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ON THE LIMIT IN THE CLT FOR A FIELD OF MARTINGALE DIFFERENCES WITH RESPECT TO A COMPLETELY COMMUTING INVARIANT FILTRATION

SUR LA LIMITE DANS LE TLC POUR UN CHAMP D'ACCROISSEMENTS DE MARTINGALES RELATIVEMENT À UNE FILTRATION COMPLÈTEMENT COMMUTANTE

Abstract. — The now classical convergence in distribution theorem for well normalized sums of stationary martingale increments has been extended to multi-indexed martingale increments. In the present article we make progress in the identification of the limit law.

In dimension one, as soon as the stationary martingale increments form an ergodic process, the limit law is normal, and it is still the case for multi-indexed martingale increments when one of the processes defined by one coordinate of the *multidimensional time* is ergodic. In the general case, the limit may be non normal.

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In the present paper we establish links between the dynamical properties of the $\mathbb{Z}^d$-measure preserving action associated to the stationary random field (like the positivity of the entropy of some factors) and the existence of a non normal limit law. The identification of a *natural* factor on which the $\mathbb{Z}^d$-action is *of product type* is a crucial step in this approach.

Résumé. — La convergence en loi d'une martingale à accroissements stationnaires convenablement normalisée est un résultat maintenant classique qui a été étendu aux martingales à indices multi-dimensionnel. Dans cet article, nous présentons des avancées sur l'identification de la loi limite.

En dimension un, si la suite strictement stationnaire d'accroissements de martingale est ergodique, la limite est la loi normale et c'est encore le cas pour les martingales à indice multiple quand le processus défini par une des dimensions du temps multi-dimensionnel est ergodique. Dans le cas général, la limite peut ne pas être la loi normale.

Dans cet article, nous établissons des liens entres les propriétés des actions de $\mathbb{Z}^d$ préservant la mesure associées au champ aléatoire stationnaire, comme la stricte positivité de l'entropie de certains facteurs, et l'existence d'une limite non normale. L'identification d'un facteur *naturel* pour lequel l'action de $\mathbb{Z}^d$ est *de type produit* est une étape cruciale de cette approche.

1. Introduction

We study here limit theorems of CLT (Central Limit Theorem) type for stationary multiparameters martingale indexed by $\mathbb{Z}^d$. A part of our results is valid in any dimension $d \geqslant 2$ (Sections 2 to 4), while another part is limited to dimension $d = 2$ (Section 5).

The framework of our work has been already described in several articles (see [Vol19] and references therein) and we recall it now.

For $\underline{i} = (i_1, i_2, \ldots, i_d)$ and $\underline{j} = (j_1, j_2, \ldots, j_d) \in \mathbb{Z}^d$, we use the two following notations:

- $\underline{i} \wedge \underline{j} = (\min(i_1, j_1), \min(i_2, j_2), \ldots, \min(i_d, j_d))$;
- $\underline{i} \leqslant \underline{j} \Leftrightarrow i_1 \leqslant j_1$ and $i_2 \leqslant j_2$ and,..., and $i_d \leqslant j_d$.

Moreover $\underline{0}$ stands for the element $(0, 0, \ldots, 0)$ of $\mathbb{Z}^d$.

We consider a $\mathbb{Z}^d$ measure preserving action $T = (T_{\underline{i}})_{\underline{i}\in\mathbb{Z}^d}$ on a probability space $(\Omega, \mathcal{A}, \mu)$, equipped with a *completely commuting invariant filtration* $(\mathcal{F}_{\underline{i}})_{\underline{i}\in\mathbb{Z}^d}$, that is a family of sub-σ-algebras of $\mathcal{A}$ satisfying for all $\underline{i}$ and $\underline{j}$ in $\mathbb{Z}^d$,

(1) $\mathcal{F}_{\underline{i}} = T_{-\underline{i}}\mathcal{F}_{\underline{0}}$;

(2) For all integrable function f, $\mathbb{E}[\mathbb{E}[f \mid \mathcal{F}_{\underline{i}}] \mid \mathcal{F}_{\underline{j}}] = \mathbb{E}[f \mid \mathcal{F}_{\underline{i}\wedge\underline{j}}]$.

Note that property (2) implies that $\mathcal{F}_{\underline{i}} \cap \mathcal{F}_{\underline{j}} = \mathcal{F}_{\underline{i}\wedge\underline{j}}$ and in particular $\mathcal{F}_{\underline{i}} \subset \mathcal{F}_{\underline{j}}$ when $\underline{i} \leqslant \underline{j}$.

We will use classical notations for the limit sub-σ-algebras when parameters go to $\pm\infty$:

$$\mathcal{F}_{-\infty,i_2,\ldots,i_d} = \bigcap_{i_1\in\mathbb{Z}} \mathcal{F}_{i_1,i_2,\ldots,i_d} \quad \text{and} \quad \mathcal{F}_{\infty,i_2,\ldots,i_d} = \bigvee_{i_1\in\mathbb{Z}} \mathcal{F}_{i_1,i_2,\ldots,i_d},$$

and so on.

A *field of martingale differences adapted to the filtration* $(\mathcal{F}_{\underline{i}})_{\underline{i}\in\mathbb{Z}^d}$ is a field of random variables $(X_{\underline{i}})_{\underline{i}\in\mathbb{Z}^d}$ of the type

$$X_{\underline{i}} = f \circ T_{\underline{i}}$$

where $f \in \mathbb{L}^2(\Omega, \mathcal{F}_{\underline{0}}, \mu)$ satisfies

(1.1) $$\mathbb{E}[f \mid \mathcal{F}_{-1,\infty,\dots,\infty}] = \mathbb{E}[f \mid \mathcal{F}_{\infty,-1,\infty,\dots,\infty}] = \cdots = \mathbb{E}[f \mid \mathcal{F}_{\infty,\infty,\dots,\infty,-1}] = 0.$$

(We will say simply that f is a martingale difference adapted to the filtration $(\mathcal{F}_{\underline{i}})$.)

From a previous article [Vol19], we know that, for any such field, we have convergence in law of

(1.2) $$\frac{1}{\sqrt{n_1 n_2 \dots n_d}} \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_d=1}^{n_d} X_{i_1,i_2,\dots,i_d}$$

when $\min(n_1, n_2, \dots, n_d)$ goes to infinity.

Moreover the limit is normal as soon as one of the d transformations

$$T_{1,0,\dots,0},\ T_{0,1,0,\dots,0}, \dots \ \text{or}\ \ T_{0,0,\dots,0,1}$$

is ergodic (this was already established in [Vol15]), but some easy examples show that the limit is not normal in general. Indeed, following the example in [WW13], we can take $X_{i_1,i_2,\dots,i_d} = X_{i_1}^{(1)} X_{i_2}^{(2)}, \dots, X_{i_d}^{(d)}$, where $(X_i^{(1)})_{i\in\mathbb{Z}}$, $(X_i^{(2)})_{i\in\mathbb{Z}}$,...and $(X_i^{(d)})_{i\in\mathbb{Z}}$ are d mutually independent i.i.d. sequences of standard normal random variables, so that for each $(n_1, n_2, \dots, n_d)$, the random variable defined by (1.2) has the same distribution as the product of d independent random variables having standard normal distribution. (Note that this example is produced by an ergodic $\mathbb{Z}^d$-action.)

Let us also mention the papers [CDV15, Gir18], which bring further results and examples.

In all the sequel, we suppose that the $\mathbb{Z}^d$-action T is ergodic on $(\Omega, \mathcal{A}, \mu)$. Moreover, we suppose that $\mathcal{F}_{\infty,\infty,\dots,\infty} = \mathcal{A}$ which does not cost anything since the whole process we are interested in is $\mathcal{F}_{\infty,\infty,\dots,\infty}$-measurable.

Here is the general organization of this article.

In Section 2 we describe a particular factor $\mathcal{I}$ on which the $\mathbb{Z}^d$-action is of *product type*. In Section 3, we show that the martingale structure is preserved by projection on this factor. For a product type action, the possible limit distributions of (1.2) are fully understood, as described in Section 4.

In Section 5, where we restrict to the case of $\mathbb{Z}^2$-actions, we study what can happen on the orthocomplement of the factor $\mathcal{I}$. In particular we obtain the following results:

- if the transformation $T_{1,0}$ acting on the factor of $T_{0,1}$-invariants has zero entropy (or if the transformation $T_{0,1}$ acting on the factor of $T_{1,0}$-invariants has zero entropy), then for any square integrable martingale difference the limit distribution in the CLT is normal.
- if the transformation $T_{1,0}$ acting on the factor of $T_{0,1}$-invariants and the transformation $T_{0,1}$ acting on the factor of $T_{1,0}$-invariants have positive entropies, then there exists a square integrable martingale difference for which the limit distribution in the CLT is not normal.

Proposition 5.2 in Section 5.1 gives a sufficient condition for the limit distribution to be normal. In our opinion, the most interesting questions we leave opened in our study concern this proposition: weakening of the moment condition ($\mathbb{L}^4$); extension to higher dimension actions.

Warnings

- In the Sections 2, 3 and 4, stated results are valid for $\mathbb{Z}^d$ actions for arbitrary $d \geqslant 2$. We write down statements and proofs in the case $d = 3$ because it seems sufficient to perfectly understand the general case and it makes for lighter drafting.
- On the contrary, Section 5 is specific to the case $d = 2$.
- All identities between σ-algebras have to be understood modulo the measure.

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2. A factor of product type

Let us denote by $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ the σ-algebras of, respectively $T_{1,0,0}$, $T_{0,1,0}$, $T_{0,0,1}$ invariant sets in $\mathcal{A}$.

