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NOISE STABILITY ON HYPERBOLIC GROUPS

STABILITÉ AU BRUIT DANS LES GROUPES HYPERBOLIQUES

Abstract. — We show that symmetric random walks on non-elementary hyperbolic groups with non-zero homomorphisms into the reals are noise stable at linear scale under finite exponential moment condition.

Résumé. — Nous montrons que les marches aléatoires symétriques sur les groupes hyperboliques non élémentaires et admettant des homomorphismes non triviaux vers les réels sont stables au bruit à échelle linéaire, sous une condition de moment exponentiel fini.

1. Introduction

Let Γ be a countable group endowed with a probability measure μ. A *μ-random walk* on Γ starting at the identity id is a sequence of random variables $(w_n)_{n\geqslant 0}$ of the form $w_n := \gamma_1 \cdots \gamma_n$ (with $w_0 := \mathrm{id}$) where the γ_i are i.i.d. variables of law μ. The *noise sensitivity and stability problem* for a μ-random walk asks the following. For each fixed real $\rho \in [0,1]$, let $w_n^\rho := \gamma_1^\rho \cdots \gamma_n^\rho$ where γ_i^ρ is resampled according to μ

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with probability ρ, or retained as γ_i with probability $1-\rho$, independently in every step. Do w_n and w_n^ρ become independent as n tends to infinity?

More precisely, for each real $\rho \in [0,1]$, consider on $\Gamma \times \Gamma$ the probability measure

$$\pi^\rho := \rho\mu \otimes \mu + (1-\rho)\mu_{\mathrm{diag}},$$

where $\mu \otimes \mu$ is the product measure and $\mu_{\mathrm{diag}}((\gamma, \gamma')) = \mu(\gamma)$ if $\gamma = \gamma'$ and 0 else. We write $\nu_n := \nu^{*n}$ for the n-fold convolution of a probability measure ν on a group. Note that the n-step distribution of a π^ρ-random walk starting at $(\mathrm{id}, \mathrm{id})$ is given by π_n^ρ, which coincides with the law of (w_n, w_n^ρ) on $\Gamma \times \Gamma$. Moreover, w_n and w_n^ρ have the same distribution μ_n on Γ. The μ-random walk on Γ is called *noise sensitive* in total variation if for every $\rho \in (0,1]$,

$$\|\pi_n^\rho - \mu_n \otimes \mu_n\|_{\mathrm{TV}} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

In the above the total variation distance is defined for probability measures $\nu^{(i)}$, $i = 1, 2$ by

$$\left\|\nu^{(1)} - \nu^{(2)}\right\|_{\mathrm{TV}} := \sup_{A \subset \Gamma \times \Gamma} \left|\nu^{(1)}(A) - \nu^{(2)}(A)\right|,$$

see also Remark 2.10. In contrast, the μ-random walk on Γ is called *noise stable* in total variation if for every $\rho \in [0,1)$,

$$\|\pi_n^\rho - \mu_n \otimes \mu_n\|_{\mathrm{TV}} \longrightarrow 1 \quad \text{as} \quad n \longrightarrow \infty.$$

Informally, on the one hand, if a μ-random walk is noise sensitive, then even though only a small number of increments are resampled (say, $\rho = 0.01$), the elements of the pair (w_n, w_n^ρ) become independent as n tends to infinity. On the other hand, if a μ-random walk is noise stable, then even though a large number of increments are resampled (say, $\rho = 0.99$), the pair (w_n, w_n^ρ) is still distinguishable from an independent pair. Note that w_n and w_n^ρ are independent when $\rho = 1$.

The study of noise sensitivity for random walks on groups was proposed by Benjamini and Brieussel in [BB23]. They observe that if Γ is a finite group, then every μ-random walk is noise sensitive [BB23, Proposition 5.1]. They also prove noise sensitivity for some (generating) μ-random walk on the infinite dihedral group [BB23, Theorem 1.4]. Later, this result has been generalized in [Tan25]. It is still a challenge to understand which finitely generated infinite groups admit a noise sensitive random walk. See also related questions in [BB23] and [Kal18, Section 3.3.4].

Noise stability is a strong negation of noise sensitivity. It has been observed that standard random walks on integer lattices, i.e. non trivial free abelian groups of finite rank are neither noise sensitive nor noise stable [Tan25, Appendix A]. An inspection shows that the simple random walk on a free semigroup of rank at least two is noise stable. In a more general setting, the following has been shown for every word hyperbolic group: if a probability measure μ is non-elementary (cf. Section 2.1) with finite first moment, then there exists $\rho_0 \in (0,1]$ such that $\|\pi_n^\rho - \mu_n \otimes \mu_n\|_{\mathrm{TV}} \to 1$ as $n \to \infty$ for all $\rho \in [0, \rho_0)$ [Tan24, Theorem 1.3]. It is not known whether $\rho_0 = 1$ in this generality. We show that this is the case for a certain class of word hyperbolic groups and probability measures.

THEOREM 1.1. — *Let Γ be a non-elementary word hyperbolic group which admits a non-zero homomorphism to $\mathbb{R}$. Let μ be a symmetric probability measure on Γ with* *finite exponential moment and whose support is not included in a proper subgroup of* Γ. *It holds that for every* $\rho \in [0,1)$,

$$\|\pi_n^\rho - \mu_n \otimes \mu_n\|_{\mathrm{TV}} \longrightarrow 1 \quad \text{as} \quad n \longrightarrow \infty,$$

i.e. the μ*-random walk on* Γ *is noise stable in total variation.*

Recall that a measure μ is called symmetric if $\mu(\gamma^{-1}) = \mu(\gamma)$ for all $\gamma \in \Gamma$. Moreover, as a hyperbolic group is finitely generated, the condition that Γ admits a non-zero homomorphism to $\mathbb{R}$ is equivalent to the requirement that its abelianization $\Gamma/[\Gamma,\Gamma]$ is infinite. Examples include finitely generated non-abelian free groups and fundamental groups of closed Riemann surfaces different from the sphere and the torus.

