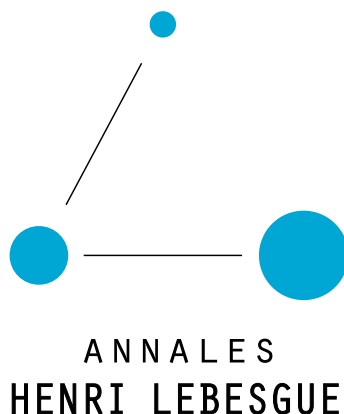




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HYPERTORIC FUKAYA CATEGORIES AND CATEGORIES $\mathcal{O}$

CATÉGORIES DE FUKAYA HYPERTORIQUES
ET CATÉGORIES $\mathcal{O}$

ABSTRACT. — To a conical symplectic resolution with Hamiltonian torus action, Braden–Proudfoot–Licata–Webster associate a category $\mathcal{O}$, defined using deformation quantization (DQ) modules. It has long been expected, though not stated precisely in the literature, that category $\mathcal{O}$ also admits a “Betti-type” realization as the Fukaya–Seidel category of a Lefschetz fibration.

In this paper, we confirm that the category $\mathcal{O}$ associated to a toric hyperkähler manifold is equivalent to the partially wrapped Fukaya category of a Liouville manifold stopped by the fiber of a J -holomorphic moment map. The proof involves relating earlier DQ-module computations

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to a new computation of microlocal perverse sheaves. Leveraging known results on (de Rham) hypertoric category $\mathcal{O}$, we deduce several Floer-theoretic consequences, including formality of simple objects and Koszul duality for the (fully) wrapped Fukaya category; conversely, by applying results about microlocal sheaves, we produce a relative Calabi–Yau structure on category $\mathcal{O}$.

RÉSUMÉ. — À une résolution symplectique conique munie d’une action hamiltonienne d’un tore, Braden, Proudfoot, Licata et Webster associent une catégorie $\mathcal{O}$, définie au moyen de modules de quantification par déformation (DQ-modules). Il était attendu de longue date, bien qu’aucun énoncé précis ne figure dans la littérature, que la catégorie $\mathcal{O}$ admette également une réalisation « de type Betti » comme catégorie de Fukaya–Seidel d’une fibration de Lefschetz.

Dans cet article, nous confirmons que la catégorie $\mathcal{O}$ associée à une variété hyperkählérienne torique est équivalente à la catégorie de Fukaya partiellement enroulée d’une variété de Liouville stoppée par la fibre d’une application moment J -holomorphe. La preuve consiste à relier des calculs antérieurs sur les DQ modules à un nouveau calcul de faisceaux pervers microlocaux. En tirant parti de résultats connus sur la catégorie $\mathcal{O}$ hypertorique (de Rham), nous en déduisons plusieurs conséquences en théorie de Floer, notamment la formalité des objets simples et la dualité de Koszul pour la catégorie de Fukaya (pleinement) enroulée ; inversement, en appliquant des résultats sur les faisceaux microlocaux, nous obtenons une structure de Calabi–Yau relative sur la catégorie $\mathcal{O}$.

1. Introduction

1.1. Category $\mathcal{O}$

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. The classical BGG category $\mathcal{O}$ [BGG76] is a certain abelian subcategory of the category of all $\mathfrak{g}$ -representations which is of great interest in representation theory. It is big enough to contain the highest weight modules and Verma modules but it is also small enough to enjoy many wonderful categorical properties [Hum08].

Category $\mathcal{O}$ also admits a purely geometric description, from which can be derived many of its useful properties. Namely, by Beilinson–Bernstein localization [BB81], category $\mathcal{O}$ is equivalent to the category of $\mathcal{D}$ -modules on the flag variety G/B constructible with respect to the Schubert stratification.

By the Riemann–Hilbert correspondence, category $\mathcal{O}$ can equivalently be described as the category of Schubert-constructible perverse sheaves on G/B . This perspective opens category $\mathcal{O}$ to the use of powerful tools from mixed Hodge theory, and it has been understood since [BGS96] that mixed geometry explains many properties of the Koszul algebras governing categories $\mathcal{O}$.

Flag varieties (or rather their cotangent bundles) are the paradigmatic example of a somewhat loosely defined class of holomorphic symplectic manifolds called *symplectic resolutions*. An important development in geometric representation theory over the past twenty years has been to generalize the tight relationship between the representation theory of category $\mathcal{O}$ and the geometry of flag varieties to other symplectic resolutions. An early generalization of category $\mathcal{O}$ appeared for Cherednik algebras in [BEG03, DO03, GGOR03a]; the hypertoric category $\mathcal{O}$, of interest in this paper, appeared in [BLPW10] with geometric interpretation given in [BLPW12],

followed by a general geometric construction of categories $\mathcal{O}$ for conical symplectic resolutions in [BLPW16], with explicit computations appearing in [Web17].

For the purpose of this introduction, we shall only consider symplectic resolutions which arise from hyperkähler reduction. Let G be a reductive complex Lie group acting on a vector space E . The induced action on T^*E is Hamiltonian and comes with a complex moment map $\mu^{\mathbb{C}} : T^*E \rightarrow \mathfrak{g}^{\vee}$. As reviewed in Section 2.2, one can form the hyperkähler reduction $\mathfrak{X}_G(t) := (\mu^{\mathbb{C}})^{-1}(0) //_t G$ at a generic parameter $t \in \mathfrak{g}_{\mathbb{R}}^{\vee}$. As described in [KR08, Section 2], the reduction carries a sheaf $\mathcal{W}$ of $\mathbb{C}((\hbar))$ -algebras, which can be viewed as a microlocalization of the sheaf of differential operators.

To define category $\mathcal{O}$, one considers two $\mathbb{C}^{\times}$ actions on $\mathfrak{X}_G(t)$. First, consider the action on the vector space $T^*E = E \times E^{\vee}$ given by scalar multiplication. Second, fix a Hamiltonian $\mathbb{C}^{\times}$ -action m on $\mathfrak{X}_G(t)$ (induced for instance from a $\mathbb{C}^{\times}$ -action on E which normalizes the action of G), which we assume to have finitely many fixed points on $\mathfrak{X}_G(t)$. To avoid confusing these actions, we follow the convention of [BPW16] and denote the first copy of $\mathbb{C}^{\times}$ acting holomorphically by $\mathbb{S}$; we will often refer to the second as $\mathbb{C}_m^{\times}$.

We now form the *category $\mathcal{O}$ skeleton*

$$(1.1) \quad \mathbb{L}_G(t, m) := \{p \in \mathfrak{X}_G(t) \mid \lim_{s \rightarrow 0} s \cdot_m p \text{ exists}\} \subset \mathfrak{X}_G(t).$$

We expect that this Lagrangian may be understood as the relative skeleton associated to the Lefschetz fibration given by the J -holomorphic moment map for the $\mathbb{C}_m^{\times}$ action. For the hypertoric case, we prove that statement in Section 2.6 below.

Example 1.1. — Let $E = \mathbb{C}^2$, and G the diagonal action of $\mathbb{C}^{\times}$, so that for generic t we have $\mathfrak{X}_G(t) \simeq T^*\mathbb{P}^1$. We may take the Hamiltonian action m on T^*E to be induced from the $\mathbb{C}^{\times}$ -action on $E = \mathbb{C}^2$ which scales either the first or second coordinate (acting trivially on the other.) In this case there are two fixed points for the induced action of $\mathbb{C}_m^{\times}$ on $T^*\mathbb{P}^1$, and the union of their attracting manifolds $\mathbb{L}_G(t, m)$ is the union of the zero section with the cotangent fiber to either 0 or ∞ (depending on the choice of m).

DEFINITION 1.2 ([BLPW16]). — *The (de Rham) category $\mathcal{O}$ is the category of coherent, $\mathbb{S}$ -equivariant $\mathcal{W}$ -modules which are supported in $\mathbb{L}_G(t, m)$ and admit a good filtration.⁽¹⁾ We denote it by $\mathcal{O}_G^{\text{dR}}(t, m)$.*

It was proved in [BLPW16], which builds on contributions of many authors, including [BGK08, BK12, BLPW12, GGOR03b, KR08, Los12], that this abelian category enjoys many of the same wonderful properties as the classical BGG category $\mathcal{O}$. It also recovers the BGG category $\mathcal{O}$ in the case where $\mathfrak{X}_G(t)$ is the cotangent bundle of a flag variety (in which case the category $\mathcal{O}$ skeleton is the union of conormals to Schubert strata).

Absent so far from this theory, outside the setting of $T^*(G/B)$, has been a Betti counterpart to the de Rham category $\mathcal{O}$. (See [Jin19] for an early Fukaya-categorical perspective on the classical category $\mathcal{O}$.) One reason for this absence is that there

⁽¹⁾In [BLPW16], this category is called the “geometric category $\mathcal{O}$ ” and denoted by $\mathcal{O}_g$. There is also an “algebraic category $\mathcal{O}$ ” considered there which will not be discussed in this paper. For the notion of a good filtration, see [BPW16, Section 2.5].

did not exist a general theory of “microlocal perverse sheaves” on which to base this definition. This missing ingredient has now been supplied by [CKNS25], which allows us to consider the following:

DEFINITION 1.3. — *The Betti category $\mathcal{O}$ is the category*

$$\mathcal{O}_G^{\text{Bet}}(t, m) := (\mu\text{Sh}_{\mathbb{L}_G(t, m)}(\mathbb{L}_G(t, m))^c)^\heartsuit$$

of microlocal perverse sheaves on $\mathfrak{X}_G(t)$ microsupported in $\mathbb{L}_G(t, m)$.⁽²⁾

1.2. Hyperkähler toric manifolds

We shall prove that the Betti and de Rham category $\mathcal{O}$ are equivalent for an important class of symplectic resolutions called *hyperkähler toric manifolds* (or hypertoric manifolds).

Hyperkähler toric manifolds are defined by performing hyperkähler reduction on a torus action on $T^*\mathbb{C}^n$. More precisely, consider an exact sequence of complex tori

$$(1.2) \quad 1 \longrightarrow G \longrightarrow D \longrightarrow F \longrightarrow 1,$$

and fix an identification $D \simeq (\mathbb{C}^\times)^n$. The standard linear action of D on $\mathbb{C}^n$ induces a linear action of G . Now consider the induced Hamiltonian action on $T^*\mathbb{C}^n$ and form the hyperkähler reduction $\mathfrak{X}_G(t)$ with respect to a stability parameter $t = (t, 0, 0) \in \mathfrak{d}_{\mathbb{R}}^\vee \otimes \mathbb{R}^3$, where $\mathfrak{d}$ is the Lie algebra of D . Observe that $\mathfrak{X}_G(t)$ admits a residual Hamiltonian F action. We finally fix a subtorus $\mathbb{C}^\times \subset F$, determined by a cocharacter m of F , which allows us to define the category $\mathcal{O}$ skeleton $\mathbb{L}_G(t, m) \subset \mathfrak{X}_G(t)$.

The de Rham category $\mathcal{O}$ in this setting was studied in [BLPW10, BLPW12], and it was shown [BLPW12, Theorems 4.7 & 4.8] that $\mathcal{O}_G^{\text{dR}}(t, m)$ was equivalent to the category of modules of a certain (discrete, i.e., non-derived) $\mathbb{C}$ -algebra, which we will denote henceforth by $A_G^{\text{dR}}(t, m)$ and which may be understood as the endomorphism algebra of the projective objects. The main result of this paper is an identification of their calculation with our Betti category $\mathcal{O}$.

THEOREM 1.4. — *With respect to the above data for a hyperkähler toric manifold, there is an equivalence of stable categories*

$$(1.3) \quad \mu\text{Sh}_{\mathbb{L}_G(t, m)}(\mathbb{L}_G(t, m)) \simeq \text{Mod}_{A_G^{\text{dR}}(t, m)}.$$

This equivalence intertwines the (microlocal) perverse t -structure on the left hand side with the natural t -structure on the right hand side.

Passing to compact objects and taking hearts, one obtains:

COROLLARY 1.5. — $\mathcal{O}_G^{\text{Bet}}(t, m) \simeq \mathcal{O}_G^{\text{dR}}(t, m)$.

⁽²⁾The superscript $(-)^c$ denotes passage to compact objects. We refer to Section 5 for a discussion of (microlocal perverse) sheaves. Throughout the introduction, we also suppress in our notation the polarization data used to define $\mu\text{Sh}_{\mathbb{L}_G(t, m)}$, which we describe in Definition 6.2.

Remark 1.6. — In the above results, in order to compare with the de Rham category, we have implicitly taken $\mathbb{C}$ as the coefficient ring for our Betti category $\mathcal{O}$. However, unlike in the de Rham setting, this choice is not forced upon us: the left-hand side of Theorem 1.4 may be defined with integral coefficients. This is a major source of added richness in the Betti theory, and it hints at the presence of a categorification, constructed in [GH26]. The equivalence with the de Rham theory is recovered after base change to $\mathbb{C}$, by a map which is a priori complex-analytic but actually, due to strong finiteness properties of category $\mathcal{O}$, turns out to be algebraic.

Remark 1.7. — The presence of a t-structure on the left-hand category in (1.3) follows from [CKNS25], but *it does not follow from general principles* that this category is the derived category of its heart. As a result, one cannot expect to deduce Theorem 1.4 directly from [BLPW12] and a Riemann–Hilbert correspondence such as [CKNS24]. (One can, however, deduce Corollary 1.5.)

1.3. Symplectic duality/3d mirror symmetry

If we replace the exact sequence (1.2) with its dual

$$1 \longleftarrow G^\vee \longleftarrow D^\vee \longleftarrow F^\vee \longleftarrow 1,$$

then the parameters $(t, m) \in \mathfrak{g}_{\mathbb{R}}^\vee \times \mathfrak{f}_{\mathbb{R}}$ switch their roles; this is an incarnation of the combinatorial phenomenon of *Gale duality*. Let S^{dR} be the direct sum of simple objects in $\mathcal{O}_G^{\mathrm{dR}}(t, m)$ and P^{dR} the direct sum of their projective covers, so that there is an isomorphism of algebras $A_G^{\mathrm{dR}}(t, m) \simeq \mathrm{end}(P^{\mathrm{dR}})$; dually, let $(S^\vee)^{\mathrm{dR}}$ and $(P^\vee)^{\mathrm{dR}}$ be the same for $\mathcal{O}_{F^\vee}^{\mathrm{dR}}(m, t)$. Then the main results of [BLPW10] are:

- (1) The algebras $\mathrm{end}(P^{\mathrm{dR}})$ and $\mathrm{end}(S^{\mathrm{dR}})$ are Koszul dual.
- (2) The algebras $\mathrm{end}(P^{\mathrm{dR}})$ and $\mathrm{end}((S^\vee)^{\mathrm{dR}})$ are equivalent (up to a grading shear).

We now have a clearer understanding of such dualities, thanks to recent developments [BDGH16] situating the above in the theory of supersymmetric 3d gauge theories. We can now refine the second equivalence from [BLPW10]; in order to achieve compatibility with our current understanding of 3d mirror symmetry, it turns out that it is preferable to state the equivalence as a relation between a de Rham category $\mathcal{O}$ and the Gale dual Betti category $\mathcal{O}$. The following is a combination of our main theorem with [BLPW10, Theorems A,B]:

COROLLARY 1.8. — *Let P^{Bet} be the direct sum of projective objects in $\mathcal{O}_G^{\mathrm{Bet}}(t, m)$ and $(S^\vee)^{\mathrm{dR}}$ the direct sum of simple objects in $\mathcal{O}_{F^\vee}^{\mathrm{dR}}(m, t)$. Then the two algebras $\mathrm{end}_{\mathcal{O}_G^{\mathrm{Bet}}(t, m)}(P^{\mathrm{Bet}})$ and $\mathrm{end}_{\mathcal{O}_{F^\vee}^{\mathrm{dR}}(m, t)}((S^\vee)^{\mathrm{dR}})$ are equivalent, up to a grading shear.*

Our insistence on the form of Corollary 1.8 comes from the theory of 2-categorical invariants of hyperkähler manifolds. The homological 3d mirror symmetry program [GH26, GMGH23] predicts that if $\mathfrak{X}, \mathfrak{X}^\vee$ are a dual pair [BLPW16] of symplectic resolutions, then there is an equivalence

$$(1.4) \quad \mathcal{A}_{\mathfrak{X}} \simeq \mathcal{B}_{\mathfrak{X}^\vee},$$

between a pair of 2-categories with the following properties:

- we can reduce (1.4) to an equivalence of 1-categories by applying the periodic cyclic homology functor $\mathbf{HP}(-)$. The resulting category $\mathbf{HP}(\mathcal{A}_{\mathfrak{X}})$ will be a category of microlocal (perverse) sheaves on $\mathcal{X}$, while $\mathbf{HP}(\mathcal{B}_{\mathfrak{X}^\vee})$ will be a category of holonomic DQ-modules on $\mathfrak{X}^\vee$;
- under the equivalence (1.4), the objects of $\mathcal{A}_{\mathfrak{X}}$ which become projective objects in the microsheaf category $\mathbf{HP}(\mathcal{A}_{\mathfrak{X}})$ are sent to objects which become simple objects in the DQ-module category $\mathbf{HP}(\mathcal{B}_{\mathfrak{X}^\vee})$.

We conclude that it is Corollary 1.8 (or at least a $\mathbb{Z}/2$ -graded version) which naturally follows from the extended 3d mirror equivalence (1.4) and the above facts. We refer to [GH26] for more details about this proposal, and a proof of the above facts in the hypertoric setting.

1.4. Fukaya categories

The celebrated work of Ganatra–Pardon–Shende [GPS24a] furnishes an (anti-) equivalence of categories

$$(1.5) \quad \mu\mathbf{Sh}_{\mathbb{L}_G(t,m)}(\mathbb{L}_G(t,m))^c \simeq \mathcal{W}(\mathfrak{X}_G(t), \partial_\infty \mathbb{L}_G(t,m))^{\mathrm{op}},$$

where, for X a Weinstein manifold and $\Lambda \subset \partial_\infty X$ a Legendrian, we write $\mathcal{W}(X, \Lambda)$ for the idempotent-completed, pretriangulated closure of the Fukaya category of X stopped at Λ , as defined in [GPS20].

Combining (1.5) with Theorem 1.4, we may immediately read off structural properties of Fukaya categories from known statements about hypertoric category $\mathcal{O}$ established in [BLPW12].

COROLLARY 1.9. — *There is an equivalence of stable categories*

$$\mathcal{W}(\mathfrak{X}_G(t), \partial_\infty \mathbb{L}_G(t,m))^{\mathrm{op}} \simeq \mathbf{Perf}_{A_G^{\mathrm{dR}}(t,m)}$$

which intertwines the perverse t -structure with the standard t -structure. As a result, we deduce the following facts from [BLPW12]:

- (1) $\mathcal{W}(\mathfrak{X}_G(t), \partial_\infty \mathbb{L}_G(t,m))$ admits a canonical full exceptional collection (corresponding heuristically to the thimbles of a Lefschetz fibration).
- (2) The complexified Grothendieck group of $\mathcal{W}(\mathfrak{X}_G(t), \partial_\infty \mathbb{L}_G(t,m))$ is isomorphic to $H_{\mathbb{C}^\times}^{2n}(\mathfrak{X}_G(t))$.
- (3) $\mathcal{W}(\mathfrak{X}_G(t), \partial_\infty \mathbb{L}_G(t,m))$ admits two natural collections of commuting auto-equivalences called twisting and shuffling, which involve moving in the respective (complexified) moduli spaces for regular parameters t and m .

As $A_G^{\mathrm{dR}}(t,m)$ is known to be finite dimensional over $\mathbb{C}$ and of finite global dimension, Corollary 1.9 is a very strong finiteness condition on the Fukaya category.

On the other hand, we can also leverage our Fukaya-categorical perspective for new insights about category $\mathcal{O}$; for instance, by combining the relative Calabi–Yau structure on microsheaf restrictions proved in [ST25] with the microsheaf-to-Fukaya equivalence of [GPS24a], we may deduce the following fact, which does not seem to have been noted before in the literature (possibly because it involves the Fukaya

category of a real symplectic manifold). Let $\mathfrak{F}$ be a fiber of the Lefschetz fibration $\mathfrak{X}_G(t) \rightarrow \mathbb{C}$ described in Section 2.6.

COROLLARY 1.10. — *The category $\mathcal{O}_G^{\text{Bet}}(t, m)$ (hence also $\mathcal{O}_G^{\text{dR}}(t, m)$) has a restriction morphism to the wrapped Fukaya category $\mathcal{W}(\mathfrak{F})$, and this functor has a relative Calabi–Yau structure. By [BD21], we conclude that the moduli space of objects in $\mathcal{O}_G^{\text{Bet}}(t, m)$ admits a Lagrangian morphism to the moduli space of objects in $\mathcal{W}(\mathfrak{F})$.*

We also note a Fukaya-categorical consequence of the 3d mirror duality discussed above. Let $\overline{\mathcal{W}}(-)$ denote the “2-periodicized” Fukaya category, obtained from $\mathcal{W}(-)$ by base-change along $\mathbb{Z} \rightarrow \mathbb{Z}[u^{\pm}]$, where u has cohomological degree 2.

COROLLARY 1.11. — *There is an equivalence*

$$(1.6) \quad \overline{\mathcal{W}}(\mathfrak{X}_G(t), \partial_{\infty} \mathbb{L}_G(t, m)) \simeq \overline{\mathcal{W}}(\mathfrak{X}_{F^{\vee}}(m), \partial_{\infty} \mathbb{L}_{F^{\vee}}(m, t))$$

between the (2-periodicized) partially-wrapped Fukaya categories associated to Gale dual hypertoric varieties.

Example 1.12. — If $G = \mathbb{C}^{\times} \xrightarrow{\Delta} (\mathbb{C}^{\times})^{n+1} = D$ is the diagonal torus, then there are natural identifications $\mathfrak{X}_G(t) = T^*\mathbb{P}^n$, $\mathfrak{X}_{F^{\vee}}(m) = \mathbb{C}^2/(\mathbb{Z}/n)$. These varieties are not even of the same dimension, and their compact skeleta are very different: the former is just $\mathbb{P}^n$, whereas the latter is a nodal chain of n $\mathbb{P}^1$ ’s. On the other hand, after equipping each manifold with a natural Lefschetz fibration, the former skeleton acquires n more noncompact components, namely conormals to the Schubert varieties $\{0\} \subset \mathbb{P}^1 \subset \dots \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n$, and the latter acquires a single noncompact component. 3d mirror symmetry explains the miraculous fact that the Ext algebra of one collection of these simple Lagrangians is equivalent, after a grading shear, to the endomorphism algebra of cocores in the Gale dual category.

Remark 1.13. — The equivalence (1.6) comes from the equivalence between endomorphism algebras of respective generators P^{Bet} and $(S^{\vee})^{\text{Bet}}$ (in the notation of Section 1.3). The need for 2-periodicization arises because these algebras are isomorphic only after a shear of grading. (The existence of such a grading shear, putting the whole Ext algebra of simples into degree 0, is rather remarkable and heavily constrains the structure of the algebra, as we see in the following corollary.)

In Section 7.1, we also appeal to the Koszulity established in [BLPW12] to deduce the formality of the Floer–Fukaya algebra for components of the skeleton of a hypertoric variety.

COROLLARY 1.14. — *Let S be the direct sum of the irreducible components of $\mathbb{L}_G(t, m)$. Then $CF^*(S, S)$ is formal in characteristic zero.*

Finally, in Section 7.2, we prove a new Koszul duality statement about the wrapped Fukaya category of a hypertoric variety $\mathfrak{X}_G(t)$. As previously mentioned, the Koszul duality statements of [BLPW10, BLPW12] are between the simple and projective objects of the category $\mathcal{O}_G^{\text{dR}}(t, m)$; using our main theorem, we may reinterpret this as a Koszul duality in the partially wrapped Fukaya category $\mathcal{W}(\mathfrak{X}_G(t), \partial_{\infty} \mathbb{L}_G(t, m))$

between the components of $\mathbb{L}_G(t, m)$ and their cocores. However, by an analysis in Section 3.2 of how Koszul duality interacts with idempotents, we may deduce from this a Koszul duality within the fully wrapped Fukaya category $\mathcal{W}(\mathfrak{X}_G(t))$.

