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LARGE PLANAR POISSON–VORONOI CELLS CONTAINING A GIVEN CONVEX BODY

GRANDES CELLULES DE
POISSON–VORONOI DANS LE PLAN
CONTENANT UN CORPS CONVEXE DONNÉ

ABSTRACT. — Let K be a convex body in \mathbb{R}^2 . We consider the Voronoi tessellation generated by a homogeneous Poisson point process of intensity λ conditional on the existence of a cell K_λ which contains K . When $\lambda \rightarrow \infty$, this cell K_λ converges *from above* to K and we provide the precise asymptotics of the expectation of its defect area, defect perimeter and number of vertices. As in Rényi and Sulanke’s seminal papers on random convex hulls, the

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regularity of K has crucial importance and we deal with both the smooth and polygonal cases. Techniques are based on accurate estimates of the area of the Voronoi flower and of the support function of K_λ as well as on an Efron-type relation. Finally, we show the existence of limiting variances in the smooth case for the defect area and the number of vertices as well as analogous expectation asymptotics for the so-called Crofton cell.

RÉSUMÉ. — Soit K un corps convexe de \mathbb{R}^2 . Nous considérons la mosaïque de Voronoï engendrée par un processus ponctuel de Poisson homogène d'intensité λ conditionné par l'existence d'une cellule K_λ contenant K . Quand $\lambda \rightarrow \infty$, la cellule K_λ converge *en décroissant* vers K et nous donnons les estimées asymptotiques précises des espérances de la différence d'aire, de la différence de périmètre et du nombre de sommets. Comme dans les articles fondateurs de Rényi et Sulanke sur les enveloppes convexes aléatoires, la régularité de K a une importance cruciale et nous traitons séparément le cas lisse et le cas polygonal. Les méthodes sont basées sur des estimées fines de l'aire de la fleur de Voronoï et de la fonction de support de K_λ et sur une relation de type Efron. Enfin, nous montrons l'existence de variances limites dans le cas lisse pour la différence d'aire et le nombre de sommets ainsi que des résultats analogues d'espérances asymptotiques pour le modèle de la cellule de Crofton.

1. Introduction

For any $\lambda > 0$, let \mathcal{P}_λ be a homogeneous Poisson point process on \mathbb{R}^2 with intensity λ . The Poisson–Voronoi tessellation generated by the set of nuclei \mathcal{P}_λ is the collection of all cells

$$\{y \in \mathbb{R}^2 : \|y - x\| \leq \|y - x'\| \text{ for all } x' \in \mathcal{P}_\lambda\}, \quad x \in \mathcal{P}_\lambda,$$

where $\|\cdot\|$ denotes the Euclidean norm of \mathbb{R}^2 , see e.g. [Mø194, OBSC00].

Let K be a convex body of \mathbb{R}^2 with non-empty interior and let K_λ be the particular Voronoi cell containing K when \mathcal{P}_λ is conditioned on the existence of such a cell. We are interested in studying the geometrical properties of this random polygon K_λ . In particular, we aim at providing exact asymptotics of several mean characteristics of K_λ , see Theorem 1.3.

This problem falls naturally within the general literature on the asymptotic description of large cells from random tessellations. The breakthrough paper [HRS04] proves and extends the famous conjecture stated by D. G. Kendall in the 40's and which asserts that large cells from a stationary and isotropic Poisson line tessellation are close to the circular shape. Thereafter, the work [CS05] investigates the mean defect area and mean number of vertices of the typical Poisson–Voronoi cell and of the zero-cell, i.e. the cell containing the origin, of a stationary and isotropic Poisson line tessellation conditioned on containing a disk of radius r when $r \rightarrow \infty$. More recently, [HS14] provides an estimate of the Hausdorff distance between K and its random polyhedral approximation in the slightly different model of a zero-cell from a stationary Poisson hyperplane tessellation in any dimension.

In a first step, we study a different random polygon. Let o be the origin of \mathbb{R}^2 and let us assume that o is included in the interior of K . We then construct the Voronoi tessellation generated by the set of nuclei $\mathcal{P}_\lambda \cup \{o\}$ and we denote by K_λ^o the Voronoi cell associated with o when \mathcal{P}_λ is conditioned on the event that this Voronoi cell contains K . In that situation, K_λ^o is equal in distribution to the Voronoi cell of o

associated with the set of nuclei $(\mathcal{P}_\lambda \cap (\mathbb{R}^2 \setminus 2\mathcal{F}_o(K))) \cup \{o\}$, where $\mathcal{F}_o(K)$ is the Voronoi flower of K with respect to o (see (2.1) for a precise definition).

In Theorems 1.1 and 1.2 below, we provide limiting expectations up to proper rescalings of its area $\mathcal{A}(K_\lambda^o)$, perimeter $\mathcal{U}(K_\lambda^o)$ and number of vertices $\mathcal{N}(K_\lambda^o)$ when K has a smooth boundary or is a polygon.

In the sequel, we set $f(u) \sim g(u)$ (resp. $f(u) = O(g(u))$) when the ratio

$$\frac{f(u)}{g(u)} \rightarrow 1$$

(resp. $\frac{f(u)}{g(u)}$ is bounded from above) when the variable $u \rightarrow \infty$ or $u \rightarrow 0$ according to the situation.

In all this work, a set K is said to be a smooth convex body if ∂K is of class \mathcal{C}^2 with positive curvature which is bounded from above and from below by positive constants.

THEOREM 1.1 (Smooth case). — *Let K be a smooth convex body containing o in its interior. For $s \in \partial K$ we denote by r_s and n_s respectively the radius of curvature and the outer unit normal vector of ∂K at point s . The mean defect area, defect perimeter and number of vertices of K_λ^o have respectively the following asymptotics when the intensity $\lambda \rightarrow \infty$:*

$$\begin{aligned} \text{(i)} \quad & \mathbb{E}(\mathcal{A}(K_\lambda^o)) - \mathcal{A}(K) \underset{\lambda \rightarrow \infty}{\sim} \lambda^{-\frac{2}{3}} 2^{-2} 3^{-\frac{1}{3}} \Gamma\left(\frac{2}{3}\right) \int_{\partial K} r_s^{\frac{1}{3}} \langle s, n_s \rangle^{-\frac{2}{3}} ds \\ \text{(ii)} \quad & \mathbb{E}(\mathcal{U}(K_\lambda^o)) - \mathcal{U}(K) \underset{\lambda \rightarrow \infty}{\sim} \lambda^{-\frac{2}{3}} 3^{-\frac{4}{3}} \Gamma\left(\frac{2}{3}\right) \int_{\partial K} r_s^{-\frac{2}{3}} \langle s, n_s \rangle^{-\frac{2}{3}} ds \\ \text{(iii)} \quad & \mathbb{E}(\mathcal{N}(K_\lambda^o)) \underset{\lambda \rightarrow \infty}{\sim} \lambda^{\frac{1}{3}} 2^2 3^{-\frac{4}{3}} \Gamma\left(\frac{2}{3}\right) \int_{\partial K} r_s^{-\frac{2}{3}} \langle s, n_s \rangle^{\frac{1}{3}} ds. \end{aligned}$$

THEOREM 1.2 (Polygonal case). — *Let K be a convex polygon containing o in its interior, with $n_K \geq 3$ consecutive vertices in anticlockwise order denoted by a_1, \dots, a_{n_K} and set $a_{n_K+1} = a_1$. We denote by o_i the orthogonal projection of o onto the line (a_i, a_{i+1}) . The mean defect area, defect perimeter and number of vertices of K_λ^o have respectively the following asymptotics when the intensity $\lambda \rightarrow \infty$:*

$$\begin{aligned} \text{(i)} \quad & \mathbb{E}(\mathcal{A}(K_\lambda^o)) - \mathcal{A}(K) \underset{\lambda \rightarrow \infty}{\sim} \lambda^{-\frac{1}{2}} 2^{-\frac{9}{2}} \pi^{\frac{3}{2}} \sum_{i=1}^{n_K} \|o_i\|^{-\frac{1}{2}} \|a_{i+1} - a_i\|^{\frac{3}{2}} \\ \text{(ii)} \quad & \mathbb{E}(\mathcal{U}(K_\lambda^o)) - \mathcal{U}(K) \underset{\lambda \rightarrow \infty}{\sim} \left(\lambda^{-1} \log \lambda \right) \cdot 2^{-1} 3^{-1} \sum_{i=1}^{n_K} \|o_i\|^{-1} \\ \text{(iii)} \quad & \mathbb{E}(\mathcal{N}(K_\lambda^o)) \underset{\lambda \rightarrow \infty}{\sim} (\log \lambda) \cdot 2 \cdot 3^{-1} n_K. \end{aligned}$$

Theorems 1.1 and 1.2 are reminiscent of the famous results obtained by Rényi and Sulanke [RS63, RS64] in the study of the approximation of a convex body K by the convex hull of $\mathcal{P}_\lambda \cap K$. In particular, in the smooth case, the exponents of λ coincide but the geometric quantities involved in the constants differ. In particular, these quantities in Rényi and Sulanke’s theorem above are intrinsic, equal to the classical affine perimeter of K in the case of the mean defect area and mean number of vertices. In Theorem 1.1, they are reminiscent of the L^p -centroaffine surface area

defined notably in [Hug96] and [Lut96]. They depend not only on K but also on the origin through the variable s present in the integral terms of our results.

In the polygonal case, the respective growth rates of the mean defect area and mean defect perimeter in Theorem 1.2 are switched with respect to their analogues in Rényi and Sulanke's work. This can be explained by the fact that the duality between the two models implies a correspondence between the defect area (resp. defect perimeter) of the Voronoi cell outside of K and the defect perimeter (resp. defect area) of the random convex hull inside K . This duality will also be at play when using the inversion technique in Section 6.

Again, the constants in points (i) and (ii) of Theorem 1.2 depend on the position of the origin inside K . Surprisingly, the limiting mean number of vertices in point (iii) of Theorem 1.2 does not depend on that position and it even coincides with Rényi and Sulanke's corresponding result. To the best of our knowledge, there is no easy explanation of this feature.

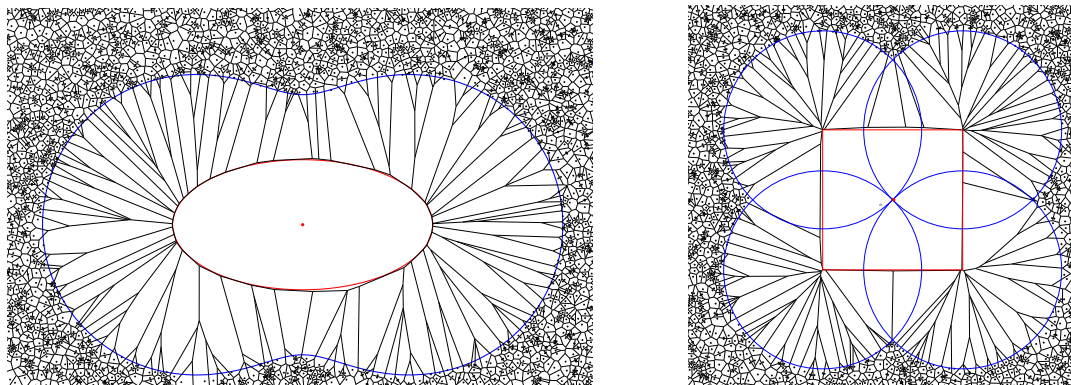


Figure 1.1. Voronoi cells (red) generated by the origin and a Poisson point process of intensity 10000 outside the double of the Voronoi flower (blue), with respect to the origin, of an ellipse (left) and of a square (right).

In a second step, we go back to the original problem, i.e. studying the asymptotic mean characteristics of the initial random polygon K_λ . We show that when $\lambda \rightarrow \infty$, the nucleus of K_λ concentrates around a point whose associated Voronoi flower has the smallest area. This point is known as the Steiner point of K . Consequently, we derive in Theorem 1.3 below the required asymptotic results for the mean characteristics of K_λ .

THEOREM 1.3. — *Let K be a convex body with non-empty interior and with its Steiner point at the origin. The asymptotics of $\mathbb{E}(\mathcal{A}(K_\lambda)) - \mathcal{A}(K)$, $\mathbb{E}(\mathcal{U}(K_\lambda)) - \mathcal{U}(K)$ and $\mathbb{E}(\mathcal{N}(K_\lambda))$ are then provided by Theorem 1.1 when K is smooth and by Theorem 1.2 when K is a polygon.*

The convergence in mean to zero of the defect area of K_λ° (resp. K_λ) allows us to show that both random polygons converge in probability to K for the Hausdorff distance. Indeed since K_λ° (resp. K_λ) is a convex body which contains K , any tail probability of the Hausdorff distance between K and K_λ° (resp. K_λ) is bounded by a tail probability of the defect area which in turn is upper-bounded by its expectation

up to a multiplicative constant, thanks to Markov’s inequality. Moreover, when the processes \mathcal{P}_λ , $\lambda > 0$, are coupled, the convergence holds almost surely. The results are stated in Corollary 1.4 below.

COROLLARY 1.4. — *Let \mathcal{P} be the homogeneous Poisson point process of intensity 1 in \mathbb{R}^3 . When $\mathcal{P}_\lambda = \{x \in \mathbb{R}^2 : \exists t \in (0, \lambda) \text{ such that } (x, t) \in \mathcal{P}\}$, the random polygons K_λ^o and K_λ converge almost surely to K for the Hausdorff distance when $\lambda \rightarrow \infty$.*

We also collect further results in two directions. First, we obtain explicit limiting variances in the smooth case for the defect area and the number of vertices. The results are provided in Theorem 1.5 below.

THEOREM 1.5. — *Let K be a smooth convex body. There exists two positive constants c_A and c_N , independent of K , such that the variances of the defect area and the number of vertices of K_λ^o have respectively the following asymptotics when the intensity $\lambda \rightarrow \infty$:*

$$\begin{aligned} \text{(i)} \quad \text{Var}(\mathcal{A}(K_\lambda^o)) &\underset{\lambda \rightarrow \infty}{\sim} \lambda^{-\frac{5}{3}} 2^{-\frac{13}{3}} \pi^{-1} c_A \int_{\partial K} r_s^{\frac{4}{3}} \langle s, n_s \rangle^{-\frac{5}{3}} \, ds \\ \text{(ii)} \quad \text{Var}(\mathcal{N}(K_\lambda^o)) &\underset{\lambda \rightarrow \infty}{\sim} \lambda^{\frac{1}{3}} 2^{-\frac{1}{3}} \pi^{-1} c_N \int_{\partial K} r_s^{-\frac{2}{3}} \langle s, n_s \rangle^{\frac{1}{3}} \, ds. \end{aligned}$$

These results are on par with the explicit limiting variances which were obtained in [CY14] for the number of k -dimensional faces and volume of random convex hulls inside a smooth convex body. It comes as no surprise as the proof of Theorem 1.5 relies on a polarisation argument followed by a general technique already used in [CY14]. In both models, the growth rate of the variance of each functional coincides with the rate of its expectation. The case of the perimeter seems more intricate and is briefly discussed at the end of Section 6.

Second, analogues of Theorems 1.1 and 1.2 are derived when the Poisson–Voronoi cell is replaced by the so-called Crofton cell.

The paper is structured as follows. We start by presenting in Section 2 preliminary results related to the so-called support function of K_λ^o . The proofs of Theorems 1.1 and 1.2 are presented in Section 3 and Section 4 for the smooth and polygonal cases respectively. The answers to the original question are summarized in Section 5. In Section 6, the additional results on the limiting variances are proved and followed by a general discussion on possible extensions of the method. Finally, Section 7 contains the proofs for the analogues of the limiting expectations for the Crofton cell.

2. The key role of the support function

In this section, we rewrite in a tractable way the three expectations which appear in Theorems 1.1 and 1.2, i.e. we aim at getting the three relations (2.3), (2.5) and (2.6). We also emphasize the basic ideas and guidelines of the proofs from Sections 3 and 4.

Let us introduce the Voronoi flower of a compact set L with respect to a point $x \in \mathbb{R}^2$ as the set defined by

$$(2.1) \quad \mathcal{F}_x(L) = \bigcup_{s \in L} B_{\frac{1}{2}\|s-x\|} \left(\frac{1}{2}(s+x) \right)$$

where $B_r(z)$ denotes the closed ball centered at $z \in \mathbb{R}^2$ and of radius $r \geq 0$. This set is central in the study of Voronoi tessellations for the reasons described below and has sometimes been renamed as the *fundamental region*, see e.g. [Zuy92]. A quick study of the literature reveals that in most cases the set $2\mathcal{F}_x(L)$ instead of $\mathcal{F}_x(L)$ is considered. We have chosen this arbitrary normalization of the Voronoi flower in this work in order to connect it to the notion of pedal curve, see e.g. [Zwi63, Chapter 11]. In other words, the boundary of $\mathcal{F}_x(L)$ is the pedal curve associated with the boundary of L with respect to the pedal point at x .

We notice in particular that $\mathcal{F}_x(L) = \mathcal{F}_x(\text{conv}(L))$ where $\text{conv}(\cdot)$ denotes the convex hull. The basic equivalence

$$x \in K_\lambda^o \setminus K \iff \mathcal{P}_\lambda \cap 2(\mathcal{F}_o(K \cup \{x\}) \setminus \mathcal{F}_o(K)) = \emptyset$$

and the equality

$$(2.2) \quad \mathbb{P}(\mathcal{P}_\lambda \cap 2(\mathcal{F}_o(K \cup \{x\}) \setminus \mathcal{F}_o(K)) = \emptyset) = \exp(-4\lambda \mathcal{A}(\mathcal{F}_o(K \cup \{x\}) \setminus \mathcal{F}_o(K)))$$

imply that

$$(2.3) \quad \mathbb{E}(\mathcal{A}(K_\lambda^o)) - \mathcal{A}(K) = \int_{\mathbb{R}^2 \setminus K} \exp(-4\lambda \mathcal{A}(\mathcal{F}_o(K \cup \{x\}) \setminus \mathcal{F}_o(K))) dx.$$

Thus, our problem consists in providing accurate estimates for the extra area of the Voronoi flower of K when adding to K a single point x outside of it. We will need to treat separately the case where K has a smooth boundary (Lemma 3.1) and the case where K is a convex polygon (Lemma 4.1). In particular, Lemma 3.1 will be proved using the Cauchy–Crofton formula (2.4) involving the so-called support function of K .

For every $\theta \in [0, 2\pi)$, let us denote by (u_θ, v_θ) the orthonormal basis in direction θ , i.e. $u_\theta = (\cos \theta, \sin \theta)$ and $v_\theta = (-\sin \theta, \cos \theta)$. The support function of K with respect to a point $x \in \mathbb{R}^2$ (see e.g. [Sch93, Section 1.7]) is the function defined for $z \in \mathbb{R}^2$ by

$$p_x(K, z) = \sup_{y \in K} \langle y - x, z \rangle.$$

Observe that $p_x(K, \cdot)$ is homogeneous of degree 1. We will denote by $p_x(K, \theta)$ the quantity $p_x(K, u_\theta)$ and we will use indifferently both notations in the sequel, depending on the context. In particular, when $x \in K$, the distance from x to the boundary of $\mathcal{F}_x(K)$ in direction u_θ is precisely $p_x(K, \theta)$, which implies in turn that

$$(2.4) \quad \mathcal{A}(\mathcal{F}_x(K)) = \frac{1}{2} \int_0^{2\pi} p_x^2(K, \theta) d\theta.$$

The support function also makes it possible to rewrite the defect perimeter as an integral using the well-known Cauchy–Crofton formula

$$(2.5) \quad \mathbb{E}(\mathcal{U}(K_\lambda^o)) - \mathcal{U}(K) = \int_0^{2\pi} \mathbb{E}(p_o(K_\lambda^o, \theta) - p_o(K, \theta)) d\theta.$$

Therefore, in order to deal with this expectation we aim to determine the distribution of the point which achieves the support function into a fixed direction (see Propositions 3.3 and 4.2). The strategy will consist in using nontrivial changes of variable according to the case where K is smooth or not. The main difficulty will be

in the computation of its Jacobian, the determination of the domain of integration for it and finally its integration.

In the next proposition, we prove a relation in the same spirit as the well-known Efron’s relation for convex hulls of random inputs, see e.g. [Efr65], which connects the mean number of sides of K_λ^o either to the mean defect area of the flower or to the mean defect support function. Efron-type identities for random tessellations have been recently derived in [Sch09, Section 5] and [HHRT15, Theorem 3.1]. The two formulas below are new though the method to get (i) is very similar to these two references. Only the asymptotic relation (2.6) will be used in the sequel.