Let us point out that Theorem 1.1 is already new for the simple random walk on a free group (an explicit form of the distribution π_n^ρ does not seem available even in this special case). We show Theorem 1.1 in a stronger form Theorem 1.2, stating that the mass distributions π_n^ρ and $\mu_n \otimes \mu_n$ not only do not intersect much, but in fact separate at linear scale. For this, we fix a word metric d on Γ, and set $\lambda_\mu > 0$ the associated escape rate of the μ-walk on Γ, see Section 2.1. We equip Γ^2 with the l_∞-metric with respect to Γ-coordinates, i.e. $d_\infty(g,h) = \max\{d(g_1,h_1), d(g_2,h_2)\}$ for $g,h \in \Gamma^2$. We consider a relaxed notion of distance between two probability measures $(\nu^{(i)})_{i=1,2}$ on Γ^2, which allows for perturbations up to scale $s \in \mathbb{R}^+$. Namely, we set

$$\mathcal{U}^s\Big(\nu^{(1)}, \nu^{(2)}\Big) := \inf_{\left(Z_1, Z_1', Z_2, Z_2'\right)} \mathbf{P}\left(Z_1' \neq Z_2'\right)$$

where Z_i, Z_i' are Γ^2-valued random variables such that Z_i has law $\nu^{(i)}$ and $d_\infty(Z_i, Z_i') \leqslant s$ almost surely. Note that for $s = 0$, this recovers the total variation distance. We show the following.

Theorem 1.2. — *Let* Γ *be a non-elementary word hyperbolic group endowed with a word metric, let* μ *be a non-elementary probability measure with finite exponential moment and such that* $\varphi_\star\mu$ *is centered for some homomorphism* $\varphi : \Gamma \to \mathbb{R}$ *which is not identically zero on the support of* μ.

Then for every $\rho, \rho' \in [0,1]$ *with* $\rho \neq \rho'$, *every* $\alpha \in (0, \lambda_\mu)$, *we have*

$$\mathcal{U}^{\alpha n}\left(\pi_n^\rho, \pi_n^{\rho'}\right) \longrightarrow 1 \quad \text{as} \quad n \longrightarrow \infty.$$

Note that the separation rate is essentially optimal. Indeed for $\beta > \lambda_\mu$, we have $\mathcal{U}^{\beta n}(\pi_n^\rho, \pi_n^{\rho'}) \to 0$ due to the fact that π_n^ρ and $\pi_n^{\rho'}$ are asymptotically supported on the ball of radius βn. For $\beta = \lambda_\mu$, it is reasonable to expect the mixed behavior $\mathcal{U}^{\lambda_\mu n}(\pi_n^\rho, \pi_n^{\rho'}) \to 1/2$, at least provided the central limit theorem for the distance between id and the μ-random walk has non-degenerate limiting variance, see [BQ16] for details on that condition.

Theorem 1.2 is related to [BB23] which exploits different methods to show positive linear separation for some small proportion of the measures π_n^ρ and $\mu_n \otimes \mu_n$ in the case where Γ is a free group [BB23, Theorem 4.2]. A generalization of Theorem 1.2, considering groups Γ that may not be hyperbolic but act by isometries on a hyperbolic space is also given later in our paper, see Theorem 2.13.

1.1. On the proof

In the proof of Theorems 1.1 and 1.2, we study the corresponding harmonic measure ν_{π^ρ} associated to the π^ρ-random walk on $\Gamma \times \Gamma$. The probability measure ν_{π^ρ} is defined on the product $(\partial\Gamma)^2$ of Gromov boundaries as the distribution of the almost sure limiting point of (w_n, w_n^ρ) in $(\Gamma \cup \partial\Gamma)^2$ (cf. Section 2.1). We show that two different values for $\rho \in [0,1]$ yield two mutually singular harmonic measures ν_{π^ρ}. This is achieved by projecting to the abelianization $\Gamma^2/[\Gamma^2, \Gamma^2]$ a typical pair of geodesic rays selected by ν_{π^ρ}, and highlighting distinct asymptotic behaviors depending on ρ. Our approach is inspired by [Bén24] where several limit theorems have been obtained for abelianized geodesic rays in the context of a hyperbolic group. Dealing with a *product* of hyperbolic groups requires to develop those tools further. We now give more details.

Let $\varphi : \Gamma \to \mathbb{R}$ be a homomorphism from a hyperbolic group Γ to $\mathbb{R}$. Given $\xi \in \partial\Gamma$, denote by $r_\xi : \mathbb{N} \to \Gamma$ a unit speed geodesic ray originating from id and converging to ξ in $\partial\Gamma$. Consider a driving measure μ on Γ which is non-elementary and has finite exponential moment. Let ν be the associated harmonic measure on $\partial\Gamma$. Viewing ξ as a random variable on $\partial\Gamma$ distributed according to ν, it is established in [Bén24, Theorem 1.4] that the random variables $(\varphi \circ r_\xi(n))_{n \geqslant 1}$ satisfy a strong law of large numbers (LLN), a central limit theorem (CLT) and a law of the iterated logarithm (LIL) as n goes to infinity. The (LLN) and the (LIL) imply that the mean and the variance appearing in those limit theorems depend only on the measure class of ν. Moreover, it is shown in [Bén24] that those quantities have explicit formulas in terms of the driving measure μ. Notably, the formulas provide numerical invariants of all possible μ such that $\mu * \nu = \nu$ (see the more precise discussion in Section 2.1).

We apply the method to a product group $\Gamma \times \Gamma$ and show a joint version of the (LIL) in the present setting. More precisely, provided the pushforward measure $\varphi_\star\mu$ is centered and not a Dirac mass, we determine explicitly the set of accumulation points of the sequence

$$(2n \log\log n)^{-1/2} \left(\varphi \circ r_{\xi^{(1)}}(n),\ \varphi \circ r_{\xi^{(2)}}(n)\right)$$

where $(\xi^{(1)}, \xi^{(2)})$ is any (typical) pair of points selected by ν_{π^ρ} on $(\partial\Gamma)^2$. It appears that those accumulation sets are distinct for π^ρ-random walks corresponding to different $\rho \in [0,1]$, thus justifying the mutual singularity of harmonic measures ν_{π^ρ} on $(\partial\Gamma)^2$. This implies noise stability with linear separation as in Theorem 1.2 since any perturbation of π_n^ρ at scale αn ($\alpha \in (0, \lambda_\mu)$) converges weakly to ν_{π^ρ} on $(\Gamma \cup \partial\Gamma)^2$ as n tends to infinity. Theorem 1.1 follows at once from Theorem 1.2.

