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HOLOMORPHIC VOLUME FORMS ON REPRESENTATION VARIETIES OF SURFACES WITH BOUNDARY

FORMES DE VOLUME HOLOMORPHES SUR LES VARIÉTÉS DE REPRÉSENTATIONS DES SURFACES À BORD

ABSTRACT. — For closed and oriented hyperbolic surfaces, a formula of Witten establishes an equality between two volume forms on the space of representations of the surface in a semisimple Lie group. One of the forms is a Reidemeister torsion, the other one is the power of the Atiyah–Bott–Goldman–Narasimhan symplectic form. We introduce an holomorphic volume form on the space of representations of the circle, so that, for surfaces with boundary, it appears as peripheral term in the generalization of Witten’s formula. We compute explicit volume and symplectic forms for some simple surfaces and for the Lie group $SL_N(\mathbb{C})$.

RÉSUMÉ. — Pour les surfaces hyperboliques fermées et orientées, une formule de Witten établit une égalité entre deux formes de volume sur l’espace de représentations des groupes de surface dans un groupe de Lie semi-simple. Une de ces formes est une torsion de Reidemeister, l’autre est la forme de volume canoniquement associée à la forme symplectique d’Atiyah–Bott–Goldman–Narasimhan. Nous introduisons une forme de volume holomorphe sur l’espace des représentations du cercle, de sorte que, pour les surfaces à bord, elle apparaisse comme terme

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périphérique dans la généralisation de la formule de Witten. Pour certaines surfaces simples et pour le groupe de Lie $\mathrm{SL}_N(\mathbb{C})$ nous calculons explicitement les formes volume et les formes symplectiques.

1. Introduction

Along this paper $S = S_{g,b}$ denotes a compact, oriented, connected surface with nonempty boundary, of genus g and with $b \geq 1$ boundary components. We assume that $\chi(S) = 2 - 2g - b < 0$. The fundamental group $\pi_1(S)$ is a free group F_k of rank $k = 1 - \chi(S) \geq 2$.

Let G be a connected, semisimple, complex, linear group with compact real form $G_{\mathbb{R}}$, e.g. $G = \mathrm{SL}_N(\mathbb{C})$ and $G_{\mathbb{R}} = \mathrm{SU}(N)$. We also assume that G is simply connected; notice that since $\pi_1(S)$ is free, its representations lift to the universal covering of the Lie group.

Fix a nondegenerate symmetric bilinear G -invariant form on the Lie algebra

$$\mathcal{B}: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C},$$

such that the restriction of \mathcal{B} to $\mathfrak{g}_{\mathbb{R}}$, the Lie algebra of $G_{\mathbb{R}}$, is positive definite. This means that \mathcal{B} is a negative multiple of the Killing form.

Let $\mathcal{R}(S, G)$ denote the set of conjugacy classes of representations of $\pi_1(S) \cong F_k$ into G . We are only interested in irreducible representations for which the centralizer coincides with the center of G . Following Johnson and Millson [JM87] we call such representations *good* (see Definition 2.3), and we use the notation $\mathcal{R}^*(S, G)$ to denote the corresponding open subset of $\mathcal{R}(S, G)$.

For a closed surface Σ , the bilinear form \mathcal{B} induces two \mathbb{C} -valued differential forms on $\mathcal{R}^*(\Sigma, G)$, a holomorphic volume form Ω_{Σ} defined as a Reidemeister torsion and the Atiyah–Bott–Goldman–Narasimhan (holomorphic) symplectic form ω . Witten has shown the following theorem for compact groups, here we state its complexification:

THEOREM 1.1 ([Wit91, Witten]). — *If Σ is a closed, oriented and hyperbolic surface, then*

$$\Omega_{\Sigma} = \frac{\omega^n}{n!}$$

on $\mathcal{R}^*(\Sigma, G)$, where $n = \frac{1}{2} \dim \mathcal{R}^*(\Sigma, G) = -\frac{1}{2} \chi(\Sigma) \dim(G)$.

A proof of this theorem as well as an introduction to Reidemeister torsion and representation spaces of surface groups can be found in Labourie's book [Lab13].

For surfaces with boundary S , we need to consider also $\mathcal{R}(S, \partial S, G)_{\rho_0}$, the relative set of conjugacy classes of representations (for each peripheral curve we require its image to be in a fixed conjugacy class), see Subsection 2.2. Let $\mathcal{R}^*(S, \partial S, G)_{\rho_0}$ denote the corresponding open subset of good representations. The holomorphic volume form Ω_S is defined on $\mathcal{R}^*(S, G)$ but the holomorphic symplectic form ω is defined on $\mathcal{R}^*(S, \partial S, G)_{\rho_0}$.

The geometric idea is to think of $\mathcal{R}(S, \partial S, G)_{\rho_0}$ as the fibre of the restriction map $\mathcal{R}(S, G) \rightarrow \mathcal{R}(\partial S, G)$. We aim to express the volume form (Reidemeister torsion) on

$\mathcal{R}(S, G)$ as the product of a volume form on $\mathcal{R}(S, \partial S, G)_{\rho_0}$ (induced by the symplectic form) and a volume form on $\mathcal{R}(\partial S, G)$ that we construct in the next paragraph.

To obtain a volume form on $\mathcal{R}(\partial S, G)$, we deal with each component of ∂S , which is a circle. We identify the variety of representations of the circle S^1 with G , by mapping each representation to the image of a fixed generator of $\pi_1(S^1)$. We restrict to regular representations namely, those which map the generators of $\pi_1(S^1)$ to regular elements. Then $\mathcal{R}^{reg}(S^1, G) \cong G^{reg}/G$. Using that G is simply connected (see Remark 2.13 when G is not simply connected), one of the consequences of Steinberg’s Theorem [Ste65] is that

$$\mathcal{R}^{reg}(S^1, G) \cong G^{reg}/G \cong \mathbb{C}^r,$$

where $r = \text{rank } G$, and that there is a natural isomorphism (Corollary 2.12):

$$H^1(S^1; \text{Ad } \rho) \cong T_{[\rho]} \mathcal{R}^{reg}(S^1, G).$$

In Section 4.3 we show the existence of a form $\nu: \wedge^r H^1(S^1, \text{Ad } \rho) \rightarrow \mathbb{C}$ defined by the formula

$$(1.1) \quad \nu(\wedge \mathbf{v}) = \pm \sqrt{\text{TOR}(S^1, \text{Ad } \rho, \mathbf{o}, \mathbf{u}, \mathbf{v})} \langle \wedge \mathbf{v}, \wedge \mathbf{u} \rangle.$$

Here \mathbf{u} and \mathbf{v} denote bases of $H^0(S^1, \text{Ad } \rho)$ and $H^1(S^1, \text{Ad } \rho)$ respectively and $\wedge \mathbf{u}$ and $\wedge \mathbf{v}$ their exterior product. Moreover, TOR denotes the Turaev’s sign refined torsion [Tur86, Section 3], \mathbf{o} an *homology orientation* of $H^*(S^1; \mathbb{R})$ (see Section 4.2), and $\langle \cdot, \cdot \rangle$ the duality pairing $H^1(S^1, \text{Ad } \rho) \times H^0(S^1, \text{Ad } \rho) \rightarrow \mathbb{C}$. We prove in Lemma 4.5 that the value $\nu(\wedge \mathbf{v}) \in \mathbb{C}$ is independent of \mathbf{u} .

Steinberg’s Theorem ([Ste65], see also [Ste74, Pop11]) provides an isomorphism

$$(\sigma_1, \dots, \sigma_r): G^{reg}/G \xrightarrow{\cong} \mathbb{C}^r,$$

where $\sigma_1, \dots, \sigma_r$ denotes a system of fundamental characters of G , which also proves the isomorphism $G//G \cong \mathbb{C}^r$.

When $G = \text{SL}_N(\mathbb{C})$, then $r = N - 1$ and $(\sigma_1, \dots, \sigma_r)$ are the coefficients of the characteristic polynomial.

PROPOSITION 1.2. — *When G is simply connected, then*

$$\nu = \pm C \, d\sigma_1 \wedge \dots \wedge d\sigma_r,$$

for some constant $C \in \mathbb{C}^*$ depending on G and \mathcal{B} . In addition, for $G = \text{SL}_N(\mathbb{C})$ and $\mathcal{B}(X, Y) = -\text{tr}(XY)$ for $X, Y \in \text{SL}_N(\mathbb{C})$,

$$C = \pm (-1)^{(N-1)(N-2)/4} \sqrt{N}.$$

Let $\rho_0 \in \mathcal{R}^*(S, G)$ be ∂ -regular, i.e. the image of each peripheral curve is a regular element of G (Definitions 2.8 and 2.9). We have an exact sequence (Corollary 2.12):

$$0 \rightarrow T_{[\rho]} \mathcal{R}^*(S, \partial S, G)_{\rho_0} \rightarrow T_{[\rho]} \mathcal{R}^*(S, G) \rightarrow \bigoplus_{i=1}^b T_{[\rho(\partial_i)]} \mathcal{R}^{reg}(\partial_i, G) \rightarrow 0,$$

where $\partial S = \partial_1 \sqcup \dots \sqcup \partial_b$ denote the boundary components of S . For a ∂ -regular representation $\rho: \pi_1(S) \rightarrow G$ we let ν_i denote the form corresponding to the restriction $\rho|_{\pi_1(\partial_i)}: \pi_1(\partial_i) \rightarrow G$ as in (1.1) on $\partial_i \cong S^1$.

Set $d = \dim G$, $r = \text{rank } G$, and $b > 0$ be the number of components of ∂S . The following generalizes Theorem 1.1 to surfaces with boundary, [Wit91] see also [BL99, Theorem 5.40] and Remark 4.9.

THEOREM 1.3. — *Let $\rho_0 \in R(S, G)$ be a good, ∂ -regular representation. Then on $T_{[\rho_0]} \mathcal{R}^*(S, G)$ we have:*

$$\Omega_{\pi_1(S)} = \pm \frac{\omega^n}{n!} \wedge \nu_1 \wedge \cdots \wedge \nu_b,$$

where $n = \frac{1}{2} \dim \mathcal{R}^*(S, \partial S, G)_{\rho_0} = \frac{1}{2}(-\chi(S)d - br)$.

Notice that we write $\Omega_{\pi_1(S)}$ instead of Ω_S , as the simple homotopy type of S only depends on $\pi_1(S)$. Following Witten [Wit91] in the closed case, the proof of Theorem 1.3 is based on Franz–Milnor duality for Reidemeister torsion.

The formula of Theorem 1.3 is homogeneous in the bilinear form $\mathcal{B}: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$: if \mathcal{B} is replaced by $\lambda^2 \mathcal{B}$ for some $\lambda \in \mathbb{C}^*$, then ω is replaced by $\lambda^2 \omega$, ν_i by $\lambda^r \nu_i$ and $\Omega_{\pi_1(S)}$ by $\lambda^{2n+br} \Omega_{\pi_1(S)}$, as $2n + br = -\chi(S)d = \dim \mathcal{R}(S, G)$.

We focus now on $G = \text{SL}_N(\mathbb{C})$, which is simply connected and has rank $r = N - 1$. We fix a bilinear form on the Lie algebra:

CONVENTION 1.4. — *Along this paper, when $G = \text{SL}_N(\mathbb{C})$ we always assume $\mathcal{B}(X, Y) = -\text{tr}(XY)$ for $X, Y \in \mathfrak{sl}_N(\mathbb{C})$.*

We compute explicit volume forms for spaces of representations of free groups in $\text{SL}_2(\mathbb{C})$ and $\text{SL}_3(\mathbb{C})$. We start with a pair of pants $S_{0,3}$. The fundamental group $\pi_1(S_{0,3}) \cong F_2$ is free on two generators γ_1 and γ_2 . By Fricke–Klein–Vogt Theorem [Fri96, FK97, Vog89] (see [Gol09, Mag80] for a modern treatment), $X(F_2, \text{SL}_2(\mathbb{C})) \cong \mathbb{C}^3$ and the coordinates are precisely the traces of the peripheral elements γ_1 , γ_2 , and $\gamma_1 \gamma_2$, denoted by t_1 , t_2 , and t_{12} respectively. In this case the relative character variety is just a point, and the symplectic form is trivial. Thus, by applying Theorem 1.3 and equality $\nu = \pm \sqrt{2} d \text{tr}_\gamma$ (Proposition 1.2), we have

$$\Omega_{F_2} = \Omega_{\pi_1(S_{0,3})} = \pm 2\sqrt{2} dt_1 \wedge dt_2 \wedge dt_{12},$$

on $\mathcal{R}^*(F_2, \text{SL}_2(\mathbb{C}))$ (see also [Mar16, 4.3.1]).

By [GAMA93], for $k \geq 3$, the $3k-3$ trace functions $t_1, t_2, t_{12}, t_3, t_{13}, t_{23}, \dots, t_k, t_{1k}, t_{2k}$ define a local parameterization

$$T: \mathcal{R}^*(F_k, \text{SL}_2(\mathbb{C})) \setminus \text{crit}(T) \rightarrow \mathbb{C}^{3k-3},$$

where $\text{crit}(T) = \bigcup_{i \geq 3} \{t_{12i} = t_{21i}\} \cup \{t_{12\bar{1}\bar{2}} = 2\}$. Here, $t_{i_1 \dots i_i}: \mathcal{R}^*(F_k, \text{SL}_2(\mathbb{C})) \rightarrow \mathbb{C}$ stands for the trace function tr_γ if $\gamma = \gamma_{i_1} \dots \gamma_{i_i}$ with the convention $\gamma_{\bar{i}} = \gamma_i^{-1}$.

THEOREM 1.5. — *The holomorphic volume form on $\mathcal{R}^*(F_k, \text{SL}_2(\mathbb{C})) \setminus \text{crit}(T)$ is $\Omega_{F_k}^{\text{SL}_2(\mathbb{C})} = \pm T^* \Omega$, where*

$$\Omega = \pm 2\sqrt{2} dt_1 \wedge dt_2 \wedge dt_{12} \bigwedge_{i=3}^k \sqrt{2} \frac{dt_i \wedge dt_{1i} \wedge dt_{2i}}{t_{12i} - t_{21i}}.$$

Next we deal with $\text{SL}_3(\mathbb{C})$. To avoid confusion with $\text{SL}_2(\mathbb{C})$, the trace functions in $\text{SL}_3(\mathbb{C})$ are denoted by $\tau_{i_1 \dots i_k}$; notice that $\tau_{\bar{i}} \neq \tau_i$. Lawton obtains in [Law07] an

explicit description of the variety of characters $X(F_2, \text{SL}_3(\mathbb{C}))$. It follows from his result that

$$\mathcal{T} := (\tau_1, \tau_{\bar{1}}, \tau_2, \tau_{\bar{2}}, \tau_{12}, \tau_{\bar{1}\bar{2}}, \tau_{\bar{1}2}) : \mathcal{R}^*(F_2, \text{SL}_3(\mathbb{C})) \setminus \{\tau_{12\bar{1}\bar{2}} = \tau_{21\bar{2}\bar{1}}\} \rightarrow \mathbb{C}^8$$

defines a local parameterization. Using the computation of the symplectic form in [Law09], we prove in Proposition 5.7 that, on $\mathcal{R}^*(F_2, \text{SL}_3(\mathbb{C})) \setminus \{\tau_{21\bar{2}\bar{1}} = \tau_{12\bar{1}\bar{2}}\}$ the volume form is $\Omega_{F_2}^{\text{SL}_3(\mathbb{C})} = \mathcal{T}^*\Omega$, where

$$\Omega = \pm \frac{3\sqrt{-3}}{\tau_{21\bar{2}\bar{1}} - \tau_{12\bar{1}\bar{2}}} d\tau_1 \wedge d\tau_{\bar{1}} \wedge d\tau_2 \wedge d\tau_{\bar{2}} \wedge d\tau_{12} \wedge d\tau_{\bar{1}\bar{2}} \wedge d\tau_{\bar{1}2} \wedge d\tau_{\bar{1}2}.$$

This is generalized to a free group of arbitrary rank. The next proposition is a special case of a theorem of Lawton [Law10, Theorem 6]:

PROPOSITION 1.6. — *For $k \geq 3$, the $8k - 8$ trace functions*

$$\mathcal{T} = (\tau_1, \tau_{\bar{1}}, \tau_2, \tau_{\bar{2}}, \dots, \tau_k, \tau_{\bar{k}}, \tau_{12}, \tau_{\bar{1}\bar{2}}, \tau_{13}, \tau_{\bar{1}\bar{3}}, \dots, \tau_{1k}, \tau_{\bar{1}\bar{k}}, \tau_{23}, \tau_{\bar{2}\bar{3}}, \dots, \tau_{2k}, \tau_{\bar{2}\bar{k}}, \tau_{1\bar{2}}, \tau_{\bar{1}2}, \tau_{1\bar{3}}, \tau_{\bar{1}3}, \dots, \tau_{1\bar{k}}, \tau_{\bar{1}k})$$

define a local parameterization $\mathcal{T} : \mathcal{R}^*(F_k, \text{SL}_3(\mathbb{C})) \setminus \text{crit}(\mathcal{T}) \rightarrow \mathbb{C}^{8k-8}$, with

$$\begin{aligned} \text{crit}(\mathcal{T}) &= \bigcup_{i \geq 2} \{\tau_{1i\bar{i}} = \tau_{i1\bar{i}\bar{1}}\} \cup \bigcup_{i \geq 3} \{\Delta_{2i}^1 = 0\}, \\ \Delta_{2i}^1 &= (\tau_{12i} - \tau_{1i2})(\tau_{\bar{1}\bar{2}i} - \tau_{\bar{1}i\bar{2}}) - (\tau_{1\bar{2}i} - \tau_{1i\bar{2}})(\tau_{\bar{1}2i} - \tau_{\bar{1}i2}). \end{aligned}$$

Next we provide the holomorphic volume form:

THEOREM 1.7. — *The volume form on $\mathcal{R}^*(F_k, \text{SL}_3(\mathbb{C})) \setminus \text{crit}(\mathcal{T})$ is $\Omega_{F_k}^{\text{SL}_3(\mathbb{C})} = \pm \mathcal{T}^*\Omega$, for*

$$\Omega = \omega_{12} \wedge \nu_1 \wedge \nu_2 \wedge \nu_{12} \bigwedge_{i=3}^k \frac{\omega_{1i} \wedge \nu_i \wedge \nu_{1i} \wedge \nu_{2i}}{3\Delta_{2i}^1}$$

where

$$\nu_i = \sqrt{-3} d\tau_i \wedge d\tau_{\bar{i}}, \quad \nu_{\ell i} = \sqrt{-3} d\tau_{\ell i} \wedge d\tau_{\bar{\ell}\bar{i}}, \quad \omega_{1i} = \frac{1}{\tau_{1i\bar{i}} - \tau_{i1\bar{i}\bar{1}}} d\tau_{1\bar{i}} \wedge d\tau_{\bar{1}i},$$

and Δ_{2i}^1 is as in Proposition 1.6.

Remark 1.8. — Notice that all the formulas for the volume and symplectic forms are rational. Rationality of the volume forms coming from Reidemeister torsion has been addressed by several authors [Bén17, Bén16, DG16, Mar16, Por97], and rationality of the symplectic form can be deduced from Goldman’s works [Gol86, Gol04].

The paper is organized as follows. In Section 2 we review the results on spaces of representations that we need, in particular we describe the relative variety of representations. In Section 3 we recall the tools of Reidemeister torsion, including the duality formula, on which Theorem 1.3 is based. In Section 4 we describe all forms and we prove Theorem 1.3. Section 5 is devoted to formulas for $\text{SL}_N(\mathbb{C})$, the form ν and as well as the volume form for the free groups of rank 2 in $\text{SL}_2(\mathbb{C})$ and $\text{SL}_3(\mathbb{C})$. In Section 6 we use Goldman’s formula for the Poisson bracket to give the

symplectic form in terms of trace functions for the relative varieties of representations of $S_{0,4}$ and $S_{1,1}$ in $\mathrm{SL}_2(\mathbb{C})$. Finally, in Section 7 we compute volume forms on spaces of representations of free groups of higher rank in $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{SL}_3(\mathbb{C})$.

