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MONOPOLE FLOER HOMOLOGY AND SOLV GEOMETRY

HOMOLOGIE DE FLOER DES MONOPÔLES ET GÉOMÉTRIE SOLV

Abstract. — We study the monopole Floer homology of a Solv rational homology sphere Y from the point of view of spectral theory. Applying ideas of Fourier analysis on solvable groups, we show that for suitable Solv metrics on Y, small regular perturbations of the Seiberg–Witten equations do not admit irreducible solutions; in particular, this provides a geometric proof that Y is an L-space.

Résumé. — On étudie l'homologie de Floer des monopôles d'une sphère d'homologie rationnelle Y de type Solv du point de vue de la théorie spectrale. En appliquant des idées d'analyse de Fourier sur les groupes résolubles, on montre que pour des métriques Solv convenables sur Y, les petites perturbations régulières des équations de Seiberg–Witten n'admettent pas de solutions irréductibles ; en particulier ceci fournit une preuve géométrique du fait que Y n'est pas un L-espace.

1. Introduction

Among the three-dimensional model geometries, Solv, i.e. $\mathbb{R}^3$ equipped with the metric $e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$, is the least symmetric one [Sco83]. This makes Solv-manifolds (i.e. compact 3-manifolds admitting a Solv metric) a very special class

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within the classification scheme of Thurston's geometrization theorem; in fact, they can be characterized as the geometric manifolds which are neither Seifert nor hyperbolic. From a historical perspective, their importance stems from the fact that many Solv manifolds arise as cusps of Hirzebruch modular surfaces [Hir73]; and the understanding of their signature defect was the main motivation behind the discovery of the Atiyah–Patodi–Singer index theorem for manifolds with boundary [APS75], see [ADS83]. In a related fashion, three-dimensional Solv manifolds are also among the simplest examples where non-abelian Fourier analysis can be performed [Bre77]. More recently, the computation of their Heegaard Floer homology has provided evidence for the far-reaching L-space conjecture [BGW13].

In this paper we study the monopole Floer homology of a Solv rational homology sphere Y from a geometric viewpoint. Monopole Floer homology is a package of invariants of three-manifolds introduced by Kronheimer and Mrowka in [KM07] obtained by studying the Seiberg–Witten equations (see also [Lin16] for a friendly introduction). While monopole Floer homology is a topological invariant, and can be therefore computed in many cases using tools such as surgery exact triangles [KMOS07], it is interesting to understand its relation with special geometric structures on the space, the case of Seifert fibered spaces [MOY97] being the prototypical example. In our case, a Solv-rational homology sphere Y has the structure of a torus semibundle, and admits several different Solv-metrics obtained by rescaling the metrics along the fibers (see Section 2 for a more detailed discussion of Solv geometry). Our main result is then the following.

Theorem 1.1. — *Let Y be a Solv-rational homology sphere, equipped with a Solv metric. If the fibers are small enough, then there are small regular perturbations for which the Seiberg–Witten equations on Y do not admit irreducible solutions.*

The following is an immediate consequence of the Theorem 1.1. Recall that a rational homology sphere Y is an *L-space* if $\widehat{HM}_*(Y,\mathfrak{s}) = \mathbb{Z}[U]$ as a $\mathbb{Z}[U]$-module for each spin^c structure $\mathfrak{s}$.

Corollary 1.2. — *Let Y be a Solv-rational homology sphere. Then Y is an L-space.*

The analogous result in the setting of Heegaard Floer homology (which is known to yield isomorphic invariants, see [CGH12, KLT11] and subsequent papers) was proved by topological means in [BGW13] with $\mathbb{Z}/2\mathbb{Z}$-coefficients, and extended to $\mathbb{Z}$-coefficients in [RR17]. Let us also point out that compact Solv manifolds have either $b_1 = 0$ or 1; in the latter case, they are Anosov torus bundles over the circle, and their Heegaard Floer homology (with $\mathbb{Z}$ coefficients) was computed in [Bal08].

In our approach, we look at the monopole Floer homology of Solv-manifolds from the point of view of spectral geometry. The main ingredient in the proof of Theorem 1.1 is the following relation, for a rational homology sphere, between the existence of irreducible solutions to the Seiberg–Witten equations and the first eigenvalue λ_1^* of the Hodge Laplacian on coexact 1-forms (which improves on the main result of [Lin17]).

Theorem 1.3 (Theorem 3 of [LL18]). — *Let Y be a rational homology sphere equipped with a metric g. Denote by $\widetilde{s}(p)$ the sum of the two least eigenvalues of the Ricci curvature at the point p. If the inequality*

$$\lambda_1^* \geqslant - \inf_{p\in Y} \widetilde{s}\frac{(p)}{2}$$

holds, then the Seiberg–Witten equations do not admit irreducible solutions.

In the case of a Solv-metric, $\widetilde{s} = -2$ at every point, so in order to prove Theorem 1.1, we need to show that for suitable Solv-metrics on Y, $\lambda_1^* \geqslant 1$. Let us describe the strategy behind the proof of this by discussing the content of each section.