The singularity of the harmonic measures raises the following question on their dimensions. Let μ be a non-elementary probability measure with finite first moment on a word hyperbolic group Γ. It has been shown that the harmonic measure ν_{π^ρ} is exact dimensional with respect to a quasi-metric in $(\partial\Gamma)^2$ (e.g., the maximum of quasi-metrics in factors) [Tan24, Theorem 3.1]. Furthermore, the Hausdorff dimension is computed as

$$\dim \nu_{\pi^\rho} = \frac{h(\pi^\rho)}{\lambda_\mu}$$

where $h(\pi^\rho)$ is the asymptotic entropy of the π^ρ-random walk. It would be interesting to know whether it holds that $\dim \nu_{\pi^\rho} < \dim \nu_{\mu\otimes\mu}$ for all $\rho \in [0,1)$.

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2. Singularity of harmonic measures and statistics in the abelianization

2.1. Preliminaries and limit laws on single hyperbolic groups

In this section, we fix notations for the rest of the paper and recall some results from [Bén24]. For a complementary background on hyperbolic geometry, see the original paper [Gro87] and the survey [Cal13].

We let Γ be a hyperbolic (finitely generated) group, endowed with some left-invariant word metric. Given any $\gamma \in \Gamma$, we denote by $|\gamma|$ the distance from γ to the identity for the word metric. We let $\partial\Gamma$ denote the Gromov boundary of Γ and for every $\xi \in \partial\Gamma$, we choose some geodesic ray $r_\xi : \mathbb{N} \to \Gamma$ such that $r_\xi(0) = \mathrm{id}$ and with limiting point ξ. Here geodesic ray means that $|r_\xi(k)^{-1} r_\xi(l)| = l - k$ for every $k, l \in \mathbb{N}$ with $k \leqslant l$. On occasion, given $t \in \mathbb{R}^+$, we will write $r_\xi(t) := r_\xi(\lfloor t \rfloor)$.

We let μ denote a probability measure on Γ. We set $\Omega = \Gamma^{\mathbb{N}}$, endowed with the probability measure $\mathbb{P}_\mu$, which is the pushforward of $\mu^{\otimes\mathbb{N}^*}$ by $(\gamma_i)_{i\geqslant 1} \mapsto (\gamma_1, \dots, \gamma_n)_{n\geqslant 0}$. A $\mathbb{P}_\mu$-typical sequence $(w_n)_{n\geqslant 0}$ represents a (right) μ-trajectory on Γ. We assume that μ has *finite exponential moment*, i.e. $\sum_{\gamma\in\Gamma} e^{c|\gamma|}\mu(\gamma) < \infty$ for some $c > 0$. We also suppose that μ is *non-elementary.* This means that the semigroup generated by the support of μ contains at least two loxodromic elements with disjoint pairs of fixed points in the Gromov boundary $\partial\Gamma$. In particular Γ is non-elementary, i.e. it is neither finite nor virtually cyclic. Conversely, assuming Γ non-elementary, any distribution whose support generates Γ as a group is non-elementary. As μ is non-elementary, $\mathbb{P}_\mu$-almost every trajectory $w = (w_n) \in \Omega$ converges to a point ξ_w in $\partial\Gamma$. Moreover, the moment condition on μ guarantees that the rate of escape to the boundary is linear for the word metric on Γ. Namely, $\mathbb{P}_\mu$-almost surely and in L^1, we have

$$\frac{1}{n}|w_n| \longrightarrow \lambda_\mu \quad \text{as} \quad n \longrightarrow \infty, \tag{2.1}$$

where $\lambda_\mu > 0$ is some constant depending only on $\Gamma, |\cdot|, \mu$ (cf. [BMSS23, Section 7.1]).

The distribution ν_μ of the limiting point ξ_w as w varies with law $\mathbb{P}_\mu$ is called the *harmonic measure* (or the *Furstenberg measure*) associated to μ. It is the unique probability measure on $\partial\Gamma$ satisfying that $\mu * \nu_\mu = \nu_\mu$, where $\mu * \nu_\mu = \sum_{\gamma\in\Gamma} \mu(\gamma)\nu_\mu \circ \gamma^{-1}$. The harmonic measure ν_μ encodes the asymptotic properties of the μ-random walk trajectories. It is, however, not clear how much information on μ can be extracted from (the measure class of) ν_μ. In [Bén24], it is shown that ν_μ conveys information regarding the projection of μ to the abelianization. Among other results, one reads the following.

Theorem 2.1 ([Bén24, Theorem 1.4]). — *Keep the above notations and let* $\varphi : \Gamma \to \mathbb{R}$ *be a homomorphism.*

(LLN) *For* ν_μ*-almost every* $\xi \in \partial\Gamma$*, we have as* $n \to \infty$,

$$\frac{\varphi \circ r_\xi(n)}{n} \longrightarrow \lambda_\mu^{-1}\mathbf{E}\,(\varphi_\star\mu).$$

(CLT) *Assume* $\mathbf{E}\,(\varphi_\star\mu) = 0$ *and* ξ *is a random variable distributed according to* ν_μ*. Then, as* $n \to +\infty$*, the sequence of random variables*

$$\frac{\varphi \circ r_\xi(n)}{\sqrt{n}}$$

converges in law to the centered Gaussian distribution on $\mathbb{R}$ *with variance given by* $\lambda_\mu^{-1}\operatorname{Var}(\varphi_\star\mu) =: \kappa_{\nu_\mu}^2 \in \mathbb{R}^+$ *(where* $\kappa_{\nu_\mu} \geqslant 0$*).*

(LIL) *Assume* $\mathbf{E}\,(\varphi_\star\mu) = 0$*. For* ν_μ*-almost every* $\xi \in \partial\Gamma$*, the set of accumulation points of the sequence*

$$\frac{\varphi \circ r_\xi(n)}{\sqrt{2n\log\log n}} \quad \text{for } n \geqslant 3$$

is equal to $[-\kappa_{\nu_\mu}, \kappa_{\nu_\mu}]$.

