**ANNALES
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A DOLBEAULT LEMMA FOR TEMPERATE CURRENTS

UN LEMME DE DOLBEAULT POUR LES COURANTS TEMPÉRÉS

Dedicated to the memory of Pierre Dolbeault

ABSTRACT. — We consider a bounded open Stein subset Ω of a complex Stein manifold X of dimension n. We prove that if f is a current on X of bidegree $(p, q+1)$, $\overline{\partial}$-closed on Ω, we can find a current u on X of bidegree (p, q) which is a solution of the equation $\overline{\partial} u = f$ in Ω. In other words, we prove that the Dolbeault complex of temperate currents on Ω (i.e. currents on Ω which extend to currents on X) is concentrated in degree 0. Moreover if f is a current on $X = \mathbb{C}^n$ of order k, then we can find a solution u which is a current on $\mathbb{C}^n$ of order $k + 2n + 1$.

RÉSUMÉ. — On considère un ouvert de Stein borné Ω d'une variété de Stein X de dimension complexe n. Nous montrons que si f est un courant sur X de bidegré $(p, q+1)$, $\overline{\partial}$-fermé sur Ω, il existe un courant u sur X de bidegré (p, q), solution sur Ω de l'équation $\overline{\partial} u = f$. En d'autres termes, nous prouvons que le complexe de Dolbeault des courants tempérés sur Ω (*i.e.* les courants sur Ω qui se prolongent en courants sur X) est concentré en degré 0. De plus si f est un courant sur $X = \mathbb{C}^n$ d'ordre k, nous montrons qu'il existe une solution u qui est un courant sur $\mathbb{C}^n$ d'ordre $k + 2n + 1$.

1. Introduction

We will prove the following result in the same way as the famous "Dolbeault–Grothendieck" lemma for $\bar{\partial}$.

THEOREM 1.1. — *Let Ω be a bounded Stein open subset of $\mathbb{C}^n$ and let f be a given current of bidegree $(p, q+1)$ on $\mathbb{C}^n$ (with compact support) which is $\bar{\partial}$-closed on Ω. Then there exists a current u of bidegree (p, q) (with compact support) in $\mathbb{C}^n$ such that:*

$$(1.1) \qquad\qquad\qquad \bar{\partial}u = f,$$

in Ω.

Moreover if f is of order k (resp. if $f \in H_{(p,q+1)}^{-s}(\mathbb{C}^n)$ for some $s > 0$), we can find a solution u is of order at most $k+2n+1$ (resp. $u \in H_{(p,q)}^{-s-2n-1}(\mathbb{C}^n)$, more precisely if k is the integer such that: $s \leqslant k < s+1$, for every $r > k$, we can find $u \in H_{(p,q)}^{-r-2n}(\mathbb{C}^n)$).

We say that a current T on $\Omega \subset^n$ is temperate if and only if it can be extended to $\mathbb{C}^n$. In other words, we have:

COROLLARY 1.2. — *For a given relatively compact open Stein subset of $\mathbb{C}^n$, the Dolbeault $\bar{\partial}$-cohomology of temperate currents on Ω vanishes.*

As usual, we denote by $H_{(p,q)}^s(\mathbb{C}^n)$ the space of current on $\mathbb{C}^n$ of bidegree (p, q) the coefficients of which are distributions in the Sobolev space $H^s(\mathbb{C}^n)$. A distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ is of order $k \in \mathbb{N}$ if it is locally a finite linear combination of derivatives of order at most k of Radon measures on $\mathbb{R}^n$ or equivalently if T can be extended as a continuous linear form defined on all functions of class C^k with compact support in $\mathbb{R}^n$ or equivalently if for every relatively compact open subset $\Omega \subset \mathbb{R}^n$, all functions $\phi \in D(\Omega)$ verify an inequality: $|\langle T, \phi \rangle| \leqslant C(\Omega, T) \sup_{x \in \Omega} \Sigma_{|\alpha| \leqslant k} |D_x^\alpha \phi(x)|$, in which the constant $C(\Omega, T)$ only depends on Ω and T. Of course a current is of order k, if its coefficients are distributions of order k.

The preceding results are still valid replacing $\mathbb{C}^n$ by a Stein manifold (Section 4, Theorem 4.3) and for currents taking their values in a given holomorphic vector bundle. But for the sake of simplicity, we begin with the case of $\mathbb{C}^n$ as in the Dolbeault–Grothendieck lemma: the general case of a Stein manifold does only need more difficult technical tools but no truly new ideas or methods. In the case of a Stein manifold the loss of regularity is larger than $2n+1$ because we have to iterate several times the construction made in the case of $\mathbb{C}^n$.

This result answers a question raised by Pierre Schapira in a personal discussion. He hopes it can be useful to make significant progress in the Microlocal Analysis theories highlighted for instance in the papers of M. Kashiwara and P. Schapira, [KS96, Sch17] in which such a temperate cohomology naturally appears.

Even though the result is essentially a consequence of L. Hörmander's L^2 estimates for $\bar{\partial}$ (Corollary 2.2), it seems that it can not be explicitly found in the literature on the subject (with complete proof). Let us observe the following features of the result. No assumption of smoothness is required for Ω. The given current f and the solution u have coefficients in spaces of distribution $H^s(\mathbb{C}^n)$ with $s < 0$. Hence they

are never supposed to be smooth but with temperate singularities as for instance derivatives of Dirac measures and the result is quite different from the most usual regularity results for $\bar\partial$ involving C^k regularity up to the boundary of Ω ($k \geqslant 0$) both for Ω and for the given differential forms on Ω. If $f \in H^s_{(p,\,q+1)}(\mathbb{C}^n)$ for some $s \geqslant 0$, then $f \in L^2_{(p,\,q+1)}(\Omega)$ and the result is an immediate consequence of Hörmander's theorem which provides a solution u in $L^2_{(p,\,q)}(\Omega)$. Then u has a trivial extension in $L^2_{(p,\,q)}(\mathbb{C}^n)$ (by 0 outside Ω).

The gap $2n + 1$ of regularity for the solution u does not depend on Ω. In the basic example $\Omega = B(0, R) \setminus H$ in which H is a complex analytic hypersurface of the ball $B(0, R)$ of center 0 and radius R, the result does not depend at all on the complexity of the singularities of H (and on the degree of H when H is algebraic). The gap $2n + 1$ is an automatic consequence of the method of proof. To improve the gap $2n + 1$ does not seem to immediately have a major interest for the purpose in [Sch17].

We need four steps to prove Theorem 1.1. At first, as P. Dolbeault in [Dol56, Dol57], by solving an appropriate Laplacian equation $\frac{1}{2}\Delta v = \bar\partial^\star f$ on $\mathbb{C}^n$ (Δ is the usual Laplacian on $\mathbb{C}^n$ defined on differential forms and currents and $\bar\partial^\star$ is the operator adjoint of $\bar\partial$ for the usual Hermitian structure on $\mathbb{C}^n$) and replacing f by $f - \bar\partial v$, we reduce the problem to the case of a current f which has harmonic coefficients on Ω. As f is temperate, the mean value properties of harmonic functions imply that f grows at the boundary of Ω like a negative power of the distance $d(z, \partial\Omega)$ to the boundary of Ω (for $z \in \Omega$). Then Hörmander's L^2 estimates for $\bar\partial$ give a solution u of $\bar\partial u = f$ such that $\int_\Omega |u|^2 [d(z, \partial\Omega)]^{2l} d\lambda(z) < +\infty$ (for some $l > 0$). Finally using an extension theorem of L. Schwartz [Sch50] for distributions, u can be extended as a current on $\mathbb{C}^n$.

Similar methods, were already used by P. Lelong [Lel64] for the Lelong–Poincaré $\partial\bar\partial$-equation and by H. Skoda [Sko71] for the $\bar\partial$-equation to obtain solutions explicitly given on $\mathbb{C}^n$ by integral representations and with precise polynomial estimates. Y.T. Siu has already studied holomorphic functions of polynomial growth on bounded open domain of $\mathbb{C}^n$ using Hörmander's L^2 estimates for $\bar\partial$ in [Siu70].

We establish the preliminary results we need in Section 2 and we prove Theorem 1.1 in Section 3. We extend the results to a Stein manifold in Section 4 Theorem 4.3 (using J-P. Demailly's Theorem 4.1 extending Hörmander's results to manifolds). A first purely analytic proof of Theorem 4.3 (of independent interest) was given in ArXiv [Sko20] on may 2020. According to constructive comments of the referee, we have finally preferred to use a more sheaf theoretic method which follows the proof of the Dolbeault isomorphism [Dol56, Dol57].

In the case of a subanalytic bounded open Stein subset Ω in a Stein manifold X, Pierre Schapira in [Sch21] gives independently a proof of Theorem 1 (i.e. of Corollary 1.2) and of Theorem 4.3. His proof is basically founded on cohomological methods which are particularly well adapted to the subanalytic case. It also heavily depends on Hörmander's L^2 estimates for $\bar\partial$ that he uses in the case of a bounded Stein open subset of $\mathbb{C}^n$ after embedding the given Stein manifold in some space

$\mathbb{C}^n$. He also uses Lojasiewicz inequalities and another Hörmander's inequality for subanalytic subsets.