Let us denote by $\overline{\mathcal{I}_1}$, $\overline{\mathcal{I}_2}$, $\overline{\mathcal{I}_3}$ the σ-algebras of invariant sets under, respectively, the $\mathbb{Z}^2$-actions (T_{0,i_2,i_3}), $(T_{i_1,0,i_3})$, $(T_{i_1,i_2,0})$. (For an arbitrary value of d, we consider here $\mathbb{Z}^{d-1}$-actions.) In other words, we have

$$\overline{\mathcal{I}_t} = \bigcap_{1\leqslant s\leqslant 3,\, s\neq t} \mathcal{I}_s, \quad 1 \leqslant t \leqslant 3.$$

(Note that in the particular case $d = 2$, we have $\overline{\mathcal{I}_1} = \mathcal{I}_2$.)

Finally we note $\mathcal{I}$ the σ-algebra generated by the union of the $\overline{\mathcal{I}_t}$:

$$\mathcal{I} = \bigvee_{t=1}^{3} \overline{\mathcal{I}_t} \ .$$

Note that all these σ-algebras are invariant under the action T, hence they define factors of the dynamical system $(\Omega, \mathcal{A}, \mu, T)$.

Proposition 2.1. — *The σ-algebras $\overline{\mathcal{I}_t}$, $1 \leqslant t \leqslant 3$ are mutually independent.*

As a consequence of this proposition we can state that the action T on the probability space $(\Omega, \mathcal{I}, \mu)$ is of *product type*, which means that it is isomorphic to a $\mathbb{Z}^3$-action defined on the product of 3 probability spaces $(\Omega_t, \mathcal{A}_t, \mu_t)$ by a formula of the type

$$T_{i_1,i_2,i_3}(\omega_1, \omega_2, \omega_3) = \left(T_1^{i_1}\omega_1, T_2^{i_2}\omega_2, T_3^{i_3}\omega_3\right)$$

where, for each t, T_t is an invertible ergodic measure preserving transformation of $(\Omega_t, \mathcal{A}_t, \mu_t)$.(1)

Proposition 2.1 is a straightforward consequence of the following general lemma, brought to our attention by the referee of this article.

(1) The fact that the sub-σ-algebras $\overline{\mathcal{I}_1}$, $\overline{\mathcal{I}_2}$ et $\overline{\mathcal{I}_3}$ are independent and generate the σ-algebra $\mathcal{I}$ tells us that the measure algebra dynamical systems

$$(\mathcal{I}, \mu, T) \quad \text{and} \quad (\overline{\mathcal{I}_1}, \mu, T) \times (\overline{\mathcal{I}_2}, \mu, T) \times (\overline{\mathcal{I}_3}, \mu, T)$$

Lemma 2.2. — *Let G and H be two measure preserving group actions on a probability space $(X, \mathcal{A}, \mu)$ which commute and such that the action generated by both G and H is ergodic. Then the σ-algebras of, respectively, G-invariant sets and H-invariant sets are independent.*

Proof of Lemma 2.2. — Denote by $\mathcal{G}$ and $\mathcal{H}$, respectively, the σ-algebras of G-invariant sets and H-invariant sets. Let u be a bounded $\mathcal{G}$-measurable function on X. Since the actions commute the σ-algebra $\mathcal{H}$ is G-invariant. Since u and $\mathcal{H}$ are both G-invariant, $\mathbb{E}[u \mid \mathcal{H}]$ is also G-invariant. The ergodicity of the action generated by G and H means that $\mathcal{G} \cap \mathcal{H}$ is trivial, and we conclude that $\mathbb{E}[u \mid \mathcal{H}]$ is constant. This establishes the independence property. □

3. Martingale property preserved by projection

Our aim here is to show that the martingale property is preserved by projection on the factor $\mathcal{I}$. We begin by a general abstract result, which is stated for a $\mathbb{Z}^2$-measure preserving action. (For an arbitrary value of d, we have to consider here a $\mathbb{Z}^{d-1}$-action.)

Let $(X, \mathcal{B}, \nu)$ be a probability space and $U = (U_{\underline{i}})$ a $\mathbb{Z}^2$-measure preserving action on this space. We denote by $\mathcal{J}$ the σ-algebra of U-invariant elements of the σ-algebra $\mathcal{B}$.

We denote by $\mathcal{F}$ a U-invariant sub-σ-algebra of $\mathcal{B}$, meaning that $U_{\underline{i}}(\mathcal{F}) = \mathcal{F}$ for all $\underline{i} \in \mathbb{Z}^2$; we denote by $\mathcal{C}$ a sub-σ-algebra of $\mathcal{F}$.

Proposition 3.1. —

(i) *For all $f \in \mathbb{L}^2(\mathcal{J} \vee \mathcal{C})$, $\mathbb{E}[f \mid \mathcal{F}]$ is $\mathcal{J} \vee \mathcal{C}$-measurable.*

(ii) *For all $f \in \mathbb{L}^2(\mathcal{F})$, $\mathbb{E}[f \mid \mathcal{J} \vee \mathcal{C}]$ is $\mathcal{F}$-measurable.*

Corollary 3.2. — *The conditional expectations with respect to $\mathcal{F}$ and to $\mathcal{J} \vee \mathcal{C}$ are commuting: for all $f \in \mathbb{L}^2(\mathcal{B})$,*

$$\mathbb{E}\Big[\mathbb{E}[f \mid \mathcal{J} \vee \mathcal{C}] \;\Big|\; \mathcal{F}\Big] = \mathbb{E}\Big[\mathbb{E}[f \mid \mathcal{F}] \;\Big|\; \mathcal{J} \vee \mathcal{C}\Big] = \mathbb{E}[f \mid \mathcal{F} \cap (\mathcal{J} \vee \mathcal{C})].$$

Proof of Corollary 3.2. — Suppose that $\mathcal{D}$ and $\mathcal{D}'$ are two sub-σ-algebras such that, for all $f \in \mathbb{L}^2(\mathcal{D})$, $\mathbb{E}[f \mid \mathcal{D}']$ is $\mathcal{D}$-measurable. By ordinary properties of projections, we have, for any $f \in \mathbb{L}^2(\mathcal{B})$

$$\mathbb{E}[f \mid \mathcal{D} \cap \mathcal{D}'] = \mathbb{E}[\mathbb{E}[\mathbb{E}[f \mid \mathcal{D}] \mid \mathcal{D}'] \mid \mathcal{D} \cap \mathcal{D}']$$

which implies

$$\mathbb{E}[f \mid \mathcal{D} \cap \mathcal{D}'] = \mathbb{E}[\mathbb{E}[f \mid \mathcal{D}] \mid \mathcal{D}']$$

since $\mathbb{E}[\mathbb{E}[f \mid \mathcal{D}] \mid \mathcal{D}']$ is $\mathcal{D} \cap \mathcal{D}'$-measurable. □

are isomorphic (in the sense of [Gla03, Definition 2.14] in Glasner book, which corresponds to the notion of conjugacy in the classical Billingsley book [Bil65]). The methods presented in [Gla03, Chapter 2], can be used to build Lebesgue measure preserving dynamical systems models for these measure algebra dynamical systems.

Proof of Proposition 3.1. — Let $f \in \mathbb{L}^2(\mathcal{F})$; by the ergodic theorem, we have

$$\mathbb{E}[f \mid \mathcal{J}] = \lim_{\ell,m\to\infty} \frac{1}{\ell m} \sum_{i=1}^{\ell} \sum_{j=1}^{m} f \circ U_{i,j},$$

hence $\mathbb{E}[f \mid \mathcal{J}]$ is $\mathcal{F}$-measurable, since $\mathcal{F}$ is U-invariant.

For any $f \in \mathbb{L}^2(\mathcal{B})$, we have

$$\mathbb{E}[f \mid \mathcal{F}] \circ U_{i,j} = \mathbb{E}\Big[f \circ U_{i,j} \,\Big|\, U_{i,j}^{-1}(\mathcal{F})\Big] = \mathbb{E}[f \circ U_{i,j} \,|\, \mathcal{F}],$$

hence if f is $\mathcal{J}$-measurable, then $\mathbb{E}[f \mid \mathcal{F}]$ is $\mathcal{J}$-measurable.

Consider now $g \in \mathbb{L}^2(\mathcal{J})$ and $h \in \mathbb{L}^2(\mathcal{C})$. Since $\mathcal{C} \subset \mathcal{F}$, we have $\mathbb{E}[gh \mid \mathcal{F}] = \mathbb{E}[g \mid \mathcal{F}]h$, so $\mathbb{E}[gh \mid \mathcal{F}]$ is $\mathcal{J} \vee \mathcal{C}$-measurable. But the functions of the form gh with $g \in \mathbb{L}^\infty(\mathcal{J})$ and $h \in \mathbb{L}^\infty(\mathcal{C})$ generate a dense subspace of $\mathbb{L}^2(\mathcal{J} \vee \mathcal{C})$, so that assertion (i) is proved.

Consider now $f \in \mathbb{L}^2(\mathcal{F})$, $g \in \mathbb{L}^2(\mathcal{J})$ and $h \in \mathbb{L}^2(\mathcal{C})$. We have

$$\begin{aligned}
\langle \mathbb{E}[f \mid \mathcal{J} \vee \mathcal{C}], gh\rangle &= \langle f, gh\rangle \quad \text{(because } gh \text{ is } \mathcal{J} \vee \mathcal{C}\text{-measurable)},\\
&= \langle f, \mathbb{E}[gh \mid \mathcal{F}]\rangle \quad \text{(because } f \text{ is } \mathcal{F}\text{-measurable)},\\
&= \langle f, h\mathbb{E}[g \mid \mathcal{F}]\rangle \quad \text{(because } h \text{ is } \mathcal{F}\text{-measurable)},\\
&= \langle \mathbb{E}[f \mid \mathcal{J} \vee \mathcal{C}], h\mathbb{E}[g \mid \mathcal{F}]\rangle \quad \text{(because } \mathbb{E}[g \mid \mathcal{F}] \text{ is } \mathcal{J}\text{-measurable)},\\
&= \langle \mathbb{E}[f \mid \mathcal{J} \vee \mathcal{C}], \mathbb{E}[gh \mid \mathcal{F}]\rangle.
\end{aligned}$$

By the density argument, we conclude that, for all $k \in \mathbb{L}^2(\mathcal{J} \vee \mathcal{C})$,

$$\langle \mathbb{E}[f \mid \mathcal{J} \vee \mathcal{C}], k)\rangle = \langle \mathbb{E}[f \mid \mathcal{J} \vee \mathcal{C}],\ \mathbb{E}[k \mid \mathcal{F}]\rangle.$$

This identity applied to the function $k = \mathbb{E}[f \mid \mathcal{J} \vee \mathcal{C}]$ shows that this function is $\mathcal{F}$-measurable. This is assertion (ii). □

Another result we need is the following classical lemma (cf for example [Par69]) and we give a short proof for the sake of completeness.