THEOREM 1.15. — *Let S_{skel} be the direct sum in $\mathcal{W}(\mathfrak{X}_G(t))$ of components of the skeleton of $\mathfrak{X}_G(t)$, and P_{skel} the direct sum of the corresponding cocores. Then the algebras $\text{end}_{\mathcal{W}_G(t)}(S_{\text{skel}})$ and $\text{end}_{\mathcal{W}_G(t)}(P_{\text{skel}})$ are Koszul bidual in the sense of Definition 3.6.*

When $\mathfrak{X}_G(t)$ is the resolution of an A_n -singularity, Theorem 1.15 recovers a result of [EL17] in type A_n . (Note that [EL17] also establishes Koszul duality in type D_n , and calculates the Fukaya category for general tree plumbings.) One virtue of this theorem is that while the endomorphism algebra of cocores, which involves wrapping, may be difficult to compute, the endomorphism algebra of S_{skel} is a finite-dimensional and formal dg-algebra: if we write $S_{\text{skel}} = \bigoplus_{\alpha} S_{\alpha}$ for the components of the skeleton, then this algebra may be written as a matrix algebra with its entries given by cohomology of intersections,

$$(1.7) \quad \text{end}_{\mathcal{W}_G(t)}(S_{\text{skel}}) \simeq \bigoplus_{\alpha, \alpha'} H^*(S_{\alpha} \cap S'_{\alpha'}),$$

where multiplication is given by a convolution using triple intersections. Analogously to [EL17], we learn from Theorem 1.15 that the symplectic cohomology of the manifold $\mathfrak{X}_G(t)$ may be computed in terms of the algebra (1.7).

COROLLARY 1.16. — *There is an equivalence*

$$SH^{*+\dim(\mathfrak{X}_G(t))}(\mathfrak{X}_G(t)) \simeq HH_{-*}(\text{end}_{\mathcal{W}_G(t)}(S_{\text{skel}}))^{\vee}$$

between the symplectic cohomology of $\mathfrak{X}_G(t)$ and the linear dual of the Hochschild homology of the algebra (1.7).

We note that the phenomenon of Koszul duality in symplectic geometry has been studied earlier in e.g. [EL17, EL23, Li19a, Li19b]; guided by those investigations, we outline in Section 7 several further consequences of Theorem 1.15. One such consequence which may appeal to geometrically-minded readers is the following.

COROLLARY 1.17. — *$\mathfrak{X}_G(t)$ does not contain an exact Lagrangian $K(\pi, 1)$.*

This is a standard consequence of the existence of a dilation on $SH^*(\mathfrak{X}_G(t))$, which we will deduce from Theorem 1.15. In dimension 4, Corollary 1.17 recovers a result of Ritter [Rit10], who proved more generally that the only exact Lagrangians in ALE spaces are spheres.

Remark 1.18. — Forthcoming work of Lee, Li, Mak [LLM] will provide an alternative approach to studying Fukaya categories of hypertoric varieties which does not go through [GPS24a]. Applications of their work include a different proof of Corollary 1.14, a computation of the endomorphism algebra of the simples in $\mathcal{W}(\mathfrak{X}_G(t), \partial_{\infty}\mathbb{L}_G(t, m))$, and a proof that they generate the infinitesimally wrapped Fukaya category.

1.5. The Fukaya–Seidel category

There is a well-established heuristic for interpreting the generalized category $\mathcal{O}$ of [BLPW16] as the Fukaya–Seidel category of a Lefschetz fibration. In the case of a 2-block nilpotent-slice in type A, this was made precise in [MS22]. In the present context, we learn from Lemma 2.33 that the partially wrapped Fukaya category $\mathcal{W}(\mathfrak{X}_G(t), \partial_\infty \mathbb{L}_G(t, m))$ should be interpreted as a Fukaya–Seidel category of $\mathfrak{X}_G(t)$ equipped with the Lefschetz fibration given by the J -holomorphic moment map for $\mathbb{C}_m^\times$. We explain here how certain representation-theoretic features of hypertoric category $\mathcal{O}$ should be understood from this perspective.

We have seen that $\mathcal{O}_G^{\mathrm{dR}}(t, m)$ is a highest-weight category. As explained in Section 3.3, it comes equipped distinguished objects $\{L_\alpha\}$, $\{P_\alpha\}$, $\{V_\alpha\}$ (resp. the simples, projectives and standards) indexed by a poset $\mathcal{J}$, each of which forms a basis for the Grothendieck group. In Table 1.1, we recall how these bases are expected to relate to objects of study in symplectic topology.

Table 1.1. A (well-known, and partly heuristic) dictionary

Representation theory	Symplectic topology
simple modules	irreducible components of the skeleton
indecomposable projective modules	cocores / linking disks
standard (or Verma) modules	Lefschetz thimbles

The first two lines of Table 1.1 are strictly true as written, and follow from tracing through the equivalences which enter into the proof of Corollary 1.9. The third line is only a heuristic. First, we have not even discussed how to make thimbles into objects of our partially wrapped model for the Fukaya–Seidel category (although see [GPS24b, Section 8.6] for one approach.) More seriously, our method only gives an algebraic description of Verma modules as iterated cones on the irreducible components of the category $\mathcal{O}$ skeleton (see Lemma 4.11). Geometrically, one may think of these objects as arising as an “adiabatic limit” of thimbles. It would be interesting to reverse this process, and prove that these algebraic objects in the Fukaya category are represented by honest Lagrangians.

Remark 1.19. — Another approach to the Fukaya–Seidel category of a hyperkähler manifold equipped with a J -holomorphic moment map is described in [Kha23]. However, that approach, based on the “algebra of the infrared” formulated in [GMW15], requires that the critical values of the moment map are in general position, and thus it cannot be directly applied to category $\mathcal{O}$, for which all critical values lie on a line. This restriction can be eliminated using the formulation of the Fukaya–Seidel category in [Hay15].

1.6. Outline of the proof

Our calculation of the Betti category $\mathcal{O}$ proceeds in several steps. First, we recall the classical calculation (phrased in the language of sheaves rather than microsheaves)

of microsheaves on the Lagrangian $\mathbb{L} \subset T^*\mathbb{C}^n$ given by the union of conormals to toric strata. The G -action on this category is also well understood, so that we may present the equivariant microsheaf category

$$\mu\mathrm{Sh}_{\mathbb{L}}^G(\mathbb{L}) \simeq \mathrm{Mod}_{A_G^{\mathrm{Bet}}}$$

as the category of modules over an algebra A_G^{Bet} with 2^n commuting idempotents, corresponding to the components of the Lagrangian $\mathbb{L}_G := \mathbb{L}/G$.

Next, we pass from the stacky Lagrangian $\mathbb{L}_G$ to an open non-stacky subset $\mathbb{L}_G(t, -)$ of points which are stable with respect to the reduction parameter $t \in \mathfrak{g}_{\mathbb{R}}^{\vee}$. Writing $e_{\mathcal{F}}$ for the sum of idempotents in A_G^{Bet} corresponding to components of $\mathbb{L}_G(t, -) \subset \mathbb{L}_G$, we obtain a new algebra $A_G^{\mathrm{Bet}}(t, -) := e_{\mathcal{F}} A_G^{\mathrm{Bet}} e_{\mathcal{F}}$, and we exhibit an equivalence

$$(1.8) \quad \mu\mathrm{Sh}_{\mathbb{L}_G}(\mathbb{L}_G(t, -)) \simeq \mathrm{Mod}_{A_G^{\mathrm{Bet}}(t, -)}.$$

We establish (1.8) by exhibiting both sides as the global sections of equivalent sheaves of categories, partly imitating a similar argument from [GMW25, Section 4.3]. The key technical step is to prove that the presheaf of categories $\mathrm{Mod}_{e_{\alpha} A_G^{\mathrm{Bet}}(t, -) e_{\alpha}}$ satisfies descent. This statement is established in a joint work of the authors with Michael McBreen and Ben Webster, located in [GMW25, Appendix C].

We pause here to advertise one corollary of the above computations.

THEOREM 1.20 (Corollary 6.11). — *The restriction functor*

$$\mu\mathrm{Sh}_{\mathbb{L}}^G(\mathbb{L}) = \mu\mathrm{Sh}_{\mathbb{L}_G}(\mathbb{L}_G) \longrightarrow \mu\mathrm{Sh}_{\mathbb{L}_G}(\mathbb{L}_G(t, -))$$

admits a fully faithful left adjoint.

A de Rham analogue of Theorem 1.20 is proved for general conic symplectic resolutions in [BPW16, Section 5.5]. In either case, such a “categorical Kirwan surjectivity theorem” plays a major role in the computation of category $\mathcal{O}$, and it would be very desirable to know whether such a result could be proved in generality on the Betti side. We expect that the answer is yes.

CONJECTURE 1.21. — *In the setting of Section 1.1, let $\mathbb{L} \subset T^*E$ be the union of attracting sets for all different cocharacters m ; $\mathbb{L}_G$ its quotient by G ; and let $\mathbb{L}_G(t, -)$ be the intersection of $\mathbb{L}/G$ with the stable locus $\mathfrak{X}_G(t)$. Then the restriction functor*

$$\mu\mathrm{Sh}_{\mathbb{L}}^G(\mathbb{L}) = \mu\mathrm{Sh}_{\mathbb{L}_G}(\mathbb{L}_G) \longrightarrow \mu\mathrm{Sh}_{\mathbb{L}_G}(\mathbb{L}_G(t, -))$$

admits a fully faithful left adjoint.

Returning to the calculation of Betti category $\mathcal{O}$, we now pass from $\mathbb{L}_G(t, -)$ to the closed subset $\mathbb{L}_G(t, m)$ defined by (1.1): by stop removal, note that the category $\mu\mathrm{Sh}_{\mathbb{L}_G}(\mathbb{L}_G(t, m))$ may be obtained from $\mu\mathrm{Sh}_{\mathbb{L}_G}(\mathbb{L}_G(t, -)) \simeq \mathrm{Mod}_{A_G^{\mathrm{Bet}}(t, -)}$ as the categorical quotient by the objects corepresenting microstalks at the components of the complement $\mathbb{L}_G(t, m) \setminus \mathbb{L}_G(t, -)$. Writing $e_{\mathcal{F} \cap \mathcal{U}} \in A_G^{\mathrm{Bet}}(t, -)$ for the idempotent corresponding to these components, we define

$$(1.9) \quad A_G^{\mathrm{Bet}}(t, m) := \frac{A_G^{\mathrm{Bet}}(t, -)}{A_G^{\mathrm{Bet}}(t, -) e_{\mathcal{F} \cap \mathcal{U}} A_G^{\mathrm{Bet}}(t, -)}$$

to be the quotient algebra.

This definition does not yet guarantee us an equivalence $\mu\mathrm{Sh}_{\mathbb{L}_G}(\mathbb{L}_G(t, m)) \simeq \mathrm{Mod}_{A_G^{\mathrm{Bet}}(t, m)}$, since (1.9) is defined as a naive (underived) quotient of algebras, whereas the categorical quotient category $\mu\mathrm{Sh}_{\mathbb{L}_G}(\mathbb{L}_G(t, m))$ is controlled by a derived quotient described in Definition 3.1. We must therefore establish that the derived and underived quotients agree in this case.

To understand the quotient (1.9), we write down a Riemann–Hilbert-type map, defined on a certain completion of A_G^{Bet} , to identify it with the completion of another algebra A_G^{dR} related to the category of G -equivariant DQ-modules on $\mathbb{L}$. The completion is necessary because the Riemann–Hilbert correspondence is controlled by a map which is not algebraic but only complex-analytic. Luckily, this completion still contains enough information to recover the construction of $A_G^{\mathrm{Bet}}(t, m)$ and (by following the Riemann–Hilbert isomorphism) to produce an isomorphism

$$(1.10) \quad A_G^{\mathrm{Bet}}(t, m) \simeq A_G^{\mathrm{dR}}(t, m)$$

with the de Rham category $\mathcal{O}$ algebra studied in [BLPW10, BLPW12]. We can then leverage many of the properties of de Rham category established in [BLPW12], deducing in particular that the quotient (1.9) is in fact a derived quotient. Miraculously, although the original Riemann–Hilbert map we wrote down was not algebraic, its restriction to (1.10) does in fact define an isomorphism of $\mathbb{C}$ -algebras (without formal completion), due to the strong finiteness properties of category $\mathcal{O}$.

1.7. Examples

We end the introduction with some extended examples.

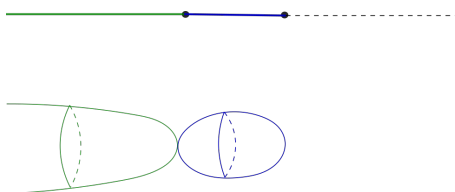


Figure 1.1. A hyperplane arrangement and the corresponding category $\mathcal{O}$ skeleton. The underlying Weinstein manifold is $T^*\mathbb{P}^1$. There are two simple objects, namely the cotangent fiber over $\infty \in \mathbb{P}^1$ and the zero section.

Example 1.22. — Consider the hyperplane arrangement drawn in Figure 1.1. This can be realized concretely by the exact sequence

$$1 \longrightarrow G = \mathbb{C}^\times \xrightarrow{\Delta} (\mathbb{C}^\times)^2 \longrightarrow F = (\mathbb{C}^\times)^2 / \mathbb{C}^\times \longrightarrow 1,$$

where $t : G \rightarrow \mathbb{C}^\times$ and $m : F \rightarrow \mathbb{C}^*$ are nonzero characters. In this case, $\mathfrak{X}_G(t) = T^*\mathbb{P}^1$, and the Hamiltonian action of F on $T^*\mathbb{P}^1$ is induced by the standard $\mathbb{C}^\times$ -action on $\mathbb{P}^1$. The Lagrangian $\mathbb{L}_G(t, m)$ is the conormal to the Schubert stratification $\mathbb{P}^1 = \{0\} \sqcup \mathbb{C}$.

We have $\mu\mathrm{Sh}_{\mathbb{L}_G(t,m)}(\mathbb{L}_G(t,m)) \simeq \mathrm{Sh}_{\mathbb{L}_G(t,m)}(\mathbb{P}^1)$, and the equivalence respects the perverse t-structure. This is the classical category $\mathcal{O}$ for $SL(2)$, which may be written as

$$\mathcal{O}_G^{\mathrm{Bet}}(t,m) \simeq \left\{ V \begin{array}{c} \xrightarrow{x} \\ \xleftarrow{y} \end{array} W \mid xy = 0 \right\}.$$

Let L_1 be the cotangent fiber (green) and let L_2 be the zero section (blue). Viewed as objects of $\mathcal{O}_G^{\mathrm{Bet}}(t,m)$, they correspond to the respective representations $\mathbb{C} \rightleftharpoons 0$ and $0 \rightleftharpoons \mathbb{C}$. We have $V_1 = L_1$, $V_2 = P_2$. These fit into nontrivial extensions $0 \rightarrow V_1 \rightarrow V_2 \rightarrow L_2 \rightarrow 0$ and $0 \rightarrow V_2 \rightarrow P_1 \rightarrow V_1 \rightarrow 0$.

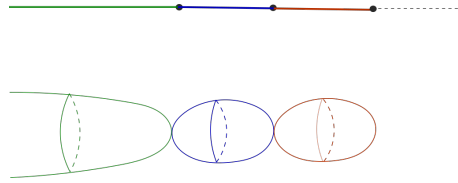


Figure 1.2. A hyperplane arrangement and the corresponding category $\mathcal{O}$ skeleton. The underlying Weinstein manifold is a plumbing of two copies of $T^*\mathbb{P}^1$. The category $\mathcal{O}$ skeleton has three irreducible components (the zero sections and a cotangent fiber), corresponding to the three simple objects of category $\mathcal{O}$.

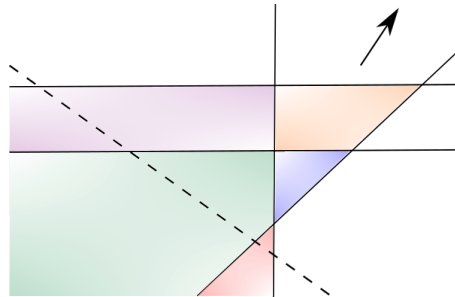


Figure 1.3. The mass parameter m is represented by an arrow. The compact skeleton of $\mathfrak{X}_G(t)$ is the union of a $\mathbb{P}^2$ and of a Hirzebruch surface, meeting cleanly along a $\mathbb{P}^1$. The five simple objects in $\mathcal{O}_G^{\mathrm{Bet}}(t,m)$ correspond to shaded components. The dotted line is the image of a fiber of the J -holomorphic moment map, which is symplectically a plumbing of three copies of T^*S^3 along circles.

Example 1.23. — Now consider the arrangement drawn in Figure 1.2, for which $\mathfrak{X}_G(t)$ is a plumbing of two copies of T^*S^2 , and $\mathbb{L}_G(t,m)$ is the union of both zero sections and a cotangent fiber. In this case, the category

$$\mathcal{O}_G^{\mathrm{Bet}}(t,m) \simeq \left\{ U \begin{array}{c} \xrightarrow{x_1} \\ \xleftarrow{y_1} \end{array} V \begin{array}{c} \xrightarrow{x_2} \\ \xleftarrow{y_2} \end{array} W \mid x_1y_1 = x_2y_2, x_2y_2 = 0 \right\}$$

has three simples $L_1 < L_2 < L_3$ (green, blue, red). All nontrivial morphisms between the simples are in degree 1. We have $V_1 = L_1$ and nontrivial extensions $0 \rightarrow L_1 \rightarrow V_2 \rightarrow L_2 \rightarrow 0$ and $0 \rightarrow V_2 \rightarrow V_3 \rightarrow L_3[-1] \rightarrow 0$. Alternatively, we can write $V_2 = \mathrm{Cone}(L_2[-1] \rightarrow L_1 = V_1)$ and $V_3 = \mathrm{Cone}(L_3[-2] \rightarrow V_2)$; this description may

be more compelling to symplectic geometers, since it gives a recipe for representing the V_i by honest Lagrangians by surgery.

Example 1.24. — Consider now the hyperplane arrangement drawn in Figure 1.3, where we have used the natural metric on $\mathbb{R}^2$ to identify the codirection m with the direction indicated by the arrow. As a Weinstein manifold, the space $\mathfrak{X}_G(t)$ may be obtained by plumbing $T^*\mathbb{P}^2$ with the cotangent bundle to a Hirzebruch surface along a shared copy of $\mathbb{P}^1$. With respect to the ordering induced by m , the five simplices are L_1 (red), L_2 (green), L_3 (blue), L_4 (purple), and L_5 (orange).

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2. Hyperkähler toric geometry

2.1. Hyperplane arrangements

We review hyperplane arrangements for the purpose of setting our notation. We mostly follow [BLPW10, Section 2], to which we refer the reader for details and motivation.

DEFINITION 2.1. — Let $n \geq 1$. A sign vector is an element of $2^{[n]} \simeq \{+, -\}^n \simeq \mathcal{P}([n])$, where the latter isomorphism takes $\alpha = (\alpha_1, \dots, \alpha_n) \in \{+, -\}^n$ to the subset $\{k \in [n] \mid \alpha_k = +\}$.

DEFINITION 2.2. — A hyperplane arrangement is an n -dimensional, oriented, real vector space V and a collection $\{H_1, \dots, H_k\}$ of $k \geq 0$ oriented affine-linear hyperplanes (empty if $k = 0$).

- The connected components of $V - \bigcup_i H_i$ are called chambers.
- Given $S \subset [k] = \{1, \dots, k\}$, the intersection $H_S := \bigcap_{i \in S} H_i$ is called the flat spanned by S .
- A hyperplane arrangement is simple if H_S has codimension $|S|$ for all $S \subset [k]$.

Example 2.3. — The coordinate hyperplanes $H_i := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i = 0\}$ form a hyperplane arrangement in $\mathbb{R}^n$. This arrangement is simple, and the chambers are naturally indexed by $2^{[n]}$. We will write Δ_α for the chamber corresponding to $\alpha \in 2^{[n]}$.

DEFINITION 2.4. — A polarized hyperplane arrangement (indexed by $[n]$) is a triple (V, t, m) , where $V \subset \mathbb{R}^n$ is a linear subspace, $t \in \mathbb{R}^n/V$, $m \in V^\vee$. Throughout this paper, we will make the assumption that the inclusion $V \hookrightarrow \mathbb{R}^n$ is induced (by $\otimes_{\mathbb{Z}} \mathbb{R}$) from an inclusion of lattices $V_{\mathbb{Z}} \hookrightarrow \mathbb{Z}^n$.

The underlying hyperplane arrangement corresponding to the above data (V, t, m) lives in the affine space $V(t) := V + t \subset \mathbb{R}^n$, and it is given by the n affine-linear hyperplanes $H_i \subset V(t)$ defined by $H_i = \{v = (v_1, \dots, v_n) \in V(t) \subset \mathbb{R}^n \mid v_i = 0\}$.

DEFINITION 2.5. — Let (V, t, m) be a polarized hyperplane arrangement.

- We say that t is regular if the corresponding hyperplane arrangement is simple, and we say that m is regular if it is not constant on any 1-dimensional flat. We call the polarized arrangement (V, t, m) regular if both t and m are regular.
- A sign vector is said to be (t) -feasible if the corresponding chamber in $\mathbb{R}^n$ has non-empty intersection with V_t ; otherwise it is called infeasible. We write $\mathcal{F} \subset 2^{[n]}$ for the collection of feasible sign vectors.
- A sign vector is said to be (m) -bounded if the restriction of m to the corresponding chamber in V is bounded; otherwise we say it is unbounded. We write $\mathcal{B} \subset 2^{[n]}$ for the collection of bounded sign vectors.
- The hyperplane arrangement is unimodular if for all $I \subset [n]$, the image of $V_{\mathbb{Z}} \hookrightarrow \mathbb{Z}^n \twoheadrightarrow \mathbb{Z}^I$ is a direct summand of $\mathbb{Z}^I$.

2.2. Quaternionic vector spaces and reduction

The spaces we study will be obtained from hyperkähler reductions of quaternionic vector spaces. We write $\mathbb{H} = \{x_0 + Ix_1 + Jx_2 + Kx_3 \mid x_i \in \mathbb{R}\}$ for the quaternions. The coordinates x_i give an isomorphism $\mathbb{H} \simeq \mathbb{R}^4$. By distinguishing the complex structure I , we can identify $\mathbb{H} \simeq \mathbb{C}^2$ by declaring that $(z, w) := (x_0 + ix_1, x_2 + ix_3)$ are holomorphic coordinates. These constructions naturally extend to $\mathbb{H}^n \simeq \mathbb{H} \otimes_{\mathbb{R}} \mathbb{R}^n$.

We write $g(-, -)$ for the metric on $\mathbb{H}^n$ induced by the Euclidean metric on $\mathbb{R}^{4n}$. This is a hyperkähler metric; for $\mathcal{J} \in \{I, J, K\}$ we write $\omega_{\mathcal{J}} := g(\mathcal{J}-, -)$ for the corresponding symplectic form, which is the standard Kähler form on the $\mathcal{J}$ -complex Kähler manifold $\mathbb{C}^{2n}$. We write $\Omega_I := \omega_J + i\omega_K$ for the I -holomorphic symplectic form. When we consider real symplectic geometry in this paper, it will be with respect to the real symplectic form $\omega_J := \text{Re}(\Omega_I)$.

Remark 2.6. — Although the complex structure I has a privileged status in our constructions, the form ω_J is not actually distinguished, and could be replaced throughout this paper by any $\omega_{J_\theta} := \text{Re}(e^{i\theta}\Omega_I)$. Indeed, it is expected that the richest content of holomorphic Floer theory is only visible when considering this whole S^1 family of symplectic forms. (Proposals for many structures expected to exist in such a theory can be found in [DR23, KS24].)

It will be convenient to identify $T^*\mathbb{C}^n \simeq \mathbb{C}^{2n}$ via the map

$$(2.1) \quad (q_1, \dots, q_n, p_1, \dots, p_n) \mapsto (q_1, -p_1, \dots, q_n, -p_n).$$

Note that per our conventions, the symplectic form on T^*Q is $d\lambda_{\text{can}}$ (if we had used instead $-d\lambda_{\text{can}}$, then there would be no minus signs in (2.1)).

Let G be a complex reductive group acting linearly on $\mathbb{C}^n$, with maximal compact subgroup G_c . The following is well-known:

LEMMA 2.7. — *The action of G_c on $T^*\mathbb{C}^n$ is trihamiltonian, in the sense that the action is Hamiltonian with respect to each of the symplectic forms $\omega_I, \omega_K, \omega_J$.*

Using the I -complex structure, we may split the resulting triple of moment maps

$$(2.2) \quad \mu : \mathbb{H}^n \longrightarrow \mathfrak{d}_{\mathbb{R}}^{\vee} \otimes \mathbb{R}^3$$

into a real and complex part,

$$\mu^{\mathbb{R}} : \mathbb{H}^n \longrightarrow \mathfrak{d}_{\mathbb{R}}^{\vee} \quad \text{and} \quad \mu^{\mathbb{C}} : \mathbb{H}^n \longrightarrow \mathfrak{d}^{\vee}.$$

Example 2.8. — Let $G = (\mathbb{C}^{\times})^n$, with its standard action on $\mathbb{C}^n$. with its standard action on $\mathbb{C}^n$. The real and complex moment maps for G may be written in coordinates as

$$\begin{aligned} (x_1, \dots, x_n, y_1, \dots, y_n) &\longmapsto (|x_1|^2 - |y_1|^2, \dots, |x_n|^2 - |y_n|^2), \\ (x_1, \dots, x_n, y_1, \dots, y_n) &\longmapsto (x_1 y_1, \dots, x_n y_n), \end{aligned}$$

respectively.