In the above statement, $\mathbf{E}$ and Var refer to the mean and variance of a probability measure on $\mathbb{R}$.

A large deviation principle and a gambler's ruin estimate are also obtained [Bén24, Theorem 1.4]. Note that the (CLT) and the (LIL) assume that the pushforward measure $\varphi_\star\mu$ is centered. Such limit laws are still available in the non-centered case. However, the involved variance does not have a simple formulation in terms of $\varphi_\star\mu$. It may even happen that the limiting Gaussian distribution for geodesic rays is non-degenerate while that of $\varphi_\star\mu$ is degenerate, see [Bén24, Lemma 4.2].

Concerning the displacements $|w_n|$, we have the corresponding (LIL).

Proposition 2.2. — *Keep the above notations. There exists a constant* $\sigma_\mu \geqslant 0$ *such that for* $\mathbb{P}_\mu$*-almost every trajectory* $(w_n)_{n\geqslant 0}$*, the set of accumulation points of the sequence*

$$\frac{|w_n| - \lambda_\mu n}{\sqrt{2n\log\log n}} \quad \text{for } n \geqslant 3,$$

is equal to $[-\sigma_\mu, \sigma_\mu]$ *in* $\mathbb{R}$.

Proof. — This is a direct application of the (LIL) for cocycles obtained in [Bén24, Proposition A.5], which applies thanks to [Bén24, Section 3.1] and Remark (2) at the beginning of [Bén24, Appendix A]. See also [Bjö10, BQ16, Cho23]. ☐

2.2. Stopping times and technical lemmas

We keep the notations $(\Gamma, |\cdot|, r_\xi, \mu, \nu_\mu)$ from Section 2.1. We consider $\varphi : \Gamma \to \mathbb{R}$ such that $\mathbf{E}\,\varphi_\star\mu = 0$.

The goal of this section is to establish Proposition 2.3, which tells us that the image under φ of a ν_μ-typical geodesic ray on Γ behaves like a random walk with centered i.i.d. increments at scale $(n\log\log n)^{1/2}$.

Proposition 2.3. — *For $\mathbb{P}_\mu$-almost every trajectory $w \in \Omega$, denoting by $\xi_w \in \partial\Gamma$ its limiting point, we have as $n \to \infty$,*

$$\frac{1}{\sqrt{n \log\log n}} \left|\varphi\left(r_{\xi_w}(\lambda_\mu n)\right) - \varphi\left(w_n\right)\right| \longrightarrow 0.$$

For the proof, we first study the sequence of stopping times $\tau_n : \Omega \to \mathbb{N} \cup \{+\infty\}$, where τ_n is the exit time of the centered ball of radius $\lambda_\mu n$ in Γ. More precisely, we define

$$\tau_n(w) := \inf\{k \geqslant 1 \,:\, |w_k| \geqslant \lambda_\mu n\}.$$

The next lemma gives a first basic estimate on τ_n and on the overshoot.

Lemma 2.4. — *We have $\mathbb{P}_\mu$-almost surely $\lim_{n\to+\infty} \tau_n/n = 1$. Moreover, there exists $C > 0$ such that for $\mathbb{P}_\mu$-almost every $w \in \Omega$, for large enough n,*

$$||w_{\tau_n(w)}| - \lambda_\mu n| \leqslant C \log(n).$$

Proof. — The first claim follows from (2.1). For the second claim, note that the Borel–Cantelli Lemma and our finite exponential moment assumption on μ together guarantee that for some C depending on $(\Gamma, |\cdot|, \mu)$, for $\mathbb{P}_\mu$-almost every $w \in \Omega$, for large enough n, we have $|w_{n-1}^{-1} w_n| \leqslant C \log n$. On the other hand, by definition of τ_n, we have

$$|w_{\tau_n(w)}| \geqslant \lambda_\mu n > |w_{\tau_n(w)-1}|.$$

Observing that $|w_{\tau_n(w)}| \leqslant |w_{\tau_n(w)-1}| + |w_{\tau_n(w)-1}^{-1} w_{\tau_n(w)}|$, the claim follows. ☐

We deduce that the stopping times τ_n are approximated by n up to an error of order $(n \log\log n)^{1/2}$. Here we use the (LIL) for displacements $|w_n|$ cited in Proposition 2.2.

Lemma 2.5. — *There exists a constant $C > 0$ such that for $\mathbb{P}_\mu$-almost every $w \in \Omega$, for all large enough n,*

$$|\tau_n(w) - n| \leqslant C\sqrt{n \log\log n}.$$

Remark 2.6. — Recall the constant σ_μ from Proposition 2.2. A law of the iterated logarithm with limiting interval $[-\sigma_\mu/\lambda_\mu, \sigma_\mu/\lambda_\mu]$ can in fact be obtained for the sequence $(2n \log\log n)^{-1/2}(\tau_n(w) - n)$. The proof below shows all accumulation points belong to this interval. The converse inclusion can be handled by a thickening argument as in [Bén24, Section 4.4]. For conciseness, we do not pursue this direction.

Proof. — By Proposition 2.2, there exists a constant $C > 0$ satisfying that for $\mathbb{P}_\mu$-almost every $w \in \Omega$, for all large enough n,

$$||w_n| - \lambda_\mu n| \leqslant C\sqrt{n \log\log n}.$$

Restricting to the subsequence (τ_n), we get for large n,

$$||w_{\tau_n(w)}| - \lambda_\mu \tau_n(w)| \leqslant C\sqrt{\tau_n(w) \log\log \tau_n(w)}.$$

By Lemma 2.4, we deduce that for $\mathbb{P}_\mu$-almost every $w \in \Omega$, for all large enough n,

$$\begin{aligned}|\lambda_\mu n - \lambda_\mu \tau_n(w)| &\leqslant ||w_{\tau_n(w)}| - \lambda_\mu n| + ||w_{\tau_n(w)}| - \lambda_\mu \tau_n(w)| \\ &\leqslant (1 + o(1))C\sqrt{n \log\log n}.\end{aligned}$$

This concludes the claim by redefining the constant C to be $2C/\lambda_\mu$. ☐

We will use the following general lemma to control the partial increments of a sum of i.i.d. variables in $\mathbb{R}$ along a short time interval of the order $(n \log\log n)^{1/2}$.