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2. Varieties of representations

Throughout this article G denotes a simply-connected semisimple complex linear Lie group. We let d denote the dimension of G , and r its rank. Also recall that along this paper $S = S_{g,b}$ denotes a compact, oriented, connected surface with nonempty boundary, of genus g and with $b \geq 1$ boundary components, $\partial S = \partial_1 \sqcup \cdots \sqcup \partial_b$. We assume that $\chi(S) = 2 - 2g - b < 0$. The fundamental group of S is a free group F_k of rank $k = 1 - \chi(S) \geq 2$.

2.1. The variety of good representations

The set of representations of $\pi_1(S) \cong F_k$ in G is

$$R(S, G) = \mathrm{hom}(\pi_1(S), G) \cong G^k.$$

It follows from [OV90, Chapter 4, Section 1.2] that G is algebraic, and hence $R(S, G)$ is an affine algebraic set (it has a natural algebraic structure independent of the choice of the isomorphism $\pi_1(S) \cong F_k$).

The group G acts on $R(S, G)$ by conjugation and we are interested in the quotient

$$\mathcal{R}(S, G) = R(S, G)/G.$$

This is not a Hausdorff space, so we need to restrict to representations with some regularity properties. Following [JM87], we define:

DEFINITION 2.1. — *A representation $\rho \in R(S, G)$ is irreducible if its image is not contained in a proper parabolic subgroup of G .*

For $\rho \in R(S, G)$, its centralizer is

$$Z(\rho) = \{g \in G \mid g\rho(\gamma) = \rho(\gamma)g, \forall \gamma \in \pi_1(S)\}.$$

PROPOSITION 2.2 ([JM87, Proposition 1.1]). — *The representation $\rho \in R(S, G)$ is irreducible if and only if the orbit $\mathcal{O}(\rho)$ is closed in $R(S, G)$ and $Z(\rho)$ is finite.*

DEFINITION 2.3. — *A representation $\rho \in R(S, G)$ is good if it is irreducible and its centralizer $Z(\rho)$ is the center of the group G , i.e. $Z(\rho) = Z(G)$.*

The set of good representations is denoted by $R^*(S, G)$, and its orbit space by $\mathcal{R}^*(S, G) = R^*(S, G)/G$.

PROPOSITION 2.4 ([JM87, Propositions 1.2 and 1.3]). — *The set of good representations $R^*(S, G)$ is a Zariski open subset of $R(S, G)$. Furthermore the action of G on $R^*(S, G)$ is proper.*

The *variety of characters* is the quotient in the algebraic category:

$$X(S, G) = R(S, G)//G.$$

Namely, it is an algebraic affine set defined by its ring of polynomial functions, as the ring of functions on $R(S, G)$ invariant by conjugation.

The projection $R(S, G) \rightarrow X(S, G)$ factors through a surjective map $\mathcal{R}(S, G) \rightarrow X(S, G)$. For good representations we have:

PROPOSITION 2.5. — *The natural map restricts to an injection*

$$\mathcal{R}^*(S, G) \hookrightarrow X(S, G)$$

whose image is a Zariski open subset and a smooth complex manifold.

For the proof, see for instance [JM87, Section 1], or [New78, Proposition 3.8] for injectivity, as irreducibility is equivalent to stability in GIT [JM87, Section 1]. For smoothness see [Gol84].

Given a representation $\rho \in R(S, G)$, the Lie algebra \mathfrak{g} turns into an $\pi_1(S)$ -module via $\text{Ad} \circ \rho$. If there is no ambiguity this module is denoted just by \mathfrak{g} , and the coefficients in cohomology are denoted by $\text{Ad } \rho$.

PROPOSITION 2.6. — *Let $\rho \in R^*(S, G)$ be a good representation. Then there is a natural isomorphism*

$$T_{[\rho]} \mathcal{R}^*(S, G) \cong H^1(S; \text{Ad } \rho).$$

In particular the dimension of $\mathcal{R}^*(S, G)$ is $-\chi(S) d$.

This proposition can be found for instance in [Sik12, Corollary 50], but we sketch the proof as it may be useful for the relative case.

Proof. — Let $Z^1 = Z^1(S; \text{Ad } \rho)$ denote the space of crossed morphisms from $\pi_1(S)$ to \mathfrak{g} , i.e. maps $d: \pi_1(S) \rightarrow \mathfrak{g}$ satisfying $d(\gamma\mu) = d(\gamma) + \text{Ad}_{\rho(\gamma)} d(\mu)$, $\forall \gamma, \mu \in \pi_1(S)$. Let $B^1 = B^1(S; \text{Ad } \rho)$ denote the subspace of inner crossed morphisms: for $a \in \mathfrak{g}$ the corresponding inner morphism maps $\gamma \in \pi_1(S)$ to $\text{Ad}_{\rho(\gamma)}(a) - a$. Weil’s construction identifies Z^1 with $T_\rho R(S, G)$ (usually Z^1 is the Zariski tangent space of a scheme, possibly non-reduced, but as $\pi_1(S)$ is free, $R(S, G)$ is a smooth algebraic variety). The subspace B^1 corresponds to the tangent space of the orbit $\text{Ad}_G(\rho)$. Then, in order to identify the tangent space of $\mathcal{R}^*(S, G)$ with the cohomology group $H^1(S; \text{Ad } \rho) = Z^1/B^1$, we use a slice, for instance an étale slice provided by Luna’s Theorem [PS94, Theorem 6.1], or an analytic slice (cf. [JM87]). In the setting of a good representation ρ , a slice is a subvariety $\mathcal{S} \subset R(S, G)$ containing ρ , invariant by $Z(\rho) = Z(G)$, such that the conjugation map

$$G/Z(G) \times \mathcal{S} \rightarrow R(S, G)$$

is locally bi-analytic at (e, ρ) and the projection $\mathcal{S} \rightarrow X(S, G)$ is also bi-analytic at ρ . (If ρ was not good, we should take care of the action of $Z(\rho)/Z(G)$. In addition, for Γ not a free group the description is more involved). Then the assertion follows easily from the properties of the slice. \square

2.2. The relative variety of representations

Let

$$\partial S = \partial_1 \sqcup \cdots \sqcup \partial_b$$

denote the decomposition in connected components. By abuse of notation, we also let ∂_i denote an element of the fundamental group represented by the corresponding oriented peripheral curve. This is well-defined only up to conjugacy in $\pi_1(S)$, but our constructions do not depend on the representative in the conjugacy class.

DEFINITION 2.7 ([Kap01, Section 4.3]). — For $\rho_0 \in R(S, G)$, the relative variety of representations is

$$\mathcal{R}(S, \partial S, G)_{\rho_0} = \{[\rho] \in \mathcal{R}(S, G) \mid \rho(\partial_i) \in \mathcal{O}(\rho_0(\partial_i)), i = 1, \dots, b\}.$$

Here $\mathcal{O}(\rho_0(\partial_i))$ denotes the conjugacy class of $\rho_0(\partial_i)$. We also denote

$$\mathcal{R}^*(S, \partial S, G)_{\rho_0} = \mathcal{R}(S, \partial S, G)_{\rho_0} \cap \mathcal{R}^*(S, G).$$

Besides considering good representations, we restrict our attention to representations which map peripheral elements to regular elements of G .

DEFINITION 2.8 ([Ste74, Section 3.5]). — An element $g \in G$ is called regular if its centralizer $Z(g)$ has minimal dimension among centralizers of elements of G . Equivalently, its conjugacy class $\mathcal{O}(g)$ has maximal dimension.

This minimal dimension is r the rank of G [Ste74, Section 3.5, Proposition 1]. In $\mathrm{SL}_N(\mathbb{C})$, a diagonal matrix is regular if and only if all eigenvalues are different. More generally, $g \in \mathrm{SL}_N(\mathbb{C})$ is regular if and only if its minimal polynomial is of degree N [Ste74, Section 3.5, Proposition 2]. In particular the companion matrix of a monic polynomial is regular.

DEFINITION 2.9. — A representation $\rho \in R(S, G)$ is called ∂ -regular if the elements $\rho(\partial_1), \dots, \rho(\partial_b)$ are regular.

PROPOSITION 2.10. — Let $\rho_0 \in R^*(S, G)$ be a good, ∂ -regular representation.

(a) $\mathcal{R}^*(S, \partial S, G)_{\rho_0}$ is a complex smooth manifold of dimension

$$d_0 = -d \chi(S) - b r = d(2g(S) - 2) + b(d - r).$$

(b) For $[\rho] \in \mathcal{R}^*(S, \partial S, G)_{\rho_0}$, there is a natural isomorphism:

$$T_{[\rho]} \mathcal{R}^*(S, \partial S, G)_{\rho_0} \cong \ker \left(H^1(S; \mathrm{Ad} \rho) \rightarrow H^1(\partial S; \mathrm{Ad} \rho) \right).$$

Proof. — We first show that the map $H^1(S; \mathrm{Ad} \rho) \rightarrow H^1(\partial S; \mathrm{Ad} \rho)$ is a surjection. By Poincaré duality $H^2(S, \partial S; \mathrm{Ad} \rho) \cong H^0(S; \mathrm{Ad} \rho) \cong \mathfrak{g}^{\mathrm{Ad} \rho(\pi_1(S))}$, that vanishes because $Z(\rho)$ is finite. Thus, by the long exact sequence of the pair $(S, \partial S)$, the map $H^1(S; \mathrm{Ad} \rho) \rightarrow H^1(\partial S; \mathrm{Ad} \rho)$ is a surjection.

We use a slice at ρ_0 , $\mathcal{S} \subset R(M)$ as in the proof of Proposition 2.6. The fact that $H^1(S; \text{Ad } \rho) \rightarrow H^1(\partial S; \text{Ad } \rho)$ is a surjection means that the restriction map

$$\text{res}|_{\mathcal{S}}: \mathcal{S} \rightarrow R(\partial S, G) = \prod_{i=1}^b R(\partial_i, G) = G^b$$

is transverse to the products of orbits by conjugation

$$O = \prod_{i=1}^b \mathcal{O}(\rho(\partial_i)).$$

Namely, $(\text{res}|_{\mathcal{S}})_*(T_{\rho}\mathcal{S}) + T_{\text{res}(\rho)}O = T_{\text{res}(\rho)}G^b$. It follows from the rank Theorem [Loj91, Chapter 4.1] that $\mathcal{O}(\rho(\partial_i)) \subset G$ is a complex analytic subvariety of dimension $d - r$ because ρ is ∂ -regular. Thus $(\text{res}|_{\mathcal{S}})^{-1}(O)$ is an analytic \mathbb{C} -submanifold of codimension

$$\dim G^b - \dim O = \sum_{i=1}^b (\dim G - \dim \mathcal{O}(\rho(\partial_i))) = br.$$

Now the proposition follows from the properties of the slice. □

2.3. Steinberg map

In order to understand the space of conjugacy classes of regular representations of \mathbb{Z} we identify each representation with the image of its generator, so that

$$R^{reg}(\mathbb{Z}, G) = G^{reg} \quad \text{and} \quad \mathcal{R}^{reg}(\mathbb{Z}, G) = G^{reg}/G.$$

Consider the Steinberg map

$$(2.1) \quad (\sigma_1, \dots, \sigma_r): G \rightarrow \mathbb{C}^r$$

where $\sigma_1, \dots, \sigma_r$ denote the characters corresponding to a system of fundamental representations (for $\text{SL}_N(\mathbb{C})$ those are the coefficients of the characteristic polynomial).

THEOREM 2.11 ([Ste65]). — *If G is simply connected, then the map (2.1) is a surjection and has a section $s: \mathbb{C}^r \rightarrow G^{reg}$ so that $s(\mathbb{C}^r)$ is a subvariety that intersects each orbit by conjugation in G^{reg} precisely once.*

For instance, when $G = \text{SL}_N(\mathbb{C})$ the section in Theorem 2.11 can be chosen to be the companion matrix (see [Ste74, p. 120] and [Hum95, Section 4.15]).

COROLLARY 2.12. — *If G is simply connected, then:*

- (i) *The map (2.1) induces natural isomorphisms between the space of regular orbits by conjugation, the variety of characters, and \mathbb{C}^r :*

$$\mathcal{R}^{reg}(S^1, G) \cong X(S^1, G) \cong \mathbb{C}^r.$$

- (ii) *The Steinberg map induces a natural isomorphism*

$$H^1(S^1, \text{Ad } \rho) \cong T_{[\rho]} \mathcal{R}^{reg}(S^1, G) \cong \mathbb{C}^r.$$

Moreover, for each good, ∂ -regular representation $\rho_0 \in R^*(S, G)$ and $[\rho] \in \mathcal{R}^*(S, \partial S, G)_{\rho_0}$ there is an exact sequence

$$0 \rightarrow T_{[\rho]} \mathcal{R}^*(S, \partial S, G)_{\rho_0} \rightarrow T_{[\rho]} \mathcal{R}^*(S, G) \rightarrow \bigoplus_{i=1}^b T_{[\rho(\partial_i)]} \mathcal{R}^{reg}(\partial_i, G) \rightarrow 0.$$

Proof.

- (i) Notice that what we aim to prove is the isomorphism $G^{reg}/G \cong G//G \cong \mathbb{C}^r$; which is straightforward from the existence of the section in Theorem 2.11.
- (ii) By the existence of the section we also know that the differential of Steinberg’s map $Z^1(\mathbb{Z}, \text{Ad } \rho) \cong \mathfrak{g} \rightarrow \mathbb{C}^r$ is surjective whenever ρ is regular [Hum95, Section 4.19]. In addition it maps $B^1(\mathbb{Z}, \text{Ad } \rho)$ to 0, because Steinberg map is constant on orbits by conjugation. Thus we have a well-defined surjection $H^1(S^1, \text{Ad } \rho) \rightarrow \mathbb{C}^r$, which is an isomorphism because of the dimension. The exact sequence follows from the long exact sequence in cohomology of the pair $(S, \partial S)$ and the identification of cohomology groups with tangent spaces, cf. Proposition 2.10. □

Remark 2.13. — When G is not simply connected, then the universal covering $\tilde{G} \rightarrow G$ is finite and $\pi_1(G)$ can be identified with a (finite) central subgroup Z of \tilde{G} . The center of \tilde{G} acts on the quotient $\tilde{G}//\tilde{G}$ and we obtain a commutative diagram

$$\begin{array}{ccc} \tilde{G} & \longrightarrow & \tilde{G}//\tilde{G} \\ \downarrow & & \downarrow \varphi \\ G & \longrightarrow & G//G \end{array}$$

where $(G//G, \varphi)$ is a quotient for the action of Z on $\tilde{G}//\tilde{G}$ (see [Pop11, Lemma 2.5]). Notice that φ is a finite branched covering.

Then part (ii) of Corollary 2.12 can be easily adapted for those $[g] \in G//G$ which are outside the branch set of φ .

3. Reidemeister torsion

Let $\rho \in R(S, G)$ be a representation; recall that we consider the action of $\pi_1(S)$ on \mathfrak{g} via the adjoint of ρ . Most of the results in this section apply not only to \mathfrak{g} but to its real form $\mathfrak{g}_{\mathbb{R}}$, provided that the image of the representation is contained in $G_{\mathbb{R}}$. Recall also that we assume that \mathcal{B} restricted to the compact real form $\mathfrak{g}_{\mathbb{R}}$ is positive definite.

Consider a cell decomposition K of S . If $C_*(\tilde{K}; \mathbb{Z})$ denotes the cellular chain complex on the universal covering, one defines

$$(3.1) \quad C^*(K; \text{Ad } \rho) = \text{hom}_{\pi_1(S)}(C_*(\tilde{K}; \mathbb{Z}), \mathfrak{g}).$$

We consider the so called geometric basis. Start with a \mathcal{B} -orthonormal \mathbb{C} -basis $\{m_1, \dots, m_d\}$ of \mathfrak{g} . For each i -cell e_j^i of K we choose a lift \tilde{e}_j^i to the universal covering \tilde{K} , then

$$\mathbf{c}^i = \{(\tilde{e}_j^i)^* \otimes m_k\}_{jk}$$

is a basis of $C^i(K; \text{Ad } \rho)$, called the *geometric basis*. Here, $(\tilde{e}_j^i)^* \otimes m_k : C_*(\widetilde{K}; \mathbb{Z}) \rightarrow \mathfrak{g}$ is the unique $\pi_1(S)$ -homomorphism given by $(\tilde{e}_j^i)^* \otimes m_k(\tilde{e}_l^i) = \delta_{jl} m_k$.

On the other hand, if $B^i = \text{Im}(\delta : C^{i-1}(K; \text{Ad } \rho) \rightarrow C^i(K; \text{Ad } \rho))$ is the space of coboundaries, choose \mathbf{b}^i a basis for $B^i \subset C^i$ and choose lifts $\widetilde{\mathbf{b}}^i$ to C^{i-1} of the coboundary map. For a basis \mathbf{h}^i of $H^i(K; \text{Ad } \rho)$, consider also representatives $\widetilde{\mathbf{h}}^i \in C^i(K; \text{Ad } \rho)$. Then the disjoint union

$$\widetilde{\mathbf{b}}^{i+1} \sqcup \widetilde{\mathbf{h}}^i \sqcup \mathbf{b}^i$$

is also a basis for $C^i(K; \text{Ad } \rho)$. Notice that we are interested in the case where the zero and second cohomology groups vanish, so we assume that $\widetilde{\mathbf{h}}^0 = \widetilde{\mathbf{h}}^2 = \emptyset$.

Reidemeister torsion is defined as

$$(3.2) \quad \text{tor}(S, \text{Ad } \rho, \mathbf{h}^1) = \frac{[\widetilde{\mathbf{b}}^2 \sqcup \widetilde{\mathbf{h}}^1 \sqcup \mathbf{b}^1 : \mathbf{c}^1]}{[\widetilde{\mathbf{b}}^1 : \mathbf{c}^0][\mathbf{b}^2 : \mathbf{c}^2]} \in \mathbb{C}^* / \{\pm 1\}$$

Here, for two bases \mathbf{a} and \mathbf{b} of a vector space, $[\mathbf{a} : \mathbf{b}]$ denotes the determinant the matrix whose colons are the coefficients of the vectors of \mathbf{a} as linear combination of \mathbf{b} .

Remark 3.1. — The choice of the bilinear form \mathcal{B} is relevant, as we use a \mathcal{B} -orthonormal basis for \mathfrak{g} and $\chi(S) \neq 0$. Namely, if we replace \mathcal{B} by $\lambda^2 \mathcal{B}$, then the orthonormal basis will be $\frac{1}{\lambda} \{m_1, \dots, m_d\}$ and the torsion will be multiplied by a factor $\lambda^{-\chi(S)d} = \lambda^{\dim \mathcal{R}(S,G)}$.

For an ordered basis $\mathbf{a} = \{a_1, \dots, a_m\}$ of a vector space, denote

$$\wedge \mathbf{a} = a_1 \wedge \dots \wedge a_m.$$

Since $\wedge \mathbf{a} = [\mathbf{a} : \mathbf{b}](\wedge \mathbf{b})$, the notation

$$[\mathbf{a} : \mathbf{b}] = \wedge \mathbf{a} / \wedge \mathbf{b}$$

is often used in the literature (cf. [Mil62]).

3.1. The holomorphic volume form

The tangent space of $\mathcal{R}^*(S, G)$ at $[\rho]$ is identified with $H^1(S; \text{Ad } \rho)$, by Proposition 2.6. There is a natural holomorphic volume form on $H^1(S; \text{Ad } \rho)$:

$$\Omega_S(\wedge \mathbf{h}) = \pm \text{tor}(S, \text{Ad } \rho, \mathbf{h})$$

where \mathbf{h} is a basis for $H^1(S; \text{Ad } \rho)$.

The surface S has the simple homotopy type of a graph. Moreover, graphs that are homotopy equivalent are also simple-homotopy equivalent, thus this volume form depends only on the fundamental group

$$\Omega_{\pi_1(S)} = \Omega_S.$$

The bilinear form \mathcal{B} defines a bi-invariant volume form θ_G on the Lie group G in the usual way. Hence $(\theta_G)^k$ is a volume form on $R(\pi_1(S), G) \cong G^k$.