LEMMA 3.3. — *Let S be a measure preserving transformation of the probability space $(X, \mathcal{B}, \nu)$, and $\mathcal{F}_n = S^{-n}\mathcal{F}_0$ be an increasing filtration in $\mathcal{B}$. Denote by $\mathcal{K}$ the sub-σ-algebra of S invariant sets. Then $\mathcal{K} \cap \mathcal{F}_\infty = \mathcal{K} \cap \mathcal{F}_{-\infty}$. In particular, if $\mathcal{K} \subset \mathcal{F}_\infty$ then $\mathcal{K} \subset \mathcal{F}_{-\infty}$.*

Proof of Lemma 3.3. — Let $f \in \mathbb{L}^2(\nu)$. By purely Hilbert space arguments, we know that, in the space $\mathbb{L}^2(X)$,

$$\lim_{n\to-\infty} \mathbb{E}[f \mid \mathcal{F}_n] = \mathbb{E}[f \mid \mathcal{F}_{-\infty}] \quad \text{and} \quad \lim_{n\to\infty} \mathbb{E}[f \mid \mathcal{F}_n] = \mathbb{E}[f \mid \mathcal{F}_\infty].$$

Now suppose that f is invariant under S, that is f is $\mathcal{K}$-measurable. Then

$$\mathbb{E}[f \mid \mathcal{F}_n] = \mathbb{E}\Big[f \circ S^n \mid S^{-n}\mathcal{F}_0\Big] = \mathbb{E}[f \mid \mathcal{F}_0] \circ S^n.$$

Thus $\|\mathbb{E}[f \mid \mathcal{F}_{-m}]\|_2 = \|\mathbb{E}[f \mid \mathcal{F}_n]\|_2$ and with $n, m \to \infty$, we obtain $\|\mathbb{E}[f \mid \mathcal{F}_{-\infty}]\|_2 = \|\mathbb{E}[f \mid \mathcal{F}_\infty]\|_2$. Since $\mathcal{F}_{-\infty} \subset \mathcal{F}_\infty$, this implies $\mathbb{E}[f \mid \mathcal{F}_{-\infty}] = \mathbb{E}[f \mid \mathcal{F}_\infty]$. In particular, if f is $\mathcal{F}_\infty$-measurable, then it is $\mathcal{F}_{-\infty}$-measurable, which is what we had to prove. □

We now come back to the situation described in the preceding section where the factor $\mathcal{I}$ is defined.

Theorem 3.4. — *Let f be a martingale difference adapted to the filtration $(\mathcal{F}_{\underline{i}})_{\underline{i}\in\mathbb{Z}^3}$. Then its projection $\mathbb{E}[f \mid \mathcal{I}]$ is a martingale difference adapted to the filtration $(\mathcal{F}_{\underline{i}})_{\underline{i}\in\mathbb{Z}^3}$, as well adapted to the filtration $(\mathcal{I}\cap\mathcal{F}_{\underline{i}})_{\underline{i}\in\mathbb{Z}^3}$. Moreover $f-\mathbb{E}[f \mid \mathcal{I}]$ is also a martingale difference adapted to the filtration $(\mathcal{F}_{\underline{i}})_{\underline{i}\in\mathbb{Z}^3}$.*

Proof. — We want to apply Proposition 3.1 to the $\mathbb{Z}^2$-action $U_{j,k}=T_{0,j,k}$ on the space $(\Omega,\mathcal{A},\mu)$. So we have $\mathcal{J}=\overline{\mathcal{I}}_1$. We consider also $\mathcal{C}=\overline{\mathcal{I}_2}\vee\overline{\mathcal{I}_3}$ and $\mathcal{F}=\mathcal{F}_{i,\infty,\infty}$, for a given i. We know that $\mathcal{F}$ is U-invariant and, thanks to Lemma 3.3, we have $\mathcal{I}_1\subset\mathcal{F}$, hence $\mathcal{C}\subset\mathcal{F}$ since $\mathcal{C}\subset\mathcal{I}_1$.

Note that $\mathcal{J}\vee\mathcal{C}=\overline{\mathcal{I}}_1\vee\overline{\mathcal{I}_2}\vee\overline{\mathcal{I}_3}=\mathcal{I}$.

Now Corollary 3.2 tells us that the conditional expectation with respect to $\mathcal{I}$ commutes with the conditional expectations with respect to $\mathcal{F}_{i,\infty,\infty}$. Of course, we can exchange the roles of i, j and k and we find as well that the conditional expectation with respect to $\mathcal{I}$ commutes with the conditional expectations with respect to $\mathcal{F}_{\infty,j,\infty}$ and with the conditional expectations with respect to $\mathcal{F}_{\infty,\infty,k}$.

By the complete commuting property of the filtration, we have

$$\mathbb{E}[\cdot\,|\,\mathcal{F}_{i,j,k}]=\mathbb{E}[\mathbb{E}[\mathbb{E}[\cdot\,|\,\mathcal{F}_{i,\infty,\infty}]\,|\,\mathcal{F}_{\infty,j,\infty}]\,|\,\mathcal{F}_{\infty,\infty,k}],$$

and we conclude that the conditional expectation with respect to $\mathcal{I}$ commutes with the conditional expectations with respect to $\mathcal{F}_{i,j,k}$: for all $f\in\mathbb{L}^2(\mathcal{A})$, for all $i,j,k\in\mathbb{Z}\cup\{\infty\}$,

$$\mathbb{E}[\mathbb{E}[f\,|\,\mathcal{F}_{i,j,k}]\mid\mathcal{I}]=\mathbb{E}[\mathbb{E}[f\,|\,\mathcal{I}]\,|\,\mathcal{F}_{i,j,k}]=\mathbb{E}[f\,|\,\mathcal{F}_{i,j,k}\cap\mathcal{I}]. \tag{3.1}$$

The first thing we want to see now is that $(\mathcal{I}\cap\mathcal{F}_{i,j,k})$ is a completely commuting invariant filtration. The first point is

$$T_{-i,-j,-k}(\mathcal{I}\cap\mathcal{F}_{0,0,0})=\mathcal{I}\cap\mathcal{F}_{i,j,k}$$

which is true thanks to (i) and the fact that $\mathcal{I}$ is invariant. The second point is that, for all integrable function f,

$$\mathbb{E}[\mathbb{E}[f\,|\,\mathcal{I}\cap\mathcal{F}_{i,j,k}]\,|\,\mathcal{I}\cap\mathcal{F}_{i',j',k'}]=\mathbb{E}\Big[f\,\Big|\,\mathcal{I}\cap\mathcal{F}_{\min(i,i'),\min(j,j'),\min(k,k')}\Big],$$

which, thanks to (3.1), can be written

$$\mathbb{E}[\mathbb{E}[\mathbb{E}[f\mid\mathcal{F}_{i,j,k}]\,|\,\mathcal{F}_{i',j',k'}]\,|\,\mathcal{I}]=\mathbb{E}\Big[\mathbb{E}\Big[f\,\Big|\,\mathcal{F}_{\min(i,i'),\min(j,j'),\min(k,k')}\Big]\,\Big|\,\mathcal{I}\Big]$$

and is true thanks to (ii).

The second thing to verify is that if f satisfies the martingale condition (1.1) then $\mathbb{E}[f\mid\mathcal{I}]$ satisfies it. The facts that conditional expectations with respect to $\mathcal{I}$ and $\mathcal{F}_{0,0,0}$ commute and that f is $\mathcal{F}_{0,0,0}$-measurable imply that $\mathbb{E}[f\mid\mathcal{I}]$ is $\mathcal{I}\cap\mathcal{F}_{0,0,0}$-measurable. Moreover we have, thanks to (3.1) and (1.1)

$$\mathbb{E}[\mathbb{E}[f\mid\mathcal{I}]\mid\mathcal{F}_{-1,\infty,\infty}]=\mathbb{E}[\mathbb{E}[f\mid\mathcal{F}_{-1,\infty,\infty}]\mid\mathcal{I}]=0$$

and similarly

$$\mathbb{E}[\mathbb{E}[f\mid\mathcal{I}]\mid\mathcal{I}\cap\mathcal{F}_{-1,\infty,\infty}]=0.$$

We conclude that $\mathbb{E}[f\mid\mathcal{I}]$ is a martingale difference (for any of the two filtrations), and $f-\mathbb{E}[f\mid\mathcal{I}]$ is also a martingale difference adapted to the filtration $(\mathcal{F}_{i,j,k})$, just as difference of two martingale differences. ☐

4. Limit law in the case of a product type action

In this section, we describe the limit distribution of (1.2) in the particular case of a product type action equipped with a *product type filtration*, that is a filtration of the type (4.1), as defined below. We explain in Section 4.2 that this result applies to our factor $\mathcal{I}$.

