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EXTENSIONS OF MAXIMAL SYMPLECTIC ACTIONS ON K3 SURFACES

EXTENSIONS D' ACTIONS SYMPLECTIQUES
MAXIMALES SUR LES SURFACES K3

ABSTRACT. — We classify pairs (X, G) consisting of a complex K3 surface X and a finite group $G \leq \text{Aut}(X)$ such that the subgroup $G_s \leq G$ consisting of symplectic automorphisms is among the 11 maximal symplectic ones as classified by Mukai.

RÉSUMÉ. — Nous classifions les paires (X, G) formées d'une surface K3 complexe X et d'un groupe fini $G \leq \text{Aut}(X)$ pour lesquelles le sous-groupe $G_s \leq G$ des automorphismes symplectiques appartient aux 11 sous-groupes symplectiques maximaux classifiés par Mukai.

1. Introduction

A (complex) *K3 surface* is a compact, complex manifold X of dimension 2 which is simply connected and admits a nowhere degenerate holomorphic symplectic form

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$\sigma_X \in H^0(X, \Omega_X^2)$ unique up to scaling. An automorphism of a K3 surface is called *symplectic* if it leaves the 2-form invariant and non-symplectic else.

Finite groups of symplectic automorphisms of K3 surfaces were classified by Mukai up to isomorphism of groups. Namely, a group acts faithfully and symplectically on some complex K3 surface if and only if it admits an embedding into the Mathieu group M_{24} which decomposes the 24 points into at least 5 orbits and fixes a point (in particular it is contained in M_{23}) [Kon98, Muk88]. This leads to a list of 11 maximal subgroups (with 5 orbits) among the subgroups of M_{24} meeting these conditions. A finer classification, namely up to equivariant deformation, was obtained in [Has12]. There are 14 maximal finite symplectic group actions (see Table 4.1).

However not every automorphism of a K3 surface is symplectic. Let X be a K3 surface and $G \leq \text{Aut}(X)$ a group of automorphisms. We remark that G is finite if and only if there is an ample class on X invariant under G . Denote by G_s the normal subgroup consisting of symplectic automorphisms. Let G be finite. Then we have a natural exact sequence

$$1 \rightarrow G_s \rightarrow G \xrightarrow{\rho} \mu_n \rightarrow 1$$

where $n \in \{k \in \mathbb{N} \mid \varphi(k) \leq 20\}$ and φ is the Euler totient function. The homomorphism ρ is defined by $g^*\sigma_X = \rho(g) \cdot \sigma_X$. In the present paper, we classify finite groups G of automorphisms of K3 surfaces, under the condition that G_s is among the 11 maximal groups and $G_s \subsetneq G$. As it turns out, this forces the underlying K3 surface X to have maximal Picard number 20, i.e. it is a singular K3 surface. In particular it has infinite automorphism group. Moreover, those K3 surfaces (with G) are rigid (i.e. not deformable). Let (X, G) and (X', G') be two pairs of K3 surfaces with a group of automorphisms. They are called isomorphic if there is an isomorphism $f: X \rightarrow X'$ with $fGf^{-1} = G'$.

THEOREM 1.1. — *Let X be a K3 surface and $G \leq \text{Aut}(X)$ a maximal finite group of automorphisms such that the symplectic part G_s is isomorphic to one of the 11 maximal groups and $G_s \subsetneq G$. Then the pair (X, G) is isomorphic to one of the 42 pairs listed in Section 6.*

The representations of the groups G on the K3 lattices $\Lambda \cong H^2(X, \mathbb{Z})$ are given in the ancillary file to [BH19] on arXiv.

The proof goes via a classification, up to conjugacy, of suitable finite subgroups of the orthogonal group of the K3 lattice. Then the strong Torelli type theorem [PŠŠ71, BR75] and the surjectivity of the period map [Tod80] abstractly provide the existence and uniqueness of the pairs (X, G) .

Note that a K3 surface admitting a non-symplectic automorphism of finite order must be projective [Nik79a, Theorem 3.1]. Thus, at least in principle, it is possible to find projective models of the K3 surfaces and the automorphisms. For 25 out of the 42 pairs (X, G) we list explicit equations in Section 6. Using the Torelli-type theorem Kondō proved in [Kon99] that the maximal order of a finite group of automorphisms of a K3 surface equals 3840. Section 7 is devoted to deriving its equations for the first time.

For the full story of symplectic groups of automorphisms we recommend the excellent survey [Kon18]. Non-symplectic automorphisms of prime order are treated

in [AST11]. In [Fra11], a similar classification with different methods is carried out, albeit under the restrictive condition that $G = G_s \times \mu_2$ and the action by μ_2 has fixed points. Note that the author misses cases **70d** and **76a** (see Section 6).

Remark 1.2. — Let (X, G) be as in Theorem 1.1. It turns out that the non-symplectic part $G/G_s \cong \mu_n$ is of even order and the pair (X, G) is determined up to isomorphism already by G_s and any involution in G/G_s . See Section 5 for details.

Open problems

We close this section with some interesting problems concerning groups of automorphisms of K3 surfaces.

- (1) Find the remaining 17 missing equations among the 42 K3 surfaces and their automorphisms.
- (2) Give generators of the full automorphism group of the corresponding K3 surfaces. Since a Conway chamber in the nef cone of this surface has large symmetry, chances are that one can find a nice generating set for the automorphism group.
- (3) Find a projective model of the K3 surface with a linear action by M_{22} in characteristic 11. Its existence is proven by Kondō [Kon06] using the crystalline Torelli type theorem.
- (4) Use the present classification to study finite groups of automorphisms of Enriques surfaces beyond the semi-symplectic case [MO14].

Finding equations for the surface is often much easier than for the automorphisms. Should you find equations or relevant publications on one of the surfaces treated here, please notify the first author. We will update the arXiv version of this paper with your findings.

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2. Lattices

In this section we recall the basics on integral lattices (equivalently quadratic forms) and fix notation. The results are found in [CS99, Nik79b].

A *lattice* consists of a finitely generated free \mathbb{Z} -module L and a non-degenerate integer valued symmetric bilinear form

$$\langle \cdot, \cdot \rangle: L \times L \rightarrow \mathbb{Z}.$$

Given a basis (b_1, \dots, b_n) of L , we obtain the *Gram matrix* $Q = (\langle b_i, b_j \rangle)_{1 \leq i, j \leq n}$. The determinant $\det Q$ is independent of the choice of basis and called the *determinant* of the lattice L ; it is denoted by $\det L$. We display lattices in terms of their Gram matrices. The *signature* of a lattice is the signature of its Gram matrix. We denote it by (s_+, s_-) where s_+ (respectively s_-) is the number of positive (respectively negative) eigenvalues. We define the *dual lattice* L^\vee of L by $L^\vee = \{x \in L \otimes \mathbb{Q} \mid \langle x, L \rangle \subseteq \mathbb{Z}\} \cong \text{Hom}(L, \mathbb{Z})$. The *discriminant group* L^\vee/L is a finite abelian group of cardinality $|\det L|$. We call a lattice *unimodular* if $L = L^\vee$, and we call it *even* if $\langle x, x \rangle$ is even for all $x \in L$. The discriminant group of an even lattice carries the *discriminant form*

$$q_L: L^\vee/L \rightarrow \mathbb{Q}/2\mathbb{Z}, \quad \bar{x} \mapsto \langle x, x \rangle + 2\mathbb{Z}.$$

An *isometry* of lattices is a linear map compatible with the bilinear forms. The *orthogonal group* $O(L)$ is the group of isometries of L and the special orthogonal group $SO(L)$ consists of the isometries of determinant 1. Discriminant forms are useful to describe embeddings of lattices and extensions of isometries. A sublattice $L \subseteq M$ is called *primitive*, if $L = (L \otimes \mathbb{Q}) \cap M$. By definition, the orthogonal complement $S^\perp \subseteq M$ of a (not necessarily primitive) sublattice S is a primitive sublattice. For L_1 primitive and $L_2 = L_1^\perp$ we call $L_1 \oplus L_2 \subseteq M$ a *primitive extension*. Now, suppose that M is even, unimodular, then

$$H_M = M/(L_1 \oplus L_2) \subseteq (L_1^\vee/L_1) \oplus (L_2^\vee/L_2)$$

is the graph of a so called *glue map* $\phi_M: L_1^\vee/L_1 \rightarrow L_2^\vee/L_2$, that is, any element in H_M is of the form $x \oplus \phi_M(x)$ for $x \in L_1^\vee/L_1$. This isomorphism is an anti-isometry, namely, it satisfies $q_{L_2} \circ \phi_M = -q_{L_1}$. Conversely given such an anti-isometry ϕ , its graph H_ϕ defines a primitive extension $L_1 \oplus L_2 \subseteq M_\phi$ with M_ϕ even, unimodular.

Given an isometry $f \in O(L_1)$, it induces an isometry $\bar{f} \in O(L_1^\vee/L_1)$ of the discriminant group. Let $g \in O(L_2)$ be an isometry on the orthogonal complement. Then $f \oplus g \in O(L_1 \oplus L_2)$ extends to M if and only if $(\bar{f} \oplus \bar{g})(H_M) = H_M$, or equivalently, $\phi_M \circ \bar{f} = \bar{g} \circ \phi_M$.