For a good representation ρ the form θ_G induces also a form $\theta_{\mathcal{O}(\rho)}$ on the orbit $\mathcal{O}(\rho)$ by push-forward: the orbit map $f_\rho : G \rightarrow R(\pi_1(S), G)$, $f_\rho(g) = \text{Ad}_g \circ \rho$ factors

through the quotient $G/Z(G)$. The quotient map $G \rightarrow G/Z(G)$ is a Lie group covering and we get an isomorphism $\bar{f}_\rho: G/Z(G) \rightarrow \mathcal{O}(\rho)$, and hence

$$(3.3) \quad \theta_{\mathcal{O}(\rho)} = (f_\rho)_*(\theta_G).$$

The next lemma justifies why Reidemeister torsion is the natural choice of volume form on the variety of representations up to conjugation.

LEMMA 3.2. — *Let $\pi: R^*(S, G) \rightarrow \mathcal{R}^*(S, G)$ denote the projection. Then at $\rho \in R^*(S, G)$ we have:*

$$(\theta_G)^k = \pm \theta_{\mathcal{O}(\rho)} \wedge \pi^* \Omega_S.$$

Proof. — We use a graph \mathcal{G} with one vertex and k edges to compute the torsion of S . The Reidemeister torsion of this graph is $\text{tor}(\mathcal{G}, \text{Ad } \rho, \mathbf{h}) = \pm [\mathbf{b}^1 \sqcup \tilde{\mathbf{h}} : \mathbf{c}^1] / [\tilde{\mathbf{b}}^1 : \mathbf{c}^0]$. If we make the choice $\tilde{\mathbf{b}}^1 = \mathbf{c}^0$, which is a basis for \mathfrak{g} , then

$$\text{tor}(\mathcal{G}, \text{Ad } \rho, \mathbf{h}) = [\delta \mathbf{c}^0 \sqcup \tilde{\mathbf{h}} : \mathbf{c}^1] = (\wedge \delta \mathbf{c}^0 \wedge \tilde{\mathbf{h}}) / \wedge \mathbf{c}^1.$$

We identify the 1-cells with the generators of F_k , so that every element in \mathbf{c}^1 is viewed as a tangent vector of the variety of representations, and \mathbf{c}^1 has volume one,

$$(\theta_G)^k(\wedge \mathbf{c}^1) = 1$$

because we started with an \mathcal{B} -orthonormal basis for \mathfrak{g} . Thus

$$(3.4) \quad (\theta_G)^k(\wedge \delta \mathbf{c}^0 \wedge \tilde{\mathbf{h}}) = \pm \text{tor}(\mathcal{G}, \text{Ad } \rho, \mathbf{h}) = \pm \Omega_S(\wedge \mathbf{h}).$$

As $\delta \mathbf{c}^0$ is a basis of the tangent space of the orbit $\pi_*(\delta \mathbf{c}^0) = 0$. Moreover, using $\pi_*(\tilde{\mathbf{h}}) = \mathbf{h}$:

$$(3.5) \quad (\theta_{\mathcal{O}(\rho)} \wedge \pi^* \Omega_S)(\wedge \delta \mathbf{c}^0 \wedge \tilde{\mathbf{h}}) = \theta_{\mathcal{O}(\rho)}(\wedge \delta \mathbf{c}^0) \Omega_S(\wedge \mathbf{h}).$$

By (3.4) and (3.5), to conclude the proof of the Lemma 3.2 we claim that $\theta_{\mathcal{O}(\rho)}(\wedge \delta \mathbf{c}^0) = 1$. For that purpose, we use the canonical identification $T_\rho \mathcal{O}(\rho) \cong B^1(\pi_1(S); \text{Ad } \rho)$. Using this identification, the tangent map of the orbit map $f_\rho: G \rightarrow \mathcal{O}(\rho)$ at $e \in G$, $df_\rho(e): \mathfrak{g} \rightarrow T_\rho \mathcal{O}(\rho)$, corresponds to

$$df_\rho(e)(X) = -\delta(X),$$

where $\delta: \mathfrak{g} \rightarrow B^1(\pi_1(S); \text{Ad } \rho)$ denotes the coboundary operator

$$\delta(X)(\gamma) = \text{Ad}_{\rho(\gamma)}(X) - X, \quad \text{for } \gamma \in \pi_1(S).$$

Therefore for the basis $\delta \mathbf{c}^0$ of $B^1(\pi_1(S); \text{Ad } \rho)$ we obtain by (3.3):

$$\theta_{\mathcal{O}(\rho)}(\wedge \delta \mathbf{c}^0) = \theta_{\mathcal{O}(\rho)}(\wedge df_\rho(e) \mathbf{c}^0) = \pm \theta_G(\wedge \mathbf{c}^0) = 1.$$

This concludes the proof of the claim and the Lemma 3.2. □

3.2. The nondegenerate pairing

Consider K' the cell decomposition dual to K : for each i -dimensional cell e_j^i of K there exists a dual $(2 - i)$ -dimensional cell f_j^{2-i} of the dual complex $(K', \partial K')$. The complex $C^*(K', \partial K'; \mathbb{Z})$ yields the relative cohomology of the pair $(S, \partial S)$. This can be generalized to cohomology with coefficients. If $C_*(\widetilde{K}; \mathbb{Z})$ denotes the simplicial chain complex on the universal covering, recall from (3.1) that

$$C^*(K; \text{Ad } \rho) = \text{hom}_{\pi_1(S)}(C_*(\widetilde{K}; \mathbb{Z}), \mathfrak{g}),$$

and we similarly define

$$C^*(K', \partial K'; \text{Ad } \rho) = \text{hom}_{\pi_1(S)}(C_*(\widetilde{K}', \partial \widetilde{K}'; \mathbb{Z}), \mathfrak{g})$$

where $\pi_1(S)$ acts on \mathfrak{g} by the adjoint representation.

Following Milnor [Mil62], there is a pairing

$$[\cdot, \cdot]: C_i(\widetilde{K}; \mathbb{Z}) \times C_{2-i}(\widetilde{K}', \partial \widetilde{K}'; \mathbb{Z}) \rightarrow \mathbb{Z}\pi_1(S)$$

defined by

$$[c, c'] := \sum_{\gamma \in \pi_1(S)} (c \cdot \gamma c') \gamma.$$

Here “ \cdot ” denotes the intersection number in the universal covering. The main properties of this pairing are that for $\eta \in \mathbb{Z}\pi_1(S)$ we have:

$$(3.6) \quad [\eta c, c'] = \eta [c, c'], \quad [c, \eta c'] = [c, c'] \bar{\eta} \quad \text{and} \quad [\partial c, c'] = \pm [c, \partial c'].$$

Here the bar $\bar{\cdot} : \mathbb{Z}\pi_1(S) \rightarrow \mathbb{Z}\pi_1(S)$ denotes the anti-involution that extends \mathbb{Z} -linearly the anti-morphism of $\pi_1(S)$ that maps $\gamma \in \pi_1(S)$ to γ^{-1} . Notice that the sign \pm in equation (3.6) depends only on the dimension of the chains.

For each i -dimensional cell e_j^i we fix a lift \tilde{e}_j^i to \widetilde{K} . Also, we choose a $(2 - i)$ -dimensional cell \tilde{f}_j^{2-i} which projects to f_j^{2-i} . By replacing \tilde{f}_j^{2-i} by a translate, we can assume that

$$\tilde{e}_j^i \cdot \tilde{f}_k^{2-i} = \delta_{jk}.$$

We obtain, for each i -chain $c \in C_i(\widetilde{K}; \mathbb{Z})$ and each $(2 - i)$ -chain $c' \in C_{2-i}(\widetilde{K}', \partial \widetilde{K}'; \mathbb{Z})$ that

$$c = \sum_j [c, \tilde{f}_j^{2-i}] \tilde{e}_j^i \quad \text{and} \quad c' = \sum_j [\tilde{e}_j^i, c'] \tilde{f}_j^{2-i}.$$

Given $\alpha \in C^i(K; \text{Ad } \rho)$ and $\alpha' \in C^{2-i-1}(K', \partial K'; \text{Ad } \rho)$ the formula

$$(\alpha, \alpha') \mapsto \sum_j \mathcal{B}(\alpha(\tilde{e}_j^i), \alpha'(\tilde{f}_j^{2-i}))$$

defines a nondegenerate pairing

$$(3.7) \quad \langle \cdot, \cdot \rangle: C^i(K; \text{Ad } \rho) \times C^{2-i-1}(K', \partial K'; \text{Ad } \rho) \rightarrow \mathbb{C}.$$

By using equation (3.6), it is easy to see that this pairing satisfies

$$(3.8) \quad \langle \delta \alpha, \alpha' \rangle = \pm \langle \alpha, \delta \alpha' \rangle,$$

and therefore it induces a non-singular pairing in cohomology

$$(3.9) \quad \langle \cdot, \cdot \rangle: H^1(S; \text{Ad } \rho) \times H^1(S, \partial S; \text{Ad } \rho) \rightarrow \mathbb{C}.$$

Given a basis $\mathbf{h} = \{h_i\}_i$ of $H^1(S; \text{Ad } \rho)$ and $\mathbf{h}' = \{h'_i\}_i$ a basis of $H^1(S, \partial S; \text{Ad } \rho)$, we introduce the notation

$$(3.10) \quad \langle \wedge \mathbf{h}, \wedge \mathbf{h}' \rangle := \det \left(\langle h_i, h'_j \rangle_{ij} \right)$$

which is the natural extension of the pairing (3.9) to

$$\bigwedge^d H^1(S; \text{Ad } \rho) \otimes \bigwedge^d H^1(S, \partial S; \text{Ad } \rho) \rightarrow \mathbb{C},$$

where $d = -\chi(S) \dim G$.

3.3. The duality formula

Let $\rho \in R(\pi_1(S), G)$ be a representation.

PROPOSITION 3.3 (Duality formula). — *Let $\mathbf{h} = \{h_i\}_i$ be a basis for $H^1(S; \text{Ad } \rho)$, and let $\mathbf{h}' = \{h'_i\}_i$ be a basis for $H^1(S, \partial S; \text{Ad } \rho)$. Assume that the cohomology groups $H^k(S; \text{Ad } \rho)$ and $H^k(S, \partial S; \text{Ad } \rho)$ vanish in dimension $k = 0, 2$. Then*

$$\text{tor}(S, \text{Ad } \rho, \mathbf{h}) \text{tor}(S, \partial S, \text{Ad } \rho, \mathbf{h}') = \pm \langle \wedge \mathbf{h}, \wedge \mathbf{h}' \rangle$$

This is E. Witten’s generalization of the duality formula of W. Franz and J. Milnor. We reproduce the proof for completeness. In Witten’s article [Wit91] the proof of this particular formula is only given in the closed case, and Milnor [Mil62] and Franz [Fra36] consider only the acyclic case.

Proof. — We choose the geometric basis of $C^i(K; \text{Ad } \rho)$ and $C^{2-i}(K', \partial K'; \text{Ad } \rho)$ to be dual to each other, by choosing dual lifts of the cells and a \mathcal{B} -orthonormal basis of the Lie algebra \mathfrak{g} . In this way, the matrix of the intersection form (3.7) with respect the geometric basis is the identity, in particular its determinant is 1: $\langle \wedge \mathbf{c}^i, \wedge (\mathbf{c}^{2-i})' \rangle = 1$. Thus we view the product of torsions in the statement of the proposition as three changes of basis, one for each intersection form:

$$\begin{aligned} (3.11) \quad & \text{tor}(S, \text{Ad } \rho, \mathbf{h}) \text{tor}(S, \partial S, \text{Ad } \rho, \mathbf{h}') \\ &= \pm \text{tor}(S, \text{Ad } \rho, \mathbf{h}) \text{tor}(S, \partial S, \text{Ad } \rho, \mathbf{h}') \frac{\langle \wedge \mathbf{c}^1, \wedge \mathbf{c}^1 \rangle'}{\langle \wedge \mathbf{c}^0, \wedge (\mathbf{c}^2)' \rangle \langle \wedge \mathbf{c}^2, \wedge (\mathbf{c}^0)' \rangle} \\ &= \pm \frac{[\widetilde{\mathbf{b}^2} \sqcup \mathbf{b}^1 \sqcup \widetilde{\mathbf{h}} : \mathbf{c}^1] [\widetilde{(\mathbf{b}^2)'} \sqcup (\mathbf{b}^1)' \sqcup \widetilde{\mathbf{h}}' : (\mathbf{c}^1)'] \langle \wedge \mathbf{c}^1, \wedge (\mathbf{c}^1)' \rangle}{[\widetilde{\mathbf{b}^1} : \mathbf{c}^0][\mathbf{b}^2 : \mathbf{c}^2] [(\widetilde{\mathbf{b}^1})' : (\mathbf{c}^0)'] [(\mathbf{b}^2)' : (\mathbf{c}^2)'] \langle \wedge \mathbf{c}^0, \wedge (\mathbf{c}^2)' \rangle \langle \wedge \mathbf{c}^2, \wedge (\mathbf{c}^0)' \rangle} \\ &= \pm \frac{\langle \wedge \widetilde{\mathbf{b}^2} \wedge \mathbf{b}^1 \wedge \widetilde{\mathbf{h}}, \wedge (\widetilde{(\mathbf{b}^2)'} \wedge (\mathbf{b}^1)' \wedge \widetilde{\mathbf{h}}') \rangle}{\langle \wedge \widetilde{\mathbf{b}^1}, \wedge (\mathbf{b}^2)' \rangle \langle \wedge \mathbf{b}^2, \wedge (\widetilde{\mathbf{b}^1})' \rangle}. \end{aligned}$$

Next, following Witten, we may choose the lift of the coboundaries to be orthogonal to the lift of the cohomology of the other complex:

$$\langle \widetilde{h}_i, (\widetilde{b}_j^2)' \rangle = \langle \widetilde{b}_i^2, \widetilde{h}'_j \rangle = 0.$$

In addition, by direct application of (3.8):

$$\langle b_i^1, (b_j^1)' \rangle = \langle b_i^1, \widetilde{h}'_j \rangle = \langle \widetilde{h}_i, (b_j^1)' \rangle = 0.$$

Thus the numerator in (3.11) is the determinant of a matrix with some vanishing blocks, and (3.11) becomes:

$$(3.12) \quad \pm \frac{\langle \wedge \widetilde{\mathbf{b}}^2, \wedge (\mathbf{b}^1)' \rangle \langle \wedge \mathbf{b}^1, \wedge (\widetilde{\mathbf{b}}^2)' \rangle \langle \wedge \mathbf{h}, \wedge \mathbf{h}' \rangle}{\langle \wedge \widetilde{\mathbf{b}}^1, \wedge (\mathbf{b}^2)' \rangle \langle \wedge \mathbf{b}^2, \wedge (\widetilde{\mathbf{b}}^1)' \rangle}.$$

Finally, since $\delta \widetilde{\mathbf{b}}^i = \mathbf{b}^i$ and $\delta (\widetilde{\mathbf{b}}^i)' = (\mathbf{b}^i)'$, $\langle \wedge \widetilde{\mathbf{b}}^2, \wedge \mathbf{b}^1 \rangle = \pm \langle \wedge \mathbf{b}^2, \wedge (\widetilde{\mathbf{b}}^1)' \rangle$ and $\langle \wedge \mathbf{b}^1, \wedge \widetilde{\mathbf{b}}^2 \rangle = \pm \langle \wedge \widetilde{\mathbf{b}}^1, \wedge (\mathbf{b}^2)' \rangle$, by (3.8). Hence (3.12) equals $\pm \langle \wedge \mathbf{h}, \wedge \mathbf{h}' \rangle$, concluding the proof of Proposition 3.3. \square

Remark 3.4. — Notice that the proof generalizes in any dimension, after changing the product by a quotient in the odd dimensional case, and taking care of the intersection product in all cohomology groups.

4. Symplectic form and volume forms

4.1. The symplectic form on the relative variety of representations

For a good and ∂ -regular representation ρ_0 , the tangent space of $\mathcal{R}^*(S, \partial S, G)_{\rho_0}$ is the kernel of the map $i: H^1(S; \text{Ad } \rho) \rightarrow H^1(\partial S; \text{Ad } \rho)$ induced by inclusion (Proposition 2.10). The long exact sequence in cohomology of the pair is:

$$0 \rightarrow H^0(\partial S; \text{Ad } \rho) \xrightarrow{\beta} H^1(S, \partial S; \text{Ad } \rho) \xrightarrow{j} H^1(S; \text{Ad } \rho) \xrightarrow{i} H^1(\partial S; \text{Ad } \rho) \rightarrow 0$$

For $a, b \in \ker(i)$, we define

$$(4.1) \quad \omega(a, b) = \langle \widetilde{a}, b \rangle = \langle a, \widetilde{b} \rangle$$

where $\widetilde{a}, \widetilde{b} \in H^1(S, \partial S; \text{Ad } \rho)$ satisfy $j(\widetilde{a}) = a$, $j(\widetilde{b}) = b$. This form is well-defined (independent of the lift), because i and β are dual maps with respect to the pairing (3.9), that is $\langle \beta(\cdot), \cdot \rangle = \langle \cdot, i(\cdot) \rangle$. Moreover we have:

THEOREM 4.1 ([Gol84, GHJW97, Law09]). — *Assume that ρ_0 is a good and ∂ -regular. Then the form ω is symplectic on $\mathcal{R}^*(S, \partial S, G)_{\rho_0}$.*

The fact that ω is bilinear and alternating is clear from construction, non-degeneracy follows from Poincaré duality, and the deep result is to prove $d\omega = 0$. When S is closed this was proved by Goldman in [Gol84]. When $\partial S \neq \emptyset$, the result with real coefficients is due to Guruprasad, Huebschmann, Jeffrey, and Weinstein [GHJW97], and in [Law09] Lawton explains why it applies also in the complex case.

4.2. Sign refined Reidemeister torsion for the circle

Let V be a finite dimensional real or complex vector space, and

$$\varphi: \pi_1(S^1) \rightarrow \text{SL}(V)$$

be a representation. In what follows we use the refined torsion with sign due to Turaev, that we denote $\text{TOR}(S^1, \varphi, \mathfrak{o}, \mathbf{u}, \mathbf{v})$ [Tur86, Section 3]. This torsion depends

on the choice of an orientation \mathfrak{o} in cohomology with constant coefficients of S^1 and the choice of respective basis \mathbf{u} for $H^0(S^1; \varphi)$ and \mathbf{v} for $H^1(S^1; \varphi)$. For a circle S^1 , the choice of an orientation determines a fundamental class, hence an orientation in homology.

We start with a cell decomposition K of S^1 , with i -cells e_1^i, \dots, e_a^i , $i = 0, 1$, and a (real or complex) basis $\{m_1, \dots, m_k\}$ for the vector space V . The geometric basis for $C^i(K; \varphi)$ is then $\mathbf{c}^i = \{(\tilde{e}_1^i)^* \otimes m_1, (\tilde{e}_1^i)^* \otimes m_2, \dots, (\tilde{e}_a^i)^* \otimes m_k\}$. As before, let $B^1 = \text{Im}(\delta: C^0(K; \varphi) \rightarrow C^1(K; \varphi))$ denote the coboundary space and choose \mathbf{b}^1 as basis for B^1 and lift it to $\tilde{\mathbf{b}}^1$ in $C^0(K; \varphi)$. Consider also $\tilde{\mathbf{v}} \subset C^1(K; \varphi)$ a representative of \mathbf{v} and similarly $\tilde{\mathbf{u}} \subset C^0(K; \varphi)$ for \mathbf{u} . Then we define the torsion:

$$\text{tor}(S^1, \varphi, \mathbf{u}, \mathbf{v}, \mathbf{c}^0, \mathbf{c}^1) = \frac{[\tilde{\mathbf{v}} \sqcup \mathbf{b}^1 : \mathbf{c}^1]}{[\tilde{\mathbf{b}}^1 \sqcup \tilde{\mathbf{u}} : \mathbf{c}^0]} \in \mathbb{C}^*.$$

Notice that there is no sign indeterminacy, because we include \mathbf{c}^i in the notation. In fact sign indeterminacy comes from changing the order or the orientation of the cells of K . The sign is not affected by the choice of a basis for V , because $\chi(S^1) = 0$.