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I thank Pierre Schapira very much for raising his insightful question which has strongly motivated this research.

2. Preliminary definitions and results

Before proving Theorem 1.1, we need to remind several classical results. We have sometimes given direct proof to establish the results in the appropriate form we wish.

An open subset of $\mathbb{C}^n$ is called Stein if it is holomorphically convex: for all compact K in Ω the holomorphic hull $\widehat{K}_\Omega$ of K is compact ($x \in \widehat{K}_\Omega$ if and only if $x \in \Omega$ and for all holomorphic function f on Ω, $|f(x)| \leqslant \max_{\xi \in K} |f(\xi)|$).

Let us recall the following fundamental Hörmander's L^2 existence theorem for $\bar{\partial}$ [Hör90] or [Hör65]. We can also use J.P. Demailly's book [Dem12, Chapter VIII, Paragraph 6, Theorem 6.9, p. 379]. We denote by $L^2_{(p,q)}(\Omega, loc)$ the vector space of current of bidegree (p, q) in Ω the coefficients of which are in $L^2(\Omega, loc)$ for the usual Lebesgue measure $d\lambda$ on $\mathbb{C}^n$.

THEOREM 2.1. — *Let Ω be an open pseudoconvex subset of $\mathbb{C}^n$ and ϕ a plurisubharmonic function defined on Ω. For every $g \in L^2_{(p,q+1)}(\Omega, loc)$ with $\bar{\partial}g = 0$ such that: $\int_\Omega |g|^2 e^{-\phi} d\lambda < +\infty$, there exists $u \in L^2_{(p,q)}(\Omega, loc)$ such that:*

$$(2.1) \qquad\qquad\qquad \bar{\partial}u = g$$

in Ω and:

$$(2.2) \qquad \int_\Omega |u|^2 e^{-\phi} \left(1 + |z|^2\right)^{-2} d\lambda \leqslant \frac{1}{2} \int_\Omega |g|^2 e^{-\phi} d\lambda.$$

If Ω is bounded, u verifies the L^2 estimate:

$$(2.3) \qquad\qquad \int_\Omega |u|^2 e^{-\phi} d\lambda \leqslant C(\Omega) \int_\Omega |g|^2 e^{-\phi} d\lambda$$

with $C(\Omega) := \frac{1}{2}(1 + \max_{z \in \Omega} |z|^2)^2$.

The classical Oka–Norguet–Bremerman theorem ([Hör90, Paragraph 2.6 and Theorem 4.2.8]) claims that the following assertions are equivalent:

   (1) Ω is Stein,
   (2) Ω is pseudoconvex: i.e. there exists a plurisubharmonic function ϕ on Ω which is exhaustive (for all $c \in \mathbb{R}$ the subset $\{z \in \Omega|\ \phi(z) < c\}$ is relatively compact in Ω),
   (3) the function $-\log d(z, \partial\Omega)$ is plurisubharmonic in Ω.

Therefore for a given $k \geqslant 0$, we can choose $\phi(z) = -k \log d(z, \partial\Omega)$ in the inequality (2.3) and we will only need to use the following special case of Theorem 2.1 (see also [Hör65, Theorem 2.2.1']).

Corollary 2.2. — *Let Ω be a bounded Stein open subset of $\mathbb{C}^n$ and $k \geqslant 0$ be a given real number. Then for every $g \in L^2_{(p,q+1)}(\Omega, loc)$ with $\bar{\partial}g = 0$ such that: $\int_\Omega |g|^2 [d(z, \partial\Omega)]^k d\lambda < +\infty$, there exists $u \in L^2_{(p,q)}(\Omega, loc)$ such that:*

$$\bar{\partial} u = g \tag{2.4}$$

in Ω and:

$$\int_\Omega |u|^2 [d(z, \partial\Omega)]^k d\lambda \leqslant C(\Omega) \int_\Omega |g|^2 [d(z, \partial\Omega)]^k d\lambda \tag{2.5}$$

If we denote by $L^{2,k}_{(p,q)}(\Omega)$ the space of $u \in L^2_{(p,q)}(\Omega, loc)$ such that

$$\int_\Omega |u|^2 [d(z, \partial\Omega)]^{2k} d\lambda < +\infty,$$

by:

$$L^{2,k}_{0,(p,q)}(\Omega) := \left\{ u \in L^{2,k}_{(p,q)}(\Omega) \middle| \; \bar{\partial}u \in L^{2,k}_{(p,q+1)}(\Omega) \right\} \tag{2.6}$$

and by $\mathcal{O}^{2,k}_p(\Omega) := \{ u \in L^{2,k}_{(p,0)}(\Omega) | \; \bar{\partial}u = 0 \}$, Corollary 2.2 means that the following Dolbeault-complex is exact:

$$
\begin{aligned}
0 \to \mathcal{O}^{2,k}_p(\Omega) \to L^{2,k}_{0,(p,0)}(\Omega) &\xrightarrow{\bar{\partial}} L^{2,k}_{0,(p,1)}(\Omega) \xrightarrow{\bar{\partial}} \ldots \xrightarrow{\bar{\partial}} L^{2,k}_{0,(p,q)}(\Omega) \\
&\xrightarrow{\bar{\partial}} L^{2,k}_{0,(p,q+1)}(\Omega) \xrightarrow{\bar{\partial}} \ldots \xrightarrow{\bar{\partial}} L^{2,k}_{0,(p,n)}(\Omega) \to 0.
\end{aligned}
\tag{2.7}
$$

We also need two results of real analysis.

Lemma 2.3. — *Let w be a distribution on $\mathbb{R}^n$ of order k which is harmonic (for the usual Laplacian on $\mathbb{R}^n$) on the bounded open subset Ω of $\mathbb{R}^n$. Then w is of polynomial growth on Ω: $|w(z)| \leqslant C(\Omega, w) \left[d(z, \mathbb{R}^n \setminus \Omega) \right]^{-k-n}$ where the constant $C(\Omega, w)$ only depends on Ω and w.*

If $w \in H^{-s}(\mathbb{R}^n)$ for $s \geqslant 0$ we have: $|w(z)| \leqslant C(\Omega, w) \left[d(z, \mathbb{R}^n \setminus \Omega) \right]^{-k-\frac{n}{2}}$ where k is the integer such that $s \leqslant k < s+1$.

Proof. — Let ρ be a non negative regularizing function in $\mathcal{D}(\mathbb{R}^n)$ which only depends on $|\zeta|$, has its support in the Euclidean ball of radius 1 and verifies: $\int_{\mathbb{R}^n} \rho(\zeta) d\lambda(\zeta) = 1$ where $d\lambda$ is the Lebesgue measure on $\mathbb{R}^n$.

Let $\rho_\epsilon(\zeta) := \frac{1}{\epsilon^n} \rho(\frac{\zeta}{\epsilon})$ be the associated family of regularizing functions in $\mathcal{D}(\mathbb{R}^n)$ so that ρ_ϵ has its support in the ball of radius ϵ and verifies too $\int_{\mathbb{R}^n} \rho_\epsilon(\zeta) d\lambda(\zeta) = 1$.

As w is harmonic in Ω, for every $z \in \Omega$, $w(z)$ coincide with its mean-value on every Euclidean sphere of center z and radius $r < d(z, \partial\Omega)$. Therefore using Fubini's theorem we get for every $\epsilon < d(z, \partial\Omega)$:

$$w(z) = \int_{\mathbb{R}^n} w(z + \zeta) \rho_\epsilon(\zeta) d\lambda(\zeta) = \int_{\mathbb{R}^n} w(\zeta) \rho_\epsilon(z - \zeta) d\lambda(\zeta). \tag{2.8}$$

i.e. $w = w \star \rho_\epsilon$ on $\Omega_\epsilon := \{ z | \; d(z, \partial\Omega) < \epsilon \}$ (in which $\star$ represents a convolution product).

Testing w as a distribution on the test function (in the variable ζ): $\rho_\epsilon(z - \zeta)$ with $\epsilon < d(z, \partial\Omega) \leqslant 1$, equation (2.8) becomes:

$$w(z) = \langle w(\zeta), \rho_\epsilon(z - \zeta) \rangle_\zeta. \tag{2.9}$$

As w is a distribution of order k, we have for every function $\phi \in \mathcal{D}(\mathbb{R}^n)$ an inequality:

$$(2.10) \qquad |\langle w, \phi \rangle| \leqslant C_1(w) \sup_{\zeta \in \mathbb{C}^n} \Sigma_{|\alpha| \leqslant k} \left| D_\zeta^\alpha \phi(\zeta) \right|,$$

in which $C_1(w) > 0$ is a constant only depending on w.

Taking $\phi(\zeta) = \rho_\epsilon(z - \zeta)$, we get:

$$(2.11) \qquad |w(z)| \leqslant C_1(w) \sup_{|\zeta| \leqslant \epsilon} \Sigma_{|\alpha| \leqslant k} \left| D_\zeta^\alpha \rho_\epsilon(z - \zeta) \right|,$$

and:

$$(2.12) \qquad |w(z)| \leqslant C_2(w) \epsilon^{-n-k} ,$$

for some constant $C_2(w) > 0$.