4.1. Convergence theorem

In this section, we deal with the case when the probability space $(\Omega, \mathcal{A}, \mu)$ has a product structure: $\Omega = \Omega_1 \times \Omega_2 \times \Omega_3$, $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_3$ and $\mu = \mu_1 \otimes \mu_2 \otimes \mu_3$, where each $(\Omega_t, \mathcal{A}_t, \mu_t)$ is a probability space, equipped with an ergodic invertible measure preserving transformation T_t. We consider the action T of $\mathbb{Z}^3$ on Ω given by

$$T_{i_1,i_2,i_3}(\omega_1, \omega_2, \omega_3) = \left(T_1^{i_1}\omega_1, T_2^{i_2}\omega_2, T_3^{i_3}\omega_3\right), \quad i_1, i_2, i_3 \in \mathbb{Z}.$$

This type of $\mathbb{Z}^3$ measure preserving action will be called a *product type action.*

For each $t \in \{1, 2, 3\}$, consider a sub-σ-algebra $\mathcal{F}_0^{(t)}$ of $\mathcal{A}_t$ such that $T_t\mathcal{F}_0^{(t)} \subset \mathcal{F}_0^{(t)}$. For each $\underline{i} = (i_1, i_2, i_3) \in \mathbb{Z}^3$, define the sub-$\sigma$-algebra $\mathcal{F}_{\underline{i}}$ of $\mathcal{A}$ by

$$\mathcal{F}_{\underline{i}} = \left(T_1^{-i_1}\mathcal{F}_0^{(1)}\right) \otimes \left(T_2^{-i_2}\mathcal{F}_0^{(2)}\right) \otimes \left(T_3^{-i_3}\mathcal{F}_0^{(3)}\right). \tag{4.1}$$

Proposition 4.1. — *The filtration $(\mathcal{F}_{\underline{i}})_{\underline{i}\in\mathbb{Z}^3}$ is a completely commuting invariant filtration.*

We omit the proof of this proposition which is relatively straightforward and similar to the proof of [Gir24, Proposition 1.2].

In order to investigate the convergence of the partial sum process given by (1.2), we will need to decompose the considered function f as a sum of functions which can be expressed as a product of functions of a single ω_t.

Lemma 4.2. — *Denote by Δ the set of square integrable functions satisfying the martingale property defined as in (1.1), that is Δ is the space of functions $f \in \mathbb{L}^2(\mathcal{F}_{\underline{0}})$ such that*

$$\mathbb{E}[f \mid \mathcal{F}_{-1,\infty,\infty}] = \mathbb{E}[f \mid \mathcal{F}_{\infty,-1,\infty}] = \mathbb{E}[f \mid \mathcal{F}_{\infty,\infty,-1}] = 0. \tag{4.2}$$

There exist random variables $v_{a_t,t} \in \mathbb{L}^2(\mathcal{A}_t)$, $1 \leqslant t \leqslant 3$, $a_t \in \mathbb{N}$, such that the collection of random variables $(\omega_1, \omega_2, \omega_3) \mapsto v_{a_1,1}(\omega_1)v_{a_2,2}(\omega_2)v_{a_3,3}(\omega_3)$ is a Hilbert basis of the space Δ.

Proof of Lemma 4.2. — Denote $H^{(t)} = \mathbb{L}^2(\mathcal{F}_0^{(t)}) \ominus \mathbb{L}^2(\mathcal{F}_{-1}^{(t)})$. The three orthogonal decompositions

$$\mathbb{L}^2\left(\mathcal{F}_0^{(t)}\right) = H^{(t)} \oplus \mathbb{L}^2\left(\mathcal{F}_{-1}^{(t)}\right), \quad 1 \leqslant t \leqslant 3,$$

give a decomposition of their tensor product:

$$\begin{aligned}\mathbb{L}^2\Big(\mathcal{F}_{\underline{0}}\Big) = &\Big[\mathbb{L}^2\Big(\mathcal{F}_{-1}^{(1)}\Big) \otimes \mathbb{L}^2\Big(\mathcal{F}_0^{(2)}\Big) \otimes \mathbb{L}^2\Big(\mathcal{F}_0^{(3)}\Big)\Big] \\ &\oplus \Big[H^{(1)} \otimes \mathbb{L}^2\Big(\mathcal{F}_{-1}^{(2)}\Big) \otimes \mathbb{L}^2\Big(\mathcal{F}_0^{(3)}\Big)\Big] \\ &\oplus \Big[H^{(1)} \otimes H^{(2)} \otimes \mathbb{L}^2\Big(\mathcal{F}_{-1}^{(3)}\Big)\Big] \\ &\oplus \Big[H^{(1)} \otimes H^{(2)} \otimes H^{(3)}\Big].\end{aligned}$$

Condition (4.2) means that the function f is orthogonal to the third first spaces appearing in this decomposition. In other words, we have

$$\Delta = H^{(1)} \otimes H^{(2)} \otimes H^{(3)}$$

and we can choose in each space $H^{(t)}$ a Hilbert base $(v_{a,t})_{a\in\mathbb{N}}$. □

Theorem 4.3. — *We consider a product type $\mathbb{Z}^3$-action as defined at the beginning of the section, equipped with a filtration defined as in (4.1). Let f be a square integrable function on Ω such that $(f \circ T_{\underline{i}})_{\underline{i}\in\mathbb{Z}^3}$ is a martingale difference random field.*

There exists a family of real numbers $(\lambda_{a_1,a_2,a_3}(f))_{a_t\geqslant 1, 1\leqslant t\leqslant 3}$ such that

$$\sum_{a_t\geqslant 1, 1\leqslant t\leqslant 3} \lambda^2_{a_1,a_2,a_3}(f) < \infty$$

and such that if $(N_a^{(t)})_{a\geqslant 1}$, $1 \leqslant t \leqslant 3$, are 3 i.i.d. and mutually independent sequences of standard normal random variables, then

$$\frac{1}{\sqrt{n_1 n_2 n_3}} \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \sum_{i_3=1}^{n_3} f \circ T_{(i_1,i_2,i_3)} \longrightarrow \sum_{a_t\geqslant 1, 1\leqslant t\leqslant 3} \lambda_{a_1,a_2,a_3}(f) N_{a_1}^{(1)} N_{a_2}^{(2)} N_{a_3}^{(3)}$$

in distribution as $\min\{n_1, n_2, n_3\} \to \infty$.

Proof. — We know by Lemma 4.2 that we can express f as

$$f(\omega_1,\omega_2,\omega_3) = \sum_{a,b,c=1}^{\infty} \lambda_{a,b,c}(f) v_{a,1}(\omega_1) v_{b,2}(\omega_2) v_{c,3}(\omega_3),$$

where the convergence takes place in $\mathbb{L}^2(\mu)$.

Define

$$f_K(\omega_1,\omega_2,\omega_3) := \sum_{a,b,c=1}^{K} \lambda_{a,b,c}(f) v_{a,1}(\omega_1) v_{b,2}(\omega_2) v_{c,3}(\omega_3).$$

Note that $(f_K \circ T^{i,j,k})_{i,j,k\in\mathbb{Z}}$ is also a martingale difference random field. Suppose that we proved for each $K \geqslant 1$ Theorem 4.3 with f replaced by f_K. By orthogonality of increments, for all $\ell, m, n > 0$,

$$\mathbb{P}\left(\left|\frac{1}{\sqrt{\ell m n}} \sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f \circ T_{i,j,k} - \frac{1}{\sqrt{\ell m n}} \sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f_K \circ T_{i,j,k}\right| > \varepsilon\right) \leqslant \frac{1}{\varepsilon^2}\|f - f_K\|_2^2$$

hence

$$\lim_{K\to\infty} \sup_{\ell,m,n} \mathbb{P}\left(\left|\frac{1}{\sqrt{\ell m n}} \sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f \circ T_{i,j,k} - \frac{1}{\sqrt{\ell m n}} \sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f_K \circ T_{i,j,k}\right| > \varepsilon\right) = 0.$$

Moreover,

$$\lim_{K\to\infty}\left\|\sum_{a,b,c=1}^{\infty}\lambda_{a,b,c}(f)N_a^{(1)}N_b^{(2)}N_c^{(3)}-\sum_{a,b,c=1}^{K}\lambda_{a,b,c}(f)N_a^{(1)}N_b^{(2)}N_c^{(3)}\right\|_2^2 = \lim_{K\to\infty}\sum_{a,b,c=1}^{\infty}\mathbb{1}_{\max\{a,b,c\}\geqslant K+1}\,\lambda_{a,b,c}^2(f)=0$$

hence we would get the conclusion of Theorem 4.3 by an application of [Bil68, Theorem 4.2]. We thus have to prove that for each K,

$$\frac{1}{\sqrt{\ell mn}}\sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n}f_K\circ T_{i,j,k}\longrightarrow\sum_{a,b,c=1}^{K}\lambda_{a,b,c}(f)N_a^{(1)}N_b^{(2)}N_c^{(3)}\tag{4.3}$$

in distribution as $\min\{\ell,m,n\}\to\infty$. By definition of f_K,

$$\frac{1}{\sqrt{\ell mn}}\sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n}f_K\circ T_{i,j,k} = \sum_{a,b,c=1}^{K}\lambda_{a,b,c}(f)\left(\frac{1}{\sqrt{\ell}}\sum_{i=1}^{\ell}v_{a,1}\circ T_1^i\right)\left(\frac{1}{\sqrt{m}}\sum_{j=1}^{m}v_{b,2}\circ T_2^j\right)\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}v_{c,3}\circ T_3^k\right).$$