LEMMA 2.1. — *Let $L \subseteq M$ be a primitive sublattice of an even unimodular lattice M . Set $O(M, L) = \{f \in O(M) \mid f(L) = L\}$ and $K = L^\perp$. If the natural map $O(K) \rightarrow O(K^\vee/K)$ is surjective, then the restriction map $O(M, L) \rightarrow O(L)$ is surjective. In other words: any isometry of L can be extended to an isometry of M .*

Proof. — Denote the glue map by $\phi = \phi_M$, and let $g \in O(K)$ be a preimage of $\phi \circ \bar{f} \circ \phi^{-1}$. Then $\phi \circ \bar{f} = \bar{g} \circ \phi$. Hence, $f \oplus g$ extends to M . \square

Let L be a lattice and $G \leq O(L)$. We define the *invariant* and *coinvariant lattices* respectively by

$$L^G = \{x \in L \mid \forall g \in G: g(x) = x\} \quad \text{and} \quad L_G = (L^G)^\perp.$$

Then, by definition, $L^G \oplus L_G \subseteq L$ is a primitive extension. Two lattices are said to be in the same genus, if they become isometric after tensoring with the p -adics \mathbb{Z}_p for all primes p and the reals \mathbb{R} . A genus is denoted in terms of the Conway–Sloane symbols [CS99, Chapter 15]. For instance the genus of even unimodular lattices of signature $(3, 19)$ is denoted by $\text{II}_{3,19}$. In fact all lattices in this genus are isometric.

3. K3 surfaces and the Torelli type theorem

In this section we recall standard facts about complex K3 surfaces. All results can be found in the textbooks [BHPVdV04, Huy16].

Let X be a K3 surface. Its second integral cohomology group $H^2(X, \mathbb{Z})$ together with the cup product is an even unimodular lattice of signature $(3, 19)$. It comes equipped with an integral weight 2 Hodge structure. Such a Hodge structure is given by its Hodge decomposition

$$H^2(X, \mathbb{Z}) \otimes \mathbb{C} = H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

with $H^{i,j}(X) = \overline{H^{j,i}(X)}$ and natural isomorphisms $H^{i,j} \cong H^j(X, \Omega_X^i)$. The corresponding Hodge numbers are $h^{2,0} = h^{0,2} = 1$ and $h^{1,1} = 20$. We can recover the entire Hodge structure from $H^{2,0}(X)$ via $H^{0,2}(X) = \overline{H^{2,0}(X)}$ and $H^{1,1}(X) = (H^{2,0}(X) \oplus H^{0,2}(X))^{\perp}$.

The *transcendental lattice* of a K3 surface is defined as the smallest primitive sublattice T_X of $H^2(X, \mathbb{Z})$ such that $T_X \otimes \mathbb{C}$ contains the period $H^{2,0}(X) = \mathbb{C}\sigma_X$. By the Lefschetz theorem on $(1, 1)$ -classes, the *Néron–Severi lattice* NS_X of a K3 surface is given by $H^{1,1}(X) \cap H^2(X, \mathbb{Z})$. Note that NS_X and T_X can be degenerate [Nik79a, (3.5)]. But if X is projective, then they are (non-degenerate) lattices of signatures $(1, \rho - 1)$ and $(2, 20 - \rho)$ respectively, and we have $\text{NS}_X = T_X^{\perp}$.

As a next step we want to compare Hodge structures of different K3 surfaces. For this we fix a reference frame, namely a lattice $\Lambda \in \text{II}_{3,19}$.

DEFINITION 3.1. — A marked K3 surface is a pair (X, η) consisting of a complex K3 surface X and an isometry $\eta: H^2(X, \mathbb{Z}) \rightarrow \Lambda$. We call η a marking.

We associate a marked K3 surface (X, η) with its *period*

$$\eta_{\mathbb{C}}(H^{2,0}(X)) \in \mathcal{P}_{\Lambda} := \{\mathbb{C}\sigma \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid \langle \sigma, \bar{\sigma} \rangle > 0, \langle \sigma, \sigma \rangle = 0\}.$$

Here we extend the bilinear form on Λ linearly to that on $\Lambda \otimes \mathbb{C}$. We call \mathcal{P}_{Λ} the *period domain*. As it turns out, the concept of marking works well in families. This allows one to define the moduli space \mathcal{M}_{Λ} of marked K3 surfaces and a period map

$$\mathcal{M}_{\Lambda} \rightarrow \mathcal{P}_{\Lambda}, \quad (X, \eta) \mapsto \eta_{\mathbb{C}}(H^{2,0}(X)).$$

The period map is in fact holomorphic, and it turns out to be surjective as well (the surjectivity of the period map for K3 surfaces [Tod80]). The moduli space \mathcal{M}_{Λ} is not very well behaved. For example it is not Hausdorff. This can be healed by taking into account the Kähler (resp. ample) cone.

The *positive cone* μ_X is the connected component of the set

$$\{x \in H^{1,1}(X, \mathbb{R}) \mid \langle x, x \rangle > 0\}$$

which contains a Kähler class. Set $\Delta_X = \{x \in \text{NS}_X \mid \langle x, x \rangle = -2\}$. An element in Δ_X is called a root. For $\delta \in \Delta_X$, either δ or $-\delta$ is an effective class by the Riemann–Roch theorem. In fact the effective cone is generated by the effective classes in Δ_X and the divisor classes in the closure of the positive cone (i.e. $\text{NS}_X \cap \overline{\mu_X}$). The connected components of $\mu_X \setminus \bigcup_{\delta \in \Delta_X} \delta^\perp$ are called the *chambers*. The hyperplanes δ^\perp for $\delta \in \Delta_X$ are called the *walls*. One of the chambers is the Kähler cone. For a root $\delta \in \Delta_X$, the reflection with respect to the wall δ^\perp is given by $r_\delta(x) = x + \langle x, \delta \rangle \delta$. The *Weyl group* is the subgroup of $O(H^2(X, \mathbb{Z}))$ generated by the reflections r_δ for $\delta \in \Delta_X$. The action of the Weyl group on the chambers is simply transitive. So by composing the marking with an element of the Weyl group, we can ensure that any given chamber in the positive cone of Λ corresponds to the Kähler cone.

DEFINITION 3.2. — *Let X, X' be K3 surfaces. An isometry $\phi: H^2(X, \mathbb{Z}) \rightarrow H^2(X', \mathbb{Z})$ is called a Hodge isometry if $\phi_{\mathbb{C}}(H^{i,j}(X)) \subseteq H^{i,j}(X')$ for all i, j . It is called effective, if it maps effective (resp. Kähler, resp. ample) classes on X to effective (resp. Kähler, resp. ample) classes on X' .*

The following Torelli type theorem for K3 surfaces is the key tool for our classification of automorphisms.

THEOREM 3.3 ([BR75, PŠŠ71]). — *Let X and X' be complex K3 surfaces. Let*

$$\phi: H^2(X, \mathbb{Z}) \rightarrow H^2(X', \mathbb{Z})$$

be an effective Hodge isometry. Then there is a unique isomorphism $f: X' \rightarrow X$ with $f^ = \phi$.*

We thus obtain a Hodge theoretic characterization of the automorphism group of a K3 surface.

COROLLARY 3.4. — *Let X be a complex K3 surface. Then the image of the natural homomorphism*

$$\text{Aut}(X) \rightarrow O(H^2(X, \mathbb{Z}))$$

consists of the isometries preserving the period and the Kähler cone.

4. Symplectic automorphisms

In this section we review known facts on symplectic automorphisms needed later on.

Let X be a complex K3 surface. We obtain an exact sequence

$$1 \rightarrow \text{Aut}(X)_s \rightarrow \text{Aut}(X) \xrightarrow{\rho} \text{GL}(\mathbb{C}\sigma_X).$$

(Recall that we have $\mathbb{C}\sigma_X = H^0(X, \Omega_X^2)$.) The elements of the kernel $\text{Aut}(X)_s$ of ρ are the symplectic automorphisms. An automorphism which is not symplectic is called non-symplectic. If $G \leq \text{Aut}(X)$ is a group of automorphisms, we denote by G_s

the kernel of $\rho|_G$ and call it the symplectic part of G . In order to keep the notation light, we identify G and its isomorphic image in $O(H^2(X, \mathbb{Z}))$.

Recall that if L is a lattice and $G \leq O(L)$, then L^G is the invariant and $L_G = (L^G)^\perp$ the coinvariant lattice. For the sake of completeness we give a proof of the following essential lemma.

LEMMA 4.1 (cf. [Nik79a]). — *Let $G_s \leq \text{Aut}(X)_s$ be a finite group of symplectic automorphisms of some K3 surface X . Then*

- (1) $T_X \subseteq H^2(X, \mathbb{Z})^{G_s}$ and $H^2(X, \mathbb{Z})_{G_s} \subseteq \text{NS}_X$;
- (2) $H^2(X, \mathbb{Z})^{G_s}$ is of signature $(3, k)$ for some $k \leq 19$;
- (3) $H^2(X, \mathbb{Z})_{G_s}$ is negative definite;
- (4) $H^2(X, \mathbb{Z})_{G_s}$ contains no vectors of square -2 ;
- (5) if G_s is maximal (that is, G_s is isomorphic to one of the 11 maximal finite groups of symplectic automorphisms), then $G_s \cong \ker(O(H) \rightarrow O(H^\vee/H))$ where $H = H^2(X, \mathbb{Z})_{G_s}$.

Proof.

