\infty([0,T],H^s(\mathbb{R}^3))}$ which depends only, and continuously, on T and $\|u\|_{L^\infty([0,T],H^1(\mathbb{R}^3))}$ (non-decreasingly in both).

Step 2: Yudovitch argument. — To apply the above to a non-smooth solution, we will use a refinement of the Yudovitch argument [Yud67] used to show the uniqueness of $L^\infty H^1$ solutions in [BGT04b, p. 591-592]. Namely:

Lemma 2.6. — *The following holds.*

(i) *For any $M > 0$, any $T > 0$, and any $\epsilon > 0$, there exists $\delta > 0$ so that, if u and $\widetilde{u}$ are solutions to the integral equations*

$$u(t) = e^{it\Delta_G}u_0 - i\int_0^t e^{i(t-s)\Delta_G}\left(|u(s)|^2u(s) - iau(s)\right)ds, \tag{2.9}$$

$$\widetilde{u}(t) = e^{it\Delta_G}\widetilde{u_0} - i\int_0^t e^{i(t-s)\Delta_G}\left(|\widetilde{u}(s)|^2\widetilde{u}(s) - ia\widetilde{u}(s)\right)ds, \tag{2.10}$$

defined on $[0,T]$ and so that

$$\|u\|_{L^\infty((0,T)H^1(\mathbb{R}^3))} \leqslant M, \quad \|\widetilde{u}\|_{L^\infty((0,T)H^1(\mathbb{R}^3))} \leqslant M,$$

and

$$\|\widetilde{u}(0)-u(0)\|_{L^2(\mathbb{R}^3)}\leqslant\delta,$$

then

$$\sup_{t\in[0,T]}\|\widetilde{u}(t)-u(t)\|_{L^2(\mathbb{R}^3)}\leqslant\epsilon.$$

(ii) *For any* $T>0$*, solutions to* (2.9) *on* $[0,T]$ *are unique subject to the condition* $u\in L^\infty([0,T],H^1(\mathbb{R}^3))$.

Point (ii) follows from almost the same proof as in [BGT04b], while for (i), the supplementary ingredient is to track the dependency on the initial data. We now give the details. To show (i), we will show that there exists $\tau_0>0$ depending only on M so that for any $\epsilon>0$ there is a $\delta>0$ so that

$$\|\widetilde{u}(0)-u(0)\|_{L^2(\mathbb{R}^3)}\leqslant\delta,\qquad\Longrightarrow\sup_{t\in[0,\tau_0]}\|\widetilde{u}(t)-u(t)\|_{L^2(\mathbb{R}^3)}\leqslant\epsilon.$$

As $\tau_0>0$ depends only on $M>0$, the result follows by iterating on $\sim\tau_0^{-1}T$ subintervals. From (2.7) it follows as in [BGT04b, (3.28) p. 591] that, if $\|u\|_{L^\infty((0,T)H^1(\mathbb{R}^3))}\leqslant M$, we have for any $0\leqslant t\leqslant T$, with a constant depending only on $M>0$, for any $p\geqslant 6$

$$\|u\|_{L^2((0,t),L^p(\mathbb{R}^3))}\leqslant C\left(\sqrt{tp}+1\right).\tag{2.11}$$

Introduce

$$g(t):=\|\widetilde{u}(t)-u(t)\|^2_{L^2(\mathbb{R}^3)}=\left\|e^{-it\Delta_G}\left(\widetilde{u}(t)-u(t)\right)\right\|^2_{L^2(\mathbb{R}^3)}.$$

Now, compute

$$\begin{aligned}\partial_t g&\leqslant 2\operatorname{Im}\left\langle|u(t)|^2u(t)-|\widetilde{u}(t)|^2\widetilde{u}(t),u(t)-\widetilde{u}(t)\right\rangle_{L^2}\\&\leqslant C\int\left(|u(t)|^2+|\widetilde{u}(t)|^2\right)|u(t)-\widetilde{u}(t)|^2\\&\leqslant C\left(\|u(t)\|^2_{L^{2p}(\mathbb{R}^3)}+\|\widetilde{u}(t)\|^2_{L^{2p}(\mathbb{R}^3)}\right)\|u(t)-\widetilde{u}(t)\|^2_{L^{2\bar{p}}(\mathbb{R}^3)},\end{aligned}$$

with $p<\infty$ big and $\bar{p}=\frac{p}{p-1}$ (observe that this is the same as [BGT04b, p. 591], but with an inequality in the first line instead of the equality, the damping iau introducing only a dissipative term). From there, it follows as in [BGT04b, p. 591], interpolating the $L^{2\bar{p}}$-norm between the L^2-norm and the L^6-norm, which is bounded by Sobolev embedding, that

$$\partial_t g\leqslant C\left(\|u(t)\|^2_{L^{2p}(\mathbb{R}^3)}+\|\widetilde{u}(t)\|^2_{L^{2p}(\mathbb{R}^3)}\right)g(t)^{1-\frac{3}{2p}},$$

hence, integrating

$$pg(t)^{\frac{3}{2p}}\leqslant C\left(\|u\|^2_{L^2((0,t)L^{2p}(\mathbb{R}^3))}+\|\widetilde{u}\|^2_{L^2((0,t)L^{2p}(\mathbb{R}^3))}\right)+pg(0)^{\frac{3}{2p}}$$

and from (2.11)

$$pg(t)^{\frac{3}{2p}}\leqslant C(tp+1)+pg(0)^{\frac{3}{2p}},$$

from which

$$g(t) \leqslant \left[C\left(t+\frac{1}{p}\right)+g(0)^{\frac{3}{2p}}\right]^{\frac{2p}{3}}$$
$$\leqslant \left[2C\left(t+\frac{1}{p}\right)\right]^{\frac{2p}{3}}+2^{\frac{2p}{3}}g(0),$$

(we used the inequality $(a+b)^q \leqslant 2^q(a^q+b^q)$ for $a,b \geqslant 0$). Now, first fix $\tau_0 > 0$ small enough so that $2C\tau_0 \leqslant \frac{1}{2}$. It then follows that, for any $t \in [0,\tau_0]$

$$\left[2C\left(t+\frac{1}{p}\right)\right]^{\frac{2p}{3}} \leqslant \left[\frac{1}{2}+\frac{2C}{p}\right]^{\frac{2p}{3}},$$

and the right-hand side goes to zero as $p \to \infty$. Let us fix $p \gg 6$ big enough so that

$$\left[\frac{1}{2}+\frac{C}{p}\right]^{\frac{2p}{3}} \leqslant \frac{\epsilon}{2}.$$

With such a p being fixed, we now take $\delta > 0$ small enough so that

$$2^{\frac{2p}{3}}\delta^2 \leqslant \frac{\epsilon}{2}.$$

Part (i) of the claim follows. Part (ii) is shown similarly with $g(0) = 0$, exactly as in [BGT04b].

Step 3: Conclusion. — Let now $u_0 \in H^s(\mathbb{R}^3)$, $s = 1+\epsilon$, and $u \in C([0,T_{\max}), H^s(\mathbb{R}^3))$ the associated maximal forward solution. We approximate u_0 in $H^s(\mathbb{R}^3)$ by a sequence of functions $(u_0^n)_{n\geqslant 1}$ in $H^m(\mathbb{R}^3)$, with m big enough. Let u^n be the associated solutions. From the first step, they are global, and in $L^\infty H^1$ and $L^\infty_{\text{loc}} H^s$, both uniformly in n. From the claim, point (i), for any $T > 0$, u^n is Cauchy in $C([0,T], L^2(\mathbb{R}^3))$. It follows that there exists a function v so that

$$\forall\, T > 0, \quad \|u^n - v\|_{C([0,T],L^2(\mathbb{R}^3))} \longrightarrow 0.$$

From the above and the uniform $L^\infty H^1$ bound on u^n, it follows that $v \in L^\infty H^1$. Thus, by interpolation, we get

$$\forall\, T > 0,\ \forall\, 0 \leqslant \sigma < 1, \quad \|u^n - v\|_{C([0,T],H^\sigma(\mathbb{R}^3))} \longrightarrow 0.$$

Using the above and the uniform $L^\infty_{\text{loc}} H^s$ bound on u^n, we get $v \in L^\infty_{\text{loc}} H^s$. Interpolating one last time, we get

$$\forall\, T > 0,\ \forall\, 0 \leqslant \sigma < s, \quad \|u^n - v\|_{C([0,T],H^\sigma(\mathbb{R}^3))} \longrightarrow 0.$$

From Sobolev embedding, it follows that for any $T > 0$, $u^n|u^n|^2 \to v|v|^2$ in the space $C([0,T], L^2(\mathbb{R}^3))$, hence v is solution to (2.9). From point (ii) of the claim, it follows that $u = v$ on $[0,T_{\max})$. As $v \in L^\infty_{\text{loc}} H^s$, this ends the proof. □

3. Uniform bound on the energy and local energy decay

The main result of this section is the following.

Proposition 3.1. — *Assume that the control condition* (1.3) *holds. Then, for any* $\epsilon > 0$, *for any* $u_0 \in H^{1+\epsilon}(\mathbb{R}^3)$, *the unique maximal solution* u *to* (1.1) *with data* u_0 *is defined globally forward in time, and verifies, for any* $R > 0$

$$\sup_{t>0} E[u(t)] \lesssim \|u_0\|^2_{H^1(\mathbb{R}^3)} + \|u_0\|^4_{L^4(\mathbb{R}^3)} < \infty,$$

$$\|u\|^2_{L^2(\mathbb{R}_+,L^2(B(0,R)))} \lesssim_R \|u_0\|^2_{H^1(\mathbb{R}^3)} + \|u_0\|^4_{L^4(\mathbb{R}^3)} < \infty.$$

As preliminaries, the mass and energy laws are stated in Section 3.1, and a perturbative (with respect to $G = \mathrm{I}$) Morawetz identity is given in Section 3.2. Proposition 3.1 is then showed in Section 3.3. Local energy decay statements are given as Corollaries in Section 3.4.