In Section 2, we review some facts about the geometry and topology of Solv-manifolds. As Solv is the left-invariant metric for a solvable Lie group structure on $\mathbb{R}^3$, one can study Fourier analysis on it, and we will introduce the basic ideas behind it. In Section 3, we use the aforementioned Fourier analysis to show that, for metrics with sufficiently small fibers, $\lambda_1^* = 1$, so that the Seiberg–Witten equations do not admit irreducible solutions by Theorem 1.3. As these metrics have λ_1^* is exactly 1, they lie in the borderline case of Theorem 1.3, and transversality is a quite subtle issue. We discuss it in Section 4, where we will study explicit small perturbations of the equations and existence of harmonic spinors.

2. Compact Solvmanifolds and their Fourier analysis

We start by reviewing the basics of Solv-geometry; most of the following discussion is taken from Section 12.7 of [Mar16]. Recall that Solv is the Riemannian manifold $\mathbb{R}^3$ equipped with the metric

$$e^{2z}dx^2 + e^{-2z}dy^2 + dz^2.$$

This is the left-invariant Riemannian metric on $\mathbb{R}^3$ when equipped with the solvable Lie group structure

$$(x, y, z) \cdot (x', y', z') = (x + e^{-z}x', y + e^{z}y', z + z').$$

This can be though of as the semidirect product corresponding to the splitting of

$$0 \to \mathbb{R}^2 \to \mathsf{Solv} \xrightarrow{p} \mathbb{R} \to 0,$$

where $p(x, y, z) = z$, given by

$$z \mapsto \begin{bmatrix} e^{-z} & 0 \\ 0 & e^{z} \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R}),$$

seen as linear automorphisms of $\mathbb{R}^2$. The Ricci tensor is given in this coordinates by

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix},$$

so that both s and $\widetilde{s}$ are -2 at each point. We can see that the foliation in $\mathbb{R}^2$ by the planes with z constant descend to any compact Solv-manifold; in fact, it descends to a foliation for which all the leaves are tori or Klein bottles.

Orientable compact solvmanifolds either have $b_1 = 0$ or 1. The manifolds of the latter type, which will be denoted by $\widetilde{Y}$, arise as quotients $\Gamma \setminus \mathsf{Solv}$ for lattices $\Gamma \subset \mathsf{Solv}$. Every such lattice is a split extension

$$0 \to \Lambda \to \Gamma \xrightarrow{p} a\mathbb{Z} \to 0,$$

where $\Lambda \subset \mathbb{R}^2$ is a lattice invariant under the action of $\begin{bmatrix} e^{-a} & 0 \\ 0 & e^{a} \end{bmatrix}$. The underlying topological manifold is a torus bundle with monodromy $A \in \mathrm{SL}(2,\mathbb{Z})$; here $|\operatorname{tr} A| > 2$ (i.e. A is Anosov) and e^a and e^{-a} are its eigenvalues.

Example 2.1. — Consider $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

The mapping torus is well-known to be the zero surgery on the figure eight knot. Its eigenvalues are φ^2 and φ^{-2} where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Recall that it satisfies $\varphi^2 = \varphi + 1$. Consider the vectors

$$v = (1-\varphi, \varphi) \quad w = (1,1).$$

If S is the matrix with columns v and w, we have

$$A = S^{-1} \begin{bmatrix} \varphi^{-2} & 0 \\ 0 & \varphi^{2} \end{bmatrix} S;$$

setting Λ to be the lattice generated by v and w, and $a = \log(\varphi^2)$, we obtain the lattice Γ equipping the mapping torus of A with a Solv metric.

Remark 2.2. — We can also think about this construction from a more number theoretic viewpoint, which makes the connection with [Hir73] and [ADS83] clearer. Consider the field $k = \mathbb{Q}(\sqrt{5})$. It is totally real, and it comes with two natural embeddings ϕ_+, ϕ_- into $\mathbb{R}$ sending $\sqrt{5}$ to $\pm\sqrt{5}$. The ring of integers $\mathcal{O}_k$ is the lattice $\Lambda = \mathbb{Z}[\varphi]$ which has basis φ and 1. The group of totally positive units is generated by φ^2; and it is easy to see that its multiplication action is given in our chosen basis by A. Finally, we can embed the lattice Λ in $\mathbb{R}^2$ using (ϕ_-, ϕ_+); our basis elements are mapped to the vectors v and w.

This construction is readily generalized to any monodromy $A \in \mathrm{SL}(2,\mathbb{Z})$ by looking at the action of a totally positive or negative unit on fractional ideals in real quadratic number fields.

A Solv-manifold with $b_1 = 0$, denoted by Y, is a torus semibundle; therefore it admits a double cover $\widetilde{Y}$ which is a Solv torus bundle $\Gamma \setminus \mathsf{Solv}$. Denoting the cover involution on $\widetilde{Y}$ by τ, we can describe Y in the following way. There is a basis v, w of the lattice $\Lambda = \Gamma \cap \mathbb{R}^2$, for which τ is the order 2 isometry

$$(av + bw, z) \mapsto \left(\left(a + \frac{1}{2}\right) v - bw, -z\right).$$

Recall that there are only two orientation preserving isometries of Solv fixing the origin and inducing an orientation reversing isometry on $\mathbb{R}^2 \times \{0\}$, namely

$$(x,y,z) \mapsto (\pm y, \pm x, -z).$$

In particular, the fact that τ is an isometry implies that v and w are multiples of $(1,1)$ and $(1,-1)$, or viceversa.