Following [Tur86, Section 3] we consider

$$\alpha_i = \sum_{l=0}^i \dim C^l(K; \varphi), \quad \beta_i = \sum_{l=0}^i \dim H^l(S^1; \varphi) \quad \text{and} \quad N = \sum_{i \geq 0} \alpha_i \beta_i.$$

We define

$$\text{Tor}(S^1, \varphi, \mathbf{u}, \mathbf{v}, \mathbf{c}^0, \mathbf{c}^1) = (-1)^N \text{tor}(S^1, \varphi, \mathbf{u}, \mathbf{v}, \mathbf{c}^0, \mathbf{c}^1).$$

This quantity is invariant under subdivision of the cells of K , but it still depends on their ordering and orientation. To make this quantity invariant, Turaev introduces the notion of cohomology orientation, i.e. an orientation of the \mathbb{R} -vector space $H^0(S^1; \mathbb{R}) \oplus H^1(S^1; \mathbb{R})$. We consider a geometric basis of the complex with trivial coefficients $C^i(K; \mathbb{R})$, $\mathbf{c}^i = \{(e_1^i)^*, \dots, (e_a^i)^*\}$, with the same ordering and orientation of cells. We choose any basis h^i of $H^i(S^1; \mathbb{R})$ that yield the orientation \mathfrak{o} .

DEFINITION 4.2. — *The sign determined torsion is*

$$\text{TOR}(S^1, \varphi, \mathfrak{o}, \mathbf{u}, \mathbf{v}) = \text{Tor}(S^1, \varphi, \mathbf{u}, \mathbf{v}, \mathbf{c}^0, \mathbf{c}^1) \cdot \text{sgn} \left(\text{Tor}(S^1, 1, h^0, h^1, c^0, c^1) \right)^{\dim \varphi}$$

Let $-\mathfrak{o}$ denote the homology orientation opposite to \mathfrak{o} . It is straightforward from construction that

$$(4.2) \quad \text{TOR}(S^1, \varphi, -\mathfrak{o}, \mathbf{u}, \mathbf{v}) = (-1)^{\dim \varphi} \text{TOR}(S^1, \varphi, \mathfrak{o}, \mathbf{u}, \mathbf{v})$$

In particular, we do not need the homology orientation when $\dim \varphi$ is even. For a circle S^1 , the choice of an orientation determines a fundamental class, hence an orientation in cohomology.

Let $\varphi_i: \pi_1(S^1) \rightarrow \text{SL}(V_i)$ be representations into finite dimensional vector spaces, for $i = 1, 2$. Then $H^*(S^1; \varphi_1 \oplus \varphi_2) \cong H^*(S^1; \varphi_1) \oplus H^*(S^1; \varphi_2)$. Let \mathbf{u}_i and \mathbf{v}_i denote bases for $H^0(S^1; \varphi_i)$ and $H^1(S^1; \varphi_i)$ respectively. The following lemma reduces to an elementary calculation:

LEMMA 4.3. — *Let $\varphi_i: \pi_1(S^1) \rightarrow \mathrm{SL}(V_i)$ be representations into finite dimensional vector spaces, for $i = 1, 2$. Then*

$$\begin{aligned} \mathrm{TOR}(S^1, \varphi_1 \oplus \varphi_2, \mathfrak{o}, \mathbf{u}_1 \times \{0\} \sqcup \{0\} \times \mathbf{u}_2, \mathbf{v}_1 \times \{0\} \sqcup \{0\} \times \mathbf{v}_2) \\ = \mathrm{TOR}(S^1, \varphi_1, \mathfrak{o}, \mathbf{u}_1, \mathbf{v}_1) \cdot \mathrm{TOR}(S^1, \varphi_2, \mathfrak{o}, \mathbf{u}_2, \mathbf{v}_2). \end{aligned}$$

4.3. An holomorphic volume form on $\mathcal{R}^{reg}(S^1, G)$

As in the introduction we let G denote a simply-connected, semisimple, complex, linear Lie group, $d = \dim G$, and $r = \mathrm{rk} G$.

DEFINITION 4.4. — *We call a representation $\rho: \pi_1(S^1) \rightarrow G$ regular if the image of the generator of $\pi_1(S^1)$ is a regular element $g \in G$. The set of conjugacy classes of regular representations is denoted by $\mathcal{R}^{reg}(S^1, G)$.*

Let $\rho: \pi_1(S^1) \rightarrow G$ be regular. Then $\dim H^0(S^1; \mathrm{Ad} \rho) = r$, because $H^0(S^1; \mathrm{Ad} \rho) \cong \mathfrak{g}^{\mathrm{Ad} \rho}$. As the Euler characteristic of S^1 vanishes, $\dim H^1(S^1; \mathrm{Ad} \rho) = r$. Furthermore, since G is simply connected, we have $H^1(S^1; \mathrm{Ad} \rho) \cong T_{[\rho]} \mathcal{R}^{reg}(S^1, G)$ (Corollary 2.12).

By Poincaré duality, the pairing

$$\langle \cdot, \cdot \rangle: H^0(S^1; \mathrm{Ad} \rho) \times H^1(S^1; \mathrm{Ad} \rho) \rightarrow H^1(S^1; \mathbb{C}) \cong \mathbb{C}$$

is non degenerate.

In the next lemma we use the refined torsion with sign due to Turaev (see Section 4.2). By (4.2) changing the orientation of S^1 changes the torsion $\mathrm{TOR}(S^1, \mathrm{Ad} \rho, \mathfrak{o}, \mathbf{u}, \mathbf{v})$ by a factor $(-1)^d = (-1)^r$, as well as $\langle \wedge \mathbf{v}, \wedge \mathbf{u} \rangle$ by the same factor.

Let $G_{\mathbb{R}}$ denote the compact real form of the semisimple complex linear group G . We will assume that the restriction of the nondegenerate symmetric bilinear G -invariant form \mathcal{B} on the Lie algebra to $\mathfrak{g}_{\mathbb{R}}$ is positive definite. This means that \mathcal{B} is a negative multiple of the Killing form. In what follows we will denote by $\mathrm{Ad}_{\mathbb{R}}: G_{\mathbb{R}} \rightarrow \mathrm{Aut}(\mathfrak{g}_{\mathbb{R}})$ the restriction of Ad to the real form $G_{\mathbb{R}}$.

LEMMA 4.5. — *If $\rho: \pi_1(S^1) \rightarrow G$ is a regular representation, and if \mathbf{u} and \mathbf{v} are bases of $H^0(S^1; \mathrm{Ad} \rho)$ and $H^1(S^1; \mathrm{Ad} \rho)$ respectively, then the product*

$$\mathrm{TOR}(S^1, \mathrm{Ad} \rho, \mathfrak{o}, \mathbf{u}, \mathbf{v}) \langle \wedge \mathbf{v}, \wedge \mathbf{u} \rangle$$

is independent of \mathbf{u} .

LEMMA 4.6. — *If $\rho: \pi_1(S^1) \rightarrow G_{\mathbb{R}}$ is a regular representation and if \mathbf{u} and \mathbf{v} are bases of $H^0(S^1; \mathrm{Ad}_{\mathbb{R}} \rho)$, and $H^1(S^1; \mathrm{Ad}_{\mathbb{R}} \rho)$ respectively, then*

$$\mathrm{TOR}(S^1, \mathrm{Ad}_{\mathbb{R}} \rho, \mathfrak{o}, \mathbf{u}, \mathbf{v}) \langle \wedge \mathbf{v}, \wedge \mathbf{u} \rangle > 0.$$

Proof of Lemma 4.5. — Let \mathbf{u} and \mathbf{u}' be bases for $H^0(S^1; \mathrm{Ad} \rho)$, and \mathbf{v} and \mathbf{v}' , for $H^1(S^1; \mathrm{Ad} \rho)$. We change bases by means of the following formulas:

$$\mathrm{TOR}(S^1, \mathrm{Ad} \rho, \mathfrak{o}, \mathbf{u}', \mathbf{v}') = \mathrm{TOR}(S^1, \mathrm{Ad} \rho, \mathfrak{o}, \mathbf{u}, \mathbf{v}) \frac{[\mathbf{v}' : \mathbf{v}]}{[\mathbf{u}' : \mathbf{u}]}$$

and

$$\langle \wedge \mathbf{v}', \wedge \mathbf{u}' \rangle = \langle \wedge \mathbf{v}, \wedge \mathbf{u} \rangle [\mathbf{v}' : \mathbf{v}] [\mathbf{u}' : \mathbf{u}].$$

Hence

$$(4.3) \quad \text{TOR}(S^1, \text{Ad } \rho, \mathfrak{o}, \mathbf{u}', \mathbf{v}') \langle \wedge \mathbf{v}', \wedge \mathbf{u}' \rangle = \text{TOR}(S^1, \text{Ad } \rho, \mathfrak{o}, \mathbf{u}, \mathbf{v}) \langle \wedge \mathbf{v}, \wedge \mathbf{u} \rangle [\mathbf{v}' : \mathbf{v}]^2.$$

This proves independence of \mathbf{u} . □

Proof of Lemma 4.6. — We are assuming that the image of ρ is contained in the compact real form, $\rho(\pi_1(S^1)) \subset G_{\mathbb{R}}$. By (4.3) in the proof of Lemma 4.5, the sign is independent of \mathbf{v} . By regularity, the image $\rho(\pi_1(S^1))$ is contained in an unique maximal torus T , and $H^0(S^1; \text{Ad}_{\mathbb{R}} \rho) \subset \mathfrak{g}_{\mathbb{R}}$ is the corresponding Cartan subalgebra \mathfrak{h} . Recall that \mathcal{B} restricted to \mathfrak{h} is positive definite. Hence we may choose an \mathbb{R} -basis of $\mathfrak{g}_{\mathbb{R}}$ compatible with the orthogonal decomposition $\mathfrak{g}_{\mathbb{R}} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$. This is also a decomposition of $\pi_1(S^1)$ -modules, and by Lemma 4.3 the torsion decomposes accordingly as a product of torsions.

We compute the torsion of \mathfrak{h} first. Since the adjoint action of T on \mathfrak{h} is trivial, we have natural isomorphisms

$$(4.4) \quad H^1(S^1; \mathfrak{h}) \cong H^1(S^1, \mathbb{R}) \otimes \mathfrak{h} \quad \text{and} \quad H^0(S^1; \mathfrak{h}) \cong H^0(S^1, \mathbb{R}) \otimes \mathfrak{h}.$$

We choose a cell decomposition of S^1 with a single (positively oriented) cell in each dimension. In particular, as the adjoint action of T on \mathfrak{h} is trivial, the boundary operator $\delta: C^0(K; \mathfrak{h}) \rightarrow C^1(K; \mathfrak{h})$ vanishes. Choose a \mathcal{B} -orthonormal basis for \mathfrak{h} ; this provides geometric basis \mathbf{c}^1 and \mathbf{c}^0 , and since $\delta = 0$, those are also representatives of basis in cohomology. By choosing those bases ($\mathbf{u} = \mathbf{c}^0$ and $\mathbf{v} = \mathbf{v}^1$),

$$\text{tor}(S^1, \text{Ad } \rho|_{\mathfrak{h}}, \mathbf{c}^0, \mathbf{c}^1, \mathbf{c}^0, \mathbf{c}^1) = 1.$$

Following the construction in Section 4.2, we compute $\alpha_0 = \beta_0 = r$ and $\alpha_1 = \beta_1 = 2r \equiv 0 \pmod{2}$. Thus $N \equiv r^2 \equiv r \pmod{2}$ and

$$\text{Tor}(S^1, \text{Ad } \rho|_{\mathfrak{h}}, \mathbf{c}^0, \mathbf{c}^1, \mathbf{c}^0, \mathbf{c}^1) = (-1)^r.$$

As the torsion for the trivial representation corresponds to the case $r = 1$, the torsion “Tor” for the trivial representation is -1 and

$$(4.5) \quad \text{TOR}(S^1, \text{Ad } \rho|_{\mathfrak{h}}, \mathfrak{o}, \mathbf{c}^1, \mathbf{c}^0) = (-1)^r \cdot \text{sgn}(-1)^r = 1.$$

Also, by construction, $\langle \wedge \mathbf{c}^1, \wedge \mathbf{c}^0 \rangle = 1$.

Next we compute the torsion of \mathfrak{h}^{\perp} . We have $H^*(S^1; \mathfrak{h}^{\perp}) = 0$ and, since $\dim \mathfrak{h}^{\perp}$ is even,

$$\text{TOR}(S^1, \text{Ad } \rho|_{\mathfrak{h}^{\perp}}, \mathfrak{o}) = \text{tor}(S^1, \text{Ad } \rho|_{\mathfrak{h}^{\perp}}, \mathbf{c}^0, \mathbf{c}^1) = \det(\text{Ad}_{\mathbb{R}}(g) - \text{Id})|_{\mathfrak{h}^{\perp}},$$

where $g \in G$ is the image of a generator of $\pi_1(S^1)$. Notice that, as $\dim \mathfrak{h}^{\perp}$ is even, the sign is independent of the cohomology orientation.

Let Δ_G be the Weyl function [GW09]. Then

$$\det(\text{Ad}(g) - \text{Id})|_{\mathfrak{h}^{\perp}} = \Delta_G(g) \Delta_G(g^{-1}) = |\Delta_G(g)|^2 > 0$$

(see [GW09, (7.47)] for details). This finishes the proof of the Lemma 4.6. □

DEFINITION 4.7. — Let $\rho: \pi_1(S^1) \rightarrow G$ be a regular representation. The form

$$\nu: \bigwedge^r H^1(S^1; \text{Ad } \rho) \rightarrow \mathbb{C}$$

is defined by the formula

$$(4.6) \quad \nu(\wedge \mathbf{u}) = \pm \sqrt{\text{TOR}(S^1, \text{Ad } \rho, \mathfrak{o}, \mathbf{u}, \mathbf{v}) \langle \wedge \mathbf{v}, \wedge \mathbf{u} \rangle}$$

for any basis \mathbf{u} of $H^1(S^1; \text{Ad } \rho)$. (By Lemma 4.5, it is independent of \mathbf{u} .)

We are interested in understanding ν as a differential form on $\mathcal{R}^{reg}(S^1, G)$ for G simply connected. Recall from Section 2.3 that when G is simply connected, the Steinberg map has coordinates the fundamental characters $(\sigma_1, \dots, \sigma_r): G \rightarrow \mathbb{C}^r$.

PROPOSITION 4.8. — For G simply connected, there exists a constant $C \in \mathbb{C}^*$ and a choice of sign for ν such that

$$\nu = C \, d\sigma_1 \wedge \dots \wedge d\sigma_r.$$

Proof. — Using Steinberg’s section $s: \mathbb{C}^r \rightarrow G^{reg}$ (Theorem 2.11), consider for each $p \in \mathbb{C}^r$ the subalgebra $\mathfrak{g}^{\text{Ad } s(p)}$ of elements fixed by $\text{Ad } s(p)$. By the constant rank theorem this defines an algebraic vector bundle

$$\mathfrak{g}^{\text{Ad } \circ s} \rightarrow E(s) \rightarrow \mathbb{C}^r.$$

Since algebraic vector bundles over \mathbb{C}^r are trivial [Qui76, Sus76], there is a trivialization $\mathbf{u} = (u_1, \dots, u_r): \mathbb{C}^r \rightarrow E(s)$, so that $\{u_1(p), \dots, u_r(p)\}$ is a basis for $\mathfrak{g}^{\text{Ad } s(p)}$, for each $p \in \mathbb{C}^r$. By the identifications, $T_{[s(p)]} \mathcal{R}^{reg}(S^1, G) \cong H^1(S^1, \text{Ad } s(p))$ (Corollary 2.12), and the identification $\mathfrak{g}^{\text{Ad } s(p)} \cong H^0(S^1, \text{Ad } s(p))$, we have two $(r, 0)$ -forms on \mathbb{C}^r :

$$(4.7) \quad \langle s_*(-), \wedge \mathbf{u} \rangle \quad \text{and} \quad \text{TOR}(S^1, \text{Ad } s, \mathfrak{o}, \mathbf{u}, s_*(-)).$$

We claim that these forms are both algebraic. Assuming the claim, they are a polynomial multiple of $dz_1 \wedge \dots \wedge dz_r$, for (z_1, \dots, z_r) the standard coordinate system for \mathbb{C}^r . Since they vanish nowhere in \mathbb{C}^r , both forms in (4.7) are a constant multiple of $dz_1 \wedge \dots \wedge dz_r$. Viewed as forms on $\mathcal{R}^{reg}(S^1, G)$, they are both a constant multiple of $d\sigma_1 \wedge \dots \wedge d\sigma_r$ and the proposition follows, once we have shown the claim.

To prove that the forms in (4.7) are algebraic, use a CW-decomposition K of S^1 with a 1 and a 0-cell, so that the groups of cochains $C^i(K, \text{Ad } s(p))$, for $i = 0, 1$, are naturally identified with \mathfrak{g} . We also have a natural isomorphism $(R_{s(p)^{-1}})_*: T_{s(p)}G \rightarrow \mathfrak{g}$, which is precisely the tangent map of the right multiplication by $s(p)^{-1}$. This identification maps $s_*(\partial_{z_i})$ at $p \in \mathbb{C}^n$ to

$$v_i(p) = (R_{s(p)^{-1}})_* \left(\frac{\partial s}{\partial z_i}(p) \right) \in \mathfrak{g},$$

which is a map algebraic on $p \in \mathbb{C}^r$. Hence the intersection product is

$$\langle s_*(\partial_{z_1} \wedge \dots \wedge \partial_{z_r}), \wedge \mathbf{u} \rangle = \det(\langle s_*(\partial_{z_i}), u_j \rangle_{ij}) = \det(\mathcal{B}(v_i, u_j)_{ij}),$$

which is polynomial on $p \in \mathbb{C}^r$.

To show that the torsion is algebraic, using again triviality of algebraic bundles on \mathbb{C}^r , complete \mathbf{u} to a section of the trivial bundle $(u_1, \dots, u_r, \dots, u_d): \mathbb{C}^r \rightarrow \mathfrak{g}$.

Setting $\tilde{\mathbf{b}}^1 = \{u_{r+1}, \dots, u_d\}$, then $\mathbf{u}(p) \sqcup \tilde{\mathbf{b}}^1(p)$ is a basis for \mathfrak{g} , for each $p \in \mathbb{C}$. We view $\mathbf{u}(p) \sqcup \tilde{\mathbf{b}}^1(p)$ as a basis for $C^0(K, \text{Ad } s(p))$, so that $\mathbf{u}(p)$ projects to a basis for $H^0(S^1, \text{Ad } s(p))$, for every $p \in \mathbb{C}^r$. Fix $\mathbf{c}^0 = \mathbf{c}^1$ a basis for \mathfrak{g} . By construction:

$$\text{TOR}(S^1, \text{Ad } s, \mathfrak{o}, \mathbf{u}, s_*(\partial_{z_1} \wedge \dots \wedge \partial_{z_r})) = \pm \frac{[\mathbf{v} \sqcup \partial \tilde{\mathbf{b}}^1 : \mathbf{c}^1]}{[\mathbf{u} \sqcup \tilde{\mathbf{b}}^1 : \mathbf{c}^0]},$$

where the sign depends on the orientation in homology, but it is constant on p . Thus this is a quotient of algebraic polynomial functions on \mathbb{C}^r , but since it is defined everywhere, it is polynomial. \square

4.4. Witten’s formula

Let $\rho: \pi_1(S) \rightarrow G$ be a good ∂ -regular representation. Let ν_i denote the peripheral form of the i -th component of ∂S (Definition 4.7), and let ω denote the symplectic form of the relative character variety (4.1). We aim to prove Theorem 1.3, namely, that $\Omega_{\pi_1(S)} = \pm \frac{1}{n!} \omega^n \wedge \nu_1 \wedge \dots \wedge \nu_b$.