3.1. Mass and energy laws

Lemma 3.2. — *Let* $\epsilon > 0$, $u_0 \in H^{1+\epsilon}(\mathbb{R}^3)$, *and* $u \in C([0, T_{\max}), H^{1+\epsilon}(\mathbb{R}^3))$ *the unique maximal solution to* (1.1). *Then for any* $0 \leqslant t_0 \leqslant t < T_{\max}$,

$$\|u(t)\|^2_{L^2(\mathbb{R}^3)} - \|u(t_0)\|^2_{L^2(\mathbb{R}^3)} = -2\int_{t_0}^t \int a(x)|u(s,x)|^2 dxds, \tag{3.1}$$

in particular,

$$\int_0^{T_{\max}} \int a(x)|u(s,x)|^2 dxds \leqslant \frac{\|u_0\|^2_{L^2(\mathbb{R}^3)}}{2}. \tag{3.2}$$

In addition,

$$E[u(t)] - E[u(t_0)] \tag{3.3}$$

$$= -\int_{t_0}^t \int a\Big(|u|^4 + G\nabla u \cdot \nabla \overline{u}\Big)\, dxds + \Re \int \int_{t_0}^t G\nabla u \cdot \overline{u}\nabla a\, dxds$$

$$= -\int_{t_0}^t \int a\Big(|u|^4 + G\nabla u \cdot \nabla \overline{u}\Big)\, dxds - \frac{1}{2}\int \int_{t_0}^t |u|^2 \Delta_G a\, dxds. \tag{3.4}$$

Proof. — We begin with (3.3)–(3.4). By taking formally the scalar product of (1.1) with $\partial_t u$, we obtain (denoting $(f,g) := \Re \int f\overline{g}\, dx$):

$$0 = (\partial_t u, i\partial_t u) = \Big(\partial_t u, -\Delta_G u + |u|^2 u - iau\Big) = \frac{d}{dt}E[u(t)] - (\partial_t u, iau).$$

For the last term on the right-hand side, we observe that

$$-(\partial_t u, iau) = -\Big(i\Delta_G u - i|u|^2 u - au, iau\Big)$$
$$= (|u|^2 u, au) + (G\nabla u, a\nabla u) + (G\nabla u, u\nabla a),$$

thus we obtain (3.3) by time integration and (3.4) follows by integration by parts. A classical approximation argument can be used to justify the computations. Similarly, by taking the scalar product of (1.1) with iu, we get

$$(i\partial_t u, iu) = \frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\mathbb{R}^3)} = (i\nabla \cdot (G\nabla u), u) - (au, u) = -(au, u) - (iG\nabla u, \nabla u).$$

The last term on the right-hand side is zero because G is symmetric, and we obtain (3.1) after integration in time. □

Remark 3.3. — It is possible to obtain an exponential bound on the H^1-norm of solutions (including negative times) in the following way. On the one hand, from Gronwall's inequality and (3.1) we have

$$M[u(t)] \leqslant M[u_0]e^{2\|a\|_{L^\infty}|t|}.$$

On the other hand, (3.4) implies, formally, that

$$\frac{d}{dt}E[u(t)] \lesssim \|a\|_{C^2(\mathbb{R}^3)}\left(E[u(t)] + M[u(t)]\right).$$

By Gronwall's inequality, we thus obtain

$$\|\nabla u(t)\|_{L^2}^2 \leqslant 2E[u(t)] \lesssim e^{C|t|}E[u_0].$$

In what follows, we will show a uniform-in-time upper bound on the energy for positive times.

3.2. A perturbative Morawetz identity

Proposition 3.4. — *Let $\chi \in C^4(\mathbb{R}^3)$ be so that, for any multi-index α of size $|\alpha| \leqslant 4$*

$$|\partial^\alpha \chi(x)| \lesssim \langle x\rangle^{1-|\alpha|}.$$

Then, $C > 0$ exists so that the following holds. Let $\epsilon > 0$, $u_0 \in H^{1+\epsilon}(\mathbb{R}^3)$, and $u \in C([0,T_{\max}), H^{1+\epsilon}(\mathbb{R}^3))$ be solution to (1.1). *Then for any $0 \leqslant s \leqslant t < T_{\max}$,*

$$\begin{aligned}\operatorname{Im}\int \bar u(t)\nabla u(t)\cdot\nabla\chi dx \\ = \operatorname{Im}\int \bar u(s)\nabla u(s)\cdot\nabla\chi dx + \int_s^t\int 2D^2\chi\nabla u\cdot\nabla\bar u - \frac12\Delta^2\chi|u|^2 \\ + \frac12\Delta\chi|u|^4 - 2\operatorname{Im}(a\bar u\nabla u\cdot\nabla\chi)\,dxd\tau + \int_s^t \mathcal{E}(\tau)\,d\tau,\end{aligned}$$

where

$$|\mathcal{E}(t)| \leqslant C\int_{\operatorname{supp}(G-\mathrm{I})} |\nabla u(t,x)|^2 + |u(t,x)|^2 + |u(t,x)|^4\,dx.$$

Proof. — We observe that

$$\frac{d}{dt}\int \operatorname{Im}(\bar u\nabla u\cdot\nabla\chi)dx = -\int \operatorname{Im}\left(\bar u\partial_t u\right)\Delta\chi dx - 2\int \operatorname{Im}\left(\partial_t u\nabla\bar u\right)\cdot\nabla\chi dx =: \mathrm{I}+\mathrm{II}\,.$$

Using equation (1.1) we get

$$\mathrm{I} = \int\left(|u|^4 + |\nabla u|^2\right)\Delta\chi - \frac12|u|^2\Delta^2\chi dx + \Re\int (G-\mathrm{I})\nabla u\cdot\nabla(\bar u\Delta\chi)dx,$$

and

$$\begin{aligned}\mathrm{II} = &\int 2D^2\chi\nabla u\cdot\nabla\bar{u} - \left(\frac{1}{2}|u|^4 + |\nabla u|^2\right)\Delta\chi + 2a\,\mathrm{Im}(u\nabla\bar{u})\cdot\nabla\chi dx\\ &+\int 2\Re\left((G-\mathrm{I})\nabla u\cdot D^2\chi\nabla\bar{u}\right) - \Re\left((G-\mathrm{I})\nabla u\cdot\nabla\bar{u}\Delta\chi\right)\\ &-\Re\sum_{1\leqslant i,j,k\leqslant 3}\partial_k G_{ij}\partial_j u\partial_i\bar{u}\partial_k\chi dx.\end{aligned}$$

We get the result by summing the two equations above and observing that for any $k,i,j\in\{1,2,3\}$, $\operatorname{supp}\partial_k G_{i,j}\subset\operatorname{supp}(G-\mathrm{I})$. □

3.3. Proof of Proposition 3.1

Proof of Proposition 3.1. — We will show that there exists a constant $C>0$ so that, for any $T>0$, if u is solution to (1.1) with data u_0 defined on a time interval $[0,T]$, then

$$\sup_{t\in[0,T]} E[u(t)] + \|u\|^2_{L^2([0,T],L^2(B(0,R)))} \leqslant C\left(\|u_0\|^2_{H^1(\mathbb{R}^3)} + \|u_0\|^4_{L^4(\mathbb{R}^3)}\right).$$

The conditional global well-posedness theory given as Lemma 2.5 then ends the proof.

Observe that, from Lemma 3.2, (3.4), using the fact that G is positive

$$E[u(t)] \leqslant E[u(0)] - \frac{1}{2}\int_0^t\int |u|^2\Delta_G a\,dxds.$$

Let now $\chi(|x|) := \sqrt{|x|^2+1}$, and define

$$\lambda(t) := \int_0^t\int\left(-\Delta^2\chi\right)|u|^2 dxds. \tag{3.5}$$

As $-\Delta^2\chi = \frac{15}{\chi^7} > 0$ and a is compactly supported, $|\Delta_G a| \lesssim -\Delta^2\chi$ and therefore

$$E[u(t)] \leqslant E[u(0)] + C_0\lambda(t), \tag{3.6}$$

where $C_0>0$ is some constant depending on a and G. We will show that

$$\sup_{t>0}\lambda(t) \lesssim \|u_0\|^2_{H^1(\mathbb{R}^3)} + \|u_0\|^4_{L^4(\mathbb{R}^3)} < \infty, \tag{3.7}$$

from which the result follows thanks to (3.6). To do so, we let

$$V(t) := \mathrm{Im}\int u\nabla\bar{u}\cdot\nabla\chi dx.$$

By Cauchy–Schwartz inequality, the uniform bound on the mass and (3.6)

$$\begin{aligned}|V(t)| &\lesssim M[u(t)]^{\frac12}E[u(t)]^{\frac12}\\ &\lesssim M[u(0)]^{\frac12}\left(E[u(0)] + C_0\lambda(t)\right)^{\frac12}.\end{aligned} \tag{3.8}$$

The plan is to show that, on the other hand, $V(t) \gtrsim \lambda(t) - C(M[u(0)] + E[u(0)])$, from which the result will follow.

By Proposition 3.4,

$$V(t) \geqslant \frac{1}{2}\int_0^t \int D^2\chi \nabla u \cdot \nabla \bar{u} dx ds + \lambda(t) - C_1 \int_0^t \int_{\operatorname{supp}(G-\mathrm{I})} |\nabla u|^2 + |u|^2 + |u|^4 dx ds - 2\int_0^t \int |a||\nabla \chi||u||\nabla u| dx ds - C_2 M[u(0)]^{\frac{1}{2}} E[u(0)]^{\frac{1}{2}}, \tag{3.9}$$

with $C_1, C_2 > 0$ constants depending only on G (and our choice of χ). We first bound the $\int_{\operatorname{supp}(G-\mathrm{I})}$ error term. From Lemma 3.2, (3.3),

$$\begin{aligned}\int_0^t \int a(|u|^4 + (G\nabla u \cdot \nabla \bar{u})) dx ds &= \Re \int_0^t \int (G\nabla u \cdot \bar{u} \nabla a) dx ds + E[u(0)] - E[u(t)] \\ &\leqslant \int_0^t \int |G||\nabla u||u||\nabla a| dx ds + E[u(0)].\end{aligned} \tag{3.10}$$

Observe that (see for example [YNC21, Lemma 4.4], [LBM25, Lemma 4.1]),

$$\forall\, \epsilon > 0,\ \exists\, C_\epsilon > 0 \quad \text{s.t.} \quad \forall\, x \in \mathbb{R}^3, \quad |\nabla a(x)| \leqslant C_\epsilon a(x) + \epsilon. \tag{3.11}$$

Indeed, it suffices to show the above for $x \in \operatorname{supp} \nabla a$. By contradiction, if the claim fails, there exists $\epsilon_0 > 0$ and a sequence $x_n \in \operatorname{supp} \nabla a$ so that $|\nabla a(x_n)| \geqslant n a(x_n) + \epsilon_0$. As $\operatorname{supp} \nabla a$ is compact, there exists a subsequence of x_n converging to some $x_\infty \in \mathbb{R}^3$. As $a \geqslant 0$ and ∇a is bounded, we have necessarily $a(x_\infty) = 0$. In particular, a attains a local minimum in x_∞, hence $\nabla a(x_\infty) = 0$, which contradicts $|\nabla a(x_\infty)| \geqslant \epsilon_0 > 0$.

