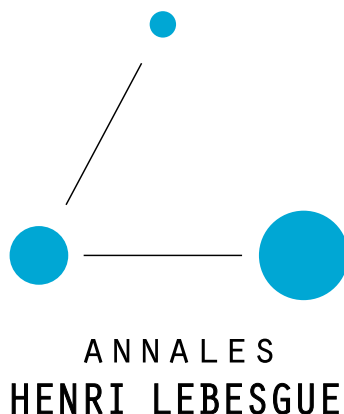




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ALICE CONTAT

BLOW-UP RATE OF SOLUTION TO GENERALISED BLASIOUS EQUATION

TAUX D'EXPLOSION DE LA SOLUTION D'UNE ÉQUATION DE BLASIOUS GÉNÉRALISÉE

ABSTRACT. — We identify the blow-up rate of a solution to a generalised Blasius equation, that we came across while studying a probabilistic model of “Poissonian burning” in Euclidean space. Our proof involves the study of the long-time behaviour of solutions to a Lotka–Volterra system.

RÉSUMÉ. — Nous identifions le taux d'explosion de la solution d'une équation de Blasius généralisée, que nous avons rencontrée en étudiant un modèle probabiliste de “burning poisson-nien” dans l'espace euclidien. Notre preuve fait intervenir l'étude du comportement en temps long des solutions d'un système de Lotka–Volterra.

Keywords: Blasius equation, blow-up, Poissonian burning.

2020 Mathematics Subject Classification: 34E05, 35B44, 60C05, 60D05.

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1. Introduction and main results

In this paper, we consider the equation

$$(1.1) \quad \begin{cases} y^{(d+1)}(t) = y(t) \cdot y^{(d)}(t), & t \in \mathbb{R}_+, \\ y(0), \dots, y^{(d-1)}(0) = 0, & \text{and } y^{(d)}(0) = 1, \end{cases}$$

where $d \in \mathbb{N}^* = \{1, 2, \dots\}$ is a fixed parameter. By the Cauchy–Lipschitz theorem, the Cauchy problem (1.1) admits a unique maximal solution $y : [0, T[\rightarrow \mathbb{R}$, where $T > 0$, and we have either $T = \infty$ (the solution is global), or $T < \infty$ and $\max_{i \in \llbracket 0, d \rrbracket} |y^{(i)}(t)| \rightarrow \infty$ as $t \rightarrow T^-$ (the solution blows up in finite time). In fact, for $d = 1$, the maximal solution of (1.1) is given explicitly by

$$y : t \in \left[0, \frac{\pi}{\sqrt{2}}\right[\mapsto \sqrt{2} \cdot \tan\left(\frac{t}{\sqrt{2}}\right).$$

In particular, this solution blows up in finite time $T = \pi/\sqrt{2}$, and its blow-up rate is given by

$$y(t) \sim \frac{2}{T-t} \quad \text{and} \quad y'(t) \sim \frac{2}{(T-t)^2} \quad \text{as } t \rightarrow T^-.$$

Here, we investigate the blow-up of the maximal solution $y : [0, T[\rightarrow \mathbb{R}$ to (1.1), for any $d \in \mathbb{N}^*$.

Equation (1.1) comes up in the work of Kuba and Panholzer [KP16], in counting increasingly $(d+1)$ -labelled unordered trees. As they notice, the maximal solution $y : [0, T[\rightarrow \mathbb{R}$ of (1.1) for $d = 2$ can be related to the solution of the Blasius equation from boundary layer theory:

$$(1.2) \quad \begin{cases} z'''(t) + z(t) \cdot z''(t) = 0, & t \in \mathbb{R}_+, \\ z(0), z'(0) = 0, & \text{and } \lim_{t \rightarrow \infty} z'(t) = 1. \end{cases}$$

The Blasius equation has been treated by several authors in the mathematics literature, including Weyl [Wey42], who first rigorously proved the existence of a solution to (1.2) by considering the Cauchy problem

$$(1.3) \quad \begin{cases} z'''(t) + z(t) \cdot z''(t) = 0, & t \in \mathbb{R}_+, \\ z(0), z'(0) = 0, & \text{and } z''(0) = 1. \end{cases}$$

In particular, Weyl gave bounds on the radius of convergence of the power series expansion at $t = 0$ of the maximal solution to (1.3), which in fact corresponds to the explosion time T of the maximal solution y of (1.1). Interestingly, Weyl also treated the following generalisation of (1.3), which is almost (1.1):

$$\begin{cases} z^{(d+1)}(t) + z(t) \cdot z^{(d)}(t) = 0, & t \in \mathbb{R}_+, \\ z(0), \dots, z^{(d-1)}(0) = 0, & \text{and } z^{(d)}(0) = 1, \end{cases}$$

where $d \in \mathbb{N}^*$ is a parameter.

1.1. Blow-up rate of y

Our main results are stated in Theorem 1.1 below. To the best of our knowledge, except for the case $d = 1$ which is explicitly solvable, such results were only available for $d = 2$, and the corresponding proofs do not straightforwardly extend to $d \geq 3$. They were obtained originally by Coppel [Cop60], and revisited more recently by Ishimura and Matsui [IM03]. For two functions $f, g : [0, T[\rightarrow \mathbb{R}$, we write $f(t) \asymp g(t)$ as $t \rightarrow T^-$ if there exists two constants $c < C \in \mathbb{R}_+^*$ and $t_0 \in [0, T[$ such that $c \cdot g(t) \leq f(t) \leq C \cdot g(t)$ for all $t \in [t_0, T[$.

THEOREM 1.1. — *Fix $d \in \mathbb{N}^*$, and let $y : [0, T[\rightarrow \mathbb{R}$ be the maximal solution of (1.1). This solution blows up in finite time: we have $T < \infty$, and for each $i \in \llbracket 0, d \rrbracket$, we have $y^{(i)}(t) \rightarrow \infty$ as $t \rightarrow T^-$. Moreover, we identify its blow-up rate in the following sense:*

- For $d \in \llbracket 1, 10 \rrbracket$, the following holds: for each $i \in \llbracket 0, d \rrbracket$, we have

$$(1.4) \quad y^{(i)}(t) \sim (d+1) \cdot \frac{i!}{(T-t)^{i+1}} \quad \text{as } t \rightarrow T^-.$$

- Regardless of the value of d , the following holds: for each $i \in \llbracket 0, d \rrbracket$, we have

$$(1.5) \quad y^{(i)}(t) \asymp \frac{1}{(T-t)^{i+1}} \quad \text{as } t \rightarrow T^-.$$

The estimates of Theorem 1.1 are of importance to us because of their consequence for a probabilistic model of “Poissonian burning” in Euclidean space, as we explain below. Contrary to (1.5), we do not know whether (1.4) holds for all values of d ; we will comment more on this once we give an overview of the proof of Theorem 1.1.

1.2. Probabilistic motivation

We came across this problem while studying the following probabilistic model of “Poissonian burning” in Euclidean space. Consider a Poisson process Π with intensity $dx \otimes f(t) dt$ on $\mathbb{R}^d \times [0, T[$, where $f : [0, T[\rightarrow \mathbb{R}_+$ is a function to be adjusted later. For each atom (x, t) of Π , think of $t \in [0, T[$ as the time at which the point $x \in \mathbb{R}^d$ is set on fire, and picture the fire as then propagating at speed $1/2$ in $\mathbb{R}^d$, see Figure 1.1.

Motivated by a discrete version of the burning model on the Euclidean lattice $\mathbb{Z}^d$, we equip $\mathbb{R}^d$ with the 1-norm $\|\cdot\|_1 : x = (x_1, \dots, x_d) \in \mathbb{R}^d \mapsto |x_1| + \dots + |x_d|$. The set of points that are burned by time $t \in [0, T[$ is

$$\mathcal{B}(t) = \bigcup_{\substack{(x,s) \in \Pi \\ s \leq t}} \bar{B}_{\|\cdot\|_1} \left(x, \frac{t-s}{2} \right),$$

where $\bar{B}_{\|\cdot\|_1}(x, r) = \{z \in \mathbb{R}^d : \|x - z\|_1 \leq r\}$ denotes the 1-norm ball with centre $x \in \mathbb{R}^d$ and radius $r > 0$. Now, we want to choose the function f in the intensity measure of Π so that at each time $t \in [0, T[$, the local intensity $f(t)$ of the process Π on $\mathbb{R}^d$ compensates exactly the mean density of points that are not burned by time t .