Proof of Theorem 1.3. — We apply the duality formula (Proposition 3.3) and the formula of the torsion for the long exact sequence of the pair, Equation (4.9) below. For this purpose we discuss the bases in cohomology. Start with \mathbf{u} a basis for $H^0(\partial S; \text{Ad } \rho)$. If β denotes the connecting map of the long exact sequence, then complete $\beta(\mathbf{u})$ to a basis for $H^1(S, \partial S; \text{Ad } \rho)$: $\beta(\mathbf{u}) \sqcup \tilde{\mathbf{h}}$. Next we choose \mathbf{v} a basis for $H^1(\partial S; \text{Ad } \rho)$ that we lift to $\tilde{\mathbf{v}}$ by i , and if we set $j(\tilde{\mathbf{h}}) = \mathbf{h}$, then $\mathbf{h} \sqcup \tilde{\mathbf{v}}$ is a basis for $H^1(S; \text{Ad } \rho)$ (and \mathbf{h} is a basis for $\ker(i) = \text{Im}(j)$). The bases are organized as follows:

$$(4.8) \quad 0 \rightarrow H^0(\partial S; \text{Ad } \rho) \xrightarrow{\beta} H^1(S, \partial S; \text{Ad } \rho) \xrightarrow{j} H^1(S; \text{Ad } \rho) \xrightarrow{i} H^1(\partial S; \text{Ad } \rho) \rightarrow 0$$

\mathbf{u} $\beta(\mathbf{u}) \sqcup \tilde{\mathbf{h}}$ $\mathbf{h} \sqcup \tilde{\mathbf{v}}$ \mathbf{v}

As the bases have been chosen compatible with the maps of the long exact sequence, the product formula for the torsion [Mil66] gives:

$$(4.9) \quad \text{tor}(S, \text{Ad } \rho, \mathbf{h} \sqcup \tilde{\mathbf{v}}) = \pm \text{tor}(S, \partial S, \text{Ad } \rho, \beta(\mathbf{u}) \sqcup \tilde{\mathbf{h}}) \text{tor}(\partial S, \text{Ad } \rho, \mathbf{u}, \mathbf{v}).$$

We shall combine (4.9) with the duality formula (Proposition 3.3):

$$(4.10) \quad \text{tor}(S, \text{Ad } \rho, \mathbf{h} \sqcup \tilde{\mathbf{v}}) \text{tor}(S, \partial S, \text{Ad } \rho, \beta(\mathbf{u}) \sqcup \tilde{\mathbf{h}}) = \pm \langle \wedge(\mathbf{h} \sqcup \tilde{\mathbf{v}}), \wedge(\beta(\mathbf{u}) \sqcup \tilde{\mathbf{h}}) \rangle.$$

We next decompose the right hand side in (4.10). By naturality of the intersection form,

$$\langle h_i, \beta(u_j) \rangle = \langle i(h_i), u_j \rangle = \langle i(j(\tilde{h}_i)), u_j \rangle = 0.$$

Hence the right hand side in (4.10) becomes:

$$\langle \wedge(\mathbf{h} \sqcup \tilde{\mathbf{v}}), \wedge(\beta(\mathbf{u}) \sqcup \tilde{\mathbf{h}}) \rangle = \langle \wedge \mathbf{h}, \wedge \tilde{\mathbf{h}} \rangle \cdot \langle \wedge \tilde{\mathbf{v}}, \wedge \beta(\mathbf{u}) \rangle.$$

Again by naturality

$$\langle \wedge \tilde{\mathbf{v}}, \wedge \beta(\mathbf{u}) \rangle = \langle \wedge i(\tilde{\mathbf{v}}), \wedge \mathbf{u} \rangle = \langle \wedge \mathbf{v}, \wedge \mathbf{u} \rangle.$$

In addition, by definition

$$\langle \wedge \mathbf{h}, \wedge \tilde{\mathbf{h}} \rangle = \omega(\wedge \mathbf{h}, \wedge \mathbf{h}).$$

Thus

$$(4.11) \quad \langle \wedge(\mathbf{h} \sqcup \tilde{\mathbf{v}}), \wedge(\beta(\mathbf{u}) \sqcup \tilde{\mathbf{h}}) \rangle = \pm \langle \wedge \mathbf{u}, \wedge \mathbf{v} \rangle \omega(\wedge \mathbf{h}, \wedge \mathbf{h}).$$

Hence by (4.9), (4.10), and (4.11):

$$\text{tor}(S, \text{Ad } \rho, \mathbf{h} \sqcup \tilde{\mathbf{v}})^2 = \pm \omega(\wedge \mathbf{h}, \wedge \mathbf{h}) \text{TOR}(\partial S, \text{Ad } \rho, \mathbf{u}, \mathbf{v}) \langle \mathbf{u}, \mathbf{v} \rangle.$$

Notice that on the right hand side we use Turaev’s sign refined torsion. Next we claim that the sign of this formula is + and not −. It suffices to determine the sign in the compact case. Then the formula will follow in the complex case by a connectedness argument (the variety of characters of a free group is connected and irreducible, and ∂-regularity and being good are Zariski open properties, hence they fail in a set of real codimension ≥ 2).

We show that the sign is + in the compact case by showing that all terms are positive. Since $\text{TOR}(\partial S, \text{Ad } \rho, \mathbf{u}, \mathbf{v}) \langle \mathbf{u}, \mathbf{v} \rangle$ is positive by Lemma 4.6, the sign will follow from the equality

$$(4.12) \quad \omega(\wedge \mathbf{h}, \wedge \mathbf{h}) = \left(\frac{1}{n!} \omega^n(\wedge \mathbf{h}) \right)^2,$$

that will also complete the proof of the Theorem 1.3.

We give a self-contained proof of (4.12) for completeness. By Darboux’s Theorem there are local coordinates so that

$$\omega = dx_1 \wedge dx_2 + \cdots + dx_{2n-1} \wedge dx_{2n}.$$

Let A be a matrix of size $2n \times 2n$ whose colons are the components of the vectors of \mathbf{h} in this coordinate system. Then, if J denotes the matrix of the standard symplectic form,

$$\omega(\wedge \mathbf{h}, \wedge \mathbf{h}) = \det(\omega(h_i, h_j)_{ij}) = \det(A^t J A) = (\det A)^2.$$

On the other hand $\omega^n = n! dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{2n}$, hence

$$\frac{1}{n!} \omega^n(\wedge \mathbf{h}) = \det A$$

and we are done. □

Remark 4.9. — Theorem 1.3 is a version for complex groups of [BL99, Theorem 5.40], which holds for compact groups. In particular, in [BL99, Definition 5.38] peripheral elements are required to be semisimple, through we only assume that the peripheral elements are regular (but perhaps not semisimple). By taking real forms, we recover [BL99, Theorem 5.40] for compact Lie groups, as we illustrate for $\text{SU}(N)$ (see Proposition 5.4), but also a formula for other real forms, like $\text{SL}_N(\mathbb{R})$ (see Remark 5.5).

5. Formulas for the group $\text{SL}_N(\mathbb{C})$

If $G = \text{SL}_N(\mathbb{C})$ we can give explicit formulas for several volume forms.

5.1. The form ν for $\mathrm{SL}_N(\mathbb{C})$

We know that ν is a constant multiple of $d\sigma_1 \wedge \cdots \wedge d\sigma_r$ and we shall determine the constant, completing the proof of Proposition 1.2.

Recall that we have fixed the \mathbb{C} -bilinear form on $\mathfrak{sl}_N(\mathbb{C})$ to be

$$\mathcal{B}(X, Y) = -\mathrm{tr}(XY) \quad \forall X, Y \in \mathrm{SL}_N(\mathbb{C}).$$

In $\mathrm{SL}_N(\mathbb{C})$ the invariant functions are the symmetric functions on the spectrum: if the eigenvalues of $A \in \mathrm{SL}_N(\mathbb{C})$ are $\lambda_1, \dots, \lambda_N$, then

$$\sigma_1(A) = \sum_i \lambda_i, \quad \sigma_2(A) = \sum_{i < j} \lambda_i \lambda_j, \quad \dots, \quad \sigma_{N-1}(A) = \sum_i \frac{1}{\lambda_i}.$$

Those symmetric functions are characterized by Cayley–Hamilton Theorem:

$$A^N - \sigma_1(A)A^{N-1} + \sigma_2(A)A^{N-2} - \cdots + (-1)^{N-1}\sigma_{N-1}(A)A + (-1)^N \mathrm{Id} = 0.$$

We identify $R(S^1, \mathrm{SL}_N(\mathbb{C}))$ with the group $\mathrm{SL}_N(\mathbb{C})$ by mapping a representation to the image of a generator of $\pi_1(S)$, so that σ_i is a function on $R(S^1, \mathrm{SL}_N(\mathbb{C}))$ invariant under conjugation. On the other hand, $\sigma_1, \dots, \sigma_{N-1}$ are the coordinates of the isomorphism:

$$\mathcal{R}(S^1, \mathrm{SL}_N(\mathbb{C})) \cong \mathrm{SL}_N(\mathbb{C}) // \mathrm{SL}_N(\mathbb{C}) \cong \mathbb{C}^{N-1}.$$

PROPOSITION 5.1. — *Let $\nu: \bigwedge^{N-1} H^1(S^1, \mathrm{Ad} \rho) \rightarrow \mathbb{C}$ denote the volume form in Definition 4.7. On $\mathcal{R}(S^1, \mathrm{SL}_N(\mathbb{C})) \cong \mathbb{C}^{N-1}$*

$$\nu = \pm (\sqrt{-1})^{\epsilon(N)} \sqrt{N} d\sigma_1 \wedge \cdots \wedge d\sigma_{N-1},$$

where $\epsilon(N) = (N - 1)(N + 2)/2$.

By direct application of the proposition, we get:

COROLLARY 5.2. — *On $\mathcal{R}^{reg}(S^1, \mathrm{SL}_2(\mathbb{C}))$*

$$\nu = \pm \sqrt{2} d \mathrm{tr}_\gamma$$

where γ is a generator of $\pi_1(S^1)$.

Proof of Proposition 5.1. — We identify the variety of representations of the cyclic group $\pi_1(S^1)$ with $\mathrm{SL}_N(\mathbb{C})$ by considering the image of a generator, that we call g . To simplify, we may assume that g is semisimple, by Proposition 4.8. After diagonalizing:

$$g = \begin{pmatrix} e^{u_1} & 0 & & 0 \\ 0 & e^{u_2} & & 0 \\ & & \ddots & \\ 0 & 0 & & e^{u_N} \end{pmatrix}$$

with $u_1 + \cdots + u_N = 0$ and all u_i are pairwise different mod $2\pi\sqrt{-1}\mathbb{Z}$. The Cartan algebra \mathfrak{h} is the subalgebra of diagonal matrices. Since the decomposition $\mathfrak{sl}_N(\mathbb{C}) = \mathfrak{h} \oplus \mathfrak{h}^\perp$ is preserved by the adjoint action of g , the torsion is the corresponding

product of torsions, by Lemma 4.3. By looking at the action on non-diagonal entries of $\mathfrak{sl}_N(\mathbb{C})$, the torsion of the adjoint representation on \mathfrak{h}^\perp is:

$$\prod_{i \neq j} (e^{u_i - u_j} - 1) = \prod_{i \neq j} (e^{u_i} - e^{u_j}) = (-1)^{N(N-1)/2} \prod_{i > j} (e^{u_i} - e^{u_j})^2,$$

which is the product $\Delta_G(g)\Delta_G(g^{-1})$ of Weyl functions [GW09, Section 7]. Thus

$$(5.1) \quad \nu = \pm (\sqrt{-1})^{N(N-1)/2} \prod_{i > j} (e^{u_i} - e^{u_j}) \theta_H,$$

where

$$(5.2) \quad \theta_H(\wedge \mathbf{v}) = \sqrt{\text{TOR}(S^1, \mathfrak{h}, \mathfrak{o}, \wedge \mathbf{v}, \wedge \mathbf{u}) \langle \wedge \mathbf{v}, \wedge \mathbf{u} \rangle}.$$

We use coordinates for the Cartan algebra via the entries of the diagonal matrices: $\mathfrak{h} \cong \{(u_1, \dots, u_N) \in \mathbb{C}^N \mid \sum u_i = 0\}$,

LEMMA 5.3. — *The form θ_H is the restriction to $\{(u_1, \dots, u_N) \in \mathbb{C}^N \mid \sum u_i = 0\}$ of the form*

$$\pm (\sqrt{-1})^{(N-1)} \frac{1}{\sqrt{N}} \sum_{i=1}^N (-1)^{N-i} du_1 \wedge \dots \wedge \widehat{du_i} \wedge \dots \wedge du_N,$$

or, equivalently, of

$$\pm (\sqrt{-1})^{(N-1)} \sqrt{N} du_1 \wedge \dots \wedge du_{N-1}.$$

Proof. — In order to compute $\text{TOR}(S^1, \mathfrak{h}, \mathfrak{o}, \wedge \mathbf{v}, \wedge \mathbf{u})$ we proceed as in the proof of Lemma 4.6. In particular we choose a cell decomposition of S^1 with a single (positively oriented) cell in each dimension, and bases in homology represented by the geometric bases. With this choice of \mathbf{u} and \mathbf{v} , by (4.5),

$$\text{TOR}(S^1, \mathfrak{h}, \mathfrak{o}, \mathbf{v}, \mathbf{u}) = 1.$$

Next we compute $\langle \wedge \mathbf{v}, \wedge \mathbf{u} \rangle$. The bases \mathbf{u} and \mathbf{v} are constructed from dual basis in $H^*(S^1; \mathbb{Z})$ tensorized by a basis of \mathfrak{h} . We choose a basis for the Cartan subalgebra, $\mathbf{e} = \{e_1, \dots, e_{N-1}\}$:

$$e_1 = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}, \quad \dots, \quad e_{N-1} = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

Since the cells of S^1 are positively oriented,

$$\langle \wedge \mathbf{v}, \wedge \mathbf{u} \rangle = \det(\mathcal{B}(e_i, e_j))_{i,j}.$$

In addition, as $\mathcal{B}(e_i, e_i) = -2$ and $\mathcal{B}(e_i, e_j) = -1$ for $i \neq j$, $\det(\mathcal{B}(e_i, e_j))_{i,j} = (-1)^{N-1}N$. Thus

$$(5.3) \quad \theta_H(\wedge \mathbf{v}) = \pm \sqrt{(-1)^{N-1}N}$$

On the other hand, direct computation yields:

$$\sum_{i=1}^N (-1)^{N-i} du_1 \wedge \dots \wedge \widehat{du_i} \wedge \dots \wedge du_N (e_1 \wedge \dots \wedge e_{N-1}) = N.$$

By the natural identification of $H^1(S^1; \mathfrak{h})$ with the Cartan algebra \mathfrak{h} we get the Lemma 5.3. □

We conclude the proof of Proposition 5.1. By (5.1) and Lemma 5.3,

$$(5.4) \quad \nu \wedge (du_1 + \dots + du_N) = \left(\sqrt{-1}\right)^{\epsilon(N)} \sqrt{N} \prod_{i>j} (e^{u_i} - e^{u_j}) du_1 \wedge \dots \wedge du_N.$$

Next we use Newton's identities:

$$\begin{aligned} \sigma_1 &= e^{u_1} + \dots + e^{u_N} \\ \sigma_2 &= -\frac{1}{2}(e^{2u_1} + \dots + e^{2u_N} - \text{Pol}(\sigma_1)) \\ &\vdots \\ \sigma_j &= (-1)^{j+1} \frac{1}{j}(e^{ju_1} + \dots + e^{ju_N} - \text{Pol}(\sigma_1, \dots, \sigma_{j-1})) \end{aligned}$$

where $\text{Pol}(\sigma_1, \dots, \sigma_{j-1})$ denotes a polynomial expression on $\sigma_1, \dots, \sigma_{j-1}$, whose precise value is not relevant here. From them we deduce

$$d\sigma_1 \wedge \dots \wedge d\sigma_{N-1} = \pm \frac{1}{(N-1)!} d(\sum e^{u_j}) \wedge d(\sum e^{2u_j}) \wedge \dots \wedge d(\sum e^{(N-1)u_j})$$

Since

$$d(\sum e^{iu_j}) = i \sum e^{iu_j} du_j,$$

for $i = 1, \dots, m-1$, Vandermonde determinant yields

$$(5.5) \quad (du_1 + \dots + du_N) \wedge d\sigma_1 \wedge \dots \wedge d\sigma_{N-1} = \pm \prod_{i>j} (e^{u_i} - e^{u_j}) du_1 \wedge \dots \wedge du_N.$$

Then combine (5.4) and (5.5) to prove the theorem, knowing that our tangent space is the kernel of $du_1 + \dots + du_N$. □

5.2. The form ν for $SU(N)$

An element in $SU(N)$ is conjugate to a diagonal element

$$\begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_N} \end{pmatrix}$$

with $\sum \theta_i \in 2\pi\mathbb{Z}$. A matrix is regular if and only if $e^{i\theta_j} \neq e^{i\theta_k}$ for $j \neq k$.

By identifying $\mathcal{R}^{reg}(S^1, SU(N))$ with the image of the generator (or its conjugacy class), functions on $\theta_1, \dots, \theta_N$ invariant under permutations are well-defined on $\mathcal{R}^{reg}(S^1, SU(N))$. Also the form $d\theta_1 \wedge \dots \wedge d\theta_{N-1}$ is well-defined up to sign by the relation $\sum \theta_i \in 2\pi\mathbb{Z}$.

PROPOSITION 5.4. — On $\mathcal{R}^{reg}(S^1, SU(N))$ (for $\mathcal{B}(X, Y) = -\text{tr}(XY)$),

$$\nu = \pm 2^{N(N-1)/2} \sqrt{N} \prod_{i<j} \sin\left(\frac{\theta_i - \theta_j}{2}\right) d\theta_1 \wedge \dots \wedge d\theta_{N-1}.$$

Proof. — From the proof of Proposition 5.1, if $g \in \text{SU}(N)$ is the image of the generator of $\pi_1(S^1)$, $\nu = |\Delta_G(g)| \theta_H$. By [GW09, Exercise 7.3.5],

$$|\Delta_G(g)| = 2^{N(N-1)/2} \prod_{i < j} \left| \sin \left(\frac{\theta_i - \theta_j}{2} \right) \right|.$$

On the other hand, by Lemma 5.3,

$$\theta_H = \pm \sqrt{N} d\theta_1 \wedge \cdots \wedge d\theta_{N-1},$$

which proves the formula. □

Remark 5.5. — We may consider also the restriction to $\text{SL}_N(\mathbb{R})$. Then the expression of the volume form is just the restriction of Proposition 5.1. It may be either real valued or $\sqrt{-1}$ times real, because \mathcal{B} is not positive definite on $\mathfrak{sl}_N(\mathbb{R})$. The restriction of \mathcal{B} to \mathfrak{so}_N is positive definite, but its restriction to its orthogonal $\mathfrak{so}_N^\perp \subset \mathfrak{sl}_N(\mathbb{R})$ is negative definite. Notice that $\dim \mathfrak{so}_N^\perp = (N - 1)(N + 2)/2 \equiv \epsilon(N) \pmod{2}$, that determines whether it is real or $\sqrt{-1}$ times real.

5.3. Volume form for representation spaces of F_2

In this subsection we compute the volume form on the space of representations of a free group of rank 2, $F_2 = \langle \gamma_1, \gamma_2 \rangle$, in $\text{SL}_2(\mathbb{C})$ and $\text{SL}_3(\mathbb{C})$. We use the notation $t_{i_1 \dots i_k}$ for the trace functions of $\gamma_{i_1} \dots \gamma_{i_k}$ in $\text{SL}_2(\mathbb{C})$, with the convention $\gamma_i = \gamma_i^{-1}$. For instance, the trace function of $\gamma_1 \gamma_2^{-1}$ will be denoted by $t_{1\bar{2}}$.