As it is true for every $\epsilon < d(z, \partial\Omega)$, we take the limit as $\epsilon \to d(z, \partial\Omega)$ and we get;

$$(2.13) \qquad |w(z)| \leqslant C_2(w) \, [d(z, \partial\Omega)]^{-l}$$

with $l = n + k$ and then:

$$(2.14) \qquad \int_\Omega |w|^2 \, [d(z, \partial\Omega)]^{2l} \, d\lambda < \infty.$$

If we now assume that $w \in H^{-s}(\mathbb{R}^n)$ for a given $s > 0$, equation (2.9) becomes:

$$(2.15) \qquad |w(z)| = \left| \langle w(\zeta), \rho_\epsilon(z - \zeta) \rangle_\zeta \right| \leqslant \|w\|_{H^{-s}(\mathbb{R}^n)} \|\rho_\epsilon(z - \zeta)\|_{H^s(\mathbb{R}^n)} .$$

Let k be the integer defined by $s \leqslant k < s + 1$ so that (denoting as usual by $\widehat{\phi}$ the Fourier transform of ϕ):

$$(2.16)$$
$$\|\phi\|_{H^s(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s |\widehat{\phi}(\xi)|^2 d\lambda(\xi) \leqslant \int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^k |\widehat{\phi}(\xi)|^2 d\lambda(\xi) = \|\phi\|_{H^k(\mathbb{R}^n)}^2$$

As k is an integer, the norm $\|\phi\|_{H^k(\mathbb{R}^n)}$ is equivalent to the sum of the L^2 norms of the derivatives of ϕ of order less or equal to k, we have:

$$(2.17) \qquad \|\phi\|_{H^k(\mathbb{R}^n)}^2 \leqslant C_2(k) \int_{\mathbb{R}^n} \sum_{|\alpha| \leqslant k} |D^\alpha \phi|^2 d\lambda$$

We replace ϕ by $\phi_\epsilon(\zeta) := \frac{1}{\epsilon^n} \phi(\frac{\zeta}{\epsilon})$ (with $\epsilon \leqslant 1$) so that we get:

$$(2.18) \qquad \|\phi_\epsilon\|_{H^k(\mathbb{R}^n)}^2 \leqslant C_2(k) \epsilon^{-2k-n} \left[\int_{\mathbb{R}^n} \sum_{|\alpha| \leqslant k} |D^\alpha \phi|^2 d\lambda \right]$$

Using (2.15), (2.16), (2.17) and (2.18) with $\phi(\zeta) = \rho(z - \zeta)$ (for a fixed $z \in \Omega$ with $\epsilon < d(z, \partial\Omega) \leqslant 1$) we finally obtain:

$$(2.19) \qquad |w(z)| \leqslant C_3(k, n) \|w\|_{H^{-s}(\mathbb{R}^n)} \epsilon^{-k-\frac{n}{2}}$$

and when $\epsilon \to d(z, \partial\Omega)$:

$$(2.20) \qquad |w(z)| \leqslant C_3(k, n) \|w\|_{H^{-s}(\mathbb{R}^n)} [d(z, \partial\Omega)]^{-k-\frac{n}{2}}$$

$$(2.21) \qquad \int_\Omega |w(z)|^2 \, [d(z, \partial\Omega)]^{2k+n} d\lambda(z) < +\infty.$$

$\square$

Remark 2.4. — Instead of using mean properties of harmonic functions, one can also use the elementary solution of Δ in $\mathbb{R}^n$ as in [KS96, Proposition 10.1., p. 53].

We also need the following theorem of L. Schwartz (in his book on distribution theory [Sch50]). We can also directly use theory of Sobolev spaces. We say that a measure μ defined on an open bounded subset Ω of $\mathbb{R}^n$, is of polynomial growth at most l in Ω, if $\int_\Omega d(z, \partial\Omega)^l d|\mu|(z) < +\infty$.

THEOREM 2.5. — *A measure of polynomial growth l defined on an open bounded subset Ω of $\mathbb{R}^n$ can be extended as a distribution on $\mathbb{R}^n$ of order at most l.*

Moreover if $w \in L^2(\Omega, loc)$ verifies the estimate: $\int_\Omega |w(z)|^2 [d(z, \partial\Omega)]^{2l} d\lambda(z) < +\infty$, with $l \in \mathbb{N}$, then for every $r > l$, w can be extended as a distribution in $H^{-r-\frac{n}{2}}(\mathbb{R}^n)$ (particularly in $H^{-l-\frac{n}{2}-1}(\mathbb{R}^n)$).

Remark 2.6. — If $\int_\Omega |w(z)|^2 [d(z, \partial\Omega)]^{2l} d\lambda(z) < +\infty$, let us observe that the extension $\widetilde{w}$ of w depends (*a priori*) on the choice of $r > l$. We will use the results of Theorem 2.5 in the case of $\mathbb{C}^n = \mathbb{R}^{2n}$ so that $\widetilde{w}$ can be constructed in H^{-l-n-1}.

Remark 2.7. — If $\int_\Omega |w(z)|^2 [d(z, \partial\Omega)]^{2l} d\lambda(z) < +\infty$, as Ω is bounded, Schwarz inequality implies that $\int_\Omega d(z, \partial\Omega)^l |w(z)| d\lambda(z) < +\infty$. w defines on Ω a measure of polynomial growth at most l. Hence the first part of Theorem 2.5 implies that w can be extended to $\mathbb{R}^n$ as a distribution of order at most l.

Proof. — In L. Schwartz's book there is no proof and no references so that we give the following proof. We consider the subspace $F \subset \mathcal{D}(\mathbb{C}^n)$ of functions the derivatives of which vanish at the order $\leqslant l - 1$ in every point $\zeta \in \partial\Omega$. For a given $z \in \Omega$, we choose a point $\zeta \in \partial\Omega$ such that $|z - \zeta| = d(z, \partial\Omega)$ and we apply Taylor's formula at the point $\zeta \in \partial\Omega$, at the order $l - 1$ (with integral remainder, cf. [Hör83, Paragraph 1.1, formula (1.1.7')]) to a function $\phi \in F$ restricted to the real interval $\{tz + (1-t)\zeta \,|\, t \in \mathbb{R}, 0 \leqslant t \leqslant 1\}$ linking in Ω the point $\zeta \in \partial\Omega$ to $z \in \Omega$, so that we obtain:

$$(2.22) \qquad \phi(z) = l \int_0^1 (1-t)^{l-1} \left[\Sigma_{|\alpha|=l} \, D^\alpha \phi \left(\zeta + t(z - \zeta) \right) \frac{(z-\zeta)^\alpha}{\alpha!} \right] dt,$$

and then:

$$(2.23) \qquad |\phi(z)| \leqslant C_4(l, n) \, [d(z, \partial\Omega)]^l \max_{\xi \in \bar{\Omega}} \left[\Sigma_{|\alpha|=l} \left| D_\xi^\alpha \phi(\xi) \right| \right].$$

For all functions $\phi \in F$ and all measures μ on Ω of polynomial growth l, i.e. $\int_\Omega d(z, \partial\Omega)^l d|\mu|(z) < +\infty$, (using (2.23)) we have:

$$(2.24) \qquad \left| \int_\Omega \phi \, d\mu \right| \leqslant C_4(l, n) \left[\int_\Omega d(z, \partial\Omega)^l d|\mu| \right] \max_{\xi \in \bar{\Omega}} \left[\Sigma_{|\alpha| \leqslant l} \left| D_\xi^\alpha \phi(\xi) \right| \right].$$

For a given measure μ of polynomial growth l, we consider the space $\mathcal{E}^l(\mathbb{C}^n)$ of functions of class C^l on $\mathbb{C}^n$. We apply Hahn Banach theorem to the linear form $\phi \to \int_\Omega \phi \, d\mu$ defined on the subspace $F \subset \mathcal{D}(\mathbb{C}^n) \subset \mathcal{E}^l(\mathbb{C}^n)$ and continuous for the seminorm $\max_{\xi \in \bar{\Omega}} [\Sigma_{|\alpha| \leqslant l} |D_\xi^\alpha \phi(\xi)|]$. This linear form can be extended in a continuous linear form T on $\mathcal{E}^l(\mathbb{C}^n)$, such that:

$$(2.25) \qquad |\langle T, \phi \rangle| \leqslant C_4(l, n) \left[\int_\Omega d(z, \partial\Omega)^l d|\mu| \right] \max_{\xi \in \bar{\Omega}} \left[\Sigma_{|\alpha| \leqslant l} \left| D_\xi^\alpha \phi(\xi) \right| \right].$$

for all $\phi \in \mathcal{E}^l(\mathbb{C}^n)$, i.e. a distribution of order l on $\mathbb{C}^n$ (with compact support).

Let us now assume that $w \in L^2(\Omega, loc)$ verifies the estimate:

$$(2.26) \qquad I_l(w) := \int_\Omega |w(z)|^2 \ [d(z, \partial\Omega)]^{2l} \ d\lambda(z) < +\infty.$$

for some integer $l \geqslant 0$.