Remark 2.3. — The existence of such an involution on Y provides non trivial constraints on the monodromy A. Among the others, we have that A is congruent to the identity modulo 2. This readily follows from the fact that the action is fixed point free (see [Sak85, Section 6]).

From this description, we see that on any Solv-manifold we obtain a one parameter family of metrics obtained by rescaling the lattice Λ; this can be seen concretely in Example 2.1.

Let us now introduce the basics of Fourier analysis on a compact Solvmanifold with $b_1(Y)=1$. We follow [Bre77, the first chapter], to which we refer for a pleasant, more thorough, discussion.

Consider a smooth function $f:\Gamma\setminus\mathsf{Solv}\to\mathbb{R}$. This can be thought (with a little abuse of notation) as a function $f:\mathsf{Solv}\to\mathbb{R}$ which is left invariant under Γ. In particular, it is invariant under the action of $\Lambda\subset\Gamma$, i.e.

$$f(\underline{x}+m,z)=f(\underline{x},z)\text{ for all }m\in\Lambda.$$

We can therefore expand f in Fourier series in the $\mathbb{R}^2\times\{0\}\subset\mathsf{Solv}$ directions

$$f(\underline{x},z)=\sum_{\mu\in\Lambda'}a_{\underline{\mu}}(z)e^{i\underline{\mu}\cdot\underline{x}}.$$

for some smooth functions $a_{\underline{\mu}}(z)$. Here Λ' is the dual lattice of Λ, where we use the convention

$$\Lambda'=\left\{\underline{\mu}\in\mathbb{R}^2\middle|\underline{\mu}\cdot m\in 2\pi\mathbb{Z}\text{ for all }m\in\Lambda\right\}.$$

We now use the fact that f is invariant by the action of $(\underline{0},a)$. Letting $A=\begin{bmatrix}e^{-a} & 0\\ 0 & e^{a}\end{bmatrix}$, we see that

$$f(\underline{x},z)=f((\underline{0},a)\cdot(\underline{x},z))=f(A\underline{x},z+a),$$

hence, after reindexing,

$$\sum_{\underline{\mu}\in\Lambda'}a_{\underline{\mu}}(z)e^{i\underline{\mu}\cdot\underline{x}}=\sum_{\underline{\mu}\in\Lambda'}a_{\underline{\mu}}(z+a)e^{i\underline{\mu}\cdot A\underline{x}}=\sum_{\underline{\mu}\in\Lambda'}a_{\underline{\mu}\cdot A}(z+a)e^{i\underline{\mu}\cdot\underline{x}}.$$

This implies that

$$a_{\underline{\mu}}(z)=a_{\underline{\mu}\cdot A}(z+a),$$

so $a_{\underline{\mu}}$ determines via translation $a_{\underline{\mu}\cdot A^n}$. In particular, the Fourier series is determined by the collection of functions for $a_{\underline{\mu}}(z)$ for $\underline{\mu}\in\Lambda'/V$, V being the group of automorphisms of the dual lattice Λ' generated by A. While a_0 is a periodic function with period a, it can be shown that the functions $a_{\underline{\mu}}(z)$ for $\underline{\mu}\neq 0$ are in the Schwartz–type space

$$\mathcal{S}=\left\{f\middle|e^{nz}f^{(m)}(z)\text{ is bounded for all }n\in\mathbb{Z},m\geqslant 0\right\},\tag{2.1}$$

where $f^{(m)}$ denotes the m^{th} derivative of f.

With this in mind, let us study as a warm-up example the Laplacian on functions on $\Gamma \setminus \mathsf{Solv}$, which can be written as

$$\Delta f = -\left(e^{-2z} f_{xx} + e^{2z} f_{yy} + f_{zz}\right).$$

Let us use the decomposition in Fourier modes discussed above. We then have a L^2-unitary decomposition

$$\Delta = \bigoplus_{\underline{\mu} \in \frac{\Lambda'}{V}} \Delta_{\underline{\mu}},$$

where Δ_0 acts on $L^2(\mathbb{R}/a\mathbb{Z})$ and $\Delta_{\underline{\mu}}$ is a diagonalizable operator on $L^2(\mathbb{R})$. In particular, if we have $\underline{\mu} = (\mu, \mu')$, the corresponding operator is given by substituting

$$\frac{d}{dx} \mapsto i\mu, \quad \frac{d}{dy} \mapsto i\mu'$$

so that

$$\Delta_{\underline{\mu}} f = -f_{zz} + \left(\mu^2 e^{-2z} + (\mu')^2 e^{2z}\right) f.$$

Therefore λ^2 is an eigenvalue of $\Delta_{\underline{\mu}}$ if and only if

$$f_{zz} = \left(\mu^2 e^{-2z} + (\mu')^2 e^{2z} - \lambda^2\right) f$$

admits a non-zero solution in $L^2(\mathbb{R})$. While this equation is not solvable in terms of elementary functions, we can still understand the basic properties of its spectrum. Let us first recall the following well-known elementary Lemma 2.4.

Lemma 2.4. — *Suppose $f : \mathbb{R} \to \mathbb{R}$ solves the second order linear ODE*

$$f_{zz} = \Phi(z) \cdot f$$

where Φ is smooth and $\Phi(z) > 0$ everywhere. If f is not identically zero, f cannot be in $L^2(\mathbb{R})$.