We start with $\mathcal{R}(F_2, \text{SL}_2(\mathbb{C}))$. By Fricke–Klein Theorem, see [Gol09], the respective trace functions of γ_1, γ_2 and $\gamma_1 \gamma_2$ define an isomorphism

$$(5.6) \quad (t_1, t_2, t_{12}) : X(F_2, \text{SL}_2(\mathbb{C})) \rightarrow \mathbb{C}^3.$$

Since F_2 is the fundamental group of a pair of pants $S_{0,3}$, and γ_1, γ_2 and $\gamma_1 \gamma_2$ correspond to the peripheral elements, by Theorem 1.3 and Corollary 5.2:

COROLLARY 5.6. — *The volume form on $\mathcal{R}^*(F_2, \text{SL}_2(\mathbb{C}))$ is*

$$\Omega_{F_2} = \pm 2\sqrt{2} dt_1 \wedge dt_2 \wedge dt_{12}.$$

We next discuss the space of representations of $F_2 = \langle \gamma_1, \gamma_2 \rangle$ in $\text{SL}_3(\mathbb{C})$.

The symmetric invariant functions σ_1 and σ_2 of a matrix in $\text{SL}_3(\mathbb{C})$ are, respectively, its trace and the trace of its inverse. Recall that the trace functions in $\text{SL}_3(\mathbb{C})$ are denoted by $\tau_{i_1 \dots i_k}$ instead of $t_{i_1 \dots i_k}$. According to [Law07], $X(F_2, \text{SL}_3(\mathbb{C}))$ is a branched covering of \mathbb{C}^8 with coordinates

$$\mathcal{T} = (\tau_1, \tau_{\bar{1}}, \tau_2, \tau_{\bar{2}}, \tau_{12}, \tau_{\bar{1}\bar{2}}, \tau_{1\bar{2}}, \tau_{\bar{1}2}) : X(F_2, \text{SL}_3(\mathbb{C})) \rightarrow \mathbb{C}^8.$$

The branching is given by the trace of the commutators $\tau_{12\bar{1}\bar{2}}$ and $\tau_{21\bar{2}\bar{1}}$ that are solutions of a quadratic equation

$$z^2 - Pz + Q = 0$$

for some polynomials P and Q on the variables $\tau_1, \tau_{\bar{1}}, \tau_2, \tau_{\bar{2}}, \tau_{12}, \tau_{\bar{1}\bar{2}}, \tau_{1\bar{2}}, \tau_{\bar{1}2}$, the expression of P and Q can be found in [Law07, Law09]. Notice that $P = \tau_{12\bar{1}\bar{2}} + \tau_{21\bar{2}\bar{1}}$ and $Q = \tau_{12\bar{1}\bar{2}} \tau_{21\bar{2}\bar{1}}$.

Thus, as γ_1, γ_2 and $\gamma_1\gamma_2$ represent the peripheral elements of a pair of pants $S_{0,3}$, a generic subset of the relative variety of representations is locally parameterized by $(\tau_{1\bar{2}}, \tau_{\bar{1}2})$; in the subset of points where there is no branching, i.e. $\tau_{12\bar{1}\bar{2}} \neq \tau_{21\bar{1}\bar{2}}$. Lawton has computed in [Law09, Theorem. 25] the Poisson bracket:

$$\{\tau_{1\bar{2}}, \tau_{\bar{1}2}\} = \tau_{21\bar{1}\bar{2}} - \tau_{12\bar{1}\bar{2}}.$$

As $(\tau_{1\bar{2}}, \tau_{\bar{1}2})$ are local coordinates, an elementary computation yields

$$(5.7) \quad \omega = -\frac{1}{\{\tau_{1\bar{2}}, \tau_{\bar{1}2}\}} d\tau_{1\bar{2}} \wedge d\tau_{\bar{1}2}.$$

Therefore

$$(5.8) \quad \omega = \frac{d\tau_{1\bar{2}} \wedge d\tau_{\bar{1}2}}{\tau_{21\bar{1}\bar{2}} - \tau_{12\bar{1}\bar{2}}}.$$

On the other hand, by Proposition 5.1, the form ν_1 corresponding to γ_1 is

$$\nu_1 = \pm\sqrt{-3} d\tau_1 \wedge d\tau_{\bar{1}},$$

and similarly for γ_2 and γ_{12} . Using Theorem 1.3 and these computations we get:

PROPOSITION 5.7. — For

$$\mathcal{T} = (\tau_1, \tau_{\bar{1}}, \tau_2, \tau_{\bar{2}}, \tau_{12}, \tau_{\bar{1}\bar{2}}, \tau_{1\bar{2}}, \tau_{\bar{1}2}) : \mathcal{R}^*(F_2, \mathrm{SL}_3(\mathbb{C})) \setminus \{\tau_{21\bar{1}\bar{2}} = \tau_{12\bar{1}\bar{2}}\} \rightarrow \mathbb{C}^8$$

the restriction of the holomorphic volume form on $\mathcal{R}^*(F_2, \mathrm{SL}_3(\mathbb{C})) \setminus \{\tau_{21\bar{1}\bar{2}} = \tau_{12\bar{1}\bar{2}}\}$ is $\Omega_{F_2}^{\mathrm{SL}_3(\mathbb{C})} = \pm\mathcal{T}^*\Omega$ where

$$\Omega = \pm \frac{3\sqrt{-3}}{\tau_{21\bar{1}\bar{2}} - \tau_{12\bar{1}\bar{2}}} d\tau_1 \wedge d\tau_{\bar{1}} \wedge d\tau_2 \wedge d\tau_{\bar{2}} \wedge d\tau_{12} \wedge d\tau_{\bar{1}\bar{2}} \wedge d\tau_{1\bar{2}} \wedge d\tau_{\bar{1}2}.$$

6. Symplectic forms

Let $\rho_0 \in R^*(S, \mathrm{SL}_2(\mathbb{C}))$ be a good, ∂ -regular representation. In this section we discuss the symplectic form on the relative character variety $\mathcal{R}^*(S, \partial S, \mathrm{SL}_2(\mathbb{C}))_{\rho_0}$ for the two surfaces $S_{1,1}$ and $S_{0,4}$, which are the surfaces with 2-dimensional relative character variety $\mathcal{R}^*(S, \partial S, \mathrm{SL}_2(\mathbb{C}))_{\rho_0}$. We use Goldman’s *product formula* for the Poisson bracket for surfaces [Gol86], as well as Lawton’s generalization [Law09, Section 4] to the relative character variety.

For this purpose, let $f: G \rightarrow \mathbb{C}$ be an *invariant function* (i.e. a function on G invariant under conjugation). Following Goldman [Gol04], its *variation function* (relative to \mathcal{B}) is defined as the unique map $F: G \rightarrow \mathfrak{g}$ such that for all $X \in \mathfrak{g}, A \in G$,

$$(6.1) \quad \frac{d}{dt} f(A \exp(tX)) \Big|_{t=0} = \mathcal{B}(F(A), X).$$

When $G = \mathrm{SL}_2(\mathbb{C})$ and $f = \mathrm{tr}$, the corresponding variation formula $\mathbf{T}: \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathfrak{sl}_2$ must satisfy, by (6.1), $\mathrm{tr}(AX) = -\mathrm{tr}(\mathbf{T}(A)X), \forall X \in \mathfrak{sl}_2$ and $\forall A \in \mathrm{SL}_2(\mathbb{C})$. Thus

$$\mathbf{T}(A) = \frac{\mathrm{tr}A}{2} \mathrm{Id} - A = -\frac{1}{2}(A - A^{-1}) \quad \text{for } A \in \mathrm{SL}_2(\mathbb{C}).$$

Notice that $\mathbf{T}(A) \in \mathfrak{sl}_2(\mathbb{C})$ is invariant by the adjoint action of A , and $\mathbf{T}(A) \neq 0$ for $A \neq \pm \text{Id}$.

PROPOSITION 6.1 ([Gol04, Law09]). — *Let α, β be oriented, simple closed curves meeting transversally in double points $p_1, \dots, p_k \in S$. For $[\rho] \in \mathcal{R}^*(S, \partial S, \text{SL}_2(\mathbb{C}))_{\rho_0}$ and each p_i , choose representatives*

$$\rho_i: \pi_1(S, p_i) \rightarrow \text{SL}_2(\mathbb{C})$$

of $[\rho]$. Let α_i, β_i be elements in $\pi_1(S, p_i)$ representing α, β respectively. For the bilinear form $\mathcal{B}(X, Y) = -\text{tr}(XY)$, the Poisson bracket of the trace functions t_α and t_β is

$$\begin{aligned} \{t_\alpha, t_\beta\}([\rho]) &= \sum_{i=1}^k \epsilon(p_i, \alpha, \beta) \mathcal{B}\left(\mathbf{T}(\rho_i(\alpha_i)), \mathbf{T}(\rho_i(\beta_i))\right) \\ &= -\sum_{i=1}^k \epsilon(p_i, \alpha, \beta) \text{tr}\left(\mathbf{T}(\rho_i(\alpha_i)) \mathbf{T}(\rho_i(\beta_i))\right) \end{aligned}$$

where $\epsilon(p_i, \alpha, \beta)$ denotes the oriented intersection number of α and β at p_i .

For later computations, it is useful to recall (cf. [GAMA93]) that for all $A, B \in \text{SL}_2(\mathbb{C})$

$$(6.2) \quad \text{tr}(A) \text{tr}(B) = \text{tr}(AB) + \text{tr}(AB^{-1}),$$

and a direct calculation gives

$$(6.3) \quad \text{tr}(\mathbf{T}(A) \mathbf{T}(B)) = \frac{1}{2} \text{tr}(AB - AB^{-1}).$$

6.1. A torus minus a disc

Let $S_{1,1}$ denote a surface of genus 1 with a boundary component. Its fundamental group is freely generated by two elements γ_1 and γ_2 that are represented by curves that intersect at one point. The peripheral element is the commutator $[\gamma_1, \gamma_2] = \gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1}$. The variety of characters $X(S_{1,1}, \text{SL}_2(\mathbb{C}))$ is the variety of characters of the free group on two generators, and it is isomorphic to \mathbb{C}^3 with coordinates t_1, t_2, t_{12} , by Fricke–Klein (5.6). Equality (6.2) implies that $t_1 t_2 = t_{12} + t_{\bar{1}\bar{2}}$.

Generically, the relative character variety is the hypersurface of \mathbb{C}^3 that is a level set of the trace of the commutator, $t_{12\bar{1}\bar{2}} = c$ for some $c \in \mathbb{C}$, where

$$(6.4) \quad t_{12\bar{1}\bar{2}} = t_1^2 + t_2^2 + t_{12}^2 - t_1 t_2 t_{12} - 2.$$

Therefore, given a good representation ρ_0 the variables (t_1, t_2) define local coordinates of $\mathcal{R}^*(S_{1,1}, \partial S_{1,1}, \text{SL}_2(\mathbb{C}))_{\rho_0}$ precisely when $\frac{\partial}{\partial t_{12}} t_{12\bar{1}\bar{2}} \neq 0$, ie. when

$$(6.5) \quad 2t_{12} - t_1 t_2 = t_{12} - t_{\bar{1}\bar{2}} \neq 0,$$

where $t_{\bar{1}\bar{2}} = t_{12}$ is the trace function of $\gamma_1 \gamma_2^{-1}$. Hence we obtain a local parametrization

$$T = (t_1, t_2): \mathcal{R}^*(S_{1,1}, \partial S_{1,1}, \text{SL}_2(\mathbb{C}))_{\rho_0} \setminus \{t_{12} = t_{\bar{1}\bar{2}}\} \rightarrow \mathbb{C}^2.$$

We compute next the symplectic form.

PROPOSITION 6.2. — *Let $\rho_0 \in R^*(S, \text{SL}_2(\mathbb{C}))$ be a good, ∂ -regular representation such that $t_{12}(\rho_0) \neq t_{1\bar{2}}(\rho_0)$. Then the symplectic form on $\mathcal{R}^*(S_{1,1}, \partial S_{1,1}, \text{SL}_2(\mathbb{C}))_{\rho_0} \setminus \{t_{12} = t_{1\bar{2}}\}$ is the pull-back $T^*\omega$, where*

$$\omega = \pm 2 \frac{dt_1 \wedge dt_2}{t_{12} - t_{1\bar{2}}}.$$

Proof. — For $[\rho] \in \mathcal{R}^*(S_{1,1}, \partial S_{1,1}, \text{SL}_2(\mathbb{C}))_{\rho_0} \setminus \{t_{12} = t_{1\bar{2}}\}$ we put $A = \rho(\gamma_1)$ and $B = \rho(\gamma_2)$. As γ_1 and γ_2 intersect in a single point, by Proposition 6.1 and (6.3) the Poisson bracket, for $\mathcal{B}(X, Y) = -\text{tr}(XY)$, between trace functions is

$$\{t_1, t_2\}([\rho]) = \pm \text{tr}(\mathbf{T}(A) \mathbf{T}(B)) = \pm \frac{1}{2}(t_{12} - t_{1\bar{2}}).$$

The proposition follows from Equation (5.7). □

Remark 6.3. — From Proposition 6.2 we can compute again the volume form on $\mathcal{R}^*(F_2, \text{SL}_2(\mathbb{C}))$, already found in Corollary 5.6. Namely, by Theorem 1.3, Proposition 6.2, and Corollary 5.2, since the commutator $\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$ is the peripheral element,

$$(6.6) \quad \Omega_{F_2} = \Omega_{S_{1,1}} = \pm 2\sqrt{2} \frac{dt_1 \wedge dt_2 \wedge dt_{12\bar{1}\bar{2}}}{t_{12} - t_{1\bar{2}}}.$$

Differentiating (6.4), we get

$$(6.7) \quad dt_{12\bar{1}\bar{2}} = (2t_1 - t_2t_{12})dt_1 + (2t_2 - t_1t_{12})dt_2 + (2t_{12} - t_1t_2)dt_{12},$$

thus, as $t_1t_2 = t_{12} + t_{1\bar{2}}$, by replacing (6.7) in (6.6):

$$\Omega_{F_2} = \pm 2\sqrt{2} dt_1 \wedge dt_2 \wedge dt_{12}.$$

6.2. A planar surface with four boundary components

Let $S_{0,4}$ denote the planar surface with four boundary components and let λ and μ be two simple closed curves so that each one divides $S_{0,4}$ in two pairs of pants and they intersect in precisely two points. Choose also one of the intersections points as a base point for the fundamental group.

Orient the curves λ and μ and obtain two new oriented curves α and β , by changing both crossings in a way compatible with the orientation, according to Figures 6.1 and 6.2.

Since the curves are oriented, we may talk about the elements they represent in $\pi_1(S_{0,4})$, in particular the products $\lambda\mu$ and $\alpha\beta$ and their trace functions, $t_{\lambda\mu}$ and $t_{\alpha\beta}$, that depend on the orientations.

LEMMA 6.4. — *Up to sign, the difference $t_{\lambda\mu} - t_{\alpha\beta}$ is independent of the choice of orientations of λ and μ . The sign depends on whether we change one (−) or both (+) orientations.*

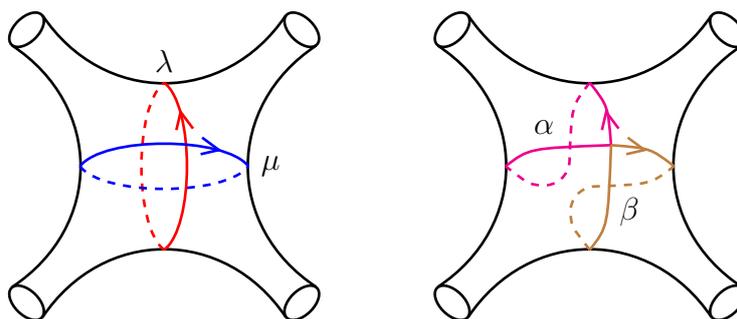


Figure 6.1. Construction of α and β from an orientation of λ and μ

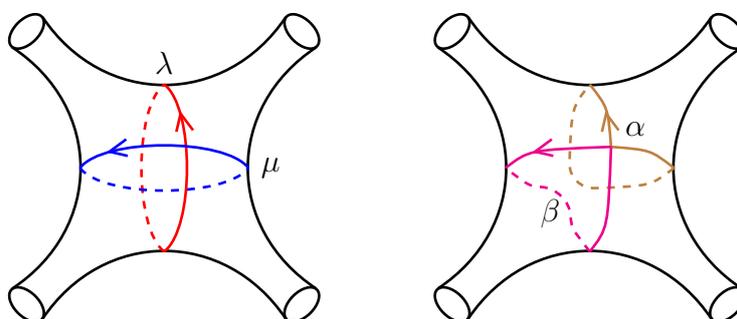


Figure 6.2. Construction of α and β from another orientation of λ and μ

PROPOSITION 6.5. — Let $\rho_0 \in R^*(S_{0,4}, \text{SL}_2(\mathbb{C}))$ be a good, ∂ -regular representation such that $t_{\lambda\mu}(\rho_0) \neq t_{\alpha\beta}(\rho_0)$. Then:

- (1) the map $T = (t_\lambda, t_\mu): \mathcal{R}^*(S_{0,4}, \partial S_{0,4}, \text{SL}_2(\mathbb{C}))_{\rho_0} \setminus \{t_{\lambda\mu} = t_{\alpha\beta}\} \rightarrow \mathbb{C}^2$ is a local parameterization.
- (2) the symplectic form on $\mathcal{R}^*(S_{0,4}, \partial S_{0,4}, \text{SL}_2(\mathbb{C}))_{\rho_0} \setminus \{t_{\lambda\mu} = t_{\alpha\beta}\}$ is the pullback $T^*\omega$ where

$$\omega = \pm \frac{dt_\lambda \wedge dt_\mu}{t_{\lambda\mu} - t_{\alpha\beta}}.$$

We fix the notation for both proofs. The fundamental group of $S_{0,4}$ is freely generated by three elements γ_1, γ_2 , and γ_3 , and the peripheral curves are represented by $\gamma_1, \gamma_2, \gamma_3$, and $\gamma_1\gamma_2\gamma_3$, see Figure 6.3. We shall assume that the orientations are so that $\lambda = \gamma_1\gamma_2$ and $\mu = \gamma_2\gamma_3$. With this choice of orientation, $\alpha = \gamma_1\gamma_2\gamma_3$ and $\beta = \gamma_2$, so

$$t_{\lambda\mu} - t_{\alpha\beta} = t_{1223} - t_{1232}.$$

Proof of Lemma 6.4. — It suffices to change the orientation of a single curve, so we follow the examples of Figures 6.1 and 6.2. If we change the orientation of μ then $\mu = \gamma_3^{-1}\gamma_2^{-1}$, $\alpha = \gamma_1$, and $\beta = \gamma_3^{-1}$. We aim to prove

$$(6.8) \quad t_{1223} - t_{1232} = -(t_{12\bar{3}\bar{2}} - t_{1\bar{3}})$$

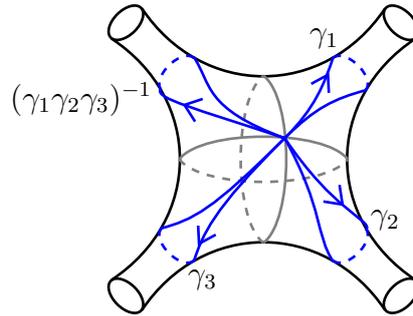


Figure 6.3. The loops γ_1 , γ_2 , and γ_3 that represent the generators of $\pi_1(S_{0,4})$.

(with negative sign, because we change the orientation of a single curve). From Equality (6.2) we have:

$$(6.9) \quad \begin{aligned} t_{12}t_{23} &= t_{1223} + t_{12\bar{3}\bar{2}} \\ t_{12}t_{32} &= t_{1232} + t_{1\bar{3}}, \end{aligned}$$

Then equality (6.8) follows by subtracting in (6.9) and using $t_{32} = t_{23}$. □

Proof of Proposition 6.5.