For every $\phi \in F$, Cauchy–Schwarz inequality gives:

$$(2.27) \qquad |\langle w, \phi \rangle|^2 = \left| \int_\Omega w\phi \ d\lambda \right|^2 \leqslant I_l(w) \ \int_\Omega |\phi(z)|^2 \ [d(z, \partial\Omega)]^{-2l} \ d\lambda(z).$$

Using inequality (2.23), (2.27) becomes:

$$(2.28) \qquad |\langle w, \phi \rangle|^2 \leqslant C_5(l, n, \Omega) \ I_l(w) \ \left[\max_{\xi \in \bar\Omega} \Sigma_{|\alpha|=l} \ \left| D_\xi^\alpha \phi(\xi) \right| \right]^2.$$

with $C_5(l, n, \Omega) := [C_4(l, n)]^2 \int_\Omega d\lambda$.

For every $r > l$, we use classical Sobolev inequality:

$$(2.29) \qquad \max_{\xi \in \mathbb{R}^n} \ \Sigma_{|\alpha| \leqslant l} \ \left| D_\xi^\alpha \phi(\xi) \right| \leqslant C_6(r) \ \|\phi\|_{H^{r+\frac{n}{2}}}$$

and inequality (2.28), so that we obtain:

$$(2.30) \qquad |\langle w, \phi \rangle| \leqslant C_7(l, n, r, \Omega) \ [I_l(w)]^{\frac{1}{2}} \ \|\phi\|_{H^{r+\frac{n}{2}}}.$$

with $C_7(l, n, r, \Omega) := C_6(r) \, [C_5(l, n, \Omega)]^{\frac{1}{2}}$.

Using still Hahn–Banach Theorem for the linear form $\phi \to \langle w, \phi \rangle$ defined on the subspace F of $H^{r+\frac{n}{2}}(\mathbb{R}^n)$ and continuous for the norm of $H^{r+\frac{n}{2}}$, we extend w as a distribution $T \in H^{-r-\frac{n}{2}}$ such that: $\|T\|_{H^{-r-\frac{n}{2}}} \leqslant C_7(l, n, r, \Omega) \, [I_l(w)]^{\frac{1}{2}}$ (of course we can also do this extension by using orthogonal projection on the closed subspace $\bar{F}$ in the Hilbert space $H^{r+\frac{n}{2}}(\mathbb{R}^n)$). $\qquad \square$

We can now prove Theorem 1.1.

3. Proof of Theorem 1.1

We follow P. Dolbeault's proof of the Dolbeault–Grothendieck lemma. A. Grothendieck's proof was different, (in some sense) more elementary than P. Dolbeault's proof but not useful for our present purpose. Of course we can suppose (w.l.o.g.) that f has compact support in $\mathbb{C}^n$ (using a cutoff function in $\mathcal{D}(\mathbb{R}^n)$ equal to 1 in a neighborhood of $\bar\Omega$). Let us remind that $\mathbb{C}^n$ being equipped with its usual flat Hermitian metric, the Laplacian acting on differential forms and currents is defined on $\mathbb{C}^n$ by:

$$(3.1) \qquad \frac{1}{2}\Delta = \frac{1}{2}(dd^\star + d^\star d) = \bar\partial\bar\partial^\star + \bar\partial^\star\bar\partial = \partial\partial^\star + \partial^\star\partial,$$

so that $\frac{1}{2}\Delta f$ is the usual Laplacian on $\mathbb{C}^n$ acting on each coefficient of the current f. $\bar\partial^\star$ (resp. $\partial^\star$) (resp. $d^\star$) is the adjoint of $\bar\partial$ (resp. ∂) (resp. $d := \partial + \bar\partial$) for the same constant metric on $\mathbb{C}^n$ (there is no weight function).

At first we solve in $\mathbb{C}^n$ the Laplacian equation:

$$(3.2) \qquad \frac{1}{2}\Delta v := \left(\bar{\partial}\bar{\partial}^\star + \bar{\partial}^\star\bar{\partial}\right) v = \bar{\partial}^\star f.$$

(v and $\bar{\partial}^\star f$ are of bidegree (p,q).)

If we write: $f = \sum'_{|I|=p,\,|J|=q+1} f_{I,J} dz_I \wedge d\bar{z}_J$ (Σ' means that we only sum on strictly increasing multi-indices I and J), we have (cf. [Hör90, Paragraph 4.1, p. 82 or 85] or [Dem12, Chapter 6, Paragraph 6.1]) :

$$(3.3) \qquad \bar{\partial}^\star f = (-1)^{p-1}\Sigma'_{|I|=p,\,|K|=q} \left[\sum_{j=1}^{j=n} \frac{\partial}{\partial z_j}(f_{I,jK})\right] dz_I \wedge d\bar{z}_K.$$

If f is of bidegree $(0,1)$, we simply have : $f = \Sigma_{j=1}^{j=n} f_j d\bar{z}_j$, $\bar{\partial}^\star f = -\Sigma_{j=1}^{j=n}\frac{\partial f_j}{\partial z_j}$ and (3.2) is the Laplace equation in $\mathbb{C}^n$:

$$\sum_{j=1}^{j=n} \frac{\partial^2}{\partial z_j \partial \bar{z}_j} v = \sum_{j=1}^{j=n} \frac{\partial f_j}{\partial z_j}.$$

v is obtained by convolution of each coefficient $\frac{\partial f_{I,jK}}{\partial z_j}$ of $\bar{\partial}^\star f$ in (3.3) with the elementary solution E of the usual Laplacian in $\mathbb{C}^n$.

We set:

$$(3.4) \qquad g := f - \bar{\partial}v$$

As $\bar{\partial}^2 = 0$, we have (by usual computation):

$$(3.5) \qquad \frac{1}{2}\Delta(\bar{\partial}v) = \left(\bar{\partial}\bar{\partial}^\star + \bar{\partial}^\star\bar{\partial}\right)\bar{\partial}v = \bar{\partial}\bar{\partial}^\star\bar{\partial}v = \bar{\partial}\left(\bar{\partial}\bar{\partial}^\star + \bar{\partial}^\star\bar{\partial}\right)v = \bar{\partial}\left(\frac{1}{2}\Delta v\right)$$

i.e. $\bar{\partial}$ commutes with Δ on $\mathbb{C}^n$. Using (3.2), we have: $\frac{1}{2}\Delta(\bar{\partial}v) = \bar{\partial}\bar{\partial}^\star f$, and:

$$(3.6) \qquad \frac{1}{2}\Delta g = \frac{1}{2}\Delta f - \frac{1}{2}\Delta(\bar{\partial}v) = \left(\bar{\partial}\bar{\partial}^\star + \bar{\partial}^\star\bar{\partial}\right) f - \bar{\partial}\bar{\partial}^\star f = \bar{\partial}^\star\bar{\partial}\,f.$$

Hence (as $\bar{\partial}f = 0$ on Ω), g is harmonic on Ω:

$$(3.7) \qquad\qquad\qquad\qquad \Delta g = 0.$$

f being of order k with compact support, $\bar{\partial}^\star f$ is of order $k+1$ with the same support (the coefficients of $\bar{\partial}^\star f$ are linear combination of derivatives $\frac{\partial}{\partial z_j}$ of the coefficients of f). Therefore the solution v of the Laplacian equation (3.2) is of order at most k. Indeed it is obtained by convolution: $E \star \frac{\partial f_{I,jK}}{\partial z_j} = \frac{\partial E}{\partial z_j} \star f_{I,jK}$ of each coefficient $\frac{\partial f_{I,jK}}{\partial z_j}$ of $\bar{\partial}^\star f$ in (3.3) with the elementary solution $E := -C_n|z|^{-2n+2}$ of Δ, the first derivatives $\frac{\partial E}{\partial z_j}$ of E are $O(|z|^{-2n+1})$, then in $L^1(\mathbb{C}^n, loc)$ and the convolution $\frac{\partial E}{\partial z_j} \star f_{I,jK}$ of a function in $L^1(\mathbb{R}^{2n}, loc)$ with a distribution of order k and compact support is still of order at most k. Hence $\bar{\partial}v$ is of order at most $k+1$ and $g = f - \bar{\partial}v$ is too of order at most $k+1$.

We write:
$$g = {\sum_{|I|=p,\,|J|=q+1}}' g_{I,J} dz_I \wedge d\bar{z}_J$$
with strictly increasing multi-indices I and J. Let $g_{I,J}$ be a coefficient of g. As $g_{I,J}$ is harmonic in Ω, we can apply Lemma 2.3 in $\mathbb{R}^{2n}$ to $g_{I,J}$ which is a distribution of order at most $k+1$, we get an inequality:

$$(3.8) \qquad\qquad |g_{I,J}(z)| \leqslant C_1(\Omega, g_{I,J})\, [d(z, \partial\Omega)]^{-2n-k-1}.$$

Hence :

$$(3.9) \qquad\qquad |g(z)| \leqslant C_2(\Omega, g)\, [d(z, \partial\Omega)]^{-l}$$

for some constant $C_2(\Omega, g) > 0$ and $l := 2n + k + 1$ (and $z \in \Omega$) and then:

$$(3.10) \qquad\qquad \int_\Omega |g|^2\, [d(z, \partial\Omega)]^{2l} d\lambda < \infty$$

where $d\lambda$ is the Lebesgue measure on $\mathbb{C}^n$.