Lemma 2.7. — *Consider a sequence of i.i.d. real-random variables $(X_n)_{n\geqslant 1}$ with finite exponential moment and zero average. Fix $M > 0$. Set $S_n := \sum_{i=1}^n X_i$ and $k_n := M\sqrt{n\log\log n}$. Then we have almost surely as $n \to +\infty$*

$$\frac{1}{k_n}\max\Big\{|S_i - S_n| \,:\, i \text{ such that } |i-n| \leqslant k_n\Big\} \longrightarrow 0.$$

Proof. — Since $\mathbf{E}\,X_i = 0$, the classical large deviation estimate shows that for every $\varepsilon > 0$ there exist constants $c_\varepsilon > 0$ and N_ε such that for all $n \geqslant N_\varepsilon$,

$$\mathbf{P}\Big(|S_n| \geqslant \varepsilon n\Big) \leqslant e^{-c_\varepsilon n}.$$

For every $\varepsilon > 0$, we have

$$\mathbf{P}\left(\max_{(\log n)^2\leqslant|i-n|\leqslant k_n} |S_i - S_n| \geqslant \varepsilon k_n\right) \leqslant \sum_{(\log n)^2\leqslant|i-n|\leqslant k_n} \mathbf{P}\Big(|S_i - S_n| \geqslant \varepsilon k_n\Big).$$

Noting that for $i_1 < i_2$, the distribution of $S_{i_2} - S_{i_1}$ and that of $S_{i_2-i_1}$ coincide, we find that for all n such that $(\log n)^2 \geqslant N_\varepsilon$ the right hand side of the above inequality is bounded by

$$\sum_{(\log n)^2\leqslant|i-n|\leqslant k_n} e^{-c_\varepsilon|i-n|} \leqslant 2k_n e^{-c_\varepsilon(\log n)^2} = 2M\sqrt{n\log\log n}\, e^{-c_\varepsilon(\log n)^2}.$$

Furthermore, since X_i has finite exponential moment, for every $\varepsilon > 0$ there exists a constant $c'_\varepsilon > 0$ such that $\mathbf{P}(|X_i| \geqslant \varepsilon k) \leqslant e^{-c'_\varepsilon k}$ for all large enough $k \geqslant 0$. Thus for all large enough n,

$$\begin{aligned}\mathbf{P}\left(\max_{|i-n|\leqslant(\log n)^2} |S_i - S_n| \geqslant \varepsilon k_n\right) &\leqslant \sum_{|i-n|\leqslant(\log n)^2} \mathbf{P}\Big(|S_i - S_n| \geqslant \varepsilon k_n\Big)\\ &\leqslant \sum_{|i-n|\leqslant(\log n)^2} |i-n| e^{-c'_\varepsilon k_n/(\log n)^2} \leqslant 2(\log n)^4 e^{-c'_\varepsilon k_n/(\log n)^2} \leqslant e^{-n^{1/4}}.\end{aligned}$$

Combining the above estimates shows that for every $\varepsilon > 0$, for all large enough n,

$$\mathbf{P}\left(\max_{|i-n|\leqslant k_n} |S_i - S_n| \geqslant \varepsilon k_n\right) \leqslant 2M\sqrt{n\log\log n}\, e^{-c_\varepsilon(\log n)^2} + e^{-n^{1/4}}.$$

The right hand side is summable in n for each $\varepsilon > 0$, whence the claim follows from the Borel–Cantelli lemma. □

We are now able to conclude the proof of the approximation statement Proposition 2.3.

Proof. — Recall that $\mathbf{E}\,\varphi(\gamma_i) = 0$ and μ has finite exponential moment. In particular, the sequence of random variables $(\varphi(w_n))_{w\sim\mathbb{P}_\mu}$ corresponds to the trajectory of a random walk on $\mathbb{R}$ with i.i.d. centered increments of finite exponential moment. By Lemma 2.7, for each $M > 0$, we deduce that $\mathbb{P}_\mu$-almost surely as $n \to \infty$,

$$\frac{1}{M\sqrt{n\log\log n}}\max\left\{|\varphi(w_i) - \varphi(w_n)| \,:\, i \text{ such that } |i-n| \leqslant M\sqrt{n\log\log n}\right\} \longrightarrow 0.$$

Lemma 2.5 shows that for a constant $C > 0$, $\mathbb{P}_\mu$-almost surely, for all large enough n,

$$|\tau_n(w) - n| \leqslant C\sqrt{n \log\log n}.$$

Therefore, by taking $M = C$, we obtain almost surely

$$\frac{1}{\sqrt{n \log\log n}} \left|\varphi(w_{\tau_n(w)}) - \varphi(w_n)\right| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{2.2}$$

We now relate $w_{\tau_n(w)}$ to the limiting geodesic ray r_{ξ_w} associated to w. Using that μ has finite exponential moment, there exists a constant $D > 0$ such that the following holds. For ν_μ-almost every $\xi \in \partial\Gamma$ for all large enough n,

$$\left|w_n^{-1} r_\xi(|w_n|)\right| \leqslant D \log n.$$

(A more general result has been established under a weaker moment assumption [Cho23, Theorem D].) As $\tau_n/n \to 1$ a.s., we can restrict to (τ_n) and get almost surely, for large n,

$$\left|w_{\tau_n(w)}^{-1} r_\xi(|w_{\tau_n(w)}|)\right| \leqslant 2D \log n.$$

On the other hand, by Lemma 2.4,

$$||w_{\tau_n(w)}| - \lambda_\mu n| \leqslant C \log n.$$

Therefore by the triangle inequality, we obtain

$$\left|w_{\tau_n(w)}^{-1} r_\xi(\lambda_\mu n)\right| \leqslant (2D + C) \log n. \tag{2.3}$$

Since φ is Lipschitz, the claim now follows from the combination of (2.2) and (2.3). ☐

2.3. A joint law of the iterated logarithm and singularity of harmonic measures

We keep the notations $(\Gamma, |\cdot|, r_\xi, \mu)$ from Section 2.1 and consider a homomorphism $\varphi : \Gamma \to \mathbb{R}$ such that $\mathbf{E}\,\varphi_\star\mu = 0$. We also consider π a probability measure on $\Gamma \times \Gamma$ such that both marginals are equal to μ. We show a joint (LIL) for the projection under φ of a pair of geodesic rays on Γ selected randomly according to the harmonic measure of π. The noise stability for the μ-walk on Γ follows.