PROPOSITION 2.1. —

(i) For every $\lambda > 0$, the following identity holds

$$\mathbb{E}(\mathcal{N}(K_\lambda^o)) = 4\lambda \left(\mathbb{E}(\mathcal{A}(\mathcal{F}_o(K_\lambda^o)) - \mathcal{A}(\mathcal{F}_o(K))) \right).$$

(ii) Moreover,

$$(2.6) \quad \mathbb{E}(\mathcal{N}(K_\lambda^o)) = 4\lambda \int_0^{2\pi} p_o(K, \theta) \mathbb{E}(p_o(K_\lambda^o, \theta) - p_o(K, \theta)) d\theta + \mathcal{R}_\lambda$$

$$\text{where } \mathcal{R}_\lambda = 2\lambda \int_0^{2\pi} \mathbb{E}((p_o(K_\lambda^o, \theta) - p_o(K, \theta))^2) d\theta.$$

Proof of Proposition 2.1. — We recall that $\mathcal{N}(K_\lambda^o)$ is the number of neighbors of o , i.e. the set of all $x \in \mathcal{P}_\lambda \setminus 2\mathcal{F}_o(K)$ such that the bisecting line of the segment $[o, x]$ has a non-empty intersection with the boundary of K_λ^o . Moreover, for any $x \in \mathcal{P}_\lambda \setminus 2\mathcal{F}_o(K)$, x is a neighbor of o if and only if $\frac{1}{2}x \in \mathcal{F}_o(C_x) \setminus \mathcal{F}_o(K)$, where C_x is the Voronoi cell of the origin associated with the set of nuclei $(\mathcal{P}_\lambda \setminus 2\mathcal{F}_o(K)) \setminus \{x\}$. Consequently, thanks to Mecke–Slivnyak’s formula (see [SW08, Corollary 3.2.3]) and Fubini theorem, we obtain

$$\begin{aligned} \mathbb{E}(\mathcal{N}(K_\lambda^o)) &= \mathbb{E} \left(\sum_{x \in \mathcal{P}_\lambda \setminus 2\mathcal{F}_o(K)} \mathbb{I}_{\left\{ \frac{x}{2} \in \mathcal{F}_o(C_x) \setminus \mathcal{F}_o(K) \right\}} \right) \\ &= \lambda \int_{\mathbb{R}^2 \setminus 2\mathcal{F}_o(K)} \mathbb{P}(x \in 2(\mathcal{F}_o(K_\lambda^o) \setminus \mathcal{F}_o(K))) dx \\ &= 4\lambda \mathbb{E}(\mathcal{A}(\mathcal{F}_o(K_\lambda^o)) - \mathcal{A}(\mathcal{F}_o(K))). \end{aligned}$$

Now, using (2.4), we obtain

$$\begin{aligned} \mathbb{E}(\mathcal{N}(K_\lambda^o)) &= 2\lambda \int_0^{2\pi} \mathbb{E}(p_o^2(K_\lambda^o, \theta) - p_o^2(K, \theta)) d\theta \\ &= 2\lambda \int_0^{2\pi} \mathbb{E}((2p_o(K, \theta) + (p_o(K_\lambda^o, \theta) - p_o(K, \theta)))(p_o(K_\lambda^o, \theta) - p_o(K, \theta))) d\theta. \end{aligned}$$

This completes the proof of Proposition 2.1. \square

Let us notice that Proposition 2.1 can be extended to higher dimension when the number of sides is replaced by the number of facets.

We now rewrite Proposition 2.1 in the two particular cases when K has a smooth boundary (Corollary 2.2) and when K is a convex polygon (Corollary 2.3). We will

show in both cases that the first term in the right hand side of (2.6) is the leading term while the quantity \mathcal{R}_λ is negligible.

COROLLARY 2.2. — *If K has a smooth boundary, then*

$$(i) \quad \mathbb{E}(\mathcal{U}(K_\lambda^o)) - \mathcal{U}(K) = \int_{\partial K} \mathbb{E}(Y_{s,\lambda}) r_s^{-1} ds$$

$$(ii) \quad \mathbb{E}(\mathcal{N}(K_\lambda^o)) = 4\lambda \int_{\partial K} \langle s, n_s \rangle \mathbb{E}(Y_{s,\lambda}) r_s^{-1} ds + \mathcal{R}_\lambda$$

where $Y_{s,\lambda} = p_o(K_\lambda^o, n_s) - p_o(K, n_s)$ and $\mathcal{R}_\lambda = 2\lambda \int_{\partial K} \mathbb{E}(Y_{s,\lambda}^2) r_s^{-1} ds$.

Proof of Corollary 2.2. — When K has a smooth boundary, for every $s \in \partial K$ such that $n_s = u_\theta$, we get

$$\frac{ds}{d\theta} = r_s \quad \text{and} \quad p_o(K, \theta) = p_o(K, n_s) = \langle s, n_s \rangle.$$

We obtain the two results by setting $Y_{s,\lambda} = p_o(K_\lambda^o, n_s) - p_o(K, n_s)$ and applying (2.5) and Proposition 2.1 (ii). \square

Let us now rewrite Proposition 2.1 when K is a polygon. Let us denote by u_{δ_i} , $\delta_i \in (0, 2\pi)$, the external unit normal vector to the line (a_i, a_{i+1}) (with the convention $\delta_0 = \delta_{n_K}$). We expect $\mathbb{E}(p_o(K_\lambda^o, \theta) - p_o(K, \theta))$ to be maximal in directions *close* to δ_i for every i while the remaining directions should have a negligible contribution inside the integrals on the right-hand side of (2.5) and (2.6).

COROLLARY 2.3. — *If K is a convex polygon, then*

$$(i) \quad \mathbb{E}(\mathcal{U}(K_\lambda^o)) - \mathcal{U}(K) = \sum_{i=1}^{n_K} \int_{\eta_{i,\lambda}}^{\infty} \sum_{\varepsilon \in \{+, -\}} \mathbb{E}(Z_{i,\gamma,\lambda,\varepsilon}) (\lambda^{-\gamma} \log \lambda) d\gamma$$

$$(ii) \quad \mathbb{E}(\mathcal{N}(K_\lambda^o)) = 4\lambda \sum_{i=1}^{n_K} \int_{\eta_{i,\lambda}}^{\infty} \sum_{\varepsilon \in \{+, -\}} p_o(K, \delta_i + \varepsilon \lambda^{-\gamma}) \mathbb{E}(Z_{i,\gamma,\lambda,\varepsilon}) (\lambda^{-\gamma} \log \lambda) d\gamma + \mathcal{R}_\lambda$$

where

$$\eta_{i,\lambda} = -\frac{\log\left(\frac{1}{2}(\delta_i - \delta_{i-1})\right)}{\log \lambda},$$

$$Z_{i,\gamma,\lambda,\varepsilon} = p_o(K_\lambda^o, \delta_i + \varepsilon \lambda^{-\gamma}) - p_o(K, \delta_i + \varepsilon \lambda^{-\gamma})$$

and

$$(2.7) \quad \mathcal{R}_\lambda = 2\lambda \sum_{i=1}^{n_K} \int_{\eta_{i,\lambda}}^{\infty} \sum_{\varepsilon \in \{+, -\}} \mathbb{E}(Z_{i,\gamma,\lambda,\varepsilon}^2) (\lambda^{-\gamma} \log \lambda) d\gamma.$$

Proof of Corollary 2.3.

Proof of (i). — The integral in (2.5) is equal to

$$\sum_{i=1}^{n_K} \left(\int_{\delta_{i-1}}^{\frac{1}{2}(\delta_{i-1} + \delta_i)} \mathbb{E}(p_o(K_\lambda^o, \theta) - p_o(K, \theta)) d\theta + \int_{\frac{1}{2}(\delta_{i-1} + \delta_i)}^{\delta_i} \mathbb{E}(p_o(K_\lambda^o, \theta) - p_o(K, \theta)) d\theta \right).$$

Let us only estimate the second integral above (the first one will be treated analogously). Using the change of variable $\theta = \delta_i - \lambda^{-\gamma}$, we get

$$\int_{\frac{1}{2}(\delta_{i-1}+\delta_i)}^{\delta_i} \mathbb{E}(p_o(K_\lambda^o, \theta) - p_o(K, \theta)) d\theta = \int_{\eta_{i,\lambda}}^{\infty} \mathbb{E}(Z_{i,\gamma,\lambda,-})(\lambda^{-\gamma} \log \lambda) d\gamma.$$

Proof of (ii). — This point is stated in the same way by decomposing the integral in (2.6) and using the same changes of variables. \square

3. The smooth case

In this section, K is a smooth convex body containing the origin in its interior. Every $x \in \mathbb{R}^2 \setminus K$ can be written as $x = s + hn_s = s_h$ with $s \in \partial K$ and $h > 0$. We denote by $\Delta\mathcal{F}_{s_h}$ the set $\mathcal{F}_o(K \cup \{s_h\}) \setminus \mathcal{F}_o(K)$. Because of (2.2) and (2.3), a key ingredient for proving Theorem 1.1 is the estimate of the area $\mathcal{A}(\Delta\mathcal{F}_{s_h})$ of the increase $\Delta\mathcal{F}_{s_h}$ of a Voronoi flower of K induced by the addition of a point outside K . To the best of our knowledge this estimate stated in Subsection 3.1 is new despite the natural aspect of the question. Subsections 3.2, 3.3 and 3.4 are then devoted to the asymptotic mean area of K_λ^o , the asymptotic mean support function and perimeter of K_λ^o and the asymptotic intensity and mean number of vertices of K_λ^o respectively.

3.1. Increase of the area of the Voronoi flower

The next lemma provides the exact calculation of the limiting rescaled defect area of the Voronoi flower as well as a lower-bound.

LEMMA 3.1. — *Let us assume that K is a smooth convex body containing the origin o in its interior.*

(i) *For every $s \in \partial K$, we get*

$$\mathcal{A}(\Delta\mathcal{F}_{s_h}) \underset{h \rightarrow 0}{\sim} h^{\frac{3}{2}} 2^{\frac{5}{2}} 3^{-1} r_s^{-\frac{1}{2}} \langle s, n_s \rangle.$$

(ii) *Moreover, there exists $C > 0$ such that, for every $h > 0$ and $s \in \partial K$,*

$$h^{-\frac{3}{2}} \mathcal{A}(\Delta\mathcal{F}_{s_h}) \geq C > 0.$$

Proof of Lemma 3.1.

Proof of (i). — We wish to use the simpler case where the origin o coincides with the center of curvature ω_s of ∂K at point s . In other words, our aim is to show that the area $\mathcal{A}(\Delta\mathcal{F}_{s_h})$ can be calculated as a function of $\mathcal{A}(\Delta\tilde{\mathcal{F}}_{s_h})$ where

$$\Delta\tilde{\mathcal{F}}_{s_h} = \mathcal{F}_{\omega_s}(K \cup \{s + hn_s\}) - \mathcal{F}_{\omega_s}(K).$$

We assume for the time being that ω_s belongs to K and we will explain at the end of the proof how to adapt the arguments to the general case. We then use the equality (2.4) and the relation $p_x(K, \theta) - p_o(K, \theta) = -\langle x, u_\theta \rangle$ for every $x \in \mathbb{R}^2$, to obtain

$$(3.1) \quad \mathcal{A}(\mathcal{F}_o(K)) = \mathcal{A}(\mathcal{F}_{\omega_s}(K)) + \frac{1}{2} \int_0^{2\pi} \langle \omega_s, u_\theta \rangle^2 d\theta + \int_0^{2\pi} p_{\omega_s}(K, \theta) \langle \omega_s, u_\theta \rangle d\theta.$$

Applying this formula to both K and $K \cup \{s_h\}$ yields

$$(3.2) \quad \mathcal{A}(\Delta\mathcal{F}_{s_h}) = \mathcal{A}(\Delta\tilde{\mathcal{F}}_{s_h}) + \int_0^{2\pi} \Delta p_{\omega_s}(\theta) \langle \omega_s, u_\theta \rangle d\theta$$

where $\Delta p_{\omega_s}(\theta) = p_{\omega_s}(K \cup \{s_h\}, \theta) - p_{\omega_s}(K, \theta)$.

We treat separately the two terms of the right-hand side of (3.2).

– *Estimate of $\mathcal{A}(\Delta\tilde{\mathcal{F}}_{s_h})$.*

Let us define $s(\theta)$ as the point belonging to ∂K such that $\theta \in (-\pi, \pi]$ is the angle between the two half-lines $\omega_s + \mathbb{R}_+(s - \omega_s)$ and $\omega_s + \mathbb{R}_+(s(\theta) - \omega_s)$ (see Figure 3.1). Denote by $-\theta_{s,h}^-$ and $\theta_{s,h}^+$ the two angles such that $p_{\omega_s}(K \cup \{s_h\}, \theta) = p_{\omega_s}(K, \theta)$ if and only if $\theta \notin [-\theta_{s,h}^-, \theta_{s,h}^+]$. Then we can write

$$(3.3) \quad \mathcal{A}(\Delta\tilde{\mathcal{F}}_{s_h}) = \int_{-\theta_{s,h}^-}^{\theta_{s,h}^+} \int_{p_{\omega_s}(K, \theta)}^{(r_s+h)\cos\theta} r dr d\theta.$$

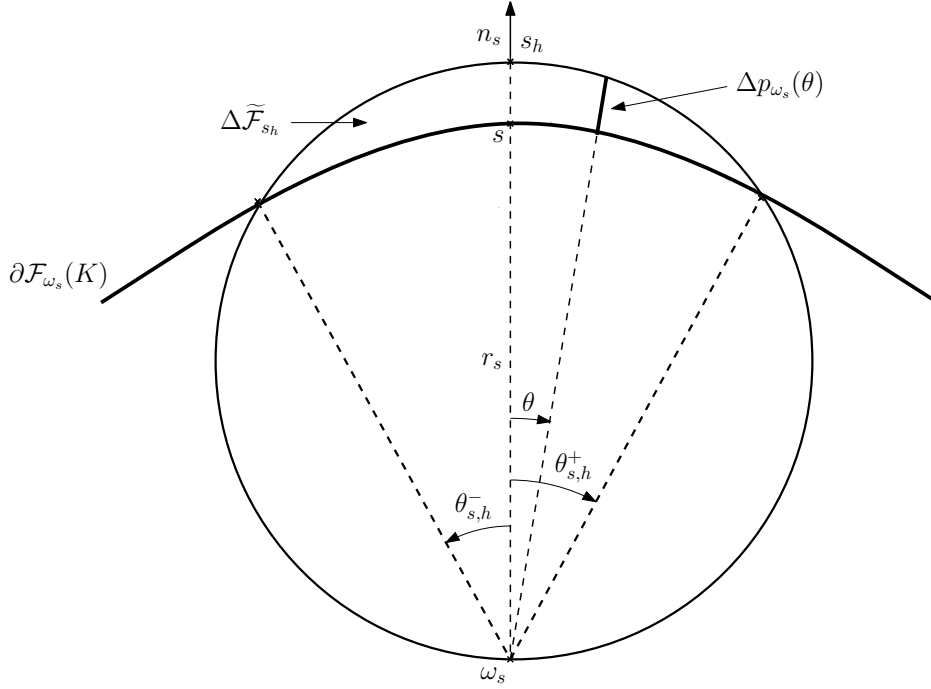


Figure 3.1. Flowers viewed from the center of curvature ω_s of ∂K at point s .

If ∂K were a perfect circle of radius r_s in the neighborhood of s we would have $\theta_{s,h}^- = \theta_{s,h}^+ = \theta_{s,h}$ with $\cos(\theta_{s,h}) = \frac{r_s}{r_s+h}$. In particular, this means that when $h \rightarrow 0$,

$$(3.4) \quad \theta_{s,h}^2 \sim 2r_s^{-1}h.$$

In the general case, we approximate locally the polar equation of ∂K with respect to the center ω_s :

$$(3.5) \quad \lim_{\theta \rightarrow 0} \theta^{-2}(\|s(\theta) - \omega_s\| - r_s) = 0.$$

Let us fix $\eta > 0$. For $h > 0$ small enough, we get from (3.5), (3.4) and the fact that r_s is lower bounded along ∂K that for any $s \in \partial K$ and $\theta \in [-\theta_{s,h}^-, \theta_{s,h}^+]$,

$||s(\theta) - \omega_s|| - r_s| \leq \eta h$. This implies that we can locally sandwich ∂K between two circles of radii $r_s + \eta h$ and $r_s - \eta h$. This being true for any $\eta > 0$, we deduce from this sandwiching that the angles $\theta_{s,h}^-$, $\theta_{s,h}^+$ and $\theta_{s,h}$ can all be written as

$$(3.6) \quad \theta_{s,h}^+ \underset{h \rightarrow 0}{=} \theta_{s,h}^- + o\left(h^{\frac{1}{2}}\right) \underset{h \rightarrow 0}{=} \theta_{s,h} + o\left(h^{\frac{1}{2}}\right) \underset{h \rightarrow 0}{=} 2^{\frac{1}{2}} r_s^{-\frac{1}{2}} h^{\frac{1}{2}} + o\left(h^{\frac{1}{2}}\right).$$

Moreover, since $\theta_{s,h}^+$ is bounded from above by its value obtained when ∂K is replaced by an outer circle of radius $r_s + Ch$, we have

$$(3.7) \quad \theta_{s,h}^+ \leq \arccos\left(1 - \frac{h}{r_s + (C+1)h}\right) \leq 2^{\frac{1}{2}} r_s^{-\frac{1}{2}} h^{\frac{1}{2}}.$$

Similarly, the defect of circularity implies that uniformly for $\theta \in [-\theta_{s,h}^-, \theta_{s,h}^+]$,

$$(3.8) \quad p_{\omega_s}(K, \theta) \underset{h \rightarrow 0}{=} r_s + o(h).$$

Indeed, by definition of the curvature radius, $p_{\omega_s}(K, \theta) = r_s + o(\theta^2)$ when $\theta \rightarrow 0$ and moreover, $\theta_{s,h}^\pm = O(\theta^{\frac{1}{2}})$ thanks to (3.6).

Let us show

$$(3.9) \quad \mathcal{A}(\Delta \tilde{\mathcal{F}}_{s_h}) \underset{h \rightarrow 0}{=} 2^{\frac{5}{2}} 3^{-1} r_s^{\frac{1}{2}} h^{\frac{3}{2}} + o\left(h^{\frac{3}{2}}\right).$$

Using (3.3), we get

$$\mathcal{A}(\Delta \tilde{\mathcal{F}}_{s_h}) = \mathcal{A}^+ + \mathcal{A}^-$$

where

$$\mathcal{A}^+ = \frac{(r_s + h)^2}{2} \left(\frac{\theta_{s,h}^+}{2} + \frac{\sin(2\theta_{s,h}^+)}{4} \right) - \int_0^{\theta_{s,h}^+} \frac{p_{\omega_s}^2(K, \theta)}{2} d\theta$$

and \mathcal{A}^- is the same with $\theta_{s,h}^+$ replaced by $\theta_{s,h}^-$. In order to get (3.9), it is enough to show that when $h \rightarrow 0$,

$$(3.10) \quad \mathcal{A}^+ = 2^{\frac{3}{2}} 3^{-1} r_s^{\frac{1}{2}} h^{\frac{3}{2}} + o\left(h^{\frac{3}{2}}\right) \quad \text{and} \quad \mathcal{A}^- = 2^{\frac{3}{2}} 3^{-1} r_s^{\frac{1}{2}} h^{\frac{3}{2}} + o\left(h^{\frac{3}{2}}\right).$$

We prove (3.10) for \mathcal{A}^+ and omit the proof for \mathcal{A}^- . We start by rewriting \mathcal{A}^+ in a more appropriate way.

$$(3.11) \quad \begin{aligned} \mathcal{A}^+ &= \left(\frac{r_s^2}{2} + r_s h + \frac{h^2}{2} \right) \left[\theta_{s,h}^+ + \left(\frac{\sin(2\theta_{s,h}^+)}{4} - \frac{\theta_{s,h}^+}{2} \right) \right] \\ &\quad - \frac{1}{2} \int_0^{\theta_{s,h}^+} (p_{\omega_s}^2(K, \theta) - r_s^2) d\theta - \frac{r_s^2}{2} \theta_{s,h}^+ \\ &= T_1 + T_2 + T_3 - T_4 \end{aligned}$$

where

$$\begin{aligned} T_1 &= \frac{r_s^2}{2} \left(\frac{\sin(2\theta_{s,h}^+)}{4} - \frac{\theta_{s,h}^+}{2} \right), \\ T_2 &= r_s h \theta_{s,h}^+, \\ T_3 &= r_s h \left(\frac{\sin(2\theta_{s,h}^+)}{4} - \frac{\theta_{s,h}^+}{2} \right) + \frac{h^2}{2} \left[\theta_{s,h}^+ + \left(\frac{\sin(2\theta_{s,h}^+)}{4} - \frac{\theta_{s,h}^+}{2} \right) \right], \\ T_4 &= \frac{1}{2} \int_0^{\theta_{s,h}^+} (p_{\omega_s}^2(K, \theta) - r_s^2) d\theta. \end{aligned}$$

Study of T_4 : because of (3.8), $p_{\omega_s}(K, \theta) - r_s = o(h)$ uniformly for $\theta \in [0, \theta_{s,h}^+]$ and

$$p_{\omega_s}(K, \theta)^2 - r_s^2 = O(p_{\omega_s}(K, \theta) - r_s) = o(h).$$

Integrating the estimate above, we obtain that when $h \rightarrow 0$

$$(3.12) \quad T_4 = o(h \theta_{s,h}^+) = o(h^{\frac{3}{2}}).$$

Study of T_3 : when $h \rightarrow 0$, $\theta_{s,h}^+ \rightarrow 0$ so using (3.6), we get when $h \rightarrow 0$

$$(3.13) \quad \frac{\sin(2\theta_{s,h}^+)}{4} - \frac{\theta_{s,h}^+}{2} = -\frac{8\theta_{s,h}^{+3}}{24} + o(\theta_{s,h}^{+3}) = -2^{\frac{3}{2}} 3^{-1} r_s^{-\frac{3}{2}} h^{\frac{3}{2}} + o(h^{\frac{3}{2}}).$$

This implies that

$$(3.14) \quad T_3 = o(h^{\frac{3}{2}}).$$

Study of T_2 : using (3.6), we get when $h \rightarrow 0$

$$(3.15) \quad T_2 = 2^{\frac{1}{2}} r_s^{\frac{1}{2}} h^{\frac{3}{2}} + o(h^{\frac{3}{2}}).$$

Study of T_1 : using the estimate (3.13) above, we obtain that when $h \rightarrow 0$

$$(3.16) \quad T_1 = -2^{\frac{1}{2}} 3^{-1} r_s^{\frac{1}{2}} h^{\frac{3}{2}} + o(h^{\frac{3}{2}}).$$

Inserting (3.16), (3.15), (3.14) and (3.12) into (3.11), we obtain the required result (3.10) and in turn (3.9).