- (1) The elements of G_s are all symplectic, i.e. they fix the 2-form σ_X . Thus $\mathbb{C}\sigma_X \subseteq H^2(X, \mathbb{Z})^{G_s} \otimes \mathbb{C}$. By minimality of the transcendental lattice and primitivity of the invariant lattice, we get $T_X \subseteq H^2(X, \mathbb{Z})^{G_s}$. Taking orthogonal complements yields the second inclusion.
- (2) Let κ' be a Kähler class. Since automorphisms preserve the Kähler cone, the class $\kappa = \sum_{g \in G} g^* \kappa'$ is a G_s -invariant Kähler class. Thus $\kappa, (\sigma_X + \bar{\sigma}_X)/2$ and $(\sigma_X - \bar{\sigma}_X)/(2i)$ span a positive definite subspace of dimension 3 of $H^2(X, \mathbb{R})^{G_s}$.
- (3) Recall that $H^2(X, \mathbb{Z})_{G_s} = (H^2(X, \mathbb{Z})^{G_s})^\perp$, and $H^2(X, \mathbb{Z})$ has signature $(3, 19)$. Now, use (2).
- (4) As before we take a G_s -invariant Kähler class κ . If $r \in \text{NS}_X$ is of square -2 , then either r or $-r$ is effective by the Riemann–Roch theorem. Thus $\langle \kappa, r \rangle \neq 0$. Since $H^2(X, \mathbb{Z})_{G_s}$ is orthogonal to κ , it cannot contain r .
- (5) Let g be an element in the kernel. Since g acts trivially on H^\vee/H , it can be extended to an isometry \tilde{g} on $H^2(X, \mathbb{Z})$ such that $\tilde{g}|_{H^\perp} = \text{id}_{H^\perp}$. As $H^\perp \otimes \mathbb{C}$ contains σ_X and a Kähler class, \tilde{g} is in fact an effective Hodge isometry. The strong Torelli type theorem implies that it is induced by a symplectic automorphism. Since the coinvariant lattice H is negative definite (by (3)), $O(H)$ is finite. In particular, the group \tilde{G} generated by G_s and g is a finite group. By the maximality of G_s , G_s must contain g . \square

THEOREM 4.2 ([Has12]). — *Let G_s be a finite group of symplectic automorphisms of a Λ -marked K3 surface. Identify G_s with its image in $O(\Lambda)$. Then the conjugacy class of G_s is determined by the isometry class of the invariant lattice Λ^{G_s} . For maximal G_s , the invariant lattices can be found in Table 4.1, and the coinvariant lattice Λ_{G_s} is uniquely determined up to isomorphism by the abstract group structure of G_s .*

For maximal G_s , we have $\text{rank } \Lambda^{G_s} = 3$ and $\text{rank } \Lambda_{G_s} = 19$ [Muk88]. The key observation we take from Lemma 4.1, is that the invariant lattice is definite and so is the coinvariant lattice. Hence, the direct product $O(H^2(X, \mathbb{Z})^{G_s}) \times O(H^2(X, \mathbb{Z})_{G_s})$

Table 4.1. Maximal finite symplectic groups of automorphisms

No.	G_s	$\#G_s$	$\det \Lambda^{G_s}$	genus of Λ^{G_s}	Λ^{G_s}	$\mathrm{SO}(\Lambda^{G_s})$	$\#O(\Lambda_{G_s})$	$\#O(q_{\Lambda_{G_s}})$
54	T_{48}	48	384	$2_1^{+1}, 8_{\mathrm{II}}^{-2}, 3^{+1}$	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 16 & 8 \\ 0 & 8 & 16 \end{pmatrix}$	D_6	9216	192
62	N_{72}	72	324	$4_7^{+1}, 3^{+2}, 9^{+1}$	$\begin{pmatrix} 6 & 0 & 3 \\ 0 & 6 & 3 \\ 3 & 3 & 12 \end{pmatrix}$	D_4	20736	288
63	M_9	72	216	$2_1^{-3}, 3^{+1}, 9^{+1}$	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 12 & 6 \\ 0 & 6 & 12 \end{pmatrix}$	D_6	5184	72
70	\mathfrak{S}_5	120	300	$4_5^{-1}, 3^{-1}, 5^{-2}$	$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 20 \end{pmatrix}$ $\begin{pmatrix} 4 & 2 & 2 \\ 2 & 6 & 1 \\ 2 & 1 & 16 \end{pmatrix}$	D_2 D_2	5760	48
74	$L_2(7)$	168	196	$4_7^{+1}, 7^{+2}$	$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 28 \end{pmatrix}$ $\begin{pmatrix} 4 & 2 & 2 \\ 2 & 8 & 1 \\ 2 & 1 & 8 \end{pmatrix}$	D_2 D_4	5376	32
76	H_{192}	192	384	$4_4^{-2}, 8_1^{+1}, 3^{+1}$	$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 12 \end{pmatrix}$	D_4	24576	128
77	T_{192}	192	192	$4_1^{-3}, 3^{-1}$	$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 8 & 4 \\ 0 & 4 & 8 \end{pmatrix}$	D_6	36864	192
78	$\mathfrak{A}_{4,4}$	288	288	$2_{\mathrm{II}}^{+2}, 8_7^{+1}, 3^{+2}$	$\begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & 2 \\ 4 & 2 & 8 \end{pmatrix}$	D_4	36864	128
79	\mathfrak{A}_6	360	180	$4_3^{-1}, 3^{+2}, 5^{+1}$	$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 8 & 0 \\ 0 & 0 & 12 \end{pmatrix}$ $\begin{pmatrix} 6 & 0 & 3 \\ 0 & 6 & 3 \\ 3 & 3 & 8 \end{pmatrix}$	D_2 D_4	11520	32
80	F_{384}	384	256	$4_1^{+1}, 8_2^{+2}$	$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$	D_4	49152	128
81	M_{20}	960	160	$2_{\mathrm{II}}^{-2}, 8_7^{+1}, 5^{-1}$	$\begin{pmatrix} 4 & 0 & 2 \\ 0 & 4 & 2 \\ 2 & 2 & 12 \end{pmatrix}$	D_4	92160	96

No. denotes the number of the group $G_s \leq O(\Lambda)$ as given in [Has12]. It is isomorphic to the corresponding group in the column G_s . See [Muk88] for the notation. The entry genus is given in Conway and Sloane's [CS99] notation. The dihedral group of order $2k$ is denote by D_k .

is a finite group. It can be computed explicitly with the Plesken–Souvignier algorithm [PS97] as implemented for instance in PARI [PAR18]. As it turns out the groups $G \leq \mathrm{Aut}(X)$ we aim to classify are subgroups of this product.

5. Non-symplectic extensions

In this section we prove the classification. The invariant lattices of the symplectic actions play a major role. For a start we observe that the cyclic group G/G_s acts on the invariant lattice. Indeed for $g \in G$ and $x \in H^2(X, \mathbb{Z})^{G_s}$, $gG_s(x) = g(x)$ is well defined and lies in the invariant lattice since G_s is normal in G and fixes x . This yields a homomorphism

$$G/G_s \rightarrow O\left(H^2(X, \mathbb{Z})^{G_s}\right)$$

of groups which turns out to be injective.

LEMMA 5.1. — *Let X be a K3 surface and $G \leq \text{Aut}(X)$ a finite group of automorphisms such that the subgroup $G_s \leq G$ of symplectic automorphisms is among the 11 maximal ones. Then the homomorphism $G/G_s \rightarrow O(H^2(X, \mathbb{Z})^{G_s})$ is injective and its image is a cyclic subgroup of $\text{SO}(H^2(X, \mathbb{Z})^{G_s})$. In particular its order is $n \in \{1, 2, 3, 4, 6\}$.*

Proof. — By our assumption G_s is maximal. Thus, by Table 4.1, $H^2(X, \mathbb{Z})^{G_s}$ is of rank 3. Hence a basis of $H^2(X, \mathbb{R})^{G_s}$ is given by a G -invariant Kähler class κ , $(\sigma_X + \bar{\sigma}_X)/2$ and $(\sigma_X - \bar{\sigma}_X)/(2i)$ (see the proof of Lemma 4.1). Since G/G_s acts on $H^{2,0}(X) = \mathbb{C}\sigma_X$ faithfully (by the definition of G_s), the injectivity in the statement of the lemma follows. By the same reason, G/G_s is cyclic.

Let gG_s be a generator of G/G_s . Then $(x-1)$ divides $\chi(x) = \det(x \text{id} - g|_{H^2(X, \mathbb{Z})^{G_s}})$. Since $H^2(X, \mathbb{R})^{G_s}$ is actually defined over \mathbb{Q} and g is of finite order, $\chi(x)$ is a product of cyclotomic polynomials. Note that the eigenvectors σ_X and $\bar{\sigma}_X$ have complex conjugate eigenvalues. Hence $\chi(x) \neq (x+1)(x-1)^2$. This leaves us with $\chi(x)/(x-1)$ to be one of $(x \pm 1)^2$ or Φ_n for $n \in \{3, 4, 6\}$. We conclude by computing the determinant from the characteristic polynomial. \square

Recall that via a marking we may identify $H^2(X, \mathbb{Z})$ and Λ .

Remark 5.2. — A choice of basis turns the groups $\text{SO}(\Lambda^{G_s})$ into subgroups of $\text{SL}(\mathbb{Z}^3)$. Finite subgroups of $\text{SO}(3)$ are an essential building block for crystallographic groups. It is known that they are isomorphic to a subgroup of a dihedral group, or one of $(\mathbb{Z}/2\mathbb{Z})^2 \rtimes C_3 \leq (\mathbb{Z}/2\mathbb{Z})^2 \rtimes S_3$ (cf. [AC91, Table I]). Up to conjugation there are exactly 3 subgroups of the second type, i.e. 3 integral representations of $(\mathbb{Z}/2\mathbb{Z})^2 \rtimes C_3$. Since the three representations are irreducible, there is, up to homothety, a unique invariant quadratic form for each. Their gram matrices are given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 4 \end{pmatrix}$$

and are of determinant 1, 2^2 and 2^4 . Obviously none of the invariant lattices in Table 4.1 is homothetic to one of these three. Thus $\text{SO}(\Lambda^{G_s})$ must be a (subgroup of) a dihedral group.