In other words, we want the mean density of points that Π puts in the unburned region $\mathbb{R}^d \setminus \mathcal{B}(t)$ to be constantly equal to 1. To formalise this, let us fix $z \in \mathbb{R}^d$, and calculate the probability that z is not burned by time t : we have

$$\begin{aligned} \mathbb{P}(z \notin \mathcal{B}(t)) &= \mathbb{P}\left(\Pi\left\{(x, s) \in \mathbb{R}^d \times [0, t] : \|x - z\|_1 \leq \frac{t-s}{2}\right\} = 0\right) \\ &= \exp\left(-\int_{\mathbb{R}^d} \int_0^t \mathbf{1}\left(\|x - z\|_1 \leq \frac{t-s}{2}\right) \cdot f(s) \, ds \, dx\right) \\ &= \exp\left(-\int_0^t \frac{2^d}{d!} \cdot \left(\frac{t-s}{2}\right)^d \cdot f(s) \, ds\right) \\ &= \exp\left(-\int_0^t \frac{(t-s)^d}{d!} \cdot f(s) \, ds\right). \end{aligned}$$

This probability does not depend on z , and corresponds to the mean density of points in $\mathbb{R}^d$ that are not burned by time t . Thus, we want to choose f so that

$$f(t) \cdot \exp\left(-\int_0^t \frac{(t-s)^d}{d!} \cdot f(s) \, ds\right) = 1 \quad \text{for all } t \in [0, T[.$$

This certainly holds for $f = y^{(d)}$, if $y : [0, T[\rightarrow \mathbb{R}$ is a solution of (1.1). Indeed, by integrating $y^{(d+1)} = y \cdot y^{(d)}$ as a first order differential equation in $y^{(d)}$, we get that

$$(1.6) \quad y^{(d)}(t) = \exp\left(\int_0^t y(s) \, ds\right) = \exp\left(\int_0^t \frac{(t-s)^d}{d!} \cdot y^{(d)}(s) \, ds\right) \quad \text{for all } t \in [0, T[,$$

where the last equality follows by successive integration by parts.

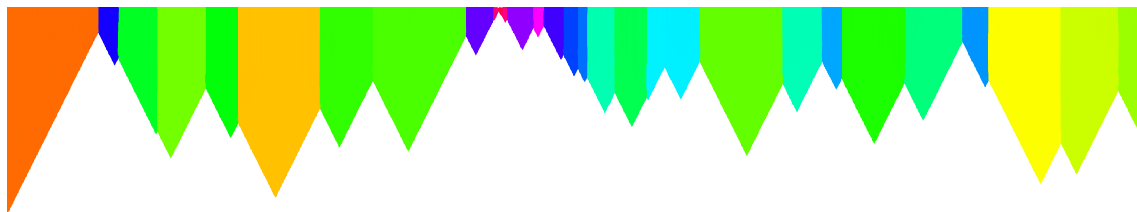


Figure 1.1. Simulation of the model of Poissonian burning for $d = 1$, in the window $[-5, 5]$. Space is depicted horizontally, whereas time evolves vertically. At each time, so for each horizontal layer, we color the points that are burned with the same color as the first point of the process that set them on fire.

Ultimately, we set the intensity measure of Π to be $dx \otimes y^{(d)}(t) \, dt$ on $\mathbb{R}^d \times [0, T[$, where $y : [0, T[\rightarrow \mathbb{R}$ is the maximal solution of (1.1). For this continuous model of Poissonian burning, the blow-up rate of y is relevant to determine whether the whole of $\mathbb{R}^d$ is burned eventually by the process. More precisely, Theorem 1.1 has the following corollary (see Section 2 for more details).

COROLLARY 1.2. — *Almost surely, the whole of $\mathbb{R}^d$ is burned eventually by the process, i.e, we have*

$$\bigcup_{t \in [0, T[} \mathcal{B}(t) = \mathbb{R}^d.$$

Finally, let us mention that we plan on interpreting this continuous model of Poissonian burning in $\mathbb{R}^d$ as the scaling limit of a discrete (in space and time) model of random burning in $\mathbb{Z}^d$ in a forthcoming paper [BC25].

Strategy of the proof of Theorem 1.1 and organisation of the paper

- In Section 3, we first prove that the maximal solution $y : [0, T[\rightarrow \mathbb{R}$ of (1.1) blows up in finite time, and provide rough bounds on y and its derivatives near the explosion time T , relying on the absolute monotonicity of y .
- In Section 4, we consider some functions of y and its derivatives that we know to remain bounded near the explosion time T , and observe that they satisfy a system of differential equations which, after a time change, is simplified into a Lotka–Volterra system.
- In Section 5, we study the long-time behaviour of solutions to this Lotka–Volterra system. For $d \in \llbracket 1, 10 \rrbracket$, a Lyapunov function argument shows that each solution converges to the bulk stationary point, which implies (1.4). Regardless of the value of d , the so-called method of average Lyapunov functions shows that the coordinates of each solution are bounded away from zero, which implies (1.5).

Phase transition in d

As we will see in Section 5, implementing the Lyapunov function argument mentioned above requires finding positive real numbers $\lambda_1, \dots, \lambda_d$ for which the symmetric matrix

$$(1.7) \quad \begin{bmatrix} 4\lambda_1 & \lambda_2 - \lambda_1 & \lambda_3 & \lambda_4 & \cdots & \lambda_d \\ \lambda_2 - \lambda_1 & 2\lambda_2 & -\lambda_2 & 0 & \cdots & 0 \\ \lambda_3 & -\lambda_2 & 2\lambda_3 & -\lambda_3 & \ddots & \vdots \\ \lambda_4 & 0 & -\lambda_3 & 2\lambda_4 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & -\lambda_{d-1} \\ \lambda_d & 0 & \cdots & 0 & -\lambda_{d-1} & 2\lambda_d \end{bmatrix}$$

is positive-definite. At first, we found solutions $\lambda_1, \dots, \lambda_d > 0$ by hand for $d \in \llbracket 1, 9 \rrbracket$, and naively thought that this would be possible for all values of d . To our surprise, it turns out that such solutions $\lambda_1, \dots, \lambda_d > 0$ exist for and only for $d \in \llbracket 1, 10 \rrbracket$, as can be proved using the Farkas lemma [Cri24]. Now, while this Lyapunov function argument fails for $d \geq 11$, it could still be the case that all solutions of the Lotka–Volterra system converge to the bulk stationary point, which would imply (1.4). However, numerical simulations suggest that for $d = 11$, generic solutions of the Lotka–Volterra system oscillate around the bulk stationary point without converging (see Figure 1.2 and Figure 1.3). Again, even without having the convergence of all solutions of the Lotka–Volterra system to the bulk stationary point, it could still be the case that (1.4) holds (for instance, if the precise solution corresponding to

the time-changed function of y and its derivatives converges to the bulk stationary point). It would be interesting to know what happens exactly, but this is beyond the scope of this paper.

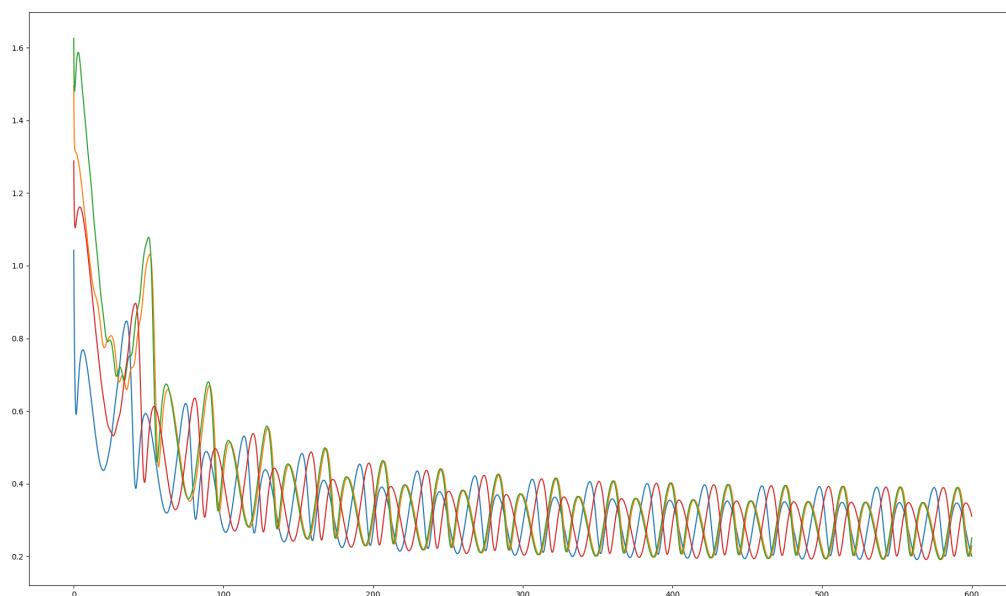


Figure 1.2. Euclidean distance to the bulk stationary point for four generic solutions of the Lotka–Volterra system (5.1) for $d = 11$.