Consider the random vector $V_{\ell,m,n}$ of dimension $3K$, where the first K entries are $\frac{1}{\sqrt{\ell}}\sum_{i=1}^{\ell}v_{a,1}\circ T_1^i$, $1\leqslant a\leqslant K$, the entries of index between $K+1$ and $2K$ are $\frac{1}{\sqrt{m}}\sum_{j=1}^{m}v_{b,2}\circ T_2^j$ and the last K are $\frac{1}{\sqrt{n}}\sum_{k=1}^{n}v_{c,3}\circ T_3^k$. By the Cramer–Wold device, the Billingsley–Ibragimov *Central Limit Theorem* for martingale differences and the fact that $\|v_{a,1}\|_2=\|v_{b,2}\|_2=\|v_{c,3}\|_2=1$, the vector $V_{\ell,m,n}$ converges in distribution as $\min\{\ell,m,n\}\to\infty$ to

$$V:=\left(N_1^{(1)},\dots,N_K^{(1)},N_1^{(2)},\dots,N_K^{(2)},N_1^{(3)},\dots,N_K^{(3)}\right),$$

where $N_a^{(1)}$, $N_b^{(2)}$ and $N_c^{(3)}$ are like in the statement of the theorem. Now, (4.3) follows from the continuous mapping theorem, that is, $g(V_{\ell,m,n})\to g(V)$, where $g\colon\mathbb{R}^{3K}\to\mathbb{R}$ is defined as

$$g(x_1,\dots,x_K,y_1,\dots,y_K,z_1,\dots,z_K)=\sum_{a,b,c=1}^{K}\lambda_{a,b,c}(f)x_ay_bz_c.$$

This ends the proof of Theorem 4.3. □

4.2. Application to our situation

We come back to the general situation described in the introduction and we consider the factor $\mathcal{I}$ defined in Section 2. We claim that Theorem 4.3 applies to the martingale difference random field $(\mathbb{E}[f\mid\mathcal{I}]\circ T_{\underline{i}})_{\underline{i}\in\mathbb{Z}^3}$. We know already that the action of $\mathbb{Z}^3$ on this factor is of product type (cf. Section 2), and that $\mathbb{E}[f\mid\mathcal{I}]$ is a martingale difference adapted to the completely commuting invariant filtration $(\mathcal{I}\cap\mathcal{F}_{\underline{i}})_{\underline{i}\in\mathbb{Z}^3}$ (cf. Theorem 3.4). We just have to verify that this filtration has the product structure described in formula (4.1). Next proposition will apply to the original $\mathbb{Z}^3$-action restricted to the space $(\Omega, \mathcal{I}, \mu)$; the key argument will be that $\mathcal{I} \subset \mathcal{F}_{\infty,\infty,\infty}$.

Consider a product type $\mathbb{Z}^3$ measure preserving action as defined at the beginning of Section 4. We denote by $\mathcal{I}_1$ (resp. $\mathcal{I}_2$, resp. $\mathcal{I}_3$) the sigma-algebra of $T_{1,0,0}$(resp. $T_{0,1,0}$, resp. $T_{0,0,1}$)-invariant events, that is

$$\mathcal{I}_1 = \{\Omega_1, \emptyset\} \otimes \mathcal{A}_2 \otimes \mathcal{A}_3, \quad \mathcal{I}_2 = \mathcal{A}_1 \otimes \{\Omega_2, \emptyset\} \otimes \mathcal{A}_3, \quad \mathcal{I}_1 = \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \{\Omega_3, \emptyset\}.$$

We denote by $\overline{\mathcal{I}_1}$ (resp. $\overline{\mathcal{I}_2}$, resp. $\overline{\mathcal{I}_3}$) the sigma-algebra of events invariant under the $\mathbb{Z}^2$-action $T_{0,j,k}$ (resp. $T_{i,0,k}$, resp. $T_{i,j,0}$) that is

$$\overline{\mathcal{I}_t} = \bigcap_{s \neq t} \mathcal{I}_s.$$

These σ-algebras $\overline{\mathcal{I}_t}$ are mutually independent and $\mathcal{A} = \bigvee_{t=1}^{d} \overline{\mathcal{I}_t}$.

Proposition 4.4. — *Suppose that $(\mathcal{G}_{\underline{i}})_{\underline{i} \in \mathbb{Z}^3}$ is a completely commuting invariant filtration such that*

$$\mathcal{A} = \mathcal{G}_{\infty,\infty,\infty}.$$

Then, for each $\underline{i} = (i_1, i_2, i_3) \in \mathbb{Z}^3$,

$$\mathcal{G}_{\underline{i}} = \bigvee_{t=1}^{3} \left(\mathcal{G}_{\underline{i}} \cap \overline{\mathcal{I}_t}\right).$$

Note that the conclusion of the proposition can be written as the identity

$$\mathcal{G}_{\underline{i}} = \bigvee_{t=1}^{3} T_{(t)}^{-i_t} \left(\mathcal{G}_{\underline{0}} \cap \overline{\mathcal{I}_t}\right),$$

which, thanks to the product structure, is of the type (4.1).

Lemma 4.5. — *For all $\underline{i} \in \mathbb{Z}^3$ the sigma-algebras $\overline{\mathcal{I}_t}$ $(1 \leqslant t \leqslant 3)$ are conditionally independent with respect to $\mathcal{G}_{\underline{i}}$.*

Proof of Lemma 4.5. — By hypothesis, $\mathcal{I}_2 \subset \mathcal{G}_{\infty,\infty,\infty}$; thanks to Lemma 3.3, we know that, for all $j \in \mathbb{Z}$, $\mathcal{I}_2 \subset \mathcal{G}_{\infty,j,\infty}$. Similarly, for all $k \in \mathbb{Z}$, $\mathcal{I}_3 \subset \mathcal{G}_{\infty,\infty,k}$. Thus

$$\overline{\mathcal{I}_1} = \mathcal{I}_2 \cap \mathcal{I}_3 \subset \mathcal{G}_{\infty,j,\infty} \cap \mathcal{G}_{\infty,\infty,k} = \mathcal{G}_{\infty,j,k}.$$

Similarly we have $\overline{\mathcal{I}_2} \subset \mathcal{G}_{i,\infty,k}$ and $\overline{\mathcal{I}_3} \subset \mathcal{G}_{i,j,\infty}$, for all $i, j, k \in \mathbb{Z}$.

Since the σ-algebra $\mathcal{G}_{i,\infty,\infty}$ is $T_{0,j,k}$-invariant we have:

$$\text{if } u \in \mathbb{L}^2\left(\overline{\mathcal{I}_1}\right) \text{ then } \mathbb{E}[u \,|\, \mathcal{G}_{i,\infty,\infty}] \in \mathbb{L}^2\left(\overline{\mathcal{I}_1}\right). \tag{4.4}$$

Let $u \in \mathbb{L}^2(\overline{\mathcal{I}_1})$, $v \in \mathbb{L}^2(\overline{\mathcal{I}_2})$ and $w \in \mathbb{L}^2(\overline{\mathcal{I}_3})$. Using the complete commuting property of the filtration $\mathcal{G}$, we write:

$$\mathbb{E}[uvw \,|\, \mathcal{G}_{i,j,k}] = \mathbb{E}[\mathbb{E}[uvw \,|\, \mathcal{G}_{i,\infty,\infty}] \,|\, \mathcal{G}_{\infty,j,k}].$$

As we saw just above, v and w are $\mathcal{G}_{i,\infty,\infty}$-measurable. Hence

$$\mathbb{E}[uvw \,|\, \mathcal{G}_{i,j,k}] = \mathbb{E}[vw\mathbb{E}[u \,|\, \mathcal{G}_{i,\infty,\infty}] \,|\, \mathcal{G}_{\infty,j,k}],$$

But we know by (4.4) that $\mathbb{E}[u \mid \mathcal{G}_{i,\infty,\infty}]$ is $\overline{\mathcal{I}_1}$-measurable, hence $\mathcal{G}_{\infty,j,k}$-measurable, and we obtain

$$\mathbb{E}[uvw \,|\, \mathcal{G}_{i,j,k}] = \mathbb{E}[u \,|\, \mathcal{G}_{i,\infty,\infty}]\, \mathbb{E}[vw \,|\, \mathcal{G}_{\infty,j,k}].$$

Repeating similar arguments, we can write

$$\begin{aligned}\mathbb{E}[vw \,|\, \mathcal{G}_{\infty,j,k}] &= \mathbb{E}[\mathbb{E}[vw \,|\, \mathcal{G}_{\infty,j,\infty}] \,|\, \mathcal{G}_{\infty,\infty,k}] \\ &= \mathbb{E}[w\mathbb{E}[v \,|\, \mathcal{G}_{\infty,j,\infty}] \,|\, \mathcal{G}_{\infty,\infty,k}] \\ &= \mathbb{E}[v \,|\, \mathcal{G}_{\infty,j,\infty}]\, \mathbb{E}[w \,|\, \mathcal{G}_{\infty,\infty,k}].\end{aligned}$$

We obtain

$$\mathbb{E}[uvw \,|\, \mathcal{G}_{i,j,k}] = \mathbb{E}[u \,|\, \mathcal{G}_{i,\infty,\infty}]\, \mathbb{E}[v \,|\, \mathcal{G}_{\infty,j,\infty}]\, \mathbb{E}[w \,|\, \mathcal{G}_{\infty,\infty,k}].$$

In particular

$$\mathbb{E}[u \,|\, \mathcal{G}_{i,j,k}] = \mathbb{E}[u \,|\, \mathcal{G}_{i,\infty,\infty}],\ \mathbb{E}[v \,|\, \mathcal{G}_{i,j,k}] = \mathbb{E}[v \,|\, \mathcal{G}_{\infty,j,\infty}]$$

and

$$\mathbb{E}[w \,|\, \mathcal{G}_{i,j,k}] = \mathbb{E}[w \,|\, \mathcal{G}_{\infty,\infty,k}]$$

and we conclude that

$$\mathbb{E}[uvw \,|\, \mathcal{G}_{i,j,k}] = \mathbb{E}[u \,|\, \mathcal{G}_{i,j,k}]\, \mathbb{E}[v \,|\, \mathcal{G}_{i,j,k}]\mathbb{E}[w \,|\, \mathcal{G}_{i,j,k}]. \qquad \square$$

Proof of Proposition 4.4. — Fix $(i,j,k) \in \mathbb{Z}^3$. The conclusion of the proof of Lemma 4.5 shows that, for any function of the type $f = uvw$ with $u \in \mathbb{L}^2(\overline{\mathcal{I}_1})$, $v \in \mathbb{L}^2(\overline{\mathcal{I}_2})$ and $w \in \mathbb{L}^2(\overline{\mathcal{I}_3})$, the function $\mathbb{E}[f \mid \mathcal{G}_{i,j,k}]$ is the product of a function $\mathcal{G}_{i,j,k} \cap \overline{\mathcal{I}_1}$-measurable, a function $\mathcal{G}_{i,j,k} \cap \overline{\mathcal{I}_2}$-measurable and a function $\mathcal{G}_{i,j,k} \cap \overline{\mathcal{I}_3}$-measurable.