– *Estimate of $\int_0^{2\pi} \Delta p_{\omega_s}(\theta) \langle \omega_s, u_\theta \rangle d\theta$.*

We have, uniformly for every $\theta \in [-\theta_{s,h}^-, \theta_{s,h}^+]$,

$$\langle \omega_s, u_\theta \rangle \underset{h \rightarrow 0}{=} \langle \omega_s, n_s \rangle + o(1).$$

Moreover, since $p_{\omega_s}(K \cup \{s_h\}, \theta) = (r_s + h) \cos \theta$, we can write successively

$$\begin{aligned}
& \int_0^{2\pi} \Delta p_{\omega_s}(\theta) \langle \omega_s, u_\theta \rangle d\theta \\
&= \int_{-\theta_{s,h}^-}^{\theta_{s,h}^+} \Delta p_{\omega_s}(\theta) \langle \omega_s, u_\theta \rangle d\theta \\
&\underset{h \rightarrow 0}{=} (\langle \omega_s, n_s \rangle + o(1)) \int_{-\theta_{s,h}^-}^{\theta_{s,h}^+} \Delta p_{\omega_s}(\theta) d\theta \\
&\underset{h \rightarrow 0}{=} (\langle \omega_s, n_s \rangle + o(1)) \int_{-\theta_{s,h}^-}^{\theta_{s,h}^+} ((r_s + h) \cos \theta - r_s) d\theta \\
&\underset{h \rightarrow 0}{=} (\langle \omega_s, n_s \rangle + o(1)) \left((r_s + h) (\sin(\theta_{s,h}^+) + \sin(\theta_{s,h}^-)) - r_s (\theta_{s,h}^+ + \theta_{s,h}^-) \right) \\
(3.17) \quad &\underset{h \rightarrow 0}{=} 2^{\frac{5}{2}} 3^{-1} r_s^{-\frac{1}{2}} \langle \omega_s, n_s \rangle h^{\frac{3}{2}} + o(h^{\frac{3}{2}})
\end{aligned}$$

where the last line is deduced from the penultimate one by methods similar to those used to prove statements (3.12) to (3.16).

– *Conclusion when $\omega_s \in K$.*

Inserting (3.9) and (3.17) into (3.2) and using $s = \omega_s + r_s n_s$, we get

$$\begin{aligned}
\mathcal{A}(\Delta \mathcal{F}_{s_h}) &\underset{h \rightarrow 0}{=} 2^{\frac{5}{2}} 3^{-1} r_s^{\frac{1}{2}} \left(1 + r_s^{-1} \langle \omega_s, n_s \rangle \right) h^{\frac{3}{2}} + o(h^{\frac{3}{2}}) \\
&\underset{h \rightarrow 0}{=} 2^{\frac{5}{2}} 3^{-1} r_s^{-\frac{1}{2}} \langle s, n_s \rangle h^{\frac{3}{2}} + o(h^{\frac{3}{2}})
\end{aligned}$$

that gives the desired result.

– *Case when $\omega_s \notin K$.*

The equality (3.1) is still valid as soon as the term $\mathcal{A}(\mathcal{F}_{\omega_s}(K))$ in the right-hand side of (3.1) is replaced by $\frac{1}{2} \int_0^{2\pi} p_{\omega_s}^2(K, \theta) d\theta$. Applying this new equality to both K and $K \cup \{s_h\}$, we get that

$$(3.18) \quad \mathcal{A}(\Delta \mathcal{F}_{s_h}) = \frac{1}{2} \int_0^{2\pi} (p_{\omega_s}^2(K \cup \{s_h\}, \theta) - p_{\omega_s}^2(K, \theta)) d\theta + \int_0^{2\pi} \Delta p_{\omega_s}(\theta) \langle \omega_s, u_\theta \rangle d\theta$$

We claim that for $h > 0$ small enough,

$$(3.19) \quad \frac{1}{2} \int_0^{2\pi} (p_{\omega_s}^2(K \cup \{s_h\}, \theta) - p_{\omega_s}^2(K, \theta)) d\theta = \mathcal{A}(\Delta \tilde{\mathcal{F}}_{s_h}).$$

Indeed, for $h > 0$ small enough, the area $\mathcal{A}(\Delta \tilde{\mathcal{F}}_{s_h})$ stays the same when K is replaced by $K \cup \{\omega_s\}$. Moreover, on the left-hand side, we recall that $p_{\omega_s}(K \cup \{s_h\}, \theta) = p_{\omega_s}(K, \theta)$ as soon as $\theta \notin [-\theta_{s,h}^-, \theta_{s,h}^+]$ and when $\theta \in [-\theta_{s,h}^-, \theta_{s,h}^+]$, $p_{\omega_s}(K \cup \{s_h\}, \theta)$ and $p_{\omega_s}(K, \theta)$ stay the same when K is replaced by $K \cup \{\omega_s\}$. Consequently, we can proceed as if ω_s would belong to K , which allows us to derive (3.19) from (2.4). Combining now (3.18) and (3.19), we get (3.2) and then Lemma 3.1 (i) by repeating verbatim the rest of the arguments.

Proof of (ii). — Thanks to (i), the function which to any couple (s, h) associates either

$$\mathcal{A}(\Delta \mathcal{F}_{s_h}) h^{-\frac{3}{2}} \text{ if } h > 0 \text{ or } 2^{\frac{5}{2}} 3^{-1} r_s^{-\frac{1}{2}} \langle s, n_s \rangle$$

if $h = 0$ is continuous on the compact set $\partial K \times [0, 1]$ so it is bounded from below by a positive constant $C > 0$. When $h \geq 1$, we notice that the region $\Delta\mathcal{F}_{s_h}$ contains a disk of radius proportional to h , which means that there exists $C' > 0$ such that $\mathcal{A}(\Delta\mathcal{F}_{s_h}) \geq C'h^2 \geq C'h^{\frac{3}{2}}$. \square

3.2. Proof of Theorem 1.1 (i): the defect area

Proof of Theorem 1.1 (i). — Every $x \in \mathbb{R}^2 \setminus K$ can be written as $x = s + \lambda^{-\frac{2}{3}}hn_s = s_{\lambda^{-\frac{2}{3}}h}$ with $s \in \partial K$ and $h > 0$, the Jacobian of this change of variables being given by

$$\frac{dx}{dsdh} = \lambda^{-\frac{2}{3}} \left| 1 + \lambda^{-\frac{2}{3}}hr_s^{-1} \right|.$$

Thus we get from (2.3) that

$$\mathbb{E}(\mathcal{A}(K_\lambda^o)) - \mathcal{A}(K) = \lambda^{-\frac{2}{3}} \int_{\partial K} \int_0^\infty \exp\left(-4\lambda\mathcal{A}(\Delta\mathcal{F}_{s_{\lambda^{-2/3}h}})\right) \left| 1 + \lambda^{-\frac{2}{3}}hr_s^{-1} \right| dh ds.$$

Thanks to Lemma 3.1, we get, for $h > 0$ fixed,

$$4\lambda\mathcal{A}(\Delta\mathcal{F}_{s_{\lambda^{-2/3}h}}) \underset{\lambda \rightarrow \infty}{\sim} 4\left(\lambda^{-2/3}h\right)^{-\frac{3}{2}} \mathcal{A}(\Delta\mathcal{F}_{s_{\lambda^{-2/3}h}}) h^{\frac{3}{2}} \underset{\lambda \rightarrow \infty}{\sim} C_s h^{\frac{3}{2}}$$

where $C_s = 2^{\frac{9}{2}}3^{-1}r_s^{-\frac{1}{2}}\langle s, n_s \rangle$ and the existence of a constant $C > 0$ such that, for all $\lambda > 0$ and $s \in \partial K$,

$$4\lambda\mathcal{A}(\Delta\mathcal{F}_{s_{\lambda^{-2/3}h}}) \geq CC_s h^{\frac{3}{2}}.$$

Consequently, we can apply Lebesgue's dominated convergence theorem to obtain

$$\lambda^{\frac{2}{3}}(\mathbb{E}(\mathcal{A}(K_\lambda^o)) - \mathcal{A}(K)) \underset{\lambda \rightarrow \infty}{\sim} \int_{\partial K} \int_0^\infty \exp\left(-C_s h^{\frac{3}{2}}\right) dh ds = \frac{2}{3}\Gamma\left(\frac{2}{3}\right) \int_{\partial K} C_s^{-\frac{2}{3}} ds$$

which provides assertion (i) of Theorem 1.1. \square

3.3. Proof of Theorem 1.1 (ii): support points and defect perimeter

The strategy of the proofs is to apply Corollary 2.2 hence we only need now to explain how to estimate the mean defect support function in a fixed direction. To do so, let us introduce the support point $m_{s,\lambda}$ on ∂K_λ^o in direction n_s , i.e. the point which satisfies $\langle m_{s,\lambda}, n_s \rangle = p_o(K_\lambda^o, n_s)$. Denoting by $X_{s,\lambda} = \langle m_{s,\lambda} - s, t_s \rangle$ and $Y_{s,\lambda} = p_o(K_\lambda^o, n_s) - p_o(K, n_s)$, we can write $m_{s,\lambda} = s + X_{s,\lambda}t_s + Y_{s,\lambda}n_s$ where (t_s, n_s) stands for the Frenet frame at point s . Thus this subsection is devoted to investigate the asymptotic distribution of the couple $(X_{s,\lambda}, Y_{s,\lambda})$ and provide the required asymptotic estimate for $\mathbb{E}(Y_{s,\lambda})$. A key step in the proof of Proposition 3.3 will be the use of the following lemma providing a change of variables formula which may be understood as a classical formula *à la* Blaschke–Petkantschin (see e.g. [SW08, Theorem 7.3.1]). It consists essentially in the computation of the Jacobian of a four dimensional transformation. We omit the calculation which is analogous to the proof of the classical Blaschke–Petkantschin's formula (see the previous reference again).

LEMMA 3.2. — Let $x = ru_\theta$, $r > 0$, $\theta \in (0, 2\pi)$ and $\theta - \pi < \theta_1 < \theta_2 < \theta$. Let $x_1 = 2r \sin(\theta - \theta_1)u_{\theta_1 + \frac{\pi}{2}}$ and $x_2 = 2r \sin(\theta - \theta_2)u_{\theta_2 + \frac{\pi}{2}}$ be the symmetric points of the origin o with respect to the lines $x + \mathbb{R}u_{\theta_1}$ and $x + \mathbb{R}u_{\theta_2}$ respectively. Then the Jacobian of the change of variables $(r, \theta, \theta_1, \theta_2) \mapsto (x_1, x_2)$ is given by

$$\frac{dx_1 dx_2}{r dr d\theta d\theta_1 d\theta_2} = 16r^2 J(\theta, \theta_1, \theta_2)$$

with $J(\theta, \theta_1, \theta_2) = |\sin(\theta_1 - \theta_2) \sin(\theta - \theta_2) \sin(\theta - \theta_1)|$.

PROPOSITION 3.3. —

(i) For every $s \in \partial K$, the couple $(\lambda^{\frac{1}{3}} X_{s,\lambda}, \lambda^{\frac{2}{3}} Y_{s,\lambda})$ converges in distribution when $\lambda \rightarrow \infty$ to the distribution with density function f_s given by

$$f_s(x, y) = 2^{\frac{11}{2}} \langle s, n_s \rangle^2 r_s^{-\frac{3}{2}} \exp \left(-2^{\frac{9}{2}} 3^{-1} r_s^{-\frac{1}{2}} \langle s, n_s \rangle \left(\frac{x^2}{2r_s} + y \right)^{\frac{3}{2}} \right) \left(\frac{x^2}{2r_s} + y \right)^{\frac{1}{2}} y \mathbb{I}_{\{y > 0\}}.$$

(ii) There exists $C > 0$ such that for every $s \in \partial K$ and $\lambda > 0$, $\lambda^{\frac{2}{3}} \mathbb{E}(Y_{s,\lambda}) \leq C$. Moreover, for every $s \in \partial K$,

$$\mathbb{E}(Y_{s,\lambda}) = \mathbb{E}(p_o(K_\lambda^o, n_s) - p_o(K, n_s)) \underset{\lambda \rightarrow \infty}{\sim} \lambda^{-\frac{2}{3}} 3^{-\frac{4}{3}} \Gamma\left(\frac{2}{3}\right) r_s^{\frac{1}{3}} \langle s, n_s \rangle^{-\frac{2}{3}}.$$

Proof of Proposition 3.3.

Proof of (i). — We first notice that the point $m_{s,\lambda}$ is necessarily one of the vertices of K_λ^o , i.e. is at the intersection of two bisecting lines between o and two Voronoi neighbors of o . For $x_1, x_2 \in \mathcal{P}_\lambda \setminus 2\mathcal{F}_o(K)$, we denote by c_{x_1, x_2} the intersection point of the two bisecting lines of the segments $[o, x_1]$ and $[o, x_2]$. In particular,

$$(c_{x_1, x_2} = m_{s,\lambda}) \iff \begin{cases} c_{x_1, x_2} \text{ is extreme in direction } n_s \\ B_{\|c_{x_1, x_2}\|}(c_{x_1, x_2}) \cap (\mathcal{P}_\lambda \setminus 2\mathcal{F}_o(K)) = \emptyset \end{cases}$$

From a given $m_{s,\lambda}$ emanate two segments, one on the left of the half-line $\mathbb{R}_+ m_{s,\lambda}$, one on the right. The symmetric points of o with respect to these two segments define the right and the left Poisson–Voronoi neighbors of o with respect to $m_{s,\lambda}$. They will be denoted by $x^+(m_{s,\lambda})$ and $x^-(m_{s,\lambda})$ respectively. Consequently, by Mecke–Slivnyak’s formula, for every positive and measurable function $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}_+$,

$$\begin{aligned} & \mathbb{E} \left(\varphi \left(\lambda^{\frac{1}{3}} X_{s,\lambda}, \lambda^{\frac{2}{3}} Y_{s,\lambda} \right) \right) \\ &= \mathbb{E} \left(\sum_{(x_1, x_2) \in (\mathcal{P}_\lambda \setminus 2\mathcal{F}_o(K))^2} \mathbb{I}_{\left\{ \begin{array}{l} c_{x_1, x_2} = m_{s,\lambda} \\ x_1 = x^+(m_{s,\lambda}) \\ x_2 = x^-(m_{s,\lambda}) \end{array} \right\}} \varphi \left(\lambda^{\frac{1}{3}} \langle c_{x_1, x_2} - s, t_s \rangle, \lambda^{\frac{2}{3}} \langle c_{x_1, x_2} - s, n_s \rangle \right) \right) \\ &= \lambda^2 \int_{\mathbb{R}^2 \times \mathbb{R}^2} \exp \left(-4\lambda \mathcal{A} \left(B_{\frac{\|c_{x_1, x_2}\|}{2}} \left(\frac{c_{x_1, x_2}}{2} \right) \setminus \mathcal{F}_o(K) \right) \right) \mathbb{I}_{\left\{ \begin{array}{l} c_{x_1, x_2} \text{ is extreme in direction } n_s \\ x_1 = x^+(c_{x_1, x_2}), x_2 = x^-(c_{x_1, x_2}) \end{array} \right\}} \\ & \quad \times \varphi \left(\lambda^{\frac{1}{3}} \langle c_{x_1, x_2} - s, t_s \rangle, \lambda^{\frac{2}{3}} \langle c_{x_1, x_2} - s, n_s \rangle \right) dx_1 dx_2. \end{aligned}$$

We now apply two consecutive changes of variables in the integral above.

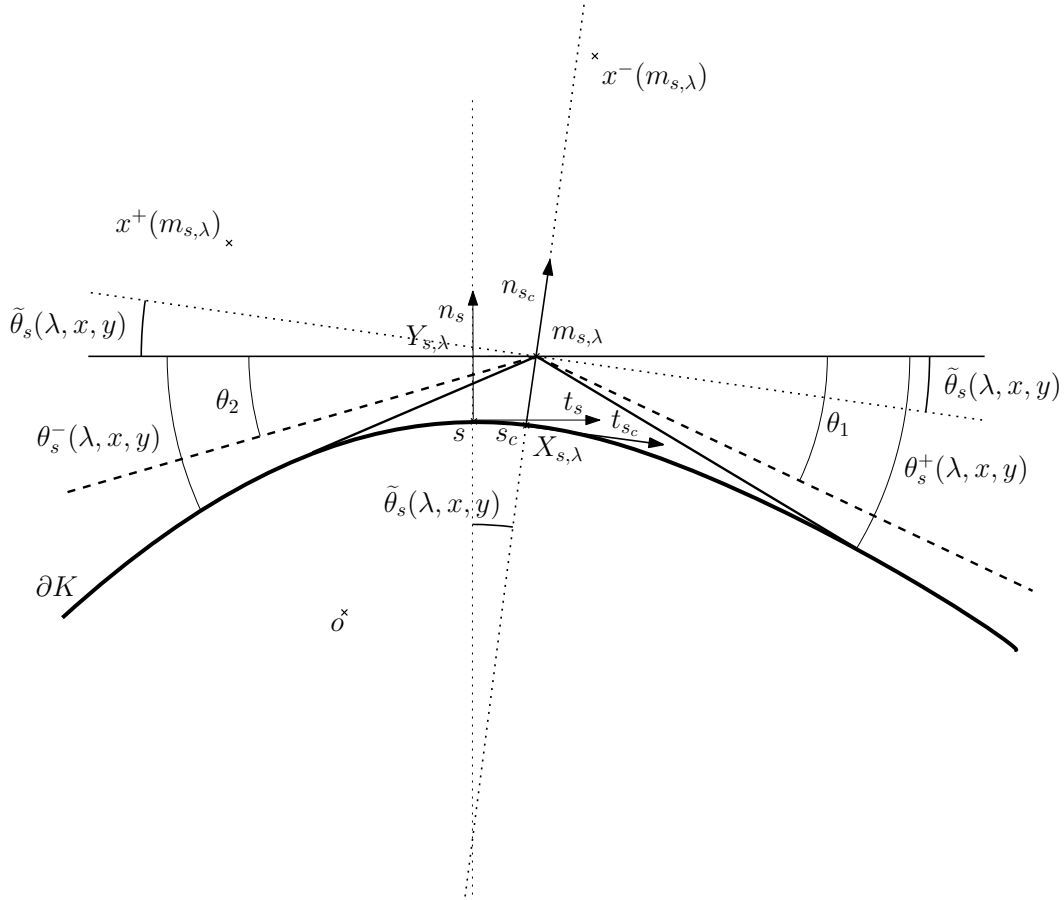


Figure 3.2. Realisation of the support function: $m_{s,\lambda}$ denotes the support point in the direction n_s .

First, we write $c_{x_1,x_2} = ru_\theta$ and denote by θ_1 and θ_2 the angles between one of the two bisecting lines emanating from c_{x_1,x_2} and t_s . We then use Lemma 3.2 to compute the corresponding Jacobian.

Secondly, we replace the couple (r, θ) by (x, y) defined by

$$x = \lambda^{\frac{1}{3}} \langle c_{x_1,x_2} - s, t_s \rangle \quad \text{and} \quad y = \lambda^{\frac{2}{3}} \langle c_{x_1,x_2} - s, n_s \rangle.$$

We get in particular

$$(3.20) \quad \begin{cases} r &= \rho_s(\lambda, x, y) = \left(\left(\langle s, t_s \rangle + \lambda^{-\frac{1}{3}} x \right)^2 + \left(\langle s, n_s \rangle + \lambda^{-\frac{2}{3}} y \right)^2 \right)^{\frac{1}{2}} \\ \theta &= \theta_s(\lambda, x, y) = \arccos \left(\left\langle \frac{s + \lambda^{-\frac{1}{3}} x t_s + \lambda^{-\frac{2}{3}} y n_s}{\|s + \lambda^{-\frac{1}{3}} x t_s + \lambda^{-\frac{2}{3}} y n_s\|}, t_s \right\rangle \right) \end{cases}$$

where ρ_s and θ_s are two functions of λ , x and y , and a Jacobian given by

$$\frac{r dr d\theta}{dx dy} = \lambda^{-1}.$$

Consequently, we deduce that

$$(3.21) \quad \mathbb{E} \left(\varphi \left(\lambda^{\frac{1}{3}} X_{s,\lambda}, \lambda^{\frac{2}{3}} Y_{s,\lambda} \right) \right) \\ = 16 \int_{\mathbb{R} \times \mathbb{R}_+} \exp(-\Delta_s(\lambda, x, y)) \varphi(x, y) \rho_s^2(\lambda, x, y) J_s^{\text{supp}}(\lambda, x, y) dx dy$$

where

$$\Delta_s(\lambda, x, y) = 4\lambda \mathcal{A} \left(B_{\frac{\rho_s}{2}} \left(\frac{\rho_s u_{\theta_s}}{2} \right) \setminus \mathcal{F}_o(K) \right)$$

and

$$J_s^{\text{supp}}(\lambda, x, y) = \lambda \int_{E_{s,x,y}^{\text{supp}}} J(\theta_s(\lambda, x, y), \theta_1, \theta_2) d\theta_1 d\theta_2$$

where $E_{s,x,y}^{\text{supp}}$ stands for the set of couples (θ_1, θ_2) which satisfy that c_{x_1, x_2} is extremal in the direction of n_s and that the two bisecting lines of $[o, x_1]$ and $[o, x_2]$ do not intersect K .

Let us make the set $E_{s,x,y}^{\text{supp}}$ explicit. Let s_c be the orthogonal projection of c_{x_1, x_2} onto K and t_{s_c} be the unit outer normal vector of ∂K at s_c . We introduce the angles $\theta_s^+(\lambda, x, y)$, $\theta_s^-(\lambda, x, y)$ and $\tilde{\theta}_s(\lambda, x, y)$ as, respectively, the angle of aperture at the point $c_{x_1, x_2} = s + \lambda^{-\frac{1}{3}} x t_s + \lambda^{-\frac{2}{3}} y n_s$ on the right, the angle of aperture at c_{x_1, x_2} on the left and the angle between the vectors n_s and n_{s_c} (see Figure 3.2). We obtain

$$(3.22) \quad (\theta_1, \theta_2) \in E_{s,x,y}^{\text{supp}} \iff -\theta_s^+(\lambda, x, y) - \tilde{\theta}_s(\lambda, x, y) \\ < \theta_1 < 0 < \theta_2 < \theta_s^-(\lambda, x, y) - \tilde{\theta}_s(\lambda, x, y).$$

In order to show the required convergence in distribution, we are going to use Lebesgue's dominated convergence theorem. To do so, we need to prove the convergence of the integrand in (3.21) and that it is dominated.