We denote by $\mathbb{P}_\pi$ the induced probability measure on the space of (double) trajectories $\boldsymbol{w} = (w_n^{(1)}, w_n^{(2)})_{n \geqslant 0}$ (defined similarly to $\mathbb{P}_\mu$ from Section 2.1). Note that $\mathbb{P}_\pi$-almost every trajectory $\boldsymbol{w}$ converges in $\partial\Gamma \times \partial\Gamma$. We write ν_π for the distribution of the limiting point, and call it the harmonic measure for the π-random walk. We also write $\varphi \times \varphi : \Gamma \times \Gamma \to \mathbb{R}^2; (\gamma_1, \gamma_2) \mapsto (\varphi(\gamma_1), \varphi(\gamma_2))$.

Proposition 2.8. — *Let A_{ν_π} be the 2×2 matrix given by*

$$A_{\nu_\pi} := \lambda_\mu^{-1} \operatorname{Cov}((\varphi \times \varphi)_\star \pi).$$

Then for ν_π-almost every $\boldsymbol{\xi} = (\xi^{(1)}, \xi^{(2)}) \in \partial\Gamma \times \partial\Gamma$, the set of accumulation points *of the sequence*

$$\frac{\left(\varphi \circ r_{\xi^{(1)}}(n),\ \varphi \circ r_{\xi^{(2)}}(n)\right)}{\sqrt{2n \log\log n}} \quad \text{for } n > 2$$

is equal to $A_{\nu_\pi}^{1/2}\overline{B}(0,1)$.

It follows that A_{ν_π} *depends only on the measure class of* ν_π. *Furthermore, if* π *and* π' *are probability measures on* $\Gamma \times \Gamma$ *such that both marginals are equal to* μ, *and that* $A_{\nu_\pi} \neq A_{\nu_{\pi'}}$, *then* ν_π *and* $\nu_{\pi'}$ *are mutually singular.*

In the above, $\overline{B}(0,1) = \{(t_1,t_2) \in \mathbb{R}^2 \ :\ t_1^2 + t_2^2 \leqslant 1\}$, and $\mathrm{Cov}(\,\cdot\,)$ refers to the covariance matrix for probability measures on $\mathbb{R}^2$.

Proof. — Proposition 2.3 yields that $\mathbb{P}_\pi$-almost surely, for each $i = 1, 2$,

$$\frac{1}{\sqrt{n \log\log n}} \left|\varphi\left(w_n^{(i)}\right) - \varphi\left(r_{\xi^{(i)}}(\lambda_\mu n)\right)\right| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{2.4}$$

By the classical (LIL) on $(\varphi(w_n^{(1)}), \varphi(w_n^{(2)}))$ for $n \geqslant 0$, the following holds: for the covariance matrix $\Sigma_\pi := \mathrm{Cov}((\varphi \times \varphi)_\star \pi)$, for $\mathbb{P}_\pi$-almost every $(w_n^{(1)}, w_n^{(2)})$, the set of accumulation points of the sequence

$$\frac{1}{\sqrt{2n \log\log n}} \left(\varphi\left(w_n^{(1)}\right), \varphi\left(w_n^{(2)}\right)\right) \quad \text{for } n \geqslant 3,$$

is equal to $\Sigma_\pi^{1/2}\overline{B}(0,1)$ in $\mathbb{R}^2$. In view of (2.4), this estimate still holds with $\varphi(r_{\xi^{(i)}}(\lambda_\mu n))$ in the place of $\varphi(w_n^{(i)})$. Noting that $|n - \lambda_\mu \lfloor n/\lambda_\mu \rfloor| \leqslant \lambda_\mu$ and that $\varphi : \Gamma \to \mathbb{R}$ is Lipschitz, we deduce that for ν_π-almost every $\boldsymbol{\xi} \in (\partial\Gamma)^2$, the sequence

$$\frac{1}{\sqrt{2n \log\log n}} \left(\varphi\left(r_{\xi^{(1)}}(n)\right),\ \varphi\left(r_{\xi^{(2)}}(n)\right)\right) \quad \text{for } n \geqslant 3, \tag{2.5}$$

has its set of accumulation points equal to $A_{\nu_\pi}^{1/2}\overline{B}(0,1)$ where $A_{\nu_\pi} := \lambda_\mu^{-1}\Sigma_\pi$. This shows the first claim.

To obtain the second part of the proposition, we first check that the knowledge of the image set $A_{\nu_\pi}^{1/2}\overline{B}(0,1)$ characterizes A_{ν_π}. Indeed, if A_1, A_2 are positive semi-definite symmetric matrices such that $A_1^{1/2}\overline{B}(0,1) = A_2^{1/2}\overline{B}(0,1)$, then up to restricting to their common image, we may assume that the A_i are invertible. We then get that $A_1^{-1/2}A_2^{1/2}$ is orthogonal. Comparing with its transpose and using that the A_i are symmetric, we get $A_1 = A_2$, whence the claim.

We may now conclude. Given a positive semi-definite symmetric matrix $A \in M_2(\mathbb{R})$, let E_A be the set of $\boldsymbol{\xi} \in (\partial\Gamma)^2$ such that the sequence (2.5) has its set of accumulation points equal to $A^{1/2}\overline{B}(0,1)$. The two preceding paragraphs respectively justify that $\nu_\pi(E_{A_{\nu_\pi}}) = 1$ and $\nu_\pi(E_A) = 0$ for every $A \neq A_{\nu_\pi}$. The second part of the Proposition 2.8 follows. □

We deduce that noisy pairs (w_n, w_n^ρ) corresponding to different parameters ρ have mutually singular limiting harmonic measures ν_{π^ρ}.