THEOREM 2.9 ([HKLR87, Theorem 3.1]). — *Suppose that $\xi \in \mathfrak{d}^{\vee} \otimes \mathbb{R}^3$ is fixed by the coadjoint action of G . If G acts freely on $\mu^{-1}(\xi)$, then there is a canonical hyperkähler structure $(\bar{\omega}_I, \bar{\omega}_J, \bar{\omega}_K)$ on $\mu^{-1}(\xi)/G_c$. It is uniquely determined by the property that so that $\pi^* \bar{\omega}_I = \iota^* \omega_I$ (resp. for J, K), where $\iota : \mu^{-1}(0) \hookrightarrow \mathbb{H}^n$ is the inclusion and $\pi : \mu^{-1}(0) \rightarrow \mu^{-1}(0)/G_c$ is the projection.*

By splitting the moment maps up using the I -complex structure as above, we recover an I -complex algebraic perspective on the hyperkähler reduction described in Theorem 2.9.

PROPOSITION 2.10. — *As an I -complex algebraic variety, the hyperkähler reduction $\mu^{-1}(\xi)/G_c$ is isomorphic to the GIT quotient $(\mu^{\mathbb{C}})^{-1}(\xi_2 + i\xi_3) //_{\xi_1} G_c$.*

As a result, we have a useful embedding of the hyperkähler reduction $\mu^{-1}(\xi_1, 0, 0)$ into an Artin stack:

COROLLARY 2.11. — *The I -holomorphic symplectic manifold $\mu^{-1}(\xi_1, 0, 0)/G_c$ admits an open symplectic embedding*

$$\mu^{-1}(\xi_1, 0, 0)/G_c \hookrightarrow T^*(\mathbb{C}^n/G) = (\mu^{\mathbb{C}})^{-1}(0)/G$$

as the ξ_1 -semistable locus inside the stack $(\mu^{\mathbb{C}})^{-1}(0)/G$.

2.3. Toric hyperkähler manifolds

We now specialize to the case of interest in this paper. Fix an exact sequence of tori

$$(2.3) \quad 1 \longrightarrow G \xrightarrow{i} D \xrightarrow{p} F \longrightarrow 1,$$

together with an identification $D \simeq (\mathbb{C}^\times)^n$. The exact sequence (2.3) induces exact sequences

$$\begin{aligned} 0 &\longrightarrow \mathfrak{g}_{\mathbb{Z}} \xrightarrow{i_{\mathbb{Z}}} \mathfrak{d}_{\mathbb{Z}} \xrightarrow{p_{\mathbb{Z}}} \mathfrak{f}_{\mathbb{Z}} \longrightarrow 0, \\ 0 &\longleftarrow \mathfrak{g}_{\mathbb{Z}}^{\vee} \xleftarrow{i_{\mathbb{Z}}^{\vee}} \mathfrak{d}_{\mathbb{Z}}^{\vee} \xleftarrow{p_{\mathbb{Z}}^{\vee}} \mathfrak{f}_{\mathbb{Z}}^{\vee} \longleftarrow 0 \end{aligned}$$

of cocharacter and character lattices, respectively. The identification $D \simeq (\mathbb{C}^\times)^n$ determines identifications $\mathfrak{d}_{\mathbb{Z}} \simeq \bigoplus_{i=1}^n \mathbb{Z}e_i$ and $\mathfrak{d}_{\mathbb{Z}}^{\vee} \simeq \bigoplus_{i=1}^n \mathbb{Z}e^i$. In the rest of this subsection we will abbreviate tensor products $(-) \otimes_{\mathbb{Z}} \mathbb{R}$ by changing the subscript $\mathbb{Z}$ to $\mathbb{R}$.

DEFINITION 2.12. — A stability parameter is an element $t \in \mathfrak{g}_{\mathbb{R}}^{\vee}$. A mass parameter is an element $m \in \mathfrak{f}_{\mathbb{R}}$.

The exact sequence (2.3) determines the embedding $\mathfrak{f}_{\mathbb{R}}^{\vee} \xrightarrow{p_{\mathbb{R}}^{\vee}} \mathbb{R}^n$; together with the parameters t, m , it therefore determines a polarized hyperplane arrangement $(\mathfrak{f}_{\mathbb{R}}^{\vee}, t, m)$.

DEFINITION 2.13. — We write $\mathfrak{f}_{\mathbb{R}}^{\vee}(t) := \mathfrak{f}_{\mathbb{R}}^{\vee} + t$ for the affine-linear subspace of $\mathbb{R}^n$ determined by the parameter t , and $\mathcal{H}(t)$ for the hyperplane arrangement in $\mathfrak{f}_{\mathbb{R}}^{\vee}(t)$ induced by the coordinate hyperplane arrangement on $\mathfrak{d}_{\mathbb{R}}^{\vee} \simeq \mathbb{R}^n$.

DEFINITION 2.14. — A (hypertoric) category $\mathcal{O}$ datum is a polarized hyperplane arrangement indexed by $[n]$, which we take to be determined by the exact sequence (2.3) together with stability and mass parameters $(t, m) \in \mathfrak{g}_{\mathbb{R}}^{\vee} \times \mathfrak{f}_{\mathbb{R}}$, which is both regular and unimodular.

For the remainder of the paper, we fix a category $\mathcal{O}$ datum (G, t, m) . Now observe that the inclusion $G_c \hookrightarrow D_c$ defines a trihamiltonian action of G_c on $T^*\mathbb{C}^n$, whose moment maps may be recovered from those described in Example 2.8 by composition with the quotient map $\mathfrak{d}_{\mathbb{R}}^{\vee} \rightarrow \mathfrak{g}_{\mathbb{R}}^{\vee}$.

DEFINITION 2.15. — The toric hyperkähler manifold (or hypertoric variety) associated to the exact sequence (2.3) and the regular stability parameter t is the hyperkähler reduction

$$\mathfrak{X}_G(t) := \mu^{-1}(t, 0, 0)/G_c.$$

THEOREM 2.16 ([BD00, Theorem 3.2]). — The space $\mathfrak{X}_G(t)$ is a complete hyperkähler manifold.

The varieties $\mathfrak{X}_G(t)$ were first introduced in [Got92] and subsequently studied in [BD00, HS02, Kon99].

Remark 2.17. — The condition of unimodularity in the definition of a category $\mathcal{O}$ datum ensures that the absence of any strictly t -semistable points; without this condition, $\mathfrak{X}_G(t)$ would no longer be guaranteed to be a smooth manifold, but it would still be a smooth orbifold (or DM stack) by [BD00, Theorem 3.2]. Most of the results in this paper would still hold in this case, but we would no longer be able to make a comparison with Fukaya categories.

2.4. Distinguished Lagrangians

Every hyperkähler toric manifold $\mathfrak{X}_G(t)$ admits a distinguished collection of subvarieties which are I -holomorphic and Lagrangian with respect to the complex-symplectic form Ω_I . These are the images of the conormals to coordinate strata in $\mathbb{C}^n$.

For $\alpha \in 2^{[n]}$, let $Z_\alpha \subset \mathbb{C}^n$ be the toric subvariety defined by

$$Z_\alpha := \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_{i_k} = 0 \text{ if } i_k = -\}.$$

DEFINITION 2.18. — We write $\mathbb{L}^\alpha := T_{Z_\alpha}^* \mathbb{C}^n \subset T^* \mathbb{C}^n$ for the conormal to the toric subvariety Z_α . We will denote by $\mathbb{L}_n$ (or just $\mathbb{L}$ if n is understood) the union of conormals to toric subvarieties: $\mathbb{L}_n := \bigcup_{\alpha \in 2^{[n]}} \mathbb{L}^\alpha$.

Consider the standard hyperplane arrangement on $\mathfrak{d}_{\mathbb{R}}^\vee \simeq \mathbb{R}^n$ given by the coordinate hyperplanes, cooriented positively. We write Δ_α for the chamber with sign vector $\alpha \in 2^{[n]}$.

LEMMA 2.19. — The chamber Δ_α is the image of the Lagrangian $\mathbb{L}^\alpha$ under the real moment map $\mu^{\mathbb{R}}$.

Proof. — The restriction of the D_c -action, and the real Kähler form, to a Lagrangian $\mathbb{L}_\alpha$, is the standard torus action on the toric variety $\mathbb{L}_\alpha \simeq \mathbb{C}^n$, with some signs reversed as indicated in the sign vector α , and the chamber Δ_α is its moment polytope. $\square$

Now we pass to the G quotient.

DEFINITION 2.20. — We consider the following Lagrangians obtained from $\mathbb{L} = \mathbb{L}_n$:

- We write $\mathbb{L}_G := \mathbb{L}/G \subset (\mu^{\mathbb{C}})^{-1}(0)/G = T^*(\mathbb{C}^n/G)$ for the Lagrangian substack of $T^*(\mathbb{C}^n/G)$.
- We write $\mathbb{L}_G(t, -) := (\mathbb{L} \cap \mu^{-1}(t, 0, 0))/G_c \subset \mathfrak{X}_G(t)$. Equivalently, we can write this Lagrangian as the intersection $\mathbb{L}_G \cap \mathfrak{X}_G(t)$, taken in $T^*(\mathbb{C}^n/G)$ using the embedding of Corollary 2.11.
- For $\alpha \in 2^{[n]}$, we write $\mathbb{L}_G^\alpha(t, -) := (\mathbb{L}^\alpha \cap \mu^{-1}(t, 0, 0))/G_c \subset \mathfrak{X}_G(t)$.

From Lemma 2.19 we can see that $\mathbb{L}_G^\alpha(t, -)$ will be nonempty precisely when $\alpha \in \mathcal{F}$ is a feasible sign vector, so that $\mathbb{L}_G(t, -)$ may be written

$$\mathbb{L}_G(t, -) = \bigcup_{\alpha \in \mathcal{F}} \mathbb{L}_G^\alpha(t, -).$$

For $\alpha \in \mathcal{F}$, we can describe the Lagrangian $\mathbb{L}_G^\alpha(t, -)$ explicitly. Observe that the manifold $\mathfrak{X}_G(t)$, obtained from $\mathbb{H}^n$ through hyperkähler reduction by $G_c \subset D_c$, has a residual trihamiltonian action of $F_c = D_c/G_c$, from which it has a real moment map

$$(2.4) \quad \mu^{\mathbb{R}} : \mathfrak{X}_G(t) \longrightarrow \mathfrak{f}_{\mathbb{R}}^\vee(t) \subset \mathfrak{d}_{\mathbb{R}}^\vee,$$

whose codomain carries the hyperplane arrangement $\mathcal{H}(t)$.

PROPOSITION 2.21. — For $\alpha \in \mathcal{F}$, the Lagrangian $\mathbb{L}_G^\alpha(t, -)$ is a toric variety, and the restriction of (2.4) is a moment map for its dense F_c action, with image given by the polytope $\Delta_\alpha \cap \mathfrak{f}_{\mathbb{R}}^\vee(t)$ in the hyperplane arrangement $\mathcal{H}(t)$.

Proof. — The assertions of the proposition all follow from the fact that $\mathbb{L}_G^\alpha(t, -)$ may be identified with the G_c -Hamiltonian reduction of the variety $\mathbb{L}^\alpha \simeq \mathbb{C}^n$ at parameter t , compatibly with the F_c -action. $\square$

We have not yet taken into account the mass parameter $m \in \mathfrak{f}_\mathbb{R}$, which we now use to single out certain components of the Lagrangian $\mathbb{L}_G(t, -)$.

DEFINITION 2.22. — *The category $\mathcal{O}$ skeleton is the Lagrangian*

$$\mathbb{L}_G(t, m) := \bigcup_{\alpha \in \mathcal{F} \cap \mathcal{B}} \mathbb{L}_G^\alpha(t, -) \subset \mathfrak{X}_G(t)$$

given by those components of $\mathbb{L}_G(t, -)$ corresponding to bounded sign vectors α .

Observe that if the mass parameter is integral, it can be understood as a cocharacter $m : \mathbb{C}^\times \rightarrow F$ of the torus F which acts on $\mathfrak{X}_G(t)$, determining an action of a 1-dimensional torus $\mathbb{C}_m^\times$ on $\mathfrak{X}_G(t)$; in the non-integral case, it still specifies a complex vector field on $\mathfrak{X}_G(t)$. To simplify some statements below, we will assume the mass parameter is integral, although the generalization to the non-integral case should be clear. The action of the torus $\mathbb{C}_m^\times$ on $\mathfrak{X}_G(t)$ allows us to give a geometric meaning to the category $\mathcal{O}$ skeleton:

LEMMA 2.23 ([BD00, Section 6.5], [BLPW12, Proposition 5.5]). — *The category $\mathcal{O}$ skeleton $\mathbb{L}_G(t, m)$ is equal to the stable set*

$$\{x \in \mathfrak{X}_G(t) \mid \lim_{z \rightarrow 0} z \cdot x \text{ exists}\}$$

for the action of $z \in \mathbb{C}_m^\times$.

In Section 2.6, we will explain how the geometry of the $\mathbb{C}_m^\times$ action may be used to reinterpret $\mathbb{L}_G(t, m)$ as the relative skeleton for a Lefschetz fibration.

Remark 2.24. — An explicit cocore to $\mathbb{L}_G^\alpha(t, m)$ was described physically in [BDGH16, Sections 6.2.3 & 6.4.3] and using the BFN construction in [HKW23, Section 6.6]. This cocore is a section of the complex moment map $\mu^\mathbb{C} : \mathfrak{X}_G(t) \rightarrow \mathfrak{f}_\mathbb{C}^\vee$ that is expected to encode the quantum cohomology of the corresponding component of the 3d mirror Lagrangian skeleton [DGGH20, Tel21]. A geometric flow relating a quantization of this cocore to a projective module in $\mathcal{O}^{\text{dR}}(t, m)$ was described in [Hil16].

For future use, we record a partial ordering which plays a major role in the study of category $\mathcal{O}$.

DEFINITION 2.25. — *Let $\alpha, \beta \in \mathcal{F} \cap \mathcal{B} \subset 2^{[n]}$ be bounded and feasible sign vectors. By feasibility, they correspond to chambers $\Delta_\alpha, \Delta_\beta$ in the hyperplane arrangement $\mathcal{H}(t)$. By boundedness, the restriction of the functional m to each chamber takes a maximal value on some vertex, which we denote $p_\alpha \in \Delta_\alpha$ and $p_\beta \in \Delta_\beta$, respectively. Then we define a partial order on $\mathcal{F} \cap \mathcal{B}$ by declaring that $\alpha \leq \beta$ precisely when $m(p_\alpha) \leq m(p_\beta)$.*

2.5. Liouville geometry

An excellent source for the Liouville geometry of symplectic resolutions is [Živ22]. The following is an elaboration of the Remark following [She25, Theorem 7].

LEMMA 2.26. — *Let $\mathfrak{X}$ be a hyperkähler manifold which is equipped with an I -holomorphic action $\mathbb{C}^\times \curvearrowright \mathfrak{X}$ of $\mathbb{C}^\times$, such that for all $x \in \mathfrak{X}$, the limit $\lim_{\lambda \rightarrow 0} \lambda \cdot x$ exists.*

- (1) *Suppose the holomorphic symplectic form Ω_I has positive weight for this action. Then up to rescaling, the vector field generated by the action of $\mathbb{R}_{>0} \subset \mathbb{C}^\times$ is a Liouville vector field on $\mathfrak{X}$ (for real symplectic form ω_J).*
- (2) *Suppose moreover that the $S^1 \subset \mathbb{C}^\times$ action is Hamiltonian for the Kähler form ω_I . Then the Hamiltonian function underlies a (generalized, i.e., Morse–Bott type) Weinstein structure on $(\mathfrak{X}, \omega_J)$.*

Proof. — The assumption that the symplectic form has positive weight for the symplectic form guarantees that the vector field for the $\mathbb{R}_{>0}$ action scales it by a positive factor, so that some rescaling V of this vector field satisfies the Liouville condition $\mathcal{L}_V(\Omega_I) = \Omega_I$. This establishes (1).

For (2), observe that holomorphicity of the $\mathbb{C}^\times$ action means that the vector field for the S^1 action may be written as $I \cdot V$, where V is the Liouville vector field described above. Using the relation $\omega_I(-, I \cdot V) = g(-, -)$ where g is the hyperkähler metric, we conclude that V agrees with the gradient vector field for a Hamiltonian function for $I \cdot V$. Since this function is the moment map for a $\mathbb{C}^\times$ -action, it is Morse–Bott. $\square$

To equip $\mathfrak{X}_G(t)$ with a Weinstein structure, it will suffice to construct a $\mathbb{C}^\times$ action satisfying the hypotheses of Lemma 2.26. To this end: let $\mathbb{S} \simeq \mathbb{C}^\times$ be the 1-dimensional torus which acts on $T^*\mathbb{C}^n \simeq \mathbb{C}^n \times \mathbb{C}^n$ by scaling each factor with weight 1. The action of $\mathbb{S}$ on $T^*\mathbb{C}^n$ descends to an action on $\mathfrak{X}_G(t)$.

PROPOSITION 2.27. — *The induced action $\mathbb{S} \curvearrowright \mathfrak{X}_G(t)$ satisfies the hypotheses of Lemma 2.26.*

Proof. — The condition that the action of $\mathbb{S}$ on $\mathfrak{X}_G(t)$ dilates the holomorphic symplectic form, and that $S^1 \subset \mathbb{S}$ is Hamiltonian for the real Kähler form, follow from the analogous facts about the action of $\mathbb{S}$ on $T^*\mathbb{C}^n$. $\square$

Recall that the *skeleton* of a Liouville manifold $\mathfrak{X}$ is the stable set for the Liouville vector field, i.e., the set of points which do not escape to infinity under Liouville flow. If $\mathfrak{X}$ is moreover assumed to be Weinstein, then this is the set of points $x \in \mathfrak{X}$ for which the limit $\lim_{s \rightarrow \infty} \phi^s(x)$ under the Liouville flow exists.

LEMMA 2.28 ([BLPW10, Proposition 5.5]). — *The skeleton of $\mathfrak{X}_G(t)$ is the union of compact components in the extended core $\mathbb{L}_G(t, -)$.*

Proof. — There is an $\mathbb{S}$ -equivariant map $\nu : \mathfrak{X}_G(t) \rightarrow \mathfrak{X}_G(0)$, and the latter is an affine variety with a single fixed point o . The skeleton of $\mathfrak{X}_G(t)$ is the preimage $\nu^{-1}(0)$, which is the union of compact components of $\mathbb{L}_G(t, -)$ by [BD00, Theorem 6.5]. $\square$

2.6. A Lefschetz fibration

In this section, which is not used in the main results of our paper, we record an alternative perspective on the category $\mathcal{O}$ skeleton $\mathbb{L}_G(t, m)$ as the relative skeleton for a Lefschetz fibration on the hypertoric variety $\mathfrak{X}_G(t)$. Recall that we write $\mathbb{C}_m^\times$ for the 1-dimensional torus which acts on $\mathfrak{X}_G(t)$ through the cocharacter given by the mass parameter $m : \mathbb{C}^\times \rightarrow F$.

DEFINITION 2.29. — We will write $W_J^m : \mathfrak{X}_G(t) \rightarrow \mathbb{C}$ for the J -holomorphic moment map (for the holomorphic symplectic form Ω_J) for the $\mathbb{C}_m^\times$ -action on $\mathfrak{X}_G(t)$.

PROPOSITION 2.30. — The function W_J^m is a Lefschetz fibration. Its critical points are the set of 0-strata in $\mathbb{L}_t$.

Proof. — The critical points of the $\mathbb{C}_m^\times$ -moment map W_J^m are the fixed points of the $\mathbb{C}_m^\times$ action on $\mathfrak{X}$. By regularity of m , these are the same as the fixed points of the torus F , which are precisely the 0-strata in $\mathbb{L}_t$.

Let $p \in \mathbb{L}_t$ be such a critical point, corresponding to a $\dim(F)$ -dimensional stratum in $\mathbb{L} \subset T^*\mathbb{C}^n$, of the form $((\mathbb{C}^\times)^{\dim(G)} \times \{0\}^{n-\dim(G)}) \subset \mathbb{C}^n \subset T^*\mathbb{C}^n$. This stratum has a neighborhood $U_p = T^*(\mathbb{C}^\times)^{\dim(G)} \times \mathbb{C}^{n-\dim(G)}$; the image of this neighborhood under hyperkähler reduction gives a holomorphic Darboux chart $\bar{U}_p \subset \mathfrak{X}_G(t)$ for the critical point p , $\bar{U}_p \simeq T^*V$, where $V \simeq \mathbb{C}^{\dim(F)}$ is the F -representation with weights $(1, \dots, 1)$. In these coordinates, the map W_J^m becomes $(x_1, \dots, x_{\dim(F)}, y_1, \dots, y_{\dim(F)}) \mapsto \sum x_i y_i$. $\square$

For the rest of this section, fix $N \in \mathbb{R}$ with $N \gg 0$.

DEFINITION 2.31. — We will write $\mathfrak{F} := (W_J^m)^{-1}(N) \subset \mathfrak{X}_G(t)$ for the fiber of this Lefschetz fibration. We will understand it as a Liouville manifold, equipped with the restriction of Liouville structure λ_J .

We will want to think of $\mathfrak{F}$ as the fiber of W_J^m “at infinity”. Observe that the real part $\operatorname{Re} W_J^m$ of the function W_J^m is the real moment map, with respect to symplectic form ω_I , for the S_m^1 action.

DEFINITION 2.32. — Let $C \subset \mathfrak{X}_G(t)$ be the real hypersurface

$$C := (\operatorname{Re} W_J^m)^{-1}(N).$$

LEMMA 2.33. — The skeleton of the fiber $\mathfrak{F}$ is precisely the intersection $\mathbb{L}_{t,m} \cap C$.

Proof. — Observe that the cocharacter m determines a Hamiltonian S^1 -action on the symplectic manifold $\mathfrak{F}$, and the Hamiltonian reduction $\mathfrak{F} // S^1$ is canonically isomorphic, as a hyperkähler manifold with $\mathbb{S}$ -action (and hence in particular as a real Liouville manifold, where the Liouville structure on $\mathfrak{F} // S^1$ is descended from $\mathfrak{F}$), to the hypertoric variety associated to the torus $\tilde{G} := p^{-1}(\mathbb{C}_m^\times) \subset D$, where $p : D \rightarrow F$ is the projection from (2.3), with stability parameter (t, N) . By Lemma 2.28, the skeleton of $\mathfrak{F} // S^1$ is therefore the image under Hamiltonian reduction of the components of $\mathbb{L}_t$ which become compact and nonempty after this Hamiltonian reduction. These are precisely the noncompact m -bounded chambers in $\mathbb{L}_G(t, -)$, or in other words the noncompact components in $\mathbb{L}_G(t, m)$. $\square$

Example 2.34. — Consider Figure 1.3 in Section 1.7. Under the real moment map, the fiber $\mathfrak{F}$ projects to the dotted line in the figure. Its skeleton projects to the intersection of this line with the three colored noncompact chambers. The skeleton of $\mathfrak{F}/S^1$ is thus seen to consist of three $\mathbb{P}^1$'s, this manifold being the hypertoric variety associated to the induced hyperplane arrangement on the dotted line.

Remark 2.35. — The proof of Lemma 2.33 shows that a ribbon for the boundary of a hypertoric category $\mathcal{O}$ skeleton is given by a $\mathbb{C}^\times$ -bundle over another hypertoric variety one dimension lower. This suggests that it might be fruitful to search for similar interpretations of ribbons of boundaries of noncompact skeleta (e.g., nilpotent cones) occurring elsewhere in geometric representation theory.

In Section 1.5, we offer a reinterpretation of the Fukaya-categorical computations in this paper through the perspective of the above Lefschetz fibration. Also see Figure 1.3 in Section 1.7 for an illustration of the relation of the fiber $\mathfrak{F}$ to the category $\mathcal{O}$ skeleton.

3. Algebraic Preliminaries

In this section we collect some algebraic facts that we will use in our proof.