– *Convergence and domination of $\rho_s(\lambda, x, y)$.*

We deduce from (3.20) that

$$(3.23) \quad \rho_s(\lambda, x, y) \xrightarrow{\lambda \rightarrow \infty} \|s\|.$$

Moreover, by triangular inequality, we get for all $\lambda \geq 1$,

$$(3.24) \quad \rho_s(\lambda, x, y) \leq \|s\| + \left(\lambda^{-\frac{2}{3}} x^2 + \lambda^{-\frac{4}{3}} y^2 \right)^{\frac{1}{2}} \leq \|s\| + \|(x, y)\|.$$

– *Convergence and domination of $\exp(-\Delta_s(\lambda, x, y))$.*

We denote by h the distance from $c_{x_1, x_2} = s + \lambda^{-\frac{1}{3}} x t_s + \lambda^{-\frac{2}{3}} y n_s$ to K . Then the following relation holds, uniformly in s ,

$$(3.25) \quad h \underset{\lambda \rightarrow \infty}{=} \lambda^{-\frac{2}{3}} \left(\frac{x^2}{2r_s} + y \right) + O(\lambda^{-\frac{4}{3}}).$$

Moreover, there exists $C > 0$ such that for every $s \in \partial K$ and all $\lambda \geq 1$,

$$(3.26) \quad C^{-1} \min \left(\lambda^{-\frac{1}{3}} (x^2 + 2y r_s)^{\frac{1}{2}}, \lambda^{-\frac{2}{3}} \left(\frac{x^2}{r_s} + 2y \right) \right) \leq h \leq C \lambda^{-\frac{2}{3}} \left(\frac{x^2}{2r_s} + y + \frac{y^2}{2r_s} \right).$$

Indeed, let us prove first (3.25) and (3.26) when K is a disk of radius r_s . We find on the one hand by Pythagoras' theorem

$$\|c_{x_1, x_2}\|^2 - r_s^2 = \left(r_s + \lambda^{-\frac{2}{3}}y\right)^2 + \lambda^{-\frac{2}{3}}x^2 - r_s^2 = \lambda^{-\frac{2}{3}}\left(x^2 + \lambda^{-\frac{2}{3}}y^2 + 2r_sy\right)$$

and on the other hand

$$\|c_{x_1, x_2}\|^2 - r_s^2 = (\|c_{x_1, x_2}\| - r_s)(\|c_{x_1, x_2}\| + r_s) = h(2r_s + h).$$

Combining these two equalities, we get

$$h = r_s \left(1 + \lambda^{-\frac{2}{3}} \left(\frac{x^2}{r_s^2} + \frac{2y}{r_s} + \lambda^{-\frac{2}{3}} \frac{y^2}{r_s^2}\right)\right)^{\frac{1}{2}} - r_s.$$

Using the estimate $\frac{1}{4} \min(u, u^{\frac{1}{2}}) \leq (1+u)^{\frac{1}{2}} - 1 \leq \frac{u}{2}$ for every $u > 0$ we obtain that the previous equality implies both (3.25) and (3.26) when K is a disk.

When K is a smooth convex body, we sandwich its boundary between two disks of radii $r_s + C'\lambda^{-\frac{2}{3}}$ and $r_s - C'\lambda^{-\frac{2}{3}}$ for a fixed positive constant $C' > 0$ and we deduce from the previous case both (3.25) and (3.26) for K .

Moreover, using the regularity assumptions on the boundary ∂K , we get, uniformly in s ,

$$(3.27) \quad |\langle s_c, n_{s_c} \rangle - \langle s, n_s \rangle| \underset{\lambda \rightarrow \infty}{=} O\left(\lambda^{-\frac{1}{3}}\right).$$

Thanks to Lemma 3.1, (3.25) and (3.27), we get, uniformly in s ,

$$(3.28) \quad \Delta_s(\lambda, x, y) \underset{\lambda \rightarrow \infty}{\sim} 2^{\frac{3}{2}} 3^{-1} r_s^{-\frac{1}{2}} \langle s, n_s \rangle \left(\frac{x^2}{2r_s} + y\right)^{\frac{3}{2}}.$$

In particular, thanks to Lemma 3.1(ii) and (3.26), there exists $C > 0$, uniform in s , such that $\Delta_s(x, y)$ satisfies, for all $\lambda \geq 1$:

$$(3.29) \quad \Delta_s(\lambda, x, y) \geq C \min \left((x^2 + 2r_sy)^{\frac{3}{4}}, \left(\frac{x^2}{2r_s} + y\right)^{\frac{3}{2}} \right).$$

– *Convergence and domination of $J_s^{\text{supp}}(\lambda, x, y)$.*

We start by estimating the function $\theta_s(\lambda, x, y)$. We get

$$(3.30) \quad \theta_s(\lambda, x, y) \underset{\lambda \rightarrow \infty}{=} \arcsin \left(\left\langle \frac{s}{\|s\|}, n_s \right\rangle \right) + O\left(\lambda^{-\frac{1}{3}}\right).$$

We now estimate the three angles $\theta_s^+(\lambda, x, y)$, $\theta_s^-(\lambda, x, y)$ and $\tilde{\theta}_s(\lambda, x, y)$. Using (3.6) and (3.25), we get

$$(3.31) \quad \theta_s^+(\lambda, x, y) \underset{\lambda \rightarrow \infty}{=} \theta_s^-(\lambda, x, y) + o\left(\lambda^{-\frac{1}{3}}\right) \underset{\lambda \rightarrow \infty}{=} \lambda^{-\frac{1}{3}} 2^{\frac{1}{2}} r_s^{-\frac{1}{2}} \left(\frac{x^2}{2r_s} + y\right)^{\frac{1}{2}} + o\left(\lambda^{-\frac{1}{3}}\right).$$

Thanks to (3.7) and (3.26), we have additionally the inequality, for some $C > 0$,

$$(3.32) \quad \theta_s^+(\lambda, x, y) \leq 2^{\frac{1}{2}} r_s^{-\frac{1}{2}} h^{\frac{1}{2}} \leq C r_s \lambda^{-\frac{1}{3}} \left(\frac{x^2}{2r_s} + y + \frac{y^2}{2r_s}\right)^{\frac{1}{2}}.$$

The same inequality holds for $\theta_s^+(\lambda, x, y)$. We turn now our attention to $\tilde{\theta}_s(\lambda, x, y)$. When K is a disk we get

$$\tilde{\theta}_s(\lambda, x, y) = \arctan \left(\frac{\lambda^{-\frac{1}{3}} x}{r_s + \lambda^{-\frac{2}{3}} y} \right) \underset{\lambda \rightarrow \infty}{=} \lambda^{-\frac{1}{3}} r_s^{-1} x + o \left(\lambda^{-\frac{1}{3}} \right).$$

When K is a smooth convex body, we sandwich again its boundary between two disks of radii $r_s + C\lambda^{-\frac{2}{3}}$ and $r_s - C\lambda^{-\frac{2}{3}}$ and we obtain

$$\tilde{\theta}_s(\lambda, x, y) \underset{\lambda \rightarrow \infty}{=} \lambda^{-\frac{1}{3}} r_s^{-1} x + o \left(\lambda^{-\frac{1}{3}} \right).$$

Consequently, we deduce from (3.30), (3.22) and (3.31) that

$$\begin{aligned} & J_s^{\text{supp}}(\lambda, x, y) \\ & \underset{\lambda \rightarrow \infty}{\sim} \lambda \int_{E_{s,x,y}^{\text{supp}}} |\sin(\theta_2 - \theta_1) \sin(\theta_s(\lambda, x, y) - \theta_2) \sin(\theta_s(\lambda, x, y) - \theta_1)| d\theta_1 d\theta_2 \\ & \underset{\lambda \rightarrow \infty}{\sim} \lambda \sin^2 \left(\arcsin \left(\left\langle \frac{s}{\|s\|}, n_s \right\rangle \right) \right) \\ & \quad \cdot \int_{\theta_1 = -\theta_s^+(\lambda, x, y) - \tilde{\theta}_s(\lambda, x, y)}^0 \int_{\theta_2 = 0}^{\theta_s^-(\lambda, x, y) - \tilde{\theta}_s(\lambda, x, y)} (\theta_2 - \theta_1) d\theta_1 d\theta_2 \\ & \underset{\lambda \rightarrow \infty}{\sim} \lambda \left\langle \frac{s}{\|s\|}, n_s \right\rangle^2 \theta_s^+(\lambda, x, y) \left(\theta_s^+(\lambda, x, y)^2 - \tilde{\theta}_s(\lambda, x, y)^2 \right) \\ (3.33) \quad & \underset{\lambda \rightarrow \infty}{\sim} \left\langle \frac{s}{\|s\|}, n_s \right\rangle^2 2^{\frac{3}{2}} r_s^{-\frac{3}{2}} y \left(\frac{x^2}{2r_s} + y \right)^{\frac{1}{2}}. \end{aligned}$$

Moreover, thanks to (3.32), we get for some $C > 0$,

$$(3.34) \quad J_s^{\text{supp}}(\lambda, x, y) \leq C r_s^3 \left(\frac{x^2}{2r_s} + y + \frac{y^2}{2r_s} \right)^{\frac{3}{2}}.$$

– *Conclusion.*

Combining (3.23), (3.28) and (3.33), we obtain that the integrand in (3.21) converges to

$$8\|s\|^2 \varphi(x, y) \exp \left(-2^{-\frac{9}{2}} 3^{-1} r_s^{-\frac{1}{2}} \langle s, n_s \rangle \left(\frac{x^2}{2r_s} + y \right)^{\frac{3}{2}} \right) \left\langle \frac{s}{\|s\|}, n_s \right\rangle^2 2^{\frac{3}{2}} r_s^{-\frac{3}{2}} y \left(\frac{x^2}{2r_s} + y \right)^{\frac{1}{2}}.$$

Now the estimates (3.24), (3.29) and (3.34) show that we can apply Lebesgue's dominated convergence theorem for any function φ bounded by a polynomial of x and y , say. This proves assertion (i) of Proposition 3.3.

Proof of (ii). — We start by rewriting (3.21) when $\varphi(x, y) = y$:

$$(3.35) \quad \lambda^{\frac{2}{3}} \mathbb{E}(Y_{s,\lambda}) = 16\lambda^{\frac{2}{3}} \int_{\mathbb{R} \times \mathbb{R}_+} \exp(-\Delta_s(\lambda, x, y)) \rho_s^2(\lambda, x, y) J_s^{\text{supp}}(\lambda, x, y) y dx dy.$$

Thanks to (i) applied to $\varphi(x, y) = y$, we can apply Lebesgue's dominated convergence theorem to get that

$$\lambda^{\frac{2}{3}} \mathbb{E}(Y_{s,\lambda}) \xrightarrow{\lambda \rightarrow \infty} 2^{\frac{11}{2}} r_s^{-\frac{3}{2}} \langle s, n_s \rangle^2 I_s$$

where

$$I_s = \int_{\mathbb{R} \times \mathbb{R}_+} \exp \left(-2^{\frac{9}{2}} 3^{-1} r_s^{-\frac{1}{2}} \langle s, n_s \rangle \left(\frac{x^2}{2r_s} + y \right)^{\frac{3}{2}} \right) \left(\frac{x^2}{2r_s} + y \right)^{\frac{1}{2}} y^2 dx dy.$$

It remains to make the integral I_s explicit. Recalling that $C_s = 2^{\frac{9}{2}} 3^{-1} r_s^{-\frac{1}{2}} \langle s, n_s \rangle$, we get

$$I_s = \frac{4}{3C_s} \int_0^\infty \left(\int_0^\infty e^{-u} \left(\frac{u^{\frac{2}{3}}}{C_s^{\frac{2}{3}}} - \frac{x^2}{2r_s} \right)^2 \mathbb{I}_{\left\{0 < x < 2^{\frac{1}{2}} r_s^{\frac{1}{2}} C_s^{-\frac{1}{3}} u^{\frac{1}{3}}\right\}} dx \right) du = I_s^{(1)} + I_s^{(2)} - I_s^{(3)}$$

where

$$I_s^{(1)} = \frac{4}{3C_s} \int_0^\infty e^{-u} C_s^{-\frac{4}{3}} u^{\frac{4}{3}} 2^{\frac{1}{2}} r_s^{\frac{1}{2}} C_s^{-\frac{1}{3}} u^{\frac{1}{3}} du = 2^{\frac{5}{2}} 3^{-2} 5 C_s^{-\frac{8}{3}} r_s^{\frac{1}{2}} \Gamma\left(\frac{5}{3}\right),$$

$$I_s^{(2)} = \frac{4}{3C_s} \int_0^\infty e^{-u} 20^{-1} r_s^{-2} \left(2^{\frac{1}{2}} r_s^{\frac{1}{2}} C_s^{-\frac{1}{3}} u^{\frac{1}{3}} \right)^5 du = 2^{\frac{5}{2}} 3^{-2} C_s^{-\frac{8}{3}} r_s^{\frac{1}{2}} \Gamma\left(\frac{5}{3}\right),$$

and

$$I_s^{(3)} = \frac{4}{3C_s} \int_0^\infty e^{-u} C_s^{-\frac{2}{3}} r_s^{-1} 3^{-1} u^{\frac{2}{3}} \left(2^{\frac{1}{2}} r_s^{\frac{1}{2}} C_s^{-\frac{1}{3}} u^{\frac{1}{3}} \right)^3 du = 2^{\frac{7}{2}} 3^{-3} 5 C_s^{-\frac{8}{3}} r_s^{\frac{1}{2}} \Gamma\left(\frac{5}{3}\right).$$

Finally, inserting these equalities into I_s yields the required result. \square

Proof of Theorem 1.1(ii). — We go back to the exact formula (3.35). Since K is a compact convex set with bounded positive curvature and containing the origin in its interior, the non-negative quantities $\|s\|$ and r_s are bounded from above and from below. Consequently, the estimates (3.24), (3.29) and (3.34) imply that the integral on the right-hand side of (3.35) is bounded independently of s , i.e. that there exists $C > 0$ such that $\lambda^{\frac{2}{3}} \mathbb{E}(Y_{s,\lambda}) \leq C$ for every $s \in \partial K$.

Consequently, we use Proposition 3.3(ii) and Lebesgue's dominated convergence theorem to get that

$$\int_{\partial K} \mathbb{E}(Y_{s,\lambda}) r_s^{-1} ds \underset{\lambda \rightarrow \infty}{\sim} \lambda^{-\frac{2}{3}} 3^{-\frac{4}{3}} \Gamma\left(\frac{2}{3}\right) \int_{\partial K} r_s^{-\frac{2}{3}} \langle s, n_s \rangle^{-\frac{2}{3}} ds.$$

The proof is completed thanks to Corollary 2.2(i). \square

3.4. Proof of Theorem 1.1(iii): intensity and number of vertices

Proof of Theorem 1.1(iii). — Again, thanks to the convergence from Proposition 3.3(ii) and Lebesgue's dominated convergence theorem, we obtain that

$$4\lambda \int_{\partial K} \langle s, n_s \rangle \mathbb{E}(Y_{s,\lambda}) r_s^{-1} ds \underset{\lambda \rightarrow \infty}{\sim} \lambda^{\frac{1}{3}} 2^{\frac{2}{3}} 3^{-\frac{4}{3}} \Gamma\left(\frac{2}{3}\right) \int_{\partial K} r_s^{-\frac{2}{3}} \langle s, n_s \rangle^{\frac{1}{3}} ds.$$

Looking at Corollary 2.2(ii), we observe that it is enough to show that

$$(3.36) \quad \lim_{\lambda \rightarrow \infty} \lambda^{-\frac{1}{3}} \mathcal{R}_\lambda = \lim_{\lambda \rightarrow \infty} 2\lambda^{\frac{2}{3}} \int_{\partial K} \mathbb{E}(Y_{s,\lambda}^2) r_s^{-1} ds = 0.$$

Applying (3.21) to $\varphi(x, y) = y^2$ and using the estimates (3.24), (3.29) and (3.34), we get that there exists $C > 0$ such that $\lambda^{\frac{4}{3}} \mathbb{E}(Y_{s,\lambda}^2) \leq C$ for every $s \in \partial K$. This implies (3.36) and completes the proof of Theorem 1.1 (iii). \square

Actually, we can provide a more precise result on the asymptotic intensity of the point process of vertices of K_λ^o . This new result that we describe below could alternatively be used to get Theorem 1.1 (iii) via an integration of the intensity given in the next proposition.

PROPOSITION 3.4. — *Let $s \in \partial K$ and consider the point process (x_v, y_v) of the Cartesian coordinates of the vertices $v \in V_\lambda$ of K_λ^o in the Cartesian orthonormal frame (s, t_s, n_s) . Let A be a bounded Borel set of $\mathbb{R}^2 \setminus K$ and denote by $\mathcal{N}_s(A)$ the number of points of the rescaled point process*

$$\left(\lambda^{\frac{1}{3}} x_v, \lambda^{\frac{2}{3}} y_v \right)_{v \in V_\lambda}$$

belonging to the set A . Then

$$\mathbb{E}(\mathcal{N}_s(A)) \xrightarrow{\lambda \rightarrow \infty} \int_A \sigma_s(x, y) dx dy$$

where

$$\sigma_s(x, y) = 2^{\frac{15}{2}} 3^{-1} \exp \left(-2^{\frac{9}{2}} 3^{-1} \left(\frac{x^2}{2r_s} + y \right)^{\frac{3}{2}} r_s^{-\frac{1}{2}} \langle s, n_s \rangle \right) \langle s, n_s \rangle^2 r_s^{-\frac{3}{2}} \left(\frac{x^2}{2r_s} + y \right)^{\frac{3}{2}}.$$

Proof of Proposition 3.4. — The strategy is very similar to the proof of Proposition 3.3, i.e. it consists in applying Mecke–Slivnyak’s formula, then the change of variables provided by Lemma 3.2 and finally Lebesgue’s dominated convergence theorem.

Consequently, we deduce that

$$(3.37) \quad \mathbb{E}(\mathcal{N}_s(A)) = 16 \int_A \exp(-\Delta_s(\lambda, x, y)) \rho_s^2(\lambda, x, y) J_s^{\text{vert}}(\lambda, x, y) dx dy$$

where

$$J_s^{\text{vert}}(\lambda, x, y) = \lambda \int_{E_{s,x,y}^{\text{vert}}} J(\theta_s(\lambda, x, y), \theta_1, \theta_2) d\theta_1 d\theta_2$$

and $E_{s,x,y}^{\text{vert}}$ is the set of couples (θ_1, θ_2) which satisfy that the two bisecting lines of $[o, x_1]$ and $[o, x_2]$ do not intersect K .

Let us make the set $E_{s,x,y}^{\text{vert}}$ explicit:

$$(3.38) \quad (\theta_1, \theta_2) \in E_{s,x,y}^{\text{vert}} \iff -\theta_s^+(\lambda, x, y) < \theta_1 < \theta_2 < \theta_s^-(\lambda, x, y).$$

The convergence and domination of $\rho_s(\lambda, x, y)$, $\exp(-\Delta_s(\lambda, x, y))$, $\theta_s^+(\lambda, x, y)$ and $\theta_s^-(\lambda, x, y)$ is identical to what has been done in the proof of Proposition 3.3. We

turn our attention to the convergence of $J_s^{\text{vert}}(\lambda, x, y)$.

$$\begin{aligned}
 & J_s^{\text{vert}}(\lambda, x, y) \\
 & \underset{\lambda \rightarrow \infty}{\sim} \lambda \int_{E_{s,x,y}^{\text{vert}}} |\sin(\theta_1 - \theta_2) \sin(\theta_s(\lambda, x, y) - \theta_2) \sin(\theta_s(\lambda, x, y) - \theta_1)| d\theta_1 d\theta_2 \\
 & \underset{\lambda \rightarrow \infty}{\sim} \lambda \sin^2 \left(\arcsin \left(\left\langle \frac{s}{\|s\|}, n_s \right\rangle \right) \right) \int_{\theta_1 = -\theta_s^+(\lambda, x, y)}^{\theta_s^-(\lambda, x, y)} \int_{\theta_2 = \theta_1}^{\theta_s^-(\lambda, x, y)} (\theta_2 - \theta_1) d\theta_2 d\theta_1 \\
 & \underset{\lambda \rightarrow \infty}{\sim} \frac{4}{3} \lambda \left\langle \frac{s}{\|s\|}, n_s \right\rangle^2 \theta_s^+(\lambda, x, y)^3 \\
 (3.39) \quad & \underset{\lambda \rightarrow \infty}{\sim} 2^{\frac{7}{2}} 3^{-1} \left\langle \frac{s}{\|s\|}, n_s \right\rangle^2 r_s^{-\frac{3}{2}} \left(\frac{x^2}{2r_s} + y \right)^{\frac{3}{2}}.
 \end{aligned}$$

Moreover, thanks to (3.32), we get for some positive constant $C > 0$,

$$(3.40) \quad J_s^{\text{vert}}(\lambda, x, y) \leq C \left\langle \frac{s}{\|s\|}, n_s \right\rangle^2 r_s^{-\frac{3}{2}} \left(\frac{x^2}{2r_s} + y \right)^{\frac{3}{2}}.$$

Combining (3.23), (3.28) and (3.39), we obtain that the integrand in (3.37) converges to

$$2^{\frac{15}{2}} 3^{-1} \exp \left(-2^{\frac{9}{2}} 3^{-1} \left(\frac{x^2}{2r_s} + y \right)^{\frac{3}{2}} r_s^{-\frac{1}{2}} \langle s, n_s \rangle \right) \langle s, n_s \rangle^2 r_s^{-\frac{3}{2}} \left(\frac{x^2}{2r_s} + y \right)^{\frac{3}{2}}.$$

Now the estimates (3.24), (3.29) and (3.40) show that we can apply Lebesgue's dominated convergence theorem. The result follows. \square

Proposition 3.4 provides an extra valuable information on the point process of vertices which is clearly not of Poisson type. To some extent, this is also reminiscent of the description of the rescaled point process of vertices of random polytopes in the unit-ball or random Gaussian polytopes, as a sub-product of a growth parabolic process based on a Poisson point process, see e.g. [CSY13, Theorem 4.1]. We skip the proof of Proposition 3.4 as it is in the same spirit as the proof of Proposition 3.3.