Part 2 of the next lemma will be used later in the proof of Proposition 5.9.

LEMMA 5.3. — *Let H be one of the 14 symplectic fixed lattices and $g \in \text{SO}(H)$ an involution. Then*

- (1) $\text{SO}(H)$ is isomorphic to a dihedral group D_k of order $2k$ with $k \in \{2, 4, 6\}$;
- (2) there is another involution $f \in \text{SO}(H)$ in the centralizer of g with $\det f|_{H_g} = -1$ where H_g denotes the coinvariant lattice of the group generated by g .

Proof. — The proof of (1) is by a direct computation of $\text{SO}(H)$ for each case using a computer. For the reader's entertainment we calculate No. 63 by hand. The fixed lattice is given as the orthogonal direct sum $(2) \oplus A_2(6)$ where $A_2(6)$ is a rescaled hexagonal lattice. The orthogonal group of the hexagonal lattice is that of the hexagon, i.e. it is the dihedral group D_6 . Since the decomposition of a lattice into irreducible lattices is unique up to ordering, the orthogonal group preserves this

decomposition and is isomorphic to $\{\pm 1\} \times D_6$. It contains the special orthogonal group with index 2. The restriction of the SO action to the hexagonal plane is by D_6 . Thus it is in fact isomorphic to D_6 .

For (2) fix $k \in \{2, 4, 6\}$ and let $g \in D_k$ be an involution. Then it is not hard to check, that there exists an involution f different from g and commuting with g . Now view f and g as elements of $\mathrm{SO}(H)$. The characteristic polynomials of both are equal to $(x-1)(x+1)^2$. If $\det f|_{H_g} = 1$, then $f|_{H_g} = -\mathrm{id}$. This implies $f = g$, which we excluded. \square

Recall the exact sequence

$$1 \rightarrow G_s \rightarrow G \rightarrow \mu_n \rightarrow 1.$$

From the proof of Lemma 5.1 we obtain that if G_s is maximal, then $n \in \{1, 2, 3, 4, 6\}$. We want to reconstruct $G \leq O(\Lambda)$ knowing G_s and $G/G_s \cong \mu_n$. In this situation one speaks of an extension of groups. We are interested not only in the group structure, but also in its action on the K3 lattice. This motivates the next definition.

DEFINITION 5.4. — *Let L be a lattice, $G \leq O(L)$ a group of isometries and $N \leq G$ a normal subgroup with cyclic quotient $G/N = \langle gN \rangle$. We say that G is an extension of N by $g|_{L^N}$ where L^N denotes the invariant lattice of $N \leq O(L)$.*

Remark 5.5. — In our setting L is unimodular and N coincides with the kernel of the natural map $O(L, L_N) \rightarrow O(L_N^\vee/L_N)$. In this case, $G \leq O(L)$ is uniquely determined by N and $g|_{L^N}$.

Before extending the group, we first have to extend single elements. We are in the luxurious position that every element extends:

LEMMA 5.6 ([Has12, Theorem 5.1]). — *Let Λ_{G_s} be the coinvariant lattice for one of the 11 maximal finite groups. Then the natural map*

$$\psi: O(\Lambda_{G_s}) \rightarrow O(\Lambda_{G_s}^\vee/\Lambda_{G_s})$$

is surjective. In particular any isometry of $O(\Lambda^{G_s})$ can be extended to an element in $O(\Lambda)$ normalizing G_s .

Remark 5.7. — One may double check the theorem as follows: first compute $O(\Lambda_{G_s})$ with the Plesken–Souvignier backtracking algorithm. Then check by a direct computation that the natural map is surjective. For the reader's convenience we list the orders of the groups involved in Table 4.1. Note that by Lemma 4.1 we have $\#G_s \cdot \#O(q_{\Lambda_{G_s}}) = \#O(\Lambda_{G_s})$, if and only if the natural map ψ is surjective.

In general extensions of a given group of isometries are not unique, not even up to conjugacy. But we are in a particularly nice situation.

LEMMA 5.8. — *Let $G_s \leq O(\Lambda)$ be one of the 11 maximal symplectic groups. Let $g \in O(\Lambda^{G_s})$ be an isometry. Then*

- (1) *there is a unique extension of G_s by g ;*
- (2) *if $\tilde{g} \in O(\Lambda^{G_s})$ is conjugate to g , then the corresponding extensions are conjugate in $O(\Lambda)$.*

Proof. — Recall that G_s is a subgroup of the orthogonal group of the K3 lattice Λ . In particular we have a primitive extension $\Lambda^{G_s} \oplus \Lambda_{G_s} \subseteq \Lambda$. Since the K3 lattice is unimodular, this primitive extension is determined by an anti-isometry

$$\phi: \Lambda^{G_s^\vee} / \Lambda^{G_s} \longrightarrow \Lambda_{G_s}^\vee / \Lambda_{G_s}.$$

The natural map $\psi: O(\Lambda_{G_s}) \rightarrow O(\Lambda_{G_s}^\vee / \Lambda_{G_s})$, $f \mapsto \bar{f}$ is surjective (Lemma 5.6). Hence, we find an $h \in O(\Lambda_{G_s})$ such that $\bar{h} = \phi \circ \bar{g} \circ \phi^{-1}$. This means that $\tilde{g} = g \oplus h$ extends to an isometry of Λ . We set $G = \langle G_s, \tilde{g} \rangle$. Any other choice of h is of the form $h \cdot (\text{id}_{\Lambda_{G_s}} \oplus f)$ with $f \in \ker \psi \cong G_s$ (Lemma 4.1 (5)). Then G remains unchanged.

We turn to the second claim. Let $f \in O(\Lambda^{G_s})$ and let $g^f = f^{-1}gf$ be a conjugate of g . Take an extension \tilde{g} of g to an isometry of Λ . We can extend f to an isometry $\tilde{f} = f \oplus f'$ of Λ as well (Lemma 2.1). Since the restriction $G_s|_{\Lambda_{G_s}}$ is a normal subgroup of $O(\Lambda_{G_s})$, conjugation by f preserves G_s . Further the restriction of $\tilde{g}^{\tilde{f}}$ to Λ^{G_s} is equal to g^f . Hence, by part 1, the extensions $G^{\tilde{f}}$ and $\langle \tilde{g}^{\tilde{f}}, G_s \rangle$ are equal. \square

If $(X, G) \cong (X', G')$ are isomorphic pairs consisting of a Λ -marked K3 surface with a group of automorphisms, then G and G' (viewed in $O(\Lambda)$ via the marking) are conjugate. In our case the pairs do not deform, so there is hope for the converse statement to hold.

PROPOSITION 5.9. — *Let (X, η) and (X', η') be marked K3 surfaces and $G \leq \text{Aut}(X)$, $G' \leq \text{Aut}(X')$ finite subgroups such that G_s and G'_s are isomorphic to one of the 11 maximal groups. Suppose that $\eta G \eta^{-1}$ and $\eta' G' \eta'^{-1}$ are conjugate in $O(\Lambda)$, then there is an isomorphism $f: X \rightarrow X'$ with $G = f^{-1} G' f$, i.e. the pairs (X, G) and (X', G') are isomorphic.*

Proof. — Changing the marking η conjugates $\eta G \eta^{-1}$ in $O(\Lambda)$. To ease notation, we identify G, G' with their image in $O(\Lambda)$. In order to use the strong Torelli type Theorem, we have to produce an effective Hodge isometry conjugating G and G' .

Let n be the order of G/G_s . We choose a primitive n th root of unity $\zeta \in \mathbb{C}$. Then G/G_s comes with a distinguished generator gG_s given by $g(\eta_{\mathbb{C}}(\sigma_X)) = \zeta \sigma_X$. And likewise $g'G'_s$. By assumption G and G' are conjugate via some $f \in O(\Lambda)$. If $n = 2$, then the generators gG_s and $g'G'_s$ are unique. Otherwise $n = 3, 4, 6$, and then $\text{SO}(\Lambda^{G_s})$ is a dihedral group of order 8 or 12 (Lemma 5.3). In any case there is a unique conjugacy class of order n . Since we can extend any conjugator of the dihedral group to an element of $O(\Lambda)$ (Lemma 2.1) preserving G_s , we may modify the conjugator f in such a way that it conjugates the distinguished generators gG_s and $g'G'_s$ as well. So after conjugation, we may assume that $G' = G$ and further that $g'G'_s = gG_s$.

Suppose that $n > 2$. Then the periods of X and X' are uniquely determined by the distinguished generators as the (1-dimensional!) eigenspaces with eigenvalue ζ of $g|H(X, \mathbb{C})^G$, respectively $g'|H(X', \mathbb{C})^{G'}$. And we are done. (Note that if σ is an eigenvector for $\zeta \neq \pm 1$, then $\langle \sigma, \sigma \rangle = \zeta^2 \langle \sigma, \sigma \rangle$ implies $\sigma^2 = 0$.) If $n = 2$ then the eigenspace for -1 of $g|H(X, \mathbb{C})^G$ is of dimension 2. However, the period is of square zero. Thus the period is one of the two isotropic lines in the eigenspace. These correspond to the two orientations of the transcendental lattice. By Lemma 5.3 one can find an isometry f of Λ^G centralizing g and reversing the orientation. This

f extends to an isometry of Λ preserving G . Thus we have obtained a Hodge isometry conjugating G and G' .