3.1. Idempotents and recollements

Let A be a differential graded $\mathbb{C}$ -algebra, and let $e \in A^0$ be an idempotent. The naive quotient algebra A/AeA is not well-behaved from a homotopical point of view. To remedy this, Braun–Chuang–Lazarev defined in [BCL18, Section 9] a derived quotient algebra $A/^L(AeA)$ and gave an explicit dg-model of this algebra by adapting the categorical dg-quotient construction from [Dri04]. We will take this dg-model as our definition of the derived quotient:

DEFINITION 3.1. — *The derived quotient of A by the ideal AeA is the dg-algebra*

$$A/^L AeA := (A\langle h \rangle / (eh = h = he), dh = e)$$

obtained by adjoining a degree-(-1) variable h , satisfying the relations $eh = h = he$, whose differential is equal to the idempotent e .

PROPOSITION 3.2 ([BCL18, Remark 9.5], [HKLY17, Section 2.1]). — *There is a recollement diagram*

$$(3.1) \quad \begin{array}{ccccc} & & i^* & & j_! \\ & \swarrow & & \searrow & \\ \mathrm{Mod}_{(A/^L AeA)} & \xleftarrow{i_* = i_!} & \mathrm{Mod}_A & \xrightarrow{j^! = j^*} & \mathrm{Mod}_{eAe} \\ & \nwarrow & & \nearrow & \\ & & i_! & & j_* \end{array}$$

of stable categories where

$$\begin{aligned} i^* &:= (-) \otimes_A A/^L AeA, & j_! &:= (-) \otimes_{eAe} eA, \\ i_* &:= \operatorname{hom}_{A/^L AeA}(A/^L AeA, -), & j^! &:= \operatorname{hom}_A(eA, -), \\ i_! &:= (-) \otimes_{A/^L AeA} A/^L AeA, & j^* &:= (-) \otimes_A Ae, \\ i^! &:= \operatorname{hom}_A(A/^L AeA, -), & j_* &:= \operatorname{hom}_{eAe}(Ae, -). \end{aligned}$$

In other words, the following properties hold:

- (i) $(i^*, i_*, i^!)$ and $(j_!, j^*, j_*)$ are adjoint triples,
- (ii) $i_* = i_!, j_*, j_!$ are fully faithful,
- (iii) $j^* \circ i_* = 0$ (hence by adjointness $i^* j_! = 0$ and $i^! j_* = 0$),
- (iv) for all $X \in \mathbf{Mod}_A$, the unit can be completed to an exact triangle

$$i_* i^! X \longrightarrow X \longrightarrow j_* j^* X \longrightarrow i_* i^! X[1]$$

and the counit can be completed to an exact triangle

$$j_! j^* X \longrightarrow X \longrightarrow i_* i^* X \longrightarrow j_! j^* X[1].$$

When A is an ordinary algebra, the recollement (3.1) appeared in the work of Cline, Parshall, and Scott [CPS88]. Rather than using the derived quotient, they assumed an extra hypothesis which guarantees that the derived quotient agrees with the naive one.

DEFINITION 3.3. — *Let A be an algebra and let $e \in A$ be an idempotent. The ideal AeA is stratifying if any of the following equivalent conditions holds:*

- (1) $Ae \otimes_{eAe} eA \simeq H^0(Ae \otimes_{eAe} eA) = AeA$.
- (2) The functor $\tilde{i}_! : \mathbf{Mod}_{A/AeA} \rightarrow \mathbf{Mod}_A$ defined by $\tilde{i}_! := (-) \otimes_{A/AeA} A/AeA$ is fully faithful.
- (3) $A/^L AeA \simeq H^0(A/^L AeA) = A/AeA$.

When this is the case, we will not distinguish between $i_!$ and $\tilde{i}_!$.

Remark 3.4. — There is also a notion of a recollement diagram for abelian categories. When A is an ordinary algebra, the diagram obtained from (3.1) by taking hearts of the standard t-structures is always a recollement of *abelian* categories, regardless of whether or not AeA is stratifying.

3.2. Koszul duality

Unlike [BGS96, BLPW10], which consider mixed Koszul duality between ordinary graded algebras, we will be interested in derived Koszul duality between differential graded algebras or, more generally, associative ring spectra [DGI06], [Lur18, Section 14].

Remark 3.5. — In this section we will need to distinguish between left and right modules. Our convention is that endomorphisms of a right module act on the left and vice versa.

Let $S = (\mathbb{C})^{\times n}$. An *augmented differential graded S -ring* is a differential graded k -algebra A equipped with $\mathbb{C}$ -algebra homomorphisms $S \rightarrow A$ and $A \rightarrow S$ such that the composition $S \rightarrow A \rightarrow S$ is the identity. The canonical example of an augmented dg S -ring is the *tensor algebra* $T_S(V) = \bigoplus_{n \geq 0} V^{\otimes_S n}$ of a finite dimensional differential graded S -bi-module V . Note that S is not required to be central in A . A reference where standard results on Koszul duality for augmented dg k -algebras are extended to dg S -rings is [HLW23].

Suppose that we have two augmented dg S -rings A and B^{op} such that there exists an augmentation $A \otimes_{\mathbb{C}} B^{\text{op}} \rightarrow S$ of the tensor product algebra extending the augmentations on A and B^{op} . This is equivalent to giving maps

$$(3.2) \quad B \longrightarrow \text{hom}_A(S, S),$$

$$(3.3) \quad A \longrightarrow \text{hom}_B(S, S),$$

where we are considering S as a right A -module and a left B -module.

DEFINITION 3.6. — We say that B is Koszul dual to A if (3.2) is an isomorphism. We say that A is Koszul dual to B if (3.3) is an isomorphism. When both of these conditions are satisfied, we will say that A and B are Koszul bidual.

DEFINITION 3.7. — A differential graded algebra A is *connective* if $H^k(A) = 0$ for all $k > 0$. It is *locally finite* if $H^k(A)$ is finite dimensional for all k and *finite* if $H^*(A)$ is finite dimensional.

PROPOSITION 3.8 ([Boo22, Theorem 4.2.8, Section 8.1]). — Suppose that A is connective, locally finite, and that the augmentation ideal of $H^0(A) \rightarrow S$ is nilpotent. Then $A \simeq \text{hom}_{\text{hom}_A(S, S)}(S, S)$. In particular, Koszul duality is an involution.

Remark 3.9. — For a different finiteness condition that also ensures that Koszul duality implies Koszul biduality see [Lur18, Section 14.1.3].

Remark 3.10. — In general, one expects the double Koszul dual of A to be the derived completion of A at the augmentation ideal, as explained in [Efi10, PSY14]. For example, the Koszul dual of $\mathbb{C}[x]$ with $|x| = 0$ is $\mathbb{C}[\epsilon]$ with $|\epsilon| = 1$, and taking the Koszul dual again gives $\mathbb{C}[[x]]$. One way to avoid this difficulty is to impose the finiteness hypotheses of Proposition 3.8; another approach is to work in the mixed setting, since the completion of $\mathbb{C}[x]$ as a graded algebra is equivalent to $\mathbb{C}[x]$. By applying the degrading functor to the mixed equivalence one gets [Che23, Section 2].

The augmentation $A \otimes_{\mathbb{C}} B^{\text{op}} \rightarrow S$ gives rise to contravariant functors

$$(3.4) \quad \text{RMod}_A \begin{array}{c} \xrightarrow{\text{hom}_A(-, S)} \\ \xleftarrow{\text{hom}_B(-, S)} \end{array} \text{LMod}_B$$

which are adjoint on the right, in the sense that there are natural equivalences

$$(3.5) \quad \text{hom}_A(M, \text{hom}_B(N, S)) \simeq \text{hom}_{A \otimes_{\mathbb{C}} B^{\text{op}}}(M \otimes_{\mathbb{C}} N, S) \simeq \text{hom}_B(N, \text{hom}_A(M, S)).$$

Remark 3.11. — To speak of the left or right adjoint of a contravariant functor $F : \mathcal{C} \rightarrow \mathcal{D}$ one needs to decide whether to view it as a covariant functor $\mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$ or $\mathcal{C} \rightarrow \mathcal{D}^{\text{op}}$. A left adjoint from one perspective is a right adjoint from the other.

On the other hand, saying that two contravariant functors are adjoint on the left or adjoint on the right is unambiguous.

When A is Koszul dual to B or vice versa, we will see that the Koszul duality functors (3.4) restrict to equivalences between certain subcategories of $\mathbf{RMod}_A$ and $\mathbf{LMod}_B$.

DEFINITION 3.12. — *A subcategory of a stable category $\mathcal{C}$ is thick if it is a stable subcategory that is closed under retracts. For X an object of $\mathcal{C}$, we write $\mathcal{T}_{\mathcal{C}}(X)$ for the smallest thick subcategory of $\mathcal{C}$ containing X .*

Remark 3.13. — The category $\mathcal{T}_{\mathbf{RMod}_A}(A)$ (resp., $\mathcal{T}_{\mathbf{LMod}_B}(B)$) is also known as the category of perfect modules and denoted by $\mathbf{RPerf}_A$ (resp. $\mathbf{LPerf}_B$).

The following proposition is proved under the assumption of Koszul biduality in [Cam14]. But by examining the proof, it is easy to see that one can split the result. Also see [Lur18, Section 14.6] for a similar result after ind-completion.

PROPOSITION 3.14 ([Cam14, Theorem 2.43]). — *If B is Koszul dual to A (resp., A is Koszul dual to B) the functors (3.4) restrict to an equivalence $\mathcal{T}_{\mathbf{RMod}_A}(S) \simeq \mathbf{LPerf}_B$ (resp., $\mathbf{RPerf}_A \simeq \mathcal{T}_{\mathbf{LMod}_B}(S)$).*

Now suppose that $e \in S$ is an idempotent and $e^c = 1 - e$ is the complementary idempotent. Let $T = e^c S e^c \simeq S / S e S$. The derived quotient $A/L(AeA)$ and the cornering⁽³⁾ $(e^c B e^c)^{\mathrm{op}}$ are both T -augmented. Moreover, the augmentation $A \otimes_S B^{\mathrm{op}} \rightarrow S$ induces an augmentation $A/L(AeA) \otimes_T (e^c B e^c)^{\mathrm{op}} \rightarrow T$, so one can ask whether $A/L(AeA)$ and $e^c B e^c$ are Koszul dual.

First, recall the recollement (3.1) associated to $e \in A$. There is an analogous recollement associated to $e^c \in B$ with functors $(i^c)^* \dashv (i^c)_* = (i^c)_! \dashv (i^c)^!$ and $(j^c)_! \dashv (j^c)^! = (j^c)^* \dashv (j^c)_*$.

PROPOSITION 3.15. —

- (1) *Suppose B is Koszul dual to A . Then $e^c B e^c$ is Koszul dual to $A/L(AeA)$ and the following diagram commutes:*

$$(3.6) \quad \begin{array}{ccc} \mathcal{T}_{\mathbf{RMod}_A}(S) & \xrightarrow{\mathrm{hom}_A(-, S)} & (\mathbf{LPerf}_B)^{\mathrm{op}} \\ \uparrow i_* & & \uparrow ((j^c)_!)^{\mathrm{op}} \\ \mathcal{T}_{\mathbf{RMod}_{A/L(AeA)}}(T) & \xrightarrow{\mathrm{hom}_{A/L(AeA)}(-, T)} & (\mathbf{LPerf}_{e^c B e^c})^{\mathrm{op}}. \end{array}$$

- (2) *Suppose A is Koszul dual to B . Then $A/L(AeA)$ is Koszul dual to $e^c B e^c$ if and only if the following diagram commutes:*

$$(3.7) \quad \begin{array}{ccc} \mathbf{RPerf}_A & \xleftarrow{\mathrm{hom}_B(-, S)} & (\mathcal{T}_{\mathbf{LMod}_B}(S))^{\mathrm{op}} \\ \downarrow i^* & & \downarrow ((j^c)_!)^{\mathrm{op}} \\ \mathbf{RPerf}_{A/L(AeA)} & \xleftarrow{\mathrm{hom}_{e^c B e^c}(-, T)} & (\mathcal{T}_{\mathbf{LMod}_{e^c B e^c}}(T))^{\mathrm{op}}. \end{array}$$

⁽³⁾This terminology is taken from [CIK18] by way of [Boo22].

Proof.

(1) It is sufficient to check commutativity on generators. First, notice that

$$\mathrm{hom}_A(i_*(T), S) \simeq \mathrm{hom}_A(e^c S, S) \simeq \mathrm{hom}_A(S, S)e^c \simeq Be^c \simeq (j^c)_!(e^c Be^c).$$

Thus, (3.6) commutes if and only if $e^c Be^c$ is Koszul dual to $A/L(AeA)$. But since i_* is fully faithful, we have

$$\mathrm{hom}_{A/L(AeA)}(T, T) \simeq \mathrm{hom}_A(i_*(T), i_*(T)) \simeq \mathrm{hom}_B(Be^c, Be^c) \simeq e^c Be^c.$$

(2) Diagram (3.7) commutes if and only if

$$A/L(AeA) \simeq i^*(A) \simeq i^* \mathrm{hom}_B(S, S) \simeq \mathrm{hom}_{e^c Be^c}((j^c)^!(S), T) \simeq \mathrm{hom}_{e^c Be^c}(T, T). \quad \square$$

COROLLARY 3.16. — *Suppose B is Koszul dual to A and $e \in S$ is an idempotent. Assume $A/L(AeA)$ is connective, locally finite, and that the augmentation ideal of $H^0(A/L(AeA)) \rightarrow T$ is nilpotent. In this case, $A/L(AeA)$ and $e^c Be^c$ are Koszul bidual.*

COROLLARY 3.17. — *Suppose A and B are Koszul bidual and $e \in S$ is an idempotent. If the diagram*

$$(3.8) \quad \begin{array}{ccc} \mathrm{RMod}_A & \xrightarrow{\mathrm{hom}_A(-, S)} & (\mathrm{LMod}_B)^{\mathrm{op}} \\ i_* \uparrow & & \uparrow ((j^c)_!)^{\mathrm{op}} \\ \mathrm{RMod}_{A/L(AeA)} & \xrightarrow{\mathrm{hom}_{A/L(AeA)}(-, T)} & (\mathrm{LMod}_{e^c Be^c})^{\mathrm{op}} \end{array}$$

commutes, then the differential graded algebras $A/L(AeA)$ and $e^c Be^c$ are Koszul bidual.

Proof. — If (3.8) commutes then so does the diagram obtained by passing to left adjoints. The resulting diagram restricts to (3.7). Note that $(-)^{\mathrm{op}}$ swaps left and right adjoints. $\square$

Remark 3.18. — Under certain finiteness conditions, the converse to Corollary 3.17 is true in the mixed setting [BLPW12, Theorem 8.23]. In the derived setting it is sufficient to assume that $\mathrm{RPerf}_A \simeq \mathcal{T}_{\mathrm{RMod}_A}(S)$ and $\mathrm{RPerf}_{A/L(AeA)} \simeq \mathcal{T}_{\mathrm{RMod}_{A/L(AeA)}}(T)$. Is this necessary?

Now assume that $A = A^0$ is concentrated in cohomological degree 0 and that it is equipped with an additional grading $A_\bullet$ (conventionally referred to as the *mixed grading* to distinguish it from the cohomological grading) such that the augmentation induces an isomorphism $A_0 \simeq S$. We will also assume that A is *mixed co-connective* and *mixed locally finite*, i.e., that $A_i = 0$ for $i < 0$ and A_i is finite dimensional for all $i \geq 0$. Priddy discovered a class of such graded algebras whose Koszul duals are especially easy to describe [Pri70]. These were later studied by Beilinson, Ginzburg, and Soergel [BGS96].

DEFINITION 3.19. — *An S -augmented graded algebra A is Koszul if it satisfies the graded Tor-vanishing condition $H^i(S \otimes_A S)_j = 0$ for $i \neq -j$.*

DEFINITION 3.20. — *The S -augmented graded algebra A is quadratic if there exists a finite dimensional S -bi-module V and a subspace $R \subseteq V \otimes V$ such that $A \simeq T_S(V/\langle R \rangle)$. The quadratic dual of A is the graded algebra $Q(A) = T_S(V^\vee)/\langle R^\perp \rangle$. Here both V and $V^\vee$ are in mixed degree 1.*

Remark 3.21. — Note that $Q(A)$ is also mixed locally finite and mixed co-connective.

PROPOSITION 3.22. —

- (1) Every Koszul algebra is quadratic [BGS96, Proposition 1.2.3].
- (2) Quadratic duality is involutive, i.e., $A \simeq Q(Q(A))$. Moreover, $Q(A)$ is Koszul if and only if A is [BGS96, Theorem 1.2.5].

The following proposition compares mixed Koszul duality of graded algebras as studied in [BGS96] with Koszul duality of differential graded algebras.

PROPOSITION 3.23 ([Bra19, Theorem 7.12]). — *If A is Koszul then its Koszul dual is the dg-algebra $A^! = (Q(A)^{\text{op}}, 0)$, where the mixed grading on $Q(A)^{\text{op}}$ becomes the homological grading on $A^!$. In particular, we see that $\text{hom}_A(S, S)$ is formal.*

Remark 3.24. — The Koszul dual of the algebra $Q(A)^{\text{op}}$ is the dga $(A, 0)$ but the Koszul dual of the dga $A^! = (Q(A)^{\text{op}}, 0)$ is only the algebra A when A is finite dimensional. In general the Koszul dual of $A^!$ is the derived completion as described in Remark 3.10.

Now we will use Koszul duality to give a criterion for proving that $A/AeA \simeq A/L(AeA)$. The key observation, Proposition 3.26, is a slight variant of [BLPW12, Proposition 8.23].

LEMMA 3.25 ([BLPW12, Lemma 8.22]). — *Suppose that A is a quadratic algebra and $e \in A$ is an idempotent such that $e^c Q(A) e^c \subseteq Q(A)$ is quadratic as a $T = e^c S e^c$ -augmented algebra. Then $Q(A/(AeA)) \simeq e^c Q(A) e^c$. In particular, if A/AeA is Koszul as a T -augmented algebra its Koszul dual is $e^c A^! e^c$.*

PROPOSITION 3.26. — *Suppose that A is Koszul and $e \in A$ is an idempotent such that $e^c Q(A) e^c$ is quadratic and $A/(AeA)$ is Koszul. Then there is a commutative diagram*

$$(3.9) \quad \begin{array}{ccc} \mathcal{T}_{\text{RMod}_A}(S) & \xrightarrow{\text{hom}_A(-, S)} & (\text{LPerf}_{A^!})^{\text{op}} \\ i_* \uparrow & & \uparrow ((j^c)_!)^{\text{op}} \\ \mathcal{T}_{\text{RMod}_{A/(AeA)}}(T) & \xrightarrow{\text{hom}_{A/(AeA)}(-, T)} & (\text{LPerf}_{e^c A^! e^c})^{\text{op}}. \end{array}$$

In particular, i_* is fully faithful since $(j^c)_!$ is.

Proof. — The same argument that proved commutativity of (3.6) (with $B = A^!$ and $A/(AeA)$ in place of $A/L(AeA)$) proves commutativity of (3.9). In particular, we know that $i_*(T) = e^c S$ and we know that $\text{hom}_{A/(AeA)}(T, T) \simeq e^c A^! e^c$ by Lemma 3.25. $\square$

COROLLARY 3.27. — Suppose that A is Koszul and $e \in A$ is an idempotent such that $e^c Q(A) e^c$ is quadratic and $A/(AeA)$ is Koszul. Moreover, assume that $A/(AeA) \in \mathcal{T}_{\mathrm{RMod}_{A/(AeA)}}(T)$ and $S \in \mathrm{RPerf}_A$. Then $A/(AeA) \simeq A^L(AeA)$.

Proof. — By Definition 3.3 it suffices to show that $i_* : \mathrm{RMod}_{A/(AeA)} \rightarrow \mathrm{RMod}_A$ is fully faithful. By our assumptions $\mathrm{RPerf}_{A/(AeA)} \subseteq \mathcal{T}_{\mathrm{RMod}_{A/(AeA)}}(T)$ and $\mathcal{T}_{\mathrm{RMod}_A}(S) \subseteq \mathrm{RPerf}(A)$. Thus we see that i_* is the ind-completion of the fully faithful embedding

$$\mathrm{RPerf}_{A/(AeA)} \subseteq \mathcal{T}_{\mathrm{RMod}_{A/(AeA)}}(T) \xrightarrow{i_*} \mathcal{T}_{\mathrm{RMod}_A}(S) \subseteq \mathrm{RPerf}(A). \quad \square$$

3.3. Highest weight categories and semi-orthogonal decompositions

Let $\mathcal{C}$ be an abelian category. An object is *simple* if it has no subobjects. We say that $\mathcal{C}$ is Artinian (resp. Noetherian) if any descending (resp. increasing) sequence of subobjects stabilizes. Given an object $X \in \mathcal{C}$, a *composition series* is a finite sequence of subobjects $0 \rightarrow Y_1 \hookrightarrow \cdots \hookrightarrow Y_k \hookrightarrow X$ such that each subquotient is simple and nonzero.

It is straightforward to verify that an abelian category is Artinian and Noetherian iff every object admits a composition series. In that case, the Jordan–Hölder lemma implies that the multiset of subquotients is unique.

We now recall the notion of a highest weight category, following [BLPW16]. For more details, we refer to [CPS88] (noting that the definition there is slightly more general than that used in [BLPW16]).

DEFINITION 3.28 ([BLPW16, Definition 4.11]). — Let $\mathcal{C}$ be a $\mathbb{C}$ -linear category which is abelian and artinian. Let $\{S_\alpha\}_{\alpha \in \mathcal{I}}$ and $\{P_\alpha\}_{\alpha \in \mathcal{I}}$ be an enumeration of the simples and their projective covers and fix a partial order on $\mathcal{I}$. We say that $\mathcal{C}$ is highest weight with respect to the poset $\mathcal{I}$ if there is a collection of objects $\{V_\alpha\}_{\alpha \in \mathcal{I}}$ and epimorphisms $\{P_\alpha \xrightarrow{\Pi_\alpha} V_\alpha \xrightarrow{\pi_\alpha} S_\alpha\}$ so that for each α ,

- $\ker \pi_\alpha$ has a finite filtration whose subquotients are isomorphic to S_β for some $\beta < \alpha$, and
- $\ker \Pi_\alpha$ has a finite filtration whose subquotients are isomorphic to V_γ for some $\gamma > \alpha$.

The V_α are called standard objects.

DEFINITION 3.29. — An algebra is quasi-hereditary if its category of finitely generated right modules is highest weight, for some ordering on the simple modules.

Remark 3.30. — Let $\mathcal{C}$ be a highest-weight category. Then the $\{S_\alpha\}, \{V_\alpha\}, \{P_\alpha\}$ each form a basis for the Grothendieck group $K_0(\mathcal{C})$; dually, so do the collections of indecomposable injectives $\{I_\alpha\}$, costandards $\{\Lambda_\alpha\}$, and tilting objects $\{T_\alpha\}$. Moreover, these objects satisfy *BGG reciprocity*: the multiplicity of V_α in P_β coincides with the multiplicity of S_β in V_α .

The feature of a highest-weight category $\mathcal{C}$ most salient to us here is the presence of a full exceptional collection. As shown in [Kra17], this is essentially equivalent to the highest weight structure.

THEOREM 3.31 ([DR92, Kra17]). — *Let $\mathcal{C}$ be a highest weight category. Then the standard objects V_α form a full exceptional collection: $\mathrm{Ext}^*(V_\alpha) = \mathbb{C} \cdot \mathrm{id}_{V_\alpha}$ for all α , and for $i > 0$, $\mathrm{Ext}^i(V_\alpha, V_\beta) = 0$ unless $\alpha < \beta$.*

Remark 3.32. — To a symplectic geometer, the above should indicate that quasi-hereditary algebras naturally arise as the endomorphism algebras of cocores in the Fukaya–Seidel category of a Lefschetz fibration, and that the structural properties their module categories enjoy admit geometric explanations, with Lefschetz thimbles playing the role of standard objects. It is indeed possible to understand category $\mathcal{O}$ in these terms; we will explain a heuristic dictionary in Section 1.5.

3.4. Betti and de Rham G -actions

In geometric representation theory it is common to consider several different notions of an action of a reductive group G on a category: any covariantly functorial sheaf theory SHV satisfying the Künneth formula gives rise to a convolution monoidal structure

$$\begin{aligned} \mathrm{SHV}(G) \otimes \mathrm{SHV}(G) &\simeq \mathrm{SHV}(G \times G) \xrightarrow{m_*} \mathrm{SHV}(G), \\ \mathrm{Vect} &\simeq \mathrm{SHV}(1) \xrightarrow{u_*} \mathrm{SHV}(G), \end{aligned}$$

and one can consider module categories for $(\mathrm{SHV}(G), m_*, u_*)$. For the sheaf theories we will consider, there is a duality $\mathrm{SHV}(G)^\vee \simeq \mathrm{SHV}(G)$ identifying $(m_*)^\vee$ and $(u_*)^\vee$ with contravariant functoriality $m^!$ and $u^!$. In particular, modules for $(\mathrm{SHV}(G), m_*, u_*)$ are equivalent to co-modules for $(\mathrm{SHV}(G), m^!, u^!)$. In this section we will collect some well-known facts about categories G -actions surveyed in [Dhi22, Gai15] and facts about the Mellin transform from [Gan22, Tel14].