4. The polygonal case

In this section K is a convex polygon with vertices a_1, a_2, \dots, a_{n_K} containing the origin o in its interior. We denote by α_i the interior angle at vertex a_i and we recall that o_i is the orthogonal projection of o onto the line (a_i, a_{i+1}) . A point outside K will be located by its polar coordinates from one vertex a_i (see Figure 4.1), i.e. we consider a point $s_{a_i, \rho, \alpha} = a_i + \rho u_{\pi - \alpha}$ in the neighborhood of a_i , with $\rho > 0$ and $\alpha \in (0, \alpha_i)$.

The different proofs of the results of this Section will require to decompose the set $\mathbb{R}^2 \setminus K$ into several regions, namely n_K cones above the vertices of K and n_K strips above the edges of K . More precisely, for every $1 \leq i \leq n_K$, let us define the cones

$$\mathcal{G}_i = \left\{ s_{a_i, \rho, \alpha} : \rho > 0 \text{ and } \alpha \in \left(\frac{\pi}{2}, \frac{3\pi}{2} - \alpha_i \right) \right\}$$

and the strip

$$\mathcal{S}_i = \{s_{a_i, \rho, \alpha} : \rho > 0 \text{ and } \rho \cos \alpha \in (0, \|a_{i+1} - a_i\|)\}$$

as the connected component of $\mathbb{R}^2 \setminus (K \cup (\cup_{i=1}^{n_K} \mathcal{G}_i))$ with (a_i, a_{i+1}) on its boundary.

4.1. Increase of the area of the Voronoi flower

First, the following geometric lemma provides accurate estimates of the area of the set

$$\Delta \mathcal{F}_{a_i, \rho, \alpha} = \mathcal{F}_o(K \cup \{s_{a_i, \rho, \alpha}\}) \setminus \mathcal{F}_o(K).$$

LEMMA 4.1. — Assume that K is a convex polygon and let $a_i \in \partial K$, $1 \leq i \leq n_K$, be a fixed vertex of K .

(i) We get, uniformly in $\rho > 0$,

$$\mathcal{A}(\Delta \mathcal{F}_{a_i, \rho, \alpha}) \mathbb{I}_{\{s_{a_i, \rho, \alpha} \in \mathcal{S}_i\}} \underset{\alpha \rightarrow 0}{\sim} \alpha^2 \frac{\|o_i\|}{2} \frac{\rho \|a_{i+1} - a_i\|}{\|a_{i+1} - a_i\| - \rho}.$$

(ii) Moreover, there exists $C > 0$ such that, for all (ρ, α) such that $s_{a_i, \rho, \alpha}$ belongs to $\mathcal{G}_i \cup \mathcal{S}_i$,

$$\mathcal{A}(\Delta \mathcal{F}_{a_i, \rho, \alpha}) \geq C \max(1, \rho) \rho \alpha^2.$$

Proof of Lemma 4.1.

Proof of (i). — For α small enough, the set $\Delta \mathcal{F}_{a_i, \rho, \alpha}$ is nothing but

$$\Delta \mathcal{F}_{a_i, \rho, \alpha} = B_{\frac{1}{2}\|s_{a_i, \rho, \alpha}\|} \left(\frac{1}{2} s_{a_i, \rho, \alpha} \right) \setminus \left(B_{\frac{1}{2}\|a_i\|} \left(\frac{1}{2} a_i \right) \cup B_{\frac{1}{2}\|a_{i+1}\|} \left(\frac{1}{2} a_{i+1} \right) \right),$$

that is a curvilinear triangle with vertices o_i , a'_i and a'_{i+1} where a'_i and a'_{i+1} are respectively the intersection of

$$\partial B_{\frac{1}{2}\|s_{a_i, \rho, \alpha}\|} \left(\frac{1}{2} s_{a_i, \rho, \alpha} \right) \text{ with } \partial B_{\frac{1}{2}\|a_i\|} \left(\frac{1}{2} a_i \right) \text{ and } \partial B_{\frac{1}{2}\|a_{i+1}\|} \left(\frac{1}{2} a_{i+1} \right)$$

(see Figure 4.1).

We aim at computing estimates for the area $\mathcal{A}(\Delta \mathcal{F}_{a_i, \rho, \alpha})$ of this curvilinear triangle. To do this, we split it into the curvilinear triangles with vertices o_i , $s'_{a_i, \rho, \alpha}$, a'_i and o_i , $s'_{a_i, \rho, \alpha}$, a'_{i+1} respectively, where $s'_{a_i, \rho, \alpha}$ is the intersection of the line (o, o_i) with the circle

$$\partial B_{\frac{1}{2}\|s_{a_i, \rho, \alpha}\|} \left(\frac{1}{2} s_{a_i, \rho, \alpha} \right).$$

Let us focus on the first curvilinear triangle. As $\alpha \rightarrow 0$, it tends to a straight triangle whose area is given by

$$\mathcal{A}(o_i s'_{a_i, \rho, \alpha} a'_i) \underset{\alpha \rightarrow 0}{\sim} \frac{1}{2} \|s'_{a_i, \rho, \alpha} - o_i\| \left(\left\| \widehat{a'_i o_i} \right\| \sin \beta_i \right)$$

where β_i is the angle between the line $(o_i, s'_{a_i, \rho, \alpha})$ and the tangent line to the disk $B_{\frac{1}{2}\|a_i\|} \left(\frac{1}{2} a_i \right)$ at o_i .

Observe first that we get by symmetry $\|s'_{a_i, \rho, \alpha} - o_i\| = \rho \sin \alpha$.

Therefore,

$$\mathcal{A}(o_i s'_{a_i, \rho, \alpha} a'_{i+1}) \underset{\alpha \rightarrow 0}{\sim} \frac{1}{2} \frac{\rho^2 \alpha^2 \|o_i\|}{\|a_{i+1} - a_i\| - \rho}.$$

Finally, summing the areas of each triangle, we obtain

$$\mathcal{A}(\Delta \mathcal{F}_{a_i, \rho, \alpha}) \underset{\alpha \rightarrow 0}{\sim} \frac{1}{2} \rho \alpha^2 \|o_i\| \left(1 + \frac{\rho}{\|a_{i+1} - a_i\| - \rho} \right) \underset{\alpha \rightarrow 0}{\sim} \frac{1}{2} \rho \alpha^2 \|o_i\| \frac{\|a_{i+1} - a_i\|}{\|a_{i+1} - a_i\| - \rho}$$

so that (i) holds.

Proof of (ii). — We first assume that $s_{a_i, \rho, \alpha} \in \mathcal{S}_i$. Because of point (i), there exists $\alpha_0 \in (0, \frac{\pi}{2})$ such that for every $\alpha < \alpha_0$, $\mathcal{A}(\Delta \mathcal{F}_{a_i, \rho, \alpha}) \geq \frac{\|o_i\|}{4} \rho \alpha^2$ and $\rho \leq \frac{\|a_{i+1} - a_i\|}{\cos \alpha_0}$. This proves the result as soon as $\alpha < \alpha_0$. When $\alpha > \alpha_0$, it is enough to show that $\mathcal{A}(\Delta \mathcal{F}_{a_i, \rho, \alpha}) \geq C \max(\rho, \rho^2)$ for some positive constant C . This last inequality comes now from the fact that $\Delta \mathcal{F}_{a_i, \rho, \alpha}$ contains both a disk of radius proportional to ρ and an angular sector with thickness ρ and constant angular width. Finally, the exact same argument holds when $s_{a_i, \rho, \alpha} \in \mathcal{G}_i$ so this completes the proof. \square

4.2. Proof of Theorem 1.2(i): the defect area

Proof of Theorem 1.2(i). — Recalling the notation of Section 4.1 and using (2.3), we can write $\mathbb{E}(\mathcal{A}(K_\lambda^o)) - \mathcal{A}(K)$ as

$$\sum_{i=1}^{n_K} \left(\int_{\mathcal{S}_i} \exp(-4\lambda \mathcal{A}(\Delta \mathcal{F}_{a_i, \rho, \alpha})) \rho d\rho d\alpha + \int_{\mathcal{G}_i} \exp(-4\lambda \mathcal{A}(\Delta \mathcal{F}_{a_i, \rho, \alpha})) \rho d\rho d\alpha \right).$$

It is then enough to show that, for every $1 \leq i \leq n_K$,

$$(4.1) \quad \lambda^{\frac{1}{2}} \int_{\mathcal{S}_i} \exp(-4\lambda \mathcal{A}(\Delta \mathcal{F}_{a_i, \rho, \alpha})) \rho d\rho d\alpha \xrightarrow{\lambda \rightarrow \infty} 2^{-\frac{9}{2}} \pi^{\frac{3}{2}} \|o_i\|^{-\frac{1}{2}} \|a_{i+1} - a_i\|^{\frac{3}{2}}$$

and

$$(4.2) \quad \lambda^{\frac{1}{2}} \int_{\mathcal{G}_i} \exp(-4\lambda \mathcal{A}(\Delta \mathcal{F}_{a_i, \rho, \alpha})) \rho d\rho d\alpha \xrightarrow{\lambda \rightarrow \infty} 0.$$

Let us prove (4.1) first. Let us fix $1 \leq i \leq n_K$. The change of variables $\beta = \lambda^{\frac{1}{2}} \alpha$ yields

$$\begin{aligned} \lambda^{\frac{1}{2}} \int_{\mathcal{S}_i} \exp(-4\lambda \mathcal{A}(\Delta \mathcal{F}_{a_i, \rho, \alpha})) \rho d\rho d\alpha \\ = \int_0^{\rho_i(\lambda^{-1/2}\beta)} \int_0^{\lambda^{1/2} \frac{\pi}{2}} \exp(-4\lambda \mathcal{A}(\Delta \mathcal{F}_{a_i, \rho, \lambda^{-1/2}\beta})) \rho d\rho d\beta \end{aligned}$$

where $\rho_i(\cdot)$ denotes the equation of the line containing a_{i+1} and orthogonal to (a_i, a_{i+1}) with respect to the polar coordinates (ρ, α) , i.e. $\rho_i(\alpha)$ is the distance from the origin to the intersection between that line and $\mathbb{R}_+ u_\alpha$.

Thanks to Lemma 4.1, we have

$$4\lambda \mathcal{A}(\Delta \mathcal{F}_{a_i, \rho, \lambda^{-1/2}\beta}) \xrightarrow{\lambda \rightarrow \infty} 2 \|o_i\| \frac{\rho \|a_{i+1} - a_i\|}{\|a_{i+1} - a_i\| - \rho} \beta^2$$

and, for all $\lambda > 0$,

$$\lambda \mathcal{A} \left(\Delta \mathcal{F}_{a_i, \rho, \lambda^{-1/2} \beta} \right) \geq C \max(1, \rho) \rho \beta^2$$

where C is a positive constant.

Consequently, we can apply Lebesgue's dominated convergence theorem to obtain

$$\begin{aligned} & \int_0^{\rho_i(\lambda^{-1/2} \beta)} \int_0^{\lambda^{1/2} \frac{\pi}{2}} \exp \left(-4\lambda \mathcal{A} \left(\Delta \mathcal{F}_{a_i, \rho, \lambda^{-1/2} \beta} \right) \right) \rho d\rho d\beta \\ & \xrightarrow{\lambda \rightarrow \infty} \int_0^{\|a_{i+1} - a_i\|} \int_0^\infty \exp \left(-2\|o_i\| \frac{\rho \|a_{i+1} - a_i\|}{\|a_{i+1} - a_i\| - \rho} \beta^2 \right) \rho d\rho d\beta \\ & = \|a_{i+1} - a_i\|^2 \int_0^1 \left(\int_0^\infty \exp \left(-2\|o_i\| \frac{l \|a_{i+1} - a_i\|}{1-l} \beta^2 \right) d\beta \right) l dl \\ & = \|a_{i+1} - a_i\|^{\frac{3}{2}} \pi^{\frac{1}{2}} 2^{-\frac{3}{2}} \|o_i\|^{-\frac{1}{2}} \int_0^1 (l(1-l))^{\frac{1}{2}} dl \\ & = \|a_{i+1} - a_i\|^{\frac{3}{2}} \pi^{\frac{3}{2}} 2^{-\frac{9}{2}} \|o_i\|^{-\frac{1}{2}}. \end{aligned}$$

Let us turn to the proof of (4.2). Using the second part of Lemma 4.1 we have successively, for all $\lambda > 0$,

$$\begin{aligned} & \int_{\mathcal{G}_i} \exp \left(-4\lambda \mathcal{A} \left(\Delta \mathcal{F}_{a_i, \rho, \alpha} \right) \right) \rho d\rho d\alpha \\ & \leq \int_0^\infty \int_{\frac{\pi}{2}}^{\frac{3\pi}{2} - \alpha_i} \exp \left(-C\lambda \max(1, \rho) \rho \alpha^2 \right) \rho d\rho d\alpha \\ & = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2} - \alpha_i} \left(\int_0^1 \exp \left(-C\lambda \rho \alpha^2 \right) \rho d\rho \right) d\alpha + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2} - \alpha_i} \left(\int_1^\infty \exp \left(-C\lambda \rho^2 \alpha^2 \right) \rho d\rho \right) d\alpha \\ & \leq \int_{\frac{\pi}{2}}^{\frac{3\pi}{2} - \alpha_i} \lambda^{-2} \alpha^{-4} d\alpha + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2} - \alpha_i} \lambda^{-1} \alpha^{-2} d\alpha \\ & \leq C\lambda^{-1}. \end{aligned}$$

That implies (4.2) and completes the proof of Theorem 1.2(i). \square

4.3. Proof of Theorem 1.2(ii): support points and defect perimeter

As in the smooth case, the strategy consists in using Corollary 2.3 and estimating the defect support function

$$Z_{i, \gamma, \lambda, -} = p_o \left(K_\lambda^o, \delta_i - \lambda^{-\gamma} \right) - p_o \left(K, \delta_i - \lambda^{-\gamma} \right)$$

of K_λ^o in a fixed direction $\delta_i - \lambda^{-\gamma}$, $1 \leq i \leq n_K$. As emphasized in the proof of point (ii) of Theorem 1.2, we will treat separately the cases $\gamma \in (0, \frac{1}{2})$ and $\gamma \geq \frac{1}{2}$. More precisely, let us introduce the support point $m_{\lambda^{-\gamma}}$ on ∂K_λ^o which satisfies

$$\langle m_{\lambda^{-\gamma}}, u_{\delta_i - \lambda^{-\gamma}} \rangle = p_o \left(K_\lambda^o, \delta_i - \lambda^{-\gamma} \right)$$

and let us denote by $(R_{\lambda^{-\gamma}}, A_{\lambda^{-\gamma}})$ the polar coordinates of $m_{\lambda^{-\gamma}}$ with respect to the coordinate system with origin a_i and first axis (a_i, a_{i+1}) . In particular, we notice that $A_{\lambda^{-\gamma}} \geq \lambda^{-\gamma}$ almost surely since $p_o(K_\lambda^o, \delta_i - \lambda^{-\gamma}) \geq p_o(K, \delta_i - \lambda^{-\gamma}) \geq \langle a_i, u_{\delta_i - \lambda^{-\gamma}} \rangle$.

The next Proposition investigates the asymptotic distribution of the couple $(R_{\lambda^{-\gamma}}, A_{\lambda^{-\gamma}})$ for $\gamma \in (0, \frac{1}{2})$.

PROPOSITION 4.2. — *Let $a_i \in \partial K$, $1 \leq i \leq n_K$, be a fixed vertex of K .*

- (i) *For every $\gamma \in (0, \frac{1}{2})$, the couple $(\lambda^{1-2\gamma} R_{\lambda^{-\gamma}}, \lambda^\gamma A_{\lambda^{-\gamma}})$ converges in distribution when $\lambda \rightarrow \infty$ to the distribution with density function f_i given by*

$$f_i(\rho, \alpha) = 8\|o_i\|^2 \exp(-2\|o_i\|\rho\alpha^2) \alpha(\alpha-1)\rho \mathbb{I}_{\{\rho>0\}} \mathbb{I}_{\{\alpha>1\}}.$$

- (ii) *There exists $C > 0$ such that for every $\gamma \in (0, \frac{1}{2})$ and $\lambda > 0$,*

$$\lambda^{1-\gamma} \mathbb{E} \left(R_{\lambda^{-\gamma}} \sin \left(A_{\lambda^{-\gamma}} - \lambda^{-\gamma} \right) \right) \leq C.$$

- (iii) *Moreover, for every $\gamma \in (0, \frac{1}{2})$,*

$$\mathbb{E}(Z_{i,\gamma,\lambda,-}) = \mathbb{E} \left(R_{\lambda^{-\gamma}} \sin \left(A_{\lambda^{-\gamma}} - \lambda^{-\gamma} \right) \right) \underset{\lambda \rightarrow \infty}{\sim} \lambda^{\gamma-1} \frac{1}{6\|o_i\|}.$$

Proof of Proposition 4.2.

Proof of (i). — Without loss of generality, we can assume in the proof that $\delta_i = \frac{\pi}{2}$. Once again, the strategy of the proof consists in going along the same lines as for the smooth case. We start with the same identity but written in polar coordinates, that is $c_{x_1, x_2} = m_{\lambda^{-\gamma}} = a_i + ru_\theta$ (see Figure 4.2). Notice that $(r, \theta) \in \mathcal{S}_i$ when $\lambda \rightarrow \infty$.

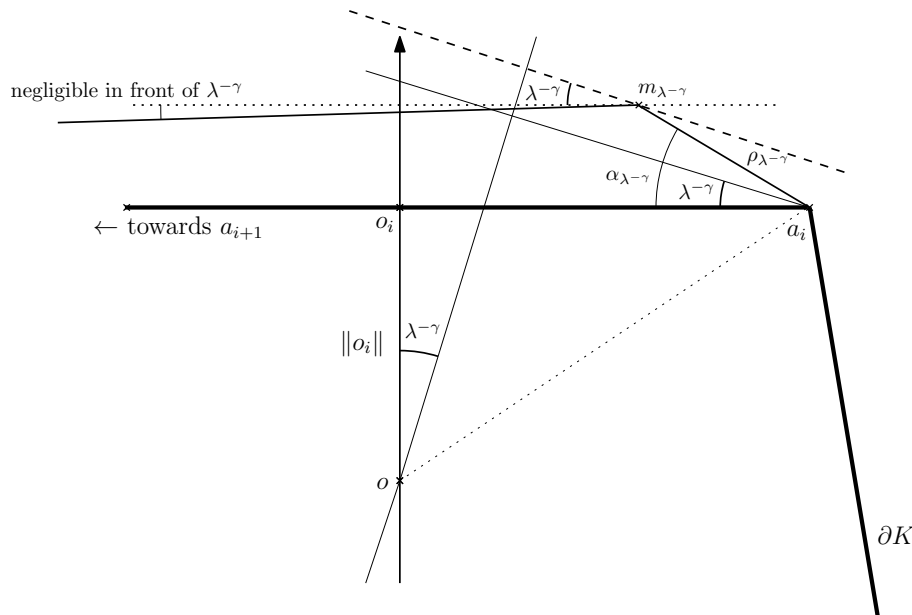


Figure 4.2. The analogue of Figure 3.2 in the polygonal case: $m_{\lambda^{-\gamma}}$ denotes the support point in the direction $\frac{\pi}{2} - \lambda^{-\gamma}$.

We then proceed with two consecutive changes of variables. First, denoting by θ_1 and θ_2 the angles between the two bisecting lines emanating from c_{x_1, x_2} corresponding to the right and left neighbor of o respectively, we use Lemma 3.2. Secondly, we replace the couple (r, θ) by (ρ, α) defined by

$$\rho = \lambda^{1-2\gamma} r \quad \text{and} \quad \alpha = \lambda^\gamma \theta.$$

We get in particular

$$\begin{cases} r &= r_i(\lambda, \rho, \alpha) = \|s_{a_i, \lambda^{2\gamma-1}\rho, \lambda^{-\gamma}\alpha}\| \\ \theta &= \theta_i(\lambda, \rho, \alpha) = \arcsin \left(\left\langle \frac{s_{a_i, \lambda^{2\gamma-1}\rho, \lambda^{-\gamma}\alpha}}{\|s_{a_i, \lambda^{2\gamma-1}\rho, \lambda^{-\gamma}\alpha}\|}, \frac{o_i}{\|o_i\|} \right\rangle \right) \end{cases}$$

and a Jacobian given by $\frac{rdrd\theta}{\rho d\rho d\alpha} = \lambda^{3\gamma-2}$. Consequently, as in the proof of point (i) of Proposition 3.3, we deduce that for every positive and measurable function $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}_+$

$$\begin{aligned} & \mathbb{E} \left(\varphi \left(\lambda^{1-2\gamma} R_{\lambda^{-\gamma}}, \lambda^\gamma A_{\lambda^{-\gamma}} \right) \right) \\ &= 16 \int_1^{\lambda^\gamma \pi + 1} \int_0^{\lambda^{1-2\gamma} \rho_i(\alpha)} \exp(-\Delta_i(\lambda, \rho, \alpha)) r_i^2(\lambda, \rho, \alpha) J_i^{\text{supp}}(\lambda, \rho, \alpha) \varphi(\rho, \alpha) \rho d\alpha d\rho \end{aligned}$$

where

$$\begin{aligned} & \Delta_i(\lambda, \rho, \alpha) \\ &= 4\lambda \mathcal{A} \left(B_{\|s_{a_i, \lambda^{2\gamma-1}\rho, \lambda^{-\gamma}\alpha}\|} (s_{a_i, \lambda^{2\gamma-1}\rho, \lambda^{-\gamma}\alpha}) \setminus \mathcal{F}_o(K) \right) = 4\lambda \mathcal{A}(\Delta \mathcal{F}_{a_i, \lambda^{2\gamma-1}\rho, \lambda^{-\gamma}\alpha}) \end{aligned}$$

and

$$J_i^{\text{supp}}(\lambda, \rho, \alpha) = \lambda^{3\gamma} \int_{E_i(\lambda, \rho, \alpha)} J(\theta_i(\lambda, \rho, \alpha), \theta_1, \theta_2) d\theta_1 d\theta_2$$

where $E_i^{\text{supp}}(\lambda, \rho, \alpha)$ stands for the set of couples (θ_1, θ_2) which satisfy that the two bisecting lines of $[o, x_1]$ and $[o, x_2]$ do not intersect K .