L. Hörmander's L^2 estimates for $\bar{\partial}$ (Corollary 2.2) provide a solution u in Ω of the equation:

$$(3.11) \qquad\qquad \bar{\partial} u = g$$

with the L^2 estimate:

$$(3.12) \qquad\qquad \int_\Omega |u|^2\, [d(z, \partial\Omega)]^{2l} d\lambda < \infty$$

As Ω is bounded, Cauchy–Schwarz inequality gives the following L^1 estimate:

$$(3.13) \qquad\qquad \int_\Omega |u|\, [d(z, \partial\Omega)]^l d\lambda < \infty$$

Therefore a coefficient $u_{I,J}$ of u defines a measure of polynomial growth l on Ω. Using L. Schwartz's Theorem 2.5, such a measure (of polynomial growth l) defined on Ω can be extended as a distribution on $\mathbb{C}^n$ (of order at most l) so that u can be extended as a current on $\mathbb{C}^n$ of order at most l. Then $u + v$ is a current on $\mathbb{C}^n$ verifying:

$$(3.14) \qquad\qquad \bar{\partial}(u + v) = f$$

on Ω. Moreover $u + v$ has order at most $l = k + 2n + 1$.

We now consider the case of a given $f \in H_{(p,q+1)}^{-s}(\mathbb{C}^n)$ for some $s \geqslant 0$. Then $\bar{\partial}^\star f \in H_{(p,q)}^{-s-1}(\mathbb{C}^n)$. Classically we can find a solution v of the Laplace equation (3.2) in $H_{(p,q)}^{-s+1}(\mathbb{C}^n)$ so that $g = f - \bar{\partial}v$ is also in $H_{(p,q+1)}^{-s}(\mathbb{C}^n)$. We apply Lemma 2.3 to every coefficient $g_{I,J}$ of g in $\mathbb{C}^n = \mathbb{R}^{2n}$ so that $|g(z)| \leqslant C\, [d(z, \mathbb{C}^n \setminus \Omega)]^{-k-n}$ where k is the integer such that $s \leqslant k < s + 1$ and $C = C(\Omega, g)$ is a constant. Therefore we obtain:

$$(3.15) \qquad\qquad \int_\Omega |g|^2\, [d(z, \partial\Omega)]^{2k+2n} d\lambda < +\infty.$$

Corollary 2.2 implies we can solve $\bar{\partial} u = g = f - \bar{\partial}v$ with the estimate:

$$(3.16) \qquad\qquad \int_\Omega |u|^2\, [d(z, \partial\Omega)]^{2k+2n} d\lambda < +\infty.$$

We now apply Theorem 2.5 in $\mathbb{C}^n = \mathbb{R}^{2n}$ to every coefficient of u with $l = k + n$ ($l + n = k + 2n$) so that for every $r > k$, u can be extended as a current in $\mathbb{C}^n$ in $H_{(p,q)}^{-r-2n}(\mathbb{C}^n)$. As $v \in H_{(p,q)}^{-s+1}(\mathbb{C}^n)$, $u + v$ is too in $H_{(p,q)}^{-r-2n}(\mathbb{C}^n)$ and verifies $\bar{\partial}(u + v) = f$ in Ω.

Remark 3.1. — We have a little more precise result: $f = \bar{\partial}(u + v)$ in Ω with $u \in L_{(p,q)}^{2,k+n}(\Omega) \cap H_{(p,q)}^{-r-2n}(\mathbb{C}^n)$ (for every $r > k$ particularly for $r = k + 1$) and $v \in H_{(p,q)}^{-s+1}(\mathbb{C}^n)$. Let us observe that $v \in H_{(p,q)}^{-s+1}(\mathbb{C}^n)$ has a better regularity than $f \in H_{(p,q)}^{-s}(\mathbb{C}^n)$.

4. Extension of Theorem 1.1 to Stein manifolds

We will now see that Theorem 1.1 remains true for a relatively compact open Stein subset Ω of a given Stein manifold X. We use the same reasoning as in the classical proof of the Dolbeault isomorphism to reduce the problem on the one hand to local solutions in charts of X for $\bar{\partial}$ and on the other hand to a global solution on X for $\bar{\partial}$. But we need much stronger technical results.

Let us recall that a complex manifold X is Stein if, by definition, global holomorphic functions $\mathcal{O}(X)$ separate the points of X, give local holomorphic coordinates on X and if X is holomorphically convex (for all compact K in X the holomorphic hull $\widehat{K}$ of K is compact with $\widehat{K} := \{x \in X | \ \forall \ f \in \mathcal{O}(X), |f(x)| \leqslant \max_{\xi \in K} |f(\xi)| \}$). Let us also remind the two following other characterizations of a Stein manifold X of complex dimension n. The first one, a complex holomorphic manifold X is Stein if and only if it can be imbedded as a closed complex submanifold of $\mathbb{C}^{2n+1}$. The second one, X is Stein if and only if there exists a strictly plurisubharmonic exhaustive function ψ on X of class C^2 (if X is a closed submanifold of $\mathbb{C}^{2n+1}$, we can take for ψ the restriction to X of the function $\|x\|^2$ defined on $\mathbb{C}^{2n+1}$).

Hence X is a Kählerian manifold [Wei58] (taking, for instance, the Kähler metric associated with the closed Kähler form $i\partial\bar{\partial}\psi$).

It is proved in [Ele75] that if we consider a relatively compact Stein open subset Ω of X and the geodesic distance associated with a given Kählerian metric on X, then the function: $-\log d(z, \partial\Omega) + C(\Omega, \psi)\,\psi$, is strictly plurisubharmonic in Ω for a constant $C(\Omega, \psi)$ large enough.

Therefore using [Dem12, Chapter VIII, Paragraph 6, Theorem 6.1 p. 376 and 6.5, p. 378] or [Dem82] the following result (similar to Corollary 2.2) still holds on a Stein manifold (we have to consider the intersection of a given Ω with a chart U of X):

THEOREM 4.1. — *Let Ω and U be two relatively compact open Stein subsets of the Stein manifold X such that $\Omega \cap U \neq \emptyset$. We consider on X a given Kähler form ω, the geodesic distance on X associated with ω and for $z \in \Omega$ the corresponding distance $d(z, \partial\Omega)$ to the boundary of Ω. Let us consider a holomorphic Hermitian vector bundle F of rank r on X and currents with values in F. Let $k \geqslant 0$ be a given real number. Then for every $g \in L_{(p,q+1)}^2(\Omega \cap U, F, loc)$ with $\bar{\partial}g = 0$ such that:*

$\int_{\Omega \cap U} |g|^2 [d(z, \partial\Omega)]^k d\lambda < +\infty$, *there exists* $u \in L^2_{(p,q)}(\Omega \cap U, F, loc)$ *such that:*

$$(4.1) \qquad\qquad\qquad\qquad \bar{\partial}u = g$$

in $\Omega \cap U$ *and:*

$$(4.2) \qquad\qquad \int_{\Omega \cap U} |u|^2 [d(z, \partial\Omega)]^k d\lambda \leqslant C(\Omega, F, k) \int_{\Omega \cap U} |g|^2 [d(z, \partial\Omega)]^k d\lambda,$$

where $d\lambda = \frac{\omega^n}{n!}$ *is the positive measure on* X *defined by the* (n, n) *form* $\frac{\omega^n}{n!}$ *($C(\Omega, F, k)$ is a constant > 0 only depending on Ω, F and k).*

Of course the result particularly holds if $\Omega = U$. Let us give more details about how to deduce Theorem 4.1 from in [Dem12, Theorems 6.1 and 6.5]. At first let us remind Demailly's Theorem 6.5 (for the sake of simplicity we state it with a little more restrictive assumption):

THEOREM 4.2. — *Let* (X, ω) *be a Stein manifold* X *of complex dimension* n *with a given Kähler metric* ω. *Let us consider a holomorphic Hermitian vector bundle* F *of rank* r *on* X *and a* C^∞ *function* ϕ *on* X *such that* $ic(F) + i\partial\bar{\partial}\phi \geqslant \mu\,\omega$ *where* $c(F)$ *is the curvature form of* F *and* $\mu > 0$ *a given constant. Then for every* $g \in L^2_{(n, q+1)}(X, F, loc)$ *with* $\bar{\partial}g = 0$ *such that:* $\int_X |g|^2 e^{-\phi} dV < +\infty$, *there exists* $u \in L^2_{(n, q)}(X, F, loc)$ *such that:*

$$(4.3) \qquad\qquad\qquad\qquad \bar{\partial}u = g$$

in X *and:*

$$(4.4) \qquad\qquad\qquad \int_X |u|^2 e^{-\phi} dV \leqslant \frac{1}{\mu} \int_X |g|^2 e^{-\phi} dV.$$

where $dV = \frac{\omega^n}{n!}$ *is the positive measure on* X *defined by the* (n, n) *form* $\frac{\omega^n}{n!}$.

In [Dem12, Theorem 6.5], F is a line bundle but the result is still valid for a vector bundle : you only need to consider the positivity of the curvature form $ic(F)$ of F in the strong sense of Nakano as explained in [Dem12, Theorem 6.1]. For (p, q)-form (with $p \neq n$ and values in F) we consider (n,q)-forms with values in the new vector bundle $F \otimes \wedge^p T^\star(X) \otimes \wedge^n T(X)$.