Note that $H^2(X, \mathbb{Z})^G$ is spanned by an ample class l and likewise for G' . Since our Hodge isometry conjugates G and G' it maps l to l' or $-l'$. In the second case our Hodge isometry is not effective. However, we may then replace it by its negative. \square

PROPOSITION 5.10. — *Let $G_s \leq O(\Lambda)$ be a maximal symplectic group. There is a one to one correspondence between conjugacy classes of non trivial cyclic subgroups of $SO(\Lambda^{G_s})$ and isomorphism classes of pairs (X, G') consisting of a K3 surface X and $G' \leq \text{Aut}(X)$ a finite subgroup with $G_s \cong G'_s < G'$.*

Proof. — It remains to show that each cyclic subgroup is actually coming from a K3 surface. To see this, choose a suitable eigenvector of $G/G_s|_{\Lambda^{G_s}}$ as period, a generator of Λ^G as Kähler class which is in fact ample since it is integral. Then use the global Torelli type theorem and surjectivity of the period map. \square

Remark 5.11. — Fix a maximal symplectic group $G_s \leq O(\Lambda)$. The non-symplectic extensions $G_s < G$ which we are classifying lie in the extension

$$(5.1) \quad SO(\Lambda^{G_s}).G_s = O(\Lambda) \cap (SO(\Lambda^{G_s}) \times O(\Lambda_{G_s})).$$

They are the subgroups $G_s < G \leq SO(\Lambda^{G_s}).G_s$ with G/G_s cyclic.

Proof of Theorem 1.1. — We use the correspondence set up in Proposition 5.10. By Lemma 5.3, $SO(\Lambda^{G_s})$ is isomorphic to a dihedral group D_n of order $2n$ for $n \in \{2, 4, 6\}$. Its maximal cyclic subgroups up to conjugacy are two groups of order 2 generated by reflections and one group of order n generated by a rotation. Thus for each of the 14 actions there are 3 maximal extensions leading to $42 = 3 \cdot 14$ cases.

It remains to derive the additional data which we provide with the tables. Let $\mu_n \cong \langle g \rangle \leq SO(\Lambda^{G_s})$ be a cyclic subgroup and G the corresponding extension. The invariant polarization Λ^G is computed as the kernel of $1 - g \in \text{End}_{\mathbb{Z}}(\Lambda^G)$ and the transcendental lattice as the orthogonal complement of Λ^G in Λ^{G_s} .

To obtain the group structure of G , we construct Λ as a primitive extension of $\Lambda^{G_s} \oplus \Lambda_{G_s}$ by calculating the corresponding glue map. Then we extend g to an isometry $g \oplus h$ of Λ as in Lemmas 2.1 and 5.8. Here the coinvariant lattices Λ_{G_s} are obtained as sublattices of the Leech lattice as tabulated in [HM16] and the invariant lattices Λ^{G_s} are tabulated in [Has12]. The gluings and extensions are carried out using the code developed by the first author for sageMath [Dev19]. We showcase the computation for **Nos. 70a, 70b, 70c** with a notebook in the ancillary files. \square

6. The classification

Using Proposition 5.10, we are ready to state the details of the classification. The tables were produced using SageMath [Dev19] and GAP [Gro19]. We denote by $\mathbb{Z}l = \Lambda^G$ the (primitive) invariant polarization of G . Then $G := \text{Aut}(X, l)$ is the full projective automorphism group. The lattice Λ^{G_s} is the fixed lattice $H^2(X, \mathbb{Z})^{G_s}$ of a maximal symplectic action. The entry “glue” denotes the index $[\Lambda^{G_s} : T_X \oplus \mathbb{Z}l]$. The GAP Id [BEO02] identifies a group up to isomorphism. If equations for the pair

(X, G) are known, then we write its identifier in bold. We set $\zeta_n = \exp(2\pi\sqrt{-1}/n)$, $\omega = \zeta_3$ and $i = \zeta_4$.

Remark 6.1. — We note that in all cases, except **62b** with $\mathbb{Z}l \cong (12)$, the exact sequence

$$1 \rightarrow G_s \rightarrow G \rightarrow \mu_n \rightarrow 1$$

splits. Namely, G is a semidirect product of G_s and μ_n .

6.1. No. 54

$G_s = T_{48}$. We have $\mathrm{SO}(\Lambda^{G_s}) \cong D_6$.

Λ^{G_s}	n	T_X	$\mathbb{Z}l$	glue	GAP Id	case
$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 16 & 8 \\ 0 & 8 & 16 \end{pmatrix}$	6	$\begin{pmatrix} 16 & 8 \\ 8 & 16 \end{pmatrix}$	(2)	1	[288, 900]	54a
	2	$\begin{pmatrix} 2 & 0 \\ 0 & 48 \end{pmatrix}$	(16)	2	[96, 193]	54b
	2	$\begin{pmatrix} 2 & 0 \\ 0 & 16 \end{pmatrix}$	(48)	2	[96, 193]	54c

A projective model of **54a** is given in [Muk88]. It is the double cover X of \mathbb{P}^2 branched over the curve defined by

$$(6.1) \quad xy(x^4 + y^4) + z^6 = 0.$$

We have $G = G_s \times \mu_6$, where μ_6 is generated by a lift to X of $(x : y : z) \mapsto (x : y : \zeta_6 z)$.

6.2. No. 62

$G_s = N_{72}$. We have $\mathrm{SO}(\Lambda^{G_s}) \cong D_4$.

Λ^{G_s}	n	T_X	$\mathbb{Z}l$	glue	GAP Id	case
$\begin{pmatrix} 6 & 0 & 3 \\ 0 & 6 & 3 \\ 3 & 3 & 12 \end{pmatrix}$	2	$\begin{pmatrix} 6 & 0 \\ 0 & 36 \end{pmatrix}$	(6)	2	[144, 186]	62a
	2	$\begin{pmatrix} 12 & 6 \\ 6 & 12 \end{pmatrix}$	(12)	2	[144, 182]	62b
	4	$\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$	(36)	2	[288, 841]	62c

A projective model of **62a** is given in [Muk88]:

$$(6.2) \quad x_1^3 + x_2^3 + x_3^3 + x_4^3 = x_1x_2 + x_3x_4 + x_5^2 = 0 \quad \text{in } \mathbb{P}^4.$$

We have $G = G_s \times \mu_2$, where μ_2 is generated by

$$(x_1 : \cdots : x_4 : x_5) \mapsto (x_1 : \cdots : x_4 : -x_5).$$

A projective model of **62c** in \mathbb{P}^5 together with a non-linear action of $N_{72} \times \mu_2$ and an invariant polarization of degree 36 is given in [MO14]:

$$X = X_{\lambda, \mu} = \begin{cases} x_0^2 - \lambda x_1 x_2 = y_0^2 - \mu y_1 y_2 \\ x_1^2 - \lambda x_0 x_2 = y_1^2 - \mu y_0 y_2 \\ x_2^2 - \lambda x_0 x_1 = y_2^2 - \mu y_0 y_1 \end{cases} \quad (\lambda = 1 + \sqrt{3}, \mu = 1 - \sqrt{3})$$

It remains to exhibit an element acting by a primitive 4th root of unity on the 2-form [Oha]: the variety is constructed using the partial derivatives of the Hesse pencil

$$z_0^3 + z_1^3 + z_2^3 - 3\kappa z_0 z_1 z_2.$$

A linear map $g \in \mathrm{PGL}_3(\mathbb{C})$ preserving this pencil acts on the base as a Möbius transformation, which is denoted by the same letter g . By [MO14, Lemma 2.1] it induces a morphism $\tilde{g}: X_{\lambda, \mu} \rightarrow X_{g(\lambda), g(\mu)}$. Take

$$g = \begin{pmatrix} 1 & 1 & \omega^2 \\ 1 & \omega & \omega \\ \omega & 1 & \omega \end{pmatrix}. \quad \text{Then } g(\kappa) = \frac{\kappa + 2\omega^2}{\omega\kappa - 1}$$

and g exchanges λ and μ . We compose $\tilde{g}: X_{\lambda, \mu} \rightarrow X_{\mu, \lambda}$ with the map $\iota: X_{\mu, \lambda} \rightarrow X_{\lambda, \mu}$ exchanging the x_i and y_i to obtain the sought for automorphism of X , which generates μ_4 with $G = N_{72} \rtimes \mu_4$. Note that μ_2 above, which is generated by $y_i \mapsto -y_i$, is not included in μ_4 . Let h be the class of hyperplane section and let f_∞ be the class represented by the smooth elliptic curve of degree 6 defined by

$$(6.3) \quad \text{rank } A \leq 1, \quad \text{where } A := \begin{pmatrix} \mu x_0 & x_2 + cy_2 & x_1 - cy_1 \\ x_2 - cy_2 & \mu x_1 & x_0 + cy_0 \\ x_1 + cy_1 & x_0 - cy_0 & \mu x_2 \end{pmatrix}, \quad c^2 = 1 - \mu^2.$$

Then we have $h, f_\infty \in \mathrm{NS}_X^H$, where $H \cong (C_3^2 \rtimes C_4) \rtimes \mu_4$ is the subgroup of G consisting of all linear transformations. Moreover, $3h - f_\infty$ is a G -invariant polarization of degree 36 (see [MO14] for details).

6.3. No. 63

$G_s = M_9$. We have $\mathrm{SO}(\Lambda^{G_s}) \cong D_6$.