Let $\epsilon > 0$ to be fixed later, and $C_\epsilon > 0$ be given by (3.11). Let now $\psi \in C_c^\infty(\mathbb{R}^3)$ be so that $\psi = 1$ on $\operatorname{supp} a$, $0 \leqslant \psi \leqslant 1$, and ψ is supported in $\operatorname{supp} a + B(0,1)$. By multiplying (3.11) by ψ, we get

$$|\nabla a| \leqslant C_\epsilon a + \epsilon \psi.$$

Now, from (3.10) and the above

$$\int_0^t \int a(|u|^4 + (G\nabla u \cdot \nabla \bar{u})) dx ds \tag{3.12}$$

$$\leqslant C_\epsilon \int_0^t \int |G||\nabla u||u||a| dx ds + \epsilon \int_0^t \int |G||\nabla u||u|\psi dx ds + E[u(0)]$$

$$\leqslant C_\epsilon \int_0^t \int |G||\nabla u||u||a| dx ds + \frac{\epsilon}{2}\int_0^t \int |G||u|^2 \psi dx ds + \frac{\epsilon}{2}\int_0^t \int |G||\nabla u|^2 \psi dx ds + E[u(0)] \tag{3.13}$$

$$\leqslant C_\epsilon \int_0^t \int |G||\nabla u||u||a| dx ds + \epsilon \sup |G| C_3 \lambda(t) + \frac{\epsilon}{2}\int_0^t \int |G||\nabla u|^2 \psi dx ds + E[u(0)], \tag{3.14}$$

where $C_3 > 0$ is a constant depending only on χ and ψ (and hence $\operatorname{supp} a$). Recall also that, from the mass law (Lemma 3.2, (3.2))

$$\int_0^t \int a|u|^2 dxds \leqslant \frac{1}{2} M[u(0)]. \tag{3.15}$$

Now, let $\delta_0 > 0$ be so that

$$\forall\, x \in \operatorname{supp}(G - \mathrm{I}), \quad a(x) \geqslant \delta_0.$$

Observe that

$$\int_{\operatorname{supp}(G-\mathrm{I})} |\nabla u|^2 + |u|^2 + |u|^4 dx \leqslant \frac{1}{\delta_0} \int a \left(|\nabla u|^2 + |u|^2 + |u|^4 \right) dx,$$

and recall that there exists $c_{\mathrm{coerc}} > 0$ so that $G\xi \cdot \xi \geqslant c_{\mathrm{coerc}} |\xi|^2$, hence denoting $\delta := \min(1, c_{\mathrm{coerc}}^{-1})\delta_0$,

$$\int_{\operatorname{supp}(G-\mathrm{I})} |\nabla u|^2 + |u|^2 + |u|^4 dx \leqslant \frac{1}{\delta} \int a \left((G\nabla u \cdot \nabla \overline{u}) + |u|^2 + |u|^4 \right) dx.$$

We therefore get, combining the above with (3.14), (3.15) and (3.9)

$$V(t) \geqslant \mathrm{I} + \mathrm{II}, \tag{3.16}$$

where, denoting

$$A := C_1 \sup |G| \delta^{-1},$$

we have

$$\mathrm{I} = \lambda(t) - \epsilon A C_3 \lambda(t), \tag{3.17}$$

and

$$\begin{aligned} \mathrm{II} = \frac{1}{2} \int_0^t \int D^2\chi \nabla u \cdot \nabla \overline{u} dxds \\ - \left(2 + C_\epsilon A \right) \int_0^t \int |\nabla u||u||a| dxds - \frac{\epsilon}{2} A \int_0^t \int |\nabla u|^2 \psi dxds \\ - E[u(0)] - C_2 M[u(0)]^{\frac{1}{2}} E[u(0)]^{\frac{1}{2}} - \frac{1}{2}\delta^{-1} M[u(0)]. \end{aligned} \tag{3.18}$$

Observe that, as $D^2\chi(x) = \frac{\mathrm{I}}{\chi(x)} - \frac{(x_i x_j)_{i,j}}{\chi(x)^3} \gtrsim \frac{\mathrm{I}}{\chi(x)^3}$ (in the sense of quadratic forms) and ψ is compactly supported we have

$$\int_0^t \int D^2\chi \nabla u \cdot \nabla \overline{u} dxds \geqslant \eta \int_0^t \int |\nabla u|^2 \psi dxds, \tag{3.19}$$

where $\eta = \eta(\chi, \psi) > 0$ depends only on ψ and χ. Let us now fix $\epsilon > 0$ small enough so that

$$\epsilon A C_3 \leqslant \frac{1}{2} \quad \text{and} \quad \epsilon A \leqslant \frac{1}{2}\eta.$$

We get from (3.16), (3.17), (3.18) together with (3.19)

$$\mathrm{I} \geqslant \frac{1}{2}\lambda(t) \tag{3.20}$$

and

$$\text{II} \geqslant \frac{1}{4}\eta \int_0^t \int |\nabla u|^2 \psi dx ds - \Big(2 + C_\epsilon A\Big) \int_0^t \int |\nabla u||u||a| dx ds \\ - E[u(0)] - C_2 M[u(0)]^{\frac{1}{2}} E[u(0)]^{\frac{1}{2}} - \frac{1}{2}\delta^{-1} M[u(0)]. \tag{3.21}$$

We will now show that

$$\text{II} \geqslant -C \times \Big(M[u(0)] + E[u(0)]\Big), \tag{3.22}$$

for a universal constant $C > 0$. Indeed, by Cauchy–Schwartz inequality and the mass law (Lemma 3.2, (3.2))

$$\int_0^t \int |\nabla u||u||a| dx ds \leqslant \Big(\int_0^t \int a|\nabla u|^2 dx ds\Big)^{\frac{1}{2}} \Big(\int_0^t \int a|u|^2 dx ds\Big)^{\frac{1}{2}} \\ \leqslant \frac{1}{2} M[u(0)]^{\frac{1}{2}} \Big(\int_0^t \int a|\nabla u|^2 dx ds\Big)^{\frac{1}{2}},$$

hence, as $\psi = 1$ on the support of a,

$$\int_0^t \int |\nabla u||u||a| dx ds \leqslant \frac{1}{2} M[u(0)]^{\frac{1}{2}} \sup |a| \Big(\int_0^t \int |\nabla u|^2 \psi dx ds\Big)^{\frac{1}{2}}. \tag{3.23}$$

Plugging the above in (3.21), we obtain

$$\text{II} \geqslant F\left(\Big(\int_0^t \int |\nabla u|^2 \psi\Big)^{\frac{1}{2}}\right),$$

where

$$F(X) := \frac{1}{4}\eta X^2 - \Big(1 + \frac{C_\epsilon A}{2}\Big) M[u(0)]^{\frac{1}{2}} \sup |a| X \\ - E[u(0)] - C_2 M[u(0)]^{\frac{1}{2}} E[u(0)]^{\frac{1}{2}} - \frac{1}{2}\delta^{-1} M[u(0)].$$

The fact that the parabola of equation $Y = F(X)$ is always above its vertex gives (3.22) (with $C > 0$ depending on A, ϵ, η, and $\sup |a|$).

Finally, combining (3.16) with (3.20), (3.22) and (3.8), we obtain

$$\lambda(t)^2 - b \times \Big(M[u(0)] + E[u(0)]\Big)\lambda(t) - c \times \Big(M[u(0)] + E[u(0)]\Big)^2 \leqslant 0,$$

with $b, c > 0$, universal constants. The estimate (3.7), and hence the result, follows. □

3.4. Local energy decay

Corollary 3.5. — *Assume that* (1.3) *holds. Then, for any $R > 0$ there is $C > 0$ so that for any solution u to* (1.1),

$$\|u\|^2_{L^2(\mathbb{R}_+, H^1(B(0,R)))} \leqslant C \Big(\|u_0\|^2_{H^1(\mathbb{R}^3)} + \|u_0\|^4_{L^4(\mathbb{R}^3)}\Big) < \infty. \tag{3.24}$$

Proof. — Let $T>0$. Observe that, by Lemma 3.2, (3.4)

$$\int_0^T\int a\left(|\nabla u|^2+|u|^4\right)dxds=E[u_0]-E[u(T)]-\frac{1}{2}\int_0^T\int(\Delta_G a)|u|^2dxds,$$

hence Proposition 3.1 implies that

$$\int_0^\infty\int a\left(|\nabla u|^2+|u|^4\right)dxds\lesssim\|u_0\|^2_{H^1(\mathbb{R}^3)}+\|u_0\|^4_{L^4(\mathbb{R}^3)}<+\infty.$$

Now, with $\chi\in C^\infty$ defined as in the proof of Proposition 3.1, observe that (3.9) together with (3.7), the above and the fact that $a(x)\geqslant\delta_0>0$ in $\operatorname{supp}(G-\mathrm{I})$ yields

$$\int_0^\infty\int D^2\chi\nabla u\cdot\nabla udxds\lesssim\|u_0\|^2_{H^1(\mathbb{R}^3)}+\|u_0\|^4_{L^4(\mathbb{R}^3)}.$$

The result follows as $D^2\chi\geqslant 0$ and $D^2\chi\geqslant C_R$ in $B(0,R)$. □

Corollary 3.6. — *Assume that* (1.3) *holds. Then, for any solution u to* (1.1) *and any $R>0$*

$$\|u(t)\|_{H^s(B(0,R))}\to 0.$$

for any $s\in[0,1)$

Proof. — Let $\chi\in C_c^\infty(\mathbb{R}^3)$ be arbitrary. Observe that, by Corollary 3.5, there exists $t_n\to+\infty$ so that

$$\|\chi u(t_n)\|_{H^1(\mathbb{R}^3)}\longrightarrow 0.$$

But

$$\partial_t\|\chi u\|^2_{L^2(\mathbb{R}^3)}=-\int\chi^2a|u|^2dx-\operatorname{Im}\int G\nabla\chi^2\cdot\nabla u\bar{u}dx,$$

from which, for any $t\geqslant t_n$

$$\|\chi u(t)\|^2_{L^2(\mathbb{R}^3)}\leqslant\|\chi u(t_n)\|^2_{L^2(\mathbb{R}^3)}+\int_{t_n}^t\int|G\nabla\chi^2\cdot\nabla u\bar{u}|dxds.$$

In particular, taking $\widetilde{\chi}\in C_c^\infty(\mathbb{R}^3)$ so that $\widetilde{\chi}=1$ on $\operatorname{supp}\chi$

$$\|\chi u(t)\|^2_{L^2(\mathbb{R}^3)}\leqslant\|\chi u(t_n)\|^2_{L^2(\mathbb{R}^3)}+C\int_{t_n}^\infty\int\widetilde{\chi}\left(|\nabla u|^2+|u|^2\right)dxds.$$

The above goes to zero as $n\to\infty$ thanks to Corollary 3.5, therefore

$$\|\chi u(t)\|_{L^2(\mathbb{R}^3)}\longrightarrow 0.$$

as $t\to\infty$. The result follows by interpolation: indeed, by Plancherel, for $t\geqslant 0$, using the fact that u is bounded in $H^1(\mathbb{R}^3)$ on $\mathbb{R}_+$

$$\|D^s(\chi u)\|^2_{L^2(\mathbb{R}^3)}\leqslant\|\nabla(\chi u)\|^{2s}_{L^2(\mathbb{R}^3)}\|\chi u\|^{2(1-s)}_{L^2(\mathbb{R}^3)}\lesssim\|\widetilde{\chi}u\|_{L^2(\mathbb{R}^3)}\longrightarrow 0,$$

so we get $\|D^s(\chi u)\|_{L^2(\mathbb{R}^3)}\to 0$ for any $0\leqslant s<1$. □

Remark 3.7 (The linear case). — Strictly analogous computations as the ones presented in this Section can be carried out in the linear case. This shows local energy decay for the linear damped flow $S(t)$, in the form that $\|S(t)u_0\|_{L^2(\mathbb{R}_+,H^1(\mathbb{R}^3))}\lesssim\|u_0\|_{H^1}$, and hence provide an alternative proof of the result of [AK07] in our setting, under the (stronger) control condition (1.3).