Corollary 2.9. — *Keep the previous notations and assume further that* φ *is not identically zero on the support of* μ. *Then the harmonic measures* ν_{π^ρ} *on* $\partial\Gamma \times \partial\Gamma$ *for* π^ρ*-random walks are mutually singular for all* $\rho \in [0,1]$.

Proof. — A direct computation yields

$$\operatorname{Cov}((\varphi\times\varphi)_\star\pi^\rho)=\operatorname{Var}(\varphi_\star\mu)\begin{pmatrix}1 & 1-\rho\\ 1-\rho & 1\end{pmatrix},$$

where $\operatorname{Var}(\varphi_\star\mu):=\mathbf{E}_\mu(\varphi(\gamma)^2)$. Note that $\operatorname{Var}(\varphi_\star\mu)>0$. Indeed, otherwise $\varphi(\gamma)=0$ for all γ in the support of μ, contradicting our assumptions. Hence, for $\rho,\rho'\in[0,1]$ with $\rho\neq\rho'$, we have $\operatorname{Cov}((\varphi\times\varphi)_\star\pi^\rho)\neq\operatorname{Cov}((\varphi\times\varphi)_\star\pi^{\rho'})$, or equivalently $A_{\nu_{\pi^\rho}}\neq A_{\nu_{\pi^{\rho'}}}$. By the second part of Proposition 2.8, the measures ν_{π^ρ} and $\nu_{\pi^{\rho'}}$ are thus mutually singular for $\rho\neq\rho'$, thus concluding the proof of Corollary 2.9. ☐

We still need to convert limiting singularity into an asymptotic separation estimate. To this end, we need a few preliminary remarks on the behavior of the total variation norm with respect to weak convergence.

Given a signed Borel measure η on a metric space Z, let

$$\|\eta\|_{\mathrm{TV}}:=\sup\{|\eta(B)|\,:\,B\text{ is Borel in }Z\}.$$

Remark 2.10. — Other standard definitions of total variation norm use a variant, which we denote by $\|\cdot\|$. Namely, let $\eta=\eta_+-\eta_-$ be the Hahn–Jordan decomposition, i.e., η_+ and η_- are non-negative Borel measures supported on disjoint Borel sets Z_+ and Z_- in Z respectively. Then $\|\eta\|:=(\eta_++\eta_-)(Z)$. Note that $\|\eta\|\leqslant 2\|\eta\|_{\mathrm{TV}}\leqslant 2\|\eta\|$. Moreover, if $\eta=\nu^{(1)}-\nu^{(2)}$ for two Borel probability measures $\nu^{(1)}$ and $\nu^{(2)}$, then(1) we have equality $\|\eta\|=2\|\eta\|_{\mathrm{TV}}$.

LEMMA 2.11. — *If a sequence of Borel probability measures $(\nu_n^{(i)})_{n\geqslant 0}$ on a metric space Z converges weakly to a Borel probability measure $\nu^{(i)}$ on Z for each $i=1,2$, then*

$$\liminf_{n\to\infty}\left\|\nu_n^{(1)}-\nu_n^{(2)}\right\|_{\mathrm{TV}}\geqslant\left\|\nu^{(1)}-\nu^{(2)}\right\|_{\mathrm{TV}}.$$

Proof. — If a sequence of signed Borel measures $\{\eta_n\}_{n\geqslant 0}$ on Z converges weakly to a signed Borel measure η, then

$$\liminf_{n\to\infty}\|\eta_n\|\geqslant\|\eta\|$$

(cf. [Bog18, Theorem 4.8.1]). The claim then follows using Remark 2.10. ☐

In order to get noise stability with *linear* separation, we also need to know that convergence toward a given boundary point is stable under a certain range of perturbations.

LEMMA 2.12. — *Consider two sequences $(\omega_n)_{n\geqslant 0},(\omega'_n)_{n\geqslant 0}\in\Gamma^{\mathbb{N}}$, and a boundary point $\xi\in\partial\Gamma$. Assume $\omega_n\to\xi$ and $|\omega_n|-|\omega_n^{-1}\omega'_n|\to+\infty$ as n goes to infinity. Then $\omega'_n\to\xi$ as well.*

Proof. — Write $(x|y):=\frac{1}{2}\left(|x|+|y|-|x^{-1}y|\right)$ the Gromov product on Γ based at the identity. We need to check that $\lim_{k,l\to\infty}(\omega'_k|\omega_l)=\infty$. By hyperbolicity, we know that

$$(\omega'_k|\omega_l)\geqslant\min\{(\omega'_k|\omega_k),(\omega_k|\omega_l)\}-O(1).$$

Moreover, each term $(\omega'_k|\omega_k)$, $(\omega_k|\omega_l)$ goes to ∞ by assumption, whence the result. ☐

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(1) To check this identity, note that $\nu^{(1)}(B)-\nu^{(2)}(B)=\nu^{(2)}(Z\setminus B)-\nu^{(1)}(Z\setminus B)$ for every measurable $B\subseteq Z$. Thus $\|\eta\|_{\mathrm{TV}}=\frac{1}{2}\sup\{|\eta(B_1)|+|\eta(B_2)|\,:\,Z=B_1\sqcup B_2\}=\frac{1}{2}\|\eta\|$.

We are finally able to conclude the proof of Theorem 1.2.

Proof of Theorem 1.2. — Note that $\Gamma \cup \partial\Gamma$ admits a compact metrizable topology which is compatible with the notion of convergence to the boundary $\partial\Gamma$, and respects the topologies on Γ and $\partial\Gamma$ individually (cf. [Gro87, 7.2.M]). Moreover, given $\alpha \in (0, \lambda_\mu)$, the definition of the escape rate λ_μ (see (2.1)) implies $\mathbb{P}_\mu(|w_n| \geqslant \alpha n) \to 1$ as $n \to \infty$.