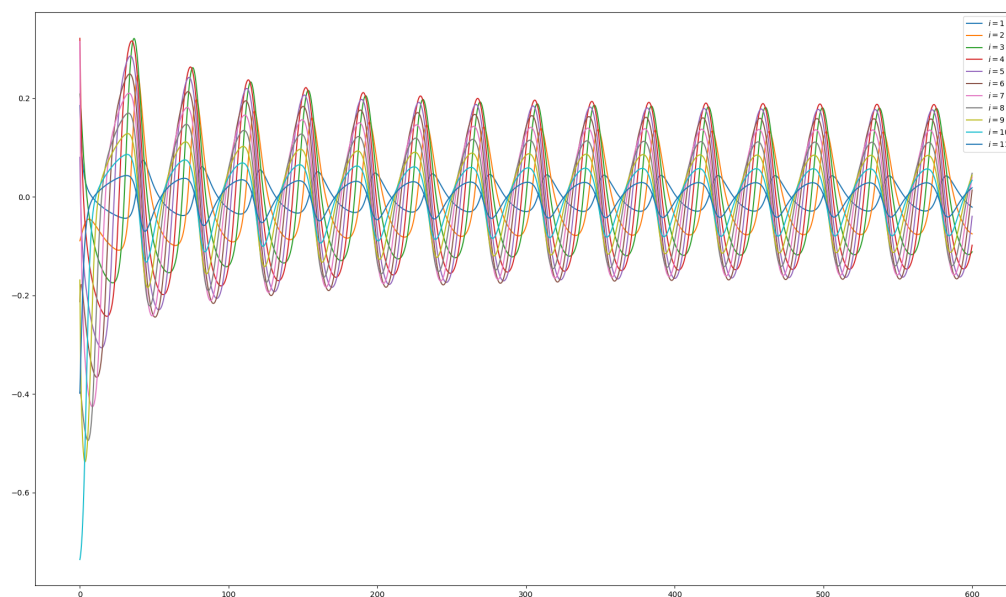


Figure 1.3. Oscillation around the bulk stationary point for some generic solution of the Lotka–Volterra system (5.1) for $d = 11$, coordinate per coordinate.

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2. Poissonian burning

We start by explaining how the results of Theorem 1.1 allow us to study the probabilistic model of Poissonian burning introduced above. Recall that Π is a Poisson process with intensity $dx \otimes y^{(d)}(t) dt$ on $\mathbb{R}^d \times [0, T[$, where $y : [0, T[\rightarrow \mathbb{R}$ is the maximal solution of (1.1). We think of each atom (x, t) of Π as a source of fire appearing at position $x \in \mathbb{R}^d$ at time $t \in [0, T[$, the fire then propagating at speed $1/2$. Equipping $\mathbb{R}^d$ with the 1-norm $\|\cdot\|_1$, we let

$$\mathcal{B}(t) = \bigcup_{\substack{(x,s) \in \Pi \\ s \leq t}} \bar{B}_{\|\cdot\|_1}\left(x, \frac{t-s}{2}\right)$$

be the set of points in $\mathbb{R}^d$ that are burned by time $t \in [0, T[$, and we let

$$\mathcal{B} = \bigcup_{t \in [0, T[} \mathcal{B}(t) = \bigcup_{(x,t) \in \Pi} B_{\|\cdot\|_1}\left(x, \frac{T-t}{2}\right)$$

be the set of points that are burned eventually, where

$$B_{\|\cdot\|_1}(x, r) = \{z \in \mathbb{R}^d : \|x - z\|_1 < r\}$$

denotes the 1-norm open ball with centre $x \in \mathbb{R}^d$ and radius $r > 0$. Thanks to Theorem 1.1, we are able to show that almost surely, the whole of $\mathbb{R}^d$ is burned eventually by the process, i.e. we have $\mathcal{B} = \mathbb{R}^d$ (Corollary 1.2).

Proof of Corollary 1.2. — The argument is similar to [BE12, Proposition 2.5], where Biermé and Estrade give a sufficient condition for “high frequency coverage” in the Mandelbrot–Shepp covering problem. For each $\varepsilon > 0$, notice that for all $z \in \mathbb{R}^d$, we have

$$\begin{aligned} \mathbb{P}\left(B_{\|\cdot\|_1}(z, \varepsilon) \text{ is not contained in } \mathcal{B}\right) &\leq \mathbb{P}(z \notin \mathcal{B}(T - \varepsilon)) \\ &= \exp\left(-\int_0^{T-\varepsilon} \frac{(T - \varepsilon - s)^d}{d!} \cdot y^{(d)}(s) ds\right) \\ &= \frac{1}{y^{(d)}(T - \varepsilon)}, \end{aligned}$$

where the last equality follows from (1.6). Now, by (1.5), we have

$$(2.1) \quad \frac{1}{y^{(d)}(T - \varepsilon)} \asymp \varepsilon^{d+1} \quad \text{as } \varepsilon \rightarrow 0.$$

We deduce that $\mathcal{B} = \mathbb{R}^d$ a.s. by the following standard argument. Fix $R > 0$, and let us show that $\bar{B}_{\|\cdot\|_1}(0, R) \setminus \mathcal{B} = \emptyset$ a.s. For each $\varepsilon > 0$, let $(B_{\|\cdot\|_1}(z, \varepsilon); z \in Z_\varepsilon)$ be a covering

of $\bar{B}_{\|\cdot\|_1}(0, R)$ by balls of radius ε , with centres $z \in \bar{B}_{\|\cdot\|_1}(0, R)$ at least ε apart from each other for the 1-norm. Since the balls $(B_{\|\cdot\|_1}(z, \varepsilon/2); z \in Z_\varepsilon)$ are disjoint and included in $B_{\|\cdot\|_1}(0, R + \varepsilon)$, a measure argument entails that $\#Z_\varepsilon \leq (2R/\varepsilon + 1)^d$. Now, notice that we have the inclusion of events

$$\left(\bar{B}_{\|\cdot\|_1}(0, R) \setminus \mathcal{B} \neq \emptyset\right) \subset \bigcup_{z \in Z_\varepsilon} \left(B_{\|\cdot\|_1}(z, \varepsilon) \text{ is not contained in } \mathcal{B}\right).$$

Thus, by the union bound, we get

$$\begin{aligned} \mathbb{P}\left(\bar{B}_{\|\cdot\|_1}(0, R) \setminus \mathcal{B} \neq \emptyset\right) &\leq \sum_{z \in Z_\varepsilon} \mathbb{P}\left(B_{\|\cdot\|_1}(z, \varepsilon) \text{ is not contained in } \mathcal{B}\right) \\ &\leq \left(\frac{2R}{\varepsilon} + 1\right)^d \cdot \frac{1}{y^{(d)}(T - \varepsilon)}. \end{aligned}$$

By (2.1), the above right hand side vanishes as $\varepsilon \rightarrow 0$, which completes the proof of Corollary 1.2. $\square$

3. Preliminary considerations

* * *

For the rest of the paper, we let $y : [0, T[\rightarrow \mathbb{R}$ be the maximal solution of (1.1).

* * *

We now move on to the proof of Theorem 1.1. In this section, we first establish some basic properties of y .

3.1. Absolute monotonicity

We start with the following “absolute monotonicity” property.