4. Bilinear estimates

The purpose of this section is to show the following:

PROPOSITION 4.1. — *Assume that* (1.3) *holds. Then, for any* $\epsilon > 0$*, any solution* $u \in C([0,\infty), H^{1+\epsilon}(\mathbb{R}^3))$ *to* (1.1) *verifies*

$$\int_0^\infty \int |u(x,t)|^4 dxdt \lesssim 1. \tag{4.1}$$

Proof. — The overall idea is to perform a bilinear virial computation similar to the one known in $\mathbb{R}^3$ (see for example [CKS$^+$04, Proposition 2.5]), perturbatively with respect to $G = \mathrm{I}$, and to handle the error terms, localized where $G \neq \mathrm{I}$, thanks to Corollary 3.5. To this purpose, for $x \in \mathbb{R}^3$, define $\rho(x) := |x|$, and

$$B(t) := \int |u(y)|^2 \int \mathrm{Im}(\bar{u}(x)\nabla u(x)) \cdot \nabla\rho(x-y)dxdy.$$

Observe for later use that

$$\nabla\rho = \frac{x}{|x|}, \quad \Delta\rho = \frac{2}{|x|}, \quad \nabla\Delta\rho = \frac{2x}{|x|^3}, \quad \Delta^2\rho = -8\pi\delta_{x=0}.$$

We have

$$\begin{aligned} \frac{d}{dt}B(t) &= 2\iint \Re(\bar{u}(y)\partial_t u(y))\,\mathrm{Im}(\bar{u}(x)\nabla u(x)) \cdot \nabla\rho(x-y)dxdy \\ &\quad + \iint |u(y)|^2\,\mathrm{Im}(\partial_t\bar{u}(x)\nabla u(x)) \cdot \nabla\rho(x-y)dxdy \\ &\quad + \iint |u(y)|^2\,\mathrm{Im}(\bar{u}(x)\nabla\partial_t u(x)) \cdot \nabla\rho(x-y)dxdy \\ &= 2\iint \Re(\bar{u}(y)\partial_t u(y))\,\mathrm{Im}(\bar{u}(x)\nabla u(x)) \cdot \nabla\rho(x-y)dxdy \\ &\quad - \iint |u(y)|^2\,\mathrm{Im}(\bar{u}(x)\partial_t u(x))\Delta\rho(x-y)dxdy \\ &\quad - 2\iint |u(y)|^2\,\mathrm{Im}(\partial_t u(x)\nabla\bar{u}(x)) \cdot \nabla\rho(x-y)dxdy \\ &=: \mathrm{I} + \mathrm{II} + \mathrm{III}\,. \end{aligned} \tag{4.2}$$

Using equation (1.1), we compute the three quantities above separately. We have

$$\begin{aligned} \frac{1}{2}\mathrm{I} &= \iint \Re\left(\bar{u}(y)\left(i\Delta_G u(y) - i|u(y)|^4u(y) - a(y)u(y)\right)\right)\mathrm{Im}(\bar{u}(x)\nabla u(x)) \\ &\qquad \cdot \nabla\rho(x-y)dxdy \\ &= -\iint a(y)|u(y)|^2\,\mathrm{Im}(\bar{u}(x)\nabla u(x)) \cdot \nabla\rho(x-y)dxdy \\ &\quad + \iint \mathrm{Im}(\bar{u}(x)\nabla u(x)) \cdot \nabla\rho(x-y)\,\mathrm{Im}\left(\nabla\bar{u}(y) \cdot (G(y)\nabla u(y))\right)dxdy \\ &\quad + \iint \mathrm{Im}(\bar{u}(y)(G(y)\nabla u(y)) \cdot \nabla_y\left(\mathrm{Im}(\bar{u}(x)\nabla u(x)) \cdot \nabla\rho(x-y)\right)dxdy \\ &=: \mathrm{Ia} + \mathrm{Ib} + \mathrm{Ic}\,. \end{aligned}$$

For the first term, we observe that

$$\begin{aligned} |\,\mathrm{Ia}\,| &\leqslant \iint a(y)|u(x)|\,|\nabla u(x)|\,|\nabla\rho(x-y)|\,|u(y)|^2dxdy \\ &\lesssim \|u\|_{H^1}^2\int a(y)|u(y)|^2dy \end{aligned}$$

as long as $|\nabla\rho(x-y)|\leqslant C<\infty$. The second term Ib equals zero since G is symmetric. For the last term, we split the integral as

$$\begin{aligned} \mathrm{Ic} &= \iint \mathrm{Im}\Big(\bar u(y)((G(y)-\mathrm{I})\nabla u(y))\cdot\nabla_y\Big(\mathrm{Im}(\bar u(x)\nabla u(x))\Big)\cdot\nabla\rho(x-y)\Big)dxdy \\ &\quad+\iint \mathrm{Im}(\bar u(y)\nabla u(y))\cdot\nabla_y\Big(\mathrm{Im}(\bar u(x)\nabla u(x))\cdot\nabla\rho(x-y)\Big)dxdy \\ &=: \mathrm{Ic1}+\mathrm{Ic2}\,. \end{aligned}$$

Here, in the first term, y is localized in space, and we compute by the Cauchy–Schwartz inequality

$$\begin{aligned} |\,\mathrm{Ic1}\,| &\lesssim \iint_{y\in\mathrm{supp}(G-\mathrm{I})}\Big|D^2\rho(x-y)\Big|\,|u(x)|\,|\nabla u(x)|\,|u(y)|\,|\nabla u(y)|dxdy \\ &\leqslant \int_{y\in\mathrm{supp}(G-\mathrm{I})}\Big\|D^2\rho(.-y)u\Big\|_{L^2}\|\nabla u\|_{L^2}|u(y)|\,|\nabla u(y)|dy \\ &\lesssim \|\nabla u\|_{L^2}^2\int_{y\in\mathrm{supp}(G-\mathrm{I})}|u(y)|\,|\nabla u(y)|dy \\ &\lesssim \|\nabla u\|_{L^2}^2\int_{y\in\mathrm{supp}(G-\mathrm{I})}|\nabla u(y)|^2+|u(y)|^2dy, \end{aligned}$$

where we used that, thanks to the Hardy inequality, $\|D^2\rho(.-y)u\|_{L^2}\lesssim\|\frac{1}{|x|}u\|_{L^2}\lesssim\|\nabla u\|_{L^2}$. For the term Ic2, from our choice of ρ, we obtain, similarly to [CKS+04, Proposition 2.5],

$$\begin{aligned} \mathrm{Ic2} &\geqslant -\iint |u(x)|\,|\nabla u(x)|\,|u(y)|\,|\nabla u(y)|\frac{dxdy}{|x-y|} \\ &\geqslant -\frac{1}{2}\iint |u(y)|^2|\nabla u(x)|^2\frac{dxdy}{|x-y|}. \end{aligned} \tag{4.3}$$

For the second term in (4.2), we use equation (1.1) to obtain

$$\begin{aligned} \mathrm{II} &= \iint |u(y)|^2|u(x)|^4\Delta\rho(x-y)dxdy \\ &\quad+\iint |u(y)|^2\Re\Big(\nabla\bar u(x)\cdot(G(x)\nabla u(x))\Big)\Delta\rho(x-y)dxdy \\ &\quad+\iint |u(y)|^2\Re\Big(\bar u(x)(G(x)\nabla u(x))\Big)\cdot\nabla_x\Delta\rho(x-y)dxdy \\ &=: \mathrm{IIa}+\mathrm{IIb}+\mathrm{IIc}\,. \end{aligned}$$

We split

$$\begin{aligned}\mathrm{IIb} &= \iint |u(y)|^2|\nabla\bar{u}(x)|^2\Delta\rho(x-y)dxdy\\ &\quad+\iint |u(y)|^2\nabla\bar{u}(x)\cdot(G(x)-\mathrm{I})\nabla u(x)\Delta\rho(x-y)dxdy\\ &=: \mathrm{IIb1}+\mathrm{IIb2}\end{aligned}$$

Now we observe that, with our choice of ρ,

$$\mathrm{IIb1} + 2\,\mathrm{Ic2} \geqslant \frac{1}{2}\,\mathrm{IIb1},$$

indeed, using (4.3) and the fact that $\Delta\rho = \frac{2}{|x|}$,

$$\begin{aligned}\mathrm{IIb1} + 2\,\mathrm{Ic2} &\geqslant \iint |u(y)|^2|\nabla\bar{u}(x)|^2\Delta\rho(x-y)dxdy - \iint |u(y)|^2|\nabla u(x)|^2\frac{dxdy}{|x-y|}\\ &= \iint |u(y)|^2|\nabla u(x)|^2\frac{dxdy}{|x-y|}\\ &= \frac{1}{2}\iint |u(y)|^2|\nabla\bar{u}(x)|^2\Delta\rho(x-y)dxdy\\ &= \frac{1}{2}\,\mathrm{IIb1}\,.\end{aligned}$$

In addition, from Hardy inequality, similar to above

$$|\mathrm{IIb2}| \lesssim \|u\|^2_{H^1(\mathbb{R}^3)}\int_{x\in\mathrm{supp}(G-\mathrm{I})}\left(|u(x)|^2+|\nabla u(x)|^2\right)dx.$$

For the third term in the equation above, we separate the integral in x and get

$$\begin{aligned}\mathrm{IIc} &= -\frac{1}{2}\iint |u(y)|^2|u(x)|^2\Delta\Delta\rho(x-y)dxdy\\ &\quad+\iint |u(y)|^2\Re\Big(\bar{u}(x)((G(x)-\mathrm{I})\nabla u(x))\Big)\cdot\nabla_x\Delta\rho(x-y)dxdy\\ &=: \mathrm{IIc1}+\mathrm{IIc2},\end{aligned}$$

and for the last term above, we have

$$\begin{aligned}|\mathrm{IIc2}| &\leqslant \int_{x\in\mathrm{supp}(G-\mathrm{I})}|u(x)||(G(x)-\mathrm{I})\nabla u(x)|\int |u(y)|^2|\nabla\Delta\rho(x-y)|dydx\\ &\lesssim \|u\|^2_{H^1(\mathbb{R}^3)}\int_{x\in\mathrm{supp}(G-\mathrm{I})}\left(|u(x)|^2+|\nabla u(x)|^2\right)dx,\end{aligned}$$

due to the Hardy inequality.