Λ^{G_s}	n	T_X	$\mathbb{Z}l$	glue	GAP Id	case
$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 12 & 6 \\ 0 & 6 & 12 \end{pmatrix}$	6	$\begin{pmatrix} 12 & 6 \\ 6 & 12 \end{pmatrix}$	(2)	1	[432, 735]	63a
	2	$\begin{pmatrix} 2 & 0 \\ 0 & 36 \end{pmatrix}$	(12)	2	[144, 182]	63b
	2	$\begin{pmatrix} 2 & 0 \\ 0 & 12 \end{pmatrix}$	(36)	2	[144, 182]	63c

A projective model of **63a** is given in [Muk88]. It is the double cover X of \mathbb{P}^2 branched over the curve defined by

$$(6.4) \quad x^6 + y^6 + z^6 - 10(x^3 y^3 + y^3 z^3 + z^3 x^3) = 0.$$

We have $G = G_s \rtimes \mu_6$, where μ_6 is generated by the covering transformation and a lift to X of $(x : y : z) \mapsto (\zeta_3 x : y : z)$.

6.4. No. 70

$G_s = \mathfrak{S}_5$. We have $\mathrm{SO}(\Lambda^{G_s}) \cong D_2$ in both cases. Since the center of G_s is trivial and there is no nontrivial outer-automorphism of G_s , we have $G = G_s \times \mu_2$ in each case.

Λ^{G_s}	n	T_X	$\mathbb{Z}l$	glue	GAP Id	case
$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 20 \end{pmatrix}$	2	$\begin{pmatrix} 10 & 0 \\ 0 & 20 \end{pmatrix}$	(6)	2	[240, 189]	70a
	2	$\begin{pmatrix} 6 & 0 \\ 0 & 20 \end{pmatrix}$	(10)	2	[240, 189]	70b
	2	$\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$	(20)	1	[240, 189]	70c
$\begin{pmatrix} 4 & 2 & 2 \\ 2 & 6 & 1 \\ 2 & 1 & 16 \end{pmatrix}$	2	$\begin{pmatrix} 20 & 10 \\ 10 & 20 \end{pmatrix}$	(4)	2	[240, 189]	70d
	2	$\begin{pmatrix} 4 & 2 \\ 2 & 16 \end{pmatrix}$	(20)	2	[240, 189]	70e
	2	$\begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}$	(60)	2	[240, 189]	70f

A projective model of **70a** is given in [Muk88]:

$$(6.5) \quad \sum_{i=1}^5 x_i = \sum_{i=1}^6 x_i^2 = \sum_{i=1}^5 x_i^3 = 0 \quad \text{in } \mathbb{P}^5,$$

where $G = \mathfrak{S}_5 \times \mu_2$ is generated by the permutations of x_1, \dots, x_5 and $(x_1 : \dots : x_5 : x_6) \mapsto (x_1 : \dots : x_5 : -x_6)$.

A projective model of **70b** is given in [Fra11, Proposition 3.16]. Let X be the minimal resolution of the double cover of \mathbb{P}^2 branched over the curve C defined by $f = 0$, where

$$\begin{aligned} f = & 2(x_1^4 x_2 x_3 + x_1 x_2^4 x_3 + x_1 x_2 x_3^4) - 2(x_1^4 x_2^2 + x_1^4 x_3^2 + x_1^2 x_2^4 + x_1^2 x_3^4 + x_2^4 x_3^2 + x_2^2 x_3^4) \\ & + 2(x_1^3 x_2^3 + x_1^3 x_3^3 + x_2^3 x_3^3) \\ & + (x_1^3 x_2^2 x_3 + x_1^3 x_2 x_3^2 + x_1^2 x_2^3 x_3 + x_1^2 x_2 x_3^3 + x_1 x_2^3 x_3^2 + x_1 x_2^2 x_3^3) - 6x_1^2 x_2^2 x_3^2. \end{aligned}$$

Let $H \cong \mathfrak{S}_4$ be the subgroup of $\mathrm{PGL}_3(\mathbb{C})$ permuting the following four points:

$$(6.6) \quad (1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (1 : 1 : 1).$$

Then the curve C is preserved by H and the following Cremona transformation (of order 5):

$$(6.7) \quad g : (x_1 : x_2 : x_3) \mapsto (x_1(x_3 - x_2) : x_3(x_1 - x_2) : x_1 x_3).$$

The actions of H and g lift to those on X . The group $G \cong \mathfrak{S}_5 \times \mu_2$ is generated by H , g and the covering transformation. Note that X/μ_2 is isomorphic to the Del Pezzo

surface of degree 5 (see [Fra11] for details). From this fact, we know the following: consider the line defined by $x_1 = 0$ in \mathbb{P}^2 and let R_1 be the strict transform of its pull-back in X . Then the orbit of R_1 (in NS_X) under the action of G consists of ten curves R_i ($1 \leq i \leq 10$), whose dual configuration is the Petersen graph with $R_i \cdot R_i = -2$ and $R_i \cdot R_j \in \{0, 2\}$ for $i \neq j$ (see also [Fra11]). Hence we have

$$(6.8) \quad R := \sum_{i=1}^{10} R_i, \quad R \cdot R = 10 \cdot (-2) + 2 \cdot 15 \cdot 2 = 40.$$

By construction (X, G) is among the surfaces in the table. Further R is invariant under G . Thus R must lie in $\mathbb{Z}l$. This is only possible if $R = 2l$ and $l^2 = 10$. Thus (X, G) is a projective model of **70b**.

Projective models of **70c** are given in [Kon86, Type VI] and [PTvdV92]. We review the one in [PTvdV92]. Let Y be the surface defined by the following equations:

$$(6.9) \quad \sum_{i=1}^5 x_i = \sum_{i=1}^5 \frac{1}{x_i} = 0 \quad \text{in } \mathbb{P}^4$$

and let X be the minimal resolution of Y , which is a K3 surface. It follows from [Has11, Theorem 4.15] that X has the same transcendental lattice as **70c**. The symmetric group \mathfrak{S}_5 acts on X by permutation of x_i . Moreover, the involution $\epsilon: (x_i) \mapsto (1/x_i)$ acts on X . The group $G \cong \mathfrak{S}_5 \times \mu_2$ is generated by \mathfrak{S}_5 and ϵ .

A projective model of **70d** is given in [Has11, Theorem 4.15]. Consider $Q := \mathbb{P}^1 \times \mathbb{P}^1$ defined by $\sum_{i=1}^5 x_i = \sum_{i=1}^5 x_i^2 = 0$ in \mathbb{P}^4 . Let X be the double cover of Q branched over the curve defined by $\sum_{i=1}^5 x_i^4 = 0$. Then X is a K3 surface and G is generated by the permutations of x_i and the covering transformation. This projective model is a (degenerate) member of the family XIII in [Smi07]. The pull-back to Q of the hyperplane class h of \mathbb{P}^4 is of bidegree $(1, 1)$, which has self intersection 2. Hence the pull-back to X of h has self intersection $2 \cdot 2 = 4$ and is invariant under G . In particular it must be equal to l , confirming that we are indeed in case **70d**. Note that the linear system $|l|$ is hyperelliptic.

Projective models of **70f** are given in [Kon86, Type VII] and [MO14, 1.2]. We review the one in [MO14], which is given as the minimal resolution of the singular surface

$$\bar{X}: \quad \sum_{1 \leq i < j \leq 5} x_i x_j = \sum_{1 \leq i < j \leq 5} \frac{1}{x_i x_j} = 0$$

in \mathbb{P}^4 . The symmetric group \mathfrak{S}_5 acts (non-symplectically) by permutations of the coordinates and a non-symplectic involution is given by the Enriques involution $\epsilon: (x_i) \mapsto (1/x_i)$. The surface \bar{X} has five A_1 singularities in the coordinate points. Denote by E_i the corresponding exceptional divisors on X . Let R_i be the strict transform of $\bar{X} \cap \{x_i = 0\}$. Then the intersection numbers are given by $R_i \cdot C_j = 2(1 - \delta_{ij})$. An invariant polarization of degree 60 is given by $h = \sum_{i=1}^5 (E_i + R_i)$.

6.5. No. 74

$G_s = L_2(7)$. We have $\mathrm{SO}(\Lambda^{G_s}) \cong D_2, D_4$, respectively, as follows.

Λ^{G_s}	n	T_X	$\mathbb{Z}l$	glue	GAP Id	case
$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 28 \end{pmatrix}$	2	$\begin{pmatrix} 14 & 0 \\ 0 & 28 \end{pmatrix}$	(2)	2	[336, 209]	74a
	2	$\begin{pmatrix} 2 & 0 \\ 0 & 28 \end{pmatrix}$	(14)	2	[336, 208]	74b
	2	$\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$	(28)	1	[336, 208]	74c
$\begin{pmatrix} 4 & 2 & 2 \\ 2 & 8 & 1 \\ 2 & 1 & 8 \end{pmatrix}$	4	$\begin{pmatrix} 14 & 0 \\ 0 & 14 \end{pmatrix}$	(4)	2	[672, 1046]	74d
	2	$\begin{pmatrix} 4 & 0 \\ 0 & 14 \end{pmatrix}$	(14)	2	[336, 208]	74e
	2	$\begin{pmatrix} 4 & 2 \\ 2 & 8 \end{pmatrix}$	(28)	2	[336, 208]	74f

A projective model of **74a** is given in [OZ02]. It is the double cover of \mathbb{P}^2 branched over the Hessian of the Klein curve defined by

$$(6.10) \quad x^5z + y^5x + z^5y - 5x^2y^2z^2 = 0.$$

We have $G = L_2(7) \times \mu_2$, where μ_2 is generated by the covering transformation.

A projective model of **74c** is given in [Uji13] and [AST11, Appendix]. It is the universal elliptic curve over $X_1(7)$. We have $G = \mathrm{PGL}_2(\mathbb{F}_7) = L_2(7) \rtimes \mu_2$.