PROPOSITION 3.1. — *We have $y(t), \dots, y^{(d-1)}(t) \geq 0$ and $y^{(d)}(t) \geq 1$ for all $t \in [0, T[$. In particular, for each $i \in \llbracket 0, d \rrbracket$, the function $y^{(i)}$ is strictly increasing on $[0, T[$, and we have $y^{(i)}(t) > 0$ for all $t \in]0, T[$.*

Proof. — Consider the set

$$O = \left\{t \in [0, T[: y(s), \dots, y^{(d-1)}(s) \geq 0 \text{ and } y^{(d)}(s) \geq 1 \text{ for all } s \in [0, t]\right\}.$$

For the first assertion, it suffices to prove that $O = [0, T[$. First, observe that O is a closed subset of $[0, T[$ that contains 0. Moreover, we claim that O is also an open subset of $[0, T[$. By the connectedness of $[0, T[$, this entails that $O = [0, T[$, as desired. To prove the claim, fix $t \in O$. Since $y^{(d)}(t) \geq 1$, we may choose $\varepsilon > 0$ small enough so that $y^{(d)}(s) \geq 0$ for all $s \in [t, t + \varepsilon]$. Now, let us show that $[0, t + \varepsilon] \subset O$. By the definition of O , we automatically have $[0, t] \subset O$, so let us prove that $[t, t + \varepsilon] \subset O$. Since $y^{(d)}(s) \geq 0$ for all $s \in [t, t + \varepsilon]$, the function $y^{(d-1)}$ is non-decreasing on $[t, t + \varepsilon]$, hence $y^{(d-1)}(s) \geq y^{(d-1)}(t) \geq 0$ for all $s \in [t, t + \varepsilon]$. In turn, the function $y^{(d-2)}$ is thus

non-decreasing on $[t, t + \varepsilon]$, hence $y^{(d-2)}(s) \geq y^{(d-2)}(t) \geq 0$ for all $s \in [t, t + \varepsilon]$, and so on. The argument iterates up to $y(s) \geq y(t) \geq 0$ for all $s \in [t, t + \varepsilon]$. Finally, since $y^{(d+1)}(s) = y(s) \cdot y^{(d)}(s) \geq 0$ for all $s \in [t, t + \varepsilon]$, the function $y^{(d)}$ is non-decreasing on $[t, t + \varepsilon]$, hence $y^{(d)}(s) \geq y^{(d)}(t) \geq 1$ for all $s \in [t, t + \varepsilon]$. This shows that $[t, t + \varepsilon] \subset O$, as we wanted to prove.

Now, let us prove the second assertion. By the previous point, we have $y^{(d)}(t) \geq 1$ for all $t \in [0, T[$, hence $y^{(d-1)}$ is strictly increasing on $[0, T[$. Since $y^{(d-1)}(0) = 0$, we deduce that $y^{(d-1)}(t) > 0$ for all $t \in]0, T[$. In turn, the function $y^{(d-2)}$ is thus strictly increasing on $[0, T[$, hence $y^{(d-2)}(t) > y^{(d-2)}(0) = 0$ for all $t \in]0, T[$, and so on. The argument iterates up to $y(t) > y(0) = 0$ for all $t \in]0, T[$. Finally, since $y^{(d+1)}(t) = y(t) \cdot y^{(d)}(t) > 0$ for all $t \in]0, T[$, we get that $y^{(d)}$ is strictly increasing on $[0, T[$, which completes the proof of Proposition 3.1. $\square$

3.2. Finite time blow-up

We now focus on the blow-up of y . Using the previous absolute monotonicity result, we start by proving that y blows up in finite time. Incidentally, this argument also provides the upper bound in (1.5). The lower bound, which is the one that we really need in view of Corollary 1.2, requires more work.

PROPOSITION 3.2. — *The solution $y : [0, T[\rightarrow \mathbb{R}$ blows up in finite time: we have $T < \infty$, and for each $i \in \llbracket 0, d \rrbracket$, we have $y^{(i)}(t) \rightarrow \infty$ as $t \rightarrow T^-$. Furthermore, for each $i \in \llbracket 0, d \rrbracket$, there exists a constant $C_i \in \mathbb{R}_+^*$ such that*

$$y^{(i)}(t) \leq \frac{C_i}{(T-t)^{i+1}} \quad \text{for all } t \in [0, T[.$$

Proof. — First, observe that $z : [0, \tau[\rightarrow \mathbb{R}$ is a (maximal) solution of (1.1) if and only if the function $x : t \in [0, \tau[\mapsto \int_0^t z(s) \, ds$ is a (maximal) solution of the Cauchy problem

$$(3.1) \quad \begin{cases} x^{(d+1)}(t) = e^{x(t)}, & t \in \mathbb{R}_+, \\ x(0), \dots, x^{(d)}(0) = 0. \end{cases}$$

Indeed, if $z : [0, \tau[\rightarrow \mathbb{R}$ is a solution of (1.1), then by integrating $z^{(d+1)} = z \cdot z^{(d)}$ as a first order differential equation in $z^{(d)}$, we get that

$$z^{(d)}(t) = \exp\left(\int_0^t z(s) \, ds\right) \quad \text{for all } t \in [0, \tau[,$$

hence the function $x : t \in [0, \tau[\mapsto \int_0^t z(s) \, ds$ is a solution of (3.1). Conversely, if $x : [0, \tau[\rightarrow \mathbb{R}$ is a solution of (3.1), then its derivative $z = x'$ satisfies $z^{(d)} = e^x$, hence $z^{(d+1)} = x' \cdot e^x = z \cdot z^{(d)}$.

Since y is the maximal solution of (1.1), it follows that the function

$$x : t \in [0, T[\mapsto \int_0^t y(s) \, ds$$

is the maximal solution of (3.1). Note that by virtue of Proposition 3.1, the function x is strictly increasing on $[0, T[$. Now, we claim that regardless of the finiteness of T ,

the function x is unbounded. Indeed:

- if $T = \infty$, then by Proposition 3.1, we have $y(t) \geq y(1) > 0$ for all $t \in [1, \infty[$, hence

$$x(t) = \int_0^t y(s) \, ds \geq \int_1^t y(1) \, ds = (t-1) \cdot y(1) \xrightarrow[t \rightarrow \infty]{} \infty.$$

- If $T < \infty$, then by a standard corollary of the Cauchy–Lipschitz theorem, the maximality of x as a solution of (3.1) entails that $\max_{i \in \llbracket 0, d \rrbracket} |x^{(i)}(t)| \rightarrow \infty$ as $t \rightarrow T^-$. Now, if x were bounded, then $x^{(d+1)} = e^x$ would be bounded, and in turn $x^{(d)}, \dots, x'$, which can be expressed as multiple integrals of $x^{(d+1)}$ on the bounded interval $[0, T[$, would be bounded. Thus, since

$$\max_{i \in \llbracket 0, d \rrbracket} |x^{(i)}(t)| \rightarrow \infty \quad \text{as } t \rightarrow T^-,$$

the function x must be unbounded.

Therefore, regardless of the finiteness of T , the function x is an increasing bijection from $[0, T[$ to $\mathbb{R}_+$. Now, consider the sequence $0 = t_0 < t_1 < \dots \in [0, T[$ defined by $x(t_n) = n$ for all $n \in \mathbb{N}$. Since $T = \sum_{n \geq 0} (t_{n+1} - t_n)$, it suffices to provide a summable bound on the $(t_{n+1} - t_n)$'s to prove the finiteness of T . To this end, we argue as follows. For each $n \in \mathbb{N}$, we have $x^{(d+1)}(t) = e^{x(t)} \geq e^n$ for all $t \in [t_n, t_{n+1}]$, hence

$$x^{(d)}(t) \geq (t - t_n) \cdot e^n \quad \text{for all } t \in [t_n, t_{n+1}].$$

We used here that $x^{(d)}(t_n) = y^{(d-1)}(t_n) \geq 0$, by Proposition 3.1. Integrating the above inequality between t_n and t and iterating the argument, we obtain that

$$x^{(d-1)}(t) \geq \frac{(t - t_n)^2}{2} \cdot e^n \quad \text{for all } t \in [t_n, t_{n+1}],$$

and so on up to

$$x'(t) \geq \frac{(t - t_n)^d}{d!} \cdot e^n \quad \text{for all } t \in [t_n, t_{n+1}].$$

Integrating one last time from t_n to t_{n+1} , we deduce that

$$1 = x(t_{n+1}) - x(t_n) \geq \frac{(t_{n+1} - t_n)^{d+1}}{(d+1)!} \cdot e^n,$$

hence

$$(3.2) \quad t_{n+1} - t_n \leq \left((d+1)! \cdot e^{-n} \right)^{1/(d+1)} =: C \cdot e^{-c \cdot n},$$

with $c = 1/(d+1)$ and $C = ((d+1)!)^{1/(d+1)}$. In particular, we obtain that

$$T = \sum_{n \geq 0} (t_{n+1} - t_n) < \infty,$$

as desired.