Thanks to Lemma 4.1, we have

$$\Delta_i(\lambda, \rho, \alpha) \xrightarrow{\lambda \rightarrow \infty} 2\|o_i\|\rho\alpha^2$$

as soon as $\lambda^{2\gamma-1}\rho \cos(\lambda^{-\gamma}\alpha) \leq \|a_{i+1} - a_i\|$ and

$$(4.3) \quad \Delta_i(\lambda, \rho, \alpha) \geq C\rho\alpha^2$$

for some constant $C > 0$. Moreover,

$$r_i(\lambda, \rho, \alpha) \xrightarrow{\lambda \rightarrow \infty} \|a_i\|$$

and for $\lambda \geq 1$,

$$(4.4) \quad r_i(\lambda, \rho, \alpha) \leq \|a_i\| + \lambda^{2\gamma-1}\rho \leq \|a_i\| + \rho.$$

Let us now turn on the term $J_i^{\text{supp}}(\lambda, \rho, \alpha)$. Using the convergence

$$\sin(\theta_i(\lambda, \rho, \alpha)) \xrightarrow{\lambda \rightarrow \infty} \frac{\|o_i\|}{\|a_i\|},$$

we get successively

$$\begin{aligned}
& J_i^{\text{supp}}(\lambda, \rho, \alpha) \\
& \underset{\lambda \rightarrow \infty}{\sim} \lambda^{3\gamma} \int_{E_i(\lambda, \rho, \alpha)} J(\theta_i(\lambda, \rho, \alpha), \theta_1, \theta_2) d\theta_1 d\theta_2 \\
& \underset{\lambda \rightarrow \infty}{\sim} \lambda^{3\gamma} \int_{(0, (\alpha-1)\lambda^{-\gamma}) \times (0, \lambda^{-\gamma})} |\sin(\theta_i(\lambda, \rho, \alpha) - \theta_1) \sin(\theta_i(\lambda, \rho, \alpha) - \theta_2) \sin(\theta_2 - \theta_1)| d\theta_1 d\theta_2 \\
& \underset{\lambda \rightarrow \infty}{\sim} \lambda^{3\gamma} \left(\frac{\|o_i\|}{\|a_i\|} \right)^2 \left(\frac{1}{2} \lambda^{-3\gamma} \alpha(\alpha-1) \right) \\
& = \frac{1}{2} \left(\frac{\|o_i\|}{\|a_i\|} \right)^2 \alpha(\alpha-1).
\end{aligned}$$

Finally, we notice that any couple $(\theta_1, \theta_2) \in E_i(\lambda, \rho, \alpha)$ satisfies that one of the two angles is at most equal to α and the other to $\rho\alpha$ up to a multiplicative constant. Consequently, we can show that for some constant $C > 0$,

$$(4.5) \quad J_i^{\text{supp}}(\lambda, \rho, \alpha) \leq C\rho\alpha^2.$$

A method based on Lebesgue's dominated convergence theorem and analogous to the proof of Proposition 3.3 will show that

$$\begin{aligned}
& \mathbb{E} \left(\varphi \left(\lambda^{1-2\gamma} R_{\lambda^{-\gamma}}, \lambda^\gamma A_{\lambda^{-\gamma}} \right) \right) \\
& \underset{\lambda \rightarrow \infty}{\sim} 16 \int_{(0, \infty)^2} \exp \left(2\|o_i\| \rho \alpha^2 \right) \|a_i\|^2 \left(\frac{1}{2} \left(\frac{\|o_i\|}{\|a_i\|} \right)^2 \alpha(\alpha-1) \mathbb{I}_{\{\alpha > 1\}} \right) \varphi(\rho, \alpha) \rho d\rho d\alpha.
\end{aligned}$$

This implies the required result.

Proof of (ii). — This is a direct consequence of the convergence in distribution proved in (i) and of the equality

$$p_o \left(K_\lambda^o, \delta_i - \lambda^{-\gamma} \right) - p_o \left(K, \delta_i - \lambda^{-\gamma} \right) = R_{\lambda^{-\gamma}} \sin \left(A_{\lambda^{-\gamma}} - \lambda^{-\gamma} \right).$$

Indeed, applying the method used in (i) to $\varphi(\rho, \alpha) = \rho(\alpha-1)$, we get the estimate in (ii) from (4.3), (4.4), (4.5) and the inequality $\sin(x) \leq x$ for $x > 0$.

Proof of (iii). — Finally, it also follows from (i) that

$$\begin{aligned}
& \lambda^{1-\gamma} \left(\mathbb{E} \left(p_o \left(K_\lambda^o, \frac{\pi}{2} - \lambda^{-\gamma} \right) - p_o \left(K, \frac{\pi}{2} - \lambda^{-\gamma} \right) \right) \right) \\
& \underset{\lambda \rightarrow \infty}{\sim} \mathbb{E} \left(\lambda^{1-2\gamma} R_{\lambda^{-\gamma}} (\lambda^\gamma A_{\lambda^{-\gamma}} - 1) \right) = \int_{\mathbb{R}^2} \rho(\alpha-1) f_i(\rho, \alpha) d\rho d\alpha = \frac{1}{6\|o_i\|}.
\end{aligned}$$

which completes the proof of Proposition 4.2. \square

We will need an analogous result for the support function of K with respect to o in a direction of the form $\tau\lambda^{-\frac{1}{2}}$, for $\tau \geq 0$. Let us introduce the point $m_{\tau\lambda^{-1/2}}$ on ∂K_λ^o which satisfies

$$\left\langle m_{\tau\lambda^{-1/2}}, u_{\frac{\pi}{2} - \tau\lambda^{-1/2}} \right\rangle = p_o \left(K_\lambda^o, \frac{\pi}{2} - \tau\lambda^{-1/2} \right)$$

and denote by $R_{\tau\lambda^{-1/2}}$ and $A_{\tau\lambda^{-1/2}}$ the polar coordinates of $m_{\tau\lambda^{-1/2}}$ with respect to a_i .

The next proposition provides the limit distribution of the couple $(R_{\tau\lambda^{-1/2}}, A_{\tau\lambda^{-1/2}})$. Since it is very similar to Proposition 4.2, the proof is omitted.

PROPOSITION 4.3. — *Let $a_i \in \partial K$, $1 \leq i \leq n_K$, be a fixed vertex of K .*

(i) *Let $\tau \geq 0$. The couple $(R_{\tau\lambda^{-1/2}}, \lambda^{-\frac{1}{2}} A_{\tau\lambda^{-1/2}})$ converges in distribution when $\lambda \rightarrow \infty$ to the distribution with density function g_i given by*

$$g_i(\rho, \alpha) = 8\|o_i\|^2 \frac{\rho\|a_{i+1} - a_i\|}{\|a_{i+1} - a_i\| - \rho} \exp\left(-2\|o_i\| \frac{\rho\|a_{i+1} - a_i\|}{\|a_{i+1} - a_i\| - \rho} \alpha^2\right) \\ \times (\alpha - \tau) \left(\frac{\rho\alpha}{\|a_{i+1} - a_i\| - \rho} + \tau \right) \mathbb{I}_{\{\rho \in (0, \|a_{i+1} - a_i\|)\}} \mathbb{I}_{\{\alpha > \tau\}}.$$

(ii) *There exists a positive constant $\tau \geq 0$ such that, for every $\gamma > \frac{1}{2}$,*

$$\mathbb{E}(Z_{i,\gamma,\lambda,-}) = \mathbb{E}\left(p_o(K_\lambda^o, \delta_i - \lambda^{-\gamma}) - p_o(K, \delta_i - \lambda^{-\gamma})\right) \leq \tau \lambda^{-\frac{1}{2}}.$$

Let us notice that the special case $\tau = 0$ provides the asymptotic distribution of the highest point of K_λ^o above the edge (a_i, a_{i+1}) . Straightforward computation show that the asymptotic distribution of R_0 admits the simple density function

$$\rho \mapsto \frac{1}{\|a_{i+1} - a_i\|} \mathbb{I}_{\{\rho \in (0, \|a_{i+1} - a_i\|)\}}$$

that is the highest point is asymptotically uniformly distributed along the edge (a_i, a_{i+1}) .

Proof of Theorem 1.2(ii). — Looking at the right hand side in the identity from Corollary 2.3(i), we concentrate on the estimation of the integral $\int_{\eta_{i,\lambda}}^\infty \mathbb{E}(Z_{i,\gamma,\lambda,-}) (\lambda^{-\gamma} \log \lambda) d\gamma$ as the other estimates will follow analogously.

First, thanks to Proposition 4.2(ii), we obtain that there exists $C > 0$ such that

$$(4.6) \quad \frac{\lambda}{\log \lambda} \int_0^{\eta_{i,\lambda}} \mathbb{E}(Z_{i,\gamma,\lambda,-}) (\lambda^{-\gamma} \log \lambda) d\gamma \leq C \eta_{i,\lambda} \xrightarrow{\lambda \rightarrow \infty} 0.$$

Thanks to (4.6), it is then enough to show that the integral $\int_0^\infty \mathbb{E}(Z_{i,\gamma,\lambda,-}) (\lambda^{-\gamma} \log \lambda) d\gamma$ is equivalent to $(\lambda^{-1} \log \lambda)$ up to a multiplicative constant. It is a consequence of the two previous Propositions that only directions $\delta_i - \lambda^{-\gamma}$ up to the critical value $\gamma = \frac{1}{2}$ will contribute. Precisely, we deduce from point (ii) of Proposition 4.2 below combined with Lebesgue's dominated convergence theorem that

$$\frac{\lambda}{\log \lambda} \int_0^{\frac{1}{2}} \mathbb{E}(Z_{i,\gamma,\lambda,-}) (\lambda^{-\gamma} \log \lambda) d\gamma \xrightarrow{\lambda \rightarrow \infty} \frac{1}{12\|o_i\|}$$

and from point (ii) of Proposition 4.3 that

$$\lambda \int_{\frac{1}{2}}^\infty \mathbb{E}(Z_{i,\gamma,\lambda,-}) \lambda^{-\gamma} d\gamma \leq \lambda \int_{\frac{1}{2}}^\infty C \lambda^{-\frac{1}{2}} \lambda^{-\gamma} d\gamma \xrightarrow{\lambda \rightarrow \infty} 0.$$

It follows that

$$\frac{\lambda}{\log \lambda} \int_0^\infty \mathbb{E}(Z_{i,\gamma,\lambda,-}) (\lambda^{-\gamma} \log \lambda) d\gamma \underset{\lambda \rightarrow \infty}{\sim} \frac{1}{12\|o_i\|}.$$

Showing the same estimate when $Z_{i,\gamma,\lambda,-}$ is replaced by $Z_{i,\gamma,\lambda,+}$ and summing over the vertices of K provide then the required result. \square

4.4. Proof of Theorem 1.2(iii): intensity and number of vertices

Proof of Theorem 1.2(iii). — Considering Corollary 2.3(ii), we aim at determining a precise estimate for the integral $\int_{\eta_{i,\lambda}}^{\infty} p_o(K, \delta_i + \varepsilon \lambda^{-\gamma}) \mathbb{E}(Z_{i,\gamma,\lambda,\varepsilon}) (\lambda^{-\gamma} \log \lambda) d\gamma$ where $\varepsilon \in \{+, -\}$. Naturally, it is enough to do it for $\varepsilon = -$ as the case $\varepsilon = +$ will follow by analogous methods. Again, thanks to Proposition 4.2(ii), we obtain that there exists $C > 0$ such that

$$\frac{\lambda}{\log \lambda} \int_0^{\eta_{i,\lambda}} p_o(K, \delta_i - \lambda^{-\gamma}) \mathbb{E}(Z_{i,\gamma,\lambda,-}) (\lambda^{-\gamma} \log \lambda) d\gamma \leq C \eta_{i,\lambda} \xrightarrow{\lambda \rightarrow \infty} 0.$$

We now use the following decomposition:

$$\int_0^{\infty} p_o(K, \delta_i - \lambda^{-\gamma}) \mathbb{E}(Z_{i,\gamma,\lambda,-}) (\lambda^{-\gamma} \log \lambda) d\gamma = I_i^{(1)}(\lambda) + I_i^{(2)}(\lambda)$$

where

$$I_i^{(1)}(\lambda) = \int_0^{\frac{1}{2}} p_o(K, \delta_i - \lambda^{-\gamma}) \mathbb{E}(Z_{i,\gamma,\lambda,-}) (\lambda^{-\gamma} \log \lambda) d\gamma$$

and

$$I_i^{(2)}(\lambda) = \int_{\frac{1}{2}}^{\infty} p_o(K, \delta_i - \lambda^{-\gamma}) \mathbb{E}(Z_{i,\gamma,\lambda,-}) (\lambda^{-\gamma} \log \lambda) d\gamma.$$

Because of point (iii) of Proposition 4.2, the integrand of $I_i^{(1)}(\lambda)$ satisfies

$$\lambda^{1-\gamma} p_o(K, \delta_i - \lambda^{-\gamma}) \mathbb{E}(Z_{i,\gamma,\lambda,-}) (\lambda^{-\gamma} \log \lambda) \underset{\lambda \rightarrow \infty}{\sim} \|o_i\| \times \frac{1}{6\|o_i\|} = \frac{1}{6}$$

and we may apply Lebesgue's dominated convergence theorem. Now point (ii) of Proposition 4.3 shows that $I_i^{(2)}(\lambda)$ is negligible. Consequently, we have obtained that

$$(4.7) \quad 4\lambda \int_{\eta_{i,\lambda}}^{\infty} p_o(K, \delta_i - \lambda^{-\gamma}) \mathbb{E}(Z_{i,\gamma,\lambda,-}) (\lambda^{-\gamma} \log \lambda) d\gamma \underset{\lambda \rightarrow \infty}{\sim} (\log \lambda) \cdot 3^{-1}.$$

We conclude by stating that the term \mathcal{R}_λ given at (2.7) is negligible in front of $\log \lambda$. This follows by extending verbatim Proposition 4.2(ii) and Proposition 4.3 to the second moment of $Z_{i,\gamma,\lambda,-}$, i.e. there exists $C > 0$ such that

$$(4.8) \quad \mathbb{E}(Z_{i,\gamma,\lambda,-}^2) \leq C \lambda^{\min(2(\gamma-1), -1)}.$$

Summing over i the result (4.7) then implies Theorem 1.2(iii). \square

We now aim at being more specific on the localization of the vertices of K_λ^o . The following statement shows a striking self-similarity of the limiting intensity of the point process of vertices around a fixed vertex of K .

PROPOSITION 4.4. — *Let $a_i \in \partial K$, $1 \leq i \leq n_K$, be a fixed vertex of K . Consider the point process $(\rho_v, \alpha_v)_{v \in V_i}$ of the polar coordinates of the vertices of K_λ^o belonging to \mathcal{S}_i . Let $R \times A$ be a bounded Borel set of $(0, \infty)^2$ and denote by $\mathcal{N}_i(R \times A)$ the*

number of points in $R \times A$ of the rescaled point process $(\lambda^{1-2\gamma}\rho_v, \lambda^\gamma\alpha_v)_{v \in V_i}$. Then we get

$$\mathbb{E}(\mathcal{N}_i(R \times A)) \xrightarrow{\lambda \rightarrow \infty} \int_A \sigma_i(\rho, \alpha) d\rho d\alpha$$

where

$$\sigma_i(\rho, \alpha) = \frac{8}{3} \|o_i\|^2 \rho \alpha^3 \exp(-2\|o_i\| \rho \alpha^2) \mathbb{I}_{\{\rho > 0\}} \mathbb{I}_{\{\alpha > 0\}}.$$

Proof of Proposition 4.4. — The strategy of the proof consists in going along the same lines as for the smooth case and proceeding exactly like for the proof of Proposition 4.2. Precisely, we obtain that

$$\mathbb{E}(\mathcal{N}_i(R \times A)) = 8 \int_{R \times A} \exp(-\Delta_i(\lambda, \rho, \alpha)) r_i^2(\lambda, \rho, \alpha) J_i^{\text{supp}}(\lambda, \rho, \alpha) \rho d\rho d\alpha$$

where

$$\begin{aligned} \Delta_i(\lambda, \rho, \alpha) &= 4\lambda \mathcal{A} \left(B_{\|s_{a_i, \lambda^{2\gamma-1}\rho, \lambda^{-\gamma}\alpha}\|} (s_{a_i, \lambda^{2\gamma-1}\rho, \lambda^{-\gamma}\alpha}) \setminus \mathcal{F}_o(K) \right) \\ &= 4\lambda \mathcal{A} (\Delta \mathcal{F}_{a_i, \lambda^{2\gamma-1}\rho, \lambda^{-\gamma}\alpha}) \end{aligned}$$

and

$$J_i^{\text{supp}}(\lambda, \rho, \alpha) = \lambda^{3\gamma} \int_{E_i(\lambda, \rho, \alpha)} J(\theta_i(\lambda, \rho, \alpha), \theta_1, \theta_2) d\theta_1 d\theta_2$$

where $E_i(\lambda, \rho, \alpha)$ stands for the set of couples (θ_1, θ_2) which satisfy that the two bisecting lines of $[o, x_1]$ and $[o, x_2]$ do not intersect K (see Figure 4.3).

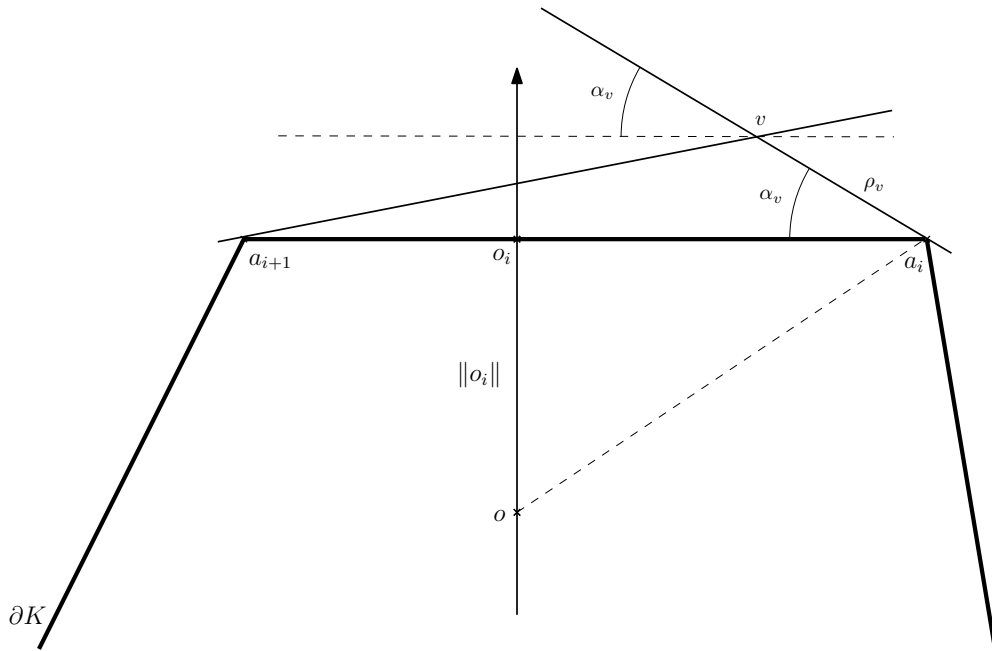


Figure 4.3. Intensity of the point process of vertices near a fixed vertex a_i of K .

Thanks to Lemma 4.1, we get

$$\Delta_i(\lambda, \rho, \alpha) \xrightarrow{\lambda \rightarrow \infty} 2\|o_i\|\rho\alpha^2.$$

Moreover,

$$r_i(\lambda, \rho, \alpha) \xrightarrow{\lambda \rightarrow \infty} \|a_i\| \quad \text{and} \quad \sin(\theta_i(\lambda, \rho, \alpha)) \xrightarrow{\lambda \rightarrow \infty} \frac{\|o_i\|}{\|a_i\|}.$$

Let us now turn on the term $J_i^{\text{supp}}(\lambda, \rho, \alpha)$. We get successively

$$\begin{aligned} J_i^{\text{supp}}(\lambda, \rho, \alpha) &\underset{\lambda \rightarrow \infty}{\sim} \lambda^{3\gamma} \int_{(0, \lambda^{-\gamma}\alpha)^2} |\sin(\theta_i(\lambda, \rho, \alpha) - \theta_1) \sin(\theta_i(\lambda, \rho, \alpha) - \theta_2) \sin(\theta_2 - \theta_1)| \, d\theta_1 d\theta_2 \\ &\underset{\lambda \rightarrow \infty}{\sim} \lambda^{3\gamma} \left(\frac{\|o_i\|}{\|a_i\|} \right)^2 \left(\frac{1}{3} (\lambda^{-\gamma}\alpha)^3 \right) = \frac{1}{3} \left(\frac{\|o_i\|}{\|a_i\|} \right)^2 \alpha^3. \end{aligned}$$

We apply now again Lebesgue’s dominated convergence theorem, omitting the domination step which is very similar to what we did in the proof of Proposition 4.2. It follows that

$$\mathbb{E}(\mathcal{N}_i(R \times A)) \xrightarrow{\lambda \rightarrow \infty} 8 \int_{R \times A} \exp\left(-2\|o_i\|\rho\alpha^2\right) \|a_i\|^2 \times \frac{1}{3} \left(\frac{\|o_i\|}{\|a_i\|} \right)^2 \alpha^3 \rho d\rho d\alpha$$

which implies the required result. \square

5. Proof of Theorem 1.3: the role of the Steiner point

In the two previous sections, the cell that we considered is associated with a deterministic nucleus at the origin which is added to the Poisson point process. In particular, the asymptotic shape of the cell depends on both the choice of the convex body K and the position of the origin o inside K . In this section, we investigate a modified question which is intrinsic in K , i.e. we ask for the behavior of the cell K_λ containing K when the Poisson point process is conditioned on its associated Voronoi tessellation to not intersect K . Since the problem is invariant under translation, we are allowed to assume that the Steiner point of K coincides with the origin, without it being a nucleus of the tessellation. More precisely, the Steiner point of K denoted by $\text{st}(K)$ is defined by the equality

$$\text{st}(K) = \frac{1}{\pi} \int_0^{2\pi} p_o(K, \theta) u_\theta d\theta.$$

When K is smooth, $\text{st}(K)$ can be rewritten as

$$\text{st}(K) = \frac{1}{\pi} \int_{\partial K} r_s^{-1} \langle s, n_s \rangle n_s ds.$$

In particular, $\text{st}(K)$ is included in the relative interior of K , see e.g. [Sch93, Section 1.7] for the definition and the general properties of $\text{st}(K)$. Note in particular that its definition is intrinsic to K , i.e. is independent of the choice of the origin o . We show as a byproduct of the proof of Proposition 5.2 the alternative characterization of the Steiner point given in Proposition 5.1 below.