Equip $(\Gamma \cup \partial\Gamma)^2$ with the product topology. It follows from the previous paragraph and Lemma 2.12 that for every $\rho \in [0,1]$, and every sequence of probability measures σ_n^ρ with $\mathcal{U}^{\alpha n}(\pi_n^\rho, \sigma_n^\rho) = 0$, the distributions σ_n^ρ converge weakly to ν_{π^ρ} in $(\Gamma \cup \partial\Gamma)^2$ as $n \to \infty$. Lemma 2.11 then implies that for $\rho, \rho' \in [0,1]$,

$$\liminf_{n\to\infty} \left\| \sigma_n^\rho - \sigma_n^{\rho'} \right\|_{\mathrm{TV}} \geqslant \|\nu_{\pi^\rho} - \nu_{\pi^{\rho'}}\|_{\mathrm{TV}} \,.$$

By Corollary 2.9, we have $\|\nu_{\pi^\rho} - \nu_{\pi^{\rho'}}\|_{\mathrm{TV}} \geqslant 1$ if $\rho \neq \rho'$. Noting that $\|\sigma_n^\rho - \sigma_n^{\rho'}\|_{\mathrm{TV}} \leqslant 1$ for all $n \geqslant 0$ (cf. Remark 2.10), we get

$$\mathcal{U}^{\alpha n}\left(\pi_n^\rho, \pi_n^{\rho'}\right) \longrightarrow 1 \quad \text{as} \quad n \longrightarrow \infty,$$

as required. ☐

Proof of Theorem 1.1. — Direct consequence of Theorem 1.2 applied to any non zero homomorphism $\varphi : \Gamma \to \mathbb{R}$. ☐

As in [Bén24], the proof could have been carried in a slightly more general setting, which allows for a finitely generated group Γ that is not hyperbolic, but only acts by isometries on a hyperbolic space. In this context, Theorem 1.2 and its proof extend verbatim as follows.

THEOREM 2.13. — *Let (X, d) be a proper geodesic hyperbolic metric space endowed with a basepoint $o \in X$. Let Γ be a finitely generated group of isometries of (X, d). Let μ be a non-elementary probability measure on Γ with finite exponential moment. Set $\lambda_\mu := \lim_{n\to\infty} \frac{1}{n}\mathbf{E}_{g\sim\mu_n}(d(g.o, o))$.*

Consider a quasi-Lipschitz map $u : X \to \mathbb{R}$ satisfying the equivariance relation $u(\gamma.x) = \varphi(\gamma) + u(x)$ for every $\gamma \in \Gamma, x \in X$ and some fixed homomorphism $\varphi : \Gamma \to \mathbb{R}$. Assume $\mathbf{E}\,\varphi_\star\mu = 0$ and φ is not identically zero on the support of μ.

Then the harmonic measures ν_{π^ρ} on the product of Gromov boundaries $\partial X \times \partial X$ associated to the π^ρ-random walks are mutually singular for all $\rho \in [0,1]$.

In particular, it holds that for every $\rho, \rho' \in [0,1]$ with $\rho \neq \rho'$, every $\alpha \in (0, \lambda_\mu)$, that

$$\mathcal{U}^{\alpha n}\left(\pi_n^\rho * \delta_{(o,o)},\ \pi_n^{\rho'} * \delta_{(o,o)}\right) \longrightarrow 1 \quad \textit{as} \quad n \longrightarrow \infty.$$

Note that the analogous statement to Theorem 1.1 also follows from Theorem 2.13.

BIBLIOGRAPHY

[BB23] Itaï Benjamini and Jérémie Brieussel, *Noise sensitivity of random walks on groups*, ALEA, Lat. Am. J. Probab. Math. Stat. **20** (2023), no. 2, 1139–1164. ↑678, 679

[Bén24] Timothée Bénard, *Winding of geodesic rays chosen by a harmonic measure*, Math. Ann. **390** (2024), no. 1, 1419–1465. ↑680, 681, 682, 683, 688

[Bjö10] Michael Björklund, *Central limit theorems for Gromov hyperbolic groups*, J. Theor. Probab. **23** (2010), no. 3, 871–887. ↑682

[BMSS23] Adrien Boulanger, Pierre Mathieu, Cagri Sert, and Alessandro Sisto, *Large deviations for random walks on Gromov-hyperbolic spaces*, Ann. Sci. Éc. Norm. Supér. (4) **56** (2023), no. 3, 885–944. ↑681

[Bog18] Vladimir I. Bogachev, *Weak convergence of measures*, Mathematical Surveys and Monographs, vol. 234, American Mathematical Society, 2018. ↑687

[BQ16] Yves Benoist and Jean-François Quint, *Central limit theorem on hyperbolic groups*, Izv. Ross. Akad. Nauk, Ser. Mat. **80** (2016), no. 1, 5–26. ↑679, 682

[Cal13] Danny Calegari, *The ergodic theory of hyperbolic groups*, Geometry and topology down under, Contemporary Mathematics, vol. 597, American Mathematical Society, 2013, pp. 15–52. ↑681

[Cho23] Inhyeok Choi, *Central limit theorem and geodesic tracking on hyperbolic spaces and Teichmüller spaces*, Adv. Math. **431** (2023), article no. 109236 (68 pages). ↑682, 685

[Gro87] Mikhael L. Gromov, *Hyperbolic groups*, Essays in group theory (Stephen M. Gersten, ed.), Mathematical Sciences Research Institute Publications, vol. 8, Springer, 1987, pp. 75–263. ↑681, 688

[Kal18] Gil Kalai, *Three puzzles on mathematics, computation, and games*, Proceedings of the International Congress of Mathematicians (ICM 2018). Vol. I. Plenary lectures, World Scientific; Sociedade Brasileira de Matemática, 2018, pp. 551–606. ↑678

[Tan24] Ryokichi Tanaka, *Non-noise sensitivity for word hyperbolic groups*, Ann. Fac. Sci. Toulouse, Math. (6) **33** (2024), no. 5, 1487–1510. ↑678, 680

[Tan25] ———, *Noise sensitivity on affine Weyl groups*, Comb. Probab. Comput. **34** (2025), no. 5, 754–779. ↑678

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