A projective model of **74d** is given in [Muk88]. Using the Klein curve with $L_2(7)$, it is defined by

$$(6.11) \quad x^3y + y^3z + z^3x + w^4 = 0 \quad \text{in } \mathbb{P}^3.$$

We have $G = L_2(7) \times \mu_4$, where μ_4 is generated by $(x : y : z : w) \mapsto (x : y : z : iw)$.

6.6. No. 76

$G_s = H_{192}$. We have $\mathrm{SO}(\Lambda^{G_s}) \cong D_2$.

Λ^{G_s}	n	T_X	$\mathbb{Z}l$	glue	GAP Id	case
$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 12 \end{pmatrix}$	2	$\begin{pmatrix} 8 & 0 \\ 0 & 12 \end{pmatrix}$	(4)	1	[384, 17948]	76a
	2	$\begin{pmatrix} 4 & 0 \\ 0 & 12 \end{pmatrix}$	(8)	1	[384, 17948]	76b
	2	$\begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix}$	(12)	1	[384, 17948]	76c

A projective model of **76a** is given as follows (this is a (degenerate) member of the family XI in [Smi07]): consider $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ defined by $\sum_{i=1}^4 x_i^2 = 0$ in \mathbb{P}^3 . Let X be the double cover of Q branched over the curve defined by $\sum_{i=1}^4 x_i^4 = 0$. Then X

is a K3 surface and $G = H_{192} \times \mu_2$ is generated by the permutations of x_i , the sign changes of x_i , and the covering transformation. Similarly to **70d**, we have $l^2 = 4$. Namely, since the pull-back to X of the hyperplane class of \mathbb{P}^3 has self intersection 4 and is invariant under G , the pair (X, G) is case **76a**.

A projective model of **76b** is given in [Muk88]:

$$(6.12) \quad x_1^2 + x_3^2 + x_5^2 = x_2^2 + x_4^2 + x_6^2, \quad x_1^2 + x_4^2 = x_2^2 + x_5^2 = x_3^2 + x_6^2 \quad \text{in } \mathbb{P}^5.$$

We have $G = H_{192} \times \mu_2$, where μ_2 is generated by $(x_1 : \cdots : x_6) \mapsto (-x_1 : x_2 : -x_3 : x_4 : -x_5 : x_6)$.

A projective model of **76c** is given as a smooth surface X of tri-degree $(2, 2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ [MO14, Example 6]:

$$(6.13) \quad v^2 w^2 + u^2 w^2 + u^2 v^2 + 1 + i(u^2 + v^2 + w^2 + u^2 v^2 w^2) = 0$$

where (u, v, w) are affine coordinates on $(\mathbb{P}^1)^3$. The surface X admits the following linear actions: the permutations of (u, v, w) , $(u, v, w) \mapsto (\pm u, \pm v, \pm w)$, and $(u, v, w) \mapsto (iu, iv, i/w)$. Those linear actions generate $G = H_{192} \times \mu_2$. Here μ_2 is generated by $(u, v, w) \mapsto (-u, -v, -w)$. The invariant polarization of degree 12 is given by $\mathcal{O}_X(1, 1, 1)$.

6.7. No. 77

$G_s = T_{192}$. We have $\mathrm{SO}(\Lambda^{G_s}) \cong D_6$.

Λ^{G_s}	n	T_X	$\mathbb{Z}l$	glue	GAP Id	case
$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 8 & 4 \\ 0 & 4 & 8 \end{pmatrix}$	6	$\begin{pmatrix} 8 & 4 \\ 4 & 8 \end{pmatrix}$	(4)	1	[1152, 157515]	77a
	2	$\begin{pmatrix} 4 & 0 \\ 0 & 24 \end{pmatrix}$	(8)	2	[384, 5602]	77b
	2	$\begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix}$	(24)	2	[384, 5608]	77c

A projective model of **77a** is given in [Muk88]:

$$(6.14) \quad x^4 + y^4 + z^4 + w^4 - 2\sqrt{-3}(x^2 y^2 + z^2 w^2) = 0 \quad \text{in } \mathbb{P}^3.$$

We have $G = (T_{24} * T_{24}) \rtimes \langle \tau, \sigma \rangle = T_{192} \rtimes \mu_6$, where $*$ denotes central product, T_{24} the binary tetrahedral group, τ the involution interchanging two copies of T_{24} and σ switches the sign of x .

6.8. No. 78

$G_s = \mathfrak{A}_{4,4}$. We have $\mathrm{SO}(\Lambda^{G_s}) \cong D_4$.

Λ^{G_s}	n	T_X	$\mathbb{Z}l$	glue	GAP Id	case
$\begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & 2 \\ 4 & 2 & 8 \end{pmatrix}$	4	$\begin{pmatrix} 12 & 0 \\ 0 & 12 \end{pmatrix}$	(8)	2	[1152, 157850]	78a
	2	$\begin{pmatrix} 8 & 0 \\ 0 & 12 \end{pmatrix}$	(12)	2	[576, 8654]	78b
	2	$\begin{pmatrix} 8 & 4 \\ 4 & 8 \end{pmatrix}$	(24)	2	[576, 8653]	78c

A projective model of **78a** is given in [Muk88]:

$$(6.15) \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} x^2 \\ y^2 \\ z^2 \end{pmatrix} = \sqrt{3} \begin{pmatrix} u^2 \\ v^2 \\ w^2 \end{pmatrix} \quad \text{in } \mathbb{P}^5.$$

We have $G = \mathfrak{A}_{4,4} \rtimes \mu_4$, where μ_4 is generated by $(x : y : z : u : v : w) \mapsto (u : v : w : x : z : y)$.

6.9. No. 79

$G_s = \mathfrak{A}_6$. We have $\mathrm{SO}(\Lambda^{G_s}) \cong D_2, D_4$, respectively, as follows.

Λ^{G_s}	n	T_X	$\mathbb{Z}l$	glue	GAP Id	case
$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 8 & 0 \\ 0 & 0 & 12 \end{pmatrix}$	2	$\begin{pmatrix} 12 & 0 \\ 0 & 30 \end{pmatrix}$	(2)	2	[720, 766]	79a
	2	$\begin{pmatrix} 2 & 1 \\ 1 & 8 \end{pmatrix}$	(12)	1	[720, 764]	79b
	2	$\begin{pmatrix} 2 & 0 \\ 0 & 12 \end{pmatrix}$	(30)	2	[720, 764]	79c
$\begin{pmatrix} 6 & 0 & 3 \\ 0 & 6 & 3 \\ 3 & 3 & 8 \end{pmatrix}$	2	$\begin{pmatrix} 6 & 0 \\ 0 & 20 \end{pmatrix}$	(6)	2	[720, 763]	79d
	2	$\begin{pmatrix} 8 & 2 \\ 2 & 8 \end{pmatrix}$	(12)	2	[720, 764]	79e
	4	$\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$	(20)	2	[1440, 4595]	79f

A projective model X of **79a** is given as follows (see also [Smi07, p. 14 (L), p. 18]). Consider the invariant curve of degree 6 by the Valentiner group in $\mathrm{GL}_3(\mathbb{C})$, which is defined in \mathbb{P}^2 by the following equation [Elk]:

$$10x^3y^3 + 9(x^5 + y^5)z - 45x^2y^2z^2 - 135xyz^4 + 27z^6 = 0.$$

The K3 surface X is defined as the double cover branched over this curve. We have $G = \mathfrak{A}_6 \times \mu_2$, where μ_2 is generated by the covering transformation.

A projective model X of **79d** is given in [Muk88]:

$$(6.16) \quad \sum_{i=1}^6 x_i = \sum_{i=1}^6 x_i^2 = \sum_{i=1}^6 x_i^3 = 0 \quad \text{in } \mathbb{P}^5.$$

The symmetric group \mathfrak{S}_6 of degree 6 acts on X . The hyperplane section of X has self intersection 6 and is invariant under \mathfrak{S}_6 . Hence $l^2 = 6$ and $G = \mathfrak{S}_6 = \mathfrak{A}_6 \rtimes \mu_2$.

In **79f**, $G/\mu_2 = G_s \rtimes (\mu_4/\mu_2)$ is isomorphic to M_{10} [KOZ07]. A projective model, which is same as **62c**, together with a (non-linear) action by $\mathfrak{A}_6 \times \mu_2$ and an invariant polarization of degree 20 is given in [MO14]. The missing automorphism acting by a primitive 4th root of unity is the one given in **62c**. The invariant polarization is given by $h + f_\infty$ using the same notation as in **62c**.

The full groups of automorphisms for **79b**, **79c** and **79f** are calculated in [Shi16].

6.10. No. 80

$G_s = F_{384}$. We have $\mathrm{SO}(\Lambda^{G_s}) \cong D_4$.

Λ^{G_s}	n	T_X	$\mathbb{Z}l$	glue	GAP Id	case
$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$	4	$\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$	(4)	1	[1536, 408544807]	80a
	2	$\begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix}$	(8)	1	[768, 1090134]	80b
	2	$\begin{pmatrix} 4 & 0 \\ 0 & 16 \end{pmatrix}$	(16)	2	[768, 1086051]	80c

A projective model of **80a** is given in [Muk88]:

$$(6.17) \quad x^4 + y^4 + z^4 + w^4 = 0 \quad \text{in } \mathbb{P}^3.$$

We have $G = F_{384} \rtimes \mu_4$, where μ_4 is generated by $(x : y : z : w) \mapsto (ix : y : z : w)$.