DEFINITION 3.33. — *A Betti (resp., de Rham, weak) G -category is a module category in Pr^{L} for the monoidal category $(\mathrm{Loc}(G), m_*)$ (resp., $(\mathrm{Dmod}(G), m_*)$, $(\mathrm{QCoh}(G), m_*)$).*

Remark 3.34. — There is a monoidal functor $\mathrm{QCoh}(G) \rightarrow \mathrm{Dmod}(G)$ given by pushforward along the homomorphism $G \rightarrow G_{dR}$. (Concretely: the ring of functions on G is a subalgebra of the ring of differential operators, and the functor can be understood as extension of scalars.) As a result, every de Rham G -category (sometimes also called a *strong* G -category) is a weak G -category in a canonical way.

DEFINITION 3.35. — *Let $\mathcal{C}$ be a Betti (resp. de Rham, weak) G -category. Then the G -invariants $\mathcal{C}^{G_\bullet}$, and G -coinvariants $\mathcal{C}_{G_\bullet}$, for $\bullet \in \{\mathrm{Bet}, dR, w\}$, are the respective categories*

$$\mathcal{C}^{G_\bullet} := \mathrm{Hom}_{\mathrm{Mod}_{\mathrm{SHV}(G)}}(\mathrm{Vect}, \mathcal{C}), \quad \mathcal{C}_{G_\bullet} := \mathrm{Vect} \otimes_{\mathrm{SHV}(G)} \mathcal{C},$$

where Vect is the trivial representation and $\mathrm{SHV} \in \{\mathrm{Loc}, \mathrm{Dmod}, \mathrm{QCoh}\}$ is the appropriate sheaf theory.

PROPOSITION 3.36. — *Let $\mathrm{SHV} \in \{\mathrm{Loc}, \mathrm{Dmod}, \mathrm{QCoh}\}$. Then the trivial $\mathrm{SHV}(G)$ -module category Vect is self-dual. As a result, for $\bullet \in \{\mathrm{Bet}, dR, w\}$ and $\mathcal{C}$ a $\bullet$ G -category, there is an equivalence $\mathcal{C}^{G_\bullet} \simeq \mathcal{C}_{G_\bullet}$ between G -invariants and coinvariants.*

PROPOSITION 3.37. — Suppose that $\mathrm{SHV}(G)$ is self-dual with $f^! = (f_*)^\vee$. Then one can identify the standard semi-co-simplicial diagram computing invariants

$$\mathrm{Hom}_{\mathrm{Mod}_{\mathrm{SHV}(G)}}(\mathrm{Vect}, \mathcal{C}) \simeq \varprojlim (\mathrm{Hom}(\mathrm{Vect}, \mathcal{C}) \rightrightarrows \mathrm{Hom}(\mathrm{SHV}(G) \otimes \mathrm{Vect}, \mathcal{C}) \rightrightarrows \dots)$$

with the cobar complex for $\mathrm{SHV}(G)$ co-acting on $\mathcal{C}$

$$\mathcal{C} \rightrightarrows \mathrm{SHV}(G) \otimes \mathcal{C} \rightrightarrows \mathrm{SHV}(G)^{\otimes 2} \otimes \mathcal{C} \rightrightarrows \dots$$

We now specialize to the abelian setting:

Notation 3.38. — For the remainder of this section, we will restrict to the case that $G = (\mathbb{C}^\times)^k$ is a torus. We will write $G^L := \mathrm{Spec}(\mathbb{C}[\pi_1 G])$ for the (Langlands) dual torus.

In this case, we can identify G -categories in terms of linear structure on the dual torus. In the de Rham setting, this equivalence is known as the geometric Mellin transform: see [BZN18, Section 2.1], or [Gan22, Appendix A] for a derived enhancement. In the Betti setting, the statement is simpler: see [Tel14, Proposition 4.1].

PROPOSITION 3.39. — There are monoidal equivalences

$$(\mathrm{Loc}(G), m_*) \simeq (\mathrm{IndCoh}(G^L), \overset{!}{\otimes}) \simeq (\mathrm{QCoh}(G^L), \otimes),$$

$$(\mathrm{Dmod}(G), m_*) \simeq (\mathrm{IndCoh}(\mathfrak{g}^L/\pi_1(G^L)), \overset{!}{\otimes}) \simeq (\mathrm{QCoh}(\mathfrak{g}^L/\pi_1(G^L)), \otimes)$$

that are functorial for surjective homomorphisms $f : G \rightarrow H$, in the sense that the following diagrams commute:

$$\begin{array}{ccc} \mathrm{Loc}(G) & \xrightarrow{f_*} & \mathrm{Loc}(H) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{IndCoh}(G^L) & \xrightarrow{(f^L)^!} & \mathrm{IndCoh}(H^L) \end{array} \quad \begin{array}{ccc} \mathrm{Dmod}(G) & \xrightarrow{f_*} & \mathrm{Dmod}(H) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{IndCoh}(\mathfrak{g}^L/\pi_1(G)) & \xrightarrow{(Df^L)^!} & \mathrm{IndCoh}(\mathfrak{h}^L/\pi_1(H)) \end{array}$$

Remark 3.40. — The exponential map gives an isomorphism between the analytification of the stack $\mathfrak{g}^L/\pi_1(G^L)$ and G^L . This is expected to be compatible with the Riemann–Hilbert correspondence.

COROLLARY 3.41. — A Betti G -action on $\mathcal{C}$ is equivalent to a $\mathbb{C}[G^L]$ -linear structure on $\mathcal{C}$.

As homomorphism $p : G \rightarrow 1$ gives rise to the trivial G -representation, the operation of taking Betti (resp., de Rham) G -invariants coincides with pulling back along $p^L : 1 \rightarrow G^L$ (resp., $Dp^L : 0 = \pi_1(G^L)/\pi_1(G^L) \rightarrow \mathfrak{g}^L/\pi_1(G^L)$):

COROLLARY 3.42. — For $\mathcal{C}$ a Betti (resp. de Rham) G -category, there are equivalences

$$\mathcal{C}^{G_{\mathrm{Bet}}} \simeq \mathrm{IndCoh}(\{1\}) \bigotimes_{\mathrm{IndCoh}(G^L)} \mathcal{C} \quad \mathcal{C}^{G_{\mathrm{dR}}} \simeq \mathrm{IndCoh}(\{0\}) \bigotimes_{\mathrm{IndCoh}(\mathfrak{g}^L/\pi_1(G^L))} \mathcal{C}.$$

We can similarly use the linear structure to describe weak invariants as the pullback along $q : \mathfrak{g}^L \rightarrow \mathfrak{g}^L/\pi_1(G^L)$:

COROLLARY 3.43. — *For $\mathcal{C}$ a weak G -category, there is an equivalence*

$$\mathcal{C}^{G_w} \simeq \mathrm{IndCoh}(\mathfrak{g}^L) \bigotimes_{\mathrm{IndCoh}(\mathfrak{g}^L/\pi_1(G^L))} \mathcal{C}.$$

Remark 3.44. — The category $\mathcal{C}^{G_w}$ is equipped with compatible actions of

$$\left(\mathrm{IndCoh}(\mathfrak{g}^L), \overset{!}{\otimes}\right) \quad \text{and} \quad \left(\mathrm{IndCoh}(\pi_1(G^L)), m_*\right).$$

These combine into an action of the monoidal category of Harish-Chandra bimodules

$$(\mathrm{HC}(G), \star) = (\mathrm{Dmod}(G)^{(G \times G)_w}, m_*).$$

An $\mathrm{HC}(G)$ -action on $\mathcal{C}^w$ is equivalent data to a de Rham G -action on $\mathcal{C}$.

In other words, passing from a de Rham G -category $\mathcal{C}$ to its weak invariants does not lose any data, so long as we remember both the $\mathrm{IndCoh}(\mathfrak{g}^L)$ - and $\mathrm{IndCoh}(\pi_1(G^L))$ -actions. We will sometimes find it useful to forget part of this data:

DEFINITION 3.45. — *A $\mathfrak{g}$ -category is a module category for $(\mathrm{IndCoh}(\mathfrak{g}^L), \overset{!}{\otimes})$, or equivalently a category equipped with $(\mathrm{Sym}(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^L])$ -linear structure. The $\mathfrak{g}$ -invariants of a $\mathfrak{g}$ -category $\mathcal{C}$ are defined by the pullback*

$$\mathcal{C}^{\mathfrak{g}} := \mathrm{IndCoh}(\{0\}) \bigotimes_{\mathrm{IndCoh}(\mathfrak{g}^L)} \mathcal{C}.$$

We will also be interested in imposing a condition slightly weaker than G -invariance, namely G -monodromicity; as above, this is most easily defined in terms of linear structure for the dual group. Let $(G^L)_1^\wedge$ (resp., $(\mathfrak{g}/\pi_1(G^L))_0^\wedge$) denote the formal completion of G^L (resp., $\mathfrak{g}/\pi_1(G^L)$) along 1 (resp., 0).

DEFINITION 3.46. — *Let $\mathcal{C}$ be a Betti (resp. de Rham) G -category. The Betti (resp. de Rham) G -monodromic category is*

$$\begin{aligned} \mathcal{C}^{G_{\mathrm{Bet}}-\mathrm{mon}} &:= \mathrm{IndCoh}\left((G^L)_1^\wedge\right) \bigotimes_{\mathrm{IndCoh}(G^L)} \mathcal{C}, \\ \mathcal{C}^{G_{\mathrm{dR}}-\mathrm{mon}} &:= \mathrm{IndCoh}\left((\mathfrak{g}^L/\pi_1(G^L))_0^\wedge\right) \bigotimes_{\mathrm{IndCoh}(\mathfrak{g}/\pi_1(G^L))} \mathcal{C}. \end{aligned}$$

PROPOSITION 3.47. — *Consider $0 \rightarrow \mathfrak{g}^L \rightarrow \mathfrak{g}^L/\pi_1(G^L)$. There is an isomorphism $(\mathfrak{g}^L)_0^\wedge \simeq (\mathfrak{g}/\pi_1(G))_0^\wedge$. Thus, for a de Rham G -category $\mathcal{C}$, there are equivalences*

$$\begin{aligned} \mathcal{C}^{G_{\mathrm{dR}}} &\simeq (\mathcal{C}^{G,w})^{\mathfrak{g}} = \mathrm{IndCoh}(\{0\}) \bigotimes_{\mathrm{IndCoh}(\mathfrak{g})} \mathcal{C}^{G,w} \\ \mathcal{C}^{G_{\mathrm{dR}}-\mathrm{mon}} &\simeq (\mathcal{C}^{G,w})^{\mathfrak{g}-\mathrm{mon}} = \mathrm{IndCoh}((\mathfrak{g}^L)_0^\wedge) \bigotimes_{\mathrm{IndCoh}(\mathfrak{g})} \mathcal{C}^{G,w}. \end{aligned}$$

The following proposition was proved for formal completions of schemes in [GR14, Section 7.4]. The case of $\mathfrak{g}^L/\pi_1(G^L)$ follows by examining the cover $q : \mathfrak{g}^L \rightarrow \mathfrak{g}^L/\pi_1(G^L)$.

PROPOSITION 3.48. — *Let $Y \rightarrow X$ be one of $1 \rightarrow G^L$, $0 \rightarrow \mathfrak{g}$, or $0 \rightarrow \mathfrak{g}/\pi_1(G^L)$ and let $U = X - Y$. There is an exact sequence*

$$\mathrm{IndCoh}(X_Y^\wedge) \longrightarrow \mathrm{IndCoh}(X) \longrightarrow \mathrm{IndCoh}(U)$$

of $\mathrm{IndCoh}(X)$ -module categories. Moreover, we have that

$$\mathrm{IndCoh}(X_Y^\wedge) = \mathrm{IndCoh}\left(\varinjlim_k N_k(Y, X)\right) \simeq \varinjlim_k \mathrm{IndCoh}(N_k(Y, X)),$$

where $N_k(Y, X)$ is the k^{th} order neighborhood of Y in X .

Remark 3.49. — The image of the embedding $\mathrm{IndCoh}(X_Y^\wedge) \rightarrow \mathrm{IndCoh}(X)$ is the category $\mathrm{IndCoh}(X)_Y$ of ind-coherent sheaves on X which are set-theoretically supported on Y .

COROLLARY 3.50. — *Let $\mathcal{C}$ be a Betti G -category. Then*

$$\mathcal{C}^{G_{\mathrm{Bet}}-\mathrm{mon}} \simeq \varinjlim_k \mathrm{IndCoh}(N_k(1, G^L)) \bigotimes_{\mathrm{IndCoh}(G)} \mathcal{C}.$$

Moreover, if $\mathcal{C}$ is dualizable as an $\mathrm{IndCoh}(G^L)$ -module then one has an exact sequence

$$\mathcal{C}^{G_{\mathrm{Bet}}-\mathrm{mon}} \longrightarrow \mathcal{C} \longrightarrow \mathrm{IndCoh}(G^L - \{1\}) \bigotimes_{\mathrm{IndCoh}(G)} \mathcal{C}$$

in Pr^{L} . In particular, the functor $\mathcal{C}^{G_{\mathrm{Bet}}-\mathrm{mon}} \rightarrow \mathcal{C}$ is fully faithful.

The analogous statements hold for de Rham G -categories and $\mathfrak{g}$ -categories.

To apply the previous corollary, we will need the following fact.

PROPOSITION 3.51 ([GR17, Corollary 8.6.3]). — *Let $(\mathcal{M}, \otimes)$ be a monoidal category in Pr^{L} and let A be an algebra object in $\mathcal{M}$. The dual of the $\mathcal{M}$ -module category RMod_A of right A -modules is the category LMod_A of left A -modules.*

4. The de Rham and Betti algebras

In this section, we define a pair of algebras $A_G^{\mathrm{dR}}(t, m)$ and $A_G^{\mathrm{Bet}}(t, m)$ which play a central role in the remainder of this paper. We define these purely algebraically, but the former was shown in [BLPW10, BLPW12] to control the category $\mathcal{O}_G^{\mathrm{dR}}(t, m)$. The latter algebra is defined in a very similar way, and we will show in Section 4.3 that these algebras are in fact isomorphic. In Section 5 and Section 6, we will see that the Betti algebra actually controls the category $\mathcal{O}_G^{\mathrm{Bet}}(t, m)$.

4.1. Recollections on the de Rham algebra

Consider the quiver

$$(4.1) \quad Q_1 := - \begin{array}{c} \xrightarrow{v} \\ \xleftarrow{u} \end{array} +.$$

Define $A_1^{\text{dR}} := P_{\mathbb{C}}(Q_1)$ to be the path algebra of Q_1 over $\mathbb{C}$. It is graded by path length. Let $e_{\pm} \in A_1^{\text{dR}}$ be the idempotents corresponding to the vertices. If we set $D = \mathbb{C}^{\times}$ to be 1-dimensional, then the inclusion

$$\begin{aligned} \mathbb{C}[\mathfrak{d}^L] &\simeq \mathbb{C}[d] \longrightarrow Z(A_1^{\text{dR}}) \\ d &\longmapsto uv + vu \end{aligned}$$

gives a $\mathfrak{d}$ -action on the category $\mathbf{Mod}_{A_1^{\text{dR}}}$.

More generally, we define $A_n^{\text{dR}} := (A_1^{\text{dR}})^{\otimes n}$. This algebra is also graded by path length. There is an isomorphism $\mathbb{C}[\mathfrak{d}^L] \simeq \mathbb{C}[d_1, \dots, d_n]$ which gives rise to a $\mathfrak{d}$ -action on $\mathbf{Mod}_{A_n^{\text{dR}}}$. The idempotents in A_n^{dR} are labeled by $\alpha = i_1 \dots i_n \in 2^{[n]}$: we write $e_{\alpha} := e_{i_1} \otimes \dots \otimes e_{i_n} \in A_n^{\text{dR}}$. Using the bijection of $2^{[n]}$ with chambers in the coordinate hyperplane arrangement on $\mathbb{R}^n$, we can think elements of A_n^{dR} as corresponding to paths between chambers in this arrangement. This perspective, which is very useful for understanding the algebra, is taken in [BLPW10]. Between any two idempotents α and β there is a minimal path

$$p^{\text{dR}}(\alpha, \beta) = \prod_{i=1}^n \begin{cases} 1 & \alpha_i = \beta_i, \\ u_i & \alpha_i = + \text{ and } \beta_i = -, \\ v_i & \alpha_i = - \text{ and } \beta_i = +. \end{cases}$$

Now (G, t, m) be a category $\mathcal{O}$ datum, determining an inclusion $G \rightarrow D$ which we use to impose G -invariants. The papers [BLPW10, BLPW12] study the rings

$$(4.2) \quad A_G^{\text{dR}} = A_n^{\text{dR}} \otimes_{\mathbb{C}[\mathfrak{g}^L]} \mathbb{C},$$

$$(4.3) \quad A_G^{\text{dR}}(t, -) = e_{\mathcal{F}} A_G^{\text{dR}}(-, -) e_{\mathcal{F}},$$

$$(4.4) \quad A_G^{\text{dR}}(t, m) = A_G^{\text{dR}}(t, -) / (A_G^{\text{dR}}(t, -) e_{\mathcal{F} \cap \mathcal{B}^c} A_G^{\text{dR}}(t, -)).$$

where in (4.2), $\mathbb{C}$ is the natural augmentation module for $\mathbb{C}[\mathfrak{g}^L]$. Note that $\mathbf{Mod}_{A_G^{\text{dR}}} \simeq (\mathbf{Mod}_{A_n^{\text{dR}}})^{\mathfrak{g}}$ has a residual $\mathfrak{f}$ -action which is inherited by $\mathbf{Mod}_{A_G^{\text{dR}}(t, -)}$ and $\mathbf{Mod}_{A_G^{\text{dR}}(t, m)}$, corresponding to the obvious $\mathbb{C}[\mathfrak{f}^L] \simeq \mathbb{C}[\mathfrak{d}^L] \otimes_{\mathbb{C}[\mathfrak{g}^L]} \mathbb{C}$ -algebra structures.

The next proposition summarizes results from [BLPW10, BLPW12] that will be used later.

PROPOSITION 4.1 ([BLPW10, Theorem B], [BLPW12, Lemma 8.25]). —

- $A_G^{\text{dR}}(t, -)$, and $A_G^{\text{dR}}(t, m)$ are Koszul. Moreover,

$$A_G^{\text{dR}}(t, m)^! = e_{\mathcal{B} \cap \mathcal{F}} A_G^{\text{dR}}(t, -)^! e_{\mathcal{B} \cap \mathcal{F}}.$$

- A_G^{dR} and $A_G^{\text{dR}}(t, -)$ are free finitely-generated $\mathbb{C}[\mathfrak{f}^L]$ -modules.
- $A_G^{\text{dR}}(t, -)$ and $A_G^{\text{dR}}(t, m)$ have finite homological dimension.
- $A_G^{\text{dR}}(t, m)$ is finite-dimensional and quasi-hereditary.

Now recall the quasi-hereditary structure on $A_G^{\text{dR}}(t, m)$ from [BLPW10, Section 5.5]. The simple modules and their projective covers are in bijection with $\mathcal{B} \cap \mathcal{F} \subseteq 2^{[n]}$, i.e., for each $\alpha \in \mathcal{B} \cap \mathcal{F}$ one has a surjection

$$(4.5) \quad P_{\alpha}^{\text{dR}} = e_{\alpha} A_G^{\text{dR}}(t, m) \longrightarrow A_G^{\text{dR}}(t, m) / \langle e_{\beta} \mid \beta \neq \alpha \rangle = S_{\alpha}^{\text{dR}}.$$

Using the partial order defined in Definition 2.25, one can define an intermediate standard module

$$(4.6) \quad V_\alpha^{\text{dR}} = P_\alpha^{\text{dR}} / (P_\alpha^{\text{dR}})^{>\alpha},$$

where $(P_\alpha^{\text{dR}})^{>\alpha}$ is the submodule generated by all paths a that pass through β with $\beta > \alpha$. Let $b_\alpha = \{i \in \{1, \dots, n\} \mid p_\alpha \in H_i\}$. It is easy to see that

$$(4.7) \quad (P_\alpha^{\text{dR}})^{>\alpha} = \{p^{\text{dR}}(\alpha, \alpha(i)) \mid i \in b_\alpha\} \cdot A_G^{\text{dR}}(t, m)$$

where $\alpha(i)$ differs from α only in position i . Define $\mathcal{B}_\alpha = \{\beta \mid \alpha_i = \beta_i \text{ for all } i \in b_\alpha\}$. Note that $\mathcal{B}_\alpha \subseteq \mathcal{B}$ and for any $\beta \in \mathcal{B}_\alpha \cap \mathcal{F}$ we have $\beta \leq \alpha$.

PROPOSITION 4.2 ([BLPW10, Lemma 5.21, Proposition 5.23]). —

- The standard module V_α^{dR} has a filtration whose associated graded is

$$\bigoplus_{\beta \in \mathcal{B}_\alpha \cap \mathcal{F}} S_\beta^{\text{dR}}.$$

- The projective module P_α^{dR} has a filtration whose associated graded is

$$\bigoplus_{\alpha \in \mathcal{B}_\beta \cap \mathcal{F}} V_\beta^{\text{dR}}.$$

COROLLARY 4.3. — There is a natural equivalence $\text{Mod}_{A_G^{\text{dR}}(t, m)} \simeq \text{Mod}_{A_G^{\text{dR}}(t, m)}^{\text{f-mon}}$.

Proof. — The simple modules are clearly f -monodromic. But since

$$A_G^{\text{dR}}(t, m) \simeq \bigoplus_\alpha P_\alpha^{\text{dR}},$$

it is a finite extension of simple modules and hence must also be f -monodromic. $\square$

PROPOSITION 4.4. — Let $i^{\text{dR}} : A_G^{\text{dR}}(t, -) \rightarrow A_G^{\text{dR}}(t, m)$. There is a recollement

$$\begin{array}{ccccc} & \swarrow & & \searrow & \\ \text{Mod}_{A_G^{\text{dR}}(t, m)} & \xrightarrow{i_*^{\text{dR}}} & \text{Mod}_{A_G^{\text{dR}}(t, -)} & \longrightarrow & \text{Mod}_{e_{\mathcal{F} \cap \mathcal{B}^c} A_G^{\text{dR}}(t, -) e_{\mathcal{F} \cap \mathcal{B}^c}} \\ & \nwarrow & & \swarrow & \end{array}$$

In particular, i_*^{dR} is fully faithful.

Proof. — Combine Proposition 4.1 and Corollary 3.27. Note that the proof of Corollary 4.3 shows that $A_G^{\text{dR}}(t, m)$ is in $\mathbf{T}_{A_G^{\text{dR}}(t, m)}(\bigoplus_{\alpha \in \mathcal{B} \cap \mathcal{F}} S_\alpha^{\text{dR}})$. $\square$

COROLLARY 4.5. — Consider the factorization

$$\begin{array}{ccc} \text{Mod}_{A_G^{\text{dR}}(t, -)}^{\text{f-mon}} & \hookrightarrow & \text{Mod}_{A_G^{\text{dR}}(t, -)} \\ \uparrow (i_*^{\text{dR}})^{\text{f-mon}} & & \uparrow i_*^{\text{dR}} \\ \text{Mod}_{A_G^{\text{dR}}(t, m)}^{\text{f-mon}} & \xrightarrow{\sim} & \text{Mod}_{A_G^{\text{dR}}(t, m)} \end{array}$$

coming from Corollary 3.50. The map $(i_*^{\text{dR}})^{\text{f-mon}}$ is fully faithful.

4.2. The Betti algebra

Consider the quiver

$$(4.8) \quad Q'_1 := - \begin{array}{c} \xrightarrow{\ell} \\ \xleftarrow{r} \end{array} +.$$

There is an inclusion

$$(4.9) \quad \begin{array}{ccc} \mathbb{C}[m] & \longrightarrow & Z(P_{\mathbb{C}}(Q'_1)) \\ m & \longmapsto & 1 - (\ell r + r \ell) \end{array}$$

Define $A_1^{\text{Bet}} = P_{\mathbb{C}}(Q'_1)[m^{-1}]$ and let $e_{\pm} \in A_1^{\text{Bet}}$ be the idempotents corresponding to the vertices. If we take $D = \mathbb{C}^{\times}$ to be 1-dimensional, then identifying $\mathbb{C}[D^L] \simeq \mathbb{C}[m^{\pm 1}]$ gives a Betti D -action on $\text{Mod}_{A_1^{\text{Bet}}}$.