Proof. — Possibly after replacing $f(z)$ by $-f(z)$ or $f(-z)$, we can assume that at x_0 both $f(x_0) = c > 0$ and $f'(x_0) \geqslant 0$. Suppose there is $t_0 > x_0$ with $0 < f(t_0) < f(x_0)$. We can also assume $f > 0$ on $[x_0, t_0]$. Then there is $x_0 < t < t_0$ with $f'(t) < 0$. Applying again the mean value theorem, there is $x_0 < t' < t$ with $f''(t') < 0$, which is contradiction as $f''(t') = \Phi(z) \cdot f > 0$. So $f(x) \geqslant c$ for $x \geqslant x_0$, and the result follows. □

We then have the following.

Lemma 2.5. — *For $\underline{\mu} \neq 0$ the first eigenvalue of $\Delta_{\underline{\mu}}$ is at least $2|\mu\mu'| \neq 0$.*

Proof. — By AM-GM, the inequality

$$\mu^2 e^{-2z} + (\mu')^2 e^{2z} \geqslant 2|\mu\mu'|,$$

holds, and the result follows from the previous Lemma 2.5. □

In terms of the number theoretic description in Remark 2.2, the quantity $\mu\mu'$ is the *norm* $N(\underline{\mu})$. The only basic property we will need is that there is $c > 0$ such that $|\mu\mu'| \geqslant c$ for all $\underline{\mu} \in \Lambda' \setminus \{0\}$; for example, we can choose $c = 1$ in Remark 2.2.

For completeness, let us conclude this section by discussing the zero mode $\underline{\mu} = 0$. In this case, we study the ODE

$$f_{zz} = -\lambda^2 f$$

with f periodic with period a. It has eigenvalues $\lambda^2 = \frac{4\pi^2}{a^2} n^2$ for $n \in \mathbb{Z}$.

3. The spectrum on coexact 1-forms

In this section we will perform the key computation behind our main result. Recall from the previous section that on a Solv-manifold there is a non-trivial family of metrics obtained by rescaling the lattice $\Lambda \subset \mathbb{R}^2$. With this in mind, we have the following.

Proposition 3.1. — *Let Y be a rational homology sphere equipped with a Solv metric such that the fibers are small enough. Then the first eigenvalue of Δ on coexact 1-forms satisfies $\lambda_1^* = 1$. Furthermore, the 1-eigenspace is one dimensional.*

In fact, our proof will provide an explicit smallness condition for the fibers.

Let us start by considering the case of a Solv-manifold $\widetilde{Y} = \Gamma \setminus \mathsf{Solv}$ with $b_1 = 1$. The 1-forms

$$\mathcal{X} = e^z dx, \quad \mathcal{Y} = e^{-z} dy \quad \mathcal{Z} = dz$$

descend to a left-invariant dual orthonormal frame on $\widetilde{Y}$. We can then write any 1-form ξ as

$$\xi = f\mathcal{X} + g\mathcal{Y} + h\mathcal{Z},$$

where f, g, h are functions on $\Gamma \setminus \mathsf{Solv}$, or equivalently left-invariant functions on Solv. We are interested in understanding for which $\lambda \in \mathbb{R}$ the equation

$$*d\xi = \lambda\xi$$

admits non-trivial solutions. Notice that, provided $\lambda \neq 0$, such a form necessarily satisfies $d{*}\xi = 0$, i.e. it is coclosed. In particular, λ^2 is an eigenvalue of the Laplacian on coexact 1-forms. We have

$$d\xi = \left(e^{-z} g_x - e^z f_y\right) \mathcal{X} \wedge \mathcal{Y} + \left(-g_z + g + e^z h_y\right) \mathcal{Y} \wedge \mathcal{Z} + \left(f_z + f - e^{-z} h_x\right) \mathcal{Z} \wedge \mathcal{X}$$

so that our equation is equivalent to the system

$$\begin{aligned} -g_z + g + e^z h_y &= \lambda f \\ f_z + f - e^{-z} h_x &= \lambda g \\ e^{-z} g_x - e^z f_y &= \lambda h, \end{aligned} \tag{3.1}$$

while coclosedness is equivalent to

$$e^{-z} f_x + e^z g_y + h_z = 0.$$

Differentiating we get

$$\begin{aligned} -e^{-2z} h_{xx} &= -e^{-z} f_{xz} - e^{-z} f_x + \lambda e^{-z} g_x \\ -e^{2z} h_{yy} &= -e^z g_{yz} + e^z g_y - \lambda e^z f_y \\ -h_{zz} &= e^{-z} f_{xz} - e^{-z} f_x + e^z g_{yz} + e^z g_y, \end{aligned}$$

therefore summing we obtain

$$\Delta h = \lambda^2 h - 2e^{-z} f_x + 2e^z g_y,$$

where Δ denotes the Laplacian on functions on $\widetilde{Y}$. Similarly for g we obtain

$$\begin{aligned} -e^{-2z} g_{xx} &= -\lambda e^{-z} h_x - f_{xy} \\ -e^{2z} g_{yy} &= f_{xy} + e^z h_{yz} \\ -g_{zz} &= \lambda f_z - g_z - e^z h_y - e^z h_{yz}, \end{aligned}$$

hence summing

$$\begin{aligned} \Delta g = \lambda(f_z - e^{-z} h_x) - g_z - e^z h_y = \lambda^2 g - \lambda f - g_z - e^z h_y = \\ = \left(\lambda^2 - 1\right) g - 2e^z h_y. \end{aligned}$$