Now we deal with the third term in (4.2). By using (1.1) we obtain

$$\begin{aligned}\frac{1}{2}\,\mathrm{III} &= \iint |u(y)|^2 a(x)\,\mathrm{Im}(\nabla\bar{u}(x)u(x))\cdot\nabla\rho(x-y)dxdy\\ &\quad-\frac{1}{4}\iint |u(y)|^2|u(x)|^4\Delta\rho(x-y)dxdy\\ &\quad+\iint |u(y)|^2\Re((G(x)\nabla u(x))\cdot\nabla_x(\nabla\bar{u}(x)\cdot\nabla\rho(x-y))dxdy\\ &=: \mathrm{IIIa}+\mathrm{IIIb}+\mathrm{IIIa}\,.\end{aligned}$$

For the first term in the above equation, we observe that

$$|\mathrm{IIIa}| \leqslant \frac{1}{2}\|u\|_{L^2}^2 \int a(x)\left(|u(x)|^2 + |\nabla u(x)|^2\right) dx.$$

For the last term,

$$\begin{aligned}\mathrm{IIIa} &= \iint |u(y)|^2 \Re(G(x)\nabla u(x)) \cdot D^2\rho(x-y)\nabla u(x) dx dy \\ &\quad + \Re \iint |u(y)|^2 \sum_{i,j,k}^{3} G_{i,j}(x)\partial_j u(x) \partial_{i,k}\bar{u}(x)\partial_k \rho(x-y) dx dy \\ &=: \mathrm{IIIc1} + \mathrm{IIIc2}\,.\end{aligned}$$

From Hardy's inequality, we get

$$\mathrm{IIIc1} = \iint |u(y)|^2 \nabla u(x) \cdot D^2\rho(x-y)\nabla u(x) dx dy + \mathrm{IIIc1(ii)},$$

with

$$|\mathrm{IIIc1(ii)}| \lesssim \|u\|_{L^2}^2 \int_{x \in \operatorname{supp}(G-\mathrm{I})} |\nabla u(x)|^2 dx.$$

For the term IIIc2, we observe that by integration by parts

$$\begin{aligned}\mathrm{IIIc2} &= -\Re \iint |u(y)|^2 \sum_{i,j,k}^{3} \partial_k (G_{i,j}(x))\partial_j u(x)\partial_i \bar{u}(x)\partial_k\rho(x-y) dx dy \\ &\quad - \iint |u(y)|^2 \Re\Big(\nabla \bar{u}(x) \cdot G(x)\nabla u(x)\Big)\Delta\rho(x-y) dx dy \\ &\quad - \iint |u(y)|^2 \sum_{i,j,k}^{3} G_{i,j}(x)\partial_{j,k} u(x)\partial_i \bar{u}(x)\partial_k\rho(x-y) dx dy.\end{aligned}$$

The last term on the equation's right-hand side above equals IIIc2. This becomes evident if one swaps the indexes i and j and later observes that $G_{i,j} = G_{j,i}$ as G is symmetric. Thus we obtain

$$\begin{aligned}\mathrm{IIIc2} &= -\frac{1}{2}\Re \iint |u(y)|^2 \sum_{i,j,k}^{3} \partial_k (G_{i,j}(x))\partial_j u(x)\partial_i \bar{u}(x)\partial_k\rho(x-y) dx dy \\ &\quad - \frac{1}{2}\iint |u(y)|^2 \Re\Big(\nabla \bar{u}(x) \cdot (G(x)-\mathrm{I})\nabla u(x)\Big)\Delta\rho(x-y) dx dy \\ &\quad - \frac{1}{2}\iint |u(y)|^2 |\nabla \bar{u}(x)|^2 \Delta\rho(x-y) dx dy \\ &=: \mathrm{IIIc2(i)} + \mathrm{IIIc2(ii)} + \mathrm{IIIc2(iii)}.\end{aligned}$$

Now we observe that IIIc2(i) is localized in space in the x variable as $\partial_k G_{i,j}(x) \equiv 0$ when $x \notin \operatorname{supp}(G-\mathrm{I})$. So we get, using Hardy's inequality for IIIc2(ii) we obtain

$$|\mathrm{IIIc2(i)}| + |\mathrm{IIIc2(ii)}| \lesssim \|u\|_{L^2}^2 \int_{x \in \operatorname{supp}(G-\mathrm{I})} |\nabla u(x)|^2 dx.$$

Putting together every term above, we reach the following conclusion: there exists a universal constant $C > 0$ such that

$$\begin{aligned}C\|u\|_{H^1(\mathbb{R}^3)}^2 \int_{\operatorname{supp} a} |\nabla u|^2 + |u|^2 dx + \frac{d}{dt}B(t)& \\ \geqslant \frac{1}{2} \iint |u(y)|^2 |\nabla \bar{u}(x)|^2 \Delta\rho(x-y) dxdy &+ 4\pi \int |u|^4 dx \\ + \frac{3}{4} \iint |u(y)|^2 |u(x)|^4 \Delta\rho(x-y) dxdy& \\ + \iint |u(y)|^2 D^2\rho(x-y) \nabla\bar{u}(x) \cdot \nabla u(x) dxdy.&\end{aligned}$$

From the positivity of $D^2\rho$, every term on the right-hand side above is non-negative. Thus, after integration in time, we obtain

$$\begin{aligned}4\pi \int_0^T \int |u|^4 dxdt &\leqslant C\|u\|_{H^1(\mathbb{R}^3)}^2 \int_0^T \int_{\operatorname{supp} a} |\nabla u|^2 + |u|^2 dxdt + B(T) - B(0) \\ &\leqslant C + \|u(T)\|_{L^2(\mathbb{R}^3)}^2 \|u(T)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2 - \|u(0)\|_{L^2(\mathbb{R}^3)}^2 \|u(0)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^2,\end{aligned}$$

where the last inequality follows from Corollary 3.5. From the uniform boundedness of the H^1-norm given by Proposition 3.1, we thus obtain (4.1). ☐

Remark 4.2 (Linear Strichartz estimates). — Similarly to Remark 3.7, the same computation can be performed in the linear case. This shows that the damped linear flow $S(t)$ verifies the global Strichartz-type estimate $\|S(t)u_0\|_{L^4(\mathbb{R}_+,L^4(\mathbb{R}^3))} \lesssim \|u_0\|_{H^1(\mathbb{R}^3)}$. As the scale-invariant estimate is

$$\left\|e^{it\Delta}u_0\right\|_{L^4(\mathbb{R}_+,L^4(\mathbb{R}^3))} \lesssim \|u_0\|_{H^{\frac{1}{4}}(\mathbb{R}^3)},$$

this corresponds to a loss of $\frac{3}{4} = \frac{1}{4} + \frac{1}{2} (= \frac{1}{p} + \frac{1}{2})$ derivatives, which is consistent with Remark 2.2.

5. End of the proof

Proof of Theorem 1.1. — We first show scattering in $H^{\frac{1}{2}}$, then upgrade it to H^s for any $s < 1$ by an interpolation argument.

Let $\chi \in C_c^\infty(\mathbb{R}^3)$ be so that $\chi = 1$ on $\operatorname{supp} a$. We let

$$w := (1-\chi)u.$$

Then, using the fact that $1-\chi = 0$ on $\operatorname{supp} a$ and hence also on $\operatorname{supp}(G-\mathrm{I})$, w solves the equation

$$i\partial_t w - \Delta w = [\Delta, \chi]u - (1-\chi)u|u|^2,$$

hence from Duhamel's formula

$$w(t) = e^{it\Delta}\left((1-\chi)u_0 + i\int_0^t e^{-is\Delta}(1-\chi)u|u|^2(s)\,ds - i\int_0^t e^{-is\Delta}[\Delta,\chi]u(s)\,ds\right).$$

Therefore, as $\|\chi u(t)\|_{H^{\frac{1}{2}}(\mathbb{R}^3)} \to 0$ thanks to Corollary 3.6, in order to conclude, it suffices to show that

$$\left\|\int_t^{\infty} e^{-is\Delta}(1-\chi)u|u|^2(s)\,ds\right\|_{H^{\frac{1}{2}}(\mathbb{R}^3)} \longrightarrow 0 \quad \text{as} \quad t \longrightarrow +\infty \tag{5.1}$$

and

$$\left\|\int_t^{\infty} e^{-is\Delta}[\Delta,\chi]u(s)ds\right\|_{H^{\frac{1}{2}}(\mathbb{R}^3)} \longrightarrow 0 \quad \text{as} \quad t \longrightarrow +\infty. \tag{5.2}$$

Indeed, the result then follows by setting

$$u_+ := (1-\chi)u_0 + i\int_0^{\infty} e^{-is\Delta}(1-\chi)u|u|^2(s)\,ds - i\int_0^{\infty} e^{-is\Delta}[\Delta,\chi]u(s)\,ds.$$