More generally, define $A_n^{\text{Bet}} = (A_1^{\text{Bet}})^{\otimes n}$. This is naturally an algebra over $\mathbb{C}[D^L] \simeq \mathbb{C}[m_1^{\pm 1}, \dots, m_n^{\pm 1}]$. Note that for each $\alpha = i_1 \dots i_n \in 2^{[n]}$, there is an idempotent $e_{\alpha} := e_{i_1} \otimes \dots \otimes e_{i_n} \in A_n^{\text{Bet}}$. Between any two idempotents α and β there is a minimal path

$$p^{\text{Bet}}(\alpha, \beta) = \prod_{i=1}^n \begin{cases} 1 & \alpha_i = \beta_i, \\ l_i & \alpha_i = + \text{ and } \beta_i = -, \\ r_i & \alpha_i = - \text{ and } \beta_i = +. \end{cases}$$

Let (G, t, m) be a category $\mathcal{O}$ datum. Define the rings

$$(4.10) \quad A_G^{\text{Bet}} = A_n^{\text{Bet}} \otimes_{\mathbb{C}[G^L]} \mathbb{C},$$

$$(4.11) \quad A_G^{\text{Bet}}(t, -) = e_{\mathcal{F}} A_G^{\text{Bet}}(-, -) e_{\mathcal{F}},$$

$$(4.12) \quad A_G^{\text{Bet}}(t, m) = A_G^{\text{Bet}}(t, -) / (A_G^{\text{Bet}}(t, -) e_{\mathcal{F} \cap B^c} A_G^{\text{Bet}}(t, -)).$$

where in (4.10), $\mathbb{C}$ is the $\mathbb{C}[G^L]$ -module corresponding to the inclusion $\{1\} \hookrightarrow G^L$. Note that $\text{Mod}_{A_G^{\text{Bet}}} \simeq (\text{Mod}_{A_n^{\text{Bet}}})^{G^{\text{Bet}}}$ has a residual Betti F -action, inherited by $\text{Mod}_{A_G^{\text{Bet}}(t, -)}$ and $\text{Mod}_{A_G^{\text{Bet}}(t, m)}$, corresponding to the obvious $\mathbb{C}[F^L] \simeq \mathbb{C}[D^L] \otimes_{\mathbb{C}[G^L]} \mathbb{C}$ -algebra structures.

The following proposition is the key technical result in this section.

PROPOSITION 4.6. — *There is a natural equivalence $\text{Mod}_{A_G^{\text{Bet}}(t, m)} \simeq \text{Mod}_{A_G^{\text{Bet}}(t, m)}^{F_{\text{Bet}} - \text{mon}}$.*

Before proving Proposition 4.6, we note that it can be combined with the algebraic Riemann–Hilbert correspondence of Corollary 4.16 to prove the following theorem.

THEOREM 4.7. — *Let $i^{\text{Bet}} : A_G^{\text{Bet}}(t, -) \rightarrow A_G^{\text{Bet}}(t, m)$. There is a recollement*

$$\begin{array}{ccccc} & \xleftarrow{\quad} & & \xleftarrow{\quad} & \\ \text{Mod}_{A_G^{\text{Bet}}(t, m)} & \xrightarrow{i_*^{\text{Bet}}} & \text{Mod}_{A_G^{\text{Bet}}(t, -)} & \longrightarrow & \text{Mod}_{e_{\mathcal{F} \cap B^c} A_G^{\text{Bet}}(t, -) e_{\mathcal{F} \cap B^c}} \\ & \xleftarrow{\quad} & & \xleftarrow{\quad} & \end{array}$$

Proof. — By Proposition 3.2, it suffices to show that i_*^{Bet} is fully faithful. However, this follows from Corollary 4.16 and Corollary 4.5 since we have a factorization

$$\begin{array}{ccccc} \text{Mod}_{A_G^{\text{dR}}(t,-)}^{\text{f-mon}} & \xrightarrow[\sim]{RH} & \text{Mod}_{A_G^{\text{Bet}}(t,-)}^{F_{\text{Bet}}\text{-mon}} & \hookrightarrow & \text{Mod}_{A_G^{\text{Bet}}(t,-)} \\ \uparrow (i_*^{\text{dR}})^{\text{f-mon}} & & \uparrow (i_*^{\text{Bet}})^{\text{f-mon}} & & \uparrow i_*^{\text{Bet}} \\ \text{Mod}_{A_G^{\text{dR}}(t,m)}^{\text{f-mon}} & \xrightarrow[\sim]{RH} & \text{Mod}_{A_G^{\text{Bet}}(t,m)}^{F_{\text{Bet}}\text{-mon}} & \xrightarrow{\sim} & \text{Mod}_{A_G^{\text{Bet}}(t,m)}. \end{array} \quad \square$$

The rest of this section is devoted to the proof of Proposition 4.6. As in Corollary 4.3, it would suffice to show that $A_G^{\text{Bet}}(t, m)$ is a finite extension of simple modules.

Remark 4.8. — In Corollary 4.16, we will establish a Riemann–Hilbert isomorphism $A_G^{\text{Bet}}(t, m) \simeq A_G^{\text{dR}}(t, m)$, from which we will deduce that the former is hence a finite dimensional quasi-hereditary algebra. Unfortunately, to prove Corollary 4.16, one needs certain power series in $(1 - m_i)$ to converge. This is the content of Proposition 4.6.

Simple modules S_α^{Bet} , projective modules P_α^{Bet} , and standard modules V_α^{Bet} can be defined by replacing dR with Bet in equations (4.5), (4.6) and (4.7). To verify that they satisfy the expected properties, we need to first establish a few facts about $A_G^{\text{Bet}}(t, m)$. The following are the Betti analogues of [BLPW10, Propositions 3.8 & 3.9, Corollary 3.10].

PROPOSITION 4.9. — *The algebra A_G^{Bet} is a free, finitely-generated $\mathbb{C}[F^L]$ -module, i.e.,*

$$A_G^{\text{Bet}} = \bigoplus_{\alpha, \beta \in 2^{[n]}} e_\alpha A_G^{\text{Bet}} e_\beta = \bigoplus_{\alpha, \beta \in 2^{[n]}} \mathbb{C}[F^L] \cdot p^{\text{Bet}}(\alpha, \beta)$$

Moreover, one has

$$p^{\text{Bet}}(\alpha, \beta) p^{\text{Bet}}(\beta, \gamma) = \prod_{i=1}^n (1 - m_i)^{\theta_i(\alpha, \beta, \gamma)} \cdot p^{\text{Bet}}(\alpha, \gamma)$$

where

$$\theta_i(\alpha, \beta, \gamma) = \begin{cases} 1 & \alpha_i = \gamma_i \neq \beta_i, \\ 0 & \text{else.} \end{cases}$$

COROLLARY 4.10. — *Suppose that*

$$a = \prod_{i=1}^n (1 - m_i)^{n_i} \cdot p^{\text{Bet}}(\alpha, \gamma)$$

and $n_i \geq \theta_i(\alpha, \beta, \gamma)$ for all $1 \leq i \leq n$. Then a can be represented by a $\mathbb{C}$ -linear combination of paths that pass through β . In particular, if $\beta \in \mathcal{B}^c \cap \mathcal{F}$ then the image of a in $A_G^{\text{Bet}}(t, m)$ is 0.

The following is the Betti analogue [BLPW10, Lemma 5.21] and is proved in the same way.

LEMMA 4.11. — *The module V_α^{Bet} has a basis consisting of all $p^{\text{Bet}}(\alpha, \beta)$ with $\beta \in \mathcal{F} \cap \mathcal{B}_\alpha$. In particular, V_α^{Bet} has a filtration whose associated graded is $\bigoplus_{\beta \in \mathcal{B}_\alpha \cap \mathcal{F}} S_\beta^{\text{Bet}}$.*

The following is the first part of [BLPW10, Proposition 5.23] and is proved in the same way. It is sufficient to complete the proof of Proposition 4.6.

LEMMA 4.12. — *The projective module P_α^{Bet} has filtration such that the associated graded is a quotient of $\bigoplus_{\alpha \in \mathcal{B}_\beta \cap \mathcal{F}} V_\beta^{\text{Bet}}$.*

Proof. — For each γ define $(P_\alpha^{\text{Bet}})^\gamma$ to be the submodule of P_α^{Bet} generated by all paths passing through γ . Moreover we can define

$$(P_\alpha^{\text{Bet}})^{\geq \beta} = \sum_{\gamma \geq \beta} (P_\alpha^{\text{Bet}})^\gamma \quad (P_\alpha^{\text{Bet}})^{> \beta} = \sum_{\gamma > \beta} (P_\alpha^{\text{Bet}})^\gamma.$$

Note that $(P_\alpha^{\text{Bet}})^{\geq \alpha} = P_\alpha^{\text{Bet}}$ and $V_\alpha^{\text{Bet}} = P_\alpha^{\text{Bet}} / (P_\alpha^{\text{Bet}})^{> \alpha}$. These give a decreasing filtration $(P_\alpha^{\text{Bet}})^{\geq \bullet}$ indexed by the partially ordered set $(\mathcal{B} \cap \mathcal{F})_{\geq \alpha}$. The associated graded pieces are the quotients

$$M_\alpha^\beta = (P_\alpha^{\text{Bet}})^{\geq \beta} / (P_\alpha^{\text{Bet}})^{> \beta}$$

for $\beta \in (\mathcal{B} \cap \mathcal{F})_{\geq \alpha}$. When $\alpha \notin \mathcal{B}_\beta \cap \mathcal{F}$ we have that $M_\alpha^\beta = 0$. Otherwise there is a surjective map

$$\begin{aligned} P_\beta^{\text{Bet}} &\longrightarrow M_\alpha^\beta \\ a &\longmapsto p^{\text{Bet}}(\alpha, \beta) \cdot a \end{aligned}$$

with kernel containing $(P_\beta^{\text{Bet}})^{> \beta}$. In particular we have a surjection $V_\beta^{\text{Bet}} \rightarrow M_\alpha^\beta$. $\square$

Remark 4.13. — The dimension estimate used to finish the proof of [BLPW10, Proposition 5.23] follows from identifying $A_G^{\text{dR}}(t, m)$ with a convolution algebra built from the homology of toric varieties. It is possible to get the same dimension estimate for $A_G^{\text{Bet}}(t, m)$ by identifying it with a convolution algebra defined using the rational K-theory of toric varieties, as is implicitly done in [GH26]. From this perspective the algebraic Riemann–Hilbert correspondence discussed below is just the Chern character.

4.3. An algebraic Riemann–Hilbert correspondence

The most primitive form of the algebraic Riemann–Hilbert correspondence is the following proposition.

PROPOSITION 4.14. — *There is an equivalence*

$$(4.13) \quad RH : \text{Mod}_{A_n^{\text{Bet}}}^{D_{\text{Bet}} - \text{mon}} \simeq \text{Mod}_{A_n^{\text{dR}}}^{\mathfrak{d} - \text{mon}}$$

compatible with the residual actions of $(\text{IndCoh}((D^L)_1^\wedge), \otimes) \simeq (\text{IndCoh}((\mathfrak{d}^L)_0^\wedge), \otimes)$.

Proof. — By Corollary 3.50, we have equivalences

$$\begin{aligned} \text{Mod}_{A_n^{\text{Bet}}}^{D_{\text{Bet}} - \text{mon}} &\simeq \varinjlim_k \text{IndCoh}(N_k(1, D^L)) \bigotimes_{\text{IndCoh}(D^L)} \text{Mod}_{A_n^{\text{Bet}}} \simeq \varinjlim_k \text{Mod}_{A_n^{\text{Bet}}/\mathcal{I}^{k+1}}, \\ \text{Mod}_{A_n^{\text{dR}}}^{\mathfrak{d} - \text{mon}} &\simeq \varinjlim_k \text{IndCoh}(N_k(0, \mathfrak{d}_n^L)) \bigotimes_{\text{IndCoh}(\mathfrak{d}^L)} \text{Mod}_{A_n^{\text{dR}}} \simeq \varinjlim_k \text{Mod}_{A_n^{\text{dR}}/\mathcal{J}^{k+1}}. \end{aligned}$$

where $\mathcal{I} \subset A_n^{\text{Bet}}$ (resp., $\mathcal{J} \subset A_n^{\text{dR}}$) is the two-sided ideal generated by $(1 - m_i)$ (resp., d_i) for $1 \leq i \leq n$.

Thus we have reduced constructing RH to the observation that in [Mal91, Section II.3.2] Malgrange constructed isomorphisms

$$(4.14) \quad \begin{array}{ccc} A_n^{\text{Bet}}/\mathcal{I}^{k+1} & \xrightarrow{rh_k} & A_n^{\text{dR}}/\mathcal{J}^{k+1} \\ \ell_i & \longmapsto & u_i \\ r_i & \longmapsto & \frac{1 - \exp(2\pi i d_i)}{d_i} v_i \end{array}.$$

Compatibility with the residual actions follows from the fact that $m_i \mapsto \exp(2\pi i d_i)$. $\square$

By taking G -invariants of (4.13), we obtain an equivalence

$$(4.15) \quad \left(\text{Mod}_{A_n^{\text{Bet}}}^{D_{\text{Bet}}^{\text{Bet}} - \text{mon}} \right)^{G_{\text{Bet}}} \simeq \left(\text{Mod}_{A_n^{\text{dR}}}^{\mathfrak{d} - \text{mon}} \right)^{\mathfrak{g}}.$$

COROLLARY 4.15. — *There is an equivalence of categories*

$$(4.16) \quad \text{Mod}_{A_G^{\text{Bet}}}^{F_{\text{Bet}}^{\text{Bet}} - \text{mon}} \simeq \text{Mod}_{A_G^{\text{dR}}}^{\mathfrak{f} - \text{mon}}.$$

Proof. — Consider the pullback squares

$$\begin{array}{ccccc} (F^L)_1^\wedge & \longrightarrow & F^L & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow \\ (D^L)_1^\wedge & \longrightarrow & D_n^L & \longrightarrow & G^L, \end{array} \quad \begin{array}{ccccc} (\mathfrak{f}^L)_0^\wedge & \longrightarrow & \mathfrak{f}^L & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ (\mathfrak{d}_n^L)_0^\wedge & \longrightarrow & \mathfrak{d}^L & \longrightarrow & \mathfrak{g}^L. \end{array}$$

The left-hand side of (4.16) may be obtained by pulling back from G^L to $(F^L)_1^\wedge$. On the other hand, this pullback may be accomplished in two steps, first pulling back to $(D^L)_1^\wedge$ and then to $(F^L)_1^\wedge$, giving an equivalence between the left-hand sides of (4.16) and (4.15). An equivalence between the right-hand categories may be produced in the same way by considering the right-hand pullback diagram. $\square$

By examining (4.14) we can easily see that the Riemann–Hilbert correspondence is compatible with idempotents.

COROLLARY 4.16. — *There is a commutative diagram*

$$\begin{array}{ccc} \text{Mod}_{A_G^{\text{dR}}(t, -)}^{\mathfrak{f} - \text{mon}} & \xrightarrow[\sim]{RH} & \text{Mod}_{A_G^{\text{Bet}}(t, -)}^{F_{\text{Bet}}^{\text{Bet}} - \text{mon}} \\ \uparrow (i_*^{\text{dR}})^{\mathfrak{f} - \text{mon}} & & \uparrow (i_*^{\text{Bet}})^{F_{\text{Bet}}^{\text{Bet}} - \text{mon}} \\ \text{Mod}_{A_G^{\text{dR}}(t, m)}^{\mathfrak{f} - \text{mon}} & \xrightarrow[\sim]{RH} & \text{Mod}_{A_G^{\text{Bet}}(t, m)}^{F_{\text{Bet}}^{\text{Bet}} - \text{mon}}. \end{array}$$

Moreover, by combining this with Corollary 4.3 and Proposition 4.6, we produce an equivalence

$$\text{Mod}_{A_G^{\text{dR}}(t, m)} \simeq \text{Mod}_{A_G^{\text{Bet}}(t, m)}.$$

5. The geometric categories

We now begin to relate the algebraic constructions of previous sections to geometry. In this section, we recall results about the category of sheaves on $\mathbb{C}^n$, stratified by its toric orbits, together with its toric-equivariant structure.

5.1. Categories of sheaves

Let M be a manifold. For consistency with parts of the literature (e.g. [GPS24a, KS94], we will always assume when discussing sheaves that M is real-analytic. Let $\mathbf{Sh}(M)$ be the category of sheaves on M valued in $\mathbf{Mod}_{\mathbb{C}}$. Given $\mathcal{F} \in \mathbf{Sh}(M)$, its *singular support* (or *microsupport*) $SS(\mathcal{F}) \subset T^*M$ is defined as in [KS94, Section 5.1]. For $\Lambda \subset T^*M$, let $\mathbf{Sh}_{\Lambda}(M) \subset \mathbf{Sh}(M)$ be the full subcategory on objects whose microsupport is contained in Λ .

Given a manifold M , one can define a presheaf of stable categories on T^*M by the prescription

$$(5.1) \quad U \longmapsto \mu\mathbf{Sh}_{T^*M}^{\text{pre}}(U) := \mathbf{Sh}(M) / \{F \in \mathbf{Sh}(M) \mid SS(F) \cap U = \emptyset\}.$$

The sheafification of this presheaf shall be denoted by $\mu\mathbf{Sh}_{T^*M}(-)$, and is called the sheaf (of categories) of *microlocal sheaves* (or *microsheaves* for short) on T^*M .

Given $\Lambda \subset T^*M$ conic, there is a sheaf of full subcategories $\mu\mathbf{Sh}_{\Lambda}(-) \subset \mu\mathbf{Sh}_{T^*M}(-)$ which is defined by the prescription

$$(5.2) \quad \mu\mathbf{Sh}_{\Lambda}(U \cap \Lambda) := \{\mathcal{F} \in \mu\mathbf{Sh}_{T^*M}(U) \mid SS(\mathcal{F}) \subset \Lambda \cap U\}.$$

Under the assumption (which will always be satisfied in this paper) that Λ is nice stratified isotropic, $\mu\mathbf{Sh}_{\Lambda}(-)$ takes values in the category $\mathbf{Pr}_{\omega}^{\text{LR}}$ (objects are compactly generated categories, morphisms have left and right adjoints).

Given an exact symplectic manifold (X, λ) equipped with a stable polarization τ and a conic (for the Liouville vector field) Lagrangian $\Lambda \subset X$, [NS20, She21] use the theory of microlocal sheaves to define a sheaf of categories $\mu\mathbf{Sh}_{\Lambda, \tau}$ on X . It has the property [NS20, Remark 9.25] that if $X = T^*M$ is a cotangent bundle and τ^{fib} is the cotangent fiber polarization, then there is an equivalence $\mu\mathbf{Sh}_{\Lambda}$ and $\mu\mathbf{Sh}_{\Lambda, \tau}$ between the two sheaves (on T^*M) of categories just discussed. We will henceforth drop the polarization τ from our notation where it is understood.

A priori, the definition of $\mu\mathbf{Sh}_{\Lambda}$ on a Lagrangian Λ polarized symplectic manifold is sensitive to the choice of primitive λ for the symplectic form on X . In this paper, we will be interested in studying the Lagrangian $\mathbb{L}_n \subset T^*\mathbb{C}^n$, which is conic for two different Liouville vector fields: the canonical Liouville structure on $T^*\mathbb{C}^n$, which dilates the cotangent fibers; and the conical Liouville structure on $\mathbb{C}^{2n}$, which dilates both base and fiber coordinates equally. The sheaf of categories $\mu\mathbf{Sh}_{\mathbb{L}_n}$ will not depend on the choice of Liouville field, thanks to the following fact. In the following lemma and its corollary, we will write $\mu\mathbf{Sh}_{\Lambda, \lambda}$ for the sheaf of categories defined using Liouville form λ (where the fixed choice of cotangent fiber polarization is assumed).

LEMMA 5.1. — *Consider the Liouville forms $\lambda_s := (1-s)p_i dq^i - sq_i dp^i = \lambda_0 - d(sp_i q^i)$ on $T^*\mathbb{R}^n$, and let $\mathbb{L} := 0_{\mathbb{R}^n} \cup T_0^* \subset T^*\mathbb{R}^n$. Then the sheaf of categories $\mu\mathbf{Sh}_{\mathbb{L}, \lambda_s}$ is independent of s .*

Proof. — Recall [NS20, p. 2] that if (X, λ) is Liouville and $L \subset X$ is conic, then $\mu\mathrm{Sh}_L$ is defined by viewing $\{0\} \times L$ as a Legendrian in the contact manifold $(\mathbb{R} \times X, dt + \lambda)$.

For any contactomorphism $(V, \xi) \rightarrow (U, \eta)$ and Legendrian $L \subset V$, we have $\mu\mathrm{Sh}_{L, \xi}(-) \simeq \mu\mathrm{Sh}_{\phi(L), \eta}$, where it is understood the Maslov data on the source is pulled back from the target. But the contactomorphism $(\mathbb{R} \times T^*\mathbb{R}^n, dt + \lambda_s) \rightarrow (\mathbb{R} \times T^*\mathbb{R}^n, dt + \lambda_0)$ taking $(t, q, p) \mapsto (t - sp_i q^i, q, p)$ fixes $\{0\} \times \mathbb{L}$ pointwise, giving the desired claim. $\square$

COROLLARY 5.2. — *Consider the family (in s) of Liouville forms $\lambda_s^{\mathbb{C}} = (1 - s)wdz - szdw$ on the holomorphic symplectic manifold $T^*\mathbb{C}$. Then $\mu\mathrm{Sh}_{\mathbb{L}_1, \mathrm{Re}(\lambda_s^{\mathbb{C}})}$ is independent of s .*

Proof. — Identify $T^*\mathbb{C} \simeq T^*\mathbb{R}^2$ as usual by the map

$$(z, w) \mapsto (\mathrm{re}(z), \mathrm{im}(z), \mathrm{re}(w), -\mathrm{im}(w)).$$

Let $\lambda_s^{\mathbb{C}} = (1 - s)wdz - szdw$. Then $\mathrm{re}(\lambda_s^{\mathbb{C}})$ is identified with λ_s , and s -independence of the category follows from Lemma 5.1. $\square$

If M is a complex manifold and $\Lambda \subset T^*M$ is a conic complex Lagrangian, then $\mu\mathrm{Sh}_{\Lambda}(-)$ pushes forwards to a sheaf of categories on the topology of $\mathbb{C}^{\times}$ -invariant conic complex open subsets. As explained in [CKNS25], for a holomorphic polarization τ , the resulting sheaf of categories $\mu\mathrm{Sh}_{\Lambda, \tau}(-)$ is equipped with a t-structures. The heart $\mu\mathrm{Sh}_{\Lambda}^{\heartsuit}(-)$ is a sheaf of abelian categories; its objects are the *perverse microsheaves*.

5.2. Group actions

We begin with the following lemma.

LEMMA 5.3. — *Let $m : G \times M \rightarrow M$ denote the action of a Lie group on a manifold M . Assume that all orbits are embedded submanifolds. Let $\Lambda \subset T^*M$ be a closed conic Lagrangian subset. Assume that Λ is contained in the conormal of the orbit stratification, and that Λ is setwise invariant under the induced G action on T^*M . Then $SS(m^*\mathcal{F}) \subset 0_G \times \Lambda$.*

Proof. — Consider

$$T^*G \times T^*M = T^*(G \times M) \xleftarrow{dm^*} (G \times M) \times_M T^*M \xrightarrow{m_{\pi}} T^*M,$$

where m_{π} is the obvious bundle projection. Since m is a submersion, [KS94, Proposition 5.4.5(i)] implies that $SS(m^*\mathcal{F}) = dm^*(m_{\pi}^{-1}(SS(\mathcal{F})))$. It suffices to show that

$$dm^*(m_{\pi}^{-1}(\Lambda)) \subset 0_G \times \Lambda \subset T^*(G \times M).$$

First we show that $dm^*(m_{\pi}^{-1}(\Lambda)) \subset 0_G \times T^*M$. If not, then there is some covector $(x, y) \in T^*G \times T^*M$, $x \neq 0$, such that $(x, y) = dm^*(q)$ for some $q \in m_{\pi}^{-1}(T_{m(\bar{x}, \bar{y})}^*M \cap \Lambda)$, where $(\bar{x}, \bar{y}) \in G \times M$ denotes the image of (x, y) under projection to the base. Choose $v \in T_{\bar{x}}G$ so that $x(v) \neq 0$. Observe that $dm(v)$ is tangent to $\mathrm{orb}(m(\bar{x}, \bar{y}))$. By assumption $m_{\pi}(q)$ is contained in the conormal of $\mathrm{orb}(m(\bar{v}))$, hence q pairs to

zero with $dm(v)$. But $q(dm(v)) = dm^*(q)(v) = x(v) \neq 0$, which is a contradiction. Finally, the fact that $dm^*(m_\pi^{-1}(SS(\mathcal{F}))) \subset 0_G \times L$ follows from our assumption that L is invariant under the G action on T^*M . $\square$

Henceforth, we let G be a complex algebraic group acting on a complex manifold M . We denote the action map by $m : G \times M \rightarrow M$. We let $\Lambda \subset T^*M$ be a G -invariant conic Lagrangian, and we assume that the assumptions of Lemma 5.3 hold.