Finally, as

$$\begin{aligned} -e^{-2z} f_{xx} &= e^{-z} h_{xz} + g_{xy} \\ -e^{2z} f_{yy} &= \lambda e^z h_y - g_{xy} \\ -f_{zz} &= f_z - e^{-z} h_{xz} + e^{-z} h_x - \lambda g_z, \end{aligned}$$

we have

$$\begin{aligned} \Delta f = \lambda(-g_z + e^z h_y) + f_z + e^{-z} h_x = \lambda^2 f - \lambda g + f_z + e^{-z} h_x = \\ = \left(\lambda^2 - 1\right) f + 2e^{-z} h_x. \end{aligned}$$

Notice that $\mathcal{Z}$ is a harmonic 1-form; as $b_1 = 1$, all harmonic forms are multiples of it.

Lemma 3.2. — *Let $\widetilde{Y}$ be a* Solv *manifold with $b_1 = 1$ equipped with a metric for which the fibers are small enough. Then $\lambda_1^* = 1$, and the 1-eigenspace is spanned by $\mathcal{X}$ and $\mathcal{Y}$.*

Proof. — We can expand f, g and h in Fourier series; the operator $*d$ decomposes accordingly in the sum of $*d_{\underline{\mu}}$, and in the $\underline{\mu}$ component our equations look like

$$\begin{aligned} \Delta_{\underline{\mu}} h &= \lambda^2 h - 2i\mu e^{-z} f + 2i\mu' e^z g \\ \Delta_{\underline{\mu}} g &= \left(\lambda^2 - 1\right) g - 2i\mu' e^z h \\ \Delta_{\underline{\mu}} f &= \left(\lambda^2 - 1\right) f + 2i\mu e^{-z} h \end{aligned}$$

with f, g and h are *complex* valued functions in the space $\mathcal{S}$.

Let us discuss first the modes $\underline{\mu} \neq 0$. By Lemma 2.5, the bottom of the spectrum of $\Delta_{\underline{\mu}}$ is bounded below by $2|\mu\mu'|$; and furthermore, by suitably rescaling the metric, we can arrange that this quantity is > 16 for all $\underline{\mu} \neq 0$. Multiplying each equation by $\bar{h}, \bar{g}$ and $\bar{f}$ respectively, and adding them together, we obtain the pointwise identity

$$\begin{aligned} &\bar{h}\Delta_{\underline{\mu}} h + \bar{g}\Delta_{\underline{\mu}} g + \bar{f}\Delta_{\underline{\mu}} f \\ &\quad = \lambda^2 |h|^2 + \left(\lambda^2 - 1\right)|g|^2 + \left(\lambda^2 - 1\right)|f|^2 + 4\,\mathrm{Re}\left(i\mu' e^z g\bar{h}\right) - 4\,\mathrm{Re}\left(2i\mu e^{-z} f\bar{h}\right). \end{aligned}$$

In particular, this implies that the left-hand side is real. By the Peter–Paul inequality, we have the pointwise inequalities

$$\left|4\operatorname{Re}(i\mu' e^{z} g\bar{h})\right| \leqslant 4\left|\mu' e^{z}\bar{h}\right|\,|g| \leqslant \frac{(\mu')^2 e^{2z}}{2}|h|^2 + 8|g|^2$$
$$\left|4\operatorname{Re}(i\mu e^{-z} f\bar{h})\right| \leqslant 4\left|\mu e^{-z}\bar{h}\right|\,|f| \leqslant \frac{\mu^2 e^{-2z}}{2}|h|^2 + 8|f|^2$$

so that

$$\bar{h}\widetilde{\Delta}_{\underline{\mu}}h + \bar{g}\Delta_{\underline{\mu}}g + \bar{f}\Delta_{\underline{\mu}}f \leqslant \lambda^2|h|^2 + (\lambda^2+7)|g|^2 + (\lambda^2+7)|f|^2 \tag{3.2}$$

where

$$\widetilde{\Delta}_{\underline{\mu}}h = -h_{zz} + \frac{1}{2}\left(\mu^2 e^{-2z} + (\mu')^2 e^{2z}\right)h$$

is still a diagonalizable operator over $L^2(\mathbb{R})$. The same argument as Lemma 2.5 implies that the first eigenvalue of $\widetilde{\Delta}_{\underline{\mu}}$ is at least $|\mu\mu'|$. Therefore, by integrating the Inequality (3.2) we have

$$|\mu\mu'|\left(\|h\|^2 + \|g\|^2 + \|f\|^2\right) \leqslant \left(\lambda^2+7\right)\left(\|h\|^2 + \|g\|^2 + \|f\|^2\right).$$

As by assumption $|\mu\mu'| > 8$, $\lambda^2 > 1$.