First, observe that (5.2) is a consequence of Corollary 3.5, thanks to the dual estimate to the smoothing effect for the free Schrödinger flow in $\mathbb{R}^3$. More precisely, selecting $\widetilde{\chi} \in C_c^{\infty}(\mathbb{R}^3)$ so that $\widetilde{\chi} = 1$ on $\operatorname{supp}\chi$ and arguing by duality,

$$\begin{aligned}
&\sup_{\substack{F\in L^2(\mathbb{R}^3),\\ \|F\|_{L^2(\mathbb{R}^3)}\leqslant 1}} \left\langle F, \langle D\rangle^{\frac{1}{2}} \int_t^{\infty} e^{-is\Delta}[\Delta,\chi]u(s)ds\right\rangle_{L^2}\\
&\qquad = \sup_{\substack{F\in L^2(\mathbb{R}^3),\\ \|F\|_{L^2(\mathbb{R}^3)}\leqslant 1}} \left\langle \langle D\rangle^{\frac{1}{2}}F, \int_t^{\infty} e^{-is\Delta}[\Delta,\chi]u(s)ds\right\rangle_{L^2}\\
&\qquad = \sup_{\substack{F\in L^2(\mathbb{R}^3),\\ \|F\|_{L^2(\mathbb{R}^3)}\leqslant 1}} \left\langle \langle D\rangle^{\frac{1}{2}}F, \int_t^{\infty} e^{-is\Delta}\widetilde{\chi}[\Delta,\chi]u(s)ds\right\rangle_{L^2}\\
&\qquad = \sup_{\substack{F\in L^2(\mathbb{R}^3),\\ \|F\|_{L^2(\mathbb{R}^3)}\leqslant 1}} \left\langle \widetilde{\chi}e^{is\Delta}\langle D\rangle^{\frac{1}{2}}F, [\Delta,\chi]u(s)\right\rangle_{L^2_s((t,+\infty),L^2(\mathbb{R}^3))}\\
&\qquad \leqslant \sup_{\substack{F\in L^2(\mathbb{R}^3),\\ \|F\|_{L^2(\mathbb{R}^3)}\leqslant 1}} \left\|\widetilde{\chi}e^{is\Delta}\langle D\rangle^{\frac{1}{2}}F\right\|_{L^2_s((t,+\infty),L^2(\mathbb{R}^3))} \|[\Delta,\chi]u(s)\|_{L^2_s((t,+\infty),L^2(\mathbb{R}^3))}\\
&\qquad \lesssim \|[\Delta,\chi]u(s)\|_{L^2_s((t,+\infty),L^2(\mathbb{R}^3))} \lesssim \|\widetilde{\chi}u(s)\|_{L^2_s((t,+\infty),H^1(\mathbb{R}^3))} \longrightarrow 0,
\end{aligned}$$

where we used Kato's smoothing effect in the last line, then Corollary 3.5 to conclude.

Finally, to show (5.1), observe that, thanks to Proposition 4.1

$$\left\|u|u|^2\right\|_{L^2(\mathbb{R}_+,W^{1,1}(\mathbb{R}^3))} \leqslant \|u\|_{L^{\infty}(\mathbb{R}_+,H^1(\mathbb{R}^3))}\|u\|^2_{L^4(\mathbb{R}_+,L^4(\mathbb{R}^3))} < +\infty. \tag{5.3}$$

But we have the Sobolev embedding $W^{1,1}(\mathbb{R}^3) \hookrightarrow W^{\frac{1}{2},\frac{6}{5}}(\mathbb{R}^3)$, and $(2,\frac{6}{5})$ is the dual of the (endpoint) Strichartz pair $(2,6)$ for the free Schrödinger flow in $\mathbb{R}^3$. Therefore, an application of the (dual) Strichartz estimate in $\mathbb{R}^3$ gives the result: more precisely, arguing by duality again

$$\begin{aligned}
&\sup_{\substack{F\in L^2(\mathbb{R}^3),\\ \|F\|_{L^2(\mathbb{R}^3)}\leqslant 1}} \left\langle F, \langle D\rangle^{\frac{1}{2}} \int_t^\infty e^{-is\Delta}(1-\chi)u|u|^2(s)\,ds\right\rangle_{L^2}\\
&\qquad = \sup_{\substack{F\in L^2(\mathbb{R}^3),\\ \|F\|_{L^2(\mathbb{R}^3)}\leqslant 1}} \left\langle e^{is\Delta}F, \langle D\rangle^{\frac{1}{2}}(1-\chi)u|u|^2\right\rangle_{L^2_s((t,+\infty),L^2(\mathbb{R}^3))}\\
&\qquad \leqslant \sup_{\substack{F\in L^2(\mathbb{R}^3),\\ \|F\|_{L^2(\mathbb{R}^3)}\leqslant 1}} \left\|e^{is\Delta}F\right\|_{L^2((t,+\infty),L^6(\mathbb{R}^3))} \left\|\langle D\rangle^{\frac{1}{2}}(1-\chi)u|u|^2\right\|_{L^2_s\left((t,+\infty),L^{\frac{6}{5}}(\mathbb{R}^3)\right)}\\
&\qquad \lesssim \left\|(1-\chi)u|u|^2\right\|_{L^2_s\left((t,+\infty),W^{\frac{1}{2},\frac{6}{5}}(\mathbb{R}^3)\right)} \longrightarrow 0,
\end{aligned}$$

where in the last line we used Strichartz estimates for the free Schrödinger flow in $\mathbb{R}^3$, then (5.3) together with the embedding $W^{1,1}(\mathbb{R}^3) \hookrightarrow W^{\frac{1}{2},\frac{6}{5}}(\mathbb{R}^3)$ to conclude. This ends the proof of scattering in $H^{\frac{1}{2}}$.

Now, as $t \to +\infty$

$$\left\|u_+ - e^{-it\Delta}u(t)\right\|_{H^{\frac{1}{2}}(\mathbb{R}^3)} \longrightarrow 0,$$

and $e^{-it\Delta}u(t)$ is uniformly bounded in H^1 forward in time, so, by uniqueness of the weak limit, $u_+ \in H^1$. By interpolation, it follows that

$$\left\|u_+ - e^{-it\Delta}u(t)\right\|_{H^s(\mathbb{R}^3)} \longrightarrow 0,$$

for any $s < 1$. This ends the proof of Theorem 1.1. □

Appendix A. The exterior of an obstacle

The purpose of this appendix is to illustrate that the non-linear arguments presented in Section 3, Section 4, Section 5, work just as well for the analogous problem posed in the exterior of a Dirichlet obstacle.

More precisely, let us consider the Cauchy problem

$$\text{(A.1)}\qquad \begin{cases} i\partial_t u + \Delta u + iau = |u|^2 u, \\ u = 0 \text{ on } \partial\Omega, \\ u(0) = u_0 \in H^s(\mathbb{R}^3), \end{cases}$$

where $\Omega := \mathbb{R}^3\backslash\Theta$, with $\Theta \subset \mathbb{R}^3$ compact with smooth boundary. The best general (i.e., not depending on the geometry of Θ) Strichartz estimates for the linear Schrödinger equation outside Θ known at the moment involve a loss of $\frac{3}{2p}$ derivatives [Ant08], or a loss of $\frac{1}{p}$ derivatives but for a restricted range of admissible couples [BSS12], and this is not enough to obtain a well-posedness theory such as presented in Section 2. On the other hand, the rest of the proof works similarly, and we obtain:

Theorem A.1. — *Assume that $\{\partial\Theta\cdot n(x)<0\}\Subset\{a>0\}$. Then, there exists a $C>0$, such that for any $T>0$, any $C([0,T],H^2(\Omega))$ solution to* (A.1) *is bounded in $L^\infty H^1$ by $C\|u_0\|_{H^1(\Omega)}$, and global $C([0,T],H^2)$ solutions scatter in $H^{1-}(\mathbb{R}^3)$.*

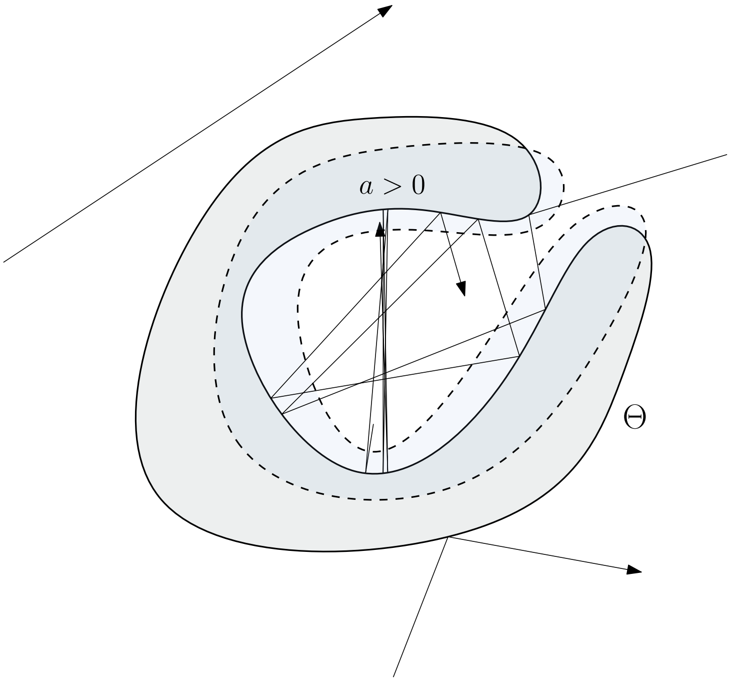

Figure A.1. Schematic representation of the control condition in the case of an obstacle. Only the non-star-shaped portion of the boundary needs to be controlled.

Sketch of the proof. — The analogous to the Morawetz computation presented in the proof of Proposition 3.4 now writes

$$\begin{aligned}\text{(A.2)}\quad \operatorname{Im}\int \bar u(t)\nabla u(t)\cdot\nabla\chi dx &= \operatorname{Im}\int(\bar u(s)\nabla u(s)\cdot\nabla\chi)dx+\int_s^t\int 2D^2\chi\nabla u\cdot\nabla\bar u dxd\tau-\frac12\Delta^2\chi|u|^2\\ &\quad+\frac12\Delta\chi|u|^4-2\operatorname{Im}(a\bar u\nabla u\cdot\nabla\chi)\,dxd\tau+\int_s^t\int_{\partial\Omega}|\partial_n u|^2\nabla\chi\cdot n\,d\sigma d\tau.\end{aligned}$$

With the above at hand, the supplementary ingredient to the computations of Section 3 and Section 4 is the following boundary hidden-regularity estimate.