If now Ω is a relatively compact open Stein subset of X, we can choose a constant $C_1(\Omega, F)$ such that $ic(F) + C_1(\Omega, F)\, i\partial\bar{\partial}\psi \geqslant \omega$ on $\bar{\Omega}$ (in the strong sense of Nakano). For every C^∞ plurisubharmonic function ϕ on Ω we can apply Theorem 4.2 restricted to the Stein manifold $\Omega \cap U$ and the function $C_1(\Omega, F)\,\psi + \phi$ so that (as ψ is bounded on Ω) we get a solution u of the equation (4.1) which satisfies the estimate:

$$(4.5) \qquad\qquad \int_{\Omega \cap U} |u|^2 e^{-\phi} dV \leqslant C_2(\Omega, F) \int_{\Omega \cap U} |g|^2 e^{-\phi} dV.$$

We can now take $\phi = -k \log d(z, \partial\Omega) + k\, C(\Omega, \psi)\, \psi$, (for a given $k \geqslant 0$) so that (as ψ is bounded on Ω) the estimate (4.5) implies the estimate (4.2) of Theorem 4.1 (for (p,q) forms with values in F). The function $\phi := -k \log d(z, \partial\Omega) + k\, C(\Omega, \psi)\, \psi$ is only continuous on Ω but as ϕ is strictly plurisubharmonic on the Stein manifold Ω it can be closely approximated by a family (ϕ_ϵ) $(0 < \epsilon < \epsilon_0)$ of C^∞ strictly plurisubharmonic functions on Ω as explained in [Dem12, Chapter 1, Paragraph 5.E, p. 42 (Richberg Theorem (5.21))] such that $\phi \leqslant \phi_\epsilon \leqslant \phi + \epsilon$. At first we obtain the estimate (4.5) for

the functions ϕ_ϵ and a solution u_ϵ of (4.1) (in $\Omega \cap U$). Taking the limit as ϵ goes to 0 and using the weak compactness of the closed ball of $L^{2,k}_{(p,q)}(\Omega \cap U, F)$, we get (4.1) and (4.5) for ϕ and a weak limit u of a subsequence of the family (u_ϵ).

We will only use Lemma 2.3 in a local chart of X (i.e. in $\mathbb{C}^n$) : we do'nt need to extend this lemma to the Riemannian Laplacian operator on X with variable coefficients. Replacing $\mathbb{C}^n = \mathbb{R}^{2n}$ by a complex Riemannian manifold X, extension Theorem 2.5 is still valid as it is a local result (alongside the boundary of Ω) using a partition of unity of class C^∞ on X.

To be able to use the same reasoning as in the classical proof of the Dolbeault isomorphism, we have to consider the following sheaves on $\bar{\Omega}$ of currents defined on open subsets of Ω with values in F. For the sake of simplicity we often omit F in the notations (F is fixed). $k \geqslant 0$ be given, we define the sheaf $\widetilde{\mathcal{D}}'_{(p,q)}$, (resp. $\widetilde{\mathcal{D}}'^k_{(p,q)}$), (resp. $\widetilde{\mathcal{O}}_p$), (resp. $\widetilde{\mathcal{O}}^k_p$) on $\bar{\Omega}$. If U is an open subset of X such that $U \cap \bar{\Omega} \neq \emptyset$, $\Gamma(\bar{\Omega} \cap U, \widetilde{\mathcal{D}}'_{(p,q)})$, (resp. $\Gamma(\bar{\Omega} \cap U, \widetilde{\mathcal{D}}'^k_{(p,q)})$), (resp. $\Gamma(\bar{\Omega} \cap U, \widetilde{\mathcal{O}}_p)$), (resp. $\Gamma(\bar{\Omega} \cap U, \widetilde{\mathcal{O}}^k_p)$) is the set of currents $T \in \mathcal{D}'_{(p,q)}(\Omega \cap U, F)$ (resp. the set of holomorphic p-form on $\Omega \cap U$ with values in F) which can be extended locally along the boundary of Ω as a current on X (resp. as a current in $H^{-k}(X, F)$) in the sense that for all $z \in \partial\Omega \cap U$ there exists an open neighborhood V of z in X ($V \subseteq U$), and a current $\widetilde{T} \in \mathcal{D}'(V, F)$ (resp. $\widetilde{T} \in H^{-k}(V, F)$ so that $T = \widetilde{T}$ on $\Omega \cap V$.

Let us observe that if $\phi \in \mathcal{D}(X)$ and if $T \in \Gamma(\bar{\Omega} \cap U, \widetilde{\mathcal{D}}'_{(p,q)})$, (resp. $T \in \Gamma(\bar{\Omega} \cap U, \widetilde{\mathcal{D}}'^k_{(p,q)})$ then $\phi T \in \Gamma(\bar{\Omega} \cap U, \widetilde{\mathcal{D}}'_{(p,q)})$ (resp. $\phi T \in \Gamma(\bar{\Omega} \cap U, \widetilde{\mathcal{D}}'^k_{(p,q)})$ so that the sheaves $\widetilde{\mathcal{D}}'_{(p,q)}$ (resp. $\widetilde{\mathcal{D}}'^k_{(p,q)}$), $0 \leqslant q \leqslant n$, are fine sheaves on $\bar{\Omega}$.

We will also need to use the associated sheaves $\widetilde{\mathcal{D}}'^k_{0,(p,q)}$ on $\bar{\Omega}$ defined by:

$$\Gamma\left(\bar{\Omega} \cap U, \widetilde{\mathcal{D}}'^k_{0,(p,q)}\right) := \left\{ T \in \Gamma\left(\bar{\Omega} \cap U, \widetilde{\mathcal{D}}'^k_{(p,q)}\right) \middle| \bar{\partial} T \in \Gamma\left(\bar{\Omega} \cap U, \widetilde{\mathcal{D}}'^k_{(p,q+1)}\right) \right\}.$$

As for all $\phi \in \mathcal{D}(X)$, we have: $\bar{\partial}(\phi T) = \bar{\partial}\phi \wedge T + \phi\,\bar{\partial}T$, the sheaves $\widetilde{\mathcal{D}}'^k_{0,(p,q)}$, $0 \leqslant q \leqslant n$, are also fine sheaves on $\bar{\Omega}$.

Let us observe that a current $T \in \Gamma(\bar{\Omega}, \widetilde{\mathcal{D}}'_{(p,q)})$ (resp. $T \in \Gamma(\bar{\Omega}, \widetilde{\mathcal{D}}'^k_{(p,q)})$ is the same as a current on Ω which can be extended to X (resp. in $H^{-k}(X, F)$). We consider indeed a finite covering of $\bar{\Omega}$ by open subsets V_j of X, $1 \leqslant j \leqslant N$, and currents $\widetilde{T}_j \in \mathcal{D}'_{(p,q)}(V_j)$ such that $\widetilde{T}_j = T$ in $\Omega \cap V_j$ and a suitable partition of unity on $\bar{\Omega}$ of functions $\phi_j \in \mathcal{D}(V_j)$, $1 \leqslant j \leqslant N$ such that $\sum_{j=1}^N \phi_j(z) = 1$ for all z in a neighborhood of $\bar{\Omega}$ so that $\widetilde{T} := \sum_{j=1}^N \phi_j \widetilde{T}_j \in \mathcal{D}'(X)$ (resp. $\widetilde{T} \in H^{-k}(X, F)$) satisfies $\widetilde{T} = T$ in Ω.

To make short, giving $z \in \Omega$, we set: $d_\Omega(z) := d(z, \partial\Omega)$. Using the geodesic distance to the boundary of Ω, we also define the following corresponding L^2-sheaves $\widetilde{\mathcal{L}}^{2,k}_{(p,q)}$ and (resp. $\widetilde{\mathcal{O}}^{2,k}_p$) on $\bar{\Omega}$. If U is an open subset of X such that $\bar{\Omega} \cap U \neq \emptyset$, $\Gamma(\bar{\Omega} \cap U, \widetilde{\mathcal{L}}^{2,k}_{(p,q)})$ (resp. $\Gamma(\bar{\Omega} \cap U, \widetilde{\mathcal{O}}^{2,k}_p)$) is the set of current $f \in L^2_{(p,q)}(\Omega \cap U, F, loc)$ (resp. the set of holomorphic forms $f \in L^2_{(p,0)}(\Omega \cap U, F, loc)$) such that $\bar{\partial}f \in L^2_{(p,q+1)}(\Omega \cap U, F, loc)$ and such that for all $z \in U \cap \partial\Omega$ there exists an open neighborhood V of z in X ($V \subseteq U$), such that $\int_{\Omega \cap V}[|f|^2 + |\bar{\partial}f|^2]\,[d_\Omega(\zeta)]^{2k}d\lambda(\zeta) < +\infty$ (resp. $\int_{\Omega \cap V}|f|^2\,[d_\Omega(\zeta)]^{2k}d\lambda(\zeta)$

$< +\infty$). Let us observe that we consider on $\Omega \cap V$ the restriction to $\Omega \cap V$ of the distance $d_\Omega(\zeta)$ (to the boundary of Ω) and not the distance $d_{\Omega \cap V}(\zeta)$ (to the boundary of $\Omega \cap V$).