A projective model of **80b** [BS] is given by

$$(6.18) \quad q_1 = 2x_2^2 - x_3^2 - ix_4^2 + ix_5^2 - x_6^2 = 0$$

$$(6.19) \quad q_2 = -x_1^2 - ix_2^2 - x_3^2 + ix_4^2 + 2x_5^2 = 0$$

$$(6.20) \quad q_3 = -x_1^2 + ix_2^2 + 2x_4^2 - ix_5^2 - x_6^2 = 0$$

with linear action generated by

$$(6.21) \quad \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -j^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & j & 0 \\ 0 & 0 & 0 & 0 & 0 & -j \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & j^2 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $j^2 = i$.

6.11. No. 81

$G_s = M_{20}$. We have $\mathrm{SO}(\Lambda^{G_s}) \cong D_4$.

Λ^{G_s}	n	T_X	$\mathbb{Z}l$	glue	GAP Id	case
$\begin{pmatrix} 4 & 0 & 2 \\ 0 & 4 & 2 \\ 2 & 2 & 12 \end{pmatrix}$	2	$\begin{pmatrix} 4 & 0 \\ 0 & 40 \end{pmatrix}$	(4)	2	[1920, 240993]	81a
	2	$\begin{pmatrix} 8 & 4 \\ 4 & 12 \end{pmatrix}$	(8)	2	[1920, 240995]	81b
	4	$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$	(40)	2	—	81c

A projective model of **81a** is given in [Muk88]:

$$(6.22) \quad x^4 + y^4 + z^4 + w^4 + 12xyzw = 0 \quad \text{in } \mathbb{P}^3.$$

We have $G = M_{20} \rtimes \mu_2$, where μ_2 is generated by $(x : y : z : w) \mapsto (y : x : z : w)$.

A projective model of **81b** is given in [BS19]:

$$(6.23) \quad q_1 = x_1^2 - x_4^2 - \phi x_5^2 + \phi x_6^2 = 0,$$

$$(6.24) \quad q_2 = x_2^2 + \phi x_4^2 + x_5^2 - \phi x_6^2 = 0,$$

$$(6.25) \quad q_3 = x_3^2 - \phi x_4^2 - \phi x_5^2 + x_6^2 = 0,$$

where $\phi = (1 + \sqrt{5})/2$ is the golden ratio. The group $G = G_s \rtimes \mu_2$ is generated by

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

In **81c**, the group G has the maximal finite order 960. Its existence is proven in [Kon99] by applying lattice theory. For equations of **81c**, see Section 7. The full automorphism group over \mathbb{C} is calculated in [KK01]; see [Shi20] for mixed characteristic.

7. The group of maximal order

In this section, we give a projective model of **81c**. Let Y be the surface in \mathbb{P}^5 defined by the following equations:

$$(7.1) \quad \begin{aligned} f_1 &= x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0, \\ f_2 &= x_3^2 + x_4^2 + x_5^2 - x_6^2 = 0, \\ f_3 &= x_1^2 - x_2^2 + x_5^2 + x_6^2 = 0. \end{aligned}$$

The surface Y has a linear action of $H_s = C_2^4 \rtimes \mathfrak{A}_4$, which is generated by

$$(7.2) \quad (x_1 : \cdots : x_6) \mapsto (x_1 : x_2 : ix_4 : ix_3 : x_6 : x_5),$$

$$(7.3) \quad (x_1 : \cdots : x_6) \mapsto (-x_1 : x_2 : -x_3 : x_4 : x_5 : x_6),$$

$$(7.4) \quad (x_1 : \cdots : x_6) \mapsto (x_3 : x_4 : x_5 : x_6 : x_1 : x_2).$$

Moreover, Y has an automorphism h of order 4:

$$(7.5) \quad h: (x_1 : \cdots : x_6) \mapsto (x_1 : ix_2 : x_5 : ix_6 : x_3 : ix_4).$$

There are 16 singular points p_1, \dots, p_{16} of Y , e.g. $(0 : 1 : 0 : 1 : 0 : 1) \in \mathbb{P}^5$. They form one orbit under the H_s -action and each of them is of type A_1 . Let $\pi: X \rightarrow Y$ be the minimal resolution. Then X is a K3 surface. The induced action of H_s on X is symplectic and we have $h^*\omega_X = i\omega_X$. Let $l' \in \text{NS}_X$ denote the pull-back of the class of hyperplane section of Y . Furthermore, let $d \in \text{NS}_X$ denote the sum of the classes of the 16 exceptional curves of π . Then we have

$$(7.6) \quad l := 3l' - d, \quad l^2 = 9 \cdot l'^2 + d^2 = 9 \cdot 8 + 16 \cdot (-2) = 40,$$

and

$$(7.7) \quad H^0(X, l) \cong \{s \in R_3 \mid s(p_1) = \cdots = s(p_{16}) = 0\} / I_3 \subset R_3 / I_3 \cong H^0(X, 3l'),$$

where R_3 and I_3 are the homogeneous parts of degree 3 of $R := \mathbb{C}[x_1, \dots, x_6]$ and the defining ideal I of Y , respectively. (Hence I_3 is spanned by $x_i f_j$ for $1 \leq i \leq 6$ and $1 \leq j \leq 3$.) We take the following basis of $H^0(X, l) \cong \mathbb{C}^{22}$:

$$\begin{aligned} (z_1, \dots, z_{22}) = & (x_1 x_2 x_3, x_1 x_2 x_4, x_3 x_4 x_5, x_3 x_4 x_6, x_1 x_5 x_6, x_2 x_5 x_6, \\ & x_1 x_2 x_5, x_1 x_2 x_6, x_1 x_3 x_4, x_2 x_3 x_4, x_3 x_5 x_6, x_4 x_5 x_6, \\ & x_1^2 x_2, x_1 x_2^2, x_3^2 x_4, x_3 x_4^2, x_5^2 x_6, x_5 x_6^2, x_1 x_3 x_5, x_1 x_4 x_6, x_2 x_3 x_6, x_2 x_4 x_5). \end{aligned}$$

(The Riemann–Roch theorem also implies $\dim H^0(X, l) = l^2/2 + 2 = 22$.) The complete linear system for l gives a smooth embedding of X into \mathbb{P}^{21} with coordinates z_1, \dots, z_{22} . Moreover, the coordinates z_1, \dots, z_6 define a non-normal model \bar{X} of X in \mathbb{P}^5 . By using [DGPS19, SINGULAR], one can check the following: the defining ideal of \bar{X} is generated by $(\bar{g}^i)^* q$ for $0 \leq i \leq 4$, where

$$(7.8) \quad q := (-z_1^2 + z_2^2 - z_3^2 - z_4^2) z_5^2 + (z_1^2 - z_2^2 - z_3^2 - z_4^2) z_6^2 + z_5^4 - z_6^4,$$

$$(7.9) \quad \bar{g}: (z_1, \dots, z_6) \mapsto (-iz_2, -z_3, -z_5, -iz_1, -z_4, z_6).$$

The automorphism of \bar{X} (of order 5) induced by \bar{g} is extended to an automorphism g of X embedded into \mathbb{P}^{21} , where g is given by

$$\begin{aligned} (7.10) \quad g: (z_1 : \cdots : z_{22}) \mapsto & (-iz_2 : -z_3 : -z_5 : -iz_1 : -z_4 : z_6 : \\ & iz_{12} : -z_{10} : z_9 : -z_7 : -z_8 : iz_{11} : \\ & -iz_{16} : iz_{22} : z_{13} : -iz_{19} : z_{18} : iz_{21} : -iz_{20} : -iz_{15} : -z_{14} : z_{17}). \end{aligned}$$

By a direct computation, one can check that the following Cremona transformation acts on Y and induces g :

$$(7.11) \quad (x_1 : \cdots : x_6) \mapsto (x_3 x_4 : -x_2 x_5 : x_1 x_2 : -ix_3 x_5 : -x_5 x_6 : -ix_2 x_3).$$

The group G_s of all symplectic automorphisms of X with polarization l is generated by H_s and g . We have $G_s \cong M_{20}$. The group $G \cong M_{20} \rtimes \mu_4$ of all automorphisms of X with polarization l is generated by G_s and h .

Remark 7.1. — The motivation for this construction is as follows. Let X be a K3 surface with an action of $G = M_{20} \rtimes \mu_4$ as in **81c**. We consider a maximal proper subgroup H_s of $G_s = M_{20}$ isomorphic to $2^4.\mathfrak{A}_4$. From [Has12, Table 10.3], we get $\text{rank } \Lambda^{H_s} = 4$ and the genus symbol of Λ^{H_s} is $2_{\text{II}}^{-2}, 8_2^{-2}$. (In [Has12], H_s is No. 75 and its structure is written as $4^2\mathfrak{A}_4$.) Consider the lattice L of rank 4 with basis (b_1, \dots, b_4) and Gram matrix

$$(7.12) \quad \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & -8 \end{pmatrix}.$$

The lattice \bar{L} generated by L and $(b_1 + b_2 + b_3)/2$ is isomorphic to Λ^{H_s} . Consider an isometry of h defined by $(b_1, \dots, b_4) \mapsto (b_2, -b_1, b_3, b_4)$, which extends to \bar{L} . By a lattice-theoretic argument, it follows that the action of μ_4 on Λ^{H_s} corresponds to $\langle h \rangle$ and there is an ample class l' of degree 8 giving rise to b_3 . This suggests that there is a complete intersection of type $(2, 2, 2)$ in \mathbb{P}^5 birational to X , which is nothing but Y above. The classes l and d correspond to $3b_3 - 2b_4$ and $2b_4$, respectively. Indeed, calculating the orthogonal complement of l' inside the Néron–Severi lattice one finds 16 vectors (up to sign) of square (-2) . These give the 16 singular points of type A_1 of Y and their sum is $2b_4$.

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