Lemma A.2. — *Let $\omega\subset\partial\Omega$. Then, for any $K\subset\mathbb{R}^3$ compact so that $\omega\Subset K$, we have*

$$\begin{aligned}\int_s^t\int_\omega|\partial_n u|^2\,d\sigma d\tau &\lesssim E[u(t)]+E[u(s)]\\ &\quad+\int_s^t\int_{K\cap\Omega}|\nabla u|^2+|u|^2+|u|^4\,dxd\tau+\int_s^t\int_\Omega|a||\nabla u||u|\,dxd\tau.\end{aligned}$$

Proof. — Let $Z \in C_c^\infty(\mathbb{R}^3)$ be so that $\nabla Z(x) = n(x)$ on $\partial\Omega$, and $\psi \in C_c^\infty(\mathbb{R}^3)$ be so that $\operatorname{supp}\psi \subset K$ and $\psi = 1$ near ω. Applying (A.2) with $\chi := Z\psi$ gives the result. □

We now let $K_0 \subset \mathbb{R}^3$ so that $\{\partial\Theta \cdot n(x) < 0\} \Subset K_0 \Subset \{a > 0\}$. Combining Lemma A.2, used with $\omega := \{\partial\Theta \cdot n(x) < 0\}$ and $K := K_0$, with (A.2), gives the following analogous to Proposition 3.4 in the case $\chi(x) = \sqrt{1 + |x|^2}$:

$$\begin{aligned}\text{(A.3)}\quad \operatorname{Im}\int \bar{u}(t)\nabla u(t) \cdot \nabla\chi dx \leqslant \operatorname{Im}\int (\bar{u}(s)\nabla u(s) \cdot \nabla\chi) \\ + \int_s^t \int 2D^2\chi\nabla u \cdot \nabla\bar{u} - \frac{1}{2}\Delta^2\chi|u|^2 + \frac{1}{2}\Delta\chi|u|^4 \, dx d\tau + \mathcal{E}(t,s),\end{aligned}$$

where

$$\begin{aligned}|\mathcal{E}(t,s)| \lesssim E[u(t)] + E[u(s)] \\ + \int_s^t \int_\Omega |a||u||\nabla u| \, dx d\tau + \int_s^t \int_{K_0 \cap \Omega} |\nabla u|^2 + |u|^2 + |u|^4 \, dx d\tau,\end{aligned}$$

note in particular that we used the sign of $\nabla\chi \cdot n$ in $\partial\Omega\backslash\omega$. Using (A.3), one obtains the analogous to Proposition 3.1 and its consequences, with the same proof, $K_0 \cap \Omega$ playing the role of $\operatorname{supp}(G - \mathrm{I})$. The analogous to Proposition 4.1 is then shown similarly by using Lemma A.2 with $K := B(0,R)$, $\partial\Omega \subset B(0,R)$, to control the boundary terms arising in the computation. Finally, the end of the proof presented in Section 5 holds in the same way, by taking the cut-off χ equal to one in a neighborhood of $\operatorname{supp} a \cup \partial\Omega$. □

Appendix B. Mildly trapping case

In this appendix, we show scattering up to H^1 in the particular case where G induces a mildly trapping Hamiltonian flow. More precisely, we assume that the undamped linear propagator verifies global Strichartz estimates without loss, and a local energy decay estimate:

Assumption B.1. — *We assume that global-in-time, non-endpoint Strichartz estimates without loss hold for $e^{it\Delta_G}$: for any (p,q) satisfying condition (2.1) with $p > 2$, for any $s \geqslant 0$, $u \in H^s(\mathbb{R}^3)$, and $f \in L^1((0,t), H^s(\mathbb{R}^3))$, the following estimates hold with implicit constants independent of t:*

$$\text{(B.1)}\qquad \left\| e^{it\Delta_G} u \right\|_{L^p((0,t),W^{s,q}(\mathbb{R}^3))} \lesssim \|u\|_{H^s(\mathbb{R}^3)},$$

$$\text{(B.2)}\qquad \left\| \int_0^t e^{i(t-s)\Delta_G} f(s) \, ds \right\|_{L^p((0,t),W^{s,q}(\mathbb{R}^3))} \lesssim \|f\|_{L^1((0,t),H^s(\mathbb{R}^3))}.$$

In addition, we assume the following local energy decay estimate: for any $\chi \in C_c^\infty(\mathbb{R}^3)$,

$$\text{(B.3)}\qquad \left\| \chi e^{it\Delta_G} u \right\|_{L^2(\mathbb{R},L^2(\mathbb{R}^3))} \lesssim \|u\|_{L^2(\mathbb{R}^3)}.$$

A primary and significant example of a matrix satisfying the above is the flat case $G = \mathrm{I}$, corresponding to the classical cubic defocusing nonlinear Schrödinger equation (NLS) with a linear, spatially localized dissipation term (see (1.1)). More generally, any non-trapping perturbation G (in the sense that all the Hamiltonian trajectories defined by (1.2) go to infinity) verifies Assumption B.1, since, as mentioned previously, the non-trapping assumption implies global-in-time local smoothing estimate [BGT04a], which, together with the frequency-localized Strichartz estimates of [BGT04b], implies global-in-time Strichartz estimates without loss [Iva10, ST02]. Note that Assumption B.1 is more general and allows some weak trapped trajectories, such as in hyperbolic trapping, for which a smoothing estimate with an arbitrary loss of $\epsilon > 0$ derivatives (hence in particular (B.3)), and global Strichartz estimates without loss, are expected to hold (see [BGH10] in the case of the Laplace operator on a manifold).

For the remainder of this section, we assume that G verifies Assumption B.1. Under this assumption:

(1) The local well-posedness result stated in Proposition 2.3 is now valid for initial data $u_0 \in H^1(\mathbb{R}^3)$, as it follows directly from the usual Kato contraction argument (e.g., see [Caz03]) thanks to the Strichartz estimates without loss of Assumption B.1.
(2) The uniform estimates on the energy in Proposition 3.1, the L^4L^4 control in Proposition 4.1, and the local energy decay in Corollary 3.5 hold with the above proofs, as they are independent of whether G is trapping or not.

We summarize (1)–(2) in the following proposition:

Proposition B.2. — *Let G verifying Assumption B.1, and a verifying the control condition* (1.3)*. Then, for any initial data $u_0 \in H^1(\mathbb{R}^3)$, there exists a unique global solution $u \in C([0,\infty); H^1(\mathbb{R}^3))$ to* (1.1)*. It satisfies the bound:*

$$\sup_{t>0} \|u(t)\|_{H^1(\mathbb{R}^3)} + \|u\|_{L^4(\mathbb{R}_+, L^4(\mathbb{R}^3))} \lesssim 1. \tag{B.4}$$

Moreover, for any $R > 0$, there is $C > 0$ so that

$$\|u\|_{L^2(\mathbb{R}_+, H^1(B(0,R)))} \leqslant C.$$

In the mildly-trapping case, we establish that it is possible to achieve *scattering in* $H^1(\mathbb{R}^3)$ rather than in $H^{1-}(\mathbb{R}^3)$, as stated in Theorem 1.1. While the result in $H^{1-}(\mathbb{R}^3)$ follows from the arguments presented in Section 5, the mildly trapping condition enables a stronger result. Specifically, the availability of global-in-time Strichartz estimates without loss for the undamped flow $e^{it\Delta_G}$ permits us to show scattering in $H^1(\mathbb{R}^3)$ from the control of a global Strichartz norm, and we obtain:

Theorem B.3. — *Let G verifies Assumption B.1, and let a satisfy condition* (1.3)*. Then, for any initial data $u_0 \in H^1(\mathbb{R}^3)$, there exists $u_+, \widetilde{u}_+ \in H^1(\mathbb{R}^3)$ so that the solution $u \in C([0,+\infty), H^1(\mathbb{R}^3))$ to* (1.1) *with data u_0 verifies*

$$\lim_{t\to+\infty} \left\|u(t) - e^{it\Delta}\widetilde{u}_+\right\|_{H^1(\mathbb{R}^3)} = \lim_{t\to+\infty} \left\|u(t) - e^{it\Delta_G}u_+\right\|_{H^1(\mathbb{R}^3)} = 0.$$

To control a global Strichartz norm at the $H^1(\mathbb{R}^3)$ level from the L^4L^4 global control, we will argue similarly as in [CKS+04]. The supplementary ingredient will be to treat iau as a source term. We will be able to do so thanks to the following:

Lemma B.4. — *Let G verifies Assumption B.1, and let a satisfy condition* (1.3). *Then, for any (p, q) Strichartz-admissible with $p > 2$ and any solution to* (1.1), *the following holds:*

$$\left\| \int_0^t e^{i(t-s)\Delta_G} au \, ds \right\|_{L^p_t(\mathbb{R}_+, W^{1,q}(\mathbb{R}^3))} \lesssim \|u_0\|_{H^1(\mathbb{R}^3)} + \|u_0\|^2_{L^4(\mathbb{R}^3)}. \tag{B.5}$$

Proof. — We will show that for any $\chi \in C_c^\infty(\mathbb{R}^3)$ the operator

$$T : f \in L^2\left(\mathbb{R}, H^1(\mathbb{R}^3)\right) \longmapsto \int_{s<t} e^{i(t-s)\Delta_G} \chi f(s) \, ds \in L^p\left(\mathbb{R}, W^{1,q}(\mathbb{R}^3)\right)$$

is bounded.

Indeed, if it is the case, take $\chi \in C_c^\infty(\mathbb{R}^3)$ be a smooth cutoff function with $\chi = 1$ on $\operatorname{supp} a$. Using the operator bound on $f(s) := au(s)\mathbf{1}_{s\geqslant 0}$, then the local energy decay property (3.24), we obtain

$$\left\| \int_0^t e^{i(t-s)\Delta_G} \chi au \, ds \right\|^2_{L^p_t(\mathbb{R}_+, W^{1,q}(\mathbb{R}^3))} \lesssim \|au\|^2_{L^2((0,\infty), H^1(\mathbb{R}^3))} \lesssim \|u_0\|^2_{H^1(\mathbb{R}^3)} + \|u_0\|^4_{L^4(\mathbb{R}^3)},$$

where the compact support of a ensures the bound, and we are done.

To this end, we first claim that, for any $s \in [0, 2]$

$$f \in L^2\left(\mathbb{R}, H^s(\mathbb{R}^3)\right) \longmapsto \int_{\mathbb{R}} e^{-is\Delta_G} \chi f(s) \, ds \in H^s(\mathbb{R}^3) \tag{B.6}$$

is bounded. Indeed, for $s = 0$, the claim is the dual estimate to local energy decay (B.3). Next, for $s = 2$, we write, letting $\widetilde{\chi} \in C_c^\infty(\mathbb{R}^3)$ be equal to one on $\operatorname{supp} \chi$ and using the claim with $s = 0$

$$\begin{aligned} \left\| \Delta_G \int_{\mathbb{R}} e^{-is\Delta_G} \chi f(s) \, ds \right\|_{L^2(\mathbb{R}^3)} &= \left\| \int_{\mathbb{R}} e^{-is\Delta_G} \widetilde{\chi} \Delta_G(\chi f(s)) \, ds \right\|_{L^2(\mathbb{R}^3)} \\ &\lesssim \|\Delta_G(\chi f(s))\|_{L^2(\mathbb{R}, L^2(\mathbb{R}^3))} \lesssim \|f\|_{L^2(\mathbb{R}, H^2(\mathbb{R}^3))}. \end{aligned}$$

But, by elliptic regularity

$$\begin{aligned} &\left\| \int_{\mathbb{R}} e^{-is\Delta_G} \chi f(s) \, ds \right\|_{H^2(\mathbb{R}^3)} \\ &\qquad \lesssim \left\| \Delta_G \int_{\mathbb{R}} e^{-is\Delta_G} \chi f(s) \, ds \right\|_{L^2(\mathbb{R}^3)} + \left\| \int_{\mathbb{R}} e^{-is\Delta_G} \chi f(s) \, ds \right\|_{L^2(\mathbb{R}^3)}, \end{aligned}$$

hence the claim for $s = 2$ follows. The claim is shown for any $s \in [0, 2]$ by interpolation.