If $\phi \in \mathcal{D}(X)$ and if $f \in \Gamma(\bar{\Omega} \cap U, \widetilde{\mathcal{L}}^{2,k}_{(p,q)})$, then $\phi f \in \Gamma(\bar{\Omega} \cap U, \widetilde{\mathcal{L}}^{2,k}_{(p,q)})$ (as $|\bar{\partial}(\phi f)|^2 = |\phi \bar{\partial} f + \bar{\partial}\phi \wedge f|^2 \leqslant 2|\phi|^2 |\bar{\partial} f|^2 + 2|\bar{\partial}\phi|^2 |f|^2$ and $|\phi|^2, |\bar{\partial}\phi|^2$ are bounded on X) so that the sheaves $\widetilde{\mathcal{L}}^{2,k}_{(p,q)}$, $0 \leqslant q \leqslant n$, are fine sheaves on $\bar{\Omega}$.

Let us also observe that if $f \in \Gamma(\bar{\Omega}, \widetilde{\mathcal{L}}^{2,k}_{(p,q)})$, then $f \in L^{2,k}_{(p,q)}(\Omega, F)$.

Moreover Lemma 2.3 implies that if $f \in \Gamma(\bar{\Omega} \cap U, \widetilde{\mathcal{O}}^k_p)$ then $f \in \Gamma(\bar{\Omega} \cap U, \widetilde{\mathcal{O}}^{2,k+n}_p)$ and for all $U' \subset\subset U$ f has a polynomial growth like $C(\Omega \cap U', f) \, d_\Omega^{-k-n}(z)$ alongside $\partial\Omega \cap U'$ and $\int_{\Omega \cap U'} |f|^2 \, [d_\Omega(z)]^{2k+2n} d\lambda(z) < +\infty$

We can now prove the following result as a consequence of the corresponding theorem proved in the case of $X = \mathbb{C}^n$. Moreover we consider currents in $\mathcal{D}'_{(p,q)}(X, F)$ with values in a given holomorphic vector bundle F (to simplify we only consider current in $H^{-k}(X, F)$, $k \in \mathbb{N}$).

THEOREM 4.3. — *Let Ω be a relatively compact open Stein subset of a Stein manifold X and F be a given Hermitian holomorphic vector bundle on X. Then for every current f of bidegree $(p, q+1)$ on X with values in F (and with compact support in X) which is $\bar{\partial}$-closed on Ω, there exists a current w of bidegree (p, q) on X with values in F (with compact support) such that:*

$$(4.6) \qquad\qquad\qquad\qquad \bar{\partial} u = f,$$

in Ω.

Moreover if $f \in H^{-k}_{(p,q+1)}(X, F, loc))$ for some integer $k > 0$, we can find a solution $u \in H^{-k-r}_{(p,q)}(X, F, loc))$ with $r = (q+2)(2n+1)$.

Proof. — Let us assume that $f \in \mathcal{D}'_{(p,q+1)}(X, F)$. We will prove that we can find a solution $u \in \mathcal{D}'_{(p,q)}(X, F)$. By considering local charts of X, we can locally reduce the problem to the case of $\mathbb{C}^n$. Using local charts on X and Borel–Lebesgue lemma, we can find a finite open covering of the compact set $\bar{\Omega}$ by relatively compact open subsets Ω_j of X, $1 \leqslant j \leqslant N$ such that every $\bar{\Omega}_j$ is contained in a geodesic chart for the given Riemannian metric and every Ω_j is biholomorphic to a bounded open ball $U_j := B_j(z_j, r_j)$ of $\mathbb{C}^n$, by a local biholomorphic map ϕ_j defined on a neighborhood of $\bar{\Omega}_j \subset X$ and taking its values into $\mathbb{C}^n$ ($z_j \in \phi_j(\bar{\Omega})$, $r_j > 0$). Moreover we can also suppose (by shrinking enough each Ω_j) that the exponential map sending the tangent space $T_{z_j} X$ (of X at z_j) into X is a diffeomorphism of an open ball in $T_{z_j} X$ onto a geodesic open ball of center z_j containing $\bar{\Omega}_j$ so that the geodesic distance and the Euclidian distance coming from $\mathbb{C}^n$ (by means of ϕ_j) are equivalent on a neighborhood of $\bar{\Omega}_j$ and so that the spaces $L^{2,k}_{(p,q)}(\Omega \cap \Omega_j, F)$ ($k \in \mathbb{N}$) associated with the geodesic distance to $\partial(\Omega \cap \Omega_j)$ or with the Euclidian distance to $\partial(\Omega \cap \Omega_j)$ coming from $\mathbb{C}^n$ (by means of ϕ_j) are the same. Finally we can also suppose that the given holomorphic vector bundle F is trivial on a neighborhood of each $\bar{\Omega}_j$.

Corollary 1.2 (applied to holomorphic charts of X) means that the following Dolbeault $\bar{\partial}$-complex of sheaves on $\bar{\Omega}$ is exact:

$$(4.7) \quad 0 \to \widetilde{\mathcal{O}}_p \to \widetilde{\mathcal{D}}'_{(p,0)} \xrightarrow{\bar{\partial}} \widetilde{\mathcal{D}}'_{(p,1)} \xrightarrow{\bar{\partial}} \ldots \xrightarrow{\bar{\partial}} \widetilde{\mathcal{D}}'_{(p,q)} \xrightarrow{\bar{\partial}} \widetilde{\mathcal{D}}'_{(p,q+1)} \xrightarrow{\bar{\partial}} \ldots \xrightarrow{\bar{\partial}} \widetilde{\mathcal{D}}'_{(p,n)} \to 0$$

The sheaves $\widetilde{\mathcal{D}}'_{(p,q)}$, $0 \leqslant q \leqslant n$, are fine sheaves on $\bar{\Omega}$. Therefore by considering the long exact sequence of cohomology associated with the sequence (4.7) we have the isomorphism:

$$(4.8) \qquad H^{q+1}\left(\bar{\Omega}, \widetilde{\mathcal{O}}_p\right) \simeq \left\{ T \in \widetilde{\mathcal{D}}'_{(p,q+1)}(\bar{\Omega}) \Big| \bar{\partial} T = 0 \right\} / \left\{ \bar{\partial} S \Big| S \in \widetilde{\mathcal{D}}'_{(p,q)}(\bar{\Omega}) \right\}$$

We use Lemma 2.3 in every chart Ω_j (for holomorphic functions) so that $\Gamma(\bar{\Omega} \cap U, \widetilde{\mathcal{O}}_p)$ is also the space of holomorphic p-forms on $\Omega \cap U$ with polynomial growth for the geodesic distance to the boundary of Ω (U is an open subset of X such that $\bar{\Omega} \cap U \neq \emptyset$).

Corollary 2.2 or Theorem 4.1 (applied to holomorphic charts of X and with the geodesic metric) means that the following Dolbeault $\bar{\partial}$-complex of sheaves on $\bar{\Omega}$ is exact:

$$(4.9) \quad 0 \to \widetilde{\mathcal{O}}_p^{2,k} \to \widetilde{\mathcal{L}}_{(p,0)}^{2,k} \xrightarrow{\bar{\partial}} \widetilde{\mathcal{L}}_{(p,1)}^{2,k} \xrightarrow{\bar{\partial}} \ldots \xrightarrow{\bar{\partial}} \widetilde{\mathcal{L}}_{(p,q)}^{2,k} \xrightarrow{\bar{\partial}} \widetilde{\mathcal{L}}_{(p,q+1)}^{2,k} \xrightarrow{\bar{\partial}} \ldots \xrightarrow{\bar{\partial}} \widetilde{\mathcal{L}}_{(p,n)}^{2,k} \to 0$$

The sheaves $\widetilde{\mathcal{L}}_{(p,q)}^{2,k}$, $0 \leqslant q \leqslant n$, are fine sheaves on $\bar{\Omega}$. Therefore by considering the long exact sequence of cohomology associated with the sequence (4.9) we have the isomorphism:

$$(4.10) \qquad H^{q+1}\left(\bar{\Omega}, \widetilde{\mathcal{O}}_p^{2,k}\right) \simeq \left\{ T \in L_{(p,q+1)}^{2,k}(\Omega) | \bar{\partial} T = 0 \right\} / \left\{ \bar{\partial} S | S \in L_{(p,q)}^{2,k}(\Omega) \right\}$$

But using now Theorem 4.1, the L^2 $\bar{\partial}$-cohomology group in the right member of (4.10) vanishes so that we get: $H^{q+1}(\bar{\Omega}, \widetilde{\mathcal{O}}_p^{2,k}) = 0$. As holomorphic p-forms with values in $\widetilde{\mathcal{O}}_p$ are the same as holomorphic p-form with values in $\widetilde{\mathcal{O}}_p^k$ for some k or the same as holomorphic p-form with values in $\widetilde{\mathcal{O}}_p^{2,k}$ for some k, we have: $H^{q+1}(\bar{\Omega}, \widetilde{\mathcal{O}}_p) = 0$ and using (4.8) that achieves the proof.