PROPOSITION 5.1. — *The Steiner point $\text{st}(K)$ is the unique point x in \mathbb{R}^2 which minimises the function $x \mapsto \mathcal{A}(\mathcal{F}_x(K))$.*

Let \mathcal{S}_λ be the event such that there is one cell \widehat{K}_λ of the Poisson–Voronoi tessellation associated with \mathcal{P}_λ which contains K . We are interested in showing that conditional on \mathcal{S}_λ , the nucleus of \widehat{K}_λ , denoted by Z_λ , is close to the Steiner point $\text{st}(K)$.

The next proposition is an intermediary result which provides a precise description of the conditional distribution of \mathcal{P}_λ given \mathcal{S}_λ as well as the explicit and limit distributions for the rescaled nucleus of K_λ . Note that Proposition 5.2 combined with Theorem 1.1 will be the key tool for proving Theorem 1.3.

PROPOSITION 5.2. —

- (i) *The conditional distribution of \mathcal{P}_λ given \mathcal{S}_λ is equal in distribution to $\{Z_\lambda\} \cup \mathcal{P}_\lambda^{(Z_\lambda)}$ where Z_λ is a random variable distributed according to a density function proportional to $x \mapsto \exp(-4\lambda\mathcal{A}(\mathcal{F}_x(K)))$ and, given $\{Z_\lambda = x\}$, $\mathcal{P}_\lambda^{(Z_\lambda)}$ is a Poisson point process of intensity $\lambda\mathbb{I}_{\mathbb{R}^2 \setminus 2\mathcal{F}_x(K)}$.*
- (ii) *Conditional on \mathcal{S}_λ , the rescaled nucleus $\lambda^{\frac{1}{2}}Z_\lambda$ converges in distribution as $\lambda \rightarrow \infty$ to the centered Gaussian distribution with covariance matrix $(4\pi)^{-1}$ times the identity matrix.*

Proof of Proposition 5.2.

Proof of (i). — Let L be a fixed compact set in \mathbb{R}^2 . Using Mecke–Slivnyak’s formula and denoting by \mathcal{C}_x the Voronoi cell associated with $x \in \mathbb{R}^2$, we get successively

$$\begin{aligned} \mathbb{E}\left(\mathbb{I}_{\mathcal{S}_\lambda} \mathbb{I}_{\{\mathcal{P}_\lambda \cap L = \emptyset\}}\right) &= \mathbb{E}\left(\sum_{x \in \mathcal{P}_\lambda} \mathbb{I}_{\{K \subset \mathcal{C}_x\}} \mathbb{I}_{\{\mathcal{P}_\lambda \cap L = \emptyset\}}\right) \\ &= \int_{\mathbb{R}^2} \mathbb{P}(K \subset \mathcal{C}_x, (\mathcal{P}_\lambda \cup \{x\}) \cap L = \emptyset) dx \\ &= \int_{\mathbb{R}^2 \setminus L} \mathbb{P}(\mathcal{P}_\lambda \cap (L \cup 2\mathcal{F}_x(K)) = \emptyset) dx \\ &= \int_{\mathbb{R}^2 \setminus L} \exp(-\lambda\mathcal{A}(L \setminus 2\mathcal{F}_x(K))) \exp(-4\lambda\mathcal{A}(\mathcal{F}_x(K))) dx. \end{aligned}$$

Dividing the last equality by $\mathbb{P}(\mathcal{S}_\lambda)$, we get the required result.

Proof of (ii). — We start by calculating both $\mathbb{P}(\mathcal{S}_\lambda)$ and the density of Z_λ conditional on \mathcal{S}_λ . For any bounded and measurable function $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$, we deduce from Mecke–Slivnyak’s formula that

$$\begin{aligned} \mathbb{E}\left(\sum_{x \in \mathcal{P}_\lambda} \mathbb{I}_{\{K \subset \mathcal{C}_x\}} \varphi(x)\right) &= \lambda \int_{\mathbb{R}^2} \mathbb{P}(K \subset \mathcal{C}_x) \varphi(x) dx \\ &= \lambda \int_{\mathbb{R}^2} \exp(-4\lambda\mathcal{A}(\mathcal{F}_x(K))) \varphi(x) dx. \end{aligned}$$

Taking $\varphi = 1$ in the last equality above, we obtain that

$$\mathbb{P}(\mathcal{S}_\lambda) = \lambda \int_{\mathbb{R}^2} \exp(-4\lambda\mathcal{A}(\mathcal{F}_x(K))) dx$$

and that the conditional density f_λ of $\lambda^{\frac{1}{2}}Z_\lambda$ is proportional to

$$x \longmapsto \exp \left(-4\lambda \mathcal{A}(\mathcal{F}_{\lambda^{-1/2}x}(K)) \right).$$

We now turn our attention to the calculation of $\mathcal{A}(\mathcal{F}_x(K))$. For any $x \in \mathbb{R}^2$, we denote by $\mathcal{E}(x) \subset [0, 2\pi]$ the set of all directions such that $p_x(K, \theta) > 0$. Denoting by

$$\mathcal{T}(x) = \frac{1}{2} \int_{[0, 2\pi] \setminus \mathcal{E}(x)} p_x^2(K, \theta) d\theta,$$

we rewrite $\mathcal{A}(\mathcal{F}_x(K))$ as

$$\mathcal{A}(\mathcal{F}_x(K)) = \frac{1}{2} \int_0^{2\pi} p_x^2(K, \theta) d\theta - \mathcal{T}(x).$$

We notice that

$$(5.1) \quad p_x(K, \theta) - p_o(K, \theta) = -\langle x, u_\theta \rangle.$$

Consequently, we get

$$\begin{aligned} \mathcal{A}(\mathcal{F}_x(K)) &= \frac{1}{2} \int_0^{2\pi} (p_o(K, \theta) - \langle x, u_\theta \rangle)^2 d\theta - \mathcal{T}(x) \\ &= \mathcal{A}(\mathcal{F}_o(K)) - \left\langle x, \int_0^{2\pi} p_o(K, \theta) u_\theta d\theta \right\rangle + \frac{1}{2} \int_0^{2\pi} \langle x, u_\theta \rangle^2 d\theta - \mathcal{T}(x) \\ (5.2) \quad &= \mathcal{A}(\mathcal{F}_o(K)) + \frac{\pi}{2} \|x\|^2 - \mathcal{T}(x) \end{aligned}$$

where we have used both the fact that o is the Steiner point of K and the equality

$$\int_0^{2\pi} \langle x, u_\theta \rangle^2 d\theta = \int_0^{2\pi} (\cos \theta)^2 \|x\|^2 d\theta = \pi \|x\|^2.$$

Let us show two basic properties of the rest $\mathcal{T}(x)$.

- When x is in the interior of K , $\mathcal{E}(x) = [0, 2\pi]$ and $\mathcal{T}(x) = 0$.
- When x is not in the interior of K , because of (5.1), $0 < p_o(K, \theta) \leq \langle x, u_\theta \rangle$ as soon as $\theta \in [0, 2\pi] \setminus \mathcal{E}(x)$ and consequently, for any $\theta \in [0, 2\pi] \setminus \mathcal{E}(x)$,

$$p_x(K, \theta)^2 = \langle x, u_\theta \rangle^2 - p_o(K, \theta)(2\langle x, u_\theta \rangle - p_o(K, \theta)) \leq \langle x, u_\theta \rangle^2.$$

Combining this inequality with the fact that $[0, 2\pi] \setminus \mathcal{E}(x)$ is an interval of length at most π implies in turn that

$$0 \leq \mathcal{T}(x) \leq \frac{1}{2} \int_{[0, 2\pi] \setminus \mathcal{E}(x)} \langle x, u_\theta \rangle^2 d\theta \leq \frac{\pi}{4} \|x\|^2.$$

In view of (5.2), this means in particular that o is the unique minimum of the function $x \longmapsto \mathcal{A}(\mathcal{F}_x(K))$.

Now, inserting (5.2) into the conditional density function f_λ of $\lambda^{\frac{1}{2}}Z_\lambda$, we obtain that $f_\lambda(x)$ is proportional to $\exp(-2\pi\|x\|^2 + 4\lambda\mathcal{T}(\lambda^{-\frac{1}{2}}x))$. Using the properties of \mathcal{T} detailed above and the fact that $\lambda^{-\frac{1}{2}}x$ lies in the interior of K for λ large enough, we get that for every $x \in \mathbb{R}^2$, $f_\lambda(x)$ converges to $2\exp(-2\pi\|x\|^2)$ and

$$\exp \left(-2\pi\|x\|^2 + 4\lambda\mathcal{T} \left(\lambda^{-\frac{1}{2}}x \right) \right) \leq \exp \left(-\pi\|x\|^2 \right).$$

Consequently, an application of Lebesgue's dominated convergence theorem shows that for any measurable function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ which is bounded by a polynomial of $\|x\|$,

$$\int_{\mathbb{R}^2} g(x) f_\lambda(x) dx \xrightarrow{\lambda \rightarrow \infty} 2 \int_{\mathbb{R}^2} g(x) \exp(-2\pi\|x\|^2) dx$$

which completes the proof of Proposition 5.2. \square

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. — We prove the result for $\mathbb{E}(\mathcal{A}(K_\lambda)) - \mathcal{A}(K)$ and explain at the end how to adapt the arguments for $\mathbb{E}(\mathcal{U}(K_\lambda)) - \mathcal{U}(K)$ and $\mathbb{E}(\mathcal{N}(K_\lambda))$.

Let \mathcal{C}_o be the Voronoi cell associated with the origin when o is added to the set of nuclei \mathcal{P}_λ . The cell K_λ containing K is distributed, up to a translation, as \mathcal{C}_o conditional on $(K + Z_\lambda) \subset \mathcal{C}_o$, where Z_λ is distributed as in Proposition 5.2(i). Recalling that f_λ is the density function of $\lambda^{\frac{1}{2}} Z_\lambda$, we obtain

$$\mathbb{E}(\mathcal{A}(K_\lambda) - \mathcal{A}(K)) = \int_{\mathbb{R}^2} \mathbb{E}(\mathcal{A}((K + \lambda^{-\frac{1}{2}}x)_\lambda^o) - \mathcal{A}(K)) f_\lambda(x) dx = I_1(\lambda) + I_2(\lambda)$$

where

$$I_1(\lambda) = \int_{\lambda^{\frac{1}{4}}K} \mathbb{E}(\mathcal{A}((K + \lambda^{-\frac{1}{2}}x)_\lambda^o) - \mathcal{A}(K)) f_\lambda(x) dx$$

and

$$I_2(\lambda) = \int_{\mathbb{R}^2 \setminus \lambda^{\frac{1}{4}}K} \mathbb{E}(\mathcal{A}((K + \lambda^{-\frac{1}{2}}x)_\lambda^o) - \mathcal{A}(K)) f_\lambda(x) dx.$$

We start by showing that

$$(5.3) \quad I_1(\lambda) \underset{\lambda \rightarrow \infty}{\sim} \mathbb{E}(\mathcal{A}(K_\lambda^o) - \mathcal{A}(K)).$$

Indeed, a method similar to what has been done in Sections 3.2 and 4.2 shows that, uniformly in $x \in \mathbb{R}^2$,

$$\mathbb{E}(\mathcal{A}((K + \lambda^{-\frac{1}{2}}x)_\lambda^o) - \mathcal{A}(K)) \mathbb{1}_{\{x \in \lambda^{\frac{1}{4}}K\}} \underset{\lambda \rightarrow \infty}{\sim} \mathbb{E}(\mathcal{A}(K_\lambda^o) - \mathcal{A}(K)).$$

Combining this with the convergence and domination of the function f_λ showed in the proof of Proposition 5.2(ii), we get (5.3).

Let us show now that the integral $I_2(\lambda)$ is negligible in front of $I_1(\lambda)$. To do so, we denote by R_x the maximal distance from o to the farthest point in $(K + \lambda^{-\frac{1}{2}}x)_\lambda^o$. We notice in particular that $\mathcal{A}((K + \lambda^{-\frac{1}{2}}x)_\lambda^o) \leq \pi R_x^2$. Moreover, by a method similar to Lemma 1 in [FZ96], we obtain that, for any $r > 0$,

$$\mathbb{P}(R_x \geq r) \leq C' \exp(-Cr^2 + C'\lambda^{-1}\|x\|^2),$$

for some positive constants $C, C' > 0$.

Consequently, $\mathbb{E}(\mathcal{A}((K + \lambda^{-\frac{1}{2}}x)_\lambda^o) - \mathcal{A}(K))$ is bounded by $1 + \lambda^{-1}\|x\|^2$ up to a multiplicative constant. Using the domination of f_λ showed in Proposition 5.2(ii), we get that $I_2(\lambda) \rightarrow 0$ exponentially fast as $\lambda \rightarrow \infty$. Combining this last result with (5.3), we obtain the required convergence for the mean defect area of K_λ .

Finally, the estimates for $\mathbb{E}(\mathcal{U}(K_\lambda) - \mathcal{U}(K))$ and $\mathbb{E}(\mathcal{N}(K_\lambda))$ follow from similar arguments, as soon as we are able to get bounds for $\mathbb{E}(\mathcal{U}((K + \lambda^{-\frac{1}{2}}x)_\lambda^o) - \mathcal{U}(K))$ and $\mathbb{E}(\mathcal{N}((K + \lambda^{-\frac{1}{2}}x)_\lambda^o))$. Using the inclusion $(K + \lambda^{-\frac{1}{2}}x)_\lambda^o \subset B_o(R_x)$, we get that, up to a multiplicative constant, $\lambda^{-\frac{1}{2}}\|x\|$ is an upper-bound of $\mathbb{E}(\mathcal{U}((K + \lambda^{-\frac{1}{2}}x)_\lambda^o) - \mathcal{U}(K))$. A use of Proposition 2.1 (i) combined with the same inclusion finally shows that, up to a multiplicative constant, $\lambda^{-1}\|x\|^2$ is an upper-bound of $\mathbb{E}(\mathcal{N}((K + \lambda^{-\frac{1}{2}}x)_\lambda^o))$. \square

6. Proof of Theorem 1.5: the question of the variance

In this section, we explain how a general method based on the application of an inversion with respect to the osculating disk at s for any $s \in \partial K$ provides an explicit calculation of the limiting variance of the defect area and the number of vertices in the smooth case. As in Theorem 1.3, these results extend to K_λ .

Proof of Theorem 1.5. — Compared to the direct calculations done in the previous sections, the method used in this proof is clearly more technical and relies on previous works for random convex hulls. Similar arguments could have also led to Theorem 1.1 but we had chosen to keep the discussion as plain as possible for the expectation asymptotics, which seems to be no longer possible for the variance asymptotics. The technique described below is reminiscent of both [CS05] for the idea of transforming through the action of an inversion a Voronoi cell into a convex hull and [CY14] for the rewriting of the variance as an integral over ∂K of a covariance of so-called *scores* and the replacement of K by a disk in the calculation of these scores. Though it is specific to the smooth case and can hardly be extended to the variance of the perimeter, it may also reinforce the parallel with Rényi and Sulanke’s work and explain to some extent why these particular functionals of K appear in the limiting expectations and variances.

We describe in detail the technique in the case of $\mathcal{N}(K_\lambda^o)$ and explain at the end how to extend it for the defect area. The first step consists in associating to any $x \in \mathcal{P}_\lambda \setminus 2\mathcal{F}_o(K)$ the point y which is the closest point to ∂K on the bisecting line of the segment $[0, x]$. We denote by \mathcal{Q}_λ the point process constituted of the coordinates (s, h) of such points y . In particular, \mathcal{Q}_λ is a Poisson point process on $\partial K \times (0, \infty)$ and its intensity with respect to the measure $dsdh$ has a density equal to $q_\lambda(s, h) = 4\lambda r_s^{-1}(\langle s, n_s \rangle + h)$ thanks to the equalities $\|x\| = 2(\langle s, n_s \rangle + h)$ and $\frac{ds}{d\theta} = r_s$ (see also Figure 6.1).

For every $(s, h) \in \mathcal{Q}_\lambda$, we denote by $\xi((s, h), \mathcal{Q}_\lambda)$ the score of (s, h) , equal to either 1 if its ancestor in \mathcal{P}_λ is a Voronoi neighbor of o and 0 otherwise. In particular,

$$\mathcal{N}(K_\lambda^o) = \sum_{(s, h) \in \mathcal{Q}_\lambda} \xi((s, h), \mathcal{Q}_\lambda) .$$

show that the new score $\tilde{\xi}((r, \theta), \tilde{\mathcal{Q}}_{\lambda, s}) = \xi(\mathcal{I}_s^{-1}(\omega_s + ru_\theta), \mathcal{Q}_\lambda)$, $(r, \theta) \in (0, 1) \times (0, 2\pi)$, is equal to 1 if (r, θ) is an extreme point of $\tilde{\mathcal{Q}}_{\lambda, s} \cup \{(r, \theta)\}$ and 0 otherwise. In the same way, we define the new pair correlation function $\tilde{c}((r, \theta), (r', \theta'), \tilde{\mathcal{Q}}_{\lambda, s})$. Applying the change of variables $r = \frac{r_s}{r_s + h}$ in $I_{1, \lambda}$ which means taking r for the distance between ω_s and the image by \mathcal{I}_s of (s, h) , we obtain that

$$I_{1, \lambda} = \lambda^{-\frac{1}{3}} \int_{\partial K} \int_{r \in (0, 1)} \mathbb{E} \left(\tilde{\xi}^2 \left((r, 0), \tilde{\mathcal{Q}}_{\lambda, s} \right) \right) 4\lambda r_s^{-1} \left(\langle s, n_s \rangle + \frac{r_s}{r} - r_s \right) \frac{r_s}{r^2} dr ds.$$

Let us define $\varepsilon_\lambda = (\frac{\log \lambda}{\lambda})^{-\frac{1}{3}}$. We claim that with high probability, only the values of r close to 1 up to a distance of order $O(\varepsilon_\lambda^2)$ will contribute significantly to the integral above and moreover, that only the points of $\tilde{\mathcal{Q}}_{\lambda, s}$ at a distance $O(\varepsilon_\lambda)$ will be needed for the computation of the scores. Indeed, this is due to a result which can be proved analogously to [CY14, Lemma 4.1 (a), Lemma 4.2] and which says the following: there exists a positive constant C such that with probability $(1 - O(\lambda^{-8}))$,

$$\tilde{\xi}((r, 0), \tilde{\mathcal{Q}}_{\lambda, s}) = \begin{cases} \tilde{\xi}((r, 0), \tilde{\mathcal{Q}}_{\lambda, s} \cap B_{C\varepsilon_\lambda}(s)) & \text{if } r \geq 1 - C\varepsilon_\lambda^2 \\ 0 & \text{otherwise.} \end{cases}$$

Note that [CY14] assumes a \mathcal{C}^3 -smooth boundary but that in particular, Lemmas 4.1 and 4.2 therein still hold with the weaker assumption of \mathcal{C}^2 -regularity. Recalling now that when $(r, \theta) \rightarrow (1, 0)$, the intensity of $\tilde{\mathcal{Q}}_{\lambda, s}$ goes to $4\lambda r_s \langle s, n_s \rangle$, we obtain that

$$(6.2) \quad I_{1, \lambda} \underset{\lambda \rightarrow \infty}{\sim} \lambda^{-\frac{1}{3}} \int_{\partial K} r_s^{-1} \cdot 4\lambda r_s \langle s, n_s \rangle \int_0^1 \mathbb{E} \left(\tilde{\xi}^2 \left((r, 0), \mathcal{P}_{4\lambda r_s \langle s, n_s \rangle} \cap B_1(o) \right) \right) dr ds.$$

We proceed analogously for $I_{2, \lambda}$, i.e. we apply the two changes of variables $r = \frac{r_s}{r_s + h}$ and (r', θ') as the polar coordinates with respect to ω_s of the image by \mathcal{I}_s of the point (s', h') . Using that with high probability $\tilde{\xi}((r, 0), \tilde{\mathcal{Q}}_{\lambda, s}) = 0$ as soon as r exceeds $O((\frac{\log \lambda}{\lambda})^{-\frac{2}{3}})$ and that $\tilde{c}((r, 0), (r', \theta'), \tilde{\mathcal{Q}}_{\lambda, s}) = 0$ as soon as the distance between the two points with polar coordinates $(r, 0)$ and (r', θ') exceeds $O((\frac{\log \lambda}{\lambda})^{-\frac{2}{3}})$, see [CY14, Lemma 4.1], we obtain that

$$(6.3) \quad \begin{aligned} \lambda^{\frac{1}{3}} I_{2, \lambda} &= \int_{\partial K} \int_{r, r' \in (0, 1), \theta' \in (0, 2\pi)} \tilde{c}((r, 0), (r', \theta'), \tilde{\mathcal{Q}}_{\lambda, s}) 4\lambda \left(\langle s, n_s \rangle + \frac{r_s}{r} - r_s \right) \\ &\quad \tilde{q}_{\lambda, s}(r', \theta') \frac{dr}{r^2} dr' d\theta' ds \underset{\lambda \rightarrow \infty}{\sim} \int_{\partial K} r_s^{-1} \cdot (4\lambda r_s \langle s, n_s \rangle)^2 \\ &\quad \int_{r, r' \in (0, 1), \theta' \in (0, 2\pi)} \tilde{c}((r, 0), (r', \theta'), \mathcal{P}_{4\lambda r_s \langle s, n_s \rangle} \cap B_1(o)) dr dr' d\theta' ds. \end{aligned}$$

Using [CSY13, Theorem 7.1], we have for fixed $s \in \partial K$

$$(6.4) \quad \begin{aligned} &\lim_{\lambda \rightarrow \infty} (4\lambda r_s \langle s, n_s \rangle)^{-\frac{1}{3}} \left(4\lambda r_s \langle s, n_s \rangle \int_0^1 \mathbb{E} \left(\tilde{\xi}^2((r, 0), \mathcal{P}_{4\lambda r_s \langle s, n_s \rangle} \cap B_1(o)) \right) dr \right. \\ &\quad \left. + (4\lambda r_s \langle s, n_s \rangle)^2 \int_{r, r' \in (0, 1), \theta' \in (0, 2\pi)} \tilde{c}((r, 0), (r', \theta'), \mathcal{P}_{4\lambda r_s \langle s, n_s \rangle} \cap B_1(o)) dr dr' d\theta' \right) = \frac{c_{\mathcal{N}}}{2\pi} \end{aligned}$$

where $c_{\mathcal{N}}$ is the positive limiting variance of the number of extreme points of a homogeneous Poisson point process inside the unit disk. Combining (6.4) with (6.1), (6.2) and (6.3), we obtain Theorem 1.5(ii).