But we know that functions of the type $f = uvw$ generate a dense subalgebra of $\mathbb{L}^2(\mathcal{A})$, so that we have proved that

$$\mathcal{G}_{i,j,k} \subset \left(\mathcal{G}_{i,j,k} \cap \overline{\mathcal{I}_1}\right) \vee \left(\mathcal{G}_{i,j,k} \cap \overline{\mathcal{I}_2}\right) \vee \left(\mathcal{G}_{i,j,k} \cap \overline{\mathcal{I}_3}\right).$$

which is the conclusion of the proposition since the other inclusion is trivial. $\square$

5. Limit law in the general case, for 2-dimensional fields

As shown in [Vol19], for a random field of martingale differences we have a CLT with convergence towards a mixture of normal laws (see Theorem below). In Section 4, for f $\mathcal{I}$-measurable it was precised which mixtures can appear as limit laws (for definition of the factor $\mathcal{I}$ see Section 2). Here we deal with the same question for the general case of a martingale difference $f \in \mathbb{L}^2$. We reduce our study to the case of (ergodic) $\mathbb{Z}^2$-actions. In many cases it is because we have not succeeded to extend the proofs to $d > 2$.

As shown in Section 3, if f is a martingale difference, so is $\mathbb{E}[f \mid \mathcal{I}]$ and also $f - \mathbb{E}[f \mid \mathcal{I}]$. Recall that the limit laws for the random field generated by $\mathbb{E}[f \mid \mathcal{I}]$ have been determined in Section 4. In Section 5.1, we give a sufficient condition guaranteeing convergence of the random field generated by $f - \mathbb{E}[f \mid \mathcal{I}]$ to a normal law. We show in Section 5.3 that, under the same condition the random field generated by f is the convolution of the preceding ones.

In Section 5.2 we establish the result announced at the end of Section 3.

Eventually in Section 5.4 we give an example of a field of martingale differences generated by $f - \mathbb{E}[f \mid \mathcal{I}]$ where the limit law is not normal. It remains an open question which mixtures of normal laws can appear as limits in the CLT for $f - \mathbb{E}[f \mid \mathcal{I}]$.

In all this section, $f \circ T_{i,j}$ is a field of martingale differences adapted to a completely commuting filtration $\mathcal{F}_{i,j}$.

5.1. Limit law for an increment orthogonal to the factor of product type

Recall that $f \circ T_{i,j}$ is a field of martingale differences and as shown in Section 3, $(f - \mathbb{E}[f \mid \mathcal{I}]) \circ T_{i,j}$ are martingale differences as well.

Let us begin b recalling [Vol19, Theorem 1] which gives information on the limit law in the CLT. It will be stated and used here for $d = 2$ but extends to any finite dimension.

Theorem 5.1. — *When $\min\{m, n\} \to \infty$ the random variables $\frac{1}{\sqrt{mn}} \sum_{i=1}^m \sum_{j=1}^n f \circ T_{i,j}$ converge in distribution to a law with characteristic function $\mathbb{E}[\exp(-\eta^2 t^2/2)]$ where η^2 is a positive random variable such that $\mathbb{E}[\eta^2] = \|f\|_2^2$. The random variables $\frac{1}{mn} \sum_{i=1}^m \Big(\sum_{j=1}^n f \circ T_{i,j}\Big)^2$ converge in distribution to η^2.*

Comment on Theorem 5.1. — In fact, by the ergodic theorem,

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^m \frac{1}{n} \left(\sum_{j=1}^n f \circ T_{i,j} \right)^2$$

exists for each n and the distribution of η^2 is the limit in distribution of this quantity when $n \to \infty$. (This can be seen as well in the proof of Theorem 5.1 or as a consequence of it.)

Proposition 5.2. — *Let $f \in \mathbb{L}^2 \ominus \mathbb{L}^2(\mathcal{I})$ be a martingale difference. If, moreover, $f \in \mathbb{L}^4$ and*

$$\lim_{\ell\to\infty} \|\mathbb{E}[f \mid \mathcal{F}_{\infty,-\ell} \vee \mathcal{I}_1]\|_2 = 0. \tag{5.1}$$

then for $\min\{m, n\} \to \infty$, $(1/\sqrt{mn}) \sum_{i=1}^m \sum_{j=1}^n f \circ T_{i,j}$ converge in distribution to a centered normal law with variance $\mathbb{E}[f^2]$.

Remark 5.3. — We have $\mathcal{I}_2 \subset \mathcal{F}_{\infty,-\ell}$ (for every ℓ) hence $\mathcal{I} \subset \mathcal{F}_{\infty,-\ell} \vee \mathcal{I}_1$. Therefore $\|\mathbb{E}[f \mid \mathcal{I}]\|_2 \leqslant \|\mathbb{E}[f \mid \mathcal{F}_{\infty,-\ell} \vee \mathcal{I}_1]\|_2$ and, for any $f \in \mathbb{L}^2$, (5.1) implies $\mathbb{E}[f \mid \mathcal{I}] = 0$.

Remark 5.4. — For any $f \in \mathbb{L}^2$, by the martingale convergence theorem, the sequence

$$\Big((\mathbb{E}[f \mid \mathcal{F}_{\infty,-\ell} \vee \mathcal{I}_1])^2 \Big)$$

converges in $\mathbb{L}^1$ hence is uniformly integrable.

As a consequence the property (5.1) is equivalent to the convergence of the sequence $(\mathbb{E}[f \mid \mathcal{F}_{\infty,-\ell} \vee \mathcal{I}_1])_{\ell\geqslant 1}$ to zero in probability. Similarly, for $f \in \mathbb{L}^4$, condition (5.1) implies that

$$\lim_{\ell\to\infty} \|\mathbb{E}[f \mid \mathcal{F}_{\infty,-\ell} \vee \mathcal{I}_1]\|_4 = 0. \tag{5.2}$$

Remark 5.5. — Of course, condition (5.1) implies

$$\mathbb{E}[f \mid \mathcal{F}_{\infty,-\infty} \vee \mathcal{I}_1] = 0. \tag{5.3}$$

In Section 5.5 we give an example where

$$\bigcap_{\ell} \mathcal{F}_{\infty,-\ell} \vee \mathcal{I}_1 \neq \mathcal{F}_{\infty,-\infty} \vee \mathcal{I}_1,$$

showing that (5.3) can be satisfied without (5.1). But we do not know if (5.3) is sufficient in order to obtain the conclusion of Proposition 5.2.

Proof of Proposition 5.2. — Define

$$V_{m,n} = \frac{1}{m}\sum_{i=1}^{m}\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{n} f\circ T_{i,j}\right)^2. \tag{5.4}$$

Following the theorem just recalled above, it is sufficient to prove that

$$\lim_n \lim_m V_{m,n} = \|f\|_2^2.$$

By the ergodic theorem

$$\lim_m V_{m,n} = \mathbb{E}\left[\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{n} f\circ T_{0,j}\right)^2 \middle| \mathcal{I}_1\right]$$

and the square terms give the expected limit; indeed, again by ergodic theorem,

$$\lim_n \mathbb{E}\left[\frac{1}{n}\sum_{j=1}^{n} f^2\circ T_{0,j} \middle| \mathcal{I}_1\right] = \mathbb{E}\Big[\mathbb{E}\big[f^2 \,\big|\, \mathcal{I}_2\big] \,\Big|\, \mathcal{I}_1\Big]$$

and $\mathbb{E}[\mathbb{E}[f^2 \mid \mathcal{I}_2] \mid \mathcal{I}_1] = \mathbb{E}[f^2]$ since the σ-algebras $\mathcal{I}_1$ and $\mathcal{I}_2$ are independent.

Thus it remains to prove that

$$\lim_n \mathbb{E}\left[\frac{1}{n}\sum_{1\leqslant j<k\leqslant n} f\circ T_{0,j}\; f\circ T_{0,k} \middle| \mathcal{I}_1\right] = 0 \quad \text{in norm } \|\cdot\|_2. \tag{5.5}$$

We'll need to work with order two moments of these sums, which are finite thanks to the assumption that $f \in \mathbb{L}^4$.