Finally, we deal with the zero mode. Suppose $0 < \lambda^2 < 1$. Then $\lambda^2 - 1 < 0$, hence

$$-g_{zz} = \left(\lambda^2 - 1\right)g, \quad -f_{zz} = \left(\lambda^2 - 1\right)f$$

have no non-zero periodic solution. It follows from Equation (3.1) that h is constant, so we have a multiple of the harmonic form $\mathcal{Z}$. Finally, the case $\lambda^2 = 1$ corresponds to the span of $\mathcal{X}$ and $\mathcal{Y}$. □

Finally, we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. — Suppose Y is a Solv-rational homology sphere. Consider its double cover $\pi : \widetilde{Y} \to Y$ where $\widetilde{Y}$ has $b_1(\widetilde{Y}) = 1$. If ξ is a λ-eigenform on Y, the $\pi^*\xi$ is a λ-eigenform on $\widetilde{Y}$. Choose a Solv-metric with fibers small enough, so that Lemma 3.2 applies. This implies that on Y we have $\lambda_1^* \geqslant 1$, and furthermore that if ξ is a 1-eigenform on Y, then $\pi^*\xi$ is a linear combination of $\mathcal{X}$ and $\mathcal{Y}$. Finally, in the notation of Section 2, if v, w is the basis of Λ, then exactly the linear combinations of $\mathcal{X}$ and $\mathcal{Y}$ that vanish on w at $z = 0$ descend to Y. □

Remark 3.3. — As the covering involution τ of $\widetilde{Y}$ sends $\mathcal{X}$ to $\pm\mathcal{Y}$, the forms that descend are multiples of either $\mathcal{X} + \mathcal{Y}$ or $\mathcal{X} - \mathcal{Y}$.

4. Transversality

In the previous section, we have exhibited a metric for which $\lambda_1^* = -\inf(\widetilde{s}/2)$. As this is the borderline case of Theorem 1.3, transversality is a quite delicate issue as small perturbation might introduce irreducible solutions. This should be compared with the discussion of flat manifolds in [KM07, Chapter 37]. As in their setting, we will show that we can achieve transversality, while still not having irreducible solutions, by considering the perturbed functional

$$\mathcal{L}(B,\Psi)-\frac{\delta}{2}\|\Psi\|^2$$

for δ sufficiently small. The corresponding equations for the critical points are

$$D_B\Psi=\delta\Psi$$
$$\frac{1}{2}\rho(F_{B^t})=(\Psi\Psi^*)_0.$$

We will denote by η the unique unit length 1-eigenform such that $\eta(v)>0$ at $z=0$ and η descends to Y. Recall ([Tur97, Lemma 1.4]) that there is a natural one-to-one correspondence between spin^c structures and unit length 1-forms up to homotopy outside balls; denote by $\mathfrak{s}_0$ the spin^c structure on Y corresponding to η. With this in mind, we have the following.

LEMMA 4.1. — *Consider a spin^c structure $\mathfrak{s}\neq\mathfrak{s}_0,\bar{\mathfrak{s}}_0$. Then, for δ small enough, the perturbed Seiberg–Witten equations do not admit irreducible solutions.*

Proof. — Suppose we have a sequence $\delta_i\to 0$ with corresponding irreducible solutions (B_i,Ψ_i); consider the corresponding configurations in the blow-up (B_i,r_i,ψ_i), where $\|\psi_i\|_{L^2}=1$. These admit (up to gauge transformations, and up to passing to a subsequence) a limit (B,r,ψ) which solves the blown-up equations with $\delta=0$; in particular, as the unperturbed equations do not admit irreducible solutions by Theorem 1.3, $r=0$, B is the flat connection, and $D_B\psi=0$. Recall that, setting $\xi=\rho^{-1}(\Psi\Psi^*)_0$, it is shown in [LL18, Proposition 1] that for solutions (B,Ψ) of the *unperturbed* Seiberg–Witten equations the pointwise identity

$$|\nabla\xi|^2+|d\xi|^2=|\Psi|^2|\nabla_B\Psi|^2$$

holds. This holds for the perturbed equations up to an error going to zero for $\delta_i\to 0$; hence it will apply to the limit form $\alpha=\rho^{-1}(\psi\psi^*)_0$. Furthermore, as it is the limit of the sequence of coexact forms $\frac{1}{2r_i^2}*F_{B^t}$, α is a coexact 1-form.

Let us study the geometry of α. As ψ is a harmonic spinor, and B is flat, the Weitzenböck formula on Y implies

$$\nabla_B^*\nabla_B\psi=\frac{1}{2}\psi,$$

hence the pointwise identity

$$\Delta|\psi|^2=2\langle\psi,\nabla_B^*\nabla_B\psi\rangle-2|\nabla_B\psi|^2=|\psi|^2-2|\nabla_B\psi|^2$$

holds. Multiplying by $|\psi|^2$ and integrating, we obtain

$$\int|\psi|^4-\int 2|\psi|^2|\nabla_B\psi|^2=\int|\psi|^2\Delta|\psi|^2\geqslant 0.$$

Recalling now that $|\alpha|^2=\frac{1}{4}|\psi|^4$, we obtain, by using the Bochner formula and $\lambda_1^*=1$, the chain of inequalities

$$\begin{aligned}2\|\alpha\|_{L^2}^2=\int\frac{1}{2}|\psi|^4\geqslant\int|\psi|^2|\nabla_B\psi|^2&=\|\nabla\alpha\|_{L^2}^2+\|d\alpha\|_{L^2}^2\\&=2\|d\alpha\|_{L^2}^2-\mathrm{Ric}(\alpha,\alpha)\geqslant 2\|d\alpha\|_{L^2}^2\geqslant 2\|\alpha\|_{L^2}^2.\end{aligned}$$