To obtain a more precise result we need to work with explicit Čech-cohomology groups $H_{\mathcal{U}}^{q+1}(\bar{\Omega}, \mathcal{G})$ associated with a sheaf $\mathcal{G}$ and with a finite Stein open covering $\mathcal{U}$ of $\bar{\Omega}$ defined by $\mathcal{U} = \{\bar{\Omega} \cap \Omega_j | 1 \leqslant j \leqslant N\}$ in which Ω_j are Stein open subsets of X and to explicitly describe the cochains and the diagram chase as in the chapter "Cohomology with bounds" of L. Hörmander's book [Hör90, Paragraphs 7.3 and 7.4]. We denote by $\delta : C^q(\mathcal{U}, \mathcal{G}) \to C^{q+1}(\mathcal{U}, \mathcal{G})$ the usual coboundary operator acting on cochains in $C^q(\mathcal{U}, \mathcal{G})$, $q \geqslant 0$ which is defined by

$$(\delta c)_I := \sum_{j=0}^{q+1} (-1)^j c_{i_0, \ldots, \hat{i}_j, \ldots, i_{q+1}}$$

if $|I| = q + 1$ with usual notations (if $|I| = q$) $I = (i_0, \ldots, i_j, \ldots, i_q)$, $\Omega_I := \bar{\Omega} \cap \Omega_{i_0} \cap \ldots \cap \Omega_{i_j} \cap \ldots \cap \Omega_{i_q}$ and $c_I \in \Gamma(\Omega_I, \mathcal{G})$. The coboundary operator δ satisfies: $\delta^2 = 0$ and $\bar{\partial}\delta = \delta\bar{\partial}$.

Let us assume that $f \in H_{(p,q+1)}^{-k}(X, F, loc)$. We will prove that we can find a solution $u \in H_{(p,q)}^{-k-r}(X, F, loc)$.

For every bounded Stein open subset $\phi_j(\Omega \cap \Omega_j) \subset \phi_j(\Omega_j) =: U_j \subset^n$ we use the construction made in the case of $\mathbb{C}^n$ (i.e. Remark 3.1, F is trivial on a neighborhood of $\bar{\Omega}_j$) so that we can construct a solution $u_j \in H_{(p,q)}^{-k-2n-1}(\Omega_j, F)$ of $f = \bar{\partial} u_j$ in $\Omega \cap \Omega_j$. The family of currents $f_{|\Omega \cap \Omega_j}$ defines a 0-cochain $c^0 \in C^0(\mathcal{U}, \widetilde{\mathcal{D}}'_{(p,q+1)})$ such that $\delta c^0 = 0$. The family of u_j defines a 0-cochain $c'^0 \in C^0(\mathcal{U}, \widetilde{\mathcal{D}}'_{(p,q)})$ such that $\bar{\partial} c'^0 = c^0$. We have $\bar{\partial}(u_j - u_l) = 0$ in $\Omega \cap \Omega_j \cap \Omega_l$ so that the family of $\bar{\partial}$-closed forms $u_j - u_l$ defines a 1-cochain $c^1 := \delta c'^0$, $c^1 \in C^1(\mathcal{U}, \widetilde{\mathcal{D}}'_{(p,q)})$ such that $\delta c^1 = 0$ and $\bar{\partial} c^1 = 0$ and then an element in the first Čech-cohomology group $H^1(\mathcal{U}, \widetilde{\mathcal{D}}'_{(p,q)})$. Moreover u_j, c'^0 and c^1 can be locally extended in the Sobolev space $H_{(p,q)}^{-k-2n-1}$ along the boundary of Ω. We iterate $q+1$ times this construction solving the equation $\bar{\partial} c'^l = c^l$ and setting $c^{l+1} := \delta c'^l$ so that $\delta c^{l+1} = 0$ and $\bar{\partial} c^{l+1} = 0$ ($\bar{\partial} c^{l+1} = \bar{\partial}(\delta c'^l) = \delta(\bar{\partial} c'^l) = \delta c^l = 0$), $0 \leqslant l \leqslant q$, to obtain a $(q+1)$-cochain $c^{q+1} \in C^{q+1}(\mathcal{U}, \widetilde{\mathcal{D}}'_{(p,0)})$ such that $\delta c^{q+1} = 0$, $\bar{\partial} c^{q+1} = 0$ and then an element in the $(q+1)$-Čech-cohomology group $H^{q+1}(\mathcal{U}, \widetilde{\mathcal{O}}_p)$. The cochain c^{q+1} can be locally extended in the Sobolev space $H_{(p,0)}^{-k-s}$ along the boundary of Ω with $s := (q+1)(2n+1)$. The cochain $c^{q+1} \in C^{q+1}(\mathcal{U}, \widetilde{\mathcal{O}}_p)$ takes its values in holomorphic p-forms and therefore it also takes its values in the sheaf $\widetilde{\mathcal{O}}_{(p,0)}^{2,k+s+n}$ (using Lemma 2.3 in $\mathbb{C}^n = R^{2n}$).

The open subsets of the covering $\mathcal{U}$ and every finite intersection of these open subsets are biholomorphic to Stein bounded open subsets of $\mathbb{C}^n$. Hence using Corollary 2.2, Theorem 4.1 and the L^2 $\bar{\partial}$-complex of sheaves (4.9) (replacing k by $k+s+n$) we have the isomorphism $H^{q+1}(\mathcal{U}, \widetilde{\mathcal{O}}_p^{2,k+s+n}) \simeq H^{q+1}(\bar{\Omega}, \widetilde{\mathcal{O}}_p^{2,k+s+n}) = 0$ (invoking Leray's theorem for acyclic covering or using direct diagram chase in (4.9)) so that we can solve the equation $\delta c''^q = c^{q+1}$ with c''^q taking its values in holomorphic p-forms and in the same sheaf $\widetilde{\mathcal{O}}_p^{2,k+s+n}$ (as c^{q+1}). Using Remark 2.6, c''^q takes its values in $\widetilde{\mathcal{O}}_p^{-k-r}$ (as $s + n + n + 1 = (q+1)(2n+1) + 2n + 1 = r$).

We have $c^{q+1} = \delta c'^q = \delta c''^q$ so that we get $\delta(c'^q - c''^q) = 0$ and $\bar{\partial}(c'^q - c''^q) = c_q$ (as $\bar{\partial} c'^q := c^q$ and c''^q takes its values in holomorphic p-forms).

We will now consider the sheaves $\mathcal{G} = \widetilde{\mathcal{D}}'^r_{0,(p,m)}$, $0 \leqslant m \leqslant n$. They are fine sheaves so that we have $H^l(\mathcal{U}, \mathcal{G}) = 0$, $1 \leqslant l \leqslant n$, i.e. for every l-cochain c such that $\delta c = 0$ we can find a (l-1)-cochain c' such that $\delta c' = c$ (c.f [Hör90, Proposition 7.3.3, p. 174 and 175]). Hence we can solve the following δ equation: $\delta(c''^{q-1}) = c'^q - c''^q$ so that $\bar{\partial} \delta(c''^{q-1}) = \bar{\partial}(c'^q - c''^q) = c^q = \delta c'^{q-1}$, i.e. $\delta[c'^{q-1} - \bar{\partial}(c''^{q-1})] = 0$. We iterate solving successively the δ equations $\delta(c''^{l-1}) = c'^l - \bar{\partial}(c''^l)$ so that $\bar{\partial} \delta(c''^{l-1}) = \bar{\partial}(c'^l - \bar{\partial}(c''^l)) = c^l := \delta c'^{l-1}$, i.e. $\delta[c'^{l-1} - \bar{\partial}(c''^{l-1})] = 0$, $1 \leqslant l \leqslant q-1$. For $l = 1$ we obtain $\delta[c'^0 - \bar{\partial}(c''^0)] = 0$, i.e. the 0-cocycle $c'^0 - \bar{\partial}(c''^0)$ defines a global current $u \in \Gamma(\bar{\Omega}, \widetilde{\mathcal{D}}'^r_{(p,q)})$ (i.e $u \in H_{(p,q)}^{-r}(X)$) such that $\bar{\partial}(c'^0 - \bar{\partial}(c''^0)) = \bar{\partial} c'^0 := c^0$, i.e. $\bar{\partial} u = f$ in Ω.

To sum up constructing u needs to solve the $\bar{\partial}$ equation $q+1$ times with a loss of $2n+1$ of regularity at each step and to use Remark 2.6 and Lemma 2.3 with still a loss of $2n+1 = n+n+1$. Then we get a solution u of $\bar{\partial} u = f$ in Ω such that u can be extended to X in $H^{-k-r}(X)$ ($r = (q+2)(2n+1)$). If f is of bidegree $(p,1)$ ($q = 0$) we get $u \in H^{-k-2(2n+1)}(X)$. $\qquad \square$

As explained in P. Schapira's article [Sch21, Remark 2.3.4], Theorem 4.3 implies the following result ([Sch21, Theorem 2.3.3]). We refer to [Sch21] for the definitions of the (derived) sheaf $\mathcal{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ of temperate holomorphic functions (defined on the subanalytic site X_{sa}) and of other objects associated with.

THEOREM 4.4 (P. Schapira). — *Let X be a complex Stein manifold and let Ω be a subanalytic relatively compact Stein open subset of X contained in a Stein compact subset K of X. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module defined on a neighborhood of K. Then $\mathrm{R}\Gamma(\Omega; \mathcal{F}^{\mathrm{tp}})$ is concentrated in degree 0.*

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