We now have everything at hand to conclude the proof. First, the finiteness of T entails that for each $i \in \llbracket 0, d \rrbracket$, we have $y^{(i)}(t) \rightarrow \infty$ as $t \rightarrow T^-$. Indeed, the function $y^{(i)}$ is increasing on $[0, T[$ by Proposition 3.1, and must be unbounded; for otherwise x , which can be expressed as a multiple integral of $y^{(i)}$ on the bounded interval $[0, T[$,

would be bounded. Finally, let us show that for each $i \in \llbracket 0, d \rrbracket$, there exists a constant $C_i \in \mathbb{R}_+^*$ such that

$$y^{(i)}(t) \leq \frac{C_i}{(T-t)^{i+1}} \quad \text{for all } t \in [0, T[.$$

It follows from (3.2) that for all $n \in \mathbb{N}$, we have

$$T - t_n = \sum_{k \geq n} (t_{k+1} - t_k) \leq \sum_{k \geq n} C \cdot e^{-c \cdot k} = C \cdot \frac{e^{-c \cdot n}}{1 - e^{-c}},$$

i.e.,

$$e^{c \cdot (n+1)} \leq \frac{C e^c}{1 - e^{-c}} \cdot \frac{1}{T - t_n} =: \frac{C'}{T - t_n}.$$

We deduce that for all $t \in [t_n, t_{n+1}]$, we have

$$e^{c \cdot x(t)} \leq e^{c \cdot (n+1)} \leq \frac{C'}{T - t_n} \leq \frac{C'}{T - t},$$

and since this holds for each $n \in \mathbb{N}$, we obtain that

$$e^{c \cdot x(t)} \leq \frac{C'}{T - t} \quad \text{for all } t \in [0, T[.$$

Now, recalling that $c = 1/(d+1)$, we get that

$$y^{(d)}(t) = x^{(d+1)}(t) = e^{x(t)} \leq \left(\frac{C'}{T-t} \right)^{d+1} =: \frac{C_d}{(T-t)^{d+1}} \quad \text{for all } t \in [0, T[,$$

with $C_d = (C')^{d+1}$. Then, by induction on i , using that for each $i \in \{d, \dots, 1\}$, we have

$$y^{(i-1)}(t) = \int_0^t y^{(i)}(s) \, ds \quad \text{and} \quad \int_0^t \frac{ds}{(T-s)^{i+1}} = \left[\frac{1/i}{(T-s)^i} \right]_0^t \leq \frac{1/i}{(T-t)^i}$$

for all $t \in [0, T[$, we get that for each $i \in \llbracket 0, d \rrbracket$, we have

$$y^{(i)}(t) \leq \frac{C_i}{(T-t)^{i+1}} \quad \text{for all } t \in [0, T[,$$

with

$$C_i = \frac{C_d}{d \cdot \dots \cdot (i+1)}.$$

This concludes the proof of Proposition 3.2. □

3.3. Rough bounds

To prepare for the core of the proof, we conclude this section by establishing the following rough bounds on y and its derivatives near the explosion time T .

PROPOSITION 3.3. — We have

$$\liminf_{t \rightarrow T^-} (T - t) \cdot y(t) > 0,$$

and for each $i \in \llbracket 1, d \rrbracket$, we have

$$\limsup_{t \rightarrow T^-} \frac{y^{(i)}(t)}{y(t) \cdot y^{(i-1)}(t)} < \infty.$$

Proof. — First, since y is a solution of (1.1), we have

$$y^{(d)}(t) = 1 + \int_0^t y(s) \cdot y^{(d)}(s) \, ds \quad \text{for all } t \in [0, T[.$$

Then, by the absolute monotonicity of y (Proposition 3.1), we get that

$$(3.3) \quad y^{(d)}(t) \leq 1 + \int_0^t y(s) \cdot y^{(d)}(s) \, ds = 1 + y(t) \cdot y^{(d-1)}(t) \quad \text{for all } t \in [0, T[.$$

Since $y(t) \cdot y^{(d-1)}(t) \rightarrow \infty$ as $t \rightarrow T^-$ by Proposition 3.2, it follows that

$$\limsup_{t \rightarrow T^-} \frac{y^{(d)}(t)}{y(t) \cdot y^{(d-1)}(t)} \leq 1,$$

as desired. Next, integrating (3.3) and iterating the argument, we get that for all $t \in [0, T[$, we have

$$y^{(d-1)}(t) \leq t + \int_0^t y(s) \cdot y^{(d-1)}(s) \, ds \leq T + \int_0^t y(s) \cdot y^{(d-1)}(s) \, ds = T + y(t) \cdot y^{(d-2)}(t).$$

Since $y(t) \cdot y^{(d-2)}(t) \rightarrow \infty$ as $t \rightarrow T^-$ by Proposition 3.2, it follows that

$$\limsup_{t \rightarrow T^-} \frac{y^{(d-1)}(t)}{y(t) \cdot y^{(d-2)}(t)} \leq 1,$$

as desired. This argument can be iterated up to obtaining the bound

$$(3.4) \quad y'(t) \leq T^{d-2} \cdot t + \int_0^t y(s) \cdot y'(s) \, ds \leq T^{d-1} + \frac{y(t)^2}{2} \quad \text{for all } t \in [0, T[.$$

Since $y(t) \rightarrow \infty$ as $t \rightarrow T^-$ by Proposition 3.2, it follows that

$$\limsup_{t \rightarrow T^-} \frac{y'(t)}{y(t)^2} \leq \frac{1}{2},$$

as desired. Finally, let us prove that $\liminf_{t \rightarrow T^-} (T - t) \cdot y(t) > 0$. To ease notation, we set $\lambda = \sqrt{2T^{d-1}}$. Equation (3.4) can be rearranged as follows:

$$\frac{y'(t)/\lambda}{1 + (y(t)/\lambda)^2} \leq \frac{\lambda}{2} \quad \text{for all } t \in [0, T[.$$

Now, in the left-hand side, we recognise the derivative of $t \in [0, T[\mapsto \arctan(y(t)/\lambda)$. Then, since $y(t) \rightarrow \infty$ as $t \rightarrow T^-$ by Proposition 3.2, integrating the above equation from t to T^- yields

$$\frac{\pi}{2} - \arctan\left(\frac{y(t)}{\lambda}\right) \leq \frac{\lambda}{2} \cdot (T - t) \quad \text{for all } t \in [0, T[.$$

Finally, since $\pi/2 - \arctan(x) = \arctan(1/x) \sim 1/x$ as $x \rightarrow \infty$, we deduce that

$$\frac{1 + o(1)}{y(t)/\lambda} \leq \frac{\lambda}{2} \cdot (T - t) \quad \text{as } t \rightarrow T^-.$$

This can be rearranged as

$$(T - t) \cdot y(t) \geq 2 + o(1) \quad \text{as } t \rightarrow T^-,$$

which completes the proof of Proposition 3.3. $\square$

4. Reduction to a Lotka–Volterra system

In this section, we consider some functions of y and its derivatives that we know to remain bounded near the explosion time T thanks to Proposition 3.3, and observe that they satisfy a system of differential equations which, after a time change, is simplified into a Lotka–Volterra system. For the sake of readability, we split this reduction argument into three straightforward claims (see Claims 4.1, 4.2 and 4.3 below).

We then conclude this section by giving the proof of Theorem 1.1, taking for granted the statement of Proposition 4.4 below about the long-time behaviour of solutions to the Lotka–Volterra system mentioned above. The latter result will be proved in the next section.

Time changes

Guided by the rough bounds obtained in Proposition 3.3, we set

$$u_0(t) = \frac{1}{(T - t) \cdot y(t)} \quad \text{for all } t \in]0, T[,$$

and for each $i \in \llbracket 1, d \rrbracket$, we let

$$u_i(t) = \frac{y^{(i)}(t)}{y(t) \cdot y^{(i-1)}(t)} \quad \text{for all } t \in]0, T[.$$

Note that by the absolute monotonicity of y (Proposition 3.1), these functions are well-defined and strictly positive.

CLAIM 4.1. — *The functions $u_0, u_1, \dots, u_d$ satisfy the following system of differential equations: for all $t \in]0, T[$, we have*

$$(4.1) \quad \begin{cases} u'_0(t) = \frac{1}{T-t} \cdot (u_0(t) - u_1(t)), \\ u'_i(t) = \frac{1}{T-t} \cdot \frac{u_i(t)(u_{i+1}(t) - u_1(t) - u_i(t))}{u_0(t)}, & i \in \llbracket 1, d-1 \rrbracket, \\ u'_d(t) = \frac{1}{T-t} \cdot \frac{u_d(t)(1 - u_1(t) - u_d(t))}{u_0(t)}. \end{cases}$$

We spare the reader the details of these calculations, which are straightforward and use the fact that y is a solution of (1.1) only for the last equation. We then let $t_0 = (1 - 1/e)T \in]0, T[$, and we introduce the time change

$$\begin{aligned}\phi : \mathbb{R}_+ &\longrightarrow [t_0, T[\\ t &\longmapsto T \cdot (1 - e^{-(1+t)}).\end{aligned}$$

For each $i \in \llbracket 0, d \rrbracket$, we set $v_i(t) = u_i(\phi(t))$ for all $t \in \mathbb{R}_+$. Note that by the rough bounds of Proposition 3.3, there exists a constant $C > 0$ such that for each $i \in \llbracket 0, d \rrbracket$, we have $v_i(t) \leq C$ for all $t \in \mathbb{R}_+$.