Let us write, for $0 < \ell < n$,

$$\begin{aligned}
&\frac{1}{n}\sum_{1\leqslant j<k\leqslant n} f\circ T_{0,j}\; f\circ T_{0,k} \\
&\qquad\qquad = \frac{1}{n}\sum_{k=\ell+1}^{n} f\circ T_{0,k}\sum_{j=1}^{k-\ell} f\circ T_{0,j} + \frac{1}{n}\sum_{k=2}^{n} f\circ T_{0,k}\sum_{j=(k-\ell+1)\vee 1}^{k-1} f\circ T_{0,j} \\
&\qquad\qquad =: I + II.
\end{aligned}$$

The sequence $(f \circ T_{0,k} \sum_{j=(k-\ell+1)\vee 1}^{k-1} f \circ T_{0,j})_{k>1}$ is a sequence of square integrable martingale differences (remember that $f \in \mathbb{L}^4$), adapted to the filtration $(\mathcal{F}_{0,k})$. Pythagoras followed by Cauchy–Schwarz gives

$$\begin{aligned}\|II\|_2^2 &= \frac{1}{n^2}\sum_{k=2}^{n}\left\| f \circ T_{0,k} \sum_{j=(k-\ell+1)\vee 1}^{k-1} f \circ T_{0,j}\right\|_2^2 \\ &\leqslant \frac{1}{n^2}\sum_{k=2}^{n}\|f \circ T_{0,k}\|_4^2 \left\|\sum_{j=(k-\ell+1)\vee 1}^{k-1} f \circ T_{0,j}\right\|_4^2.\end{aligned}$$

Burkholder's inequality (see for example [Rio09]) gives

$$\left\|\sum_{j=(k-\ell+1)\vee 1}^{k-1} f \circ T_{0,j}\right\|_4^2 \leqslant 3 \sum_{j=(k-\ell+1)\vee 1}^{k-1} \|f \circ T_{0,j}\|_4^2 \leqslant 3\ell \|f\|_4^2$$

hence

$$\|\mathbb{E}[II \mid \mathcal{I}_1]\|_2 \leqslant \|II\|_2 \leqslant \sqrt{3}\|f\|_4^2 \sqrt{\frac{\ell}{n}}. \tag{5.6}$$

The sequence $(f \circ T_{0,k} \sum_{j=1}^{k-\ell} f \circ T_{0,j})_{k>\ell}$ is a sequence of square integrable martingale differences (remember that $f \in \mathbb{L}^4$), adapted to the filtration $(\mathcal{F}_{0,k})$. Applying Proposition 3.1 to the $\mathbb{Z}$-action $(T_{k,0})$, $\mathcal{I} = \mathcal{I}_1$, $\mathcal{F} = \mathcal{F}_{\infty,k}$ and $\mathcal{C}$ trivial, we obtain that conditional expectations with respect to $\mathcal{I}_1$ and $\mathcal{F}_{\infty,k}$ commute. Moreover, since $\mathcal{I}_1 \subset \mathcal{F}_{0,\infty}$, conditional expectations with respect to $\mathcal{I}_1$ and $\mathcal{F}_{0,\infty}$ commute. Using the commuting property of the filtration $(\mathcal{F}_{i,j})$, we affirm that conditional expectations with respect to $\mathcal{I}_1$ and $\mathcal{F}_{0,k}$ commute. As a consequence the sequence

$$\left(\mathbb{E}\left[f \circ T_{0,k} \sum_{j=1}^{k-\ell} f \circ T_{0,j} \,\middle|\, \mathcal{I}_1\right]\right)_{k>\ell}$$

is a sequence of martingale differences, and in particular it is an orthogonal sequence in $\mathbb{L}^2$. Using successively Pythagoras, invariance of the measure under $T_{0,k}$, properties of the conditional expectation, Cauchy–Schwarz and Burkholder, we write

$$\begin{aligned}\|\mathbb{E}[I \mid \mathcal{I}_1]\|_2^2 &= \frac{1}{n^2}\sum_{k=\ell+1}^{n}\left\|\mathbb{E}\left[f \circ T_{0,k} \sum_{j=1}^{k-\ell} f \circ T_{0,j} \,\middle|\, \mathcal{I}_1\right]\right\|_2^2 \\ &= \frac{1}{n^2}\sum_{k=\ell+1}^{n}\left\|\mathbb{E}\left[f \sum_{j=1-k}^{-\ell} f \circ T_{0,j} \,\middle|\, \mathcal{I}_1\right]\right\|_2^2 \\ &\leqslant \frac{1}{n^2}\sum_{k=\ell+1}^{n}\left\|\mathbb{E}[f \mid \mathcal{F}_{\infty,-\ell} \vee \mathcal{I}_1] \sum_{j=1-k}^{-\ell} f \circ T_{0,j}\right\|_2^2 \\ &\leqslant \|\mathbb{E}[f \mid \mathcal{F}_{\infty,-\ell} \vee \mathcal{I}_1]\|_4^2 \,\frac{1}{n}\sum_{k=\ell+1}^{n}\left\|\frac{1}{\sqrt{n}}\sum_{j=1-k}^{-\ell} f \circ T_{0,j}\right\|_4^2 \\ &\leqslant 3\|\mathbb{E}[f \mid \mathcal{F}_{\infty,-\ell} \vee \mathcal{I}_1]\|_4^2 \|f\|_4^2.\end{aligned}$$

This estimation, associated with (5.6), gives

$$\left\|\mathbb{E}\left[\frac{1}{n}\sum_{1\leqslant j<k\leqslant n} f\circ T_{0,j}\, f\circ T_{0,k}\,\middle|\,\mathcal{I}_1\right]\right\|_2 \leqslant \sqrt{3}\left(\left\|\mathbb{E}[f\,|\,\mathcal{F}_{\infty,-\ell}\vee\mathcal{I}_1]\right\|_4\|f\|_4+\|f\|_4^2\sqrt{\frac{\ell}{n}}\right),$$

hence for each ℓ,

$$\limsup_{n\to\infty}\left\|\mathbb{E}\left[\frac{1}{n}\sum_{1\leqslant j<k\leqslant n} f\circ T_{0,j}\, f\circ T_{0,k}\,\middle|\,\mathcal{I}_1\right]\right\|_2 \leqslant \sqrt{3}\|\mathbb{E}[f\,|\,\mathcal{I}_1\vee\mathcal{F}_{\infty,-\ell}]\|_4\|f\|_4,$$

and, thanks to (5.2), we conclude that (5.5) is true. □

5.2. Entropy condition for convergence to a normal law

Theorem 5.6. — *There exists a martingale difference $f\in\mathbb{L}^2$ with non normal limit in the CLT if and only if $T_{1,0}$ is of positive entropy in $\mathcal{I}_2$ and $T_{0,1}$ is of positive entropy in $\mathcal{I}_1$.*

Proof. — 1. Suppose that $T_{0,1}$ is of positive entropy in $\mathcal{I}_1$ and $T_{1,0}$ is of positive entropy in $\mathcal{I}_2$. For a measure preserving and bimeasurable transformation S, positive entropy implies existence of a nontrivial i.i.d. sequence of the form $h\circ S^i$. In the factor of product type given by $\mathcal{I}=\mathcal{I}_1\vee\mathcal{I}_2$ we are in the situation of Theorem 4.3 and we thus get a non trivial field of Wang–Woodroofe type.

2. Suppose that the transformation $T_{0,1}$ is of zero entropy on $\mathcal{I}_1$. Then $\mathcal{I}_1$ is an invariant sub-σ-algebra of the Pinsker σ-algebra for $T_{0,1}$ hence by [Vol87, Theorem 2] we have that for any integrable and $\mathcal{F}_{\infty,0}$-measurable function g, $\mathbb{E}[g\mid\mathcal{F}_{\infty,-\ell}\vee\mathcal{I}_1]=\mathbb{E}[g\mid\mathcal{F}_{\infty,-\ell}]$ for all $\ell>0$. Hence, if g is a square integrable martingale difference, we have

$$\mathbb{E}[g\,|\,\mathcal{F}_{\infty,-\ell}\vee\mathcal{I}_1]=0\quad\text{for all }\ell>0.$$

Moreover, we know that $\mathcal{I}_2\subset\mathcal{F}_{\infty,-1}$, so that $\mathcal{I}\subset\mathcal{F}_{\infty,-1}\vee\mathcal{I}_1$ and we have also $\mathbb{E}[g\mid\mathcal{I}]=0$. Thus, by Proposition 5.2, we know that if f is a martingale difference with finite fourth moment, then the CLT applies with a normal limit. For a square integrable martingale difference f, we write f as a sum of a bounded martingale difference and a rest small in norm $\|\cdot\|_2$ (cf. [Bil68, Theorem 4.2]); we conclude that the CLT applies to f with a normal limit. □

Notice that Theorem 5.6 improves the result in [Vol15] which tells that the limit distribution is normal as soon as one of the σ-algebra $\mathcal{I}_1$ or $\mathcal{I}_2$ is trivial. However, the result in [Vol15] applies to all $d>1$ while here it applies to $d=2$ only. The case of $d>2$ remains open.

5.3. Remark on the decomposition given by the projection on the factor of product type

If f is a martingale difference, we studied in Section 4 the CLT for the martingale difference $\mathbb{E}[f \mid \mathcal{I}]$ and in Section 5 the CLT for the martingale difference $f-\mathbb{E}[f \mid \mathcal{I}]$. Using the theorem recalled p. 13, it can be seen that if the limit law for $f-\mathbb{E}[f \mid \mathcal{I}]$ is normal, then the limit law for f is the convolution of the limit laws for $\mathbb{E}[f \mid \mathcal{I}]$ and $f-\mathbb{E}[f \mid \mathcal{I}]$.

5.4. Example of a non-normal limit law, in the orthocomplement of the product factor

Here we give an example of an ergodic $\mathbb{Z}^2$-action and a martingale difference f which is orthogonal to the factor $\mathcal{I}$ but for which the limit distribution in the CLT is not normal.

Let us denote by $(\Omega, \mathcal{B}, \mu, S)$ the Bernoulli scheme $\left(\frac{1}{2}, \frac{1}{2}\right)$ on the alphabet $\{-1,1\}$. This means that Ω is the space of bilateral sequences of -1 or 1, equipped with the product σ-algebra, the probability measure which makes the coordinate maps i.i.d. with law $\left(\frac{1}{2}, \frac{1}{2}\right)$, and the shift S.

Another dynamical system is $Z=\{-1,1\}$ equipped with the permutation U and the uniform probability.