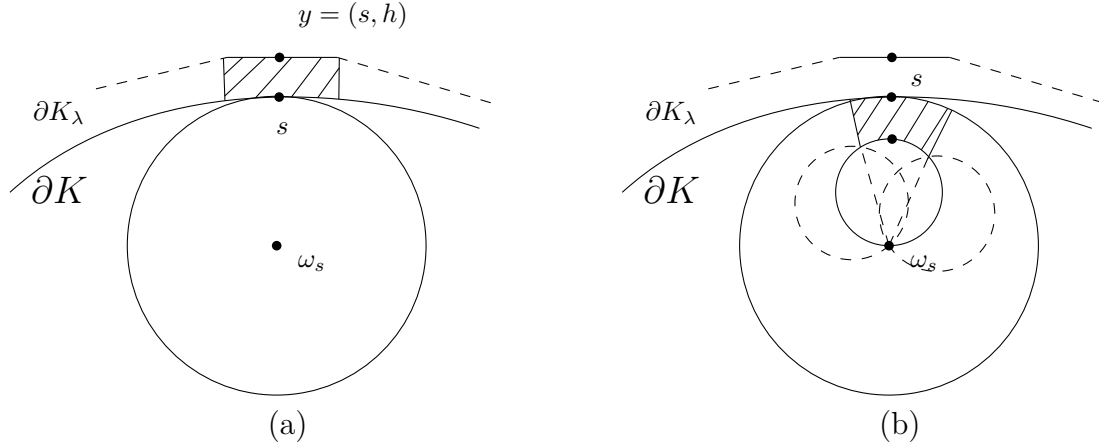


Figure 6.2. The two scores ξ (a) and $\tilde{\xi}$ (b) for the variance of $\mathcal{A}(K_\lambda^o)$

Regarding the variance of $\mathcal{A}(K_\lambda^o)$, we proceed in the exact same way, the only notable difference being that we take $\xi((s, h), \mathcal{Q}_\lambda)$ equal to the area between ∂K and the Voronoi edge contained in the bisecting line that (s, h) belongs to if such Voronoi edge exists and 0 otherwise, see the hatched region in Figure 6.2(a). Denoting by μ_s the image of the Lebesgue measure by \mathcal{I}_s , is also equal to the μ_s -measure of the hatched region on Figure 6.2(b) when $(r, 0)$ is an extreme point of $\tilde{\mathcal{Q}}_{\lambda, s} \cap B_1(o)$ and 0 otherwise. Since with high probability this hatched region is very close to the boundary of $B_1(o)$, the μ_s -measure is almost equal to r_s^2 times the Lebesgue measure. Consequently, we choose for $\tilde{\xi}((r, 0), \tilde{\mathcal{Q}}_{\lambda, s})$ the Lebesgue measure of the hatched region and define \tilde{c} accordingly.

Thanks to [CSY13, Theorem 7.1], we get that

$$(6.5) \quad \lim_{\lambda \rightarrow \infty} (4\lambda r_s \langle s, n_s \rangle)^{\frac{5}{3}} \left(4\lambda r_s \langle s, n_s \rangle \int_0^1 \mathbb{E} \left(\tilde{\xi}^2((r, 0), \mathcal{P}_{4\lambda r_s \langle s, n_s \rangle} \cap B_1(o)) \right) dr \right. \\ \left. + (4\lambda r_s \langle s, n_s \rangle)^2 \int_{r, r' \in (0, 1), \theta' \in (0, 2\pi)} \tilde{c}((r, 0), (r', \theta'), \mathcal{P}_{4\lambda r_s \langle s, n_s \rangle} \cap B_1(o)) dr dr' d\theta' \right) = \frac{c_{\mathcal{A}}}{2\pi}$$

where $c_{\mathcal{A}}$ is the positive limiting variance of the area between the boundary of $B_1(o)$ and the Voronoi flower of a homogeneous Poisson point process inside the unit disk. Incidentally, this quantity is also the limiting variance of π times the mean width of the convex hull of the same homogeneous Poisson point process. Finally, (6.5) combined with the fact that $\text{Var}(\mathcal{A}(K_\lambda^o))$ is equivalent to the integral over ∂K of r_s^3 times the quantity inside the brackets in (6.5) implies Theorem 1.5(i). \square

A byproduct of the method developed in the proof of Theorem 1.5 is that

$$\mathbb{E}(\mathcal{A}(K_\lambda^o)) - \mathcal{A}(K) \underset{\lambda \rightarrow \infty}{\sim} \lambda^{-\frac{2}{3}} 2^{-\frac{7}{3}} a_\infty \int_{\partial K} r_s^{\frac{1}{3}} \langle s, n_s \rangle^{-\frac{2}{3}} ds$$

and

$$\mathbb{E}(\mathcal{N}(K_\lambda^o)) \underset{\lambda \rightarrow \infty}{\sim} \lambda^{\frac{1}{3}} 2^{-\frac{1}{3}} \pi^{-1} n_\infty \int_{\partial K} r_s^{-\frac{2}{3}} \langle s, n_s \rangle^{\frac{1}{3}} ds$$

where a_∞ (resp. n_∞) is the normalized limiting expectation of the defect mean width (resp. number of vertices) of the convex hull of a homogeneous Poisson point process in the unit disk.

Using [RS63, Satz 3] and [RS64, Satz 1], the two equalities above imply points (i) and (iii) of Theorem 1.1. Nevertheless, we have chosen to prove them in Section 3 through a direct calculation instead because it is more natural and self-contained on one hand and easier to extend to the defect perimeter and to the polygonal case on the other hand. Besides, the method used in Section 3 implies Proposition 3.3 on the support points which is new to the best of our knowledge and interesting on its own.

The extension of the method from Section 6 to the case of the perimeter is indeed problematic. This comes from the fact that it relies in particular on the application of an inversion with respect to an osculating circle of K and the use of the known asymptotics for a random convex hull inside a disk. There have been in the past successful uses of the polarisation of a convex body in order to derive asymptotics for the perimeter or mean width of random convex hulls, see e.g. [GG97, BR04]. Only understanding how this transition to the polar body would impact the construction of the Poisson–Voronoi cell seems non-trivial. This could suggest that we should adapt the technique of [CY14] without any use of an inversion or a polarisation, i.e. show the existence of a scaling limit of the boundary of the Voronoi tessellation in the osculating disk and define limiting scores associated with the defect perimeter in the rescaled space. This program seems reachable for both the limiting expectation and limiting variance of the defect perimeter, though it would not provide any explicit calculation of the constants involved, contrary to Theorem 1.1 (ii). We leave it for future research.

We believe that the extension to higher dimension of the method from Section 6 is delicate and would require a non-trivial input which is still unclear up to now. Indeed, the technique relies on the approximation of the boundary of K near a boundary point by an osculating circle, then on the use of the inversion transformation and the required asymptotic result for the convex hull of a random set of points in the disk. In higher dimension, the boundary of K is locally approximated by an ellipsoid. In the case of the convex hull of a Poisson point process, [CY14] settles that problem by applying a volume-preserving affine transformation so that the principal curvatures of K at a fixed boundary point are all equal. This does not modify the set of extreme points, as well as the defect volume of the random convex hull. The considered functionals, number of k -dimensional faces and defect volume, are then estimated inside an *osculating ball* where there are an available global scaling transformation and asymptotics ready for use, see e.g. [CSY13, CY14]. But in our case, the Voronoi construction is not preserved by the application of an affine transformation which is not isometric. Up to now, we have not been able to overcome this new difficulty.

Regarding the polygonal case, there is no easy transformation which could play the role of the inversion, i.e. send the Voronoi cell to the convex hull of a random set of points inside the polygon and allow us then to use the known results on the random convex hull. A direct proof designed to mimic the strategy of [CY17] would induce many technical results currently missing (negligibility of the flat parts, decorrelation

of the parts around the vertices of the polygon). Moreover, it could very well crash into the problem of defining a universal scaling transformation in the vicinity of a vertex of the polygon since the affine transformations do not preserve the Voronoi construction.

7. The Crofton cell

The main issues of the paper prove to be equally appealing when the Poisson–Voronoi tessellation is replaced by any random line tessellation in the plane and in particular by the stationary and isotropic Poisson line tessellation, see e.g. [SW08, Section 10.3]. This tessellation is obtained by taking a Poisson point process $\overline{\mathcal{P}}_\lambda$ of intensity measure $\lambda\|x\|^{-1}dx$ in \mathbb{R}^2 and constructing for every x in the point process, the line L_x containing x and with normal vector x . The cell containing the origin is the so-called *Crofton cell* and is defined as the intersection of all closed half-planes containing the origin and delimited by lines L_x . We denote by \overline{K}_λ a cell distributed as the Crofton cell conditional on the event that no line crosses K , which is equivalent to say that no point from $\overline{\mathcal{P}}_\lambda$ meets $\mathcal{F}_o(K)$. We recall that thanks to the Cauchy–Crofton formula, this event has probability

$$\begin{aligned} \exp\left(-\lambda \int_{\mathcal{F}_o(K)} \|x\|^{-1} dx\right) &= \exp\left(-\lambda \int_0^{2\pi} \int_0^{p_o(K,\theta)} r^{-1} r dr d\theta\right) \\ &= \exp\left(-\lambda \int_0^{2\pi} p_o(K, \theta) d\theta\right) \\ (7.1) \qquad \qquad \qquad &= \exp(-\lambda \mathcal{U}(K)). \end{aligned}$$

This new random polygon \overline{K}_λ satisfies (2.5) and its rewritings given in Corollary 2.2 (i) for the smooth case and in Corollary 2.3 for the polygonal case. The identity (2.3) is replaced by

$$(7.2) \quad \mathbb{E}(\mathcal{A}(\overline{K}_\lambda)) - \mathcal{A}(K) = \int_{\mathbb{R}^2 \setminus K} \exp\left(-\lambda(\mathcal{U}(\text{conv}(K \cup \{x\})) - \mathcal{U}(K))\right) dx.$$

We state below the direct analogues for \overline{K}_λ of Proposition 2.1, Theorems 1.1 and 1.2 with the same notation. In particular, the Efron-type identity given at (a) can be seen as a consequence of [Sch09, penultimate display on page 693].

THEOREM 7.1. —

(a) *For every $\lambda > 0$, the following identity holds*

$$\mathbb{E}(\mathcal{N}(\overline{K}_\lambda)) = \lambda(\mathbb{E}(\mathcal{U}(\overline{K}_\lambda)) - \mathcal{U}(K)).$$

(b) Let K be a smooth convex body containing o in its interior. The mean defect area, defect perimeter and number of vertices of K_λ^o have respectively the following asymptotics when the intensity $\lambda \rightarrow \infty$:

$$\begin{aligned} \text{(i)} \quad \mathbb{E}(\mathcal{A}(\overline{K}_\lambda)) - \mathcal{A}(K) &\underset{\lambda \rightarrow \infty}{\sim} \lambda^{-\frac{2}{3}} 2^{-\frac{2}{3}} 3^{-\frac{1}{3}} \Gamma\left(\frac{2}{3}\right) \int_{\partial K} r_s^{\frac{1}{3}} ds \\ \text{(ii)} \quad \mathbb{E}(\mathcal{U}(\overline{K}_\lambda)) - \mathcal{U}(K) &\underset{\lambda \rightarrow \infty}{\sim} \lambda^{-\frac{2}{3}} 2^{\frac{4}{3}} 3^{-\frac{4}{3}} \Gamma\left(\frac{2}{3}\right) \int_{\partial K} r_s^{-\frac{2}{3}} ds \\ \text{(iii)} \quad \mathbb{E}(\mathcal{N}(\overline{K}_\lambda)) &\underset{\lambda \rightarrow \infty}{\sim} \lambda^{\frac{1}{3}} 2^{\frac{4}{3}} 3^{-\frac{4}{3}} \Gamma\left(\frac{2}{3}\right) \int_{\partial K} r_s^{-\frac{2}{3}} ds. \end{aligned}$$

(c) Let K be a convex polygon containing o in its interior. The mean defect area, defect perimeter and number of vertices of K_λ^o have respectively the following asymptotics when the intensity $\lambda \rightarrow \infty$:

$$\begin{aligned} \text{(i)} \quad \mathbb{E}(\mathcal{A}(\overline{K}_\lambda)) - \mathcal{A}(K) &\underset{\lambda \rightarrow \infty}{\sim} \lambda^{-\frac{1}{2}} 2^{-\frac{5}{2}} \pi^{\frac{3}{2}} \sum_{i=1}^{n_K} \|a_{i+1} - a_i\|^{\frac{3}{2}} \\ \text{(ii)} \quad \mathbb{E}(\mathcal{U}(\overline{K}_\lambda)) - \mathcal{U}(K) &\underset{\lambda \rightarrow \infty}{\sim} (\lambda^{-1} \log \lambda) \cdot 2 \cdot 3^{-1} n_K \\ \text{(iii)} \quad \mathbb{E}(\mathcal{N}(\overline{K}_\lambda)) &\underset{\lambda \rightarrow \infty}{\sim} (\log \lambda) \cdot 2 \cdot 3^{-1} n_K. \end{aligned}$$

Proof of Theorem 7.1. — As all these results can be derived in a very similar way to what we did in the previous sections, we only sketch the proof below.

- (a) As in Proposition 2.1, we observe that $x \in \overline{\mathcal{P}}_\lambda$ gives birth to a side of the Crofton cell \overline{K}_λ if and only if L_x intersects the Crofton cell \overline{C}_x corresponding to the point process $(\overline{\mathcal{P}}_\lambda \setminus \mathcal{F}_o(K)) \setminus \{x\}$. This is equivalent to saying that $x \in \mathcal{F}_o(\overline{C}_x) \setminus \mathcal{F}_o(K)$. We then conclude by using Mecke–Slivnyak’s formula combined with (7.1).
- (b) (i) We need an analogue of Lemma 3.1 for the quantity $\mathcal{U}(K \cup \{s_h\}) - \mathcal{U}(K)$ where we recall that $s_h = s + hn_s$ for any $s \in \partial K$ and $h > 0$. Using the Cauchy–Crofton formula, we get with the same notation as in the proof of Lemma 3.1

$$(7.3) \quad \mathcal{U}(K \cup \{s_h\}) - \mathcal{U}(K) = \int_{-\theta_{s,h}^-}^{\theta_{s,h}^+} \Delta p_{\omega_s}(\theta) d\theta \underset{h \rightarrow 0}{=} 2^{\frac{5}{2}} 3^{-1} r_s^{-\frac{1}{2}} h^{\frac{3}{2}} + o\left(h^{\frac{3}{2}}\right).$$

We then insert (7.3) into (7.2) and apply Lebesgue’s dominated convergence theorem.

- (b) (ii) We start by stating an analogue of Proposition 3.3: denoting by $(\overline{X}_{s,\lambda}, \overline{Y}_{s,\lambda})$ the support point of $\partial \overline{K}_\lambda$ in direction n_s , we can show that the couple $(\lambda^{\frac{1}{3}} \overline{X}_{s,\lambda}, \lambda^{\frac{2}{3}} \overline{Y}_{s,\lambda})$ converges in distribution to the distribution with density function equal to

$$2^{\frac{3}{2}} r_s^{-\frac{3}{2}} \exp \left(-2^{\frac{5}{2}} 3^{-1} r_s^{-\frac{1}{2}} \left(\frac{x^2}{2r_s} + y \right)^{\frac{3}{2}} \right) \left(\frac{x^2}{2r_s} + y \right)^{\frac{1}{2}} y \mathbb{I}_{\{y > 0\}}.$$

In particular, this implies

$$(7.4) \quad \mathbb{E}(\overline{Y}_{s,\lambda}) \underset{\lambda \rightarrow \infty}{\sim} \lambda^{-\frac{2}{3}} 2^{\frac{4}{3}} 3^{-\frac{4}{3}} \Gamma\left(\frac{2}{3}\right) r_s^{\frac{1}{3}}.$$

Inserting (7.4) into the equality (i) from Corollary 2.2 applied to \overline{K}_λ , we get the required result.

- (c) (i) We prove an analogue of Lemma 4.1 for the quantity $\mathcal{U}(K \cup \{s_{a_i, \rho, \alpha}\}) - \mathcal{U}(K)$ for $s_{a_i, \rho, \alpha} \in \mathcal{S}_i$ where we recall that $s_{a_i, \rho, \alpha} = a_i + \rho u_{\pi - \alpha}$. We get

$$(7.5) \quad \begin{aligned} & \mathcal{U}(K \cup \{s_{a_i, \rho, \alpha}\}) - \mathcal{U}(K) \\ &= \|a_i - s_{a_i, \rho, \alpha}\| + \|a_{i+1} - s_{a_i, \rho, \alpha}\| - \|a_{i+1} - a_i\| \\ &= \rho + \left(\rho^2 + \|a_{i+1} - a_i\|^2 - 2\rho \|a_{i+1} - a_i\| \cos \alpha \right)^{\frac{1}{2}} - \|a_{i+1} - a_i\| \\ &\underset{\alpha \rightarrow 0}{=} \frac{\alpha^2}{2} \frac{\rho \|a_{i+1} - a_i\|}{\|a_{i+1} - a_i\| - \rho} + o(\alpha^2). \end{aligned}$$

Inserting (7.5) into (7.2), applying Lebesgue's dominated convergence theorem for the integrals over the regions \mathcal{S}_i and showing that the integrals over the regions \mathcal{G}_i are negligible, we obtain (c) (i).

- (c) (ii) We need to derive an analogue of Proposition 4.2: let us denote by $(\overline{R}_{\lambda^{-\gamma}}, \overline{A}_{\lambda^{-\gamma}})$ the polar coordinates of the support point of \overline{K}_λ in direction $\delta_i - \lambda^{-\gamma}$ with respect to the coordinate system with origin a_i and first axis (a_i, a_{i+1}) . Then for every $\gamma \in (0, \frac{1}{2})$, the couple $(\lambda^{1-2\gamma} \overline{R}_{\lambda^{-\gamma}}, \lambda^\gamma \overline{A}_{\lambda^{-\gamma}})$ converges in distribution to the distribution with density function f_i given by

$$\frac{1}{2} \exp \left(-\frac{1}{2} \rho \alpha^2 \right) \alpha (\alpha - 1) \rho \mathbb{I}_{\{\rho > 0\}} \mathbb{I}_{\{\alpha > 1\}}.$$

In particular, this implies that the expectation of the defect support function $\overline{Z}_{i, \gamma, \lambda, -}$ in direction $(\delta_i - \lambda^{-\gamma})$ satisfies

$$(7.6) \quad \mathbb{E}(\overline{Z}_{i, \gamma, \lambda, -}) = \mathbb{E}(\overline{R}_{\lambda^{-\gamma}} \sin(\overline{A}_{\lambda^{-\gamma}} - \lambda^{-\gamma})) \underset{\lambda \rightarrow \infty}{\sim} \frac{2}{3} \lambda^{\gamma-1}.$$

Inserting (7.6) into the equality (i) from Corollary 2.3 applied to \overline{K}_λ and showing that the integral over $\gamma > \frac{1}{2}$ is negligible yields the desired estimate.

- (b) (iii) & (c) (iii) These estimates are direct consequences of (a) combined with (b) (ii) and (c) (ii) respectively. \square

In 1968, Rényi and Sulanke investigated a model close to the Crofton cell, save for the fact that they did not use the notion of point process in the whole plane. Instead, they fixed a domain B which includes K and they considered the polygon containing K and delimited by n random lines which intersect B without crossing K . This is on a par with the actual Crofton cell when the number of lines is Poissonized and the set B goes to \mathbb{R}^2 . In this context, they obtained the mean number of vertices in the smooth and polygonal cases, see [RS68, Sätze 4 and 5]. Replacing $\frac{n}{b-l} = \frac{n}{u(B)-u(K)}$ by λ in their formulas provides the exact same results as ours. To the best of our knowledge, they did not cover the calculations for the asymptotic mean area and mean perimeter, nor did they establish an Efron-type relation. Let us also notice that the asymptotics of Theorem 7.1(b) extend to any smooth convex body the results for the defect area and number of vertices obtained in [CS05, Theorem 2] when K is a disk. When K is any convex body, let us note that the very recent preprint [HS19] provides bounds for the defect mean width in any dimension for a more general model which includes the Crofton cell. Finally, it comes as no surprise that contrary to the Voronoi case, the limiting expectations do not depend on the position of K with respect to o . Indeed, by stationarity, the origin has no privileged status among the points of the Crofton cell.

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