CLAIM 4.2. — *The functions $v_0, v_1, \dots, v_d$ satisfy the following system of differential equations: for all $t \in \mathbb{R}_+$, we have*

$$(4.2) \quad \begin{cases} v'_0(t) = v_0(t) - v_1(t), \\ v'_i(t) = \frac{v_i(t)(v_{i+1}(t) - v_1(t) - v_i(t))}{v_0(t)}, & i \in \llbracket 1, d-1 \rrbracket, \\ v'_d(t) = \frac{v_d(t)(1 - v_1(t) - v_d(t))}{v_0(t)}. \end{cases}$$

Moreover, we have

$$(4.3) \quad v_0(t) = e^t \cdot \int_t^\infty v_1(s) \cdot e^{-s} ds \quad \text{for all } t \in \mathbb{R}_+.$$

Proof. — Observe that ϕ is a smooth increasing bijection from $\mathbb{R}_+$ to $[t_0, T[$, and that

$$\phi'(t) = T \cdot e^{-(1+t)} = T - \phi(t) \quad \text{for all } t \in \mathbb{R}_+.$$

Thus, for each $i \in \llbracket 0, d \rrbracket$, we have

$$v'_i(t) = u'_i(\phi(t)) \cdot \phi'(t) = u'_i(\phi(t)) \cdot (T - \phi(t)) \quad \text{for all } t \in \mathbb{R}_+,$$

and (4.2) readily follows from (4.1). Now, to prove (4.3), let us set $w_0(t) = v_0(t) \cdot e^{-t}$ for all $t \in \mathbb{R}_+$. By (4.2), we have

$$w'_0(t) = v'_0(t) \cdot e^{-t} - v_0(t) \cdot e^{-t} = -v_1(t) \cdot e^{-t} \quad \text{for all } t \in \mathbb{R}_+.$$

Moreover, since v_0 is bounded, we have $w_0(t) \rightarrow 0$ as $t \rightarrow \infty$. Thus, integrating the above equation between t and infinity, we obtain

$$0 - w_0(t) = - \int_t^\infty v_1(s) \cdot e^{-s} ds \quad \text{for all } t \in \mathbb{R}_+,$$

which yields (4.3). □

Finally, we introduce one last time change ψ , defined as follows. Observe that the function

$$t \in \mathbb{R}_+ \longmapsto \int_0^t \frac{ds}{v_0(s)}$$

is a smooth increasing bijection from $\mathbb{R}_+$ to $\mathbb{R}_+$ (since v_0 is strictly positive and bounded, the function $1/v_0$ is strictly positive and bounded away from zero), and let $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be its inverse function. For each $i \in \llbracket 1, d \rrbracket$, we set

$$(4.4) \quad w_i(t) = v_i(\psi(t)) \quad \text{for all } t \in \mathbb{R}_+.$$

CLAIM 4.3. — *The functions $w_1, \dots, w_d$ satisfy the following system of differential equations: for all $t \in \mathbb{R}_+$, we have*

$$(4.5) \quad \begin{cases} w'_i(t) = w_i(t)(w_{i+1}(t) - w_1(t) - w_i(t)), & i \in \llbracket 1, d-1 \rrbracket, \\ w'_d(t) = w_d(t)(1 - w_1(t) - w_d(t)). \end{cases}$$

Proof. — By construction, we have

$$\psi'(t) = v_0(\psi(t)) \quad \text{for all } t \in \mathbb{R}_+.$$

Thus, for each $i \in \llbracket 1, d \rrbracket$, we have

$$w'_i(t) = v'_i(\psi(t)) \cdot \psi'(t) = v'_i(\psi(t)) \cdot v_0(\psi(t)) \quad \text{for all } t \in \mathbb{R}_+,$$

and (4.5) readily follows from (4.2). $\square$

We recognise in (4.5) a Lotka–Volterra system, from which it is reasonable to hope obtaining useful information on $w_1, \dots, w_d$. Namely, we obtain the following information.

PROPOSITION 4.4. —

(1) *For $d \in \llbracket 1, 10 \rrbracket$, the following holds: for each $i \in \llbracket 1, d \rrbracket$, we have*

$$w_i(t) \longrightarrow \frac{i}{d+1} \quad \text{as } t \longrightarrow \infty.$$

(2) *Regardless of the value of d , there exists a constant $c > 0$ such that for each $i \in \llbracket 1, d \rrbracket$, we have $w_i(t) \geq c$ for all $t \in \mathbb{R}_+$.*

We postpone the proof of Proposition 4.4 to the next section.

4.1. Back to y

We conclude this section by explaining how the above information on $w_1, \dots, w_d$ can be traced back to y and its derivatives to yield the results of Theorem 1.1.

Proof of Theorem 1.1 assuming Proposition 4.4. —

Proof of (1). — Let $d \in \llbracket 1, 10 \rrbracket$. By Proposition 4.4, for each $i \in \llbracket 1, d \rrbracket$, we have

$$v_i(t) = w_i(\psi^{-1}(t)) \longrightarrow \frac{i}{d+1} \quad \text{as } t \longrightarrow \infty.$$

In particular, by (4.3), we deduce that $v_0(t) \rightarrow 1/(d+1)$ as $t \rightarrow \infty$. Now, for each $i \in \llbracket 0, d \rrbracket$, we get that $u_i(t) = v_i(\phi^{-1}(t)) \rightarrow (1 \vee i)/(d+1)$ as $t \rightarrow T^-$. In particular, we obtain that

$$(T-t) \cdot y(t) = \frac{1}{u_0(t)} \longrightarrow d+1 \quad \text{as } t \longrightarrow T^-.$$

Moreover, observe that for each $j \in \llbracket 1, d \rrbracket$, we have

$$\prod_{i=1}^j u_i(t) = \frac{y^{(j)}(t)}{y(t)^{j+1}} \quad \text{for all } t \in [t_0, T[.$$

Thus, we get that

$$(T-t)^{j+1} \cdot y^{(j)}(t) = ((T-t) \cdot y(t))^{j+1} \cdot \prod_{i=1}^j u_i(t) \xrightarrow{t \rightarrow T^-} (d+1)^{j+1} \cdot \prod_{i=1}^j \frac{i}{d+1} = (d+1) \cdot j!,$$

which completes the proof of (1.4).

Proof of (2). — Let $d \in \mathbb{N}^*$. By Proposition 4.4, for each $i \in \llbracket 1, d \rrbracket$, we have

$$v_i(t) = w_i(\psi^{-1}(t)) \geq c \quad \text{for all } t \in \mathbb{R}_+.$$

In particular, by (4.3), we deduce that $v_0(t) \geq c$ for all $t \in \mathbb{R}_+$. Now, for each $i \in \llbracket 0, d \rrbracket$, we get that $u_i(t) = v_i(\phi^{-1}(t)) \geq c$ for all $t \in [t_0, T[$. Moreover, recall that by Proposition 3.3, there exists a constant $C > 0$ such that $u_i(t) \leq C$ for all $t \in [t_0, T[$. In particular, we obtain that

$$(T-t) \cdot y(t) = \frac{1}{u_0(t)} \in \left[\frac{1}{C}, \frac{1}{c} \right] \quad \text{for all } t \in [t_0, T[.$$

Furthermore, for each $j \in \llbracket 1, d \rrbracket$, we get that

$$(T-t)^{j+1} \cdot y^{(j)}(t) = ((T-t) \cdot y(t))^{j+1} \cdot \prod_{i=1}^j u_i(t) \in \left[\frac{c^j}{C^{j+1}}, \frac{C^j}{c^{j+1}} \right]$$

for all $t \in [t_0, T[$, which completes the proof of (1.5). $\square$

5. Study of the Lotka–Volterra system

In this section, we study the long-time behaviour of solutions to the Lotka–Volterra system that appears in Claim 4.3:

$$(5.1) \quad \begin{cases} w'_i(t) = w_i(t)(w_{i+1}(t) - w_1(t) - w_i(t)), & i \in \llbracket 1, d-1 \rrbracket, \\ w'_d(t) = w_d(t)(1 - w_1(t) - w_d(t)), & t \in \mathbb{R}_+, \end{cases}$$

with initial condition $w(0) \in (\mathbb{R}_+^*)^d$. The main result of this section is stated in Proposition 5.2 below. Before diving into the study of (5.1), let us emphasise the following point: in the previous section, the letters $w_1, \dots, w_d$ referred to the precise functions defined in (4.4). By Claim 4.3, these functions satisfy (5.1), however with a specific initial condition in $(\mathbb{R}_+^*)^d$ on which we do not have explicit information. Therefore, in this section, we study the long-time behaviour of all solutions to the Lotka–Volterra system (5.1) with initial condition in $(\mathbb{R}_+^*)^d$. From now on, we will use the letter $w = (w_1, \dots, w_d)$ as a variable for the system (5.1), and to denote its solutions. The system (5.1) may be rewritten in a concise way as

$$w'(t) = F(w(t)), \quad t \in \mathbb{R}_+,$$

where $F = (F_1, \dots, F_d) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is given by:

$$(5.2) \quad \begin{cases} F_i(w) = w_i(w_{i+1} - w_1 - w_i), & i \in \llbracket 1, d-1 \rrbracket, \\ F_d(w) = w_d(1 - w_1 - w_d), \end{cases}$$

for all $w = (w_1, \dots, w_d) \in \mathbb{R}^d$. Contrary to the previous section where we considered a specific solution of (5.1), we do not know a priori that every maximal solution of (5.1) with initial condition in $(\mathbb{R}_+^*)^d$ is global, and has strictly positive and bounded coordinates: our first task is to prove that this is indeed the case.