On the product probability space $\Omega \times \Omega \times Z$, consider the $\mathbb{Z}^2$-action T defined by

$$T_{i,j}(\omega, \omega', z) = \left(S^i\omega, S^j\omega', U^{i+j}z\right).$$

This space is equipped with the *natural* filtration $(\mathcal{F}_{i,j})$:

$$\mathcal{F}_{i,j} = \sigma(X_k, Y_\ell, z; k \leqslant i, \ell \leqslant j) = \sigma(X_k; k \leqslant i) \otimes \sigma(Y_\ell; \ell \leqslant j) \otimes \mathcal{P}(Z).$$

where of course $X_k((\omega_n)_{n\in\mathbb{Z}}) = \omega_k$ et $Y_\ell((\omega'_m)_{m\in\mathbb{Z}}) = \omega'_\ell$ are the two independent Bernoulli processes.

The σ-algebra $\mathcal{I}_1$ of $T_{1,0}$-invariants is the σ-algebra of events depending only on the second coordinate:

$$\mathcal{I}_1 = (\text{trivial } \sigma\text{-algebra of } \Omega) \otimes \mathcal{B} \otimes (\text{trivial } \sigma\text{-algebra of } Z).$$

And symmetrically for the σ-algebra $\mathcal{I}_2$ of $T_{0,1}$-invariants. So that we have

$$\mathcal{I} = \mathcal{B} \otimes \mathcal{B} \otimes (\text{trivial } \sigma\text{-algebra of } Z).$$

We consider now the random variable $f(\omega, \omega', z) = X_0(\omega)Y_0(\omega')z$.

Since the expectation of z is zero, we have $\mathbb{E}[f \mid \mathcal{I}] = 0$. Associated to the $\mathbb{Z}^2$-action and the natural filtration, the function f is a martingale difference.

But it is easy to calculate the limit distribution of $\left(\frac{1}{\sqrt{nm}}\sum_{i=1}^n\sum_{j=1}^m f\circ T_{i,j}\right)$. Indeed,

$$\frac{1}{\sqrt{nm}}\sum_{i=1}^n\sum_{j=1}^m f\circ T_{i,j}(\omega,\omega',z) = \left(\frac{1}{\sqrt{n}}\sum_{i=1}^n(-1)^iX_i(\omega)\right)\left(\frac{1}{\sqrt{m}}\sum_{j=1}^m(-1)^jY_j(\omega')\right)z.$$

The limit distribution is the distribution of the product of two independent $\mathcal{N}(0,1)$ random variables, which is a non-normal Bessel law.

5.5. Remark on the asymptotic condition insuring a normal law

When looking to Proposition 5.2 the following question appears naturally. Is it true that

$$\bigcap_{\ell\geqslant 0}(\mathcal{F}_{\infty,-\ell}\vee\mathcal{I}_1)=\mathcal{F}_{\infty,-\infty}\vee\mathcal{I}_1? \tag{5.7}$$

We give here a negative answer by the construction of an example. (Note however that in this example the transformation $T_{2,0}$ is the identity so there will not exist any non zero martingale difference.)

The space is the bidimensional torus $\mathbb{T}^2=\mathbb{R}^2/\mathbb{Z}^2\simeq[0,1[^2$ equipped with the Borel σ-algebra and the Lebesgue measure. The transformation $T_{0,1}$ is the automorphism defined by the matrix $\left(\begin{smallmatrix}3&1\\2&1\end{smallmatrix}\right)$, that is $T_{0,1}(x,y)=(3x+y,2x+y)$, and the transformation $T_{1,0}$ is the translation $T_{1,0}(x,y)=(x+\frac{1}{2},y)$.

Transformations $T_{0,1}$ et $T_{1,0}$ are commuting:

$$T_{1,0}T_{0,1}(x,y)=T_{0,1}T_{1,0}(x,y)=\left(3x+2y+\frac{1}{2},2x+y\right),$$

so they generate a $\mathbb{Z}^2$-action denoted by T.

Let us denote by $\mathcal{P}$ the partition $\{[0,\frac{1}{2}[\times[0,1[,[\frac{1}{2},1[\times[0,1[\}$ of the torus, and by $(\mathcal{F}_j)$ the filtration generated by the transformation $T_{0,1}$ and this partition:

$$\mathcal{F}_j=\sigma(T_{0,-k}(\mathcal{P}),\ k\leqslant j)\ .$$

We know that the measure preserving dynamical system $(\mathbb{T}^2,T_{0,1})$ has the Kolmogorov property, thus the limit σ-algebra $\mathcal{F}_{-\infty}=\bigcap_{j\in\mathbb{Z}}\mathcal{F}_j$ is trivial (modulo the measure).

Note also that the partition $\mathcal{P}$ is invariant under the transformation $T_{1,0}$.

Define for all $i,j\in\mathbb{Z}$, $\mathcal{F}_{i,j}=\mathcal{F}_j$. Then $(\mathcal{F}_{i,j})_{i,j\in\mathbb{Z}}$ is a completely commuting invariant filtration. Moreover, $\mathcal{F}_{\infty,-\infty}\vee\mathcal{I}_1=\mathcal{F}_{-\infty}\vee\mathcal{I}_1=\mathcal{I}_1$.

Finally the following lemma shows that property (5.7) is not satisfied.

Lemma 5.7. — *For all $\ell\in\mathbb{Z}$, the σ-algebra $\mathcal{F}_{\infty,\ell}\vee\mathcal{I}_1$ is the whole Borel algebra.*

Proof. — Let us show that, for any ℓ, the σ-algebra generated by $\mathcal{I}_1$ and the partition $T_{0,-\ell}(\mathcal{P})$ is the whole Borel algebra.

For each integer n, consider

$$\begin{pmatrix}3&1\\2&1\end{pmatrix}^n=\begin{pmatrix}a_n&b_n\\c_n&d_n\end{pmatrix}$$

A straightforward induction shows that, for all $n\in\mathbb{Z}$, the number a_n is odd and the number c_n is even.

Any measurable function of the two-dimensional variable $(2x,y)$ is $\mathcal{I}_1$ measurable and the map $(x,y)\mapsto\mathbb{1}_{[0,\frac{1}{2}[}(a_\ell x+b_\ell y)$ is $T_{0,-\ell}(\mathcal{P})$-measurable. By multiplication, we obtain that the map $(x,y)\mapsto\mathbb{1}_{[0,\frac{1}{2}[}(x)$ is $T_{0,-\ell}(\mathcal{P})\vee\mathcal{I}_1$-measurable. Using the representation $\mathbb{T}=[0,1[$ we write

$$\exp(2i\pi x)=\exp\left(2i\pi\frac{1}{2}(2x\bmod 1)\right)\mathbb{1}_{[0,\frac{1}{2}[}(x)-\exp\left(2i\pi\frac{1}{2}(2x\bmod 1)\right)\mathbb{1}_{[\frac{1}{2},1[}(x).$$

This shows that any character of the two dimensional torus is $T_{0,-\ell}(\mathcal{P}) \vee \mathcal{I}_1$ measurable, proving that this σ-algebra is the whole Borel algebra. $\square$

Remark 5.8. — The idea behind the previous construction owes a great deal to an example attributed to Jean-Pierre Conze and communicated to us by Jean-Paul Thouvenot: if we denote by $\mathcal{B}(\alpha)$ the σ-algebra of Borel subsets of the one-dimensional torus invariant by the translation $x \mapsto x + \alpha$, we have that the algebra $\bigcap_{n\in\mathbb{N}} \mathcal{B}(2^{-n})$ and $\bigcap_{n\in\mathbb{N}} \mathcal{B}(3^{-n})$ are trivial (modulo the Lebesgue measure), but for each n the σ-algebra $\mathcal{B}(2^{-n}) \vee \mathcal{B}(3^{-n})$ is the whole Borel algebra.

BIBLIOGRAPHY

[Bil65] Patrick Billingsley, *Ergodic theory and information*, Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons, 1965. ↑5

[Bil68] ______, *Convergence of probability measures*, John Wiley & Sons, 1968. ↑10, 16

[CDV15] Christophe Cuny, Jérôme Dedecker, and Dalibor Volný, *A functional CLT for fields of commuting transformations via martingale approximation*, Zap. Nauchn. Semin. (POMI) **441** (2015), 239–262. ↑3

[Gir18] Davide Giraudo, *Invariance principle via orthomartingale approximation*, Stoch. Dyn. **18** (2018), no. 6, article no. 1850043 (29 pages). ↑3

[Gir24] ______, *Deviation inequality for Banach-valued orthomartingales*, Stochastic Processes Appl. **175** (2024), article no. 104391 (20 pages). ↑8

[Gla03] Eli Glasner, *Ergodic theory via joinings*, Mathematical Surveys and Monographs, vol. 101, American Mathematical Society, 2003. ↑5

[Par69] William Parry, *Entropy and generators in ergodic theory*, Mathematics Lecture Note Series, W. A. Benjamin, Inc., 1969. ↑6

[Rio09] Emmanuel Rio, *Moment inequalities for sums of dependent random variables under projective conditions*, J. Theor. Probab. **22** (2009), no. 1, 146–163. ↑15

[Vol87] Dalibor Volný, *Martingale decompositions of stationary processes*, Yokohama Math. J. **35** (1987), no. 1-2, 113–121. ↑16

[Vol15] ______, *A central limit theorem for fields of martingale differences*, Comptes Rendus. Mathématique **353** (2015), no. 12, 1159–1163. ↑3, 16

[Vol19] ______, *On limit theorems for fields of martingale differences*, Stochastic Processes Appl. **129** (2019), no. 3, 841–859. ↑2, 3, 12, 13

[WW13] Yizao Wang and Michael Woodroofe, *A new condition for the invariance principle for stationary random fields*, Stat. Sin. **23** (2013), no. 4, 1673–1696. ↑3

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