Consider the semi-cosimplicial diagram of categories, where the face maps $d_i^j : \mathbf{Sh}(G^{\times k} \times M) \rightarrow \mathbf{Sh}(G^{\times(k+1)} \times M)$, $0 \leq k, 0 \leq j \leq k+1$, are induced by the pullbacks of group multiplication and action maps. Specifically, for the first level ($k=0$), we have $d_0^0 = m^! : \mathbf{Sh}(M) \rightarrow \mathbf{Sh}(G \times M)$ and $d_0^1 = p_1^! : \mathbf{Sh}(M) \rightarrow \mathbf{Sh}(G \times M)$, and so on for the higher levels. By Proposition 3.37 we can define $\mathbf{Sh}(G)$ -invariants by the formula

$$(5.3) \quad \mathbf{Sh}(M)^G := \varprojlim (\mathbf{Sh}(M) \rightrightarrows \mathbf{Sh}(G \times M) \Rrightarrow \mathbf{Sh}(G \times G \times M) \Rrightarrow \dots).$$

Objects of $\mathbf{Sh}(M)^G$ are often called “ G -equivariant sheaves” in the literature. The projection onto the first factor in the limit defines a map $\mathbf{Sh}(M)^G \rightarrow \mathbf{Sh}(M)$ which, concretely, “forgets” the equivariant structure. We let $\mathbf{Sh}_\Lambda(M)^G \subset \mathbf{Sh}(M)^G$ be the full subcategory of $\mathbf{Sh}(M)^G$ on objects which have singular support contained in Λ after forgetting the equivariant structure.

Meanwhile, fixing the microsupport condition Λ , we can consider another notion of G -equivariant sheaf by acting on $\mathbf{Sh}_\Lambda(M)$ by $\mathrm{Loc}(G)$. More precisely, there is a semicosimplicial set

$$d_i^j : sh_{0_G^{\times k} \times \Lambda}(G^{\times k} \times M) \longrightarrow sh_{0_G^{\times(k+1)} \times \Lambda}(G^{\times(k+1)} \times M), \quad 0 \leq k, \quad 0 \leq j \leq k+1,$$

where as before the d_i^k are the pullbacks of the action maps and group multiplication maps. Here Lemma 5.3 ensures that the pullbacks satisfy the correct microsupport condition, so that the face maps are well defined. (In particular, we rely on our standing assumption that Λ satisfies the conditions of Lemma 5.3.)

By Proposition 3.37, the Betti G -invariants are computed by the formula

$$(5.4) \quad \mathbf{Sh}_\Lambda(M)^{G_{\mathrm{Bet}}} = \varprojlim (\mathbf{Sh}_\Lambda(M) \rightrightarrows \mathbf{Sh}_{0_G \times \Lambda}(G \times M) \Rrightarrow \mathbf{Sh}_{0_G \times 0_G \times \Lambda}(G \times G \times M) \Rrightarrow \dots).$$

At this point, we have two notions of what it means to be a G -equivariant sheaf microsupported in Λ , via the respective semicosimplicial diagrams (5.3) and (5.4). In fact, these notions agree:

LEMMA 5.4. — *There is an equivalence of categories $\mathbf{Sh}_\Lambda(M)^G \simeq \mathbf{Sh}_\Lambda(M)^{G_{\mathrm{Bet}}}$.*

Proof. — The fully faithful embeddings $\mathbf{Sh}_{0_G^{\times k} \times \Lambda}(G^{\times k} \times M) \rightarrow \mathbf{Sh}(G^{\times k} \times M)$ of the limit diagrams (5.4) and (5.3) induce a fully faithful embedding of the limit categories

$$(5.5) \quad \mathbf{Sh}_\Lambda(M)^{G_{\mathrm{Bet}}} \hookrightarrow \mathbf{Sh}(M)^G.$$

The essential image of (5.5) is contained in the subcategory $\mathbf{Sh}_\Lambda(M)^G \subset \mathbf{Sh}(M)^G$ given by the singular-support condition Λ , and we claim that the essential image of (5.5) is in fact $\mathbf{Sh}_\Lambda(M)$.

To see this, observe that the forgetful map $\mathbf{Sh}_\Lambda(M)^G \subset \mathbf{Sh}(M)^G \rightarrow \mathbf{Sh}(M)$ factors through $\mathbf{Sh}_\Lambda(M)^G \subset \mathbf{Sh}(M)^G \rightarrow \mathbf{Sh}(G^{\times k} \times M) \rightarrow \mathbf{Sh}(M)$. Per Lemma 5.3, it in fact factors as $\mathbf{Sh}_\Lambda(M)^G \rightarrow \mathbf{Sh}_{0_G^{\times k} \times \Lambda}(G^{\times k} \times M) \rightarrow \mathbf{Sh}(G^{\times k} \times M) \rightarrow \mathbf{Sh}(M)$. This induces an inverse map $\mathbf{Sh}_\Lambda(M)^G \rightarrow \mathbf{Sh}_\Lambda(M)^{G_{\text{Bet}}}$, as desired. $\square$

If $U \subset T^*M$ is conic and G -invariant, then the G action will preserve the condition that singular support is contained outside of U , thus inducing a G -action on the category $\mu\mathbf{Sh}_\Lambda^{\text{pre}}(U)$.

DEFINITION 5.5. — *Let $\Lambda \subset T^*M$ be a G -invariant conic Lagrangian. We write $\mu\mathbf{Sh}_\Lambda^G$ for the sheaf of categories on Λ/G obtained by sheafifying $U \mapsto \mu\mathbf{Sh}_\Lambda^{\text{pre}}(U)^{G_{\text{Bet}}}$ (equivalently $U \mapsto \mu\mathbf{Sh}_\Lambda^{\text{pre}}(U)^G$) on G -invariant open subsets of Λ .*

Now suppose that $\tilde{U} \subset \mu^{-1}(0)$ is an open subset on which the G -action is free, such that $U := \tilde{U}/G$ is a Liouville submanifold of the stack $\mu^{-1}(0)/G = T^*(M/G)$. As a cotangent bundle, T^*M comes equipped with a canonical polarization by cotangent fibers.

LEMMA 5.6 ([NS23, Corollary 4.5]). — *The cotangent fiber polarization on T^*M descends to a polarization on the manifold U .*

Let τ_G denote the polarization on U defined in Lemma 5.6. We learn from [NS23] a recipe for computing the microsheaf category on U in terms of microsheaves in T^*M .

PROPOSITION 5.7 ([NS23, Corollary 4.9]). — *Let $\tilde{U}$ as above, and let $\tilde{\Lambda} \subset \tilde{U} \subset T^*M$ be a conic Lagrangian. Then there is an equivalence*

$$\mu\mathbf{Sh}_\Lambda^G(\tilde{\Lambda}) \simeq \mu\mathbf{Sh}_{\Lambda, \tau_G}(\Lambda).$$

5.3. Perverse (micro)sheaves in $T^*\mathbb{C}$

We now begin to study the geometry of the Lagrangians

$$\mathbb{L}_1 := \mathbb{C} \cup T_0^*\mathbb{C} \subset T^*\mathbb{C}, \quad \mathbb{L}_n := (\mathbb{L}_1)^n \subset (T^*\mathbb{C})^n = T^*\mathbb{C}^n.$$

The following fundamental calculation was first observed by Deligne, with proofs by Verdier [Ver85], Galligo–Granger–Maisonobe [GGM85], MacPherson–Vilonen [MV86], and Beilinson [Bei87].

THEOREM 5.8. — *There is an equivalence of categories*

$$(5.6) \quad \mathbf{Sh}_{\mathbb{L}_1}(\mathbb{C}) \simeq \mathbf{Mod}_{A_1^{\text{Bet}}},$$

exchanging the perverse t -structure on the left-hand side with the usual t -structure on the right-hand side, and exchanging the Betti $\mathbb{C}^\times$ -action induced on the left-hand side by the action $\mathbb{C}^\times \curvearrowright \mathbb{C}$ with the Betti $\mathbb{C}^\times$ -action on the right-hand side described by (4.9).

Proof. — The equivalence of the abelian hearts of these categories is a direct corollary of [MV86, Theorem 3.3] (described explicitly in the example following the

proof there), except that the theorem there is stated with the assumption of finite-dimensionality. However, the assumption of finite-dimensional microstalks is never used in the proof, which continues to hold in the infinite-dimensional case.

In other words, [MV86, Theorem 3.3] proves that $\mathrm{Perv}_{\mathbb{L}_1}(\mathbb{C}) \simeq \mathrm{Mod}_{A_1^{\mathrm{Bet}}}^\heartsuit$. It remains to show that $\mathrm{Sh}_{\mathbb{L}_1}(\mathbb{C})$ is in fact the derived category of its perverse heart, for which it is sufficient to check that the objects P_Ψ, P_Φ of $\mathrm{Sh}_{\mathbb{L}_1}$ corepresenting the “generic stalk” and “generic microstalk over 0” (i.e., nearby and vanishing cycles) functors are in fact perverse. This is the case, since $P_\Psi \simeq j_!(\exp_* \underline{\mathbb{C}})$, where $\exp : \mathbb{C} \rightarrow \mathbb{C}^\times$ is the universal covering map and $j : \mathbb{C}^\times \rightarrow \mathbb{C}$ is the inclusion, and P_Φ can be obtained from P_Ψ by Fourier transform.

Finally, the statement about Betti $\mathbb{C}^\times$ -actions follows from the identification of the monodromy on $\mathrm{end}(P_\Psi) \sim \mathbb{C}[m^\pm]$ as multiplication by m . $\square$

We also want to (micro)localize the above equivalence. Let $\mathbb{L}_+ = \mathbb{C} \setminus \{0\}$, $\mathbb{L}_- = T_0^* \mathbb{C} \setminus \{0\} \subset \mathbb{L}_1$ be the open subsets obtained by deleting one or the other component of $\mathbb{L}_1$.

COROLLARY 5.9. — *The following vertical maps are an equivalence of diagrams of categories, respecting Betti $\mathbb{C}^\times$ -actions and t -structures:*

$$\begin{array}{ccccc} \mu\mathrm{Sh}_{\mathbb{L}_1}(\mathbb{L}_-) & \longleftarrow & \mu\mathrm{Sh}_{\mathbb{L}_1}(\mathbb{L}_1) & \longrightarrow & \mu\mathrm{Sh}_{\mathbb{L}_1}(\mathbb{L}_+) \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ \mathrm{Mod}_{e_- A_1^{\mathrm{Bet}} e_-} & \longleftarrow & \mathrm{Mod}_{A_1^{\mathrm{Bet}}} & \longrightarrow & \mathrm{Mod}_{e_+ A_1^{\mathrm{Bet}} e_+} \end{array}$$

By taking tensor products, we obtain a version of the above in n dimensions, which we may state as follows.

DEFINITION 5.10. — *Equip $\{1, +, -\}$ with the poset structure $- \leftarrow 1 \rightarrow +$, and write $\mathbf{P} := \{1, +, -\}^n$ for the product poset. For $\alpha \in \{1, +, -\}^n$, let $\mathbb{L}_\alpha := \prod_{i=1}^n \mathbb{L}_{\alpha_i} \subset (T^* \mathbb{C})^n \simeq T^* \mathbb{C}^n$ be the product Lagrangian.*

Observe that $\mathbb{L}_{1^n} = \mathbb{L}_n$, and that there is an open embedding $\mathbb{L}_{\alpha'} \subset \mathbb{L}_\alpha$ whenever α' is obtained from α by changing some 1's to $\pm$'s. As α varies, we thus obtain a $\mathbf{P}$ -diagram of categories $\alpha \mapsto \mu\mathrm{Sh}_{\mathbb{L}_n}(\mathbb{L}_\alpha)$, with the evident restriction maps. Similarly, let $e_1 = 1, e_+, e_-$ be the idempotents in A_1^{Bet} , and for $\alpha \in \{1, +, -\}^n$, let $e_\alpha = \otimes_i e_{\alpha_i} \in (A_1^{\mathrm{Bet}})^{\otimes n} \simeq A_n^{\mathrm{Bet}}$. Then we get another $\mathbf{P}$ -diagram of categories $\alpha \mapsto \mathrm{Mod}_{e_\alpha A_n^{\mathrm{Bet}} e_\alpha}$.

COROLLARY 5.11. — *There are equivalences of $\mathbf{P}$ -shaped diagrams of Betti $(\mathbb{C}^\times)^n$ -categories with t -structures, given by equivalences*

$$(5.7) \quad \mu\mathrm{Sh}_{\mathbb{L}_n}(\mathbb{L}_\alpha) \simeq \mathrm{Mod}_{e_\alpha A_n^{\mathrm{Bet}} e_\alpha}.$$

commuting with the restriction maps.

COROLLARY 5.12. — *Let $G \subset (\mathbb{C}^\times)^n$ be a subtorus. Then there are equivalences of $\mathbf{P}$ -shaped diagrams of categories with t -structures $\mu\mathrm{Sh}_{\mathbb{L}_n}^G(\mathbb{L}_\alpha) \simeq \mathrm{Mod}_{e_\alpha A_n^{\mathrm{Bet}}(G, -, -) e_\alpha}$.*

Proof. — This follows from Corollary 5.11 by taking G -invariants. $\square$

5.4. The de Rham category

We will now describe a portion of the Riemann–Hilbert correspondence associated to the complex manifold $\mathbb{C}$ with singular-support condition $\mathbb{L}_1 \subset T^*\mathbb{C}$. Consider the category $\mathbf{Dmod}_{\mathbb{L}_1}^{reg}(\mathbb{C})$ of regular D-modules on $\mathbb{C}$ with singular support in $\mathbb{L}_1$.

THEOREM 5.13 ([Mal91, Théorème (2.1)]). — *The compact objects in the category $\mathbf{Dmod}_{\mathbb{L}_1}^{reg}(\mathbb{C})$ are equivalent to the category of finite-dimensional A_1^{dR} -modules on which all eigenvalues of vu have real part in the interval $[0, 1)$.*

The respective categories of perverse sheaves and D-modules on $\mathbb{C}$, even after imposing the singular support condition $\mathbb{L}_1$, are not equivalent: a perfect module over A_1^{Bet} may not be finite-dimensional (as indeed the regular representation of the infinite-dimensional $\mathbb{C}$ -algebra A_1^{Bet} is not). The Riemann–Hilbert correspondence, described in [Mal91, Théorème (3.1)], gives an equivalence between the category described in Theorem 5.13 and the category $\mathbf{Mod}_{A_1^{\mathrm{Bet}}}^{fd}$ of finite-dimensional A_1^{Bet} -modules.

The de Rham category can be described more simply if we pass to monodromic modules. In our language, one of the main calculations of [MVdB98] is the following:

THEOREM 5.14 ([MVdB98, Theorem 6.3]). — *There is an equivalence*

$$\mathbf{Mod}_{A_n^{\mathrm{dR}}}^{\mathrm{d-mon}} \simeq \mathbf{Dmod}(\mathbb{C}^n)^{D_{\mathrm{dR}}-\mathrm{mon}}.$$

Using Theorem 5.14 and Theorem 5.13, we can now describe the geometric origin of the Riemann–Hilbert isomorphism of Proposition 4.14.

COROLLARY 5.15. — *There is an equivalence of categories*

$$\mathbf{Sh}_{\mathbb{L}}(\mathbb{C}^n)^{D_{\mathrm{Bet}}-\mathrm{mon}} \simeq \mathbf{Dmod}(\mathbb{C}^n)^{D_{\mathrm{dR}}-\mathrm{mon}},$$

given by the map (4.13).

6. Betti category $\mathcal{O}$

In this section, we describe the category $\mu\mathbf{Sh}_{\mathbb{L}_G(t,m)}(\mathbb{L}_G(t,m))$ of microlocal sheaves on the category $\mathcal{O}$ skeleton $\mathbb{L}_G(t,m)$ of the hypertoric variety $\mathfrak{X}_G(t)$. We will begin by describing the passage from the Lagrangian $\mathbb{L} = \mathbb{L}_n$ to the extended core $\mathbb{L}_G(t, -)$ of the hypertoric variety $\mathfrak{X}_G(t)$; afterward, we will pass from $\mathbb{L}_G(t, -)$ to the category $\mathcal{O}$ skeleton $\mathbb{L}_G(t,m)$.

For the reader's convenience, we recall the following notation from Section 2.

Notation 6.1. —

- $\mathbb{L}_G = \mathbb{L}_G(-, -)$ denotes the stacky Lagrangian $\mathbb{L}_n/G \subset T^*(\mathbb{C}^n/G)$.
- $\mathbb{L}_G(t, -) = (\mathbb{L}_n \cap \mu^{-1}(t, 0, 0))/G \subset \mathfrak{X}_G(t)$ is the extended core of $\mathfrak{X}_G(t)$; equivalently, $\mathbb{L}_G(t, -) = \mathfrak{X}_G(t) \cap \mathbb{L}_G \subset \mathbb{L}_G$ is the t -stable locus in $\mathbb{L}_G$.
- The category $\mathcal{O}$ skeleton $\mathbb{L}_G(t,m) \subset \mathbb{L}_G(t, -)$ is the subset which remains bounded under the action of $\mathbb{C}_m^\times$.

6.1. Stable microrestriction

We now begin to study the category of microsheaves on the Lagrangian $\mathbb{L}_G(t, -) \subset \mathfrak{X}_G(t)$. Our main tool will be the embedding of $\mathfrak{X}_G(t)$ in the stack $T^*(\mathbb{C}^n/G)$, to which we will apply the theory of Section 5.2.

DEFINITION 6.2. — *We will write τ_G for the polarization on $\mathfrak{X}_G(t)$ induced from the embedding $\mathfrak{X}_G(t) \subset T^*(\mathbb{C}^n/G)$ by Lemma 5.6.*

Our first goal is to describe the category $\mu\mathrm{Sh}_{\mathbb{L}_G(t, -), \tau_G}(\mathbb{L}_G(t, -))$, which we will accomplish by taking a limit over microsheaf categories on an open cover of the Lagrangian $\mathbb{L}_G(t, -)$.

Recall from Proposition 2.21 that there is a bijection between irreducible components of $\mathbb{L}_G(t, -)$ and chambers of the hyperplane arrangement $\mathcal{H}(t)$, realized by a map $\mu^{\mathbb{R}} : \mathbb{L}_G(t, -) \rightarrow \mathfrak{f}_{\mathbb{R}}^{\vee}(t)$ whose restriction to each irreducible component of $\mathbb{L}_G(t, -)$ is its toric moment map to the corresponding polytope in $\mathcal{H}(t)$. The space $\mathfrak{f}_{\mathbb{R}}^{\vee}(t)$, stratified by the hyperplane arrangement $\mathcal{H}(t)$, has an obvious open cover indexed by strata, where to each stratum S we associate the union of strata containing S in their closure.

DEFINITION 6.3. — *Let $\mathbf{P}(t)$ denote the stratum poset of $\mathfrak{f}_{\mathbb{R}}^{\vee}(t)$, stratified by $\mathcal{H}(t)$.*

The poset $\mathbf{P}(t)$ indexes the aforementioned open cover of $\mathfrak{f}_{\mathbb{R}}^{\vee}(t)$, and hence also an open cover of $\mathbb{L}_G(t, -)$, which we now describe. Observe that there is a canonical embedding of $\mathbf{P}(t)$ as a subposet of the poset $\mathbf{P}$ defined in Definition 5.10, using the natural identification of $\mathbf{P}$ as the stratum poset for the coordinate hyperplane arrangement on $\mathbb{R}^n$. Thus, to each $\alpha \in \mathbf{P}(t)$, we can associate an open subset $\mathbb{L}_{\alpha} \subset \mathbb{L}$, as described in Definition 5.10. Write $\mathbb{L}_{G, \alpha} := (\mu^{-1}(t, 0, 0) \cap \mathbb{L}_{\alpha})/G \subset \mathfrak{X}_G(t)$ for the induced Lagrangian subset of $\mathfrak{X}_G(t)$.

LEMMA 6.4. — *The $\mathbf{P}(t)$ -shaped diagram of open subsets $\mathbb{L}_{G, \alpha} \subset \mathbb{L}_G(t, -)$ and their inclusions is an open cover of the Lagrangian $\mathbb{L}_G(t, -)$.*

Proof. — The Lagrangian $\mathbb{L}_{G, \alpha}$ is precisely the preimage in $\mathbb{L}_G(t, -)$ of the open neighborhood of the stratum α in the hyperplane arrangement $\mathcal{H}(t)$. $\square$

As $\mu\mathrm{Sh}_{\mathbb{L}_G(t, -), \tau_G}$ is a sheaf of categories on $\mathbb{L}_G(t, -)$, the above open cover determines a limit description of its global sections:

COROLLARY 6.5. — *There is an equivalence*

$$(6.1) \quad \mu\mathrm{Sh}_{\mathbb{L}_G(t, -), \tau_G}(\mathbb{L}_G(t, -)) \simeq \varinjlim_{\mathbf{P}(t)} \mu\mathrm{Sh}_{\mathbb{L}_G(t, -), \tau_G}(\mathbb{L}_{G, \alpha})$$

By the results of the previous section, we can reduce the above to a purely algebraic computation.

PROPOSITION 6.6. — *There are equivalences of categories*

$$(6.2) \quad \mu\mathrm{Sh}_{\mathbb{L}_G(t, -), \tau_G}(\mathbb{L}_{G, \alpha}) \simeq \mathrm{Mod}_{e_{\alpha} A_G^{\mathrm{Bet}} e_{\alpha}},$$

commuting with restriction maps and thus determining an equivalence of $\mathbf{P}(t)$ -shaped diagrams of categories.

Proof. — Let $\mathbb{L}(t, -) = \bigcup_{\alpha \in \mathbf{P}(t)} \mathbb{L}_\alpha$. Then Proposition 5.7 determines an equivalence

$$\mu\mathrm{Sh}_{\mathbb{L}(t, -)}^G \simeq \mu\mathrm{Sh}_{\mathbb{L}_G(t, -), \tau_G}$$

of sheaves of categories on $\mathbb{L}_G(t, -)$ (where implicitly on the left-hand side we have used the identification of G -invariant open subsets of $\mathbb{L}(t, -)$ with open subsets of $\mathbb{L}_G(t, -)$). The identification of $\mathbf{P}(t)$ -shaped diagrams (6.2) now follows from Corollary 5.12. $\square$

The following lemma, which is joint with McBreen and Webster, asserts that the assignment $F \mapsto \mathrm{Mod}_{e_F A_G^{\mathrm{Bet}} e_F}$ satisfies descent.

LEMMA 6.7 ([GMW25, Lemma C.2]). — *There is an equivalence*

$$(6.3) \quad \varprojlim_{\mathbf{P}(t)} \mathrm{Mod}_{e_F A_G^{\mathrm{Bet}} e_F} \simeq \mathrm{Mod}_{A_G^{\mathrm{Bet}}(t, -)}.$$

Remark 6.8. — The statement of [GMW25, Lemma C.2] refers to an algebra $\mathcal{Q}$ associated to a hyperplane arrangement $\mathcal{H}$. When $\mathcal{H} = \mathcal{H}(t)$ is the hyperplane arrangement defined in Definition 2.13, the definition of this algebra in [GMW25] coincides with $A^{\mathrm{Bet}}(G, t, -)$. Similarly, $\mathcal{Q}^F = e_F A_G^{\mathrm{Bet}}(t, -) e_F$.

COROLLARY 6.9. — *There is an equivalence of categories*

$$(6.4) \quad \mu\mathrm{Sh}_{\mathbb{L}_G(t, -), \tau_G}(\mathbb{L}_G(t, -)) \simeq \mathrm{Mod}_{A_G^{\mathrm{Bet}}(t, -)}.$$

Proof. — The left-hand side admits a limit description given by (6.1), which is computed by the equivalences (6.2) and (6.3). $\square$

Remark 6.10. — By tracing through the above equivalences, one can see that the equivalence (6.4) is induced by an equivalence between the algebra $A_G^{\mathrm{Bet}}(t, -)$ and the endomorphism algebra of the objects corepresenting microstalk functors at smooth strata of the Lagrangian $\mathbb{L}_G(t, -)$.

We note the following corollary of our results, which ought to go by the name of “categorical Kirwan surjectivity” (a version of which was proved in the de Rham setting in [BPW16, Theorem 5.31]); it witnesses the essential surjectivity of restriction from stacky microsheaves to the t -stable locus.