PROPOSITION 5.1. — *Fix $w^0 = (w_1^0, \dots, w_d^0) \in (\mathbb{R}_+^*)^d$, and let*

$$w = (w_1, \dots, w_d) : [0, \tau[\longrightarrow \mathbb{R}^d$$

be the maximal solution of (5.1) with initial condition $w(0) = w^0$. This solution has strictly positive and bounded coordinates, and is global: we have $\tau = \infty$, and there exists a constant $C > 0$ such that for each $i \in \llbracket 1, d \rrbracket$, we have $0 < w_i(t) \leq C$ for all $t \in \mathbb{R}_+$.

Proof. — The fact that $(\mathbb{R}_+^*)^d$ is forward-invariant for (5.1) is a standard result on Lotka–Volterra systems (see, e.g., [Bai10, Theorem 6]). Now, fix

$$w^0 = (w_1^0, \dots, w_d^0) \in (\mathbb{R}_+^*)^d,$$

and let $w = (w_1, \dots, w_d) : [0, \tau[\rightarrow (\mathbb{R}_+^*)^d$ be the maximal solution of (5.1) with initial condition $w(0) = w^0$, which we know to have strictly positive coordinates by the previous point. We claim that the function

$$M : t \in [0, \tau[\longmapsto w_1(t) \vee \dots \vee w_d(t) \vee 1$$

is non-increasing. Indeed, fix $t_0 \in [0, \tau[$, and consider the set

$$O = \{t \in [t_0, \tau[: M(s) \leq M(t_0) \text{ for all } s \in [t_0, t]\}.$$

To prove the claim, it suffices to show that $O = [t_0, \tau[$. First, observe that O is a closed subset of $[t_0, \tau[$ that contains t_0 . Moreover, we claim that O is also an open subset of $[t_0, \tau[$. By the connectedness of $[t_0, \tau[$, this entails that $O = [t_0, \tau[$, as desired. To prove that O is open, fix $t \in O$, and let us show that there exists $\varepsilon > 0$ such that $[t_0, t + \varepsilon] \subset O$. By the definition of O , we automatically have $[t_0, t] \subset O$, so let us prove that there exists $\varepsilon > 0$ such that $[t, t + \varepsilon] \subset O$. We set $I = \{i \in \llbracket 1, d \rrbracket : w_i(t) = M(t)\}$. On the one hand, for each $i \in \llbracket 1, d \rrbracket \setminus I$, we have $w_i(t) < M(t)$, hence there exists $\varepsilon_i > 0$ such that $w_i(s) \leq M(t)$ for all $s \in [t, t + \varepsilon_i]$. On the other hand, for each $i \in I$, we have

$$\begin{aligned} w'_i(t) &= \begin{cases} w_i(t)(w_{i+1}(t) - w_1(t) - w_i(t)) & \text{if } i \in \llbracket 1, d-1 \rrbracket, \\ w_i(t)(1 - w_1(t) - w_i(t)) & \text{if } i = d, \end{cases} \\ &\leq w_i(t)(M(t) - w_1(t) - w_i(t)) \\ &= w_i(t) \cdot (-w_1(t)) < 0, \end{aligned}$$

hence there exists $\varepsilon_i > 0$ such that $w'_i(s) \leq 0$ for all $s \in [t, t + \varepsilon_i]$. Thus, the function w_i is non-increasing on $[t, t + \varepsilon_i]$, and we get that $w_i(s) \leq w_i(t) = M(t)$ for all $s \in [t, t + \varepsilon_i]$. Setting $\varepsilon = \varepsilon_1 \wedge \dots \wedge \varepsilon_d$, we conclude that for each $i \in \llbracket 1, d \rrbracket$, we have $w_i(s) \leq M(t) \leq M(t_0)$ for all $s \in [t, t + \varepsilon]$, which shows that $[t, t + \varepsilon] \subset O$, as we wanted to prove. In particular, we obtain that for each $i \in \llbracket 1, d \rrbracket$, we have $0 < w_i(t) \leq M(0)$ for all $t \in [0, \tau[$. Finally, since w is maximal, the fact that it

is bounded entails that it is global, i.e, that $\tau = \infty$, which completes the proof of Proposition 5.1. $\square$

Before coming to the main result of this section, let us introduce some additional notation that will be convenient for the proof. We set

$$A = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ 1 & 1 & -1 & \ddots & \vdots \\ 1 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 1 & 0 & \dots & 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

so that (5.1) may be rewritten as

$$\begin{cases} w'_i(t) = w_i(t) \left(b_i - \sum_{j=1}^d a_{i,j} w_j(t) \right), & i \in \llbracket 1, d \rrbracket, \\ t \in \mathbb{R}_+. \end{cases}$$

Observe that the matrix A is invertible. Indeed, if $w \in \mathbb{R}^d$ is such that $Aw = 0$, then we have

$$\begin{cases} 2w_1 - w_2 = 0, \\ w_1 + w_2 - w_3 = 0, \\ \dots, \\ w_1 + w_{d-1} - w_d = 0, \\ w_1 + w_d = 0, \end{cases} \quad \text{i.e.,} \quad \begin{cases} w_2 = 2w_1, \\ w_3 = w_1 + w_2 = 3w_1, \\ \dots, \\ w_d = w_1 + w_{d-1} = dw_1, \\ (d+1)w_1 = 0, \end{cases}$$

hence $w = 0$. Moreover, similar manipulations show that

$$A^{-1} \cdot b = \left(\frac{1}{d+1}, \dots, \frac{d}{d+1} \right) =: w^*.$$

The following proposition settles the long-time behaviour of solutions to (5.1). In particular, it implies Proposition 4.4, which yields the results of Theorem 1.1.

PROPOSITION 5.2. —

- (1) For $d \in \llbracket 1, 10 \rrbracket$, the following holds: for every solution $w : \mathbb{R}_+ \rightarrow (\mathbb{R}_+^*)^d$ of (5.1), we have $w(t) \rightarrow w^*$ as $t \rightarrow \infty$.
- (2) Regardless of the value of d , the following holds: for every solution

$$w : \mathbb{R}_+ \longrightarrow \left(\mathbb{R}_+^* \right)^d$$

of (5.1), there exists a constant $c > 0$ such that for each $i \in \llbracket 1, d \rrbracket$, we have $w_i(t) \geq c$ for all $t \in \mathbb{R}_+$.

Proof of Proposition 5.2.

Proof of (1). — By a theorem of Goh [Bai10, Theorem 12] (see [Goh79, Theorem 1] for the original result), it suffices to show that for $d \in \llbracket 1, 10 \rrbracket$, there exists $\lambda_1, \dots, \lambda_d > 0$ such that the symmetric matrix

$$(5.3) \quad DA + (DA)^* = \begin{bmatrix} 4\lambda_1 & \lambda_2 - \lambda_1 & \lambda_3 & \lambda_4 & \cdots & \lambda_d \\ \lambda_2 - \lambda_1 & 2\lambda_2 & -\lambda_2 & 0 & \cdots & 0 \\ \lambda_3 & -\lambda_2 & 2\lambda_3 & -\lambda_3 & \ddots & \vdots \\ \lambda_4 & 0 & -\lambda_3 & 2\lambda_4 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & -\lambda_{d-1} \\ \lambda_d & 0 & \cdots & 0 & -\lambda_{d-1} & 2\lambda_d \end{bmatrix}$$

is positive-definite, where D is the diagonal matrix with diagonal entries $\lambda_1, \dots, \lambda_d$, and $(DA)^*$ denotes the transpose of the matrix DA . The idea of Goh's theorem is that the positive-definiteness of $DA + (DA)^*$ entails that

$$V : w \in (\mathbb{R}_+^*)^d \mapsto \sum_{i=1}^d \lambda_i \cdot \left(w_i - w_i^* - w_i^* \cdot \ln \left(\frac{w_i}{w_i^*} \right) \right)$$

is a so-called Lyapunov function, in the sense that it has the following properties:

- we have $V(w) \geq 0$ for all $w \in (\mathbb{R}_+^*)^d$, with equality if and only if $w = w^*$,
- for every solution $w : \mathbb{R}_+ \rightarrow (\mathbb{R}_+^*)^d$ of (5.1), we have

$$\frac{d[V(w(t))]}{dt} = - \left\langle w(t) - w^*, \frac{DA + (DA)^*}{2} \cdot (w(t) - w^*) \right\rangle \quad \text{for all } t \in \mathbb{R}_+,$$

and by the positive-definiteness of $DA + (DA)^*$, we have

$$- \left\langle w - w^*, \frac{DA + (DA)^*}{2} \cdot (w - w^*) \right\rangle \leq 0 \quad \text{for all } w \in (\mathbb{R}_+^*)^d,$$

with equality if and only if $w = w^*$.