(1). — We shall use a computation in cohomology, first by cutting the surface $S_{0,4}$ along $\lambda = \gamma_1\gamma_2$ into two pairs of pants P_1 and P_2 , with $\pi_1(P_1) = \langle \gamma_1, \gamma_2 \rangle$ and $\pi_1(P_2) = \langle \gamma_3, \gamma_1\gamma_2 \rangle$. Notice that for $[\rho] \in \mathcal{R}^*(S_{0,4}, \partial S_{0,4}, \text{SL}_2(\mathbb{C}))_{\rho_0} \setminus \{t_{\lambda\mu} = t_{\alpha\beta}\}$ we have that $\rho|_{\pi_1(P_i)}$ is nonabelian. Suppose that, contrary to our claim, $\rho|_{\pi_1(P_1)}$ is abelian that is $\rho(\gamma_1)$ and $\rho(\gamma_2)$ commute. Then

$$t_{\lambda\mu}([\rho]) = t_{1223}([\rho]) = t_{2123}([\rho]) = t_{1232}([\rho]) = t_{\alpha\beta}([\rho]),$$

contradicting the hypothesis. This argument also shows that $\rho|_{\pi_1(P_2)}$ is nonabelian.

As $\rho|_{\pi_1(P_i)}$ is nonabelian, $H^0(\pi_1(P_i); \text{Ad } \rho) = 0$. Hence, we obtain the following Mayer–Vietoris exact sequence:

$$0 \rightarrow H^0(\lambda; \text{Ad } \rho) \xrightarrow{\beta} H^1(S; \text{Ad } \rho) \rightarrow H^1(P_1; \text{Ad } \rho) \oplus H^1(P_2; \text{Ad } \rho) \rightarrow H^1(\lambda; \text{Ad } \rho) \rightarrow 0.$$

Using the local parameterization of a pair of pants, this sequence yields that the tangent space of $\mathcal{R}^*(S, \text{SL}_2(\mathbb{C}))$ at ρ is generated by the infinitesimal deformations $\partial_{t_1}, \partial_{t_2}, \partial_{t_3}, \partial_{t_{123}}, \partial_{t_{12}}$ and $\beta(a)$, where $0 \neq a \in H^0(\lambda; \text{Ad } \rho) \cong \mathfrak{sl}_2(\mathbb{C})^{\text{Ad } \rho(\gamma)}$. Hence, the tangent space of $\mathcal{R}^*(S, \partial S, \text{SL}_2(\mathbb{C}))_{\rho_0}$ at ρ is generated by $\partial_{t_{12}} = \partial_{t_\lambda}$ and $\beta(a)$. Notice that $dt_\lambda(\beta(a)) = 0$ since $\beta(a)$ is an infinitesimal bending along λ . In order to prove that (t_μ, t_λ) are local parameters at ρ we must show that $dt_\mu(\beta(a)) \neq 0$.

Next we compute $dt_\mu(\beta(a))$. By setting $A_i = \rho(\gamma_i)$, we obtain $\rho(\lambda) = A_1A_2$ and we can choose $a = \frac{1}{2}(A_1A_2 - A_2^{-1}A_1^{-1})$. As λ is a separating curve, the infinitesimal bending is the derivative respect to ε of the path of representations:

$$\begin{aligned} \gamma_1 &\mapsto A_1 \\ \gamma_2 &\mapsto A_2 \\ \gamma_3 &\mapsto (1 + \varepsilon a + o(\varepsilon))A_3(1 - \varepsilon a + o(\varepsilon)), \end{aligned}$$

see [JM87, Lemma 5.1] for details. Since $\mu = \gamma_2\gamma_3$ is mapped to $A_2A_3 + \varepsilon(A_2aA_3 - A_2A_3a) + o(\varepsilon)$, we have

$$dt_\mu(\beta(a)) = \text{tr}(A_2aA_3 - A_2A_3a) = \frac{1}{2}(t_{2123} - t_{\bar{1}3} - t_{2312} + t_{23\bar{2}\bar{1}}).$$

By Lemma 6.4 and its proof, and using that the trace is invariant by cyclic permutations and by taking the inverse:

$$\begin{aligned} t_{\alpha\beta} - t_{\lambda\mu} &= t_{1232} - t_{1223} = t_{2123} - t_{2312} \\ &= t_{12\bar{3}\bar{2}} - t_{\bar{1}3} = t_{23\bar{2}\bar{1}} - t_{\bar{1}3}. \end{aligned}$$

Thus $dt_\mu(\beta(a)) = t_{\alpha\beta} - t_{\lambda\mu} \neq 0$. This proves Assertion (1) of the Proposition 6.5.

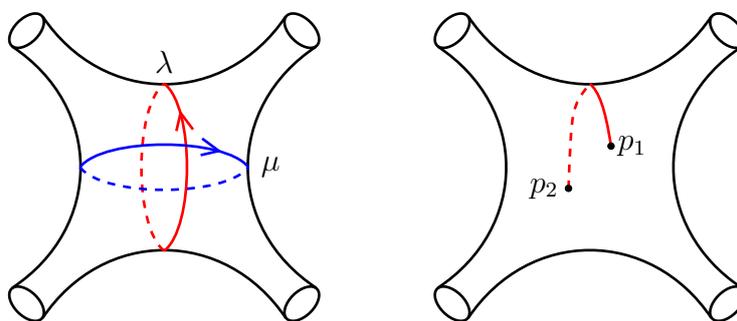


Figure 6.4. The intersection points p_1, p_2 and the arc between them.

(2). — Let $[\rho] \in \mathcal{R}^*(S_{0,4}, \partial S_{0,4}, \text{SL}_2(\mathbb{C}))_{\rho_0} \setminus \{t_{\lambda\mu} = t_{\alpha\beta}\}$, and set again $A_i = \rho(\gamma_i)$. We apply Proposition 6.1 to compute the Poisson bracket $\{t_\lambda, t_\mu\}([\rho])$.

The curves λ and μ intersect in two points, p_1 and p_2 , in Figure 6.4. Let p_1 be the base point of the fundamental group used in Figure 6.3. The contribution of p_1 is

$$\varepsilon \text{tr}(\mathbf{T}(A_1A_2) \mathbf{T}(A_2A_3)) = \frac{\varepsilon}{2}(t_{1223}([\rho]) - t_{12\bar{3}\bar{2}}([\rho]))$$

for some $\varepsilon = \pm 1$. To compute the contribution of p_2 we consider an arc from p_1 to p_2 to relate the base points between fundamental groups. Assume that this arc is half of λ , as in Figure 6.4, then $\rho_2(\lambda) = \rho(\lambda) = A_1A_2$ and $\rho_2(\mu) = \rho(\gamma_1\gamma_2\gamma_3\gamma_1^{-1}) = A_1A_2A_3A_1^{-1}$. In addition, the orientation of the intersection is opposite to the previous one, hence the contribution of p_2 is

$$-\varepsilon \text{tr}(\mathbf{T}(A_1A_2) \mathbf{T}(A_1A_2A_3A_1^{-1})) = -\frac{\varepsilon}{2}(t_{2123}([\rho]) - t_{\bar{1}3}([\rho]))$$

Hence, for $\mathcal{B}(X, Y) = -\text{tr}(XY)$ we obtain from Proposition 6.1

$$\{t_\lambda, t_\mu\} = -\frac{\varepsilon}{2}(t_{1223} - t_{12\bar{3}\bar{2}} - t_{2123} + t_{\bar{1}3}),$$

and by equation (6.8) we have

$$t_{\lambda\mu} - t_{\alpha\beta} = t_{1223} - t_{2123} = -t_{12\bar{3}\bar{2}} + t_{\bar{1}3}.$$

Finally, the formula for the symplectic form on the coordinates (t_λ, t_μ) follows again from equation (5.7). \square

7. Volume forms for free groups of higher rank

7.1. Volume form on $\mathcal{R}^*(F_k, \text{SL}_2(\mathbb{C}))$

We recall the notation $t_{i_1 \dots i_k}$ for the trace function tr_γ of $\gamma = \gamma_{i_1} \dots \gamma_{i_k}$, with the convention $\gamma_{\bar{i}} = \gamma_i^{-1}$.

We start discussing the volume form for the free group of rank three. Following [GAMA93], the variety of characters $X(F_3, \text{SL}_2(\mathbb{C}))$ is a branched covering of \mathbb{C}^6 . More precisely, the branched covering is given by trace functions:

$$(7.1) \quad T = (t_1, t_2, t_3, t_{12}, t_{13}, t_{23}) : X(F_3, \text{SL}_2(\mathbb{C})) \rightarrow \mathbb{C}^6.$$

The branching is given by the variables t_{123} and t_{213} , as they are the solutions of the quadratic equation

$$z^2 - Rz + S = 0$$

for

$$(7.2) \quad R = t_1 t_{23} + t_2 t_{13} + t_3 t_{12} - t_1 t_2 t_3$$

$$(7.3) \quad S = t_1^2 + t_2^2 + t_3^2 + t_{12}^2 + t_{13}^2 + t_{23}^2 + t_{12} t_{13} t_{23} - t_1 t_2 t_{12} - t_1 t_3 t_{13} - t_2 t_3 t_{23} - 4.$$

Recall that the trace is invariant by cyclic permutation of the group elements:

$$t_{123} = t_{231} = t_{312}.$$

The branching locus is defined by $t_{123} = t_{213}$. Away from it, the variables (7.1) define local coordinates.

PROPOSITION 7.1. — *For $T = (t_1, t_2, t_3, t_{12}, t_{13}, t_{23}) : \mathcal{R}^*(F_3, \text{SL}_2(\mathbb{C})) \rightarrow \mathbb{C}^6$, the restriction of the volume form to the open subset $\mathcal{R}^*(F_3, \text{SL}_2(\mathbb{C})) \setminus \{t_{123} = t_{213}\}$ is the pull-back form $\Omega_{F_3}^{\text{SL}_2(\mathbb{C})} = \pm T^* \Omega$, where*

$$(7.4) \quad \Omega = \pm \frac{4}{t_{123} - t_{213}} dt_1 \wedge dt_2 \wedge dt_3 \wedge dt_{12} \wedge dt_{13} \wedge dt_{23}.$$

Proof. — Consider the surface $S = S_{0,4}$. Since $\gamma_1, \gamma_2, \gamma_3$, and $\gamma_1 \gamma_2 \gamma_3$ are the peripheral elements, using Proposition 6.5 and Corollary 5.2,

$$(7.5) \quad \Omega_{F_3}^{\text{SL}_2} = \Omega_{S_{0,4}}^{\text{SL}_2} = \pm 4 \frac{dt_{12} \wedge dt_{23}}{t_{1223} - t_{1232}} \wedge dt_1 \wedge dt_2 \wedge dt_3 \wedge dt_{123}.$$

It remains to replace dt_{123} by dt_{13} in this formula. Differentiating the equality

$$t_{123}^2 - R t_{123} + S = 0,$$

where R and S are given in (7.2) and (7.3), we deduce:

$$(7.6) \quad (2t_{123} - R) dt_{123} = \sum_{\eta \in \{1,2,3,12,13,23\}} \left(\frac{\partial R}{\partial t_\eta} t_{123} - \frac{\partial S}{\partial t_\eta} \right) dt_\eta.$$

Since $R = t_{123} + t_{213}$,

$$(7.7) \quad 2t_{123} - R = t_{123} - t_{213}$$

In addition, using $t_1 t_3 = t_{13} + t_{1\bar{3}}$,

$$(7.8) \quad \frac{\partial R}{\partial t_{13}} t_{123} - \frac{\partial S}{\partial t_{13}} = t_2 t_{123} - (2t_{13} + t_{12} t_{23} - t_1 t_3) = t_2 t_{123} - t_{12} t_{23} - t_{13} + t_{1\bar{3}}$$

Using the standard relations on traces, we have:

$$(7.9) \quad t_{13} = t_{12\bar{2}3} = t_{31\bar{2}2} = t_{312}t_{\bar{2}} - t_{31\bar{2}2} = t_{123}t_2 - t_{1223}$$

$$(7.10) \quad t_{1\bar{3}} = t_{12\bar{2}\bar{3}} = t_{12}t_{3\bar{2}} - t_{123\bar{2}} = t_{12}t_{23} - t_{123\bar{2}}$$

From those equations:

$$(7.11) \quad \frac{\partial R}{\partial t_{13}}t_{123} - \frac{\partial S}{\partial t_{13}} = t_{1223} - t_{123\bar{2}}.$$

Hence, using (7.7) and (7.11), equality (7.6) becomes:

$$(7.12) \quad (t_{123} - t_{213})dt_{123} = (t_{1223} - t_{123\bar{2}})dt_{13} + \sum_{\eta \in \{1,2,3,12,23\}} p_{\eta} dt_{\eta}$$

for some polynomials p_{η} . Using (7.12) to replace dt_{123} by dt_{13} in (7.5), we finally prove (7.4). □

Proof of Theorem 1.5. — Write $F_k = \langle \gamma_1, \gamma_2, \dots, \gamma_k \rangle$ and consider the graph \mathcal{G} with one vertex and k edges, so that $\pi_1(\mathcal{G}) \cong F_k$. Consider subgraphs \mathcal{G}' and \mathcal{G}'' , so that $\pi_1(\mathcal{G}') = \langle \gamma_1, \gamma_2, \dots, \gamma_{k-1} \rangle$ and $\pi_1(\mathcal{G}'') = \langle \gamma_1, \gamma_2, \gamma_k \rangle$; therefore $\mathcal{G} = \mathcal{G}' \cup \mathcal{G}''$ and $\pi_1(\mathcal{G}' \cap \mathcal{G}'') = \langle \gamma_1, \gamma_2 \rangle$. Since we assume $t_{12\bar{1}\bar{2}} \neq 2$, $\rho(\pi_1(\mathcal{G}' \cap \mathcal{G}''))$ is irreducible, therefore, the long exact sequence of Mayer–Vietoris applied to $(\mathcal{G}', \mathcal{G}'')$ is:

$$(7.13) \quad 0 \rightarrow H^1(\mathcal{G}, \text{Ad } \rho) \rightarrow H^1(\mathcal{G}', \text{Ad } \rho) \oplus H^1(\mathcal{G}'', \text{Ad } \rho) \rightarrow H^1(\mathcal{G}' \cap \mathcal{G}'', \text{Ad } \rho) \rightarrow 0.$$

Interpreting cohomology groups as tangent spaces of spaces of representations, the assertion on the local parameterization is straightforward from the sequence. By an induction argument, the formula for the volume form is a consequence of the product of torsions, Corollary 5.6 and Proposition 7.1. □

7.2. Volume form on $\mathcal{R}^*(F_k, \text{SL}_3(\mathbb{C}))$

Before proving Proposition 1.6 and Theorem 1.7, we need two lemmas on regular elements in $\text{SL}_3(\mathbb{C})$. Recall that an element of $\text{SL}_3(\mathbb{C})$ is regular if its minimal polynomial and its characteristic polynomial have the same degree. This is the case if and only if each eigenspace is one-dimensional.

LEMMA 7.2. — *Let $A, B \in \text{SL}_3(\mathbb{C})$. If $\text{tr}(ABA^{-1}B^{-1}) \neq \text{tr}(BAB^{-1}A^{-1})$ then:*

- (i) *both A and B are regular and*
- (ii) *the subgroup $\langle A, B \rangle \subset \text{SL}_3(\mathbb{C})$ is irreducible.*

Proof.

- (i) Assume that A is not regular. Then it has an eigenvalue $\lambda \in \mathbb{C}^*$ with an eigenspace $E_{\lambda} = \ker(A - \lambda \text{Id})$ of dimension $\dim E_{\lambda} \geq 2$. Therefore $\dim(E_{\lambda} \cap B(E_{\lambda})) \geq 1$. Choose a nonzero vector $v \in E_{\lambda} \cap B(E_{\lambda})$, by construction $B^{-1}(v) \in E_{\lambda}$ and $(ABA^{-1}B^{-1})(v) = v$. This yields that 1 is an eigenvalue of the commutator $ABA^{-1}B^{-1}$, therefore it has the same eigenvalues as its inverse, which implies that $\text{tr}(ABA^{-1}B^{-1}) = \text{tr}(BAB^{-1}A^{-1})$.

- (ii) By contradiction, assume that $L \subset \mathbb{C}^3$ is a proper subspace invariant by both A and B . If $\dim L = 1$, then this is an eigenspace of $ABA^{-1}B^{-1}$ with eigenvalue 1, and if $\dim L = 2$, by looking at the action on \mathbb{C}^3/L we also deduce that 1 is an eigenvalue of $ABA^{-1}B^{-1}$. Therefore, by the discussion on the previous item, this contradicts the hypothesis $\text{tr}(ABA^{-1}B^{-1}) \neq \text{tr}(BAB^{-1}A^{-1})$. \square

LEMMA 7.3. — *Let $A \in \text{SL}_3(\mathbb{C})$. If A is regular then the Ad_A -invariant subspace of $\mathfrak{sl}_3(\mathbb{C})$ is*

$$\mathfrak{sl}_3(\mathbb{C})^{\text{Ad}_A} = \langle A - \frac{\text{tr}(A)}{3} \text{Id}, A^{-1} - \frac{\text{tr}(A^{-1})}{3} \text{Id} \rangle.$$

Proof. — It is clear from construction that both $A - \frac{\text{tr}(A)}{3} \text{Id}$ and $A^{-1} - \frac{\text{tr}(A^{-1})}{3} \text{Id}$ are Ad_A -invariant. All we need to show is that those elements are linearly independent, as by regularity $\dim \mathfrak{sl}_3(\mathbb{C})^{\text{Ad}_A} = 2$. If $A - \frac{\text{tr}(A)}{3} \text{Id}$ and $A^{-1} - \frac{\text{tr}(A^{-1})}{3} \text{Id}$ were linearly dependent, then A, Id , and A^{-1} would satisfy a nontrivial linear relation. Multiplying it by A , the same relation would be satisfied by A^2, A and Id , and hence A would have an eigenspace of dimension at least 2, contradicting regularity. \square

Remark 7.4. — It follows from Schur’s Lemma [FH91] that every irreducible representation $\rho: \Gamma \rightarrow \text{SL}_N(\mathbb{C})$ is good, that is the centralizer of $\rho(\Gamma)$ coincides with the center of $\text{SL}_N(\mathbb{C})$.

Proof of Proposition 1.6. — Assume $k = 3$, the general case follows from an induction argument as in the proof of Theorem 1.5.

We choose generators $F_3 = \langle \gamma_1, \gamma_2, \gamma_3 | - \rangle$ and we identify F_3 with $\pi_1(S_{0,4})$. We represent $S_{0,4}$ as the union of two pairs of pants P' and P'' , so that $P' \cap P''$ is a circle. Choose the generators of the fundamental group so that $\pi_1(P') = \langle \gamma_1, \gamma_2 \rangle$, $\pi_1(P'') = \langle \gamma_1, \gamma_3 \rangle$, and γ_1 is the generator of $\pi_1(P' \cap P'')$. Then the peripheral elements of P' are γ_1, γ_2 , and $\gamma_1\gamma_2$, and those of P'' , γ_1, γ_3 , and $\gamma_1\gamma_3$. The peripheral elements of S are $\gamma_2, \gamma_3, \gamma_1\gamma_2$, and $\gamma_1\gamma_3$.