This implies that all inequalities are equalities, so that in particular α is a 1-eigenform, i.e. a multiple of η. So, according to the sign of $\alpha(v)$ at $z=0$, the underlying spin^c structure is either $\mathfrak{s}_0$ or its conjugate $\bar{\mathfrak{s}}_0$. ☐

We need to understand more in detail the spin^c structure $\mathfrak{s}_0$ on Y; before doing this, let us study the spin geometry of the double cover $\widetilde{Y}$. The manifold $\widetilde{Y} = \Gamma\backslash\mathrm{Solv}$ comes with a natural spin structure $\mathfrak{s}_*$ coming from the left invariant orthonormal framing dual to $\mathcal{Z}, \mathcal{X}, \mathcal{Y}$, i.e.

$$e_1 = \frac{d}{dz}, \quad e_2 = e^{-z}\frac{d}{dx}, \quad e_3 = e^{z}\frac{d}{dy}.$$

This defines a spin structure $\mathfrak{s}_*$ by taking the trivial bundle $S = Y \times \mathbb{C}^2$ and letting these vector fields act via the Pauli matrices

$$\sigma_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

Let B_* the spin connection on Y induced by the Levi–Civita connection.

Lemma 4.2. — *The kernel of the Dirac operator D_{B_*} consists of the constant spinors.*

Proof. — Let us write explicitly the Dirac operator. Our orthonormal frame satisfies the commutation relations

$$\begin{aligned} [e_1, e_2] &= -e_2 \\ [e_1, e_3] &= e_3 \\ [e_2, e_3] &= 0. \end{aligned}$$

Setting $[e_i, e_j] = \sum_k C_{ijk} e_k$, we have that the Christoffel symbols are

$$\Gamma_{ijk} = \frac{1}{2}(C_{ijk} - C_{ikj} - C_{jki}),$$

hence in our case the non-zero ones are

$$\Gamma_{212} = -\Gamma_{221} = 1, \quad \Gamma_{313} = -\Gamma_{331} = -1.$$

The spin connection on the spinor bundle is given by

$$\nabla_{e_i}\Psi = e_i(\Psi) + \frac{1}{4}\sum_{j<k}\Gamma_{ijk}[\sigma_j, \sigma_k]\cdot\Psi,$$

see Equation (3.13) in [BGV04, Section 3.3]. Therefore, as $[\sigma_1, \sigma_2] = -2\sigma_3$ and $[\sigma_1, \sigma_3] = 2\sigma_2$, we have

$$\begin{aligned} \nabla_{e_1}\Psi &= e_1(\Psi) \\ \nabla_{e_2}\Psi &= e_2(\Psi) - \frac{1}{2}\sigma_3\cdot\Psi \\ \nabla_{e_3}\Psi &= e_3(\Psi) - \frac{1}{2}\sigma_2\cdot\Psi \end{aligned}$$

As σ_2 and σ_3 anticommute, we have

$$D_{B_*}\Psi = \sum_i \rho(e_i)\cdot\nabla_{e_i}\Psi = \sum_i \rho(e_i)\cdot e_i(\Psi).$$

Hence, writing $\Psi = (f, g)$, we have

$$D_{B_*}\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} if_z - e^{-z}g_x + ie^z g_y \\ -ig_z + e^{-z}f_x + ie^z f_y \end{bmatrix},$$

and the equations for a harmonic spinor are

$$\begin{aligned} f_z + ie^{-z}g_x + e^z g_y &= 0 \\ g_z + ie^{-z}f_x - e^z f_y &= 0. \end{aligned}$$

Let us now decompose the equations according to the eigenmodes $\underline{\mu} = (\mu, \mu') \in \Lambda'$. We obtain

$$\begin{aligned} f_z - \mu e^{-z} g + i\mu' e^z g &= 0 \\ g_z - \mu e^{-z} f - i\mu' e^z f &= 0. \end{aligned} \tag{4.1}$$

Of course for the zero mode the kernel consists of constant solutions. Let us show now that the eigenmodes with $\underline{\mu} \neq 0$ do not admit non-zero harmonic spinors. We have

$$\begin{aligned} \frac{d}{dz}|f|^2 &= 2\Re\left(f_z \bar{f}\right) = 2\operatorname{Re}\left((\mu e^{-z} g - i\mu' e^z g)\bar{f}\right) \\ \frac{d}{dz}|g|^2 &= 2\operatorname{Re}\left(\bar{g}_z g\right) = 2\operatorname{Re}\left(\left(\mu e^{-z}\bar{f} - i\mu' e^z \bar{f}\right) g\right) \end{aligned}$$

hence

$$\frac{d}{dz}(|f|^2 - |g|^2) = 0.$$

As $|f|^2 - |g|^2$ is in the class of function $\mathcal{S}$ from Equation (2.1), we have $|f|^2 = |g|^2$ everywhere. This, together with our ODE (4.1), shows that the functions f and g are never zero. We then have

$$\begin{aligned} f_z\bar{g} &= \mu e^{-z}|g|^2 - i\mu' e^z |g|^2 \\ f\bar{g}_z &= \mu e^{-z}|f|^2 - i\mu' e^z |f|^2 \end{aligned}$$

hence

$$\frac{d}{dz}\left(\frac{f}{\bar{g}}\right) = \frac{f_z\bar{g} - f\bar{g}_z}{\bar{g}^2} = 0.$$