Thus, by the LaSalle invariance principle (see, e.g. [Bai10, Theorem 10]), we get that for every solution $w : \mathbb{R}_+ \rightarrow (\mathbb{R}_+^*)^d$ of (5.1), we have $w(t) \rightarrow w^*$ as $t \rightarrow \infty$, as desired.

Now, it remains to show that for $d \in \llbracket 1, 10 \rrbracket$, there exists $\lambda_1, \dots, \lambda_d > 0$ such that the symmetric matrix in (5.3) is positive-definite. By the Sylvester criterion, it suffices to exhibit $\lambda_1, \dots, \lambda_{10} > 0$ such that

$$\Delta_k := \begin{vmatrix} 4\lambda_1 & \lambda_2 - \lambda_1 & \lambda_3 & \lambda_4 & \cdots & \lambda_k \\ \lambda_2 - \lambda_1 & 2\lambda_2 & -\lambda_2 & 0 & \cdots & 0 \\ \lambda_3 & -\lambda_2 & 2\lambda_3 & -\lambda_3 & \ddots & \vdots \\ \lambda_4 & 0 & -\lambda_3 & 2\lambda_4 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & -\lambda_{k-1} \\ \lambda_k & 0 & \cdots & 0 & -\lambda_{k-1} & 2\lambda_k \end{vmatrix} > 0 \quad \text{for all } k \in \llbracket 1, 10 \rrbracket.$$

For $(\lambda_1, \dots, \lambda_{10}) = (1024, 227, 118, 92, 89, 97, 116, 153, 232, 481)$,⁽¹⁾ we find that

$$\begin{aligned}\Delta_1 &= 4096, \\ \Delta_2 &= 1224375, \\ \Delta_3 &= 114265104, \\ \Delta_4 &= 6270814340, \\ \Delta_5 &= 280372975336, \\ \Delta_6 &= 12330415584972, \\ \Delta_7 &= 687248010753336, \\ \Delta_8 &= 69483419810465760, \\ \Delta_9 &= 12807765625815100744, \\ \Delta_{10} &= 136953089422286895648,\end{aligned}$$

which completes the proof of (1).

Proof of (2). — Let us show that for every solution $w : \mathbb{R}_+ \rightarrow (\mathbb{R}_+^*)^d$ of (5.1), there exists a constant $c > 0$ such that for each $i \in \llbracket 1, d \rrbracket$, we have $w_i(t) \geq c$ for all $t \in \mathbb{R}_+$. For Lotka–Volterra systems, this property is called *permanence*. By a theorem of Jansen [Kir86, Lemma 2] (see [Jan87, Theorem 3] for the original result, see also [HS87]), it suffices to show that the function

$$\Psi : w \in \mathbb{R}_+^d \mapsto \sum_{i=1}^d \left(b_i - \sum_{j=1}^d a_{i,j} w_j \right) = 1 - (d+1)w_1$$

has the following property: for each stationary point $w \in \partial(\mathbb{R}_+^d) = \mathbb{R}_+^d \setminus (\mathbb{R}_+^*)^d$ of (5.1), i.e, for each point $w \in \partial(\mathbb{R}_+^d)$ such that $F(w) = 0$, where F is defined in (5.2), we have $\Psi(w) > 0$. The idea of Jansen's theorem is that the assumption on Ψ entails that $P : w \in \mathbb{R}_+^d \mapsto w_1 \cdots w_d$ is a so-called average Lyapunov function, in the sense that it has the following properties:

- We have $P(w) \geq 0$ for all $w \in \mathbb{R}_+^d$, with equality if and only if $w \in \partial(\mathbb{R}_+^d)$,
- For every solution $w : \mathbb{R}_+ \rightarrow \mathbb{R}_+^d$ of (5.1), we have

$$\frac{d[P(w(t))]}{dt} = \Psi(w(t)) \cdot P(w(t)) \quad \text{for all } t \in \mathbb{R}_+,$$

and the assumption on Ψ entails that if $w(0) \in \partial(\mathbb{R}_+^d)$, then there exists $\tau > 0$ such that

$$\int_0^\tau \Psi(w(t)) dt > 0.$$

Thus, by Hofbauer's method of average Lyapunov functions [Kir86, Lemma 1] (see [Hof81, Theorem 1] for the original result), we get the permanence of (5.1), as desired.

To check the assumption on Ψ , let $w \in \mathbb{R}_+^d$ be such that $F(w) = 0$, and suppose that $w_1 > 0$. Then, since $F_1(w) = 0$, we must have $w_2 - 2w_1 = 0$, hence $w_2 = 2w_1 > 0$.

⁽¹⁾This example was communicated to us by Christopher Criscitiello [Cri24], who found it using semi-definite programming.

In turn, since $F_2(w) = 0$, we must have $w_3 - w_1 - w_2 = 0$, hence $w_3 = 3w_1 > 0$, etc. The argument iterates up to $w_d = dw_1 > 0$; and finally, since $F_d(w) = 0$, we must have $1 - w_1 - w_d = 0$, hence $(d+1)w_1 = 1$. This shows that $w = w^* \in (\mathbb{R}_+^*)^d$. In particular, if $w \in \partial(\mathbb{R}_+^d)$, then we must have $w_1 = 0$, hence $\Psi(w) = 1 > 0$, as desired. $\square$

Remark 5.3. — It follows from Propositions 5.1 and 5.2 that regardless of the value of d , the following holds: for every solution $w : \mathbb{R}_+ \rightarrow (\mathbb{R}_+^*)^d$ of (5.1), we have

$$\frac{1}{t} \cdot \int_0^t w(s) \, ds \longrightarrow w^* \quad \text{as } t \longrightarrow \infty.$$

Indeed, for each $i \in \llbracket 1, d \rrbracket$, integrating $w'_i/w_i = b_i - \sum_{j=1}^d a_{i,j} w_j$ from 0 to t yields

$$\ln w_i(t) - \ln w_i(0) = b_i \cdot t - \sum_{j=1}^d a_{i,j} \cdot \int_0^t w_j(s) \, ds \quad \text{for all } t \in \mathbb{R}_+.$$

This can be rearranged as follows:

$$\sum_{j=1}^d a_{i,j} \cdot \left(\frac{1}{t} \cdot \int_0^t w_j(s) \, ds \right) = b_i - \frac{1}{t} \cdot \ln \left(\frac{w_i(t)}{w_i(0)} \right) \quad \text{for all } t \in \mathbb{R}_+^*.$$

Now, for each $i \in \llbracket 1, d \rrbracket$, set

$$\bar{w}_i(t) = \frac{1}{t} \cdot \int_0^t w_i(s) \, ds \quad \text{and} \quad \varepsilon_i(t) = \frac{1}{t} \cdot \ln \left(\frac{w_i(t)}{w_i(0)} \right) \quad \text{for all } t \in \mathbb{R}_+^*.$$

The previous identity can be rewritten as $A \cdot \bar{w}(t) = b - \varepsilon(t)$ in matrix notation, i.e.,

$$\bar{w}(t) = A^{-1} \cdot (b - \varepsilon(t)) \quad \text{for all } t \in \mathbb{R}_+^*.$$

Finally, since for each $i \in \llbracket 1, d \rrbracket$, the function w_i is bounded away from zero and infinity by Propositions 5.1 and 5.2, we get that $\varepsilon_i(t) \rightarrow 0$ as $t \rightarrow \infty$. It follows that

$$\bar{w}(t) \longrightarrow A^{-1} \cdot b = w^* \quad \text{as } t \longrightarrow \infty,$$

as desired.

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