Let $[\rho] \in \mathcal{R}^*(F_3, \text{SL}_3(\mathbb{C})) \setminus \{ \tau_{12\bar{1}\bar{2}} = \tau_{21\bar{2}\bar{1}} \} \cup \{ \tau_{13\bar{1}\bar{3}} = \tau_{31\bar{3}\bar{1}} \} \cup \{ \Delta_{23}^1 = 0 \}$ be a representation where

$$(7.14) \quad \Delta_{23}^1 = (\tau_{123} - \tau_{132})(\tau_{1\bar{2}\bar{3}} - \tau_{1\bar{3}\bar{2}}) - (\tau_{1\bar{2}\bar{3}} - \tau_{1\bar{3}\bar{2}})(\tau_{\bar{1}23} - \tau_{\bar{1}32}).$$

We have to show that the 16 functions

$$(\tau_1, \tau_{\bar{1}}, \tau_2, \tau_{\bar{2}}, \tau_3, \tau_{\bar{3}}, \tau_{12}, \tau_{\bar{1}\bar{2}}, \tau_{13}, \tau_{\bar{1}\bar{3}}, \tau_{23}, \tau_{\bar{2}\bar{3}}, \tau_{1\bar{2}}, \tau_{\bar{1}2}, \tau_{1\bar{3}}, \tau_{\bar{1}3})$$

define a local parameterization at $[\rho]$. The hypothesis $\text{tr}(\rho([\gamma_1, \gamma_i])) \neq \text{tr}(\rho([\gamma_i, \gamma_1]))$ for $i = 2, 3$ implies that $\rho(\gamma_j), j = 1, 2, 3$, are regular elements (Lemma 7.2). It follows also that $\rho(\gamma_1\gamma_2)$ and $\rho(\gamma_1\gamma_3)$ are regular since $\text{tr}(\rho([\gamma_1\gamma_i, \gamma_1])) = \text{tr}(\rho([\gamma_i, \gamma_1]))$ and $\text{tr}(\rho([\gamma_1, \gamma_1\gamma_i])) = \text{tr}(\rho([\gamma_1, \gamma_i]))$ for $i = 2, 3$. The Mayer–Vietoris long exact sequence is:

$$(7.15) \quad 0 \rightarrow H^0(\gamma_1, \text{Ad } \rho) \xrightarrow{\beta} H^1(S, \text{Ad } \rho) \xrightarrow{j} H^1(P', \text{Ad } \rho) \oplus H^1(P'', \text{Ad } \rho) \xrightarrow{\Delta} H^1(\gamma_1, \text{Ad } \rho) \rightarrow 0.$$

Choose \mathbf{u} a basis for $H^0(\gamma_1, \text{Ad } \rho)$. We will proceed as in the proof of Proposition 6.5. Viewing the cohomology groups as tangent spaces, the proposition will follow from

the local parameterizations for the representation space of P' and P'' , and from (7.15), provided we show that

$$d\tau_{23} \wedge d\tau_{\bar{2}\bar{3}}(\wedge\beta(\mathbf{u})) \neq 0.$$

We prove below in Lemma 7.5 that $d\tau_{23} \wedge d\tau_{\bar{2}\bar{3}}(\wedge\beta(\mathbf{u})) = \pm\Delta_{23}^1$, which is nonzero by hypothesis. \square

LEMMA 7.5. — $d\tau_{23} \wedge d\tau_{\bar{2}\bar{3}}(\beta(\wedge\mathbf{u})) = \pm\Delta_{23}^1$, where Δ_{23}^1 is as in (7.14).

Proof. — Set $A_1 = \rho(\gamma_1)$. By Lemma 7.3, the elements

$$x = A_1 - \frac{\text{tr } A_1}{3} \text{Id} \quad \text{and} \quad y = A_1^{-1} - \frac{\text{tr } A_1^{-1}}{3} \text{Id}$$

form a basis of the invariant subspace $\mathfrak{sl}_3(\mathbb{C})^{\text{Ad } \rho(\gamma_1)}$. We choose $\mathbf{u} = \{x, y\}$ via the isomorphism $H^0(\gamma_1, \text{Ad } \rho) \cong \mathfrak{sl}_3(\mathbb{C})^{\text{Ad } A_1}$.

Then $\beta(x)$ is the tangent vector of the infinitesimal bending:

$$\gamma_1 \mapsto A, \quad \gamma_2 \mapsto B, \quad \gamma_3 \mapsto (\text{Id} + \varepsilon x)C(\text{Id} - \varepsilon x) = C + \varepsilon(xC - Cx) \quad \text{in } \mathbb{C}[\varepsilon]/\varepsilon^2$$

and similarly for $\beta(y)$. To compute $d\tau_{23}$ and $d\tau_{\bar{2}\bar{3}}$ on $\beta(x)$ and $\beta(y)$, we must evaluate the infinitesimal deformations on $\gamma_2\gamma_3$ and $\bar{\gamma}_2\bar{\gamma}_3$. Thus the path corresponding to $\beta(x)$ evaluated at $\gamma_2\gamma_3$ is

$$(7.16) \quad \gamma_2\gamma_3 \mapsto BC + \varepsilon(BxC - BCx) = BC + \varepsilon(BAC - BCA).$$

Therefore, taking traces we get:

$$(7.17) \quad d\tau_{23}(\beta(x)) = \tau_{213} - \tau_{231} = \tau_{132} - \tau_{123}.$$

The same argument for y instead of x gives:

$$(7.18) \quad d\tau_{23}(\beta(y)) = \tau_{2\bar{1}3} - \tau_{23\bar{1}} = \tau_{\bar{1}32} - \tau_{\bar{1}23}.$$

To evaluate $d\tau_{\bar{2}\bar{3}} = d\tau_{\bar{3}\bar{2}}$, we take inverses in (7.16)

$$(7.19) \quad (\gamma_2\gamma_3)^{-1} \mapsto C^{-1}B^{-1} + \varepsilon(xC^{-1}B^{-1} - C^{-1}xB^{-1}) = C^{-1}B^{-1} + \varepsilon(AC^{-1}B^{-1} - C^{-1}AB^{-1})$$

and taking traces we get:

$$(7.20) \quad d\tau_{\bar{2}\bar{3}}(\beta(x)) = \tau_{\bar{1}\bar{3}\bar{2}} - \tau_{\bar{3}\bar{1}\bar{2}} = \tau_{1\bar{3}\bar{2}} - \tau_{1\bar{2}\bar{3}}.$$

Again the same argument for y instead of x gives:

$$(7.21) \quad d\tau_{\bar{2}\bar{3}}(\beta(y)) = \tau_{\bar{1}\bar{3}\bar{2}} - \tau_{\bar{3}\bar{1}\bar{2}} = \tau_{1\bar{3}\bar{2}} - \tau_{1\bar{2}\bar{3}}.$$

Hence

$$\begin{aligned} d\tau_{23} \wedge d\tau_{\bar{2}\bar{3}}(\beta(x) \wedge \beta(y)) &= \pm(\tau_{123} - \tau_{132})(\tau_{1\bar{2}\bar{3}} - \tau_{1\bar{3}\bar{2}}) - (\tau_{1\bar{2}\bar{3}} - \tau_{1\bar{3}\bar{2}})(\tau_{\bar{1}23} - \tau_{\bar{1}32}) \\ &= \pm\Delta_{23}^1, \end{aligned}$$

which concludes the proof of the Lemma 7.5. \square

Proof of Theorem 1.7. — We assume again that $k = 3$. The general case follows with the same argument as in Theorem 1.5.

As in the proof of Proposition 1.6 we decompose $S = S_{0,4} = P' \cup P''$, $\gamma_1 = P' \cap P''$. Also, we choose generators of $\pi_1(P')$, $\pi_1(P'')$, and $\pi_1(P' \cap P'')$ as in the proof of Proposition 1.6. The peripheral elements of S are $\gamma_2, \gamma_3, \gamma_1\gamma_2$, and $\gamma_1\gamma_3$.

For a representation $\rho: \pi_1(S) \rightarrow \mathrm{SL}_3(\mathbb{C})$ we let $\rho': \pi_1(P') \rightarrow \mathrm{SL}_3(\mathbb{C})$ and $\rho'': \pi_1(P'') \rightarrow \mathrm{SL}_3(\mathbb{C})$ denote the restriction of ρ to $\pi_1(P')$ and $\pi_1(P'')$ respectively.

Let $[\rho] \in \mathcal{R}(S, \mathrm{SL}_3(\mathbb{C})) \setminus \{\tau_{12\bar{1}\bar{2}} = \tau_{21\bar{2}\bar{1}}\} \cup \{\tau_{13\bar{1}\bar{3}} = \tau_{31\bar{3}\bar{1}}\} \cup \{\Delta_{23}^1 = 0\}$. It follows from Lemma 7.2 and Remark 7.4 that ρ' and ρ'' are good, ∂ -regular representations. In what follows we let ω_{12} and ω_{13} denote the pullback of the symplectic form $\omega_{P'}$ on $\mathcal{R}^*(P', \partial P', \mathrm{SL}_3(\mathbb{C}))_{\rho'}$ and $\omega_{P''}$ on $\mathcal{R}^*(P'', \partial P'', \mathrm{SL}_3(\mathbb{C}))_{\rho''}$ respectively.

Given a basis \mathbf{v} for $H^1(\gamma_1, \mathrm{Ad} \rho)$ we can choose lifts $\mathbf{v}' \subset H^1(P', \mathrm{Ad} \rho)$, and $\mathbf{v}'' \subset H^1(P'', \mathrm{Ad} \rho)$ which map to \mathbf{v} . By exactness there exists $\tilde{\mathbf{v}} \subset H^1(S, \mathrm{Ad} \rho)$ which maps to $(\mathbf{v}', -\mathbf{v}'')$.

LEMMA 7.6. — *Let \mathbf{v} a basis for $H^0(\gamma_1, \mathrm{Ad} \rho)$ and \mathbf{v} a basis for $H^1(\gamma_1, \mathrm{Ad} \rho)$. Then*

$$(7.22) \quad \Omega_S = \pm \frac{\langle \wedge \mathbf{u}, \wedge \mathbf{v} \rangle}{(\nu_1 \wedge \nu_{23})(\wedge \mathbf{v} \wedge \beta(\mathbf{u}))} \omega_{12} \wedge \omega_{13} \wedge \nu_2 \wedge \nu_3 \wedge \nu_{12} \wedge \nu_{13} \wedge \nu_1 \wedge \nu_{23}.$$

Proof of Lemma 7.6. — Choose \mathbf{a}' a basis of $\ker(H^1(P', \mathrm{Ad} \rho) \rightarrow H^1(P' \cap P'', \mathrm{Ad} \rho))$ and \mathbf{a}'' a basis of $\ker(H^1(P'', \mathrm{Ad} \rho) \rightarrow H^1(P' \cap P'', \mathrm{Ad} \rho))$. Moreover, we can choose lifts $(\widetilde{\mathbf{a}'})$, $(\widetilde{\mathbf{a}''}) \subset H^1(S, \mathrm{Ad} \rho)$ which map under $j: H^1(S, \mathrm{Ad} \rho) \rightarrow H^1(P', \mathrm{Ad} \rho) \oplus H^1(P'', \mathrm{Ad} \rho)$ to $(\mathbf{a}', \mathbf{0})$ and $(\mathbf{0}, \mathbf{a}'')$ respectively.

Then, by using (7.15), $\mathbf{a}' \sqcup \mathbf{v}'$ is a basis for $H^1(P', \mathrm{Ad} \rho)$, $\mathbf{a}'' \sqcup \mathbf{v}''$ is a basis for $H^1(P'', \mathrm{Ad} \rho)$ and $(\widetilde{\mathbf{a}'}) \sqcup (\widetilde{\mathbf{a}''}) \sqcup \beta(\mathbf{u}) \sqcup \tilde{\mathbf{v}}$ is a basis for $H^1(S, \mathrm{Ad} \rho)$.

The product formula applied to (7.15) yields:

$$\begin{aligned} \Omega_S(\wedge(\widetilde{\mathbf{a}'}) \wedge (\widetilde{\mathbf{a}''}) \wedge \beta(\mathbf{u}) \wedge \tilde{\mathbf{v}}) &= \pm \frac{\Omega_{P'}(\wedge \mathbf{a}' \wedge \mathbf{v}') \Omega_{P''}(\wedge \mathbf{a}'' \wedge \mathbf{v}'')}{\mathrm{tor}(P' \cap P'', \mathrm{Ad} \rho, \mathbf{u}, \mathbf{v})} \\ &= \pm (\omega_{12} \wedge \nu_2 \wedge \nu_{12})(\wedge \mathbf{a}') (\omega_{13} \wedge \nu_3 \wedge \nu_{13})(\wedge \mathbf{a}'') \frac{\nu_1(\wedge \mathbf{v})^2}{\mathrm{tor}(P' \cap P'', \mathrm{Ad} \rho, \mathbf{u}, \mathbf{v})}. \end{aligned}$$

The last equality follows since $d\tau_2, d\tau_{\bar{2}}, d\tau_{12}, d\tau_{\bar{1}\bar{2}}, d\tau_{1\bar{2}}, d\tau_{\bar{1}\bar{2}}$ vanish on each cocycle v'_i of $\mathbf{v}' = (v'_1, v'_2)$, and $d\tau_3, d\tau_{\bar{3}}, d\tau_{13}, d\tau_{\bar{1}\bar{3}}, d\tau_{1\bar{3}}, d\tau_{\bar{1}\bar{3}}$ vanish on each cocycle v''_i of $\mathbf{v}'' = (v''_1, v''_2)$.

By Definition 4.7, $\nu_1(\mathbf{v})^2/\mathrm{tor}(P' \cap P'', \mathrm{Ad} \rho, \mathbf{u}, \mathbf{v}) = \pm \langle \wedge \mathbf{u}, \wedge \mathbf{v} \rangle$, hence

$$\begin{aligned} \Omega_S(\wedge(\widetilde{\mathbf{a}'}) \wedge (\widetilde{\mathbf{a}''}) \wedge \beta(\mathbf{u}) \wedge \tilde{\mathbf{v}}) &= \pm (\omega_{12} \wedge \nu_2 \wedge \nu_{12})(\wedge \mathbf{a}') (\omega_{13} \wedge \nu_3 \wedge \nu_{13})(\wedge \mathbf{a}'') \langle \wedge \mathbf{u}, \wedge \mathbf{v} \rangle \\ &= \pm (\omega_{12} \wedge \nu_2 \wedge \nu_{12})(\wedge \mathbf{a}') (\omega_{13} \wedge \nu_3 \wedge \nu_{13})(\wedge \mathbf{a}'') \langle \wedge \mathbf{u}, \wedge \mathbf{v} \rangle \frac{(\nu_1 \wedge \nu_{23})(\wedge \mathbf{v} \wedge \beta(\mathbf{u}))}{(\nu_1 \wedge \nu_{23})(\wedge \mathbf{v} \wedge \beta(\mathbf{u}))} \\ &= \pm \frac{\langle \wedge \mathbf{u}, \wedge \mathbf{v} \rangle}{(\nu_1 \wedge \nu_{23})(\wedge \mathbf{v} \wedge \beta(\mathbf{u}))} \omega_{12} \wedge \omega_{13} \wedge \nu_2 \wedge \nu_3 \wedge \nu_{12} \wedge \nu_{13} \wedge \nu_1 \wedge \nu_{23} \\ &\quad \left(\wedge(\widetilde{\mathbf{a}'}) \wedge (\widetilde{\mathbf{a}''}) \wedge \beta(\mathbf{u}) \wedge \tilde{\mathbf{v}} \right). \end{aligned}$$

The last equality follows since $\beta(\mathbf{u})$ is an infinitesimal bending that vanish on ν_1 , and $\beta(\mathbf{u})$ is in the kernel of j (see (7.15)). Moreover, the bases $(\widetilde{\mathbf{a}'})$ and $(\widetilde{\mathbf{a}''})$ map to $(\mathbf{a}', \mathbf{0})$ and $(\mathbf{0}, \mathbf{a}'')$ respectively. \square

To conclude the proof of Theorem 1.7 we need to compute the quotient

$$\frac{\langle \wedge \mathbf{u}, \wedge \mathbf{v} \rangle}{(\nu_1 \wedge \nu_{23})(\wedge \mathbf{v} \wedge \beta(\mathbf{u}))}$$

As $\beta(\mathbf{u})$ consist of infinitesimal bendings that vanish on $d\tau_1$ and $d\tau_{\bar{1}}$,

$$(\nu_1 \wedge \nu_{23})(\wedge \mathbf{v} \wedge \beta(\mathbf{u})) = \nu_1(\wedge \mathbf{v})\nu_{23}(\wedge \beta(\mathbf{u})).$$

Write

$$A = \rho(\gamma_1), \quad B = \rho(\gamma_2), \quad \text{and} \quad C = \rho(\gamma_3),$$

and

$$x = A - \frac{\text{tr}(A)}{3} \text{Id} \quad \text{and} \quad y = A^{-1} - \frac{\text{tr}(A^{-1})}{3} \text{Id}.$$

Hence $x, y \in \mathfrak{sl}_3(\mathbb{C})$ generate the A -invariant subspace by Lemma 7.3. By the natural identification $H^0(\gamma_1, \text{Ad } \rho) \cong \mathfrak{sl}_3(\mathbb{C})^{\text{Ad } A}$, we choose $\mathbf{u} = \{x, y\}$.

To finish the proof of Theorem 1.7, we assume semi-simplicity, so that $H^1(\gamma_1, \text{Ad } \rho) \cong H^1(\gamma_1, \mathbb{R}) \otimes_{\mathbb{R}} \mathfrak{sl}_3(\mathbb{C})^{\text{Ad } A}$ and we may choose \mathbf{v} to be $\{x, y\}$ times the fundamental class. Therefore

$$(7.23) \quad \langle \wedge \mathbf{u}, \wedge \mathbf{v} \rangle = \det \begin{pmatrix} \text{tr}(x^2) & \text{tr}(xy) \\ \text{tr}(xy) & \text{tr}(y^2) \end{pmatrix}.$$

Next we compute $\nu(\wedge \mathbf{v})$. Write $\mathbf{v} = \{v_x, v_y\}$, where v_x and v_y are the infinitesimal deformations corresponding to x and y respectively. Namely, the tangent vector of the infinitesimal paths

$$(7.24) \quad \begin{aligned} \gamma_1 &\mapsto (\text{Id} + \varepsilon x)A = A + \varepsilon x A & \text{and} \\ \gamma_1 &\mapsto (\text{Id} + \varepsilon y)A = A + \varepsilon y A & \text{in } \mathbb{C}[\varepsilon]/\varepsilon^2. \end{aligned}$$

These infinitesimal deformations evaluated at γ_1^{-1} are, respectively,

$$(7.25) \quad \begin{aligned} \gamma_1^{-1} &\mapsto A^{-1}(\text{Id} - \varepsilon x) = A^{-1} - \varepsilon A^{-1} x & \text{and} \\ \gamma_1^{-1} &\mapsto A^{-1}(\text{Id} - \varepsilon y) = A^{-1} - \varepsilon A^{-1} y & \text{in } \mathbb{C}[\varepsilon]/\varepsilon^2. \end{aligned}$$

Thus, $d\tau_1(v_x) = \text{tr}(xA)$, and as $\text{tr}(x) = 0$, $\text{tr}(xA) = \text{tr}(xA - \frac{\text{tr}(x)}{3}x) = \text{tr}(xx)$. By the very same argument, $\text{tr}(yA) = \text{tr}(A^{-1}x) = \text{tr}(xy)$ and $\text{tr}(A^{-1}y) = \text{tr}(y^2)$, and (7.24) and (7.25) yield

$$(7.26) \quad \begin{aligned} d\tau_1(v_x) &= \text{tr}(x^2), & d\tau_1(v_y) &= \text{tr}(xy), \\ d\tau_{\bar{1}}(v_x) &= -\text{tr}(xy), & d\tau_{\bar{1}}(v_y) &= -\text{tr}(y^2). \end{aligned}$$

From (7.23) and (7.26) we have

$$(7.27) \quad d\tau_1 \wedge d\tau_{\bar{1}}(\wedge \mathbf{v}) = \pm \langle \wedge \mathbf{u}, \wedge \mathbf{v} \rangle.$$

In addition, by Lemma 7.5

$$(7.28) \quad \nu_{23}(\beta(\mathbf{u})) = \sqrt{-3} d\tau_{23} \wedge d\tau_{\bar{2}\bar{3}}(\beta(\mathbf{u})) = \pm \sqrt{-3} \Delta_{23}.$$

Hence, as $\nu_1 = \sqrt{-3} d\tau_1 \wedge d\tau_{\bar{1}}$, by (7.27) and (7.28):

$$\frac{\langle \wedge \mathbf{u}, \wedge \mathbf{v} \rangle}{(\nu_1 \wedge \nu_{23})(\wedge \mathbf{v} \wedge \beta(\mathbf{u}))} = \frac{\langle \wedge \mathbf{u}, \wedge \mathbf{v} \rangle}{\nu_1(\wedge \mathbf{v})\nu_{23}(\wedge \beta(\mathbf{u}))} = \pm \frac{1}{3\Delta_{23}^1}.$$

Now the volume formula follows from Lemma 7.6, the last equation, and the expression of the symplectic forms ω_{12} and ω_{13} in (5.8). \square

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