Therefore, up to multiplying f and g by the same complex constant, we have

$$g = \bar{f}$$

and both are equations are equivalent to

$$f_z = \mu e^{-z}\bar{f} - i\mu' e^z \bar{f}.$$

Writing $f = a + ib$ for real functions a, b, this can be written as the system

$$\begin{aligned} a_z &= \mu e^{-z} a - \mu' e^z b \\ b_z &= -\mu' e^z a - \mu e^{-z} b \end{aligned}$$

Differentiating the first equation, and making some simple substitutions, we obtain the equation

$$a_{zz} = a_z + \left(\mu^2 e^{-2z} + \mu'^2 e^{2z} - 2\mu e^{-z}\right) a.$$

Then, $A = e^{-z/2}a$ (which still lies in $\mathcal{S}$) satisfies the ODE

$$A_{zz} = \left(\mu^2 e^{-2z} + \mu'^2 e^{2z} - 2\mu e^{-z} + \frac{1}{4}\right) \cdot A.$$

The claim is that under our assumption $|\mu\mu'| > 8$, the first factor on the right hand side is always strictly positive, so that by Lemma 2.4, A is zero, and so are a and b. We need to show

$$\mu^2 e^{-2z} + \mu'^2 e^{2z} - 2\mu e^{-z} + \frac{1}{4} > 0,$$

which is equivalent to

$$(\mu e^{-z} - 1)^2 + (\mu' e^z)^2 > \frac{3}{4}.$$

If $|\mu e^{-z}| \geqslant 2$, the inequality is clearly true; otherwise, we have

$$|\mu' e^z| \geqslant \frac{|\mu e^{-z}|}{2} \cdot |\mu' e^z| = \frac{|\mu\mu'|}{2} \geqslant 4,$$

so we are done. □

With this computation in mind, we will show that the Dirac operator on our rational homology sphere Y equipped with the spin^c structure $\mathfrak{s}_0$ has no kernel (the same will hold for $\bar{\mathfrak{s}}_0$).

First of all, we pull it back to $\widetilde{Y}$; suppose that this is the mapping torus of $A \in \mathrm{SL}(2;\mathbb{Z})$. The Mayer–Vietoris sequence for the mapping torus of any map f implies the exact sequence

$$H_1\left(T^2;\mathbb{Z}/2\mathbb{Z}\right) \xrightarrow{1-f_*} H_1\left(T^2;\mathbb{Z}/2\mathbb{Z}\right) \to H_1\left(M_f;\mathbb{Z}/2\mathbb{Z}\right) \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

As A is congruent to the identity modulo 2 (see Remark 2.3), we have that

$$H^1\left(\widetilde{Y};\mathbb{Z}/2\mathbb{Z}\right) \cong \mathbb{Z}/2\mathbb{Z} \quad\oplus\quad H^1\left(T^2;\mathbb{Z}/2\mathbb{Z}\right) \cong (\mathbb{Z}/2\mathbb{Z})^3,$$

so that, from the point of view of spin topology, $\widetilde{Y}$ looks like the more familiar three-torus. The pullback of $\mathfrak{s}_0$ to $\widetilde{Y}$, call it $\tilde{\mathfrak{s}}$, also corresponds to the 1-form η. Denoting by ξ the left invariant 1-form obtained from η by $\pi/2$ counterclockwise rotation within the fibers, we see that the cover involution τ sends the frame $\eta, \xi, \mathcal{Z}$ to $\eta, -\xi, -\mathcal{Z}$. Therefore $\tilde{\mathfrak{s}}$ is the spin structure obtained from the standard one $\mathfrak{s}_*$ by twisting by 2π around the class dual to $[v] \in H^1(T^2;\mathbb{Z}/2\mathbb{Z})$. In particular the holonomy of the pullback of the flat connection of $\mathfrak{s}_0$ is trivial around the generator of $b_1(\widetilde{Y})$. The sublattice of Λ spanned by $2v$ and w is preserved by A; the corresponding mapping torus $\overline{Y}$ is a double cover of $\widetilde{Y}$; and the pullback of $\tilde{\mathfrak{s}}$ is the standard spin structure $\mathfrak{s}_*$ on $\overline{Y}$. One can then identify the harmonic spinors on $(\widetilde{Y}, \tilde{\mathfrak{s}})$ as the harmonic spinors on $(\overline{Y}, \mathfrak{s}_*)$ which change sign under translation by v at $z = 0$; by Lemma 4.2, there are no such spinors. Hence, there are no harmonic spinors on the base space $(Y, \mathfrak{s}_0)$.

Putting pieces together, we finally conclude.

Proof of Theorem 1.1. — By the discussion above, we have found small perturbations for which there are no irreducible solutions and the (perturbed) Dirac operator of the reducible solution has no kernel; we can then add a further small perturbation to make all of its eigenvalues simple (while preserving these properties) as in [KM07, Chapter 12]; the proof of Theorem 1.1 is then completed. □

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