COROLLARY 6.11. — *The restriction functor*

$$\mu\mathrm{Sh}_{\mathbb{L}}^G(\mathbb{L}) \longrightarrow \mu\mathrm{Sh}_{\mathbb{L}_G(t, -), \tau_G}(\mathbb{L}_G(t, -))$$

has a fully faithful left adjoint.

Proof. — By Corollary 5.12 and Corollary 6.9, this restriction functor is equivalent to the functor $\mathrm{Mod}_{A_G^{\mathrm{Bet}}} \rightarrow \mathrm{Mod}_{A_G^{\mathrm{Bet}}(t, -)}$, which does indeed admit a fully faithful left adjoint, given by tensoring with the $A_G^{\mathrm{Bet}} - A_G^{\mathrm{Bet}}(t, -)$ -bimodule $A_G^{\mathrm{Bet}} e_{\mathcal{F}}$. $\square$

6.2. Betti category $\mathcal{O}$

We are now ready for the computation of the Betti category $\mathcal{O}$.

THEOREM 6.12. — *There is an equivalence of categories*

$$(6.5) \quad \mu\mathrm{Sh}_{\mathbb{L}_G(t,-),\tau_G}(\mathbb{L}_G(t,m)) \simeq \mathrm{Mod}_{A_G^{\mathrm{Bet}}(t,m)},$$

intertwining the microlocal perverse t -structure with the standard t -structure.

Proof. — By stop removal, the left-hand side of (6.5) can be computed as a quotient

$$\mu\mathrm{Sh}_{\mathbb{L}_G(t,-),\tau_G}(\mathbb{L}_G(t,m)) \simeq \mu\mathrm{Sh}_{\mathbb{L}_G(t,-),\tau_G}(\mathbb{L}_G(t,-))/\mathcal{B}^c,$$

where $\mathcal{B}^c \subset \mu\mathrm{Sh}_{\mathbb{L}_G(t,-),\tau_G}(\mathbb{L}_G(t,-))$ is the full subcategory generated by corepresentatives of the microstalk functors at the smooth points of m -unbounded components of $\mathbb{L}_G(t,-)$. Under the equivalence (6.4), $\mathcal{B}^c$ corresponds to the full subcategory of $\mathrm{Mod}_{A_G^{\mathrm{Bet}}(t,-)}$ split-generated by the module $e_{\mathcal{F}}A_G^{\mathrm{Bet}}e_{\mathcal{F}\cap\mathcal{B}^c}$, where we write $e_{\mathcal{F}\cap\mathcal{B}^c}$ for the idempotent corresponding to those chambers which are both t -feasible and m -unbounded.

Using the equivalence $\mu\mathrm{Sh}_{\mathbb{L}_G(t,-),\tau_G}(\mathbb{L}_G(t,-)) \simeq \mathrm{Mod}_{A_G^{\mathrm{Bet}}(t,-)}$, we thus have a recollement

$$(6.6) \quad \begin{array}{ccccc} & \swarrow & & \swarrow & \\ \mu\mathrm{Sh}_{\mathbb{L}_G(t,-),\tau_G}(\mathbb{L}_G(t,m)) & \hookrightarrow & \mathrm{Mod}_{A_G^{\mathrm{Bet}}(t,-)} & \twoheadrightarrow & \mathrm{Mod}_{e_{\mathcal{F}\cap\mathcal{B}^c}A_G^{\mathrm{Bet}}(t,-)e_{\mathcal{F}\cap\mathcal{B}^c}} \\ & \searrow & & \searrow & \end{array}$$

We conclude from Theorem 4.7 that the left-hand category in (6.6) is equivalent to $\mathrm{Mod}_{A_G^{\mathrm{Bet}}(t,m)}$. $\square$

Proof of Theorem 1.4. — By Theorem 6.12 we have

$$\mu\mathrm{Sh}_{\mathbb{L}_G(t,-),\tau_G}(\mathbb{L}_G(t,m)) \simeq \mathrm{Mod}_{A_G^{\mathrm{Bet}}(t,m)}.$$

By Corollary 4.16 $\mathrm{Mod}_{A_G^{\mathrm{Bet}}(t,m)} = \mathrm{Mod}_{A_G^{\mathrm{dR}}(t,m)}$. $\square$

REMARK 6.13. — In light of Theorem 6.12, we can now reinterpret Theorem 4.7 as the statement that the stop removal functor

$$\mu\mathrm{Sh}_{\mathbb{L}_G(t,-),\tau_G}(\mathbb{L}_G(t,-)) \longrightarrow \mu\mathrm{Sh}_{\mathbb{L}_G(t,-),\tau_G}(\mathbb{L}_G(t,m))$$

becomes fully faithful when restricted to the subcategory generated by simple objects, i.e., components of $\mathbb{L}_G(t,-)$. This is obvious for the compact components of $\mathbb{L}_G(t,-)$ but nontrivial in general. (See the following remark for a counterexample.)

REMARK 6.14. — Theorem 4.7 states that if we take $e := e_{\mathcal{F}\cap\mathcal{B}^c}$ to be the idempotent in $A := A_G^{\mathrm{Bet}}(t,-)$ corresponding to feasible unbounded sign vectors, then the ideal $AeA \subset A$ is stratifying in the sense of Definition 3.3. If instead we had taken e to be the idempotent corresponding to all noncompact components of $\mathbb{L}(t,-)$, this would no longer be true, and to obtain a recollement as in (6.6) it would be necessary to replace A/AeA with the derived quotient defined in Definition 3.1. In the case where $G = \mathbb{C}^\times \xrightarrow{\Delta} (\mathbb{C}^\times)^{n+1} = D$ is the diagonal copy of $\mathbb{C}^\times$ inside D , this construction gives an explicit presentation of the dg algebra $C_*(\Omega\mathbb{P}^n)$ governing the category $\mathrm{Loc}(\mathbb{P}^n)$ of local systems on projective space.

7. Fukaya categories

The fundamental theorem of Ganatra–Pardon–Shende relates partially wrapped Fukaya categories to microlocal sheaf categories. As a consequence:

THEOREM 7.1. — *Given a datum (G, t, m) for category $\mathcal{O}$, there is an equivalence of categories*

$$(7.1) \quad \mathcal{W}(\mathfrak{X}_G(t), \partial_\infty \mathbb{L}_G(t, m)) \simeq \left(\mu\mathrm{Sh}_{\mathbb{L}_G(t, m)}(\mathbb{L}_G(t, m))^c \right)^{\mathrm{op}}$$

compatible with the microlocal perverse t -structure.

Proof. — $\mathfrak{X}_G(t)$ is I -holomorphic and the relative skeleton $\mathbb{L}_G(t, m)$ is an analytic isotropic subvariety. $\mathfrak{X}_G(t)$ is Weinstein by Lemma 2.26. The claim follows by [GPS24a, Theorem 1.4]. $\square$

Combining Theorems 1.4 and 7.1, we deduce Corollary 1.9. In particular, there is a t -exact equivalence of categories

$$(7.2) \quad \mathcal{W}(\mathfrak{X}_G(t), \partial_\infty \mathbb{L}_G(t, m))^{\mathrm{op}} \simeq D^b(\mathcal{O}_G^{\mathrm{dR}}(t, m)).$$

Remark 7.2. — For any holomorphic Weinstein manifold X and conic holomorphic (possibly singular) Lagrangian $\mathbb{L} \subset X$, there is always a map

$$(7.3) \quad D^b((\mathcal{W}(X, \partial_\infty \mathbb{L})^c)^\vee) \longrightarrow \mathcal{W}(X, \partial_\infty \mathbb{L})^c$$

but it may fail to be an equivalence. For example, if $X = T^*\mathbb{P}^1$ and Λ is the zero section, then this map is the inclusion of $\mathrm{Mod}_{\mathbb{C}}$ into $\mathrm{Loc}(S^2)$. This example may be understood as illustrating the failure of the heart of the t -structure to capture the derived behavior of the quotient algebra described in Remark 6.14.

Remark 7.3. — The $\mathbb{S}$ -action of weight-2 described in Section 2.5 furnishes an anti-symplectic involution of $\mathfrak{X}_G(t)$ which negates the Liouville form and fixes $\mathbb{L}_G(t, m)$ set-wise. It follows that $\mathcal{W}(\mathfrak{X}_G(t), \partial_\infty \mathbb{L}_G(t, m))$ and $\mathcal{W}(\mathfrak{X}_G(t))$ are self opposite.

7.1. Formality

Recall that an A_∞ algebra is said to be *formal* if it is equivalent to its cohomology algebra. The paradigmatic example of formality is a classical result of Deligne–Griffiths–Mumford–Sullivan [DGMS75], who proved that the A_∞ algebra of chains on a compact Kähler manifold M is formal in characteristic zero. When M can be embedded as a holomorphic Lagrangian in a hyperkähler manifold, Solomon–Verbitsky proved in [SV19] that this algebra is isomorphic to the Floer cochains on M . In slogan form, “holomorphic Lagrangians generically don’t bound holomorphic disks.” This motivates the following well-known “folk-conjecture” in symplectic topology, whose statement can be found in [SV19, Question 2.9]:

CONJECTURE (Formality conjecture). — *Let $M = (M, g, I, J, K)$ be a hyperkähler manifold. Let $L_1, \dots, L_k$ be a collection of closed I -complex submanifolds which are Lagrangian with respect to $\omega_{\mathbb{C}} := \omega_J + i\omega_K$. Then the Floer–Fukaya A_∞ algebra $CF^*(\oplus_i L_i, \oplus_i L_i)$, defined with respect to ω_J for the real symplectic manifold (M, ω_J) is formal in characteristic zero.*

In our situation, we prove a slightly more general result, which also incorporates the non-compact Lagrangian components of the category $\mathcal{O}$ skeleton.

COROLLARY 7.4. — *Fix a datum (G, t, m) for category $\mathcal{O}$. Let S be the direct sum of the irreducible components of $\mathbb{L}_G(t, m)$. Then $CF^*(S, S)$ is formal in characteristic zero.*

Proof. — Note that $CF^*(S, S) \simeq \text{End}_{(\mathcal{W}(\mathfrak{X}_G(t), \partial_\infty \mathbb{L}_G(t, m)))}(S)$. Hence it is equivalent to check formality for the endomorphism algebra of the image of S under (7.2). In other words, we must verify that the direct sum of the simples in $\mathcal{O}_G^{\text{dR}}(t, m)$ has formal Yoneda-dg algebra, which is a known fact (e.g. $\mathcal{O}_G^{\text{dR}}(t, m)$ is Koszul [BLPW12, Corollary 4.10]; this implies the desired claim e.g. by [BLPW16, Remark 4.6].) $\square$

It may be useful to contrast Corollary 7.4 with earlier results on the hyperkähler formality conjecture.

The first work going beyond the case of a single Lagrangian is probably Seidel–Thomas [ST01] which established the conjecture for skeleta of ALE spaces of type A_n . This was subsequently extended to type D_n by Etgü–Lekili [EL17]. These proofs are in some sense algebraic, since they establish that the relevant A_∞ algebras are *intrinsically formal*, meaning that any A_∞ structure with the given cohomology ring is formal.

An entirely different method for proving formality was introduced by Abouzaid–Smith [AS16] and was also used by Mak–Smith [MS22]. They construct a non-commutative degree-1 Hochschild cocycle (understood as a non-commutative vector field) satisfying a purity condition. By an argument of Seidel, the existence of such a vector field implies formality of the Floer algebra of the Lagrangians, due to the presence of an extra grading induced by the cocycle.

This is the approach to the hypertoric formality conjecture taken in [LLM].

On the other hand, our Corollary 7.4 follows from Theorem 1.4, since the image of S under (1.5) and (1.3) is known to have formal endomorphism algebra. However, from our perspective, the origin of this formality is Hodge theory: the endomorphism algebra of S is equivalent to the endomorphism algebra of a D -module which admits a lift to mixed Hodge modules. Purity of these mixed Hodge modules furnishes an extra grading which precludes the existence of any higher A_∞ operations. We recommend [CG16] as a reference for this theory, and [Web17, Section 2.5] for a discussion in the context of de Rham category $\mathcal{O}$.

7.2. Koszul duality

As in the previous section, let S be the direct sum of the irreducible components of $\mathbb{L}_G(t, m)$, as an object of $\mathcal{W}(\mathfrak{X}_G(t), \partial_\infty \mathbb{L}_G(t, m))$, and let P be the direct sum of their projective covers (i.e., the direct sum of cocores to the components of S). Then by applying the main theorem of [BLPW10], we deduce a relationship between their endomorphism algebras.

THEOREM 7.5. — *The algebras $\text{end}(S)$ and $\text{end}(P)$ are Koszul bidual.*

Proof. — This is the statement of Koszulity for the algebra $A_G^{\text{Bet}}(t, m)$, which we may deduce from the same statement about $A_G^{\text{dR}}(t, m)$ in [BLPW10, Theorem B]. $\square$

However, we also prove a new result. Recall from Lemma 2.28 that the skeleton of the Liouville manifold $\mathfrak{X}_G(t)$ is equal to the union of *compact* components of $\mathbb{L}_G(t, m)$. Let $S_{\text{skel}} = \bigoplus_{\alpha} S_{\alpha}$ be the direct sum of those components, as an object of the (fully) wrapped Fukaya category $\mathcal{W}(\mathfrak{X}_G(t))$. Similarly, let $P_{\text{skel}} = \bigoplus_{\alpha} P_{\alpha}$ be the direct sum of cocores to those components.

THEOREM 7.6. — *The following algebras are Koszul bidual:*

$$\text{end}_{\mathcal{W}(\mathfrak{X}_G(t))}(P_{\text{skel}}) \quad \text{and} \quad \text{end}_{\mathcal{W}(\mathfrak{X}_G(t))}(S_{\text{skel}}).$$

Proof. — Let A denote the algebra $A_G^{\text{Bet}}(t, m)$, which is the endomorphism algebra of the indecomposable projective objects in $\mathcal{W}(\mathfrak{X}_G(t, m), \partial_{\infty}\mathbb{L}_G(t, m))^{\text{op}}$, and write $A^!$ for its Koszul dual, which, as we have seen above, is the endomorphism algebra of the simple components of $\mathbb{L}_G(t, m)$.

Let $\mathcal{N} \subset \mathcal{F} \cap \mathcal{B}$ be the set of bounded feasible sign vectors corresponding to noncompact components of $\mathbb{L}_G(t, m)$, and write $e_{\mathcal{N}} \in A$ for the corresponding idempotent (and $e_{\mathcal{N}^c}$ for the complementary idempotent). By stop removal, the first algebra mentioned in the theorem is $A^{/L}(Ae_{\mathcal{N}}A)$. The second algebra is $e_{\mathcal{N}^c}A^!e_{\mathcal{N}^c}$. The Koszul duality between these is the statement of Corollary 3.16. $\square$

The phenomenon of Koszul duality in symplectic geometry has been studied earlier in e.g. [EL17, EL23, Li19a, Li19b]; guided by those investigations, we highlight here several known Floer-theoretic consequences of Theorem 7.6.

First, we deduce that the symplectic cohomology of the Weinstein manifold $\mathfrak{X}_G(t)$ can be calculated purely in terms of compact components of the Lagrangian skeleton.

DEFINITION 7.7. — *We will write $B_G(t) := \text{end}_{\mathcal{W}(\mathfrak{X}_G(t))}(S_{\text{skel}})$ for the endomorphism algebra of compact components of the skeleton of $\mathfrak{X}_G(t)$.*

The components $S_{\alpha}, S_{\alpha'}$ intersect cleanly in the manifold $S_{\alpha} \cap S'_{\alpha'}$, and the morphisms among these Lagrangians in the Fukaya category may be computed as cohomologies

$$(7.4) \quad \text{Hom}_{\mathcal{W}(\mathfrak{X}_G(t))}(S_{\alpha}, S_{\alpha'}) \simeq H^*(S_{\alpha} \cap S_{\alpha'}).$$

The spaces (7.4) are the matrix coefficient blocks of the matrix algebra $B_G(t)$, and the compositions among them (i.e., the multiplications in the algebra $B_G(t)$) are given by cap product in the triple intersection followed by Gysin pushforward, as in [BLPW10, Section 4.3]. (A priori, it might have seemed necessary to work with some model of cochains, rather than cohomology, on the intersections $S_{\alpha} \cap S_{\alpha'}$, and to keep careful track of higher A_{∞} operations, but by Corollary 7.4 the algebra $B_G(t)$ is actually formal.)

Using the algebra $B_G(t)$, we compute $SH^*(\mathfrak{X}_G(t))$.

COROLLARY 7.8. — *There is an equivalence*

$$(7.5) \quad SH^{*+\dim_{\mathbb{C}}(\mathfrak{X}_G(t))}(\mathfrak{X}_G(t)) \simeq HH_{-*}(B_G(t))^{\vee}.$$

between the symplectic cohomology of $\mathfrak{X}_G(t)$ and the dual of the Hochschild homology of the algebra $B_G(t)$.

Proof. — By [Gan13, Theorem 1.1], the open-closed map gives an isomorphism between the left-hand side of (7.5) and the Hochschild homology $HH_*(\mathcal{W}(\mathfrak{X}_G(t)))$ of the wrapped Fukaya category $\mathcal{W}(\mathfrak{X}_G(t))$, which is the category of perfect modules over $\text{end}_{\mathcal{W}(\mathfrak{X}_G(t))}(P_{\text{skel}})$. (Here we have implicitly appealed to Remark 7.3). From the Koszul duality equivalence of Theorem 7.6, this is equivalent to the thick subcategory of Mod_B generated by the simple B -modules. However, since B is finite-dimensional, this latter category is dual to Perf_B by [Cam14, Proposition 4.9]. We conclude that the Hochschild homology of $\text{end}_{\mathcal{W}(\mathfrak{X}_G(t))}(P_{\text{skel}})$ is dual to the Hochschild homology of $B_G(t)$, as in [Cam14, Theorem 4.16]. $\square$

Example 7.9. — If $\mathfrak{X}_G(t) = T^*\mathbb{P}^1$ (see Example 1.22) then we have equivalences

$$HH_*(B_G(t))^\vee \simeq HH_*(H^*(\mathbb{P}^1))^\vee \simeq H^{-*}(L\mathbb{P}^1)^\vee.$$

Meanwhile, $SH^{*+\dim(\mathfrak{X}_G(t))}(\mathfrak{X}_G(t)) \simeq H_{-*}(L\mathbb{P}^1)$.

Remark 7.10. — The significance of Corollary 7.8 is that $B_G(t)$ is a finite dimensional associative algebra which can be read off directly from the hyperplane arrangement. Hence the right hand side of (7.5) is, at least in principle, computable (see [EL17] for explicit computations in the 4-dimensional case). Of course, by [Gan13, Theorem 1.1], one can always express the symplectic cohomology of a Weinstein manifold as the Hochschild homology of an A_∞ -algebra. However, there is no algorithm to compute the differentials, let alone the Hochschild homology.

Remark 7.11. — In fact, one can show that $SH^*(\mathfrak{X}_G(t))$ is always locally finite, so (7.5) remains true without dualizing the right hand side. Indeed, we saw in the proof of Theorem 7.6 that $\mathcal{W}(\mathfrak{X}_G(t)) \simeq \text{Perf}_{A/L Ae_{\mathcal{N}} A}$ for $A = A_G^{\text{Bet}}(t, m)$. We know from [GPS24b] that $A/L Ae_{\mathcal{N}} A$ is smooth, i.e. the diagonal bimodule $\Delta_{A/L Ae_{\mathcal{N}} A}$ is perfect. Since $A/L Ae_{\mathcal{N}} A$ and hence $(A/L Ae_{\mathcal{N}} A) \otimes (A/L Ae_{\mathcal{N}} A)^{\text{op}}$ is locally finite (Corollary 3.16), any perfect $A/L Ae_{\mathcal{N}} A$ -bimodule has locally finite endomorphism ring. But $SH^*(\mathfrak{X}_G(t)) = HH^*(\mathcal{W}(\mathfrak{X}_G(t))) = \text{end}(\Delta_{A/L Ae_{\mathcal{N}} A})$ by [Gan13, GPS24b].

Koszul duality also implies the following generation result.

COROLLARY 7.12. — *Any compact Lagrangian brane in $\mathfrak{X}_G(t)$ is split-generated by components of the compact core.*

Since a compact Lagrangian defines a proper module over $\mathcal{W}(\mathfrak{X}_G(t)) = A/L(AeA)$, Corollary 7.12 follows immediately from the following lemma:

LEMMA 7.13. — *In the notation of Proposition 3.15, $\mathcal{T}_{\text{RMod}_{A/L(AeA)}}(T)$ is the category of proper modules over $A/L(AeA)$.*

Proof. — Note that since A is a finite-dimensional quasi-hereditary algebra and has finite homological dimension we have that $\mathcal{T}_{\text{RMod}_A}(S) = \text{Perf}_A = \text{Prop}_A$. Moreover the fully faithful embedding $i_* : \text{RMod}_{A/L(AeA)} \hookrightarrow \text{RMod}_A$ sends proper modules to proper modules and $i_*(T) = Se^c$. Thus it suffices to show that

$$\text{Im } i_* \cap \text{Prop}_A = \text{Im } i_* \cap \mathcal{T}_{\text{RMod}_A}(S) = \mathcal{T}_{\text{RMod}_A}(Se^c).$$

This follows from the fact that the left inverse of i_* is i^* and $i^*(Se) = 0$. $\square$

Another application of Koszul duality is the existence of dilations in symplectic cohomology. Let $n = \dim_{\mathbb{C}}(\mathfrak{X}_G(t))$, and recall that the main results of [Gan13, ST25] furnish the algebras $A/^L(AeA) = \text{end}_{\mathcal{W}(\mathfrak{X}_G(t))}(P_{\text{skel}})$ and $B_G(t) = \text{end}_{\mathcal{W}(\mathfrak{X}_G(t))}(S_{\text{skel}})$ with smooth (respectively, proper) n -Calabi–Yau structures, which determine BV-algebras structure on their Hochschild cohomologies by [Gin06, Theorem 3.4.3] and [Tra08]. (As explained in [BR23], the identification of Hochschild cohomology, which is an $\mathbb{E}_2$ -algebra, with Hochschild homology, which carries a circle action, given by the CY structure results in a framed $\mathbb{E}_2$ -algebra, i.e., a homotopy BV algebra.) Moreover, the Koszul duality between these algebras induces a BV-algebra isomorphism of their Hochschild cohomologies by [HLW23].

Finally, since $\mathfrak{X}_G(t)$ is Weinstein, we know from [Sei08, Section 5] that the closed-open map

$$SH^*(\mathfrak{X}_G(t)) \longrightarrow HH^*(A/^L(AeA), A/^L(AeA))$$

is also a BV-algebra isomorphism. We summarize the above discussion as follows:

LEMMA 7.14. — *There is a BV-algebra isomorphism*

$$SH^*(\mathfrak{X}_G(t)) \simeq HH^*(B_G(t)).$$

Following [SS12], if B is a BV-algebra (with BV-operator Δ) and α is a 1-cocycle in B satisfying the equation $\Delta(\alpha) = 1$, then we call α a *dilation* on B . We can now state our second application of Koszul duality.

COROLLARY 7.15. — *$SH^*(\mathfrak{X}_G(t))$ admits a dilation.*

Proof. — Using the equivalence of Lemma 7.14, we need to produce a dilation in the Hochschild cohomology of $B_G(t)$. Observe that $B_G(t)$ is formal, it admits a canonical Hochschild 1-cocycle $\mathbf{eu}_{B_G(t)}$, which acts on a degree- d element of $B_G(t)$ as multiplication by d . The result now follows from the following lemma. $\square$

LEMMA 7.16. — *Suppose that B is a formal A_{∞} -algebra equipped with a proper n -Calabi–Yau structure $\langle -, - \rangle$. Then $\Delta(\frac{1}{n}\mathbf{eu}_B) = 1$.*

Proof. — Let $x \in B$ be a n -cocycle and let $\alpha \in CC^1(B, B)$ be a Hochschild 1-cocycle in cohomological degree zero. By definition of Δ [Tra08], we have $\langle \Delta(\alpha), x \rangle = \langle \alpha(x), 1 \rangle$. Meanwhile $\mathbf{eu}(x) = nx$, so we have $\langle \Delta(\mathbf{eu}_B), x \rangle = \langle nx, 1 \rangle = n\langle 1, x \rangle$. Hence $\Delta(\mathbf{eu}_B) = n \cdot 1$. $\square$

The existence of a dilation on symplectic cohomology puts strong constraints on the symplectic geometry of $\mathfrak{X}_G(t)$. Standard consequences include: non-existence of exact Lagrangian $K(\pi, 1)$ [SS12], improved intersection numbers for Lagrangians [SS12], upper bounds on the size of a collection of disjointable Lagrangian spheres [Sei14].

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