To conclude, we use a Theorem due to Christ and Kiselev [CK01], in the following form from [PV09, Lemma 5.6]:

Lemma B.5. — *Let $U(t)$ be a one-parameter group of operators, $1 \leqslant a < b \leqslant +\infty$, H a Hilbert space, and B, $\widetilde{B}$ two Banach spaces. Assume that*

$$\varphi \in H \longmapsto U(t)\varphi \in L^b(B), \qquad g \in L^a(\widetilde{B}) \longmapsto \int U(-s)g(s)ds \in H$$

are bounded. Then, the operator

$$g \in L^a(\widetilde{B}) \longmapsto \int_{s<t} U(t-s)g(s)ds \in L^b(B)$$

is bounded.

Take $U(t) = e^{it\Delta_G}$, $H = H^1(\mathbb{R}^3)$, $a = 2$, $b = p$, $B = W^{1,q}$, $\widetilde{B} = H^1_{\mathrm{comp}}$. The claim with $s = 1$ together with the Strichartz estimates (B.1) and Lemma B.5 shows that T is bounded, and the proof is completed. □

Proof of Theorem B.3. — Scattering in $H^1(\mathbb{R}^3)$ will follow once we establish a uniform bound of the form

$$Z(t) := \sup_{\substack{p,q \text{ admissible} \\ p>2}} \|u\|_{L^p((0,t),W^{1,q}(\mathbb{R}^3))} \leqslant C\left(\|u_0\|_{H^1(\mathbb{R}^3)}\right). \tag{B.7}$$

From the L^4L^4 control (B.4), we partition $[0,\infty)$ into a finite number of intervals $J_1,\dots,J_K$ such that for each $i = 1,\dots,K$,

$$\|u\|_{L^4(J_i,L^4(\mathbb{R}^3))} \leqslant \epsilon,$$

where $\epsilon = \epsilon(\|u_0\|_{H^1(\mathbb{R}^3)})$ is a small constant to be determined. For any $t \in J_i$, the Strichartz estimates (2.1) together with (B.5) yield

$$Z(t) \lesssim \||u|^2u\|_{L^{10/7}\left((0,t),W^{1,10/7}(\mathbb{R}^3)\right)} + \|u_0\|_{H^1(\mathbb{R}^3)} + \|u_0\|^2_{L^4(\mathbb{R}^3)}.$$

Using the fractional Leibniz rule, we have

$$\begin{aligned}\||u|^2u\|_{L^{10/7}\left((0,t),W^{1,10/7}(\mathbb{R}^3)\right)} &\lesssim \|u\|_{L^{10/3}\left((0,t),W^{1,10/3}\right)}\|u\|^2_{L^5((0,t),L^5(\mathbb{R}^3))} \\ &\lesssim Z(t)\|u\|^{\alpha}_{L^4((0,t),L^4(\mathbb{R}^3))}\|u\|^{\beta}_{L^6((0,t),L^6(\mathbb{R}^3)}\end{aligned}$$

for some $\alpha,\beta > 0$, where we interpolate L^5L^5 between L^4L^4 and L^6L^6. By Sobolev embedding, we bound the L^6L^6-norm as:

$$\|u\|_{L^6((0,t),L^6(\mathbb{R}^3))} \lesssim \|u\|_{L^6((0,t),W^{2/3,18/7}(\mathbb{R}^3))} \leqslant Z(t),$$

and we conclude

$$Z(t) \lesssim \|u_0\|_{H^1(\mathbb{R}^3)} + \|u_0\|^2_{L^4} + \epsilon^{\alpha}Z(t)^{1+\delta_2},$$

for some constants $\alpha,\delta_2 > 0$. Choosing $\epsilon > 0$ sufficiently small ensures that (B.7) holds on J_i. We establish the bound globally by repeating this argument over all intervals $J_1,\dots,J_K$.

Scattering in $H^1(\mathbb{R}^3)$ to a wave $e^{it\Delta_G}u_+$ now follows from the similar arguments as used in the conclusion of the proof of Theorem 1.1 in Section 5, replacing the $H^{\frac{1}{2}}$-norm with the H^1-norm, and without cutting away $\operatorname{supp} a$. More precisely, from the Duhamel formula, it suffices to show that

$$\left\|\int_t^{\infty} e^{-is\Delta_G}u|u|^2(s)\,ds\right\|_{H^1(\mathbb{R}^3)} \longrightarrow 0 \quad \text{as} \quad t \longrightarrow +\infty, \tag{B.8}$$

and

$$\left\|\int_t^{\infty} e^{-is\Delta_G}au(s)ds\right\|_{H^1(\mathbb{R}^3)} \longrightarrow 0 \quad \text{as} \quad t \longrightarrow +\infty. \tag{B.9}$$

For (B.8) we use (B.2), Hölder's inequality and Riesz–Thorin theorem to get

$$\begin{aligned}\Big\|\int_t^\infty e^{-is\Delta_G}|u|^2u(s)ds\Big\|_{H^1(\mathbb{R}^3)}&\lesssim \||u|^2u\|_{L^{\frac{8}{5}}\left((t,\infty),W^{\frac{4}{3},1}(\mathbb{R}^3)\right)}\\&\lesssim \|u\|^2_{L^8((t,\infty),L^4(\mathbb{R}^3))}\|u\|_{L^{\frac{8}{3}}((t,\infty),W^{4,1}(\mathbb{R}^3))}\\&\lesssim \|u\|_{L^\infty((t,\infty),L^4)}\|u\|_{L^4((t,\infty),L^4(\mathbb{R}^3))}\|u\|_{L^{\frac{8}{3}}((t,\infty),W^{4,1}(\mathbb{R}^3))}\longrightarrow 0\end{aligned}$$

since $(\frac{8}{3},4)$ is admissible. For (B.9), we use the operator bound (B.6) for $s=1$ and $\chi\in C_c^\infty(\mathbb{R}^3)$ so that $\chi=1$ on $\operatorname{supp} a$, on $f(s):=au(s)\mathbf{1}_{s\in[t,+\infty)}$ and obtain

$$\Big\|\int_t^\infty e^{-is\Delta_G}au(s)ds\Big\|_{H^1(\mathbb{R}^3)}\lesssim \|au\|_{L^2([t,+\infty),H^1(\mathbb{R}^3))}\longrightarrow 0$$

thanks to (3.24). Finally, to show that scattering also holds for a free wave $e^{it\Delta}\widetilde{u}_+$, we use the following linear scattering result:

Lemma B.6. — *For any $u_+\in H^1(\mathbb{R}_+)$, there exists $\widetilde{u}_+$ so that, as $t\to+\infty$*

$$\left\|e^{it\Delta}\widetilde{u}_+-e^{it\Delta_G}u_+\right\|_{H^1(\mathbb{R}^3)}\longrightarrow 0.$$

Proof. — Let $\chi\in C^\infty(\mathbb{R}^3)$ so that $\chi=1$ on $\operatorname{supp}(G-\mathrm{I})$. First, observe that $\chi e^{it\Delta_G}u_+\to 0$ in $H^1(\mathbb{R}^3)$. Indeed, one can show from (B.3), similarly to the proof of Corollary 3.6, that $\chi e^{it\Delta_G}u_+\to 0$ in $L^2(\mathbb{R}^3)$. This implies the claim by interpolation for $u_+\in H^s(\mathbb{R}^3)$ with $s>1$, and the claim then follows for any $u_+\in H^1(\mathbb{R}^3)$ by approximation. Hence, it suffices to show the existence of $\widetilde{u}_+\in H^1(\mathbb{R}^3)$ so that

$$\left\|e^{it\Delta}\widetilde{u}_+-(1-\chi)e^{it\Delta_G}u_+\right\|_{H^1(\mathbb{R}^3)}\longrightarrow 0.$$

Let $v(t):=(1-\chi)e^{it\Delta_G}u_+$. It verifies

$$i\partial_t v+\Delta v=[\Delta,\chi]u,$$

hence from Duhamel's formula

$$v(t)=e^{it\Delta}\Big((1-\chi)u_+-i\int_0^t e^{-is\Delta}[\Delta,\chi]u(s)ds\Big).$$

Observe that the resolvent estimate without loss sufficiently far from the origin of Cardoso–Vodev [CV02] implies with the arguments of [BGT04a, Proposition 2.7], the following smoothing estimate without loss far away from $\operatorname{supp}(G-I)$ for $e^{it\Delta_G}$: there exists $R\gg 1$ big enough so that for any $\psi\in C_c^\infty(\mathbb{R}^3)$ supported away from $B(0,R)$ and any $u_0\in L^2$

$$\left\|\psi e^{it\Delta_G}u_0\right\|_{L^2\left(\mathbb{R},H^{\frac{1}{2}}(\mathbb{R}^3)\right)}\lesssim\|u_0\|_{L^2}. \tag{B.10}$$

Taking $\chi=1$ on $B(0,R)$, from the dual estimate to the smoothing effect for $e^{-is\Delta}$ together with (B.10), we obtain as $[\Delta,\chi]$ is supported away from $B(0,R)$:

$$\Big\|\int_t^\infty e^{-is\Delta}[\Delta,\chi]u(s)ds\Big\|_{H^1(\mathbb{R}^3)}\longrightarrow 0,$$

hence the result follows by setting

$$\widetilde{u}_+ := (1-\chi)u_+ - i\int_0^\infty e^{-is\Delta}[\Delta,\chi]u(s)ds. \qquad \square$$

This concludes the proof of Theorem B.3. □

Acknowledgments

We are grateful to Professor Nicola Visciglia for highlighting that we could improve our initial result of scattering in $H^{\frac{1}{2}}$ to H^{1-} by interpolation.

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Manuscript received on 7th March 2025,
revised on 3rd September 2025,
accepted on 24th November 2025.

Recommended by Editors S. Vu Ngoc and P. Gérard.

eISSN: 2644-9463
This journal is a member of Centre Mersenne.

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